OPTIMAL PRICING AND BASE STOCK DETERMINATION IN A MULTICLASS M/G/1 MAKE-TO-STOCK QUEUE

by

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ABSTRACT

OPTIMAL PRICING AND BASE STOCK DETERMINATION IN A MULTICLASS M/G/1 MAKE-TO-STOCK QUEUE

In this study, the problem of profit maximization of a single-server multiclass queuing system is analyzed. Customer demands follow independent Poisson processes and service times have general distribution. The demand is assumed to decrease linearly as the price increases. For each arrival, an order is placed to the server and a single queue is formed under FIFO discipline. If the on-hand inventory is not enough to satisfy the demand, then unsatisfied orders are backordered. Different backorder costs and price levels are considered for each customer class. Holding cost is incurred for the on-hand items in the inventory. Firstly, model formulation of the average profit function is developed. It is analyzed under an M/M/1 make-to-stock system. Then, optimal base stock level of an M/G/1 make-to-stock system is formulated. The dynamics of profit functions of M/M/1 make-to-stock systems are analyzed by trying different base stock levels under a single price. A convex approximation of the sum of holding and backorder costs over arrival rates is proposed to obtain optimal pricing and base stock level. Base stock levels are determined by using approximate optimal arrival rates. An algorithm is proposed to improve the results of the approximation. Approximations are tested under various parameters and service time distributions.

ÖZET

BİR STOK İÇİN ÜRETİM YAPAN ÇOK SINIFLI M/G/1 KUYRUĞUNDA EN İYİ FİYATLAMA VE TEMEL STOK SEVİYESİNİN BELİRLENMESİ

Bu çalışmada tek üreticili çok sınıflı kuyruk sisteminin kar maksimizasyon problemi incelendi. Talepler bağımsız Poisson süreçlerine göre gerçekleşir ve üretim zamanı genel dağılımdadır. Taleplerin fiyat arttıkça doğrusal olarak azaldığı varsayılır. Her talep için üreticiye bir sipariş gönderilir ve FIFO disiplini altında bir kuyruk oluşur. Eğer envanter talep için yeterli değilse talep sonradan karşılanır. Her müşteri sınıfı için ayrı sonradan karşılanma bedeli ve fiyat seviyeleri göz önünde bulundurulmuştur. Envanterdeki ürünler için stokta tutma maliyetleri uygulanır. Öncelikle kar fonksiyonunun model formulasyonu geliştirildi ve stok için üretim yapan M/M/1 sistemi altında incelendi. Sonra stok için üretim yapan M/G/1 sisteminin en iyi temel stok seviyesi formule edilmiştir. Tek fiyat altında değişik temel stok seviyeleri kullanılarak stok için üretim yapan M/M/1 sistemlerinin dinamiği incelenmiştir. Stokta tutma ve sonradan karşılanma maliyetleri toplamının konveks olarak yaklaşık değerinin bulunması önerilmiştir. Bunun sonucunda konkav bir kar fonksiyonu elde edilir. Bu yolla yaklaşık en iyi varış oranları elde edilir. Temel stok seviyeleri de bu yaklaşık en iyi varış oranları kullanılarak elde edilir. Daha iyi yaklaşık sonuçlar bulabilmek için bir algoritma önerilmiştir. Yaklaşık değerler çeşitli parametreler ve üretim zamanı dağılımları altında test edilmiştir.

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i	Type of a customer class
n	Number of customer class types
h	Unit holding cost per unit time for the products in the inventory
I	Inventory level
b_i	Unit backorder cost per unit time for unsatisfied demands for customer class i
B_i	Backorder level for customer class i
B	Backorder level
S	Base stock level
N	Number of orders to server(including the one at the service)
p_i	Price of the product for customer class i
k_i	Maximum arrival rate of customer class i
m_i	Decrease in arrival rate of customer class i as price increases by one
K	Maximum arrival rate
M	Decrease in arrival rate as price increases by one
λ_i	Arrival rate of customer class i
λ	Sum of arrival rates
μ	Service rate
ρ	Traffic intensity
Θ_i	The probability that an arrival is from customer class i

1. INTRODUCTION

This thesis considers the problem of profit maximization of a single-server multi-class queuing system with Poisson arrivals and the service times with general distributions under FIFO discipline. Demand for each customer class depends on the price of the product. Demand is assumed to decrease linearly as the price increases. Backorder cost is incurred when the inventory is not enough for the demands. Backorder cost changes according to the type of customer classes. Holding cost is incurred for the on-hand inventory.

Increasing globalization and recent achievements of the internet technology have increased the importance of evaluating pricing decisions more frequently in the competitive markets. However, setting the right price has always been a very crucial decision for companies to obtain the profitability. But, it is not enough to obtain the highest profit in some systems. Setting the right price at a right base stock level is essential for profit maximization. Price determines both revenue and system costs. Base stock level also determines associated holding and stockout costs. Thus, joint pricing and determining base stock level is required for the profit maximization of such systems.

In a single server manufacturing facility, generally same backorder cost is used for all customer classes. However, in reality it may not be the same. Agreements with customers do not have to be same since different backorder costs may be incurred to each of customer classes. Charging different customers different prices may be preferred especially in different markets. Thus, considering different backorder costs is very essential for meeting such possible conditions.

In Chapter 3, a general multi-item model of M/G/1 make-to-stock queuing systems with different backorder costs and price levels is proposed. Then, the base model is dropped to a submodel that different customer classes are charged with same price. Stability requirements of this submodel are analyzed. The computation of steady state probabilities of M/G/1 systems are presented for the calculation of the profit functions. Then, a Multiclass M/M/1 Make-to-Stock Queuing system and the dynamics of its profit function are analyzed. Optimal base stock level of the base model with predetermined price levels is formulated. Optimal base stock level for an M/M/1 Make-to-Stock Queue is also formulated. Continuous optimal base stock level of an M/M/1 system is found. An approximation of the profit function is proposed by assuming the number of orders in the system is exponentially distributed. Even if it is difficult to compute the distribution of the number of orders in the system, it allows us to find an approximation of the profit function. Optimal base stock level for this approximation is also formulated. As well as the base model; the loss sales model, in which the number of customers cannot exceed the base stock level, is also analyzed. The optimal price levels of M/M/1 Make-to-Stock Queuing systems are analyzed for various base stock levels.

In Chapter 4, an approximation of the sum of backorder and holding costs as an appropriate convex function of traffic intensity ρ is proposed to sustain the concavity of the profit function. In order to determine the optimal prices and the base stock level jointly, firstly base stock level is determined as a function of price or demand. By using some initial values of demand, parameters of the approximate cost function are found. By using the concavity of the approximated profit function, approximated optimal demand values are determined. Also a linear part is added to the approximation. Additionally, the approximation with the highest profit value is defined as the third approximation. The quality of the approximations is tested by considering numerous scenarios.

In this thesis, we make contributions to determine the prices and the base stock levels jointly in an M/G/1 make-to-stock queuing environment with different price levels and backorder costs for each customer class. We provide optimal base stock levels as a function of prices, thus only optimal prices are needed to obtain optimal profit. We approximate cost function as a convex function of demand and obtain a concave profit of function of demand and obtain optimal prices.

This thesis is organized as follows: Chapter 2 gives literature survey about pricing and inventory control. In Chapter 3, we provide the problem description, model formulations in detail with corresponding assumptions. Approximations for the optimal price and testing quality of the approximations are provided in Chapter 4. In Chapter 5, we summarize our thesis. We provide an appendix for the proofs.

2. LITERATURE REVIEW

In this chapter, we present a literature review on related aspects of the problem. We review the papers that consider pricing, inventory control, especially the ones that consider joint pricing and inventory control of manufacturing systems. Even though we consider profit maximization of M/G/1 queuing systems, we did not restricted our survey on queuing models and also reviewed papers that consider newsvendor problems.

Rubio and Wein [1] provide a model for determining base stock levels of manufacturing systems that produce a variety of products in a make-to-stock mode. They also investigate single and multi product lost sale cases. Benjaafar *et al.* [2] offer a solution procedure to obtain optimal demand allocations and optimal inventory base stock levels of make-to-stock systems. They consider different customer classes that may vary in their demand rates, and backorder penalties.

Maoui *et al.* [3] focus on optimal pricing that maximizes the long-run average profit per unit time by investigating how optimal prices vary as system parameters change. They consider a single-server queuing system with Poisson arrivals with limited and unlimited buffer size. Backorder cost is not used in their model. Karabati *et al.* [4] present a model to approximate base stock levels in a make-to-stock M/M/1 system by replacing distribution of the number in the system(geometric distribution) by an exponential distribution. They assume price-demand curve is linear. The approximated base stock level is placed in the profit function and a numerical analysis is proposed to obtain approximated optimum price.

Petruzzi and Dada [5] provide a model for joint pricing and determining inventory level of a newsvendor problem. They consider both additive and multiplicative forms of random price-dependent demand functions. They first analyze the problem in single period, then extend it to a multiple period problem that the units left over from one period are available to meet demands in subsequent periods. Federgruen and Heching [6] study the simultaneous determination of pricing and inventory replenishment strategies of periodic review model. Unsatisfied demands are fully backordered and no setup cost is incurred in each period. They show joint concavity of the profit functions over price and base stock level. They claim the existence of an optimum inventory level and price pair when a linear demand function is used. Chen and Simchi-Levi [7] focus on pricing and inventory replenishment of such a multiperiod system by fully backordering unsatisfied demands, but setup cost is also incurred.

Yao *et al.* [8] focus on the pricing of a newsvendor problem. They divide the demand into two parts as mean demand and random demand. They prove that if the mean demand has increasing pricing elasticity and random demand has generalized strict increasing failure rate, then the expected profit is quasi-concave. Nahapetyan *et al.* [9] consider solving capacitated multi-item dynamic pricing problems. They provide a nonlinear mixed-integer formulation of the problem and they simplify it by some linearization techniques. Then, they propose a heuristic algorithm that maximizes the profit.

Yeng and Chen consider a joint pricing and Inventory Control problem. In one of their papers [10], they analyze the problem by using (s, d, D, S) policy as the pricing reference levels. According to their policy, the product is sold at high price if the inventory level is between d and D, it is sold at low price otherwise. They extend their model by adding extra reference levels and multiple prices in such a way that prices decrease as inventory levels reaches s or S. They allow for dynamically varying prices and they propose a policy that finds a class of pricing and inventory policies which is optimal among all policies. In their another paper [11] , they establish the optimality of (s, S, p) policy that the stock is replenished according to a (s, S) ordering policy and price is inventory-level dependent. Shi and Zhang [12] also consider joint pricing and Inventory Control problem. But they maximize the profit subjected to a budget constraint of inventory and assume that the demand is price-sensitive and stochastic. They propose a mixed integer nonlinear programming model and develop a Lagrangian based solution.

Gayon *et al.* [13] focus on stock rationing of make-to-stock queuing systems. In their study, a single item is demanded by several customer classes according to Poisson processes and service times are distributed with Erlang distribution. They consider different backorder costs for different customer classes. Their objective is to find the optimal stock level that minimizes the sum of backorder and holding costs.

Table 2.1. Summary of Literature Survey.

	Inventory Control	Pricing	Queuing System	Multi Class Customer
Rubio and Wein [1]	X		X	
Benjaafar *et al.* [2]	X		X	
Maoui *et al.* [3]		X	X	
Karabati *et al.* [4]	X		X	
Petruzzi and Dada [5]	X	X		
Federgruen and Heching [6]	X	X		
Chen and Simchi-Levi [7]	X	X		
Yao *et al.* [8]		X		
Nahapetyan *et al.* [9]		X		
Yeng and Chen [10]	X	X		
Yeng and Chen [11]	X	X		
Shi and Zhang [12]	X	X		
Gayon *et al.* [13]	X		X	X
This thesis	X	X	X	X

In this study, we are interested in inventory control and pricing for a queuing system that considers multiclass customers for a single item. Table 2.1 is formed according to four important topics related to our thesis.

We consider M/G/1 make-to-stock systems with $(S-1,S)$ type of policy that whenever a satisfied demand occurs, an order is placed at the same time. Production continues until on-hand inventory reaches to the base stock level. We use a linear decreasing demand function of price as used in many other papers. Considering both prices and backorder cost differences in different customer classes makes this thesis essential for meeting some conditions of the manufacturing systems. Joint pricing and base stock level determination is provided by approximating the cost function to a convex function of demand.

3. MODEL FORMULATION

3.1. Model Formulation of M/G/1 Queuing Systems

We consider a manufacturing facility that produces a single product and supplies demands arising from n different customer classes. We assume that the demand of each customer class i follows a Poisson process with rate λ_i, $i = 1, 2, ...n$. They form a single queue with an arrival rate of $\lambda = \sum_{i=1}^{n} \lambda_i$. Production process is modeled as a single server queue with an infinite waiting capacity. We use $(S-1,S)$ type of inventory policy that whenever a demand occurs, an order is placed at the same time. If the demand is satisfied, then we have positive inventory level and no customers wait in the system. If the on-hand inventory level is not enough to satisfy the demand, then the customers are backordered. These backordered customers are satisfied under FIFO discipline. Backordering perpetuates until having a nonnegative inventory level. Manufacturing times are i.i.d and distributed under general distributions with a mean denoted by $1/\mu$. We assume different prices are set according to the type of customer classes. Stability is sustained by assuming the traffic intensity is less than 1, in other words, the arrival rate is less than the service rate($\rho = \lambda/\mu < 1$ or $\lambda < \mu$).

We assume that increase in the price causes a linear decrease in the arrival rate. Equation 3.1 gives the demand function of the model.

$$\lambda_i(p_i) = k_i - m_i p_i \tag{3.1}$$

Maximum demand of customer class i is denoted by k_i. If the price increases by one, unit decrease in the arrival rate of customer class i is denoted by m_i. This function gives average number of demands if the price is set to p_i. Since the prices and the arrival rates must be nonnegative, and the sum of the arrival rates must be less than the service rate μ; arrival rates should satisfy following condition:

$$0 \leq \lambda_i(p_i) \leq \min(k_i, \mu) \tag{3.2}$$

Corresponding conditions that price should satisfy can be shown as following:

$$\max(0, \frac{k_i - \mu}{m_i}) \leq p_i \leq \frac{k_i}{m_i} \tag{3.3}$$

If all customer classes are considered, it results in $\sum_{i=1}^{n} \lambda_i(p_i) \leq \sum_{i=1}^{n} k_i$ or $\lambda \leq K$ requirement. Since the sum of the arrival rates should be less than the service rate μ, arrival rate should satisfy following condition:

$$0 \leq \lambda \leq \min(K, \mu) \tag{3.4}$$

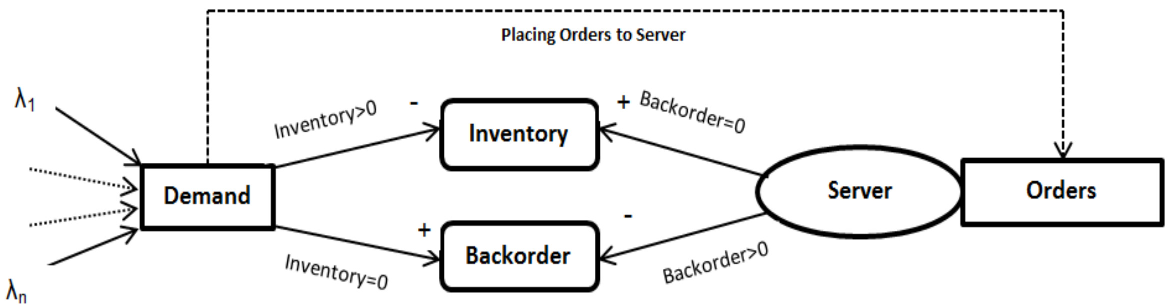


Figure 3.1. Make-to-Stock System with Backorder.

A make-to-stock system with backorders can be depicted as in Figure 3.1. Initially, inventory level is set to base stock level S. Whenever a demand occurs; an order is placed to the server, and the number of orders N increases by 1. When the server finishes an order, N decreases by 1. If there are enough products in the inventory, the demand is satisfied, otherwise it is backordered. As the server finishes the orders, possible backorders are also to be satisfied under FIFO discipline. The server resumes the production until having a net inventory of S. If there are N orders placed to the server, the net inventory is computed as $S-N$ and it is conserved in all cases. If the net inventory is positive, holding cost incurs for each product in the inventory. On-hand inventory is computed as:

$$I = (S - N)^+$$

If the net inventory is negative, then the backorder cost incurs for each backordered customer. Backorder level is computed as :

$$B = (N - S)^+$$

The number of backorders of a customer class i is denoted as B_i. Total backorder level can be expressed as $B = \sum_{i=1}^{n} B_i$. Holding cost for each product in the inventory per unit time is denoted by h. Backorder cost changes according to the type of customer classes. Backorder cost for class i per item per unit time is denoted by b_i. Expected holding cost is computed as $hE[(S - N)^+]$. Expected backorder cost is computed as $\sum_{i=1}^{n} b_i E[B_i]$. Expected revenue of the model is computed as $\sum_{i=1}^{n} \lambda_i(p_i)p_i$. Given base stock level S, price p_i for all classes, the general formula for the expected profit is computed as:

$$\Pi(p_i, S) = -\left[\sum_{i=1}^{n} b_i E[B_i] + hE[(S - N)^+]\right] + \sum_{i=1}^{n} \lambda_i(p_i)p_i \tag{3.5}$$

It is possible to obtain a simpler form of Equation 3.5 by simplifying $\sum_{i=1}^{n} b_i E[B_i]$ term. Note that, backordered customers are satisfied under FIFO discipline. The probability that an arrival is from customer class i can be determined as following:

$$\Theta_i = \frac{\lambda_i}{\lambda} \tag{3.6}$$

Let $X_{k,i}$ be a Bernoulli random variable whose value is 1, if an arrival k is from customer class i, 0 otherwise.

$$X_{k,i} = \begin{cases} 1 & \text{if} \quad k = i \\ 0 & \text{if} \quad k \neq i \end{cases}$$

Given that the backorder level is B, the backorder level for customer class i can be computed as $B_i = \sum_{k=1}^{B} X_{k,i}$. When we take expectations we get following results:

$$\begin{aligned} E[B_i] &= E\left[E[B_i|B]\right] \\ &= E\left[E[\sum_{i=1}^{B} X_{k,i}|B]\right] \\ &= E\left[B\Theta_i\right] \\ &= E[B]\Theta_i \end{aligned}$$

$E[B]\Theta_i$ term can be used instead of $E[B_i]$ variable. Buzacott and Shanthikumar use it in one of their books. [14] Thus, the general expected profit function can be expressed as follows:

$$\Pi(p_i, S) = -\left[\sum_{i=1}^{n} b_i\Theta_i E[(N-S)^+] + hE[(S-N)^+]\right] + \sum_{i=1}^{n} \lambda_i(p_i)p_i \qquad (3.7)$$

The profit function expressed in Equation 3.7 has $n+1$ inputs. n inputs for prices for all customer classes and 1 input for base stock level are used. Instead of using p_i values as inputs, corresponding λ_i values can be used by using Equation 3.1. It is also possible to use λ and Θ_i values as inputs by using Equation 3.6.

3.2. Using Single Price in the General Model

The general model is simplified by incurring same price to all customer classes. Arrival rates of all customer classes depend on only a single price. But the model still considers different backorder costs for each customer class.

Using single price leads to some important requirements that the arrival rate of any customer class i is nonnegative $\lambda_i = ki - m_i p \geq 0$. For each customer class i, a feasible price is obtained by $p \leq \frac{k_i}{m_i}$ restriction. If this requirement is considered for all customer classes, the price levels should not be more than any of those $\frac{k_i}{m_i}$ values. It can be expressed as follows:

$$p \leq \min_{i=1..n}\left(\frac{k_i}{m_i}\right)$$

Let define $K = \sum_{i=1}^{n} k_i$ and $M = \sum_{i=1}^{n} m_i$. Sum of all arrival rates arising from n customer classes can be computed as $\lambda = K - Mp$. If the arrival rate restriction $\lambda \leq \min(K, \mu)$ is considered, $p \geq \max(0, \frac{K-\mu}{M})$ restriction is obtained. So, the price levels should satisfy following condition:

$$\max(0, \frac{K-\mu}{M}) \leq p \leq \min_{i=1..n}\left(\frac{k_i}{m_i}\right)$$

Corresponding conditions that the arrival rate should satisfy can be expressed as follows:

$$K - M \min_{i=1..n}\left(\frac{k_i}{m_i}\right) \leq \lambda \leq \min(K, \mu)$$

When these stability requirements are satisfied, it is logical to express p and λ in terms of traffic intensity ρ for simplicity. Determining optimal ρ provides finding optimal price since they can be converted to each other.

$$\lambda = K - Mp \Rightarrow p = \frac{K}{M} - \frac{\lambda}{M} \Rightarrow p = \frac{K}{M} - \frac{\mu}{M}\rho$$

If single price is incurred, revenue function becomes simpler as λp. So corresponding profit function turns out to be as following:

$$\Pi(p, S) = -\left[\sum_{i=1}^{n} b_i \Theta_i E[(N-S)^+] + hE[(S-N)^+]\right] + \lambda p \qquad (3.8)$$

Profit function expressed in Equation 3.8 has only two inputs that the profit depends on only price and base stock level. It can also be computed by arrival rates or traffic intensities by a simple conversion as explained previously.

3.3. Steady State Probabilities of M/G/1 Systems

Steady state probabilities of M/G/1 systems enable us to find $E[(N-S)^+]$ and $E[(S-N)^+]$ values in order to compute the general profit function. If the number of steady state probabilities is sufficient, then the expectations can be accurately calculated. Markov chains of M/G/1 systems are analyzed in order to calculate these steady state probabilities.

State transitions occur during a service time with some arrivals. If there are no customers in the system, then j arrivals occur in a transition of $(0, j)$. Because service cannot be completed and only arrivals determine the number of the orders in system. If there are some customers in the system, then $j-r+1$ arrivals occur in a transition of (r, j) with $(r > 0)$ arrivals are added and one departure are subtracted from the number of customers in the system.

Table 3.1. Matrix of Transition Probabilities of M/G/1 Systems.

	0	1	2	3	4	...	j	...
0	q_0	q_1	q_2	q_3	q_4	...	q_j	...
1	q_0	q_1	q_2	q_3	q_4	...	q_j	...
2	0	q_0	q_1	q_2	q_3	...	q_{j+1}	...
3	0	0	q_0	q_1	q_2	...	q_{j+2}	...
4	0	0	0	q_0	q_1	...	q_{j+3}	...
.	.	.	.	.	.	$\ddots$	.	...
r	0	0	0	0	0	...	q_{j-r+1}	...
.	.	.	.	.	.	...	.	$\ddots$

Probability that an M/G/1 system to be idle is $\pi_0 = 1-\rho$. Probability that j customers arrive during a service time is denoted as q_j. Probability transition matrix is depicted at Table 3.1. Letting the probability density function of service time be

$f_s(t)$, q_j can be computed as following:

$$q_j = \int_{t=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} f_s(t) dt$$

Steady state probabilities are computed by a formula of $\pi = \pi P$ as followings:

$$\begin{aligned} \pi_j &= \pi_0 q_j + \pi_{(j+1)} q_0 + \sum_{r=1}^{j} \pi_r q_{(j+1-r)} \\ \pi_{(j+1)} &= \frac{\pi_j - \pi_0 q_j - \sum_{r=1}^{j} \pi_r q_{(j+1-r)}}{q_0} \end{aligned} \tag{3.9}$$

Since π_0 is known as $1 - \rho$, any π_j probability is found recursively by Equation 3.9. If the steady state probabilities are known, inventory and backorder levels can be computed as followings:

$$\begin{aligned} E[(S-N)^+] &= \sum_{j=0}^{S} (S-j)\pi_j \\ E[(N-S)^+] &= \sum_{j=S}^{\infty} (j-S)\pi_j \end{aligned}$$

Computing S steady state probabilities is enough to compute the expected inventory level. But all steady state probabilities should be computed for computing the exact value of the expected backorder level $E[(N-S)^+]$. It can be approximated by using sufficient number of steady state probabilities as following:

$$E[(N-S)^+] \approx \sum_{j=S}^{C} (j-S)\pi_j$$

The number of steady state probabilities C, can be determined by a small toleration parameter denoted as tol. The complementary cumulative distribution should be less than or equal to toleration. In other words, cumulative distribution should be more than or equal to $1 - tol$. The value of C variable can be determined by following formula:

$$C = \arg\min_{x \in \mathbb{N}} \{F(x) : F(x) \geq 1 - tol\}$$

3.4. Model Formulation of M/M/1 Make-to-Stock Queuing Systems

Using exponentially distributed service time in the model is a well-known example of the queuing systems. In order to compute the profit function under M/M/1 make-to-stock queuing systems, $E[B]$ and $E[(S-N)^+]$ values are required. These expectations can be calculated without using the steady state probabilities.

Since the service times are exponentially distributed, the number of orders in the system is geometrically distributed. This information enables us to determine the generating functions of backorder and inventory levels. If the derivatives of generating functions are obtained, expectations can be easily computed by setting 1 to their inputs. These calculations are shown in APPENDIX A and B. Corresponding expectations are found as followings:

$$E[B] = \frac{\rho^{S+1}}{1-\rho} \tag{3.10}$$

$$E[(S-N)^+] = S - \frac{\rho(1-\rho^S)}{1-\rho} \tag{3.11}$$

By using the values of $E[(S-N)^+]$ and $E[B]$ expressions, expected profit of an M/M/1 queuing system can be calculated as followings:

$$\begin{aligned}
\Pi(p_i, S) &= -\left[\sum_{i=1}^{n} b_i \Theta_i E[B] + hE[(S-N)^+]\right] + \sum_{i=1}^{n} \lambda_i(p_i) p_i \\
&= -\left[\sum_{i=1}^{n} b_i \Theta_i \frac{\rho^{S+1}}{1-\rho} + h(S - \frac{\rho(1-\rho^S)}{1-\rho})\right] + \sum_{i=1}^{n} \lambda_i(p_i) p_i \\
&= -\left[(\sum_{i=1}^{n} b_i \Theta_i + h)\frac{\rho^{S+1}}{1-\rho} + h(S - \frac{\rho}{1-\rho})\right] + \sum_{i=1}^{n} \lambda_i(p_i) p_i
\end{aligned} \tag{3.12}$$

If multiple prices are used, the profit function can also be defined by $\Theta_1..\Theta_n, \lambda, S$ parameters. λ_i values can be computed by $\Theta_i\lambda$ calculation. The prices can be computed from $\lambda_i = k_i - m_i p_i$ formula. The first and the second derivatives of the backorder cost part are calculated by taking $\sum_{i=1}^{n} b_i\Theta_i$ as a constant value.

$$\frac{\partial(-\sum_{i=1}^{n} b_i\Theta_i E[(N-S)^+])}{\partial\rho} = -\sum_{i=1}^{n} b_i\Theta_i \frac{\rho^S(S(1-\rho)+1)}{(1-\rho)^2}$$

$$\frac{\partial^2(-\sum_{i=1}^{n} b_i\Theta_i E[(N-S)^+])}{\partial\rho^2} = -\sum_{i=1}^{n} b_i\Theta_i \frac{\rho^{S-1}(S^2(1-\rho)^2+S(1-\rho^2)+2\rho)}{(1-\rho)^3}$$

Backorder cost part of the profit function is a decreasing concave function of ρ since the first and the second derivatives are nonpositive.

If single price is used, λ_i values are also computed from the same price as in $\lambda_i = k_i - m_i p$ calculation with single price $p = \frac{K-\lambda}{M}$. Any Θ_i value is not predetermined, because it is a function of λ. Also, $\sum_{i=1}^{n} b_i\Theta_i$ is a function of λ. It does not have a constant value. Thus, the backorder cost part becomes more complicated and the concavity is not obtained.

The first and the second derivative of the holding cost part are calculated as followings:

$$\frac{\partial(-hE[(S-N)^+])}{\partial\rho} = -h\frac{(S+1)\rho^S - S\rho^{S+1} - 1}{(1-\rho)^2}$$

$$\frac{\partial^2(-hE[(S-N)^+])}{\partial\rho^2} = -h\frac{S(S-1)\rho^{S+1} - 2(S-1)(S+1)\rho^S + S(S+1)\rho^{S-1} - 2}{(1-\rho)^3}$$

The value of $-hE[(S-N)^+]$ expression is increasing in ρ since its first derivative is nonnegative. It can be seen by letting $f(\rho) = (S+1)\rho^S - S\rho^{S+1} - 1$. $f(\rho)$ is a nondecreasing function since its first derivative is nonnegative.

$$f'(\rho) = \rho^{S-1}S(S+1)(1-\rho) \geq 0$$

Maximum value of $f(\rho)$ function is found by setting $\rho = 1$. Since the maximum value of $f(\rho)$ function is $f(1) = 0$; $f(\rho)$ is a nonpositive function and $\frac{\partial(-hE[(S-N)^{+}])}{\partial\rho}$ is a nonnegative function. Thus $-hE[(S-N)^{+}]$ is an increasing function of ρ.

The convexity of the $-hE[(S-N)^{+}]$ value can be found in a similar way. Let $g(\rho) = S(S-1)\rho^{S+1} - 2(S-1)(S+1)\rho^{S} + S(S+1)\rho^{S-1} - 2$. $g(\rho)$ is an increasing function since its first derivative is a nonnegative function.

$$g'(\rho) = S(S-1)(S+1)\rho^{S-2} \geq 0$$

Since the maximum value of $g(\rho)$ function is $g(1) = 0$, $g(\rho)$ is a nonpositive function and $\frac{\partial^2(-hE[(S-N)^{+}])}{\partial\rho^2}$ is a nonnegative function. Thus, $-hE[(S-N)^{+}]$ is a convex function of ρ. The cost part of the profit function is the sum of convex and concave functions. When the profit function is analyzed, no convexity or concavity is obtained with respect to ρ.

3.5. Base Stock Levels

3.5.1. Optimal Base Stock Levels of M/G/1 Systems

Determining optimal base stock level is an important decision to maximize the profit function if the prices for all customer classes are given. Since the base stock level S is an integer value, computing the second difference of the general expected profit function is required to be sure whether the function is concave over S. Concavity of the profit function and optimal solutions are shown in APPENDIX C. When the first and the second differences are defined as $\Delta\Pi(S) = \Pi(S+1) - \Pi(S)$ and $\Delta^2\Pi(S) = \Delta\Pi(S+1) - \Delta\Pi(S)$ respectively, they can be computed as followings:

$$\begin{aligned} \Delta\Pi(S) &= \sum_{i=1}^{n} b_i\Theta_i - P(N \leq S)\left[\sum_{i=1}^{n} b_i\Theta_i + h\right] \\ \Delta^2\Pi(S) &= -P(N = S)\left[\sum_{i=1}^{n} b_i\Theta_i + h\right] < 0 \end{aligned}$$

Concavity of the profit function holds since the second difference is nonpositive. It allows us to find the optimal base stock level by selecting the first S value that makes the first difference of the profit function nonpositive. It results in an inequality of $\Delta\Pi(S) \leq 0$. After some algebra, following inequality is obtained:

$$P(N \leq S) \geq \frac{\sum_{i=1}^{n} b_i \Theta_i}{\sum_{i=1}^{n} b_i \Theta_i + h} \tag{3.13}$$

Optimal base stock level of a make-to-stock system can be found by the cumulative distribution of the number of the orders in the system ($F(S) = P(N \leq S)$). Since the cumulative distribution function is increasing, optimal base stock level can be easily found by summing steady state probabilities one by one until $F(S)$ is greater or equal to critical value of $\frac{\sum_{i=1}^{n} b_i \Theta_i}{\sum_{i=1}^{n} b_i \Theta_i + h}$.

This critical value is sensitive to the price levels for different customer classes since Θ_i is a function of price. It is also sensitive to the unit backorder costs. If single backorder cost rate is used as $b_i = b$ for $i = 1..n$, the optimal base stock level should be selected as the first S value that satisfies following condition:

$$P(N \leq S) \geq \frac{b}{b + h}$$

However, this inequality ignores both backorder cost and price differences, since the sum of all Θ_i values add up to 1.

3.5.2. Optimal Base Stock Levels of M/M/1 Systems

If prices are given and fixed, optimal base stock level of an M/M/1 system can be found by using Equation 3.13. In order to compute the base stock level, cumulative distribution function of the number of orders is required. Since the number of orders in an M/M/1 system is geometrically distributed, $P(N \leq S)$ can be computed as followings:

$$\begin{aligned} P(N \leq S) &= \sum_{n=0}^{S} P(N=n) = \sum_{i=0}^{S} (1-\rho)\rho^{i} \\ P(N \leq S) &= 1-\rho^{S+1} \end{aligned}$$

The optimal base stock level is computed by substituting the cumilative distribution into Equation 3.13.

$$\begin{aligned} &P(N \leq S) \geq \frac{\sum_{i=1}^{n} b_i \Theta_i}{\sum_{i=1}^{n} b_i \Theta_i + h} \\ &1-\rho^{S+1} \geq \frac{\sum_{i=1}^{n} b_i \Theta_i}{\sum_{i=1}^{n} b_i \Theta_i + h} \\ &\frac{h}{\sum_{i=1}^{n} b_i \Theta_i + h} \geq \rho^{S+1} \\ &S \geq \log\left(\frac{h}{\sum_{i=1}^{n} b_i \Theta_i + h}\right) \frac{1}{\log(\rho)} - 1 \end{aligned}$$

The smallest integer that satisfies previous inequality should be selected as the optimal base stock level. It can also formulated as following:

$$S^{*} = \left\lceil \log\left(\frac{h}{\sum_{i=1}^{n} b_i \Theta_i + h}\right) \frac{1}{\log(\rho)} \right\rceil - 1 \tag{3.14}$$

Optimal base stock level is sensitive to the prices for different customer classes. Since Θ is a function of price, it is also sensitive to backorder differences. If a single backorder cost rate is used as $b_i = b$ for $i = 1..n$, optimal base stock level is computed as following:

$$S^{*} = \left\lceil \log\left(\frac{h}{b+h}\right) \frac{1}{\log(\rho)} \right\rceil - 1$$

3.5.3. Optimal Continuous Base Stock Levels of M/M/1 Systems

In the base model, base stock level is required to be an integer. If the integer restriction for base stock level is ignored, profit function becomes simpler and no ceiling operators are needed as in Equation 3.14. It can be used as an approximation of base stock level of a queuing system.

We follow a similar procedure in finding optimal base stock level as we considered the integer restriction. Now the concavity is tested by computing the second derivative of the profit function instead of controlling the second difference. Required calculations are shown in APPENDIX D. The second derivative of expected profit function of an M/M/1 make-to-stock system is computed as following:

$$\Pi''(S) \quad = \quad -(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{\rho^{S+1}(\log(\rho))^2}{1-\rho} \leq 0$$

Since the second derivative is nonpositive, the profit function is concave over S. So optimal base stock level can be computed by setting the first derivative of profit function to zero. After some algebra, optimal continuous base stock level is computed as following:

$$S^* \quad = \quad \frac{1}{\log(\rho)}\log\left(\frac{h(1-\rho)}{(-\log(\rho))(\sum_{i=1}^{n} b_i\Theta_i + h)}\right) - 1 \tag{3.15}$$

Continuous base stock level can be viewed as an approximation of the model which considers the integer restriction. By making ceiling and flooring operations, two successive integer base stock levels are obtained. The base stock level that makes the profit highest is selected as the optimal base stock level. Approximated integer base stock level X^* is determined considering the maximum profit or the minimum cost. Let $C(\lambda_i, S)$ denote the sum of the holding and backorder costs, X^* is computed by following formula:

$$X^* = \underset{X\in\{\lfloor S^*\rfloor,\lceil S^*\rceil\}}{\arg\min} \; C(\lambda_i, X)$$

3.6. Exponentially Approximated Model of M/G/1 Queuing Systems

Computing the expected profit of an M/G/1 Make-to-Stock queuing system requires the distribution of the number of orders. It is determined by the inter arrival times and the service times. Since the distribution of service times is general, we may have difficulty in determining the distribution of the number of orders. Assuming that the number of orders in the system is exponentially distributed provides us a good approximation for M/G/1 queuing systems.

Let v be the parameter of an exponential distribution with a mean of $1/v$. v can be determined as $\frac{1}{E[N]}$ calculation. The values of $E[(S-N)^+]$ and $E[(N-S)^+]$ expressions are computed in APPENDIX E by using the parameter of v as followings:

$$\begin{aligned} E[(S-N)^+] &= S + \frac{e^{-vS}}{v} - \frac{1}{v} \\ E[(N-S)^+] &= \frac{e^{-vS}}{v} \end{aligned}$$

The inventory and the backorder costs are substituted into the general profit function.

$$\hat{\Pi}(p_i, S) = -\left[(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{e^{-vS}}{v} + h(S - \frac{1}{v}) \right] + \sum_{i=1}^{n} \lambda_i(p_i)p_i$$

The concavity of the profit function is shown in APPENDIX E by testing the second derivative of the profit function. By setting the first derivative to 0, optimal base stock level is obtained as following:

$$S^* = \frac{-1}{v} \log\left(\frac{h}{\sum_{i=1}^{n} b_i\Theta_i + h} \right) \tag{3.16}$$

If the optimal base stock level is substituted into the profit function, a simpler function is obtained as following:

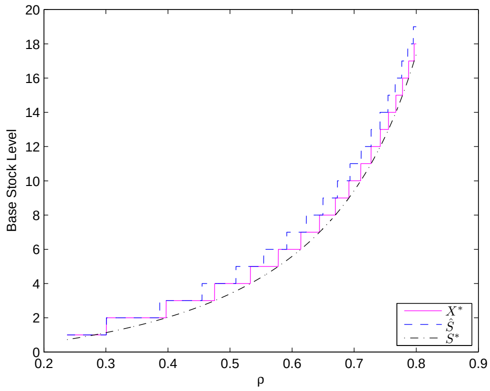
$$\hat{\Pi}(p_i, S^*) = -hS^* + \sum_{i=1}^{n} \lambda_i(p_i)p_i \tag{3.17}$$

This final equation does not only consider the holding cost. Actually, S^* is a function of the holding and backorder costs. As a result of the exponential approximation of the number of the orders, continuous approximation for base stock level is obtained. Actually, the continuous base stock level approximation can be used to approximate the integer base stock levels as mentioned in the previous section. Approximated integer base stock level X^* is determined considering the maximum profit or the minimum cost. Let $C(\lambda_i, S)$ be the sum of the holding and backorder costs, X^* is computed by following formula:

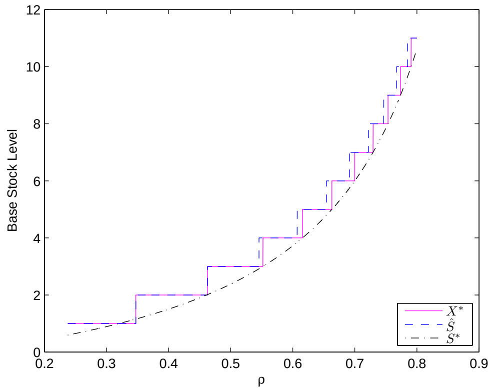
$$X^* = \underset{X \in \{\lfloor S^* \rfloor, \lceil S^* \rceil\}}{\arg\min} \; C(\lambda_i, X)$$

Approximated continuous base stock level S^* and integer base stock level X^* are graphed with optimal base stock level $\hat{S}$. M/PH/1 distribution is used to compare results under different coefficient of variations. Cost parameters are b_1=0.5, b_2=1, h=1 and μ=1.

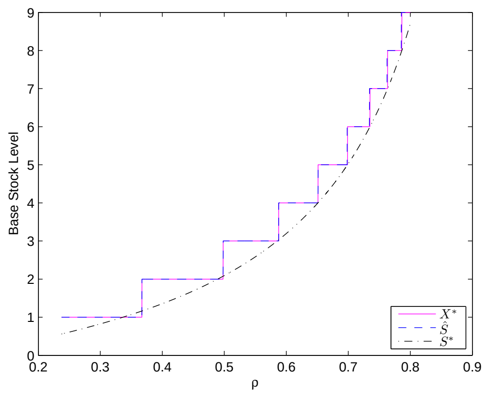
As seen in Figure 3.2; approximated integer base stock level X^* approaches to the optimal base stock level $\hat{S}$ as the coefficient of variation decreases. If the coefficient of variation is small, using exponentially approximated model allows us to find good enough solutions for the optimal base stock level.



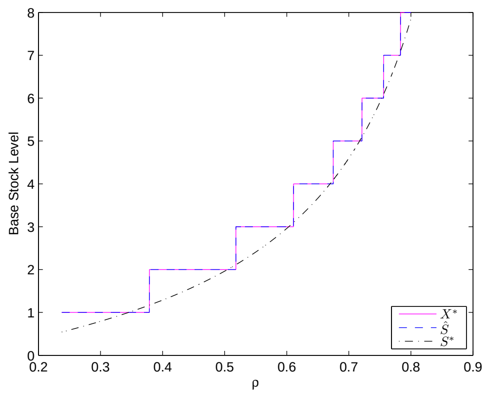
(a) $cv = 1.8700$



(b) $cv = 1.2454$



(c) $cv = 1.0000$



(d) $cv = 0.8683$

Figure 3.2. Integer and continuous base stock levels.

3.7. Loss Sales Model in M/M/1 Queuing Systems

In loss sales model, the number of orders in the system cannot exceed the base stock level S. Thus, no backorder is applied. Only holding cost is incurred. No arrival is accepted if the number of orders is equal to the base stock level($N = S$). The general profit function can be modified as following:

$$\begin{aligned}\Pi(p_i, S) &= -hE[S-N] + P(N<S)\sum_{i=1}^{n}\lambda_i(p_i)p_i \\ &= -hS + hE[N] + (1-P(N=S))\sum_{i=1}^{n}\lambda_i(p_i)p_i\end{aligned}$$

If single price level is used for all customer classes, the profit function of loss sales model can be expressed as follows:

$$\Pi(p, S) = -hS + hE[N] + (1-P(N=S))\rho\mu\frac{K-\rho\mu}{M}$$

The values of $E[N]$ and $P(N = S)$ expressions should be determined in order to compute the profit function. Note that, the system is totally changed and the distribution of the number of orders is also changed. These values for an M/M/1 system are computed in APPENDIX F as followings:

$$\begin{aligned}P(N=S) &= \frac{(1-\rho)\rho^S}{1-\rho^{S+1}} \\ 1-P(N=S) &= \frac{1-\rho^S}{1-\rho^{S+1}} \\ E[N] &= \frac{\rho(1-(S+1)\rho^S + S\rho^{S+1})}{(1-\rho)(1-\rho^{S+1})}\end{aligned}$$

The profit function $\Pi(p, S)$ is computed by substituting $E[N]$ and $P(N = S)$ computation as followings:

$$\Pi(p, S) = -hS + h\frac{\rho(1-(S+1)\rho^S + S\rho^{S+1})}{(1-\rho)(1-\rho^{S+1})} + \frac{1-\rho^S}{1-\rho^{S+1}}\rho\mu\frac{K-\rho\mu}{M} \quad (3.18)$$

In loss sales model, profit function is neither concave, nor convex over S. So, it is not possible to find any expression that represents optimal base stock level.

3.8. Analysis of Price in M/M/1 Queuing Systems with Single Price

Understanding the dynamic of the profit function in M/M/1 queuing systems requires analyzing different scenarios of the base stock levels. Possible values for the base stock levels can be classified by two special conditions such as no base stock level is used or a large base stock level is applied. For simplicity single price is used. It is expressed as a functions of traffic intensity($p = \frac{K}{M} - \frac{\mu}{M}\rho$).

3.8.1. Determining Optimal ρ if $S = 0$

If no base stock level is used; it corresponds to a make-to-order system where no inventory is kept, and each demand triggers a production order. Thus, holding cost is not incurred. Expected profit function of an M/M/1 queuing system is obtained as followings:

$$\begin{aligned}\Pi(\rho) &= -(\sum_{i=1}^{n} b_i\Theta_i)\frac{\rho}{1-\rho} + \lambda p \\ \Pi(\rho) &= -(\sum_{i=1}^{n} b_i\Theta_i)\frac{\rho}{1-\rho} + \frac{K\mu}{M}\rho - \frac{\mu^2}{M}\rho^2\end{aligned}$$

If the first derivative is set to 0, a third order equation is obtained. For simplicity, $1-\rho$ is denoted as z. Then the first derivative of the profit function can be shown to have the following form:

$$\frac{2\mu^2}{M}z^3 + (\frac{K\mu}{M} - \frac{2\mu^2}{M})z^2 + \left\{\frac{\sum_{i=1}^{n} b_i m_i}{M}(\frac{K}{\mu} - 1) - \frac{\sum_{i=1}^{n} b_i k_i}{\mu}\right\} = 0 \qquad (3.19)$$

Let z_1, z_2, and z_3 be three roots of Equation 3.19 calculated by using APPENDIX H. These roots are the critical values that can be absolute maximum point of the profit function. The concavity of the profit function over ρ is proved in APPENDIX G by showing that the second derivative is negative if $K \leq \mu$. In that case, one real global maximum point of the profit function is obtained. It can be also verified by using APPENDIX H. Otherwise all real roots should be considered.

The traffic intensities should be in $[K - M\min\limits_{i=1..n}(\frac{k_i}{m_i}), \min(K,\mu)]/\mu$ interval for stability. As well as the real roots, boundary points $(K - M\min\limits_{i=1..n}(\frac{k_i}{m_i}))/\mu$ and $(\min(K,\mu))/\mu$ are also critical. Since the profit function is smooth, optimal ρ should be one of these critical values. Let A denote the set of these critical points expressed as follows:

$$A = \left\{1 - z_1, 1 - z_2, 1 - z_3, \frac{K - M\min\limits_{i=1..n}(\frac{k_i}{m_i})}{\mu}, \frac{\min(K,\mu)}{\mu}\right\} \cap [\frac{K - M\min\limits_{i=1..n}(\frac{k_i}{m_i})}{\mu}, \frac{\min(K,\mu)}{\mu}] \tag{3.20}$$

Let price p be is a function of ρ that $p = \frac{K}{M} - \frac{\mu}{M}(\rho)$. Optimal ρ value that maximizes the profit function is computed by following formula:

$$\hat{\rho} = \arg\max_{\rho \in A} \Pi(\rho) \tag{3.21}$$

If $K < \mu$, the concavity of the profit function is sustained as proved in APPENDIX G. Since the profit function is concave, the cubic function should have one real root. It is also verified by the information given in the APPENDIX H. If the corresponding traffic intensity is in $[K - M\min\limits_{i=1..n}(\frac{k_i}{m_i}), \min(K,\mu)]/\mu$ interval, it is evident that this root is the maximizer of the profit function.

3.8.2. Determining Critical ρ Values with Zero Derivatives If S is Large

If base stock level S is not zero, we cannot prove any concavity of profit function over demand or ρ and the derivative of the profit function with respect to ρ may have many roots. If the base stock level S is large, then the server perpetuate producing the products and the server is less likely to be idle. If a large base stock level S is used in Equation 3.12, we get following profit function:

$$\Pi(p) \quad = \quad -\left[(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{\rho^{S+1}}{1-\rho} + h(S - \frac{\rho}{1-\rho})\right] + \lambda p$$

Since S is large, $(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{\rho^{S+1}}{1-\rho} \approx 0$. A simpler profit function and its derivative are expressed as follows:

$$\begin{aligned}
\Pi(p) &= -h(S - \frac{\rho}{1-\rho}) + \lambda p \\
\Pi(\rho) &= -hS + h\frac{\rho}{1-\rho} + \frac{K\mu}{M}\rho - \frac{\mu^2}{M}\rho^2 \\
\Pi'(\rho) &= \frac{h}{(1-\rho)^2} + \frac{K\mu}{M} - \frac{2\mu^2}{M}\rho
\end{aligned}$$

$1-\rho$ is denoted as z for the simplicity of the cubic function. Critical points with zero derivatives are found to analyze the maximal and minimal points.

$$\begin{aligned}
\frac{h}{z^2} + \frac{K\mu}{M} - \frac{2\mu^2}{M}(1-z) &= 0 \\
\frac{2\mu^2}{M}z^3 + (\frac{K\mu}{M} - \frac{2\mu^2}{M})z^2 + h &= 0
\end{aligned} \tag{3.22}$$

z_1, z_2, and z_2 are the three roots of Equation 3.22 calculated by using the formulas in APPENDIX H. Critical traffic intensities that are in $[K - M\min\limits_{i=1..n}(\frac{k_i}{m_i}), \min(K,\mu)]/\mu$ interval are considered. Boundary points $(K - M\min\limits_{i=1..n}(\frac{k_i}{m_i}))/\mu$ and $(\min(K,\mu))/\mu$ are also critical. As mentioned in previous subsection, A is denoted as the set of these critical points expressed in Equation 3.20. Optimal traffic intensity that maximizes the profit function is expressed as follows:

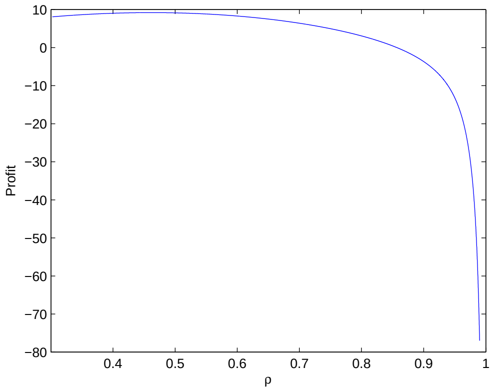
$$\hat{\rho} = \arg\max_{\rho \in A}\Pi(\rho, S) \tag{3.23}$$

3.8.3. Numerical Examples of M/M/1 Queuing Systems

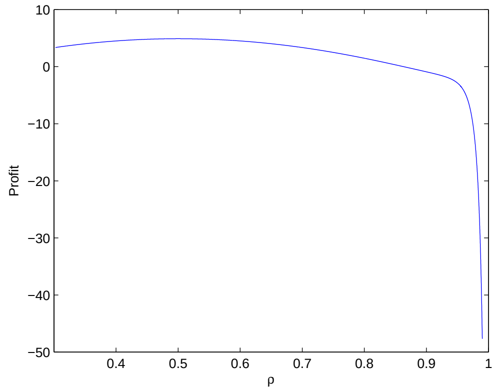
The dynamic of M/M/1 Make-to-Stock Queuing Systems is analyzed by using some numerical examples under different base stock levels. Two customer classes are considered. Profits are graphed as a function of ρ by using Equation 3.12. The parameters are selected as followings:

Table 3.2. Parameters for Numerical Examples.

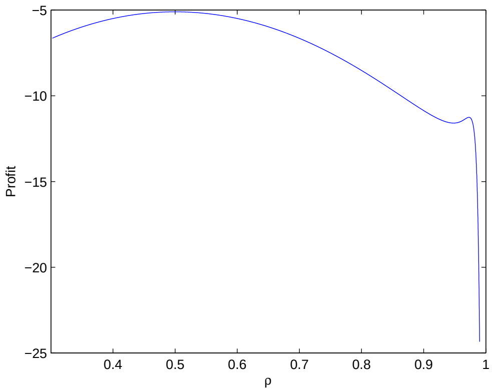
h	b_1	b_2	k_1	k_2	m_1	m_1	μ
0.1	0.5	1	0.44	0.551	0.005	0.02	1



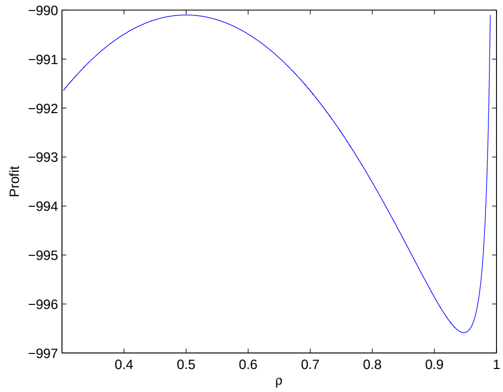
(a) S=0



(b) S=50



(c) S=150



(d) S=10000

Figure 3.3. Profit vs traffic intensity.

If base stock level is not used, only backorder cost incurs. The feasible region of the traffic intensity is $[K - M \min_{i=1..n}(\frac{k_i}{m_i}), \min(K, \mu)]/\mu$=[0.3023,0.9910] interval. When Equation 3.19 is solved, three critical traffic intensity levels are computed as 0.4619, 1.0168 - 0.1335i, 1.0168 + 0.1335i. Only one real traffic intensity is obtained in the feasible region, thus it is the global maximum of the equation. There is no need to consider boundary points 0.3023, 0.9910 as critical. It can be seen from Figure 3.3(a), that the maximum profit is obtained at ρ=0.4619.

If the base stock level is set to 50, holding cost is also incurred. As seen in the Figure 3.3(b), some reduction in the profit is obtained. Optimal traffic intensity is observed at $\rho = 0.5005$. If the base stock level is increased to 150, holding cost is also increased. Since the holding cost part of the profit is an increasing and convex function, it changed the concave shape of the graph to a more complicated one, as seen in Figure 3.3(c). Optimal traffic intensity is observed as $\rho = 0.5005$.

If the base stock level is increased to a large number as $S = 10000$, we may consider only revenue and holding cost parts of the profit function. The summation of the concave revenue part and convex holding cost part of the profit is seen in Figure 3.3(d). The optimal traffic intensity can be calculated by using Equation 3.23. The three critical points with zero derivatives are found by using Equation 3.22 as 0.50051, 0.94741, 1.0476. The boundary levels 0.30225,0.9910 are also critical. Optimal traffic intensity is found by comparing the profits of the critical points in feasible region [0.30225,0.9910] as followings:

ρ	0.30225	0.50051	0.94741	0.9910
Π	-99991,63	-99990.08	-99996.55	-99988,99

The highest profit is observed with an optimal traffic intensity of $\rho = 0.9910$. The profit is the summation of the concave revenue function and the convex and increasing holding cost. So, testing the critical values enables us to determine the optimal traffic intensities in this two special conditions of the base stock level.

4. APPROXIMATIONS FOR THE OPTIMAL PRICE

In this chapter, we aim to find optimal prices and base stock level jointly. Finding the optimal base stock levels if the prices are given is described in the previous chapter. Thus, finding the optimal prices is sufficient to maximize the profit. In our model, it is desired to the obtain optimal arrival rates that enable us to compute the optimal prices by using Equation 3.1. In order to compute the optimal arrival rates, a concave profit function is required. But, as we demonstrate in the previous chapter, the concavity of the profit function with respect to arrival rates is not guaranteed.

If the total cost of an M/G/1 make-to-stock systems is analyzed, an increasing function of ρ is observed. It starts from $C(0) = 0$ in numerical examples. For instance, a single class system with a phase type service time distribution can be used in order to analyze the dynamics of the cost function. Two phases are used in the example. $\alpha = \{0.6, 0.4\}$ is the probability vector of the server. Service starts from the first phase with a probability of 0.6, and it starts from the second phase with a probability of 0.4. Following is the rate matrix of the server that represents transition from one phase to another:

$$A = \begin{pmatrix} -8.2 & 1.025 \\ 0 & -0.5125 \end{pmatrix}$$

The service rate of this M/PH/1 system is calculated as μ=1 and coefficient of variation is calculated as 1.6341. In this example following cost and demand parameters are used:

h	b	k	m
0.01	0.05	0.75	5

The dynamics of the cost function is analyzed by using many traffic intensity (ρ) values as seen in the following figure:

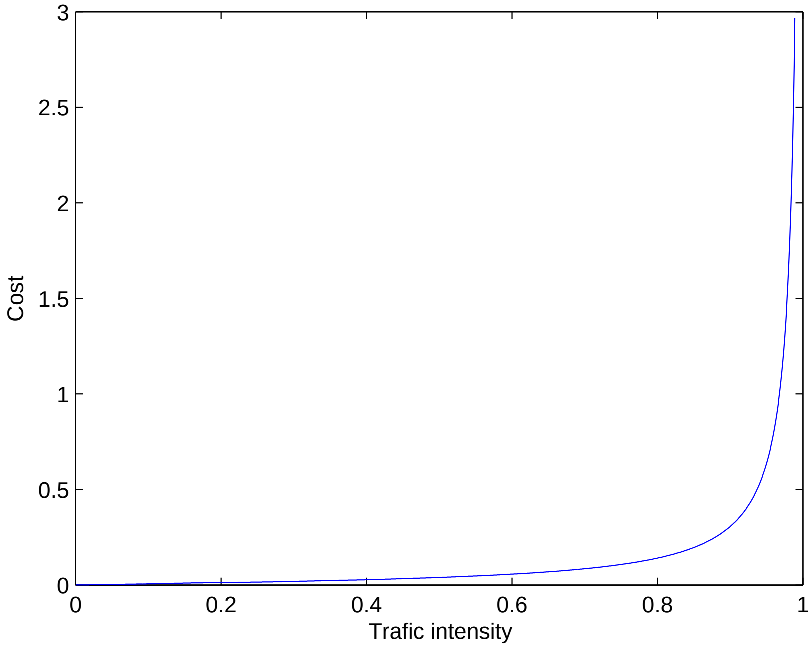


Figure 4.1. Cost Function.

As seen in Figure 4.1; as ρ increases, the total cost also increases. As ρ increases, the slope of the function also increases. Thus, it seems to be a convex function of ρ. Approximating the expected cost function as a convex function of arrival rates is proposed to sustain the concavity condition. Optimal arrival rates are formulated by using the concave profit function. Also a linear part is added to convex cost function and corresponding approximated arrival rates are formulated. Then the quality of the approximations are tested by using exact values of the profit functions. Various scenarios are considered by using appropriate parameter sets.

4.1. Approximating the Cost Function

If the expected cost function is approximated by a convex function, a concave expected profit function is obtained. It enables us to approximate the optimal backorder level and the prices for all customer classes. In this section we prove the concavity of the expected revenue function and the expected profit function with respect to arrival rates by using proposed convex approximation of the expected cost function. Then we obtain optimal arrival rates by using the concavity of the profit function.

Denoting the expected costs(backorder and holding) and the expected revenue as $C(\lambda_1 ... \lambda_n)$ and $R(\lambda_1 ... \lambda_n)$ respectively, the profit function can be expressed as follows:

$$\Pi(\lambda_1 ... \lambda_n) = -C(\lambda_1 ... \lambda_n) + R(\lambda_1 ... \lambda_n)$$

Recall that $C(\lambda_1 ... \lambda_n)$ considers the optimal base stock level, because it can be found as a function of arrival rates. Optimal base stock level is determined by using Equation 3.13 as explained in detail in the previous chapter.

The concavity of the revenue function is proven by proving the convexity of its negation. Let denote $T = -R$.

$$T = -\sum_{i=1}^{n} \lambda_i p_i = -\sum_{i=1}^{n} \lambda_i \frac{(k_i - \lambda_i)}{m_i}$$

Required differentials for the Hessian matrix of T are shown as followings:

$$\frac{\partial T}{\partial \lambda_i} = \frac{-k_i + 2\lambda_i}{m_i}$$
$$\frac{\partial T}{\partial \lambda_i^2} = \frac{2}{m_i}$$
$$\frac{\partial T}{\partial \lambda_i \partial \lambda_j} = 0 (i \neq j)$$

The Hessian matrix of T is a diagonal matrix with positive diagonal entries. Since the determinants of all leading principal minors of Hessian matrix are nonnegative, T is a convex function of arrival rates. Thus, the concavity of the revenue function R$(\lambda_1 ... \lambda_n)$ with respect to arrival rates is sustained.

If the total cost function $C(\lambda_1 ... \lambda_n)$ is approximated as a convex function $\widetilde{C}(\lambda_1 ... \lambda_n)$, the concavity of the profit function is sustained. In this way, optimal demand levels $(\lambda_1 ... \lambda_n)$ and ultimately optimal price levels can be approximated.

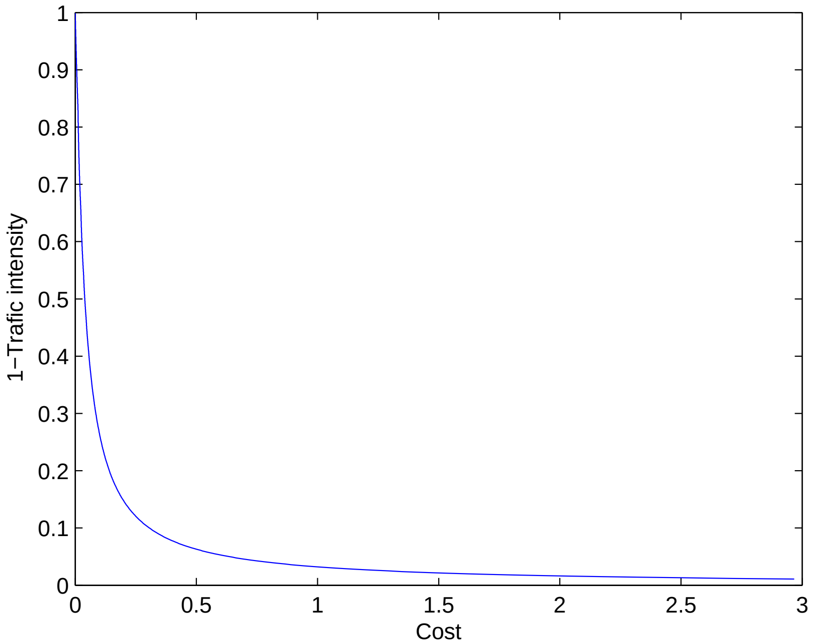


Figure 4.2. Cost vs 1-ρ.

As seen in Figure 4.1; as ρ increases, the total cost also increases. The relation is shown in a more convenient way in Figure 4.2 in order to find an approximation of the cost function. Let denote $y = 1 - \rho$ and $x = C(\lambda_1 ... \lambda_n)$. It can be seen that as x increases from 0 to ∞, y decreases from 1 to 0. One possible approximation for such functions is approximated as $y = \frac{1}{1+x/f}$ where f is a positive number. After substituting $x = \widetilde{C}(\lambda_1 ... \lambda_n)$ and $y = 1 - \rho$ values into $y = \frac{1}{1+x/f}$ function, we can approximate total cost function as followings:

$$
\begin{aligned}
y &= \frac{1}{1 + x/f} \\
1 - \rho &= \frac{1}{1 + \widetilde{C}(\lambda_1 ... \lambda_n)/f} \\
\widetilde{C}(\lambda_1 ... \lambda_n) &= f \frac{\rho}{1 - \rho} = f \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} \\
\widetilde{C}(\lambda_1 ... \lambda_n) &= f \frac{\lambda}{\mu - \lambda}
\end{aligned}
$$

The value of f can be approximated using an initial $\hat{\lambda}$ value, for instance $\frac{k_i}{2}$ which is the maximizer of the revenue function $\sum_{i=1}^{n} \lambda_i p_i = \sum_{i=1}^{n} \lambda_i \frac{k_i - \lambda_i}{m_i}$. The value of f is obtained by solving the following equation:

$$\begin{aligned} \widetilde{C}(\hat{\lambda_1}...\hat{\lambda_n}) &= C(\hat{\lambda_1}...\hat{\lambda_n}) \\ f &= \frac{\mu - \sum_{i=1}^{n} \hat{\lambda_i}}{\sum_{i=1}^{n} \hat{\lambda_i}} C(\hat{\lambda_1}...\hat{\lambda_n}) \end{aligned} \tag{4.1}$$

Note that, approximated cost function $\widetilde{C}(\hat{\lambda_1}...\hat{\lambda_n}\ )$ is a convex function. Thus, it can be used to obtain a concave profit function. Its convexity can be proved by its Hessian matrix. Required differentials for the Hessian matrix are found as followings:

$$\begin{aligned} \frac{\partial \widetilde{C}(\lambda_1...\lambda_n)}{\partial \lambda_i} &= \frac{f\mu}{(\mu - \lambda)^2} \\ \frac{\partial \widetilde{C}(\lambda_1...\lambda_n)}{\partial \lambda_i^2} &= \frac{2f\mu}{(\mu - \lambda)^3} \\ \frac{\partial \widetilde{C}(\lambda_1...\lambda_n)}{\partial \lambda_i \lambda_j} &= \frac{2f\mu}{(\mu - \lambda)^3}(i \neq j) \end{aligned}$$

The Hessian matrix of the cost function can be written as following:

$$H = \frac{2f\mu}{(\mu - \lambda)^3} \begin{pmatrix} 1 & 1..1 & 1 \\ 1 & 1..1 & 1 \\ . & .... & . \\ 1 & 1..1 & 1 \\ 1 & 1..1 & 1 \end{pmatrix}$$

Since the determinants of all leading principal minors of H matrix are nonnegative, $\widetilde{C}(\hat{\lambda}_1...\hat{\lambda}_n\ )$ is a jointly convex function with respect to arrival rates. If the convex approximation of the cost function is used in the profit function, a concave function is obtained due to the concavity of the revenue function. By setting the first derivatives of the approximated profit function $\widetilde{\Pi}(\lambda_1...\lambda_n)$ to zero, optimal arrival rates are obtained.

$$\begin{aligned}
\frac{\partial\widetilde{\Pi}(\lambda_1...\lambda_n)}{\partial\lambda_i} &= -\frac{\partial\widetilde{C}(\lambda_1...\lambda_n)}{\partial\lambda_i} + \frac{\partial R(\lambda_1...\lambda_n)}{\partial\lambda_i} = 0 \\
\frac{\partial\widetilde{C}(\lambda_1...\lambda_n)}{\partial\lambda_i} &= \frac{\partial R(\lambda_1...\lambda_n)}{\partial\lambda_i} \\
\frac{f\mu}{(\mu-\lambda)^2} &= \frac{k_i - 2\lambda_i}{m_i} \\
m_i f\mu &= (\mu-\lambda)^2(k_i - 2\lambda_i) \\
\sum_{i=1}^{n} m_i f\mu &= (\mu-\lambda)^2\sum_{i=1}^{n}(k_i - 2\lambda_i)
\end{aligned} \tag{4.2}$$

If we denote $M = \sum_{i=1}^{n} m_i$, $K = \sum_{i=1}^{n} k_i$, the equation is simplified as following:

$$\begin{aligned}
&Mf\mu = (\mu-\lambda)^2(K - 2\lambda) \\
&2\lambda^3 - (4\mu + K)\lambda^2 + (2\mu^2 + 2\mu K)\lambda + Mf\mu - K\mu^2 = 0
\end{aligned} \tag{4.3}$$

Finally a cubic function is obtained. It has at most 3 distinct roots. As stated in Appendix H; if $\Delta < 0$, the equation has one real root. Δ value of the cubic function is computed as following:

$$\begin{aligned}
\Delta &= -4Mf\mu(-K^3 + 6\mu K^2 - 12\mu^2 K + 8\mu^3 + 27Mf\mu) \\
&= 4Mf\mu\left((K - (2\mu - \frac{3}{2}\sqrt[3]{Mf\mu}))^2 + 27Mf\mu\right)\left(K - (2\mu + 3\sqrt[3]{Mf\mu}\right)
\end{aligned}$$

Note that, $4Mf\mu\left((K - (2\mu - \frac{3}{2}\sqrt[3]{Mf\mu}))^2 + 27Mf\mu\right)$ is the positive part of the Δ expression. We can find condition where the value of Δ is negative or where the cubic function has only one real root. Following condition satisfies the equation's having a single root.

$$\begin{aligned}
\Delta &< 0 \\
K - (2\mu + 3\sqrt[3]{Mf\mu}) &< 0 \\
K &< 2\mu + 3\sqrt[3]{Mf\mu}
\end{aligned}$$

It is evident that the unique real root solves problems where $K < 2\mu + 3\sqrt[3]{Mf\mu}$. Even such conditions are not satisfied, any appropriate root in the $(0, \mu)$ interval can be selected.

After finding the root of the cubic function as λ^*, λ_i^* values can be obtained from Equation 4.2 as following:

$$\lambda_i^* = \frac{k_i}{2} - \frac{m_i f \mu}{2(\mu - \lambda^*)^2} \tag{4.4}$$

Optimal price for class i can be approximated simply by $\frac{k_i - \lambda_i^*}{m_i}$ calculation.

Better results can be obtained by repeating calculations via resetting initial $\hat{\lambda}_i$ values with λ_i^* . If some λ_i^* values are found to be nonpositive, they should be set to zero. Thus the corresponding classes should be excluded from the set and calculations should be repeated. If we apply the algorithm depicted in Figure 4.3, we obtain better approximation for the optimal prices and the base stock level.

4.2. Approximating the Cost Function with a Linear Part

A linear function, which does not affect the convexity, can be added to the approximated cost function to get better results. Following equation gives us the new convex approximation for the cost function:

$$\widetilde{C}(\lambda_1 ... \lambda_n) = f\frac{\lambda}{\mu - \lambda} + \sum_{i=1}^{n} a_i \lambda_i$$

This approximation contains $n + 1$ coefficients that are computed by $n + 1$ sets of initial arrival rate values. These initial values can be selected around the revenue maximizer $\frac{k_i}{2}$ such as $(\hat{\lambda}_{1j} ... \hat{\lambda}_{nj})$ j∈(1.. n,n+1). If the sum of all arrival rates are greater than the service rate $(\sum_{i=1}^{n} \hat{\lambda}_{ij} \geq \mu)$, the values of corresponding arrival rates $\hat{\lambda}_{ij}$ $i = 1..n$ can be modified by multiplying with some appropriate values like $\frac{0.99\mu}{\sum_{i=1}^{n} \hat{\lambda}_{ij}}$. It gives an appropriate traffic intensity ρ. By setting $n + 1$ cost approximations to

- **Step 0 (Initialization)**

  Let all classes be included in the product set $ps(i) = 1$

  Set $j = 0$. For any class $i = 1...n$, set $\lambda_i(j) = 0$. Also set $\Pi(\lambda_1...\lambda_n)(j) = 0$

- **Step 1**

  Set $j = 1$. For any class $i = 1...n$, if $\lambda_i(j) > 0$, set them to $k_i/2$ else, exclude class i, set $ps(i) = 0$, $\lambda_i(j) = 0$. Determine $\lambda(j)$.

- **Step 2**

  2.0 If $\lambda(j) \geq \mu$, set $frac = \frac{\mu}{\lambda(j)} * 0.99$ Update all $\lambda_i(j) = \lambda_i(j) * frac$

  2.1 Compute f with the help of Equation 4.1. Calculate K and M values as $K = \sum_{i=1}^{n} k_i ps(i)$, $M = \sum_{i=1}^{n} m_i ps(i)$ and solve Equation 4.3 to obtain 3 optimal roots z_1,z_2,z_3

  2.2 For any real roots of Equation 4.3 make following calculations by letting $\tilde{\lambda}$ be any real root.

  2.2.1 If $\tilde{\lambda} \leq 0$, set all $\tilde{\lambda}_i = 0$,

  2.2.2 Else if $\tilde{\lambda} > 0$ and $\tilde{\lambda} < \mu$ set all $\tilde{\lambda}_i$ as in Equation 4.4,

  2.2.3 Else, set all $\tilde{\lambda}_i$ as in Equation 4.4. If any $\tilde{\lambda}_i < 0$, set them $\tilde{\lambda}_i = 0$ Update $\tilde{\lambda}$ and set $frac = \frac{\mu}{\tilde{\lambda}(j)} * 0.99$.Update all $\tilde{\lambda}_i = \tilde{\lambda}_i * frac$

  2.2.4 Compute corresponding profit $\Pi(\tilde{\lambda}_1...\tilde{\lambda}_n)$

  2.2.5 Go to **Step 2.2** if there other real roots.

  2.3 Set $\Pi(\lambda_1...\lambda_n)(j)$ as the maximum of the profits computed in **Step 2.2** as $\Pi(\tilde{\lambda}_1...\tilde{\lambda}_n)$ values. Also set $\lambda_i(j)$ with corresponding $\tilde{\lambda}_i$ values.

  2.4 If there are no real root, or all $\lambda_i(j) = 0$ set $j = 1$ and go to **Step 4**

  2.5 If $\Pi(\lambda_1...\lambda_n)(j) > \Pi(\lambda_1...\lambda_n)(j-1)$ or $j = 1$, then increase j by 1 and go to **Step 2**; else go to **Step 3**.

- **Step 3** If there are any negative $\lambda_i(j)$ values go to **Step 1**, else go to **Step 4**

- **Step 4** Stop with the index j^* that satisfies

  $\Pi(\lambda_1...\lambda_n)(j^*) = \max(\Pi(\lambda_1...\lambda_n)(j-1), \Pi(\lambda_1...\lambda_n)(0))$.

  $\lambda_i(j^*)$ values are the optimal arrival rates. Optimal base stock level is computed by Equation 3.13

Figure 4.3. Approximation Algorithm.

their exact values like $\widetilde{C}(\hat{\lambda}_{1j}...\hat{\lambda}_{nj}) = C(\hat{\lambda}_{1j}...\hat{\lambda}_{nj})$, the coefficients can be found easily. Letting $\Lambda_j = \sum_{i=1}^{n} \lambda_{ij}$, those coefficients of the approximated cost matrix can be found by following matrix form:

$$\begin{pmatrix} f \\ a_1 \\ . \\ a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} \frac{\Lambda_1}{\mu-\Lambda_1} & \hat{\lambda}_{11}.. & \hat{\lambda}_{n1} \\ \frac{\Lambda_2}{\mu-\Lambda_2} & \hat{\lambda}_{12}.. & \hat{\lambda}_{n2} \\ . & .... & . \\ \frac{\Lambda_n}{\mu-\Lambda_n} & \hat{\lambda}_{1n}.. & \hat{\lambda}_{nn} \\ \frac{\Lambda_{(n+1)}}{\mu-\Lambda_{(n+1)}} & \hat{\lambda}_{1(n+1)}.. & \hat{\lambda}_{n(n+1)} \end{pmatrix}^{-1} \begin{pmatrix} C(\hat{\lambda}_{11}...\hat{\lambda}_{n1}) \\ C(\hat{\lambda}_{12}...\hat{\lambda}_{n2}) \\ . \\ C(\hat{\lambda}_{1n}...\hat{\lambda}_{nn}) \\ C(\hat{\lambda}_{1(n+1)}...\hat{\lambda}_{n(n+1)}) \end{pmatrix}$$

If the convex approximation of the cost function is used in the profit function, a concave function is obtained due to the concavity of the revenue function. By setting the first derivatives of the approximated profit function $\widetilde{\Pi}(\lambda_1...\lambda_n)$ to zero, optimal arrival rates are obtained. Followings are the important computations for solving optimal arrival rates:

$$\begin{aligned} \frac{\partial\widetilde{\Pi}(\lambda_1...\lambda_n)}{\partial\lambda_i} &= -\frac{\partial\widetilde{C}(\lambda_1...\lambda_n)}{\partial\lambda_i} + \frac{\partial R(\lambda_1...\lambda_n)}{\partial\lambda_i} = 0 \\ \frac{\partial\widetilde{C}(\lambda_1...\lambda_n)}{\partial\lambda_i} &= \frac{\partial R(\lambda_1...\lambda_n)}{\partial\lambda_i} \\ \frac{f\mu}{(\mu-\lambda)^2} + a_i &= \frac{k_i - 2\lambda_i}{m_i} \\ m_i f\mu &= (\mu-\lambda)^2(k_i - a_i m_i - 2\lambda_i) \\ \sum_{i=1}^{n} m_i f\mu &= (\mu-\lambda)^2 \sum_{i=1}^{n}(k_i - a_i m_i - 2\lambda_i) \end{aligned} \tag{4.5}$$

Letting $M = \sum_{i=1}^{n} m_i$, $K = \sum_{i=1}^{n} k_i$, $G = K - \sum_{i=1}^{n} a_i m_i$ simplifies the equation as following:

$$Mf\mu = (\mu-\lambda)^2(G - 2\lambda)$$
$$2\lambda^3 - (4\mu + G)\lambda^2 + (2\mu^2 + 2\mu G)\lambda + Mf\mu - G\mu^2 = 0 \tag{4.6}$$

Finally a cubic function is obtained as expressed in Equation 4.6. It has at most 3 distinct real roots. As stated in Appendix H if $\Delta < 0$, the equation has one real root.

Δ value of Equation 4.6 is obtained as following:

$$\begin{aligned}\Delta &= -4Mf\mu(-G^3 + 6\mu G^2 - 12\mu^2 G + 8\mu^3 + 27Mf\mu) \\ &= 4Mf\mu\left((G - (2\mu - \frac{3}{2}\sqrt[3]{Mf\mu}))^2 + 27Mf\mu\right)\left(G - (2\mu + 3\sqrt[3]{Mf\mu}\right)\end{aligned}$$

Note that $4Mf\mu\left((G - (2\mu - \frac{3}{2}\sqrt[3]{Mf\mu}))^2 + 27Mf\mu\right)$ is the positive part of the Δ. Following expressions give the condition that the value of Δ is negative or the cubic function has one real root:

$$\begin{aligned}\Delta &< 0 \\ G - (2\mu + 3\sqrt[3]{Mf\mu}) &< 0 \\ K - \sum_{i=1}^{n} a_i m_i - (2\mu + 3\sqrt[3]{Mf\mu}) &< 0 \\ K &< \sum_{i=1}^{n} a_i m_i + 2\mu + 3\sqrt[3]{Mf\mu}\end{aligned}$$

If the linear part of the approximated cost function were not added, it is evident that an unique real root solves problems and provide a real optimal arrival rate if $K < 2\mu + 3\sqrt[3]{Mf\mu}$. But the sign of $\sum_{i=1}^{n} a_i m_i$ is not known and we cannot be sure if $K < \sum_{i=1}^{n} a_i m_i + 2\mu + 3\sqrt[3]{Mf\mu}$. Even this condition is not satisfied, any appropriate root can be selected.

After finding the roots of the cubic function λ^*, λ_i^* values can be obtained from Equation 4.5 as followings:

$$\lambda_i^* = \frac{k_i - a_i m_i}{2} - \frac{m_i f \mu}{2(\mu - \lambda^*)^2} \tag{4.7}$$

Optimal price for class i can be approximated simply $\frac{k_i - \lambda_i^*}{m_i}$

Better results can be obtained by repeating calculations via resetting initial $\hat{\lambda}_i$ values with λ_i^* . If some nonpositive λ_i^* values are found, they should be set to zero. However, it may result in some mathematical errors in solving approximated cost func-

2.1 Compute a_i and f values by solving corresponding matrix form considering the active classes in the set. Calculate K and M values as $K = \sum_{i=1}^{n} k_i ps(i)$, $M = \sum_{i=1}^{n} m_i ps(i)$ and solve Equation 4.6 to obtain 3 optimal roots z_1,z_2,z_3

2.2 For any real roots of Equation 4.6 make following calculations by letting $\tilde{\lambda}$ be any real root.

2.2.1 If $\tilde{\lambda} \leq 0$, set all $\tilde{\lambda}_i = 0$,

2.2.2 Else if $\tilde{\lambda} > 0$ and $\tilde{\lambda} < \mu$ set all $\tilde{\lambda}_i$ as in Equation 4.7,

2.2.3 Else, set all $\tilde{\lambda}_i$ as in Equation 4.7. If any $\tilde{\lambda}_i < 0$, set them $\tilde{\lambda}_i = 0$ Update $\tilde{\lambda}$ and set $frac = \frac{\mu}{\tilde{\lambda}(j)} * 0.99$. Update all $\tilde{\lambda}_i = \tilde{\lambda}_i * frac$

2.2.4 Compute corresponding profit $\Pi(\tilde{\lambda}_1 ... \tilde{\lambda}_n)$

2.2.5 Go to **Step 2.2** if there other real roots.

Figure 4.4. Modified Approximation Algorithm.

tion parameters by using the matrix form, since the inversion operation of a matrix may not be made due to the fact that we may obtain $(\hat{\lambda}_{i1} ... \hat{\lambda}_{i(n+1)})$ all zero. Thus, the algorithm in previous section can be modified to obtain accurate results. It is enough to change the algorithm proposed previously by using modifications in Figure 4.4.

4.3. Testing the Quality of Approximations

Approximations are tested with many parameter sets with many distributions. Test procedure is performed by comparing the approximations with exact values of the profits. Exact values are obtained by brute-force method. All possible arrival rates are tested by considering the stability of the system.

Expected inventory level $E[(S-N)^+]$ and expected backorder level $E[(N-S)^+]$ values are calculated by using steady state probabilities. Testing the quality of the approximations requires considering various conditions. Using M/PH/1 distributions gives us a chance for calculating these probabilities easily under different systems with various coefficients of variations.

In such a M/PH/1 system, Poisson arrivals enter to the phases of service system. These phases are transient states of a Markov Chain. Completing the services is the absorbing state a Markov Chain. The service phase that the Poisson arrival enters is determined by a probability vector α. Service time for each phase is exponentially distributed. In Coxian M/PH/1 distributions, arrival passes the other phases, after leaving current phase. After last phase, service is finished.

Let A denote the transition rate matrix of the system. Let $\mathtt{1}$ be a column vector having same values in its elements as 1 and $R = (I - \mathtt{1}\alpha - \lambda^{-1}A)^{-1}$. Probability that the system is idle is computed by $\pi_0 = (\alpha(I - R)^{-1}\mathtt{1})^{-1}$ equation. System states are defined as followings:

$\pi_{n,i} = P$ {There are n jobs in the system and the job service is in phase i }

$\pi_n = \{\pi_{n,0},\pi_{n,1},......\}$.

Other states are calculated by $\pi_n = \pi_0 R^n$ formula.

Coxian types of M/PH/1 distributions are used in all test problems. In Coxian M/PH/1 distributions, an arrival passes the next phase, after leaving current phase. After the last phase, service is finished. In testing, two service phases of Coxian M/PH/1 distributions are used. Probability vector for the distribution is set as $\alpha = \{0.6, 0.4\}$. 6 different Coxian M/PH/1 distributions are generated with an expected service time μ=1 with different coefficient of variations.

Approximations are tested by different parameters and service time distributions in two class M/G/1 make-to-stock systems. In all distributions, service rates are set to 1(μ=1). h,b_1,b_2,k_1,k_2,m_1/q, m_2/q, and q parameters are used. m_1,m_2 values are computed by m_1/q, m_2/q, and q parameters. m_1 and m_2 are obtained from $m_1 = (q)(m_1/q)$ and $m_2 = (q)(m_2/q)$ calculations. Parameter q is used to group m_1 and m_2 values simultaneously. Since as q increases, m_1 and m_2 values also increases. Recall that the revenue function can be expressed as follows:

$$\sum_{i=1}^{n} \lambda_i p_i = \sum_{i=1}^{n} \lambda_i \frac{k_i - \lambda_i}{m_i} = \sum_{i=1}^{n} \lambda_i \frac{k_i - \lambda_i}{q(m_i/q)}$$

If the factor q is used for fixed ratio m_i/q, the revenue is decreased by a factor of $\frac{1}{q}$. The revenue function variety is sustained by q parameter.

Table 4.1. Parameters for Test.

h	b_1	b_2	k_1	k_2	m_1/q	m_2/q	q
0.01	1	1	0.3	0.2	0.05	0.05	0.5
	10		0.6	0.4	0.10		1
			0.9	0.8			5
			1.5	1.2			10
			3	2			
			4	3			

As seen in Table 4.1 1x2x1x6x6x2x1x4=576 different parameter sets are tested for each Coxian M/PH/1 distribution. Totally, 576x6=3456 systems are used for testing the approximations.

Let Π_1 be the approximated profit function without a linear part, Π_2 be the approximated profit function with a linear part and Π^* be the optimum profit function. Let ρ_1, ρ_2 be the approximated traffic intensities of the approximations, ρ^* be the optimal traffic intensity. A third approximation can be defined as the best of the first and the second approximations with a profit of Π_3 and a traffic intensity of ρ_3. In all approximations, when no results are obtained, profits and traffic intensities or arrival rates are all set to zero. Also the algorithms proposed in previous sections give nonnegative profits. If the optimum profit is really zero, optimum solution is guaranteed. For the first approximation, error percentage of the profit is computed as:

$$\Delta\Pi_1 = \begin{cases} 0 & \text{if} \quad \Pi^* = 0 \\ \frac{100(\Pi^* - \Pi_1)}{\Pi^*} & \text{if} \quad \Pi^* \neq 0 \end{cases}$$

For the other approximations, corresponding computations are made. Note that maximum error percentage obtained as 100 if $\Pi^* \neq 0$ and $\Pi_1 = 0$ which is an inaccurate result of the problem.

Table 4.2. Count of Results According to Error Percentages.

	$\Delta\Pi_1 \neq 100$	$\Delta\Pi_1 = 100$	Total
$\Delta\Pi_2 \neq 100$	3094	23	3117
$\Delta\Pi_2 = 100$	294	45	339
Total	3388	68	3456

Table 4.2 gives an idea about how much of the problems can be solved with some accuracy. As seen in Table 4.2, 3388 of 3456 problems are solved with some profit by using the first approximation. On the other hand, only 3117 of 3456 problems are solved with some accuracy by using the second approximation. More problems can be solved by using the first approximation. If the third approximation is used, only 45 of 3456 problems give no accuracy. Thus, the third approximation which is the maximum of other two approximations can be used to solve various kinds of problems.

If all 3456 problems are analyzed with Table 4.3, better error percentage averages are obtained for all q values, if the first approximation is used instead of the second approximation. It is due to the fact that the second approximation contains more results like $\Pi_2 = 0$ and $\Pi^* \neq 0$ as in Table 4.2. All these results have an error percentage of 100. Despite the problems with error percentages of 100, on average we obtain acceptable errors of %3.141 in the first approximation and %2.100 in the third approximation.

Table 4.3. Averages of the Error Percentages.

q	$\Delta\Pi_1$	$\Delta\Pi_2$	$\Delta\Pi_3$	Count
0.5	0.076	0.015	0.014	864
1	0.307	0.606	0.117	864
5	4.405	15.677	3.074	864
10	7.774	27.801	5.196	864
Average	3.141	11.025	2.100	3456

If only the results that have an error percentage of 100 are ignored, still better error percentages are obtained when the first approximation is used instead of the second approximation. As seen in Figure 4.4; we obtain very small error percentages of %0.810 in the first approximation, %1.238 in the second approximation and only %0.271 in the third approximation. However, the second approximation is better than the first approximation when the value of q is low($q = 0.5$ and $q = 1$).

Table 4.4. Averages of the Error Percentages($\Delta\Pi_1 \neq 100$ and $\Delta\Pi_2 \neq 100$).

q	$\Delta\Pi_1$	$\Delta\Pi_2$	$\Delta\Pi_3$	Count
0.5	0.076	0.015	0.014	864
1	0.295	0.144	0.103	860
5	1.524	2.596	0.626	743
10	1.684	2.814	0.435	627
Average	0.810	1.238	0.271	3094

In Table 4.4 and Table 4.3, we can see that as q decreases better averages of error percentages are obtained. The profit function consists of costs and revenue parts. The cost function is the complicated part of the profit function that we can not prove its concavity or convexity. On the other hand, the revenue function is the simple part of the profit function that we can easily prove its concavity. Cost part does not change as q parameter changes. On the other hand, revenue part of the profit function decreases as q parameter increases. As q decreases; revenue function, which is the simple part of the profit function, becomes more dominant in the profit function and better averages of error percentages are obtained. We can also observe that as q increases, the number of results that satisfy the condition($\Delta\Pi_1 = 100$ and $\Delta\Pi_2 = 100$) also increases since the count of the results that satisfies ($\Delta\Pi_1 \neq 100$ and $\Delta\Pi_2 \neq 100$) condition decreases. Because the cost function, which is the complicated part of the profit function, becomes more dominant in the profit function and we obtain more inaccurate results as q increases.

Error percentages are analyzed according to the coefficient of variations in Table 4.5. If all 3456 results are considered; as the coefficient of variations increases, error percentage averages generally increase in all results. In Table 4.5, we also see that as the coefficient of variations increases, the number of results that satisfy the condition($\Delta\Pi_1 = 100$ and $\Delta\Pi_2 = 100$) also increases. It is the main reason behind the error percentage increase in all results. On the other hand; if inaccurate results are ignored, as the coefficient of variations increases, error percentage averages of the third approximation generally decrease.

Table 4.5. Coefficient of Variations and Averages of the Error Percentages.

	All results				$\Delta\Pi_1 \neq 100$ and $\Delta\Pi_2 \neq 100$			
cv	$\Delta\Pi_1$	$\Delta\Pi_2$	$\Delta\Pi_3$	Count	$\Delta\Pi_1$	$\Delta\Pi_2$	$\Delta\Pi_3$	Count
0.868	2.070	8.037	1.675	576	0.812	1.174	0.389	536
0.939	2.432	9.019	1.808	576	0.803	0.749	0.310	527
1	2.429	9.376	1.744	576	0.842	1.205	0.348	527
1.245	2.823	10.525	1.891	576	0.825	1.487	0.202	421
1.505	4.473	12.454	2.496	576	0.961	1.036	0.180	502
1.870	4.617	16.739	2.987	576	0.610	1.820	0.183	481
Average	3.141	11.025	2.100	3456	0.810	1.238	0.271	3094

Profit values found by the approximating the cost functions are very close to the optimum profit values. First approximation gives better results than the second approximation in most cases, but on average using the second approximation enables us to obtain very small error percentages.

5. CONCLUSION

In this thesis, profit maximization of M/G/1 make-to-stock systems with different customer classes is analyzed. Arrival of each customer class follows a Poisson process. Demands of customer classes are independent of each other that they form a single queue under FIFO discipline. Different backorder costs and prices are incurred to the customer classes.

Firstly, the model formulation of the profit is developed. The general model is modified by using a single price in the profit function and corresponding stability requirements are analyzed. As an example of the general model, M/M/1 make-to-stock systems are analyzed. The general formula for the optimal base stock level of M/G/1 make to stock systems is determined by using the concavity of the profit function with respect to the base stock level. The optimal base stock levels of M/M/1 make-to-stock systems are found by this formula. The model is approximated by assuming the number of the orders is exponentially distributed. This model is used to obtain an approximate continuous base stock level. By using ceiling and flooring operations, integer approximated base stock levels are also found by considering the minimum cost. Optimal and approximated base stock levels are analyzed by using many M/PH/1 distributions with different coefficient of variations. Optimal traffic intensity levels of M/M/1 make-to-stock systems are formulated when the base stock levels are zero and large.

Secondly, optimal base stock levels and prices are determined jointly by approximating the cost function with a convex function of arrival rates. It leads to a concave profit function over arrival rates. The parameters of the cost approximation is determined by an initial λ value. By using the concavity of the profit function, optimal arrival rate for a customer class λ_i is determined as a function of sum of arrival rates λ. Sum of optimal arrival rates λ is determined by solving a cubic function. Better results are obtained by using optimal λ value in determining the parameters of the approximated cost function. A corresponding algorithm is proposed to obtain better results.

It is also desired to have a better approximation for the cost function by adding a linear part to approximation. The parameters of the cost function is determined by $n+1$ sets of initial arrival rate values. Proposed algorithm is modified for this approximation. Also another approximation is proposed as the one of the other two approximations with the highest profit value. The quality of the approximations are tested by 3456 different systems under 6 different M/PH/1 distributions. Profit values found by the approximating the cost functions are very close to the optimum profit values.

APPENDIX A: Expected Backorder Levels of M/M/1 Queuing Systems

With the ease of generating functions, expected backorder level can be determined as followings:

$$
\begin{aligned}
G_B(z) &= E[z^B] = E[z^{(N-S)^+}] \\
&= E[z^{(N-S)^+}1_{\{N\le S\}}] + E[z^{(N-S)^+}1_{\{N>S\}}] \\
&= P(N \le S) + \sum_{n=S+1}^{\infty} z^{(n-S)}P(N=n) \\
&= P(N \le S) + z^{-S}\sum_{n=S+1}^{\infty} z^{n}P(N=n) \\
&= P(N \le S) + z^{-S}\sum_{n=S+1}^{\infty} z^{n}\rho^{n}(1-\rho) \\
&= P(N \le S) + (1-\rho)z^{-S}\sum_{n=S+1}^{\infty} (z\rho)^{n} \\
&= P(N \le S) + (1-\rho)z^{-S}\frac{(\rho z)^{S+1}}{1-\rho z} \\
&= P(N \le S) + (1-\rho)\rho^{S}(\frac{1}{1-\rho z} - 1) \\
G'_B(z) &= (1-\rho)\frac{\rho^{S+1}}{(1-\rho z)^2} \\
E[B] &= G'_B(1) = (1-\rho)\frac{\rho^{S+1}}{(1-\rho)^2} \\
&= \frac{\rho^{S+1}}{1-\rho}
\end{aligned}
$$

$$
\begin{aligned}
E[B_i] &= E[B]\Theta_i \\
&= \frac{\rho^{S+1}\Theta_i}{1-\rho}
\end{aligned}
$$

APPENDIX B: Expected Inventory Levels of M/M/1 Queuing Systems

With the ease of generating functions expected inventory level can be determined as followings:

$$
\begin{aligned}
G_{(S-N)^+}(z) &= E[z^{(S-N)^+}] \\
&= E[z^{(S-N)^+} 1_{\{N \le S\}}] + E[z^{(S-N)^+} 1_{\{N > S\}}] \\
&= \sum_{n=0}^{S} z^{(S-n)} P(N = n) + P(N > S) \\
&= \sum_{n=0}^{S} z^{(S-n)} (1-\rho)\rho^n + 1 - \sum_{n=0}^{S} (1-\rho)\rho^n \\
&= z^S (1-\rho) \sum_{n=0}^{S} (\frac{\rho}{z})^n + 1 - (1-\rho) \sum_{n=0}^{S} \rho^n \\
&= z^S (1-\rho) \frac{1 - (\frac{\rho}{z})^{S+1}}{1 - (\frac{\rho}{z})} + 1 - (1-\rho) \frac{1-\rho^{S+1}}{1-\rho} \\
&= (1-\rho) \frac{z^{S+1} - \rho^{S+1}}{z - \rho} + \rho^{S+1}
\end{aligned}
$$

$$
G'_{(S-N)^+}(z) = (1-\rho) \frac{z^S (zS - \rho S - \rho) + \rho^{S+1}}{(1-\rho)^2}
$$

$$
\begin{aligned}
E[(S-N)^+] &= G'{}_{(S-N)^+}(1) \\
&= (1-\rho) \frac{(S - \rho S - \rho) + \rho^{S+1}}{(1-\rho)^2} \\
&= \frac{S(1-\rho) - \rho + \rho^{S+1}}{1-\rho} \\
&= S - \frac{\rho(1-\rho^S)}{1-\rho}
\end{aligned}
$$

APPENDIX C: Optimal Base Stock Level

$$
\begin{aligned}
\Pi(S) &= -\left[\sum_{i=1}^{n} E[B] b_i \Theta_i + hE[(S-N)^+]\right] + \sum_{i=1}^{n} \lambda_i(p_i)p_i \\
\Pi(S) &= -\left[\sum_{i=1}^{n} b_i \Theta_i \sum_{j=S}^{\infty}(j-S)P(N=j) + h\sum_{j=0}^{S}(S-j)P(N=j)\right] + \sum_{i=1}^{n} \lambda_i(p_i)p_i \\
\Pi(S+1) &= -\left[\sum_{i=1}^{n} b_i \Theta_i \sum_{j=S+1}^{\infty}(j-S-1)P(N=j) + h\sum_{j=0}^{S+1}(S+1-j)P(N=j)\right] \\
&\quad + \sum_{i=1}^{n} \lambda_i(p_i)p_i
\end{aligned}
$$

$$
\begin{aligned}
\Delta\Pi(S) &= \Pi(S+1) - \Pi(S) \\
&= \sum_{i=1}^{n} b_i \Theta_i P(N > S) - hP(N \le S) \\
&= \sum_{i=1}^{n} b_i \Theta_i - P(N \le S)\left[\sum_{i=1}^{n} b_i \Theta_i + h\right]
\end{aligned}
$$

$$
\begin{aligned}
\Delta\Pi(S-1) &= \sum_{i=1}^{n} b_i \Theta_i - P(N \le S-1)\left[\sum_{i=1}^{n} b_i \Theta_i + h\right] \\
\Delta^2\Pi(S) &= \Delta\Pi(S) - \Delta\Pi(S-1) \\
&= -P(N=S)\left[\sum_{i=1}^{n} b_i \Theta_i + h\right] < 0
\end{aligned}
$$

Since the second difference $\Delta^2\Pi(S)$ is negative, $\Pi(S)$ function is concave over S. Optimal base stock level is determined as the first integer that makes the first difference $\Delta\Pi(S)$ nonpositive.

$$
\begin{aligned}
\sum_{i=1}^{n} b_i \Theta_i - P(N \le S)\left[\sum_{i=1}^{n} b_i \Theta_i + h\right] &\le 0 \\
P(N \le S) \ge \frac{\sum_{i=1}^{n} b_i \Theta_i}{\sum_{i=1}^{n} b_i \Theta_i + h}&
\end{aligned}
$$

APPENDIX D: Continuous Optimal Base Stock Level of M/M/1 Systems

$$
\begin{aligned}
\Pi(S) &= -\left[(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{\rho^{S+1}}{1-\rho} + h(S - \frac{\rho}{1-\rho})\right] + \sum_{i=1}^{n}\lambda_i(p_i)p_i \\
\Pi'(S) &= -\left[(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{\rho^{S+1}\log(\rho)}{1-\rho} + h\right] \\
\Pi''(S) &= -(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{\rho^{S+1}(\log(\rho))^2}{1-\rho} \leq 0
\end{aligned}
$$

Since $\Pi''(s)$ is nonpositive, the function is concave over S. Thus, we can find the optimal base stock level by setting the first derivative to zero.

$$
\begin{aligned}
\Pi'(S) = -\left[(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{\rho^{S+1}\log(\rho)}{1-\rho} + h\right] &= 0 \\
(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{\rho^{S+1}\log(\rho)}{1-\rho} + h &= 0
\end{aligned}
$$

$$
\begin{aligned}
\rho^{S+1} &= -\left[\frac{h(1-\rho)}{\log(\rho)(\sum_{i=1}^{n} b_i\Theta_i + h)}\right] \\
(S+1)\log(\rho) &= \log\left(\frac{h(1-\rho)}{(-\log(\rho))(\sum_{i=1}^{n} b_i\Theta_i + h)}\right) \\
S^* &= \frac{1}{\log(\rho)}\log\left(\frac{h(1-\rho)}{(-\log(\rho))(\sum_{i=1}^{n} b_i\Theta_i + h)}\right) - 1
\end{aligned}
$$

APPENDIX E: Exponentially Approximated Model

Let v=$\frac{1}{E[N]}$.

$$\begin{aligned}
E[(S-N)^+] &= \int_{x=0}^{S}(S-x)P(N=x)dx = \int_{x=0}^{S}(S-x)ve^{-vx}dx \\
&= v\left[(S-x)\frac{-e^{-vx}}{v} - \int_{x=0}^{S}(-1)\frac{-e^{-vx}}{v}\right]_0^S \\
&= \left[(x-S)e^{-vx} - \int_{x=0}^{S}e^{-vx}\right]_0^S \\
&= \left[(x-S)e^{-vx} + \frac{e^{-vx}}{v}\right]_0^S \\
&= \frac{e^{-vS}}{v} - (-S + \frac{1}{v}) \\
&= S + \frac{e^{-vS}}{v} - \frac{1}{v}
\end{aligned}$$

$$\begin{aligned}
(N-S)^+ &= (S-N)^+ + N - S \\
E[(N-S)^+] &= E[(S-N)^+] + E[N] - S \\
&= S + \frac{e^{-vS}}{v} - \frac{1}{v} + \frac{1}{v} - S \\
&= \frac{e^{-vS}}{v}
\end{aligned}$$

$$\begin{aligned}
\Pi(p_i,S) &= -\left[\sum_{i=1}^{n} b_i\Theta_i E[B] + hE[(S-N)^+]\right] + \sum_{i=1}^{n}\lambda_i(p_i)p_i \\
&= -\left[\sum_{i=1}^{n} b_i\Theta_i\frac{e^{-vS}}{v} + h(S + \frac{e^{-vS}}{v} - \frac{1}{v})\right] + \sum_{i=1}^{n}\lambda_i(p_i)p_i \\
&= -\left[(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{e^{-vS}}{v} + h(S - \frac{1}{v})\right] + \sum_{i=1}^{n}\lambda_i(p_i)p_i \\
\Pi'(p_i,S) &= -\left[-(\sum_{i=1}^{n} b_i\Theta_i + h)e^{-vS} + h\right] = (\sum_{i=1}^{n} b_i\Theta_i + h)e^{-vS} - h \\
\Pi''(p_i,S) &= -v(\sum_{i=1}^{n} b_i\Theta_i + h)e^{-vS} < 0
\end{aligned}$$

Since the second derivative is negative, the profit function is concave over S. So we can find the optimum base stock level by setting the first derivative to zero.

$$\begin{aligned}
\Pi'(p_i, S) &= (\sum_{i=1}^{n} b_i\Theta_i + h)e^{-vS} - h = 0 \\
S^* &= \frac{-1}{v} log\left(\frac{h}{\sum_{i=1}^{n} b_i\Theta_i + h}\right)
\end{aligned}$$

S^* is substituted into profit function.

$$\begin{aligned}
\Pi(p_i, S) &= -\left[(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{e^{-vS^*}}{v} + h(S^* - \frac{1}{v})\right] + \sum_{i=1}^{n} \lambda_i(p_i)p_i \\
&= -\left[(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{e^{-v\frac{-1}{v}log\left(\frac{h}{\sum_{i=1}^{n} b_i\Theta_i+h}\right)}}{v} + h(S^* - \frac{1}{v})\right] + \sum_{i=1}^{n} \lambda_i(p_i)p_i \\
&= -\left[(\sum_{i=1}^{n} b_i\Theta_i + h)\frac{\frac{h}{\sum_{i=1}^{n} b_i\Theta_i+h}}{v} + h(S^* - \frac{1}{v})\right] + \sum_{i=1}^{n} \lambda_i(p_i)p_i \\
&= -hS^* + \sum_{i=1}^{n} \lambda_i(p_i)p_i
\end{aligned}$$

APPENDIX F: Loss Sales Model

By using balance equations,probabilities can be computed as followings:

$$\begin{aligned}
\lambda P_{i-1} &= \mu P_i \\
P_i &= \frac{\lambda}{\mu} P_{i-1} \\
P_i &= (\frac{\lambda}{\mu})^i P_0 = \rho^i P_0 \\
\sum_{i=0}^{S} P_i &= \sum_{i=0}^{S} \rho^i \lambda P_0 \\
1 &= \frac{1-\rho^{S+1}}{1-\rho} P_0 \\
P_0 &= \frac{1-\rho}{1-\rho^{S+1}} \\
P_i &= \frac{(1-\rho)\rho^i}{1-\rho^{S+1}}
\end{aligned}$$

By using these probabilities, expectation can be computed as followings:

$$\begin{aligned}
E[N] &= \sum_{n=0}^{S} n P_n \\
&= \sum_{n=0}^{S} n \frac{(1-\rho)\rho^n}{1-\rho^{S+1}} \\
&= \frac{(1-\rho)}{1-\rho^{S+1}} \sum_{n=0}^{S} n \rho^n
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=0}^{S} n\rho^n &= \sum_{n=0}^{S}\sum_{k=1}^{n} \rho^n = \sum_{k=1}^{S}\sum_{n=k}^{n} \rho^n \\
&= \sum_{k=1}^{S} \frac{1-\rho^{S+1}}{1-\rho} - \frac{1-\rho^k}{1-\rho} \\
&= \frac{-S\rho^{S+1} + \sum_{k=1}^{S} \rho^k}{1-\rho} \\
&= \frac{-S\rho^{S+1} + \frac{1-\rho^{S+1}}{1-\rho} - 1}{1-\rho} \\
&= \frac{-(1-\rho)S\rho^{S+1} - 1 + \rho + 1 - \rho^{S+1}}{(1-\rho)^2} \\
&= \frac{-S\rho^{S+1} + S\rho\rho^{S+1} + \rho - \rho^{S+1}}{(1-\rho)^2} \\
&= \frac{\rho(-S\rho^S + S\rho^{S+1} + 1 - \rho^S)}{(1-\rho)^2} \\
&= \frac{\rho(1-(S+1)\rho^S + S\rho^{S+1})}{(1-\rho)^2}
\end{aligned}
$$

$$
\begin{aligned}
E[N] &= \frac{(1-\rho)}{1-\rho^{S+1}} \sum_{n=0}^{S} n\rho^n \\
&= \frac{(1-\rho)}{1-\rho^{S+1}} \frac{\rho(1-(S+1)\rho^S + S\rho^{S+1})}{(1-\rho)^2} \\
&= \frac{\rho(1-(S+1)\rho^S + S\rho^{S+1})}{(1-\rho)(1-\rho^{S+1})} \\
P(N=S) &= \frac{(1-\rho)\rho^S}{1-\rho^{S+1}} \\
1 - P(N=S) &= \frac{1-\rho^S}{1-\rho^{S+1}}
\end{aligned}
$$

We can express $\Pi(p_i, S)$ by using $E[N]$ and $P(N=S)$ as following:

$$
\Pi(p_i, S) = -hS + h\frac{\rho(1-(S+1)\rho^S + S\rho^{S+1})}{(1-\rho)(1-\rho^{S+1})} + \frac{1-\rho^S}{1-\rho^{S+1}}\rho\mu\frac{K-\rho\mu}{M}
$$

APPENDIX G: Optimal Traffic Intensity ρ of an M/M/1 System If $S = 0$

Setting $S = 0$ and expressing p as $\frac{K-\lambda}{M}$ $=\frac{K-\rho\mu}{M}$ results in a profit function as followings:

$$\Pi(\rho) = -(\sum_{i=1}^{n} b_i \Theta_i)\frac{\rho}{1-\rho} + \frac{K\mu}{M}\rho - \frac{\mu^2}{M}\rho^2$$

Defining A=$\sum_{i=1}^{n} b_i\Theta_i$, B=$\frac{\rho}{1-\rho}$ and R=$\frac{K\mu}{M}\rho - \frac{\mu^2}{M}\rho^2$ helps us to find the derivative of profit function over ρ.

$$\begin{aligned}
\Theta_i &= \frac{\lambda_i}{\lambda} \\
&= \frac{k_i - m_i p}{\rho\mu} \\
&= \frac{k_i - m_i\left(\frac{K-\rho\mu}{M}\right)}{\rho\mu} \\
&= \frac{k_i M - m_i K + \rho\mu m_i}{\rho\mu M} \\
A &= \sum_{i=1}^{n} b_i\Theta_i = \frac{M\sum_{i=1}^{n} b_i k_i - K\sum_{i=1}^{n} b_i m_i + \rho\mu\sum_{i=1}^{n} b_i m_i}{\rho\mu M} \\
A' &= \frac{\sum_{i=1}^{n} b_i m_i}{\rho M} - \frac{M\sum_{i=1}^{n} b_i k_i - K\sum_{i=1}^{n} b_i m_i + \rho\mu\sum_{i=1}^{n} b_i m_i}{\mu M}\left(\frac{1}{\rho^2}\right) \\
&= -\frac{M\sum_{i=1}^{n} b_i k_i - K\sum_{i=1}^{n} b_i m_i}{\rho^2\mu M} \\
A'' &= 2\frac{M\sum_{i=1}^{n} b_i k_i - K\sum_{i=1}^{n} b_i m_i}{\rho^3\mu M} \\
B &= \frac{\rho^{S+1}}{(1-\rho)} \\
B' &= \frac{\rho^S(S(1-\rho)+1)}{(1-\rho)^2} \\
B'' &= \frac{\rho^{S-1}(S^2(1-\rho)^2 + S(1-\rho^2) + 2\rho)}{(1-\rho)^3}
\end{aligned}$$

$$
\begin{aligned}
C' &= A'B + B'A \\
&= -\frac{M\sum_{i=1}^{n} b_i k_i - K\sum_{i=1}^{n} b_i m_i}{\rho^2 \mu M}\frac{\rho^{S+1}}{(1-\rho)} \\
&\quad + \frac{\rho^S (S(1-\rho)+1)}{(1-\rho)^2}\frac{(M\sum_{i=1}^{n} b_i k_i - K\sum_{i=1}^{n} b_i m_i + \rho\mu\sum_{i=1}^{n} b_i m_i)}{\rho\mu M} \\
&= \rho^{S-1}\frac{(S(1-\rho)+\rho)(M\sum_{i=1}^{n} b_i k_i + (\mu\rho - K)\sum_{i=1}^{n} b_i m_i) + \mu\rho(1-\rho)\sum_{i=1}^{n} b_i m_i}{M\mu(1-\rho)^2} \\
C'' &= \rho^{S-2}\frac{((1-\rho)^2 S^2 - (3\rho^2 - 4\rho + 1)S + 2\rho^2)(M\sum_{i=1}^{n} b_i k_i + (\mu\rho - K)\sum_{i=1}^{n} b_i m_i)}{M\mu(1-\rho)^3} \\
&\quad + \rho^{S-2}\frac{2\mu\rho\sum_{i=1}^{n} b_i m_i(\rho(1-\rho) + S(1-\rho)^2)}{M\mu(1-\rho)^3}
\end{aligned}
$$

If the base stock level S is set to 0, the first and the second derivatives of the backorder cost are obtained as followings:

$$
\begin{aligned}
C' &= \frac{1}{(1-\rho)^2}\left\{\frac{\sum_{i=1}^{n} b_i m_i}{M}(1-\frac{K}{\mu}) + \frac{\sum_{i=1}^{n} b_i k_i}{\mu}\right\} \\
C'' &= \frac{2}{(1-\rho)^3}\left\{\frac{\sum_{i=1}^{n} b_i m_i}{M}(1-\frac{K}{\mu}) + \frac{\sum_{i=1}^{n} b_i k_i}{\mu}\right\}
\end{aligned}
$$

If $S = 0$, and $K \leq \mu$, it is evident that the first and the second derivatives of the backorder cost is nonnegative. Thus, the convexity of the backorder cost is sustained. The first and the second derivatives of the revenue function can be found as followings:

$$
\begin{aligned}
R &= \frac{K\mu}{M}\rho - \frac{\mu^2}{M}\rho^2 \\
R' &= \frac{K\mu}{M} - 2\frac{\mu^2}{M}\rho \\
R'' &= -2\frac{\mu^2}{M} < 0
\end{aligned}
$$

Since the second derivative of the revenue function is negative, it is concave over traffic intensity. Profit function is obtained by subtracting the convex backorder cost from the concave revenue function. Thus, the profit function is also concave. The critical points are found from Π'=0 equation. For $z = 1 - \rho$ variable is defined.

$$\begin{aligned} -\frac{1}{z^2}\left\{\frac{\sum_{i=1}^{n} b_i m_i}{M}(1-\frac{K}{\mu}) + \frac{\sum_{i=1}^{n} b_i k_i}{\mu}\right\} + \frac{K\mu}{M} - 2\frac{\mu^2}{M}(1-z) &= 0 \\ 2\frac{\mu^2}{M}z^3 + (\frac{K\mu}{M} - 2\frac{\mu^2}{M})z^2 + \left\{\frac{\sum_{i=1}^{n} b_i m_i}{M}(\frac{K}{\mu}-1) - \frac{\sum_{i=1}^{n} b_i k_i}{\mu}\right\} &= 0 \end{aligned}$$

Finally a cubic function is obtained. Its roots can be evaluated with cubic function formula.

Let define z=1-ρ, X=$\frac{2\mu^2}{M}$, Y=$\frac{K\mu}{M}$, d=$\left\{\frac{\sum_{i=1}^{n} b_i m_i}{M}(\frac{K}{\mu}-1) - \frac{\sum_{i=1}^{n} b_i k_i}{\mu}\right\}$.

$$Xz^3 + (Y-X)z^2 + d = 0$$

If $K < \mu$, the concavity is sustained. So, only a single root is required to have this maximum point. It can verified by the APPENDIX H. If $K < \mu$, then $\Delta < 0$. Thus, the cubic function has only one real solution that gives optimal ρ.

APPENDIX H: The roots of a cubic function

$$f(x) = ax^3 + bx^2 + cx + d$$

Let $A = 2b^3 - 9abc + 27a^2d,\ B = 4(b^2 - 3ac)^3$

$$x_1 = -\frac{b}{3a} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[A + \sqrt{A^2 - B}\right]} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[A - \sqrt{A^2 - B}\right]}$$

$$x_2 = -\frac{b}{3a} + \frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[A + \sqrt{A^2 - B}\right]} + \frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[A - \sqrt{A^2 - B}\right]}$$

$$x_3 = -\frac{b}{3a} + \frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[A + \sqrt{A^2 - B}\right]} + \frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[A - \sqrt{A^2 - B}\right]}$$

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$$

- If $\Delta > 0$, the equation has three distinct roots.
- If $\Delta = 0$, the equation has a multiple root and all its roots are real.
- If $\Delta < 0$, the equation has one real root and two nonreal complex roots.

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