COMPUTATION OF GRID HOMOLOGY

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B.S., Mathematics, Boğaziçi University, 2015

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2017

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my thesis advisor, Assoc. Prof. Çagrı Karakurt, for his advices, numerous ideas, and his endless patience during the work. Without his support and motivation, this thesis would not exist.

I should thank to Burak Özbağcı and Ferit Öztürk for joining my thesis committee and expressing valuable and encouraging comments.

I would also like to thank Prof. Olcay Coşkun for his help, useful comments and support throughout my studies.

I am debtful to my family and to all lovely friends, especially İlkiz Bildik and Kübra Kaytancı and to my favorite couple Turan Karakurt and Sibel Kahraman for their sincere and genuine friendship.

ABSTRACT

COMPUTATION OF GRID HOMOLOGY

In this thesis, the main aim is to study a combinatorial approach of knot Floer homology for a given knot K (or a link) in S^3, called *grid homology*. We will define the bigrading structure on the chain complexes, and will show the difference and relation between the variants of grid homology. We will show, by a simple example, the invariance of simply blocked grid homology $\widehat{GH}(\mathbb{G})$ under stabilization move. We will compute the symmetrized Alexander polynomial of a knot K in S^3, a polynomial knot invariant, using grid diagrams. Finally, we will compute grid homology of *positive Hopf link*.

ÖZET

GRİD HOMOLOJİ HESABI

Bu tezde, verilen bir düğümün knot Floer homolojisini hesaplamak için kullanacağımız, grid homoloji diye adlandırılan kombinatorik bir yaklaşımı inceleyeceğiz. Zincir kompleksleri üzerindeki ikili dereceleri tanımlayacak ve grid homoloji türlerinin farklarını ve aralarındaki ilişkiyi göstereceğiz. Bir örnekle "simply blocked" grid homolojinin $\widehat{GH}(\mathbb{G})$ stabilizasyon operasyonu altında değişmezliğini göstereceğiz. Grid diyagramlarını kullanarak bir düğümün simetrize edilmiş Alexander polinomunu hesaplayacağız. Son olarak, pozitif Hopf linkinin grid homolojisini hesaplayacağız.

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LIST OF SYMBOLS

$\mathbb{G}$	Grid diagram
$GC^{-}(\mathbb{G})$	The unblocked grid chain complex associated to the grid diagram $\mathbb{G}$
$\widehat{GC}(\mathbb{G})$	The simply blocked grid chain complex associated to the grid diagram $\mathbb{G}$
$\widetilde{GC}(\mathbb{G})$	The fully blocked grid chain complex associated to the grid diagram $\mathbb{G}$
$GH^{-}(\mathbb{G})$	The unblocked grid homology of the chain complex $GC^{-}(\mathbb{G})$
$\widehat{GH}(\mathbb{G})$	The simply blocked grid homology of the chain complex $\widehat{GC}(\mathbb{G})$
$\widetilde{GH}(\mathbb{G})$	The fully blocked grid homology of the chain complex $\widetilde{GC}(\mathbb{G})$
$\mathbb{O}$	The set of O-markings on grid diagram $\mathbb{G}$
$\mathrm{Rect}(\mathbf{x},\mathbf{y})$	The set of rectangles going from $\mathbf{x}$ to $\mathbf{y}$
$\mathrm{Rect}^{\circ}(\mathbf{x},\mathbf{y})$	The set of empty rectangles going from $\mathbf{x}$ to $\mathbf{y}$
$\mathbf{S}(\mathbb{G})$	The set of grid states
$T_{2,3}$	Right handed trefoil
$T_{-2,3}$	Left handed trefoil
$\mathbb{X}$	The set of X-markings on grid diagram $\mathbb{G}$

1. INTRODUCTION

The classification of knots has always drawn topologists' attention. In the history of knot theory, one can easily see many useful knot invariants. There are numerical, polynomial, and algebraic invariants for knots. Knot Floer homology is a homology theory which associates invariants to knots. It has been discovered by Peter Ozsváth and Zoltán Szabó [1], and independently by Jacob Rasmussen [2]. Knot Floer homology can be considered as an algebraic invariant discovered through studying how Heegaard Floer homology for 3-manifolds, a 3-manifold invariant [3], changes when the 3-manifold is exposed to Dehn surgery along a knot. Knot Floer homology is extremely important for the development of knot theory since it contains many information about the corresponding knot such as its Alexander-Conway polynomial detection of unknot, Seifert genus, four-ball genus, fiberedness, the unknotting number, etc. Although knot Floer homology has several useful applications which improve the studies in the theory of knots, the computation, counting holomorphic disks, is not so easy. Grid homology is a combinatorial description of knot Floer homology, roughly a bigraded module over a multi-variable polynomial ring associated to the grid diagram, defined by Ciprian Manolescu, Peter Ozsváth and Sucharit Sarkar in [4]. As a combinatorial approach, grid homology has an important role in the computation of knot Floer homology and to prove many crucial results of knot Floer homology.

2. PRELIMINARIES

Definition 2.1. *A smootly embedded circle S^1 into S^3, up to isotopy, is called a knot K in S^3.*

A link L in S^3 can be considered as a multi-component knot.

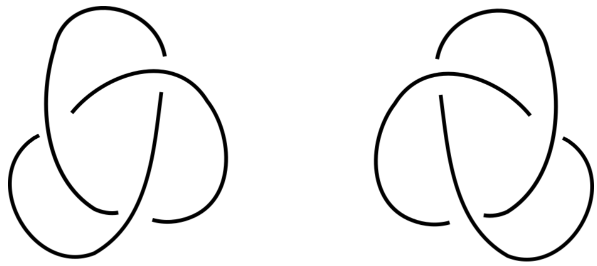

Figure 2.1. Left handed trefoil $T_{-2,3}$ and right handed trefoil $T_{2,3}$.

Theorem 2.2. *(Reidemeister) [5] Two links are equivalent if and only if their diagrams can be connected by a (finite) sequence of Reidemeister moves and (planar) isotopies.*

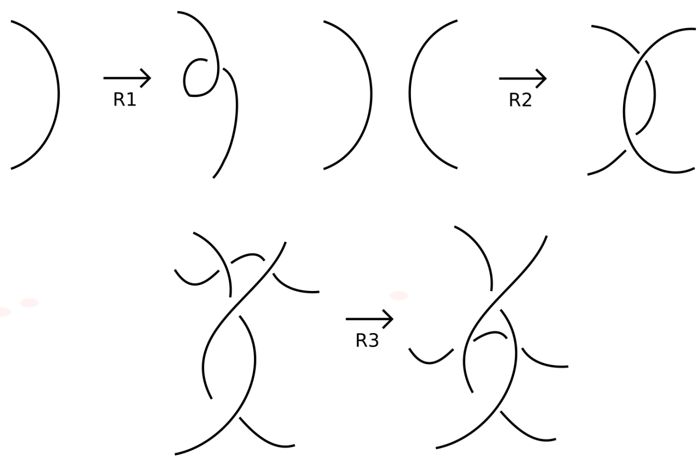

Figure 2.2. Reidemeister moves.

2.1. Grid presentation of a knot

Definition 2.3. *A planar grid diagram $\mathbb{G}$ for a knot is an $n \times n$ grid with n^2 squares and X and O markings with the following rules:*

(i) Each column and each row of grid contains exactly one X marking and one O marking.

(ii) On the columns X-markings point to O-markings while on the rows O-markings point to X-markings with the convention that vertical segments pass over horizontal segments at a crossing.

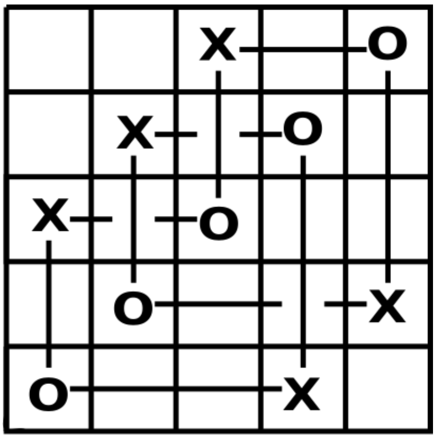

Figure 2.3. A grid representation of left handed trefoil $T_{-2,3}$.

Lemma 2.4. *Every link L in S^3 has a grid representation.*

Proof. Let L be a link in S^3 with a projection, say $p(L)$, onto $\mathbb{R}^2$. We can approximate the projection of the link L by a PL-embedding with the convention that $p(L)$ has only vertical and horizontal segments. In case of need, we must modify the crossings with the convention of grid diagrams that at a crossing vertical segments pass over horizontal ones. Finally, after marking the turning points by suitable O's and X's, we get a grid representation for the link L. See figure 2.4 to illustrate the process. □

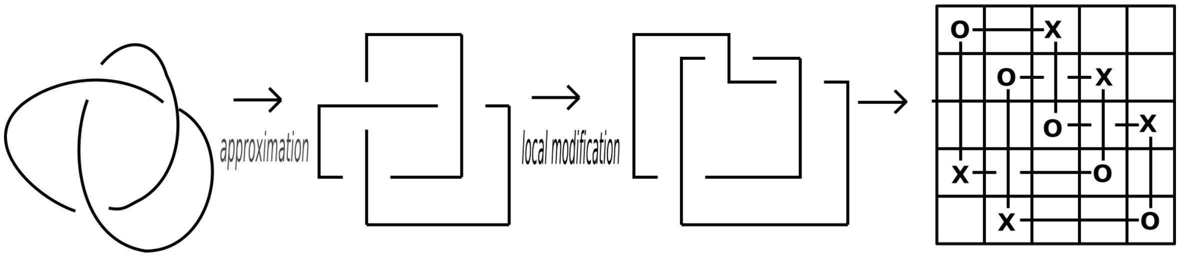

Figure 2.4. Obtaining grid diagram for a knot from its projection.

2.2. Grid Moves

A natural question that should come to mind would be how can we distuinguish whether two grid diagrams represent the same link or not. The answer is in the theorem of Cromwell, which is an analog of the theorem of Reidemeister. We will have some moves on the grid diagram called *Cromwell moves* like as *Reidemeister moves*.

It may be useful to regard our grid diagram as living on a torus by identifying the top segment with the bottom segment, and the left segment with the right segment on the boundary of the planar grid diagram. The resulting diagram will be called *toroidal grid diagram.*

2.2.1. Commutation Moves

Let $\mathbb{G}$ be a (planar) grid presentation of a knot K. Consider the grid diagram on the plane. Then the segments connecting the markings have corresponding intervals in the plane. In this sense, if two consecutive columns (or rows) are either disjoint or nested, interchanging these columns (or rows) is called *commutation move.* Notice that in the figure 2.5 the commutation move on the columns corresponds to Reidemeister move 2.

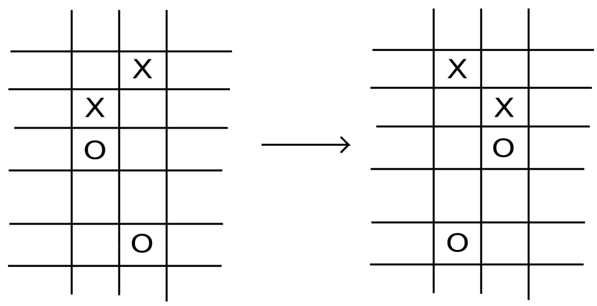

Figure 2.5. Commutation of two nested columns in the grid.

2.2.2. Stabilization / Destabilization

There is another grid move which is called *stabilization* of $\mathbb{G}$. Stabilization process on a grid diagram is handled as follows:

Start with a grid diagram $\mathbb{G}$. Choose a marked square in $\mathbb{G}$. Then delete its column and row in order to add two columns and two rows instead of the deleted ones. There are four choices to occupy some suitable squares in the new columns and rows with X- and O-markings, see figure 2.6. Finally, we get a new $(n+1) \times (n+1)$ grid diagram $\mathbb{G}'$ which is equivalent to the initial grid diagram $\mathbb{G}$. Notice that in the figure 2.6, $X : NW$ type stabilization move corresponds to Reidemeister move 1 while $X : NE$, $X : SE$ and $X : SW$ types correspond to planar isotopies. Destabilization move can be defined the inverse of the stabilization move so that it drops the gird number by one.

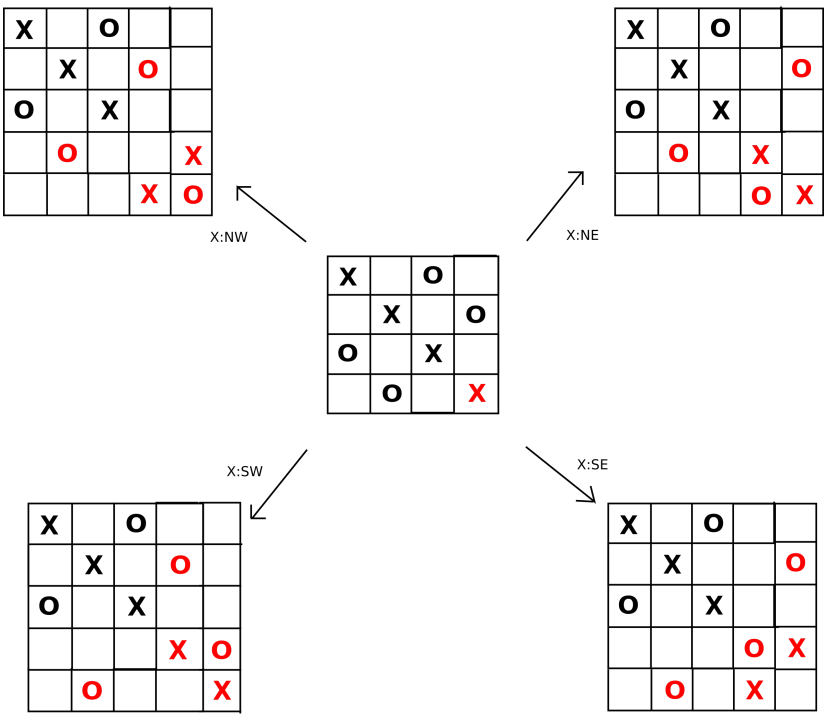

Figure 2.6. Stabilization types

For a grid torus diagram $\mathbb{G}$, we can define another grid move which is called *cyclic permutation*. Cut a grid torus through an arbitrary horizontal and vertical circle. This can be done in n^2 ways if the grid number of $\mathbb{G}$ is n. Then we call the resulting diagrams as a *planar realizations* of the grid torus $\mathbb{G}$. Permuting the columns (or rows) of a planar grid diagram is called *cyclic permutation* which has changed nothing on the associated grid torus.

Lemma 2.5. *A cyclic permutation can be expressed as a sequence of grid moves such as commutations, X:NW, X:SE, O:NW, O:SE type stabilizations, and destabilizations.*

For the proof, see the book [6].

Theorem 2.6. *(Cromwell) [7] Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two planar diagrams for links L_1 and L_2, respectively. L_1 and L_2 are equivalent links if and only if there is a finite sequence of grid moves between $\mathbb{G}_1$ and $\mathbb{G}_2$.*

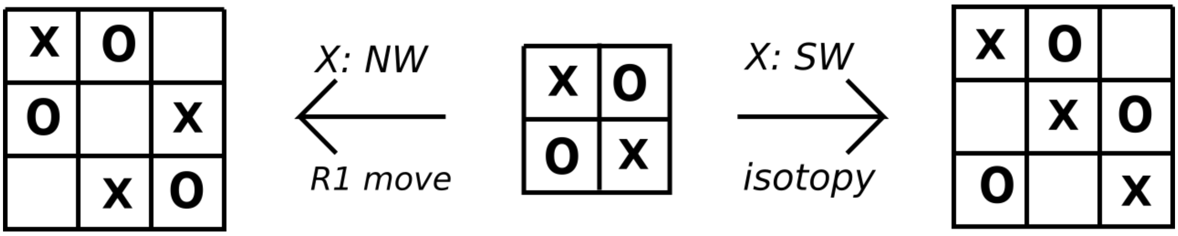

Figure 2.7. Three different grid diagrams of the unknot $\mathcal{O}$

3. GRID HOMOLOGY

The aim of this chapter is to define the grid homology. In this purpose, we need to specify our chain complex with their differential maps. In the language of grid diagrams (toroidal) for knots, our chain complexes have bigraded structure with generators which have one-to-one correspondence with grid states, and their differentials count rectangles between grid states. The way that rectangles are counted give us rise to define different versions of grid homology. There are different versions of grid homology for a knot which are determined by how the differential counts rectangles connecting two grid states.

3.1. Grid states

In this section we will define a new concept called *grid states*. The set of grid states basically forms a basis for our chain complex.

Definition 3.1. *Consider a grid torus $\mathbb{G}$ of a knot K. A grid state for an $n \times n$ grid diagram $\mathbb{G}$ is an n-tuple of points $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ such that each x_i lies on the intersection of horizontal circles $\{\alpha_i\}_{i=1}^n$ and vertical circles $\{\beta_i\}_{i=1}^n$ on the grid torus with the convention that each horizontal and each vertical circle contains exactly one of the components of $\mathbf{x}$. For illustration by an example, see the figure 3.1.*

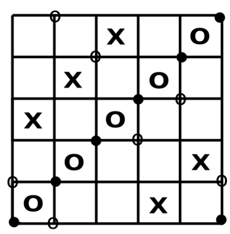

Figure 3.1. Two grid states of the left handed trefoil $T_{-2,3}$

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Let $\mathbf{S}(\mathbb{G})$ be the set of all grid states for a grid diagram. Notice that the cardinality of the set $\mathbf{S}(\mathbb{G})$ is $n!$ for an $n \times n$ grid diagram $\mathbb{G}$. This makes the computation of grid homology challenging. There are some algorithms to compute the knot Floer homology associated to a knot with at most 12 crossings using grid diagrams. For more detail you may see the joint work of Baldwin and Gillam [8].

3.2. Rectangles connecting grid states

Consider two grid states $\mathbf{x}$ and $\mathbf{y}$ which differ only in two intersection points. An embedded rectangle r in the torus has the orientation inheritted from the torus. If the four different coordinates of $\mathbf{x}$ and $\mathbf{y}$ form the corners of the rectangle r with the convention that the boundary of this rectangle lies on the union of horizontal and vertical circles on the torus, $\partial r \subset \alpha \cup \beta$, and the boundary of the horizontal segments of the rectangle $\partial\left(\partial_{\alpha} r\right) = \mathbf{y} - \mathbf{x}$ while the boundary of the vertical segments of the rectangle $\partial\left(\partial_{\beta} r\right) = \mathbf{x} - \mathbf{y}$, then we say that the rectangle r *goes* from $\mathbf{x}$ *to* $\mathbf{y}$.

The set of rectangles going from $\mathbf{x}$ to $\mathbf{y}$ is denoted by $\text{Rect}(\mathbf{x}, \mathbf{y})$. A rectangle r *goes from* $\mathbf{x}$ *to* $\mathbf{y}$ is said to be *empty* if its interior does not contain any points of $\mathbf{x}$ or $\mathbf{y}$. The set of empty rectangles going from $\mathbf{x}$ to $\mathbf{y}$ is denoted by $\text{Rect}^{\circ}(\mathbf{x}, \mathbf{y})$.

The type of the rectangle is determined by the orientation of the rectangle inherited from the torus. If the orientation of the boundary of the rectangle is counterclockwise when the horizontal segments in the boundary of rectangle point from the components of $\mathbf{x}$ to the components of $\mathbf{y}$, then we can also say that r *goes from* $\mathbf{x}$ *to* $\mathbf{y}$. An example of rectangles in the sets $\text{Rect}(\mathbf{x}, \mathbf{y})$ and $\text{Rect}(\mathbf{y}, \mathbf{x})$ on a grid diagram is illustrated in the figure 3.2.

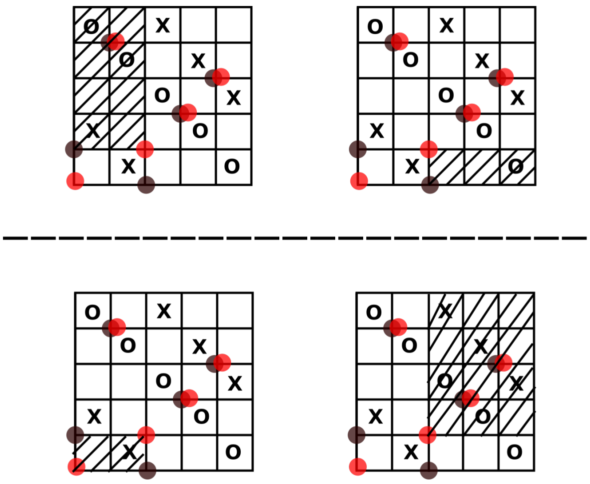

Figure 3.2. Rectangles in the top row are in the set Rect($\mathbf{x}$, $\mathbf{y}$) while rectangles in the bottom row are in the set Rect($\mathbf{y}$, $\mathbf{x}$). Black dots and red dots are $\mathbf{x}$ and $\mathbf{y}$, respectively.

3.3. Characterization of two gradings defined on $S(\mathbb{G})$

Grid chain complexes have bigraded structures induced by two integer-valued functions, *Maslov* and *Alexander functions.*

Proposition 3.2. *Consider a toroidal grid diagram* $\mathbb{G}$*. There exists a integer-valued function on* $\mathbf{S}(\mathbb{G})$

$$M_{\mathbb{O}} : \mathbf{S}(\mathbb{G}) \to \mathbb{Z}.$$

which characterized by the followings

(i) $M_{\mathbb{O}}(\mathbf{x}^{NWO}) = 0$ *where* $\mathbf{x}^{NWO}$ *represents the grid state whose components are on the northwest corner of the squares with* O*-marking.*

(ii) If the grid states $\mathbf{x}$ *and* $\mathbf{y}$ *are connected by a rectangle* $r \in$ *Rect*$(\mathbf{x}, \mathbf{y})$*, then*

$$M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{O}}(\mathbf{y}) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathbf{x} \cap Int(r)). \tag{3.1}$$

For the proof, see [6].

We will call this function $M_{\mathbb{O}} : \mathbf{S}(\mathbb{G}) \to \mathbb{Z}$ as *Maslov function.* The integer value $M_{\mathbb{O}}(\mathbf{x}) = d$ of a grid state $\mathbf{x}$ will be called as *Maslov grading of* $\mathbf{x}$.

Definition 3.3. *The Alexander function on* $\mathbf{S}(\mathbb{G})$ *can be defined using Maslov function on grid states as follows:*

$$A(\mathbf{x}) = \frac{1}{2}\left(M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{x})\right) - \left(\frac{n-1}{2}\right) \tag{3.2}$$

, where $M_{\mathbb{X}}$ *can be defined analogously, writing* $\mathbb{X}$*'s instead of* $\mathbb{O}$*'s.*

Proposition 3.4. *Consider a toroidal grid diagram* $\mathbb{G}$*. For all rectangles* r *going from* $\mathbf{x}$ *to* $\mathbf{y}$*, the Alexander function* A *is characterized, up to overall additive constant, by the following formula*

$$A(\mathbf{x}) - A(\mathbf{y}) = \#(r \cap \mathbb{X}) - \#(r \cap \mathbb{O}). \tag{3.3}$$

Moreover, for any grid state $\mathbf{x}$*,* $A(\mathbf{x}) \in \mathbb{Z}$*.*

Proof. For the proof, see [6]. □

Now we will give a combinatorial formula to compute the Maslov grading absolutely. Define a symmetric, rational-valued function $\mathcal{J}$ on $\mathbb{R}^2 \times \mathbb{R}^2$

$$\mathcal{J}(S_1, S_2) = \frac{\mathcal{I}(S_1, S_2) + \mathcal{I}(S_2, S_1)}{2}$$

where $\mathcal{I}(S_1, S_2)$ is a function that counts the number of pairs $(s_1, s_2) \in S_1 \times S_2$ with $s_1 < s_2$. The inequality $s_1 < s_2$ means that first (and second) component of s_1 is less than the first (and second) component of s_2.

Consider a toroidal grid diagram $\mathbb{G}$ on the plane $\mathbb{R}^2$ with the convention that the bottom-left corner of the grid is placed at the origin $(0,0)$. Consider the elements of the set $\mathbb{O} = \{O_i\}_{i=1}^n$ of O_i -markings have half-integer coordinates in the plane while any grid state $\mathbf{x} = \{x_1, \ldots, x_n\}$ in the set $\mathbf{S}(\mathbb{G})$ has integer coordinates.

The Maslov function $M_{\mathbb{O}} : \mathbf{S}(\mathbb{G}) \to \mathbb{Z}$ can be computed by the following formula

$$M_{\mathbb{O}}(\mathbf{x}) = \mathcal{J}(\mathbf{x}, \mathbf{x}) - 2\mathcal{J}(\mathbf{x}, \mathbb{O}) + \mathcal{J}(\mathbb{O}, \mathbb{O}) + 1. \tag{3.4}$$

3.4. The unblocked grid homology $GH^-(\mathbb{G})$

In this section we will ready to define a version of grid homology which is called *the unblocked grid homology* and denoted by $GC^-(\mathbb{G})$.

Definition 3.5. *The unblocked grid complex $GC^-(\mathbb{G})$ is the free $\mathcal{R}$-module with a generating set $\mathbf{S}(\mathbb{G})$, equipped with the $\mathcal{R}$-module endomorphism*

$$\partial_{\mathbb{X}}^- : GC^-(\mathbb{G}) \to GC^-(\mathbb{G})$$

defined as

$$\partial_{\mathbb{X}}^-(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}(\mathbb{G})} \sum_{\{r \in Rect^\circ(\mathbf{x},\mathbf{y}) | r \cap \mathbb{X} = \emptyset\}} U_1{}^{r(O_1)} \cdots U_n{}^{r(O_n)} \cdot \mathbf{y}$$

where $\mathcal{R} = \mathbb{F}[U_1, U_2, \ldots, U_n]$ is the polynomial ring over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ and $r(O_i)$ is the number of the O_i-marking in the rectangle r so that it is equal to 1 or 0.

As a $\mathbb{F}$-vector space, $GC^-(\mathbb{G})$ has a basis

$$\mathcal{B}(GC^-(\mathbb{G})) = \{U_1{}^{k_1} \cdots U_n{}^{k_n} \cdot \mathbf{x} \mid \mathbf{x} \in \mathbf{S}(\mathbb{G}) \text{ and } k_i \in \mathbb{Z}^{\geq 0}\}.$$

For this purpose, we should extend the Maslov and Alexander functions to the basis $\mathcal{B}(GC^-(\mathbb{G}))$ in order to give $GC^-(\mathbb{G})$ a bigrading $\mathcal{R}$-module structure as the following way

$$M({U_1}^{k_1}\cdots {U_n}^{k_n}\cdot \mathbf{x}) = M(\mathbf{x}) - 2k_1 - \cdots - 2k_n \tag{3.5}$$

$$A({U_1}^{k_1}\cdots {U_n}^{k_n}\cdot \mathbf{x}) = A(\mathbf{x}) - k_1 - \cdots - k_n. \tag{3.6}$$

The extended Maslov and Alexander gradings give $GC^-(\mathbb{G})$ a bigraded $\mathcal{R}$-module structure. We will prove the following theorem 3.6 by using some lemmas and basic definitions.

Theorem 3.6. *The pair $(GC^-(\mathbb{G}), \partial^-_{\mathbb{X}})$ is a bigraded chain complex over the polynomial ring $\mathcal{R} = \mathbb{F}[U_1, U_2, \ldots, U_n]$ in the sense of the following definition 3.7.*

Definition 3.7. *A bigraded $\mathcal{R}$-module M is an $\mathcal{R}$-module where $\mathcal{R} = \mathbb{F}[U_1, U_2, \ldots, U_n]$ such that*

(i) As a $\mathbb{F}$-vector space, M has a splitting as $M = \oplus_{d,s\in\mathbb{Z}} M_{d,s}$ where d and s are Maslov and Alexander functions, respectively.

(ii) Each U_i is an injective map between $\mathbb{F}$-vector subspaces $M_{d,s}$ of M such that

$$U_i : M_{d,s} \to M_{d-2,s-1}.$$

Definition 3.8. *A bigraded $\mathcal{R}$-module homomorphism that is homogeneous of degree (m, t) is a module homomorphism f: $M \to M'$ between bigraded $\mathcal{R}$-modules M and M'*

such that

$$f : M_{d,s} \to M'_{d+m,s+t}$$

for all $d, s \in \mathbb{Z}$.

Definition 3.9. *A bigraded chain complex over* $\mathcal{R} = \mathbb{F}[U_1, U_2, \ldots, U_n]$ *is a bigraded* $\mathcal{R}$*-module* C *equipped with an* $\mathcal{R}$*-module endomorphism* $\partial : C \to C$ *satisfying*

(i) $\partial \circ \partial = 0$.

(ii) ∂ *is a homogeneous map of degree* $(-1, 0)$.

Lemma 3.10. *The differential map* $\partial_{\mathbb{X}}^- : GC^-(\mathbb{G}) \to GC^-(\mathbb{G})$ *satisfies the property* $(\partial_{\mathbb{X}}^-)^2 = 0$.

Denote the horizontal boundary of the domain ψ by $\partial(\partial_\alpha\psi)$, where $\partial_\alpha\psi$ is the part of the domain ψ in $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$. Similarly denote the vertical boundary of the domain ψ by $\partial(\partial_\beta\psi)$ where $\partial_\beta\psi$ is the part of the domain ψ in $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_n$.

Definition 3.11. *A domain* ψ *going from* $\mathbf{x}$ *to* $\mathbf{y}$ *is a linear combination of the interiors of the squares in the grid diagram such that* $\partial(\partial_\alpha\psi) = \mathbf{y} - \mathbf{x}$ *and* $\partial(\partial_\beta\psi) = \mathbf{x} - \mathbf{y}$. *The set of all domains going from* $\mathbf{x}$ *to* $\mathbf{y}$ *is denoted by* $\pi(\mathbf{x}, \mathbf{y})$.

Proof of Lemma 3.10. Let $\mathbf{x}$ and $\mathbf{z}$ be two grid states in $\mathbf{S}(\mathbb{G})$. Fix a domain $\psi \in \pi(\mathbf{x}, \mathbf{z})$ such that $\psi = r_1 * r_2$ where $r_1 \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$ and $r_2 \in \mathrm{Rect}^\circ(\mathbf{y}, \mathbf{z})$. We know that if there is an empty rectangle r_1 going from $\mathbf{x}$ to $\mathbf{y}$ with $r_1 \cap \mathbb{X} = \emptyset$, the term ${U_1}^{r_1(O_1)} \cdots {U_n}^{r_1(O_n)} \cdot \mathbf{y}$ appears in $\partial_{\mathbb{X}}^-(\mathbf{x})$. We also know that if there is an empty rectangle r_2 going from $\mathbf{y}$ to $\mathbf{z}$ with $r_2 \cap \mathbb{X} = \emptyset$, consider $\partial_{\mathbb{X}}^- \circ \partial_{\mathbb{X}}^-(\mathbf{x})$ as follows:

$$\begin{aligned} \partial_{\mathbb{X}}^{-} \circ \partial_{\mathbb{X}}^{-}(\mathbf{x}) &= \partial_{\mathbb{X}}^{-}(U_1{}^{r_1(O_1)} \cdots U_n{}^{r_1(O_n)} \cdot \mathbf{y}) \\ &= U_1{}^{r_1(O_1)} \cdots U_n{}^{r_1(O_n)} \cdot \partial_{\mathbb{X}}^{-}(\mathbf{y}) \\ &= U_1{}^{r_1(O_1)} \cdots U_n{}^{r_1(O_n)} U_1{}^{r_2(O_1)} \cdots U_n{}^{r_2(O_n)} \cdot \mathbf{z} \\ &= U_1{}^{r_1(O_1)+r_2(O_1)} \cdots U_n{}^{r_1(O_n)+r_2(O_n)} \cdot \mathbf{z}. \end{aligned} \tag{3.7}$$

So the map $\partial_{\mathbb{X}}^{-} \circ \partial_{\mathbb{X}}^{-} : GC^{-}(\mathbb{G}) \to GC^{-}(\mathbb{G})$ is given by for all $\mathbf{x} \in \mathbf{S}(\mathbb{G})$,

$$\partial_{\mathbb{X}}^{-} \circ \partial_{\mathbb{X}}^{-}(\mathbf{x}) = \sum_{\mathbf{z} \in \mathbf{S}(\mathbb{G})} \sum_{\{\psi \in \pi(\mathbf{x},\mathbf{z}) | \psi \cap \mathbb{X} = \emptyset\}} N(\psi) \cdot U_1{}^{\psi(O_1)} \cdots U_n{}^{\psi(O_n)} \cdot \mathbf{z} \tag{3.8}$$

where $N(\psi)$ is the number of ways of decomposition of the domain ψ as a composite of two empty rectangles r_1 and r_2, and $\psi(O_i) = r_1(O_i) + r_2(O_i), i \in \{1, 2, \ldots, n\}$.

We want to prove that $N(\psi) \cdot U_1{}^{\psi(O_1)} \cdots U_n{}^{\psi(O_n)}$ is equal to 0. For a fixed domain $\psi \in \pi(\mathbf{x}, \mathbf{z})$ with $\psi = r_1 * r_2$, there are three possible situation for the location of rectangles r_1 and r_2 in the grid diagram:

(i) If $\mathbf{x}$ and $\mathbf{z}$ has *four* different coordinates, the rectangles r_1 and r_2 are disjoint. There exists a unique grid state $\mathbf{w} \in \mathbf{S}(\mathbb{G})$ such that there is an empty rectangle $r_1' \in \mathrm{Rect}^{\circ}(\mathbf{x}, \mathbf{w})$ with the same support with r_2 since $\mathbf{x}$ has the same coordinates with $\mathbf{y}$ on the upper-right and lower-left corners of the rectangle r_2. By the existence of r_1', $\mathbf{x}$ and $\mathbf{w}$ have the same coordinates on the upper-right and lower-left corners of the rectangle r_1. So there exists another rectangle $r_2' \in \mathrm{Rect}^{\circ}(\mathbf{w}, \mathbf{y})$ having the same support with r_1. So the domain ψ has another decomposition as $\psi = r_1' * r_2'$ which implies that in this case, $N(\psi) = 2$. Since we are working in the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, for this case we have $\partial_{\mathbb{X}}^{-} \circ \partial_{\mathbb{X}}^{-}(\mathbf{x}) = 0$.

(ii) If $\mathbf{x}$ and $\mathbf{z}$ has *three* different coordinates, see the figure for the placement of rectangles r_1 and r_2. Since $\mathbf{x}$ and $\mathbf{y}$ have the same coordinate on the upper-right corner of the rectangle r_2, there exists a unique grid state $\mathbf{w} \in \mathbf{S}(\mathbb{G})$ such that there is an empty rectangle $r_1^{'} \in \text{Rect}^{\circ}(\mathbf{x},\mathbf{w})$. By the existence of the rectangle $r_1^{'}$, and the fact that $\mathbf{y}$ and $\mathbf{z}$ have the same coordinate on the upper-left corner of the rectangle r_1, there exists another rectangle $r_2^{'} \in \text{Rect}^{\circ}(\mathbf{w},\mathbf{z})$. See the figure for the supports of the rectangles $r_1^{'}$ and $r_2^{'}$. In this case, we again conclude that $N(\psi) = 2$. Hence $\partial_{\mathbb{X}}^{-} \circ \partial_{\mathbb{X}}^{-}(\mathbf{x}) = 0$.

(iii) If $\mathbf{x}$ and $\mathbf{z}$ has no different coordinates, the rectangles r_1 and r_2 are their complements on a horizontal or a vertical strip on the grid torus. Since each horizontal and vertical strip contain an X-marking, the domain $\psi = r_1 * r_2$ violates the convention that $\psi \cap \mathbb{X} = \emptyset$. Hence, in this case, it has zero contribution to the map $\partial_{\mathbb{X}}^{-} \circ \partial_{\mathbb{X}}^{-}(\mathbf{x})$.

□

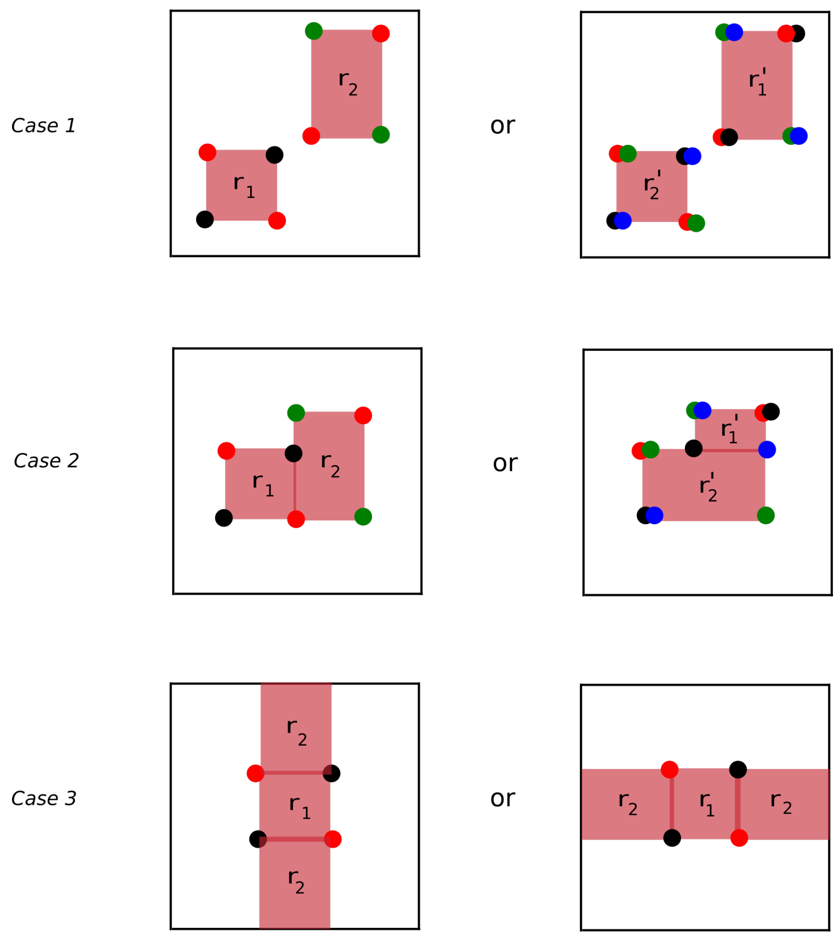

Figure 3.3. The way of decompositions of the domain ψ for each three cases. The grid states are $\mathbf{x}$(black dots), $\mathbf{y}$(red dots), $\mathbf{z}$(green dots), $\mathbf{w}$(blue dots).

Lemma 3.12. *The differential map* $\partial_{\mathbb{X}}^- : GC^-(\mathbb{G}) \to GC^-(\mathbb{G})$ *is homogeneous of degree* $(-1, 0)$.

Proof. For any two grid states $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{S}(\mathbb{G})$ which are connected by an empty rectangle r $\in Rect^\circ(\mathbf{x}, \mathbf{y})$ with $r \cap \mathbb{X} = \emptyset$, without loss of generality, $\partial_{\mathbb{X}}^-(\mathbf{x})$ contains a summand that is of form ${U_1}^{k_1} \cdots {U_n}^{k_n} \cdot \mathbf{y}$, where $k_i = r(O_i)$ for all $i \in \{1, 2, \ldots, n\}$. By the characteristic property of Maslov function M, we can say that

$$M(\mathbf{x}) - M(\mathbf{y}) = 1 - 2\#(r \cap \mathbb{O})$$

since the rectangle r is empty. We also know that

$$M({U_1}^{k_1} \cdots {U_n}^{k_n} \cdot \mathbf{y}) = M(\mathbf{y}) - 2k_1 - \cdots - 2k_n = M(\mathbf{y}) - 2\#(r \cap \mathbb{O}).$$

So we get

$$M({U_1}^{k_1} \cdots {U_n}^{k_n} \cdot \mathbf{y}) = M(\mathbf{x}) - 1.$$

So the differential $\partial_{\mathbb{X}}^-$ drops the Maslov grading by one. Similarly, by the characterization of Alexander function A, we have

$$A(\mathbf{x}) - A(\mathbf{y}) = -\#(r \cap \mathbb{O})$$

since $r \cap \mathbb{X} = \emptyset$. Therefore, we have

$$A({U_1}^{k_1} \cdots {U_n}^{k_n} \cdot \mathbf{y}) = A(\mathbf{y}) - k_1 - \cdots - k_n = A(\mathbf{y}) - \#(r \cap \mathbb{O}) = A(\mathbf{x})$$

implying Alexander grading is preserved under the differential $\partial_{\mathbb{X}}^-$.

□

Now we can prove the theorem 3.6 using the above results as follows:

Proof of theorem 3.6. First it can be easily seen that $GC^-(\mathbb{G})$ is a bigraded $\mathcal{R}$-*module* since the multiplication by U_i is a map that is homogeneous of degree $(-2,-1)$ by the equations (3.5) and (3.6). The differential map $\partial^-_{\mathbb{X}} : GC^-(\mathbb{G}) \to GC^-(\mathbb{G})$ is an $\mathcal{R}$-*module* homomorphism by its definition. We can say that the differential operator $\partial^-_{\mathbb{X}}$ is a homogeneous map of degree $(-1,0)$ by Lemma 3.12. So the differential $\partial^-_{\mathbb{X}}$ is a bigraded $\mathcal{R}$-module homomorphism that is homogeneous of degree $(-1,0)$. Finally, Lemma 3.10 concludes the proof. □

Definition 3.13. *An $\mathcal{R}$-module homomorphism $f : (C,\partial) \to (C^{'},\partial^{'})$ between two bigraded chain complexes (C,∂) and $(C^{'},\partial^{'})$ over $\mathcal{R} = \mathbb{F}[U_1,U_2,\ldots,U_n]$ is called chain map if it satisfies that $\partial^{'} \circ f = f \circ \partial$.*

A chain map f is called a bigraded chain map that is homogeneous of degree (m, t) if it is a bigraded $\mathcal{R}$-module homomorphism that is homogeneous of degree (m, t).

A bigraded chain map $f : (C,\partial) \to (C^{'},\partial^{'})$ is called an isomorphism of bigraded chain complexes if there exists a bigraded chain map $g : (C^{'},\partial^{'}) \to (C,\partial)$ such that $g \circ f = 1_C$ and $f \circ g = 1_{C^{'}}$. In the existence of such an isomorphism, we call (C,∂) and $(C^{'},\partial^{'})$ isomorphic bigraded chain complexes.

Definition 3.14. *Let f and g be bigraded chain maps that are homogeneous of degree (m, t) between two chain complexes (C,∂) and $(C^{'},\partial^{'})$. If there exists an $\mathcal{R}$-module homomorphism $h : C \to C^{'}$ that is homogeneous of degree (m+1, t), called chain homotopy from g to f, such that*

$$f - g = \partial^{'} \circ h + h \circ \partial$$

then f and g are called chain homotopic.

Lemma 3.15. *Multiplication by U_i is chain homotopic to multiplication by U_j for any pair $(i, j) \in \{1, 2, \ldots, n\}$, where these multiplication maps are the chain maps between $GC^-(\mathbb{G})$ which are homogeneous of degree $(-2, -1)$.*

For the proof see the book [6].

We should be careful about the lemma 3.15. It is true when $\mathbb{G}$ represents a knot. In the case of links, this lemma is not valid for every pair (i, j) in the set $\{1, 2, \ldots, n\}$. Observe that the multiplication by U_i and U_j are not chain homotopic if they correspond to different link components [6].

The *unblocked grid homology* $GH^-(\mathbb{G})$ is the homology of the chain complex $(GC^-(\mathbb{G}), \partial^-_{\mathbb{X}})$. This homology can be considered as a bigraded $\mathbb{F}[U]$-module, where U is induced by the chain map U_i for any $i \in \{1, 2, \ldots, n\}$, by the above lemma 3.15. Because the chain homotopic maps induce the same homomorphism on homology. The bigrading structure on the grid chain complex $(GC^-(\mathbb{G}), \partial^-_{\mathbb{X}})$ induces a bigrading structure on homology groups as follows:

$$GH^-(\mathbb{G}) = \bigoplus_{d,s \in \mathbb{Z}} GH^-_d(\mathbb{G}, s), \text{where } GH^-_d(\mathbb{G}, s) = \frac{Ker(\partial^-_{\mathbb{X}}) \cap GC^-_d(\mathbb{G}, s)}{Im(\partial^-_{\mathbb{X}}) \cap GC^-_d(\mathbb{G}, s)}. \tag{3.9}$$

There are two other grid chain complexes, associated to the given grid diagram $\mathbb{G}$, derived from the unblocked grid chain complex $(GC^-(\mathbb{G}), \partial^-_{\mathbb{X}})$. These complexes are specializations of $GC^-(\mathbb{G})$ so that they differ by how their differentials count the empty rectangles. In the case of $GC^-(\mathbb{G})$, the differential counts the empty rectangles with the property that they do not contain any X-marking.

The grid complex $\widetilde{GC}(\mathbb{G})$ is the simplest verison of the grid complex $GC^{-}(\mathbb{G})$ given as the following

$$\widetilde{GC}(\mathbb{G}) = \frac{GC^{-}(\mathbb{G})}{U_1 = \cdots = U_n = 0}.$$

$\widetilde{GC}(\mathbb{G})$ is a vector space over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ with a basis corresponding to the set $\mathbf{S}(\mathbb{G})$ of grid states of the grid diagram $\mathbb{G}$. The $\mathbb{F}$-vector space $\widetilde{GC}(\mathbb{G})$ becomes a grid chain complex, called *fully blocked grid chain complex*, when it is associated with a specific differential map $\widetilde{\partial}_{\mathbb{X},\mathbb{O}} : \widetilde{GC}(\mathbb{G}) \to \widetilde{GC}(\mathbb{G})$ which counts only empty rectangles which contains no square marked with any X or O defined as

$$\widetilde{\partial}_{\mathbb{X},\mathbb{O}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})} \sum_{\{r\in Rect^{\circ}(\mathbf{x},\mathbf{y}) | r\cap\mathbb{X}=r\cap\mathbb{O}=\emptyset\}} \mathbf{y}.$$

Maslov and Alexander functions on $\mathbf{S}(\mathbb{G})$ give this chain bigraded vector space structure as follows:

$\widetilde{GC}_d(\mathbb{G}, s)$ is the $\mathbb{F}$-vector subspaces of $\widetilde{GC}(\mathbb{G})$ with a basis

$$\mathcal{B}(\widetilde{GC}_d(\mathbb{G}, s)) = \{\mathbf{x} \in \mathbf{S}(\mathbb{G}) \mid M(\mathbf{x}) = d \text{ and } A(\mathbf{x}) = s\}.$$

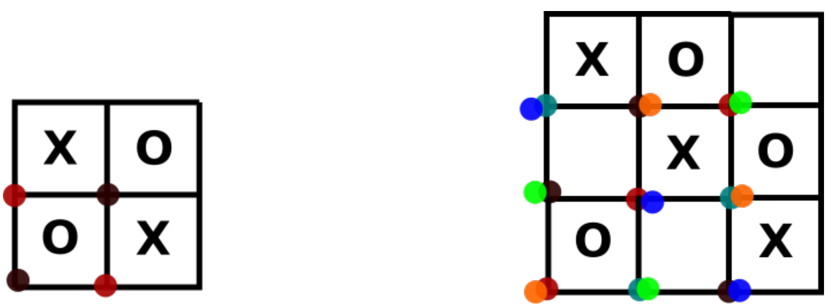

Figure 3.4. Two grid representations $\mathbb{G}_1$ and $\mathbb{G}_2$ of the unknot with their grid states.

As an example, we will compute $\widetilde{GH}(\mathbb{G})$ for two different grid diagrams for the unknot. Let $\mathbb{G}_1$ be 2×2 grid diagram of unknot, and $\mathbb{G}_2$ be 3×3 grid diagram of unknot, see Figure 2.12.

A grid state $\mathbf{x} = \{x_1, \ldots, x_n\}$ can be considered as a permutation $(1, \ldots, n) \mapsto (\alpha_{\sigma(1)} \cap \beta_1, \ldots, \alpha_{\sigma(n)} \cap \beta_n)$ where $\alpha_{\sigma(n)}$ represents the horizontal circle on the grid torus where the n^{th}-component of the grid state locates $\mathbf{x}$ and β_n represents the n^{th}-vertical circle on the grid torus.

First we will compute the grid homology $\widetilde{GH}(\mathbb{G}_1)$ associated to the grid $\mathbb{G}_1$. The grid $\mathbb{G}_1$ has exactly 2! grid states $\mathbf{x_1} = (1, 2)$ and $\mathbf{x_2} = (2, 1)$. There are exactly two rectangles going from $\mathbf{x_1}$ to $\mathbf{x_2}$, but none of them satisfies the equation $r \cap \mathbb{X} = r \cap \mathbb{O} = \emptyset$. So they have no contribution to $\widetilde{\partial}_{\mathbb{X},\mathbb{O}}$. Similar argument is valid for the rectangles going from $\mathbf{x_2}$ to $\mathbf{x_1}$. So the differential map $\widetilde{\partial}_{\mathbb{X},\mathbb{O}}(\mathbf{x}_i) = 0,\ i \in \{1, 2\}$. Thus, the kernel of the differential $\widetilde{\partial}_{\mathbb{X},\mathbb{O}}$ is obviously the whole chain complex $\widetilde{GC}(\mathbb{G}_1)$, and the image of the differential $\widetilde{\partial}_{\mathbb{X},\mathbb{O}}$ is zero.

We compute the Maslov and Alexander gradings of $\mathbf{x_1}$ and $\mathbf{x_2}$ as $M(\mathbf{x_1}) = -1$ since $\mathbf{x_1} = \mathbf{x}^{\text{SWO}}$, $A(x_1) = -1$, and $M(\mathbf{x_2}) = 0$ since $\mathbf{x_2} = \mathbf{x}^{\text{NWO}}$, $A(x_2) = 0$ computed using the equation (3.3), see the Figure 3.5 for the grading table.

There exist two 1-dimensional $\mathbb{F}$-vector subspaces of the chain complex $\widetilde{GC}(\mathbb{G}_1)$, namely $\widetilde{GC}_{-1}(\mathbb{G}_1, -1)$ and $\widetilde{GC}_0(\mathbb{G}_1, 0)$ generated by the grid states $\mathbf{x_1}$ and $\mathbf{x_2}$, respectively. So we can write $\widetilde{GC}_{-1}(\mathbb{G}_1, -1) = \langle \mathbf{x_1} \rangle$, and $\widetilde{GC}_0(\mathbb{G}_1, 0) = \langle \mathbf{x_2} \rangle$.

$$\begin{aligned}
\widetilde{GH}_{-1}(\mathbb{G}_1, -1) &= \frac{Ker(\widetilde{\partial}_{\mathbb{X},\mathbb{O}}) \cap \widetilde{GC}_{-1}(\mathbb{G}_1, -1)}{Im(\widetilde{\partial}_{\mathbb{X},\mathbb{O}}) \cap \widetilde{GC}_{-1}(\mathbb{G}_1, -1)} \\
&\cong \frac{\widetilde{GC}(\mathbb{G}_1) \cap \mathbb{F}_{(-1,-1)}}{0 \cap \mathbb{F}_{(-1,-1)}} \\
&\cong \mathbb{F}_{(-1,-1)}.
\end{aligned} \tag{3.10}$$

$$\begin{aligned} \widetilde{GH}_0(\mathbb{G}_1, 0) &= \frac{Ker(\widetilde{\partial}_{\mathbb{X},\mathbb{O}}) \cap \widetilde{GC}_0(\mathbb{G}_1, 0)}{Im(\widetilde{\partial}_{\mathbb{X},\mathbb{O}}) \cap \widetilde{GC}_0(\mathbb{G}_1, 0)} \\ &\cong \frac{\widetilde{GC}(\mathbb{G}_1) \cap \mathbb{F}_{(0,0)}}{0 \cap \mathbb{F}_{(0,0)}} \\ &\cong \mathbb{F}_{(0,0)}. \end{aligned} \tag{3.11}$$

By interpreting the definition in (3.9) for the fully blocked grid homology, we find

$$\widetilde{GH}(\mathbb{G}_1) \cong \mathbb{F}_{(-1,-1)} \oplus \mathbb{F}_{(0,0)}$$

which is a two dimensional bigraded vector space over $\mathbb{F}$ with one generator $\mathbf{x}_1$ in bigrading $(-1,-1)$ and another generator $\mathbf{x}_2$ in bigrading $(0,0)$.

	d	s
x_1	-1	-1
x_2	0	0

Figure 3.5. The table of gradings of grid states of $\mathbb{G}_1$

After applying $X : SW$ type of stabilization to the grid $\mathbb{G}_1$, we get a new 3×3 grid diagram $\mathbb{G}_2$. Now, we compute the homology of the grid $\mathbb{G}_2$ for the unknot. It has 3! different grid states, namely $\mathbf{x}_1$(red dots), $\mathbf{x}_2$(green dots), $\mathbf{x}_3$(orange dots), $\mathbf{x}_4$(black dots), $\mathbf{x}_5$(dark green dots), $\mathbf{x}_6$(blue dots). Write them in permutation notation as follows:

$$\mathbf{x}_1 = (1,2,3),\ \mathbf{x}_2 = (2,1,3),\ \mathbf{x}_3 = (1,3,2),\ \mathbf{x}_4 = (2,3,1),\ \mathbf{x}_5 = (3,1,2),\ \mathbf{x}_6 = (3,2,1).$$

We compute Maslov and Alexander gradings of each grid state $\mathbf{x}_i$, $i \in \{1,2\}$ by using the formulas in (3.1), (3.2), (3.3), and (3.4), see the figure 3.6.

Notice that $\widetilde{GC}_{-2}(\mathbb{G}_2,-2) = \langle \mathbf{x}_3 \rangle$, $\widetilde{GC}_{-2}(\mathbb{G}_2,-1) = \langle \mathbf{x}_6 \rangle$, $\widetilde{GC}_{-1}(\mathbb{G}_2,-1) = \langle \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_5 \rangle$, $\widetilde{GC}_0(\mathbb{G}_2,0) = \langle \mathbf{x}_2 \rangle$.

	d	s
x_1	-1	-1
x_2	0	0
x_3	-2	-2
x_4	-1	-1
x_5	-1	-1
x_6	-2	-1

Figure 3.6. The table of gradings of grid states of $\mathbb{G}_2$

Look at the behaviour of the differential map:

$$\widetilde{\partial}_{\mathbb{X},\mathbb{O}}(\mathbf{x}_1) = \widetilde{\partial}_{\mathbb{X},\mathbb{O}}(\mathbf{x}_4) = \widetilde{\partial}_{\mathbb{X},\mathbb{O}}(\mathbf{x}_5) = \mathbf{x}_6$$

$$\widetilde{\partial}_{\mathbb{X},\mathbb{O}}(\mathbf{x}_2) = \widetilde{\partial}_{\mathbb{X},\mathbb{O}}(\mathbf{x}_3) = \widetilde{\partial}_{\mathbb{X},\mathbb{O}}(\mathbf{x}_6) = 0.$$

Observe that under the differential map Alexander grading remains the same while Maslov grading drops by one. The kernel $\mathrm{Ker}(\widetilde{\partial}_{\mathbb{X},\mathbb{O}}) = \langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_6, \mathbf{x}_1 - \mathbf{x}_4, \mathbf{x}_1 - \mathbf{x}_5 \rangle$ and the image $\mathrm{Im}(\widetilde{\partial}_{\mathbb{X},\mathbb{O}}) = \mathbf{x}_6$. We conclude that the fully blocked grid homology associated to the grid diagram $\mathbb{G}_2$ is

$$\widetilde{GH}(\mathbb{G}_2) \cong \mathbb{F}_{(-2,-2)} \oplus \mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(-1,-1)} \oplus \mathbb{F}_{(-1,-1)}$$

which is a four dimensional bigraded vector space.

This result seems interesting because we know that $\mathbb{G}_1$ and $\mathbb{G}_2$ are equivalent grid diagrams, connected by a stabilization move. So we would expect $\widetilde{GH}(\mathbb{G}_1)$ and

$\widetilde{GH}(\mathbb{G}_2)$ to be the same. But even their dimesions are not same:

$$2 = dim_{\mathbb{F}}(\widetilde{GH}(\mathbb{G}_1)) \neq dim_{\mathbb{F}}(\widetilde{GH}(\mathbb{G}_2)) = 4$$

The fact that the homology $\widetilde{GH}(\mathbb{G})$ depends on the grid diagram $\mathbb{G}$ implies that the invariance of grid homology must be a little different.

Definition 3.16. *Suppose that V is a bigraded vector space with the splitting $V = \bigoplus_{d,s\in\mathbb{Z}} V_{(d,s)}$. For fixed two integers k and m, the shift of V, denoted by $V[\![k,m]\!]$, can be defined as a bigraded vector space which is isomorphic to V with the following bigrading structure as*

$$V[\![k,m]\!]_{(d,s)} = V_{(d+k,s+m)}.$$

Suppose W is a two dimensional vector space with a bigraded structure such as $W = \mathbb{F}_{(-1,-1)} \oplus \mathbb{F}_{(0,0)}$, and let X be an arbitrary bigraded vector space. Then the tensor product of X and W is given by

$$X \otimes W \cong X \oplus X[\![1,1]\!] \tag{3.12}$$

which is also a bigraded vector space with $X \otimes W = \bigoplus_{d,s\in\mathbb{Z}}(X \otimes W)_{d,s}$.

Proposition 3.17. *Assume $\mathbb{G}$ represents a knot K. Let W be a two dimensional bigraded vector space with $W = \mathbb{F}_{(-1,-1)} \oplus \mathbb{F}_{(0,0)}$. Then there is an isomorphism of bigraded vector spaces*

$$\widetilde{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes(n-1)} \tag{3.13}$$

where the $\widehat{GH}(\mathbb{G})$ is the homology of the chain complex $(GC^-(\mathbb{G})/U_i,\ \partial_{\mathbb{X}}^-)$ for all $i \in \{1, 2, \cdots, n\}$.

Theorem 3.18. *Let $\mathbb{G}$ be an $n \times n$ grid diagram representing a knot K. Then the integer value $dim_{\mathbb{F}}(\widetilde{GH}(\mathbb{G}))/2^{n-1}$ is a knot invariant.*

In fact, since $\widehat{GH}(\mathbb{G})$ is finite dimensional, $2^{n-1} \cdot dim_{\mathbb{F}}(\widehat{GH}(\mathbb{G})) = dim_{\mathbb{F}}(\widetilde{GH}(\mathbb{G}))$ for a grid diagram of a knot with grid number n.

Proposition 3.19. *Assume $\mathbb{G}'$ is the stabilized grid diagram obtained from the grid diagram $\mathbb{G}$. Then*

$$\widehat{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}') \text{ as bigraded vector spaces} \tag{3.14}$$

$$GH^-(\mathbb{G}) \cong GH^-(\mathbb{G}') \text{ as bigraded } \mathbb{F}[U]\text{-modules} \tag{3.15}$$

$$\widetilde{GH}(\mathbb{G}') \cong \widetilde{GH}(\mathbb{G}) \oplus \widetilde{GH}(\mathbb{G})[\![1,1]\!] \text{ as bigraded vector spaces.} \tag{3.16}$$

The proposition above shows the invariance of grid homologies under stabilization. For the proofs, see the book [6]. If we go back to our example for the fully blocked grid homology of unknot, we have

$$\begin{aligned} \widetilde{GH}(\mathbb{G}_1) &\cong \widehat{GH}(\mathbb{G}_1) \otimes W \\ &\cong \widehat{GH}(\mathbb{G}_1) \oplus \widehat{GH}(\mathbb{G})[\![1,1]\!] \\ &\cong \mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(-1,-1)}. \end{aligned} \tag{3.17}$$

Hence we get $\widehat{GH}(\mathbb{G}_1) \cong \mathbb{F}_{(0,0)}$. With a similiar process, by iterated the equation (3.12),

$$\begin{aligned} \widetilde{GH}(\mathbb{G}_2) &\cong \widehat{GH}(\mathbb{G}_2) \otimes W^{\otimes 2} \\ &\cong \widehat{GH}(\mathbb{G}_2) \oplus \widehat{GH}(\mathbb{G}_2)[\![1,1]\!] \oplus \widehat{GH}(\mathbb{G}_2)[\![1,1]\!] \oplus \widehat{GH}(\mathbb{G}_2)[\![2,2]\!]. \end{aligned}$$

Since $\widetilde{GH}(\mathbb{G}_2) \cong \mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(-1,-1)} \oplus \mathbb{F}_{(-1,-1)} \oplus \mathbb{F}_{(-2,-2)}$, we get $\widehat{GH}(\mathbb{G}_2) \cong \mathbb{F}_{(0,0)}$, and this verifies the invariance of simply blocked grid homology under stabilization.

Proposition 3.20. *If G_1 and G_2 are two grid diagrams which differ by a commutation move, then there is an isomorphism of bigraded vector spaces*

$$\widehat{GH}(\mathbb{G}_1) \cong \widehat{GH}(\mathbb{G}_2)$$

and an isomorphism of bigraded $\mathbb{F}[U]$-modules

$$GH^-(\mathbb{G}_1) \cong GH^-(\mathbb{G}_2).$$

4. APPLICATIONS

4.1. Computing Alexander polynomial using grid diagrams

Consider an $n \times n$ toroidal grid diagram for a knot. Compute the winding number of each point of intersection of horizontal α_i and vertical circles β_j of the grid torus for any pair $(i, j) \in \{1, 2, \ldots, n\}$.

We can construct a matrix whose $(i, j)^{th}$ entry is equal to $t^{-w(p)}$, where p is the $(j, n-i+1)^{th}$ lattice point, and $w(p)$ is the winding number of the grid diagram $\mathbb{G}$ around the lattice point p. This matrix is called the *grid matrix*, denoted by $M(\mathbb{G})$. The grid matrices $\boldsymbol{M}(\mathbb{G})$ and $\boldsymbol{M}(\mathbb{G}')$ of two different grid diagrams for the left-handed

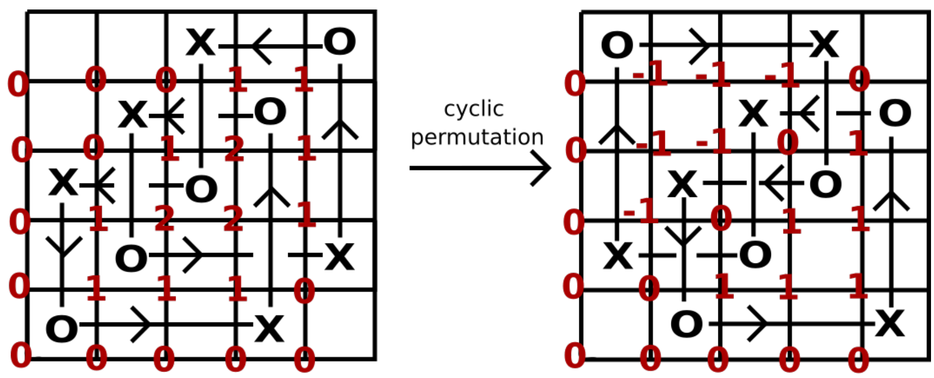

Figure 4.1. Winding numbers of intersection points in two grid diagrams $\mathbb{G}$ and $\mathbb{G}'$.

trefoil $T_{-2,3}$ are

$$M(\mathbb{G}) = \begin{pmatrix} 1 & 1 & 1 & t^{-1} & t^{-1} \\ 1 & 1 & t^{-1} & t^{-2} & t^{-1} \\ 1 & t^{-1} & t^{-2} & t^{-2} & t^{-1} \\ 1 & t^{-1} & t^{-1} & t^{-1} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, M(\mathbb{G}') = \begin{pmatrix} 1 & t & t & t & 1 \\ 1 & t & t & 1 & t^{-1} \\ 1 & t & 1 & t^{-1} & t^{-1} \\ 1 & 1 & t^{-1} & t^{-1} & t^{-1} \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The determinants $\det(\boldsymbol{M}(\mathbb{G}))$ and $\det(\boldsymbol{M}(\mathbb{G}'))$ are computed as follows:

$\det(\boldsymbol{M}(\mathbb{G}))=\frac{t^6-5t^5+11t^4-14t^3+11t^2-5t+1}{t^6}$, $\det(\boldsymbol{M}(\mathbb{G}')) = \frac{t^6-5t^5+11t^4-14t^3+11t^2-5t+1}{t^3}$.

Notice that as a function of t, $\det(M(\mathbb{G}))$ is not a knot invariant for the knot corresponding to the grid diagram $\mathbb{G}$ since it is not preserved under the grid move, that it is dependent of the choice of grid represention of a knot.

Consider the corners of the marked squares in the grid diagram $\mathbb{G}$ and sum up the winding numbers corresponding to these corners and divide this sum by 8. We will denote the resulting number by $a(\mathbb{G})$. The function of t, $D_{\mathbb{G}}(t)$, is given by

$$D_{\mathbb{G}}(t) = \epsilon(\mathbb{G}) \cdot det(M(\mathbb{G})) \cdot (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^{1-n} t^{a(\mathbb{G})} \tag{4.1}$$

where $\epsilon(\mathbb{G}) \in \{\pm 1\}$ is the sign of the permutation that connects $\sigma_{\mathbb{O}}$ and $(n, n-1, \ldots, 1)$.

We conclude that $D_{\mathbb{G}}(t) = D_{\mathbb{G}'}(t)$, and this result is not a coincidence. The function $D_{\mathbb{G}}(t)$ may be a good candidate to be a knot invariant.

Theorem 4.1. *For a grid representation $\mathbb{G}$ of the knot K, the function $D_{\mathbb{G}}(t)$ is a knot invariant which coincides with the symmetrized Alexander polynomial $\Delta_K(t)$ of the knot K.*

It is enough to show that the function $D_{\mathbb{G}}(t)$ is invariant under grid moves; in particular, it is independent of the chosen diagram. For the proof see [6].

We can compute Alexander grading by using winding numbers. This allows us to understand Alexander grading geometrically. First give the winding number of the corresponding knot diagram (which is a closed curve in the plane, say $\mathcal{C}$) to the given grid diagram $\mathbb{G}$ around a lattice point p a combinatorical definition as follows:

$$w_{\mathcal{C}}(p) = \mathcal{J}(p, \mathbb{O} - \mathbb{X}) = \mathcal{J}(p, \mathbb{O}) - \mathcal{J}(p, \mathbb{X}).$$

Consider $\mathbb{O} - \mathbb{X}$ as the set of line segments in the plane such that initial point is an X-marking while terminal point is an O-marking. We see that the lattice point p $\in \mathbb{R}^2 \setminus \mathcal{C}$ lies on the horizontal line L such that $L = L^+ \cup L^-$ where $L^+ \cap L^- = \{p\}$.The function $\mathcal{I}(p, \mathbb{O} - \mathbb{X})$ counts the number of points of intersection of the ray L^+ and the set $\mathbb{O} - \mathbb{X}$ with a sign convention such that if the line segment has an orientation upward, its contribution to $\mathcal{I}(p, \mathbb{O} - \mathbb{X})$ is $+1$ and if the line segment has an orientation downward, then its contribution to $\mathcal{I}(p, \mathbb{O} - \mathbb{X})$ is -1. On the other hand, the function $\mathcal{I}(\mathbb{O} - \mathbb{X}, p)$ counts the number of points of intersection of the ray L^- and the set $\mathbb{O} - \mathbb{X}$ with a sign convention such that if the line segment has an orientation downward, its contribution to $\mathcal{I}(\mathbb{O} - \mathbb{X}, p)$ is $+1$ and if the line segment has an orientation upward, then its contribution to $\mathcal{I}(\mathbb{O} - \mathbb{X}, p)$ is -1. To compute $\mathcal{J}(p, \mathbb{O} - \mathbb{X})$, take the average of the functions $\mathcal{I}(p, \mathbb{O} - \mathbb{X})$ and $\mathcal{I}(\mathbb{O} - \mathbb{X}, p)$ as before. Notice that the winding number of $\mathcal{C}$ around p is equal to $\mathcal{I}(p, \mathbb{O} - \mathbb{X})$ and $\mathcal{I}(\mathbb{O} - \mathbb{X}, p)$. So we can easily see $\mathcal{J}(p, \mathbb{O} - \mathbb{X}) = w(p)$.

Now consider the function A' on $\mathbf{S}(\mathbb{G})$ which is minus the sum of winding numbers of $\mathcal{C}$ around each component of the grid state $\mathbf{x}$:

$$A'(\mathbf{x}) = -\sum_{x_i \in \mathbf{x}} w_{\mathcal{C}}(x_i) \text{ for all } i \in \{1, 2, \ldots, n\}.$$

With the consideration above, Alexander function has a new definition in terms of the winding numbers $w(p)$ as follows:

$$A(\mathbf{x}) = -\sum_{x_i \in \mathbf{x}} w_{\mathcal{C}}(x_i) + \frac{1}{8}\sum_{i=1}^{8n} w_{\mathcal{C}}(c_i) - (\frac{n-1}{2}) = A^{'}(\mathbf{x}) + a(\mathbb{G}) - (\frac{n-1}{2}).$$

4.2. Computation of grid homology for links by an example

When it comes to compute grid homology for links, we should refine gradings on the chain complexes and on their homologies. Recall that Maslov grading (homology grading) can be computed only using the given grid diagram, while Alexander polynomial can be computed in terms of winding number of the closed curve associated to the given knot. But in any link diagram, there are disjoint closed curves, which are called link components. So our Alexander grading definition must be carefully adjusted for links.

Let $\mathbb{G}$ represent an l-component link L in S^3. The subsets $\mathbb{O}_i \subset \mathbb{O}$ and $\mathbb{X}_i \subset \mathbb{X}$ are the markings on the i^{th}-component L_i of the link L. Define a vector-valued function $\mathbb{A} = (A_1, \ldots, A_l)$ on $\mathbf{S}(\mathbb{G})$ where

$$A_i(x) = \mathcal{J}(\mathbf{x} - \frac{1}{2}(\mathbb{X} + \mathbb{O}), \mathbb{X}_i - \mathbb{O}_i) - \frac{n_i - 1}{2}$$

where n_i denotes the number of O-markings in the $\mathbb{O}_i \subset \mathbb{O}$. For each link component we have an Alexander grading, and call the vector-valued function $\mathbb{A}$ as *Alexander multi-grading.*

The Alexander function (Alexander grading, s) $A(\mathbf{x})$ on $\mathbf{S}(\mathbb{G})$ is related to the Alexander multi-grading with the formula

$$A(\mathbf{x}) = \sum_{i=1}^{l} A_i(\mathbf{x}), \ \textit{for any } \mathbf{x} \in \mathbf{S}(\mathbb{G}).$$

Consider the first homology $H_1(S^3 \setminus L; \mathbb{Z})$ of the link complement $S^3 \setminus L$ of the link L. Define the affine space for $H_1(S^3 \setminus L; \mathbb{Z})$ as *Alexander grading set* $\mathbf{H}(L)$ which consists of elements $\sum_{i=1}^{l} s_i \cdot \mu_i$ with the convention that $2s_i + lk(L_i, L \setminus L_i)$ must be an even integer, where $\{\mu_i\}_{i=1}^{l}$ gives a basis for the first homology and $s_i \in \frac{1}{2}\mathbb{Z}$. For the Hopf link L, s_i can not be integer. Indeed $\mathbf{H}(L) = (\frac{1}{2} + \mathbb{Z}) \oplus (\frac{1}{2} + \mathbb{Z}) \subset \mathbb{Q} \oplus \mathbb{Q}$ and $H_1(S^3 \setminus L; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

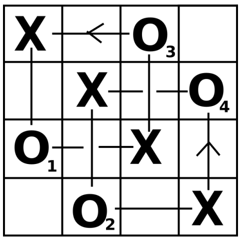

Figure 4.2. A grid diagram of the positive Hopf link with ordered markings

Consider a grid diagram $\mathbb{G}$ for the positive Hopf link $H_+ = (H_1, H_2)$ with two components H_1 and H_2, see the Figure 4.2 above. Notice that there are eight empty rectangles which does not contain any X-marking or O-marking. There are twelve grid states which are connected by differentials and all of them are in Alexander bigrading $(-\frac{1}{2}, -\frac{1}{2})$. The other twelve generators are not connected to any generator.

$$\partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_1) = \mathbf{x}_2 + \mathbf{x}_4 + \mathbf{x}_5 + \mathbf{x}_7$$

$$\partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_2) = \partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_7) = \partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_{21}) = \partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_{23}) = \mathbf{x}_{10}$$

$$\partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_4) = \partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_5) = \partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_{20}) = \partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_{22}) = \mathbf{x}_{17}$$

$$\partial_{\mathbb{X},\mathbb{O}}(\mathbf{x}_{13}) = \mathbf{x}_{20} + \mathbf{x}_{21} + \mathbf{x}_{22} + \mathbf{x}_{23}.$$

In the Alexander bigrading $(-\frac{3}{2},-\frac{3}{2})$, we have a chain complex whose homology is $\widetilde{\mathbf{GH}}_{-4}(\mathbb{G},(-\frac{3}{2},-\frac{3}{2}))=\langle\mathbf{x}_6\rangle\cong\mathbb{F}_{(-4,(-\frac{3}{2},-\frac{3}{2}))}\cong\mathbb{F}_{(-4,-3)}$ with rank 1.

In the Alexander bigrading $(-\frac{3}{2},-\frac{1}{2})$, we have a chain complex whose homology is $\widetilde{\mathbf{GH}}_{-3}(\mathbb{G},(-\frac{3}{2},-\frac{1}{2}))=\langle\mathbf{x}_9,\mathbf{x}_{15}\rangle\cong\mathbb{F}_{(-3,(-\frac{3}{2},-\frac{1}{2}))}\oplus\mathbb{F}_{(-3,(-\frac{3}{2},-\frac{1}{2}))}\cong\mathbb{F}_{-3,-2}\oplus\mathbb{F}_{-3,-2}$ with rank 2.

In the Alexander bigrading $(-\frac{1}{2},-\frac{1}{2})$, we have a chain complex whose homology is $\widetilde{\mathbf{GH}}_{-2}(\mathbb{G},(-\frac{1}{2},-\frac{1}{2}))=\frac{\langle\mathbf{x}_2-\mathbf{x}_7,\mathbf{x}_2-\mathbf{x}_{21},\mathbf{x}_2-\mathbf{x}_{23},\mathbf{x}_4-\mathbf{x}_5,\mathbf{x}_4-\mathbf{x}_{20},\mathbf{x}_4-\mathbf{x}_{22}\rangle}{\langle\mathbf{x}_2+\mathbf{x}_4+\mathbf{x}_5+\mathbf{x}_7,\mathbf{x}_{20}+\mathbf{x}_{21}+\mathbf{x}_{22}+\mathbf{x}_{23}\rangle}\cong\mathbb{F}_{(-2,(-\frac{1}{2},-\frac{1}{2}))}\oplus\mathbb{F}_{(-2,(-\frac{1}{2},-\frac{1}{2}))}\oplus\mathbb{F}_{(-2,(-\frac{1}{2},-\frac{1}{2}))}\oplus\mathbb{F}_{(-2,(-\frac{1}{2},-\frac{1}{2}))}\cong\mathbb{F}_{-2,-1}\oplus\mathbb{F}_{-2,-1}\oplus\mathbb{F}_{-2,-1}\oplus\mathbb{F}_{-2,-1}$ with rank 4.

In the Alexander bigrading $(-\frac{1}{2},-\frac{3}{2})$, we have a chain complex whose homology is $\widetilde{\mathbf{GH}}_{-3}(\mathbb{G},(-\frac{1}{2},-\frac{3}{2}))=\langle\mathbf{x}_{16},\mathbf{x}_{18}\rangle\cong\mathbb{F}_{(-3,(-\frac{1}{2},-\frac{3}{2}))}\oplus\mathbb{F}_{(-3,(-\frac{1}{2},-\frac{3}{2}))}\cong\mathbb{F}_{-3,-2}\oplus\mathbb{F}_{-3,-2}$ with rank 2.

In the Alexander bigrading $(-\frac{3}{2},\frac{1}{2})$, we have a chain complex whose homology is $\widetilde{\mathbf{GH}}_{-2}(\mathbb{G},(-\frac{3}{2},\frac{1}{2}))=\langle\mathbf{x}_{19}\rangle\cong\mathbb{F}_{(-2,(-\frac{3}{2},\frac{1}{2}))}\cong\mathbb{F}_{-2,-1}$ with rank 1.

In the Alexander bigrading $(\frac{1}{2},-\frac{3}{2})$, we have a chain complex whose homology is $\widetilde{\mathbf{GH}}_{-2}(\mathbb{G},(\frac{1}{2},-\frac{3}{2}))=\langle\mathbf{x}_{24}\rangle\cong\mathbb{F}_{(-2,(\frac{1}{2},-\frac{3}{2}))}\cong\mathbb{F}_{-2,-1}$ with rank 1.

In the Alexander bigrading $(\frac{1}{2},-\frac{1}{2})$, we have a chain complex whose homology is $\widetilde{\mathbf{GH}}_{-1}(\mathbb{G},(\frac{1}{2},-\frac{1}{2}))=\langle\mathbf{x}_8,\mathbf{x}_{14}\rangle\cong\mathbb{F}_{(-1,(\frac{1}{2},-\frac{1}{2}))}\oplus\mathbb{F}_{(-1,(\frac{1}{2},-\frac{1}{2}))}\cong\mathbb{F}_{-1,0}\oplus\mathbb{F}_{-1,0}$ with rank 2.

In the Alexander bigrading $(-\frac{1}{2},\frac{1}{2})$, we have a chain complex whose homology is $\widetilde{\mathbf{GH}}_{-1}(\mathbb{G},(-\frac{1}{2},\frac{1}{2}))=\langle\mathbf{x}_{11},\mathbf{x}_{12}\rangle\cong\mathbb{F}_{(-1,(-\frac{1}{2},\frac{1}{2}))}\oplus\mathbb{F}_{(-1,(-\frac{1}{2},\frac{1}{2}))}\cong\mathbb{F}_{-1,0}\oplus\mathbb{F}_{-1,0}$ with rank 2.

In the Alexander bigrading $(\frac{1}{2},\frac{1}{2})$, we have a chain complex whose homology is $\widetilde{\mathbf{GH}}_0(\mathbb{G},(\frac{1}{2},\frac{1}{2})) = \langle \mathbf{x}_3 \rangle \cong \mathbb{F}_{(0,(\frac{1}{2},\frac{1}{2}))} \cong \mathbb{F}_{0,1}$ with rank 1.

As we can see total rank of the multi-graded fully blocked homology is 16 with

$$\begin{aligned} \widetilde{\mathbf{GH}}(\mathbb{G}) &\cong \mathbb{F}_{(-4,(-\frac{3}{2},-\frac{3}{2}))} \oplus \mathbb{F}_{(-3,(-\frac{3}{2},-\frac{1}{2}))}^{\oplus 2} \oplus \mathbb{F}_{(-3,(-\frac{1}{2},-\frac{3}{2}))}^{\oplus 2} \oplus \mathbb{F}_{(-2,(-\frac{1}{2},-\frac{1}{2}))}^{\oplus 4} \\ &\oplus \mathbb{F}_{(-2,(-\frac{3}{2},\frac{1}{2}))} \oplus \mathbb{F}_{(-2,(\frac{1}{2},-\frac{3}{2}))} \oplus \mathbb{F}_{(-1,(\frac{1}{2},-\frac{1}{2}))}^{\oplus 2} \oplus \mathbb{F}_{(-1,(-\frac{1}{2},\frac{1}{2}))}^{\oplus 2} \oplus \mathbb{F}_{(0,(\frac{1}{2},\frac{1}{2}))}. \end{aligned} \tag{4.2}$$

This isormophism could also be written as follows

$$\widetilde{\mathbf{GH}}(\mathbb{G}) \cong \mathbb{F}_{(-4,-3)} \oplus \mathbb{F}_{(-3,-2)}^{\oplus 4} \oplus \mathbb{F}_{(-2,-1)}^{\oplus 6} \oplus \mathbb{F}_{(-1,0)}^{\oplus 4} \oplus \mathbb{F}_{(0,1)}. \tag{4.3}$$

Proposition 4.2. *Assume $\mathbb{G}$ represents an oriented l-component link. Let W_i be the two dimensional $\mathbb{Z} \oplus H_1(S^3 \setminus L; \mathbb{Z})$-graded vector space with one generator in bigrading $(0,0)$ and the other generaotr in bigrading $(-1,-\mu_i)$. Then there is an isomorphism of $\mathbb{Z} \oplus \boldsymbol{H}(L)$-graded vector spaces*

$$\widetilde{\boldsymbol{GH}}(\mathbb{G}) \cong \widehat{\boldsymbol{GH}}(\mathbb{G}) \otimes \bigotimes_{i=1}^{l} W_i^{\otimes(n_i-1)}. \tag{4.4}$$

Define the tensor product of a $\mathbb{Z} \oplus \mathbf{H}(L)$-graded vector space X and W_i described in the Proposition 4.3 as a $\mathbb{Z} \oplus \mathbf{H}(L)$-graded vector space with

$$X \otimes W_i \cong X \oplus X[\![1,\mu_i]\!] \tag{4.5}$$

where $X[\![1,\mu_i]\!]$ is the shift of X which is isomorphic to X as a bigraded vector space with $X[\![1,\mu_i]\!]_{(d,(s_1,s_2,\ldots,s_n))} = X_{(d+1,(s_1,\ldots,s_i+1,\ldots,s_n))}$.

Using Proposition 4.3, we can derive the multi-graded simply blocked grid homology $\widehat{\mathbf{GH}}(\mathbb{G})$ when $\mathbb{G}$ represents positive Hopf link in the figure 4.2.

$$\begin{aligned} \widetilde{\mathbf{GH}}(\mathbb{G}) &\cong \widehat{\mathbf{GH}}(\mathbb{G}) \otimes W_1 \otimes W_2 \\ &\cong (\widehat{\mathbf{GH}}(\mathbb{G}) \oplus \widehat{\mathbf{GH}}(\mathbb{G})[\![1, \mu_1]\!]) \otimes W_2 \\ &\cong \widehat{\mathbf{GH}}(\mathbb{G}) \oplus \widehat{\mathbf{GH}}(\mathbb{G})[\![1, \mu_1]\!] \oplus \widehat{\mathbf{GH}}(\mathbb{G})[\![1, \mu_2]\!] \oplus \widehat{\mathbf{GH}}(\mathbb{G})[\![2, \mu_1 + \mu_2]\!]. \end{aligned} \tag{4.6}$$

By using the isomorphism in (4.2), we can see that

$$\widehat{\mathbf{GH}}(\mathbb{G}) \cong \mathbb{F}_{(-2,(-\frac{1}{2},-\frac{1}{2}))} \oplus \mathbb{F}_{(-1,(\frac{1}{2},-\frac{1}{2}))} \oplus \mathbb{F}_{(-1,(-\frac{1}{2},\frac{1}{2}))} \oplus \mathbb{F}_{(0,(\frac{1}{2},\frac{1}{2}))}$$

$$\widehat{\mathbf{GH}}(\mathbb{G})[\![1, \mu_1]\!] \cong \mathbb{F}_{(-3,(-\frac{3}{2},-\frac{1}{2}))} \oplus \mathbb{F}_{(-2,(-\frac{1}{2},-\frac{1}{2}))} \oplus \mathbb{F}_{(-2,(-\frac{3}{2},-\frac{1}{2}))} \oplus \mathbb{F}_{(-1,(-\frac{1}{2},\frac{1}{2}))}$$

$$\widehat{\mathbf{GH}}(\mathbb{G})[\![1, \mu_2]\!] \cong \mathbb{F}_{(-3,(-\frac{1}{2},-\frac{3}{2}))} \oplus \mathbb{F}_{(-2,(\frac{1}{2},-\frac{3}{2}))} \oplus \mathbb{F}_{(-2,(-\frac{1}{2},-\frac{1}{2}))} \oplus \mathbb{F}_{(-1,(\frac{1}{2},-\frac{1}{2}))}$$

$$\widehat{\mathbf{GH}}(\mathbb{G})[\![2, \mu_1 + \mu_2]\!] \cong \mathbb{F}_{(-4,(-\frac{3}{2},-\frac{3}{2}))} \oplus \mathbb{F}_{(-3,(-\frac{1}{2},-\frac{3}{2}))} \oplus \mathbb{F}_{(-3,(-\frac{3}{2},-\frac{1}{2}))} \oplus \mathbb{F}_{(-2,(-\frac{1}{2}.-\frac{1}{2}))}.$$

Also we can see the multi-graded simply blocked grid homology $\widehat{\mathbf{GH}}(\mathbb{G})$ as follows, using the equation (4.3),

$$\widehat{\mathbf{GH}}(\mathbb{G}) \cong \mathbb{F}_{(-2,-1)} \oplus \mathbb{F}_{(-1,0)} \oplus \mathbb{F}_{(-1,0)} \oplus \mathbb{F}_{(0,1)}.$$

		d	h	s
x_1	(1234)	−1	(−1/2, −1/2)	−1
x_2	(2134)	−2	(−1/2, −1/2)	−1
x_3	(3214)	0	(1/2, 1/2)	1
x_4	(4231)	−2	(−1/2, −1/2)	−1
x_5	(1324)	−2	(−1/2, −1/2)	−1
x_6	(1432)	−4	(−3/2, −3/2)	−3
x_7	(1243)	−2	(−1/2, −1/2)	−1
x_8	(3124)	−1	(1/2, −1/2)	0
x_9	(2431)	−3	(−3/2, −1/2)	−2
x_{10}	(2143)	−3	(−1/2, −1/2)	−1
x_{11}	(2314)	−1	(−1/2, 1/2)	0
x_{12}	(3241)	−1	(−1/2, 1/2)	0
x_{13}	(3412)	−1	(−1/2, −1/2)	−1
x_{14}	(4213)	−1	(1/2, −1/2)	0
x_{15}	(1342)	−3	(−3/2, −1/2)	−2
x_{16}	(1423)	−3	(−1/2, −3/2)	−2
x_{17}	(4321)	−3	(−1/2, −1/2)	−1
x_{18}	(4132)	−3	(−1/2, −3/2)	−2
x_{19}	(2341)	−2	(−3/2, 1/2)	−1
x_{20}	(3421)	−2	(−1/2, −1/2)	−1
x_{21}	(2413)	−2	(−1/2, −1/2)	−1
x_{22}	(4312)	−2	(−1/2, −1/2)	−1
x_{23}	(3142)	−2	(−1/2, −1/2)	−1
x_{24}	(4123)	−2	(1/2, −3/2)	−1

Figure 4.3. Maslov and Alexander multi-gradings of the (positive) Hopf link.

5. CONCLUSION

In this thesis, we give an introduction to *grid homology* for knots by providing the preliminary definitons to understand the construction of the combinatorial approach to knot Floer homology. First, we examine the equivalence of knots using their grid diagrams, using grid moves, thanks to Cromwell [7]. Then we define the various types of grid homology for knots, and investigate their invariance under grid moves by an example. We adapt our grid homology definiton for links, and compute the multi-graded simply blocked grid homology of positive Hopf link as an example.

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