

THE BRAIDED ALGEBRA AND ITS JORDAN-SCHWINGER CONSTRUCTION
IN TERMS OF Q-DEFORMED FERMIONIC OSCILLATORS

by

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ABSTRACT**THE BRAIDED ALGEBRA AND ITS
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The standard bosonic and fermionic Jordan-Schwinger constructions for the Lie algebra of $SU(2)$ are reviewed in this thesis. It is shown that the Jordan-Schwinger constructions of the quantum group with q as deformation parameter $SU_q(2)$ are obtained by using q -deformed bosonic and fermionic oscillators. The construction of the braided algebra $BM_q(2)$ of Hermitian braided matrices in terms of two independent q -bosonic oscillators in the Fock space is studied. It is also determined that the braided algebra of $BM_q(2)$ can be constructed by a pair of q, q^{-1} deformed bosonic oscillators. By means of a similar approach we construct the braided algebra of (nonHermitian) $BM_q(2)$ braided matrices in terms of two independent q -deformed fermionic oscillators. We also observe that the representations of this algebra of q, q^{-1} deformed fermionic oscillators are constructed in a complex vector space. Finally, in the limit $q \rightarrow 1$, we show that our construction gives the Pauli exclusion principle.

ÖZET

ÖRGÜLÜ CEBİR VE BU CEBRİN Q DEFORME EDİLEN FERMİONİK SALINIMCILARIYLA JORDAN-SCHWINGER YAPISI

Bu tezde, Lie cebri $SU(2)$ ya ait olan standart bozonik ve fermionik Jordan-Schwinger yapıları incelenmiştir. Deformasyon parametresi q olan kuantum grubu $SU_q(2)$ nin Jordan - Schwinger yapıları q deforme edilen bozonik ve fermionik salınımcılarıyla elde edilebildiği gösterilmiştir. Örgülü Hermisyen matris $BM_q(2)$ nin yapısı iki bağımsız q bozonik salınımcısı açısından Fock uzayında çalışılmıştır. Ayrıca, örgülü $BM_q(2)$ cebri bir çift q, q^{-1} deforme edilen bozonik salınımcılarıyla oluşturulabileceği belirlenmiştir. Benzer yaklaşımla örgülü (Hermisyen olmayan) $BM_q(2)$ cebri iki bağımsız q fermionik salınımcısını kullanarak inşa ettik. Aynı zamanda bu cebri temsilinin q, q^{-1} deforme edilen fermionik salınımcılarıyla kompleks vektör uzayında oluşabildiğini inceledik. Son olarak, $q \rightarrow 1$ limitinde yapımızın Pauli prensibini verdiğini gösterdik.

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LIST OF SYMBOLS

\mathbf{a}	Annihilation operator of bosonic oscillator
\mathbf{a}^\dagger	Creation operator of bosonic oscillator
$\mathbf{j}^0, \mathbf{j}^\pm$	The angular momentum operators of $SU(2)$
\mathbf{j}, \mathbf{m}	The angular momentum eigenstates
\mathbf{q}	Deformation parameter
\mathbf{N}	Number operator
$\mathbf{J}^0, \mathbf{J}^\pm$	The angular momentum operators of $SU_q(2)$
$[\mathbf{n}]_b$	q-bosonic basic number
\mathbf{c}	Annihilation operator of fermionic oscillator
\mathbf{c}^\dagger	Creation operator of fermionic oscillator
$[\mathbf{n}]_f$	q-fermionic basic number
\mathbf{u}	Braided matrix
\mathbf{t}	Quantum matrix
$\mathbf{B}(\mathbf{R})$	Braided algebra obtained from R-matrix
ψ	Braiding
Δ	Coproduct map

1. INTRODUCTION

The Lie groups play a crucial role in the theory of quantum groups. Q-deformed Lie algebras and Lie groups [1 - 4] have been investigated comprehensively by a lot of scientists in many fields of physics, particularly the theory of exactly solvable quantum systems where the Yang-Baxter equation has a special role [5]. Due to the fact that for a quantum matrix group the elements of distinct matrices commute when the group product is chosen, this fact which is not valid for supermatrices must be generalized to determine the braided matrix group $BM_q(2)$ [6]. From the mathematical point of view, a quantum matrix group $SU_q(2)$ is defined as the algebra in the tensor product space which is connected with Hopf algebras while a braided matrix group $BM_q(2)$ is the quasitensor quantum algebra [7]. Majid investigated that the elements of braided matrix group $BM_q(2)$ can be obtained in terms of the generators of a quantum matrix group $SU_q(2)$ [8]. By using a q-deformed oscillator and a unitary operator the matrix elements of the quantum matrix group $SU_q(2)$ can be generalized for irreducible representations in a Hilbert space [9]. On the other hand, a realization of the quantum group with as q deformation parameter $SU_q(2)$ based on the Jordan-Schwinger construction by using a pair of commuting q-deformed bosonic oscillators has been obtained by Macfarlane and Biedernarn [10 - 11]. Using a similar way the construction of the braided algebra $BM_q(2)$ of Hermitian braided matrices in terms of a pair of independent q-bosonic oscillators has been found by Arık and Yıldız [12].

The outline of this thesis is the following. In section 2, we review the standard bosonic Jordan-Schwinger construction of $SU(2)$. We also generalize the Jordan-Schwinger constructions for q-fermionic and bosonic oscillators to realize the quantum group with q as deformation parameter $SU_q(2)$. In section 3, we review the braided algebra of Hermitian $BM_q(2)$ in terms of two independent q-bosonic oscillators in a Hilbert space. In section 4, we first construct an infinite dimensional representation of $BM_q(2)$ for the case of new values of the parameter q^2 . We show that $BM_q(2)$ can be chosen to be Hermitian for this case. We then present a construction of the braided algebra of $BM_q(2)$ in terms of a pair of independent q-fermionic oscillators. For this

construction finite dimensional representations are desired and this forces $BM_q(2)$ to be nonHermitian. We show that our braided algebra of q, q^{-1} deformed fermionic oscillators can be developed in a complex vector space. We discuss the $q \rightarrow 1$ limit of our results for construction of the braided quantum group in terms of q -deformed fermionic oscillators. Finally, in section 5 we present our conclusion.

2. THE JORDAN-SCHWINGER CONSTRUCTION OF SU(2) AND SU_q(2)

2.1. Bosonic Jordan-Schwinger Construction of SU(2)

In this section we would like to recall briefly the standard bosonic Jordan-Schwinger construction of $SU(2)$ [13]. This can be defined by the creation and the annihilation operators a^\dagger and a which satisfy a pair of bosonic oscillators.

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2 \quad (2.1)$$

$$[a_i, a_j] = 0 \quad (2.2)$$

The Jordan-Schwinger construction for the Lie algebra of $SU(2)$ can be generated by operators j^+ , j^0 , j^- as bilinears in a and a^\dagger

$$j^+ = a_1^\dagger a_2, \quad (2.3)$$

$$j^0 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), \quad (2.4)$$

$$j^- = a_2^\dagger a_1. \quad (2.5)$$

These operators described by (2.1 - 2.2) satisfy the commutation relations

$$[j^+, j^-] = 2j^0, \quad (2.6)$$

$$[j^0, j^\pm] = \pm j^\pm. \quad (2.7)$$

It is remarkable to notice that one can get both the defined algebraic relations of $SU(2)$ and the all unitary representations of $SU(2)$ by means of the Jordan-Schwinger

construction for the $SU(2)$ algebra from a single pair of commuting bosonic oscillators. To verify this, the angular momentum eigenstates $|j, m\rangle$ are described by

$$|j, m\rangle = \frac{(a_1^\dagger)^{j+m}}{\sqrt{(j+m)!}} \frac{(a_2^\dagger)^{j-m}}{\sqrt{(j-m)!}} |0\rangle \quad (2.8)$$

where $j = 0, \frac{1}{2}, 1, \dots$, $-j \leq m \leq j$ and the vacuum state $|0\rangle$ is defined as

$$a_i |0\rangle = 0, \quad i, j = 1, 2. \quad (2.9)$$

From (2.1 - 2.2) it is straightforward to verify that

$$j^0 |j, m\rangle = m |j, m\rangle, \quad (2.10)$$

$$j^\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle. \quad (2.11)$$

Hence, the states (2.8) are the ordinary angular momentum states in which j corresponds to the total spin and m to the third component of the spin. As j can choose arbitrary integers or half integers values, the full set of unitary representations of $SU(2)$ are recovered by the states $|j, m\rangle$ on which the operators are determined by (2.3 - 2.5).

2.2. Q-Bosonic Jordan-Schwinger Construction of $SU_q(2)$

As pointed out by Macfarlane and Biedernharn, the Jordan-Schwinger construction can be also generalized to realize the quantum group $SU_q(2)$, with q as deformation parameter, in terms of a pair of commuting independent q-bosonic oscillators [10, 11, 14]. To prove this, we must firstly determine the operators which satisfy the q-bosonic oscillator.

Let us now consider q- deformed bosonic oscillator which can be given by following

commutation relations

$$aa^\dagger - \sqrt{q}a^\dagger a = q^{-N/2}, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad (2.12)$$

where a , a^\dagger are the annihilation, creation operators, $q > 0$, and N is the number operator respectively [10 - 11]. It is essential to notice that N becomes $a^\dagger a$ only in the limit $q \rightarrow 1$.

Then, we can determine the representation of (2.12) in the Fock space covered by the orthonormal n-quanta eigenstates $|n\rangle$ of the number operator [15]

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}} |0\rangle, \quad a|0\rangle = 0, \quad N|n\rangle = n|n\rangle, \quad (2.13)$$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{and} \quad [n]! = [n][n-1] \dots [1]. \quad (2.14)$$

Note that q-deformed integer $[n]$ will frequently appear in the expression that follows in the limit $q \rightarrow 1$, it reduces n . The following properties follow

$$a^\dagger |n\rangle = \sqrt{[n+1]} |n+1\rangle, \quad (2.15)$$

$$a |n\rangle = \sqrt{[n]} |n-1\rangle, \quad (2.16)$$

$$a^\dagger a = [N], \quad (2.17)$$

$$aa^\dagger = [N+1]. \quad (2.18)$$

It is worthwhile to mention that there are several forms of relations (2.12). For example [16], we can define as follows

$$A = q^{N/4} a, \quad A^\dagger = a^\dagger q^{N/4}, \quad (2.19)$$

to represent new operators which verify the commutation relation

$$AA^\dagger - qA^\dagger A = 1, \quad q > 0. \quad (2.20)$$

Using (2.13) we can construct the Fock space representation by means of these new operators [17]. We get

$$|n\rangle = \frac{(A^\dagger)^n}{\sqrt{[n]_b!}} |0\rangle, \quad A|0\rangle = 0, \quad N|n\rangle = n|n\rangle, \quad (2.21)$$

where the combined q-bosonic basic number is given by

$$[n]_b = \frac{1 - q^n}{1 - q}. \quad (2.22)$$

In Fock space, it is easily found as

$$AA^\dagger = [N]_b, \quad A^\dagger A = [N + 1]_b. \quad (2.23)$$

By means of the above calculations, we can introduce a pair of mutually commuting q-bosonic oscillators with a_i and a_i^\dagger , for $i = 1, 2$, which verify the commutation relations as equation (2.12).

Thus, the angular momentum operators of $SU_q(2)$ can be described in terms of a_i , a_i^\dagger are the q-deformed oscillator annihilation and creation operators. These operators can be given in the following way

$$J^+ = a_1^\dagger a_2, \quad (2.24)$$

$$J^- = a_2^\dagger a_1, \quad (2.25)$$

$$J^0 = \frac{1}{2}(N_1 - N_2), \quad (2.26)$$

where

$$N_1 |n_1\rangle = n_1 |n_1\rangle \quad \text{and} \quad N_2 |n_2\rangle = n_2 |n_2\rangle. \quad (2.27)$$

The total number operator N can be given by [18]

$$N \equiv N_1 + N_2 = a_1^\dagger a_1 + a_2^\dagger a_2. \quad (2.28)$$

Then, action of the J^0 , J^+ and J^- operators on the states $|n_1, n_2\rangle$ one obtain

$$J^0 |n_1, n_2\rangle = \frac{1}{2} (N_1 - N_2) |n_1, n_2\rangle = \frac{1}{2} (n_1 - n_2) |n_1, n_2\rangle, \quad (2.29)$$

$$J^+ |n_1, n_2\rangle = a_1^\dagger a_2 |n_1, n_2\rangle = \sqrt{[n_2(n_1 + 1)]} |n_1 + 1, n_2 - 1\rangle, \quad (2.30)$$

$$J^- |n_1, n_2\rangle = a_2^\dagger a_1 |n_1, n_2\rangle = \sqrt{[n_1(n_2 + 1)]} |n_1 - 1, n_2 + 1\rangle. \quad (2.31)$$

We can describe $n_1 \rightarrow j + m$ and $n_2 \rightarrow j - m$ and plug these into (2.29 - 2.31) as follows

$$J^0 |j + m, j - m\rangle = m |j + m, j - m\rangle, \quad (2.32)$$

$$J^+ |j + m, j - m\rangle = \sqrt{[(j - m)(j + m + 1)]} |j + m + 1, j - m - 1\rangle, \quad (2.33)$$

$$J^- |j + m, j - m\rangle = \sqrt{[(j + m)(j - m + 1)]} |j + m - 1, j - m + 1\rangle. \quad (2.34)$$

Defining the Casimir operator J^2

$$J^2 = (J^0)^2 + \frac{1}{2}(J^+ J^- + J^- J^+), \quad (2.35)$$

can be rewritten as

$$J^2 = \frac{N}{2} \left(\frac{N}{2} + 1 \right). \quad (2.36)$$

The action of J^2 on states $|j + m, j - m\rangle$ is given by

$$J^2 |j + m, j - m\rangle = \frac{1}{2}(n_1 + n_2)[1 + \frac{1}{2}(n_1 + n_2)] |j + m, j - m\rangle. \quad (2.37)$$

It can be rewritten as

$$J^2 |j + m, j - m\rangle = j(j + 1) |j + m, j - m\rangle. \quad (2.38)$$

Thus, the angular momentum eigenstates $|j, m\rangle$ can be easily defined as follows

$$|j, m\rangle = \frac{(a_1^\dagger)^{j+m}}{\sqrt{[j+m]!}} \frac{(a_2^\dagger)^{j-m}}{\sqrt{[j-m]!}} |0\rangle \equiv |j + m\rangle \otimes |j - m\rangle \quad (2.39)$$

with $j = 1/2(n_1 + n_2)$ and $m = 1/2(n_1 - n_2)$.

Using (2.24 - 2.26), one verifies the commutation relations in the following way [10]

$$[J^+, J^-] |j, m\rangle = ([j + m][j - m + 1] - [j - m][j + m - 1]) |j, m\rangle, \quad (2.40)$$

$$[J^+, J^-] |j, m\rangle = \left(\frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}} \right) |j, m\rangle, \quad (2.41)$$

$$[J^0, J^\pm] |j, m\rangle = \pm J^\pm |j, m\rangle. \quad (2.42)$$

These commutators satisfy the defining algebraic realization of $SU_q(2)$ and hence one can decide that the Jordan-Schwinger realization for q-bosonic oscillators gives rise to the quantum group $SU_q(2)$.

2.3. Fermionic Jordan-Schwinger Construction of $SU(2)$

The Jordan-Schwinger construction can be also generalized by using fermions instead of bosons. We can consider a pair of fermionic oscillators which satisfy the

following anticommutation relations

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \quad i, j = 1, 2 \quad (2.43)$$

$$\{c_i, c_j\} = 0. \quad (2.44)$$

Then, let us define the operators of $SU(2)$ in terms of fermionic oscillators

$$j^+ = c_1^\dagger c_2, \quad (2.45)$$

$$j^0 = \frac{1}{2} \{c_1^\dagger c_1 - c_2^\dagger c_2\}, \quad (2.46)$$

$$j^- = c_2^\dagger c_1. \quad (2.47)$$

which satisfy the commutation relations in the following way

$$[j^+, j^-] = 2j^0, \quad (2.48)$$

$$[j^0, j^\pm] = \pm j^\pm. \quad (2.49)$$

In contrast to the bosonic case, in the fermionic case we cannot obtain the full set of the unitary representations of $SU(2)$. Since the anticommutation relations restrict to occupy no two fermion of each state we can constitute only the following four states,

$$|0\rangle, \quad c_1^\dagger |0\rangle, \quad c_2^\dagger |0\rangle, \quad c_1^\dagger c_2^\dagger |0\rangle, \quad (2.50)$$

and therefore only the $j = 0$ and $j = 1/2$ representations of $SU(2)$ are recovered. In other words, it can be constituted to realize the full set of unitary representations of the $SU(2)$ algebra by combining the spin 0 and spin 1/2 representations. To cope with this hardship, we can take several copies of pairs of fermionic oscillators.

2.4. Q-Fermionic Jordan-Schwinger Construction of $SU_q(2)$

The Jordan-Schwinger construction for q-fermionic oscillator can be also used to realize the quantum group with q as deformation parameter $SU_q(2)$. We can explain a realization of the $SU_q(2)$ based on the Jordan-Schwinger construction by using a pair of commuting q-fermionic oscillators.

Let us now start a non-trivial q-deformed fermionic oscillator which can be given by following commutation relations

$$cc^\dagger + \sqrt{q}c^\dagger c = q^{-N/2}, \quad [N, c^\dagger] = c^\dagger, \quad [N, c] = -c, \quad c^2 \neq 0, \quad (c^\dagger)^2 \neq 0, \quad (2.51)$$

where c , c^\dagger are the annihilation, creation operators, $q > 0$, and N is the number operator respectively. The equations $c^2 \neq 0$ and $(c^\dagger)^2 \neq 0$ allow more than two fermions in a given each state, so that these operators do not obey the Pauli exclusion principle. It is necessary to remark that N is only $c^\dagger c$ in the limit $q \rightarrow 1$.

Then, we can determine the representation of (2.51) in the Fock space covered by the orthonormalized $|n\rangle$ eigenstates of the number operator [15]

$$|n\rangle = \frac{(c^\dagger)^n}{\sqrt{[n]!}} |0\rangle, \quad c|0\rangle = 0, \quad N|n\rangle = n|n\rangle. \quad (2.52)$$

where

$$[n] = \frac{q^{-n} - (-q)^n}{q + q^{-1}} \quad \text{and} \quad [n]! = [n][n-1] \dots [1]. \quad (2.53)$$

The following properties follow

$$c^\dagger |n\rangle = \sqrt{[n+1]!} |n+1\rangle, \quad (2.54)$$

$$c |n\rangle = \sqrt{[n]!} |n-1\rangle, \quad (2.55)$$

$$cc^\dagger = [N], \quad (2.56)$$

$$c^\dagger c = [N + 1]. \quad (2.57)$$

It is of significance to maintain that equation (2.51) can be written in different forms by introducing possible transformations. To illustrate this, we can define as

$$C = q^{N/4}c, \quad C^\dagger = c^\dagger q^{N/4}, \quad (2.58)$$

and modified operators satisfy the commutation relation

$$CC^\dagger + qC^\dagger C = 1, \quad q > 0. \quad (2.59)$$

From (2.51) we can constitute the Fock space representation in terms of these new operators. We have

$$|n\rangle = \frac{(C^\dagger)^n}{\sqrt{[n]_f!}} |0\rangle, \quad C|0\rangle = 0, \quad N|n\rangle = n|n\rangle. \quad (2.60)$$

where the q-fermionic basic number is given by

$$[n]_f = \frac{1 - (-q)^n}{1 + q}. \quad (2.61)$$

In Fock space, it is easily verified as

$$C^\dagger C = [N]_f, \quad CC^\dagger = [N + 1]_f. \quad (2.62)$$

It should be stressed that $[n]_f$ given by (2.61) is quite different from the case of $[n]_b$ given by (2.22). Firstly, note that $[n]_f = n$ only for $n = 0, 1$ and $[n]_f \neq n$ for $n > 1$, which result is valid for any q . Besides, for $q < 1$, although the q-fermionic basic number never goes beyond 1 for any n , the q-bosonic basic number becomes usual basic number. Also for $n \rightarrow \infty$, $[n]_b \rightarrow 1/(1 - q)$. For $q > 1$, we get for the q-fermionic basic number, $[0]_f = 0$, $[1]_f = 1$ and $[n]_f > 0$ for n odd, $[n]_f < 0$ for n even, in which

case the Fock space is not constructed. Secondly, in the limit $q \rightarrow 1$, we see that $[n]_f \rightarrow (1/2)\{1 - (-1)^n\}$ taking the values 0, 1 for n even and odd and thus in this limit, $[n]_f = 0, 1, 0, 1 \dots$ for $n = 0, 1, 2, \dots \infty$. In contrast to the q -fermionic basic number, the q -bosonic basic number reduces n in the limit $q \rightarrow 1$ as discussed in the previous section. It is also remarkable to note that $[n]_f! = [n][n-1][n-2] \dots [1]$ reduces in the classical limit to $[n]_f! = 1$ for $n = 0, 1$ and $[n]_f! = 0$ for $n > 1$ and thus $(C^\dagger)^n |0\rangle = 0$ for $n > 1$. On the other hand, it is easily seen that for $q \neq 1$ imposing $(C^\dagger)^n = 0$ for $n > 1$ brings about inconsistencies. Therefore we take $0 < q \leq 1$ for q -deformed fermionic oscillator.

With the help of the above calculations, we can show a pair of mutually commuting fermionic q -oscillators with c_i and c_i^\dagger , for $i = 1, 2$, which satisfy the commutation relations (2.50) can be used for the Jordan-Schwinger construction.

Thus, the angular momentum operators of $SU_q(2)$ can be defined by the annihilation and creation operators c_i, c_i^\dagger . These operators can be given by

$$J^+ = c_1^\dagger c_2, \quad (2.63)$$

$$J^- = c_2^\dagger c_1, \quad (2.64)$$

$$J^0 = \frac{1}{2}(N_1 - N_2), \quad (2.65)$$

which verify the $SU(2)$ commutation relations in equations (2.48 - 2.49). Defining the Casimir operator J^2 for a pair of q -fermionic oscillators

$$J^2 = \frac{N}{2} \left(\frac{N}{2} + 1 \right) - 2N_1 N_2. \quad (2.66)$$

The allowed eigenvalues of N_1, N_2 are 0 and 1 when only the $j = 0$ and $j = 1/2$ representations of $SU(2)$ are allowed. Furthermore, the only allowed eigenvalues of J^0

as described in equation (2.65) are 0, 1/2, -1/2 [19].

$$J^0 |0, 0\rangle = 0, \quad J^0 c_1^\dagger c_2^\dagger |0, 0\rangle = 0, \quad (2.67)$$

$$J^0 c_1^\dagger |0, 0\rangle = \frac{1}{2} c_1^\dagger |0, 0\rangle, \quad J^0 c_2^\dagger |0, 0\rangle = -\frac{1}{2} c_2^\dagger |0, 0\rangle, \quad (2.68)$$

$$J^0 |1, 1\rangle = 0, \quad J^2 |1, 1\rangle = 0. \quad (2.69)$$

Thus, it is obvious to note that using two q-fermionic oscillators we can construct just one spin-1/2 state which represents $c_1^\dagger |0, 0\rangle$, $c_2^\dagger |0, 0\rangle$ and two spin-0 states ($|0, 0\rangle$, $|1, 1\rangle$).

As an application of the above, we specify the Jordan-Schwinger construction of $SU_q(2)$ in terms of a pair of commuting independent q-fermionic oscillators by defining the operators given by (2.63 - 2.65) which verify the following commutation relations

$$[J^+, J^-] = [2J^0] = \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}, \quad (2.70)$$

$$[J^0, J^\pm] = \pm J^\pm. \quad (2.71)$$

3. THE BRAIDED QUANTUM GROUP $BM_q(2)$ AND ITS JORDAN-SCHWINGER CONSTRUCTION IN TERMS OF Q-DEFORMED OSCILLATORS

As discussed in the previous section, Macfarlane and Biedernarn have obtained the Jordan-Schwinger construction for $SU_q(2)$ by using two independent q -bosonic harmonic oscillators. On the hand, the braided algebra of Hermitian $BM_q(2)$ in terms of a pair of independent q -bosonic oscillators has been achieved by Arık and Yıldız [12].

Now we would like to recall the definition braided group. The algebra $B(R)$ called the braided algebra with braided matrix generators $\mathbf{u} = (u_j^i)$ verifying the braided commutativity relations such that

$$R_{21}u_1R_{12}u_2 = u_2R_{21}u_1R_{12} \quad (3.1)$$

is quantum group covariant under the transformation $\mathbf{u}' = \mathbf{t}^{-1}\mathbf{u}\mathbf{t}$ where \mathbf{t} is a quantum matrix, $\mathbf{u}_1 = \mathbf{u} \otimes I$ and $\mathbf{u}_2 = I \otimes \mathbf{u}$. Moreover, since the usual transposition map π in tensor product space

$$(v \otimes w)(y \otimes z) = v\pi(w \otimes y)z = vy \otimes wz \quad (3.2)$$

is not in general quantum group covariant, it is replaced by a generalized map ψ called braided statistics relation which can be given by

$$(v \otimes w)(y \otimes z) = v\psi(w \otimes y)z. \quad (3.3)$$

Note that one should determine commutation relations between independent copies of algebras due to the fact that the transformed ones do not commute in a quantum group covariant formulation. Besides, as the notion of tensor product \otimes allows algebras to

live independent of each other, $\underline{\otimes}$ are used instead of \otimes . Thus, the braided-coproduct Δ is a homomorphism when $B(R)\underline{\otimes}B(R)$ has the braided tensor product algebra given by (3.3) as defined by ψ . The braiding statistics relation ψ is usually trivial.

For a simple nontrivial example of braided quantum matrix $BM_q(2)$ we use the standard $SL_q(2)$ R-matrix which satisfies the quantum Yang-Baxter equation. It is given by

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2 - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}. \quad (3.4)$$

Denoting an element $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ verifies the braided commutativity relations

$$R_{21}u_1R_{12}u_2 = u_2R_{21}u_1R_{12} \quad (3.5)$$

where $u_1 = u \otimes I$ and $u_2 = I \otimes u$. Then the braided commutativity relations turn out

$$ba = q^2ab, \quad (3.6)$$

$$ca = q^{-2}ac, \quad (3.7)$$

$$ad = da, \quad (3.8)$$

$$bc = cb + (1 - q^{-2})a(d - a), \quad (3.9)$$

$$db = bd + (1 - q^{-2})ab, \quad (3.10)$$

$$cd = dc + (1 - q^{-2})ca. \quad (3.11)$$

The braid group action on the generators is given by

$$\psi(a \otimes a) = a \otimes a + (1 - q^2)b \otimes c, \quad (3.12)$$

$$\psi(a \otimes b) = b \otimes a, \quad (3.13)$$

$$\psi(a \otimes c) = c \otimes a + (1 - q^2)(d - a) \otimes c, \quad (3.14)$$

$$\psi(a \otimes d) = d \otimes a + (1 - q^{-2})b \otimes c, \quad (3.15)$$

$$\psi(b \otimes a) = a \otimes b + (1 - q^2)b \otimes (d - a), \quad (3.16)$$

$$\psi(b \otimes b) = q^2b \otimes b, \quad (3.17)$$

$$\begin{aligned} \psi(b \otimes c) &= q^{-2}c \otimes b + (1 + q^2)(1 - q^{-2})^2b \otimes c \\ &\quad - (1 - q^{-2})(d - a) \otimes (d - a) \end{aligned} \quad (3.18)$$

$$\psi(b \otimes d) = d \otimes b + (1 - q^{-2})b \otimes (d - a), \quad (3.19)$$

$$\psi(c \otimes a) = a \otimes c, \quad (3.20)$$

$$\psi(c \otimes b) = q^{-2}b \otimes c, \quad (3.21)$$

$$\psi(c \otimes c) = q^2c \otimes c, \quad (3.22)$$

$$\psi(c \otimes d) = d \otimes c, \quad (3.23)$$

$$\psi(d \otimes a) = a \otimes d + (1 - q^{-2})b \otimes c, \quad (3.24)$$

$$\psi(d \otimes b) = b \otimes d, \quad (3.25)$$

$$\psi(d \otimes c) = c \otimes d + (1 - q^{-2})(d - a) \otimes c, \quad (3.26)$$

$$\psi(d \otimes d) = d \otimes d - q^{-2}(1 - q^{-2})b \otimes c. \quad (3.27)$$

For instance, let us take any f and g

$$(f \otimes c)(c \otimes g) = f\psi(c \otimes c)g, \quad (3.28)$$

$$= q^2fc \otimes cg, \quad (3.29)$$

$$(f \otimes a)(d \otimes g) = f\psi(a \otimes d)g, \quad (3.30)$$

$$= fd \otimes ag + (1 - q^{-2})fb \otimes cg. \quad (3.31)$$

Note that the braided determinant $\det B = ad - q^2cb$ and the quantum element $q^{-1}a + qd$ are bosonic and central such that

$$\psi((q^{-1}a + qd) \otimes f) = f \otimes (qd + q^{-1}) \quad (3.32)$$

which hold for all $f \in B(R)$.

Note that for $BM_q(2)$ the coproduct Δ is given by the matrix product

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}$$

so that for example $\Delta(a) = a \otimes a + b \otimes c$. Since Δ must be an algebra homomorphism the relations (3.6 - 3.11) must hold when a, b, c, d are respectively replaced with $\Delta(a), \Delta(b), \Delta(c), \Delta(d)$. When taking products of these the tensor products appearing should be developed using the braiding relations (3.12 - 3.27). For example,

$$\Delta(b)\Delta(a) = (a \otimes b + b \otimes d)(a \otimes a + b \otimes c), \quad (3.33)$$

$$= a\psi(b \otimes a)a + a\psi(b \otimes b)c + b\psi(d \otimes a)a + b\psi(d \otimes b)c, \quad (3.34)$$

$$= a[a \otimes b + (1 - q^2)b \otimes (d - a)]a + q^2ab \otimes bc + bb \otimes dc + b[a \otimes d + (1 - q^{-2})b \otimes c]a, \quad (3.35)$$

$$= a[a \otimes b - q^2\psi(b \otimes d) + q^2d \otimes b]a + q^2ab \otimes bc + bb \otimes dc + b[a \otimes d - q^{-2}\psi(a \otimes a) + q^{-2}a \otimes a]a, \quad (3.36)$$

$$= aa \otimes ba - q^2ad \otimes ba + q^2(1 - q^2)ab \otimes (d - a)a + q^2ad \otimes ba + q^2ab \otimes bc + bb \otimes dc + ba \otimes da - q^{-2}ba \otimes aa - q^{-2}(1 - q^2)bb \otimes ca + q^{-2}ba \otimes aa, \quad (3.37)$$

$$= q^2aa \otimes ba + q^2(1 - q^2)ab \otimes cb + q^2ab \otimes da + q^2ba \otimes cb + bb \otimes dc, \quad (3.38)$$

$$= q^2[a\psi(a \otimes a)b + a\psi(a \otimes b)d + b\psi(c \otimes a)b + b\psi(c \otimes b)d], \quad (3.39)$$

$$= q^2\Delta(a)\Delta(b). \quad (3.40)$$

We can determine a representation of u of Hermitian $BM_q(2)$ in terms of a, b, c, d as generators in a Hilbert space. The deformation parameter q is taken to be real. Taking the element u to be hermitian we get $a^\dagger = a, d^\dagger = d, \text{ and } b^\dagger = c$. It can be indicated that a commuting set which can be diagonalized is generated by the Hermitian operators $a, d, b^\dagger b$ and bb^\dagger . Using the diagonalized operators together with

the lowering and raising operators b and b^\dagger , we obtain

$$a |n\rangle = a_n |n\rangle, \quad (3.41)$$

$$d |n\rangle = d_n |n\rangle, \quad (3.42)$$

$$b |n\rangle = b_n |n-1\rangle, \quad (3.43)$$

$$b^\dagger |n\rangle = b_{n+1}^\dagger |n+1\rangle. \quad (3.44)$$

Substituting these into equations (3.6 - 3.11) we get the difference equations

$$a_{n+1} = q^2 a_n, \quad (3.45)$$

$$d_{n+1} = d_n + (1 - q^{-2})a_n, \quad (3.46)$$

$$b_{n+1}^\dagger b_{n+1} = b_n^\dagger b_n + (1 - q^{-2})a_n(d_n - a_n). \quad (3.47)$$

The solutions of these difference equations can be given by

$$a_n = q^{2n} a_0, \quad (3.48)$$

$$d_n = d_0 + a_0 q^{-2}(1 - q^{2n}), \quad (3.49)$$

$$b_n^\dagger b_n = A q^{4n} + B q^{2n} + C, \quad (3.50)$$

where

$$A = -a_0^2 q^{-4}, \quad (3.51)$$

$$B = a_0 q^{-2}(d_0 + q^{-2} a_0), \quad (3.52)$$

$$C = b_0^\dagger b_0 - q^{-2} a_0 d_0. \quad (3.53)$$

As discussed in [20], the spectrum $b^\dagger b$ is similar to the spectrum of the quadratic oscillator. We should have $b^\dagger b$ to be positive definite and this need to have the ground

state which is annihilated by the lowering operator, i.e., $b|j\rangle = 0$ for $q^2 < 1$. From equations (3.48 - 3.50), the quantum determinant $ad - q^2cb$ has only the eigenvalues a_0d_0 where a_0 and d_0 are integration constants of the equations (3.45 - 3.47). Thus the infinite dimensional representations

$$a_n = q^{2n}a_0, \quad (3.54)$$

$$d_n = d_0 + a_0q^{-2}(1 - q^{2n}), \quad (3.55)$$

$$b_n^\dagger b_n = a_0q^{-2}(1 - q^{2n})(q^{2(n-1)}a_0 - d_0), \quad (3.56)$$

$$n = 0, 1, 2, \dots \quad (3.57)$$

We take the finite (N)-dimensional representation with the states $|n\rangle = |0\rangle |1\rangle \dots |N-1\rangle$ for $d_0 = q^{2(N-1)}a_0$. The eigenvalues of diagonal Hermitian operator are given by

$$a_n = q^{2n}a_0 \quad (3.58)$$

$$d_n = a_0q^{-2}(q^{2N} - q^{2n} + 1) \quad (3.59)$$

$$b_n^\dagger b_n = a_0^2q^{-4}(1 - q^{2n})(q^{2n} - q^{2N}). \quad (3.60)$$

From the equations (3.58 - 3.60), the elements (a, b, b^\dagger and d) consist of a_0 as a multiplicative factor. Thus a_0 determines a commutative factor \mathfrak{R} . Using the quotient of the Hermitian braided matrix by \mathfrak{R} denotes to taking a particular value for a_0 . As a_0 can be based arbitrarily on any central element of the algebra, it is not a real number. Then m, j and a_0 can be defined as

$$m \equiv n - \frac{N-1}{2}, \quad j \equiv \frac{N-1}{2}, \quad a_0 = \frac{q^2}{1-q^2}A. \quad (3.61)$$

Then equations (3.58 - 3.60) can be rewritten as

$$a_{j+m} = \frac{q^{2(j+m+1)}}{1-q^2}A, \quad (3.62)$$

$$d_{j+m} = \frac{q^{2(2j+1)} - q^{2(j+m)} + 1}{1 - q^2} A, \quad (3.63)$$

$$b_{j+m} = q^{j+m} \sqrt{\left(\frac{1 - q^{2(j+m)}}{1 - q^2}\right) \left(\frac{1 - q^{2(j-m+1)}}{1 - q^2}\right)} A. \quad (3.64)$$

The Casimir operator can be defined by

$$C = \frac{q}{A(1 - q^2)} (q^{-1}a + qd) - \frac{1 + q^2}{A^2 q^4} (ad - q^2 b^\dagger b). \quad (3.65)$$

Let us consider two commuting q -bosonic oscillators which are constructed by a_1 , a_1^\dagger , N_1 , a_2 , a_2^\dagger and N_2 verifying

$$a_1 a_1^\dagger - q a_1^\dagger a_1 = 1, \quad (3.66)$$

$$a_2 a_2^\dagger - q^2 a_2^\dagger a_2 = q^{N_2+1}, \quad (3.67)$$

$$[a_1, a_2] = [a_1, a_2^\dagger] = 0, \quad (3.68)$$

$$N_1 |n_1\rangle = n_1 |n_1\rangle, \quad (3.69)$$

$$N_2 |n_2\rangle = n_2 |n_2\rangle. \quad (3.70)$$

Then we show the action of the operators on a Hilbert space as follows

$$a_1 |n_1\rangle = \sqrt{\left(\frac{1 - q^{n_1}}{1 - q}\right)} |n_1 - 1\rangle, \quad (3.71)$$

$$a_1^\dagger |n_1\rangle = \sqrt{\left(\frac{1 - q^{n_1+1}}{1 - q}\right)} |n_1 + 1\rangle, \quad (3.72)$$

$$a_2 |n_2\rangle = \sqrt{\left(\frac{q^{n_2}(1 - q^{n_2})}{1 - q}\right)} |n_2 - 1\rangle, \quad (3.73)$$

$$a_2^\dagger |n_2\rangle = \sqrt{\left(\frac{q^{n_2+1}(1 - q^{n_2+1})}{1 - q}\right)} |n_2 + 1\rangle. \quad (3.74)$$

Defining j and m as follows

$$m \equiv \frac{n_2 - n_1}{2} \quad \text{and} \quad j \equiv \frac{n_2 + n_1}{2}. \quad (3.75)$$

It can be found as

$$a_1^\dagger a_2 |j - m\rangle |j + m\rangle = \sqrt{q^{j+m} \frac{(1 - q^{j+m})}{1 - q} \frac{(1 - q^{j-m+1})}{1 - q}} |j, m - 1\rangle, \quad (3.76)$$

$$a_2^\dagger a_1 |j - m\rangle |j + m\rangle = \sqrt{q^{j+m+1} \frac{(1 - q^{j+m+1})}{1 - q} \frac{(1 - q^{j-m})}{1 - q}} |j, m + 1\rangle \quad (3.77)$$

$$\left(\frac{N_1 + N_2}{2}\right) |j - m\rangle |j + m\rangle = j |j, m\rangle, \quad (3.78)$$

$$\left(\frac{N_1 - N_2}{2}\right) |j - m\rangle |j + m\rangle = m |j, m\rangle. \quad (3.79)$$

By specifying

$$a \equiv \frac{q^{N_2+1}}{1 - q} A, \quad (3.80)$$

$$d \equiv \frac{q^{N_1+N_2+1} - q^{N_2} + 1}{1 - q} A, \quad (3.81)$$

$$b \equiv a_1^\dagger a_2 A, \quad (3.82)$$

$$b^\dagger \equiv a_2^\dagger a_1 A, \quad (3.83)$$

from equations (3.76 - 3.79), there is possible to state the representation of the braided algebra in $n_1 + n_2 + 1 = N$ by means of the q-bosonic oscillators. We can put q^2 instead of q and thus is a realization of braided matrices $BM_q(2)$ which can be defined as

$$u = A \begin{pmatrix} \frac{q^{2(N_2+1)}}{1 - q^2} & a_1^\dagger a_2 \\ a_2^\dagger a_1 & \frac{q^{2(N_1+N_2+1)} - q^{2N_2+1}}{1 - q^2} \end{pmatrix} \quad (3.84)$$

The multiplication of the braided group elements by central elements the braided algebra relations stay invariant, so that we determine $A' = q^{(N_1+N_2)}A$. From the equations (3.66 - 3.70) describing as

$$f_1 \equiv q^{-N_1}a_1, \quad (3.85)$$

$$f_1^\dagger \equiv q^{-N_1+1}a_1^\dagger, \quad (3.86)$$

$$f_2 \equiv q^{-(N_2+1)}a_2, \quad (3.87)$$

$$f_2^\dagger \equiv q^{-N_2}a_2^\dagger. \quad (3.88)$$

We get

$$f_1^\dagger f_2 \equiv q^{-(N_1+N_2)}a_1^\dagger a_2, \quad (3.89)$$

$$f_2^\dagger f_1 \equiv q^{-(N_1+N_2)}a_2^\dagger a_1, \quad (3.90)$$

$$f_1^\dagger f_1 \equiv \frac{1 - q^{-2N_1}}{1 - q^{-2}}, \quad (3.91)$$

$$f_2^\dagger f_2 \equiv \frac{1 - q^{2N_2}}{1 - q^2}, \quad (3.92)$$

and

$$f_1 f_1^\dagger - q^{-2} f_1^\dagger f_1 = 1, \quad (3.93)$$

$$f_2 f_2^\dagger - q^2 f_2^\dagger f_2 = 1. \quad (3.94)$$

It is essential to notice that there is the $q \rightarrow q^{-1}$ symmetry between these two independent q-bosonic oscillators. The u matrix which can be given by

$$u = A' \begin{pmatrix} \frac{q^{(N_2-N_1+2)}}{1-q^2} & f_1^\dagger f_2 \\ f_2^\dagger f_1 & \frac{q^{N_1+N_2+2} - q^{N_2-N_1} + q^{-(N_2+N_1)}}{1-q^2} \end{pmatrix} \quad (3.95)$$

constituted in terms of these oscillators is a Hermitian braided matrix $BM_q(2)$. The action of the Casimir operator defined by (3.65) on state $|j, m\rangle$ is given by

$$C|j, m\rangle = q^2 \frac{(q^{2(j+1)} + q^{-2j} - 1 - q^2)}{(1 - q^2)^2} |j, m\rangle. \quad (3.96)$$

To renormalize u we divide out the central element A' subtract a q dependent multiple of the identity matrix. Then, in the $q \rightarrow 1$ limit we obtain

$$\lim_{q \rightarrow 1} \left(\frac{u}{A'} - \frac{q^2}{1 - q^2} I \right) = \begin{pmatrix} -\frac{(N_2 - N_1)}{2} & a_1^\dagger a_2 \\ a_2^\dagger a_1 & \frac{(N_2 - N_1)}{2} \end{pmatrix} \quad (3.97)$$

where a_1^\dagger , a_1 , a_2^\dagger and a_2 are the creation and annihilation operators of a pair of independent q -bosonic oscillators and N_1 and N_2 are the corresponding number operators. By means of the commutation relations between these operators we can determine

$$J^- = a_1^\dagger a_2, \quad (3.98)$$

$$J^+ = a_2^\dagger a_1, \quad (3.99)$$

$$J^0 = \frac{1}{2} (N_2 - N_1), \quad (3.100)$$

which verify the commutation relation in the following way

$$[J^+, J^-] = 2J^0, \quad (3.101)$$

$$[J^0, J^\pm] = \pm J^\pm. \quad (3.102)$$

This is the Jordan-Schwinger construction for $su(2)$. In the $q \rightarrow 1$ limit, the Casimir operator given by (3.65) becomes

$$\lim_{q \rightarrow 1} C = J^+ J^- + (J^0)^2 - J^0 \quad (3.103)$$

and the action of C on state $|j, m\rangle$ can be found as

$$C |j, m\rangle = j(j+1) |j, m\rangle. \quad (3.104)$$

This means that imposing the $q \rightarrow q^{-1}$ symmetry on these oscillators in the finite dimensional representations of the Hermitian $BM_q(2)/\mathfrak{R}$ gives a one-to-one correspondence with the finite dimensional representations of $su(2)$. It is shown that the unitary condition $U^\dagger U = UU^\dagger = 1$ for $BM_q(2)$ is inconsistent due to the fact that the matrix elements of Hermitian braided matrix group $BM_q(2)$ do not satisfy the commutations relations. In contrast to the braided algebra which has finite dimensional representations, the quantum group $SU_q(2)$ consists of only a single q -deformed oscillator and the quantum algebra satisfied by the matrix elements of $SU_q(2)$ does not have finite dimensional representations. However, the case of the quantum determinant being equal to non zero for the braided matrices $BM_q(2)$ is familiar to the the quantum group $SU_q(2)$ with q as deformation parameter of the Lie algebra $su(2)$.

It is remarkable to notice that there are two significant differences between the Macfarlane-Biedenharn construction for the quantum group $SU_q(2)$ and the Jordan-Schwinger construction of the Hermitian braided matrices $BM_q(2)$ in terms of q -deformed bosonic oscillators as found by Arık-Yıldız. Firstly, the generators of the Macfarlane-Biedenharn construction of $SU_q(2)$ which are constructed by using two oscillators do not give a matrix form. Secondly, there is the $q \rightarrow q^{-1}$ symmetry between these two independent linear q -deformed bosonic oscillators.

4. Q-DEFORMED FERMIONIC OSCILLATOR CONSTRUCTION OF THE BRAIDED QUANTUM GROUP $BM_q(2)$

4.1. The Representations of The Braided Quantum Group $BM_q(2)$

In the previous section, we have mentioned the construction of the braided quantum group $BM_q(2)$ by using two commuting q-bosonic oscillators. Now we will express the representations of $BM_q(2)$ in terms of a pair of q-deformed fermionic oscillators.

We note that in the relations (3.6 - 3.11) parameter q only appears as q^2 . This was defined as q^2 rather than q to avoid writing $q^{1/2}$ in the q-deformed oscillators in terms of which $BM_q(2)$ was constructed. Here we want to consider this parameter q^2 to be negative for q-fermionic oscillators. So we set $q^2 = -Q$ ($Q > 0$). Then the relations (3.6 - 3.11) become

$$ba = -Qab, \tag{4.1}$$

$$ca = -Q^{-1}ac, \tag{4.2}$$

$$ad = da, \tag{4.3}$$

$$bc = cb + (1 + Q^{-1})a(d - a), \tag{4.4}$$

$$db = bd + (1 + Q^{-1})ab, \tag{4.5}$$

$$cd = dc + (1 + Q^{-1})ca. \tag{4.6}$$

The braid group action on the generators is given by

$$\psi(a \otimes a) = a \otimes a + (1 + Q)b \otimes c, \tag{4.7}$$

$$\psi(a \otimes b) = b \otimes a, \tag{4.8}$$

$$\psi(a \otimes c) = c \otimes a + (1 + Q)(d - a) \otimes c, \tag{4.9}$$

$$\psi(a \otimes d) = d \otimes a + (1 + Q^{-1})b \otimes c, \quad (4.10)$$

$$\psi(b \otimes a) = a \otimes b + (1 + Q)b \otimes (d - a), \quad (4.11)$$

$$\psi(b \otimes b) = -Qb \otimes b, \quad (4.12)$$

$$\begin{aligned} \psi(b \otimes c) &= -Q^{-1}c \otimes b + (1 - Q)(1 + Q^{-1})^2b \otimes c \\ &\quad - (1 + Q^{-1})(d - a) \otimes (d - a) \end{aligned} \quad (4.13)$$

$$\psi(b \otimes d) = d \otimes b + (1 + Q^{-1})b \otimes (d - a), \quad (4.14)$$

$$\psi(c \otimes a) = a \otimes c, \quad (4.15)$$

$$\psi(c \otimes b) = -Q^{-1}b \otimes c, \quad (4.16)$$

$$\psi(c \otimes c) = -Qc \otimes c, \quad (4.17)$$

$$\psi(c \otimes d) = d \otimes c, \quad (4.18)$$

$$\psi(d \otimes a) = a \otimes d + (1 + Q^{-1})b \otimes c, \quad (4.19)$$

$$\psi(d \otimes b) = b \otimes d, \quad (4.20)$$

$$\psi(d \otimes c) = c \otimes d + (1 + Q^{-1})(d - a) \otimes c, \quad (4.21)$$

$$\psi(d \otimes d) = d \otimes d + Q^{-1}(1 + Q^{-1})b \otimes c. \quad (4.22)$$

To show a representation of u of $BM_q(2)$ in a complex vector space we use a, b, c, d as operators. We look for the representation where a, d are diagonal. We also use b and c as the lowering and raising operators. We have

$$a |n\rangle = a_n |n\rangle, \quad (4.23)$$

$$d |n\rangle = d_n |n\rangle, \quad (4.24)$$

$$b |n\rangle = b_n |n - 1\rangle, \quad (4.25)$$

$$c |n\rangle = c_{n+1} |n + 1\rangle. \quad (4.26)$$

It is important to note that if the conjugate transpose of u matrix is denoted by u^\dagger , then the Hermitian property can be written concisely as $u = u^\dagger$. As we will show below, we could not obtain $a^\dagger = a, d^\dagger = d,$ and $b^\dagger = c$ for finite dimensional

representations. So our construction will be true for general $BM_q(2)$. It will not be applicable to Hermitian $BM_q(2)$. In other words, the algebra of Hermitian $BM_q(2)$ cannot be constructed in terms of a pair of q -fermionic oscillators, trying to impose $BM_q(2)$ to be Hermitian forces fermionic oscillators to have $q = 1$.

For obtaining the difference equations, the equations (4.1 - 4.6) can be rewritten with these relations

$$a_{n+1} = -Qa_n, \quad (4.27)$$

$$d_{n+1} = d_n + (1 + Q^{-1})a_n, \quad (4.28)$$

$$c_{n+1}b_{n+1} = c_nb_n + (1 + Q^{-1})a_n(d_n - a_n). \quad (4.29)$$

whose solutions can be given by

$$a_n = (-Q)^n a_0, \quad (4.30)$$

$$d_n = d_0 - a_0 Q^{-1} (1 - (-Q)^n), \quad (4.31)$$

$$c_n b_n = c_0 b_0 - a_0^2 Q^{-2} Q^{2n} - a_0 Q^{-1} (d_0 - Q^{-1} a_0) (-Q)^n + Q^{-1} a_0 d_0. \quad (4.32)$$

The infinite dimensional representation is

$$a_n = (-Q)^n a_0, \quad (4.33)$$

$$d_n = d_0 - a_0 Q^{-1} [1 - (-Q)^n], \quad (4.34)$$

$$c_n b_n = a_0 Q^{-1} [1 - (-Q)^n] [d_0 - (-Q)^{n-1} a_0]. \quad (4.35)$$

If we can write $c = b^\dagger$, the spectrum of $b^\dagger b$ will be given $\bar{b}_n b_n$ in a complex vector space

$$\bar{b}_n b_n = a_0^2 Q^{-1} (1 - (-Q)^n) \left(\frac{d_0}{a_0} - (-Q)^{n-1} \right). \quad (4.36)$$

For $0 < Q < 1$, the infinite dimensional representations will be determined by the positivity conditions of $\bar{b}_n b_n$ as follows

$$n = 0 \quad \frac{d_0}{a_0} > 1, \quad (4.37)$$

$$n = 1 \quad \frac{d_0}{a_0} > -Q, \quad (4.38)$$

$$n = 2 \quad \frac{d_0}{a_0} > Q^2. \quad (4.39)$$

In general, to take the infinite dimension of the representations $\bar{b}_n b_n$ should be positive definite and this requires the condition $\frac{d_0}{a_0} > 1$ for $0 < Q < 1$

$$a_n = (-Q)^n a_0, \quad (4.40)$$

$$d_n = d_0 - a_0 Q^{-1} [1 - (-Q)^n], \quad (4.41)$$

$$\bar{b}_n b_n = a_0^2 Q^{-1} (1 - (-Q)^n) \left(\frac{d_0}{a_0} - (-Q)^{n-1} \right), \quad (4.42)$$

$$n = 0, 1, 2, \dots \quad (4.43)$$

For $d_0 = (-Q)^{N-1} a_0$ we find the finite (N)-dimensional representation with the states $|n\rangle = |0\rangle, |1\rangle, \dots, |N-1\rangle$. The eigenvalues of diagonal operators are found as

$$a_n = (-Q)^n a_0, \quad (4.44)$$

$$d_n = a_0 (-Q)^{-1} [(-Q)^N - (-Q)^n + 1], \quad (4.45)$$

$$c_n b_n = a_0^2 Q^{-2} [Q^{2n} - (-Q)^n] [(-Q)^{N-n} - 1]. \quad (4.46)$$

We observe the spectrum of $c_n b_n$

$$n = 0 \quad c_n b_n = 0, \quad (4.47)$$

$$n = 1 \quad c_n b_n = a_0^2 Q^{-2} [Q^{2n} - (-Q)^n] [(-Q)^{N-1} - 1], \quad (4.48)$$

$$n = 2 \quad c_n b_n = a_0^2 Q^{-2} [Q^{2n} - (-Q)^n] [(-Q)^{N-2} - 1], \quad (4.49)$$

and for $Q > 1$, positivity of $c_n b_n$ can be satisfied only when n and N are odd. So that we cannot choose $c = b^\dagger$. From the equations (4.36 - 4.38), the elements (a, b, c and d) contain a_0 as a multiplicative factor. We identify

$$a_0 = \frac{-Q}{1+Q}A. \quad (4.50)$$

Then m and j can be determined as

$$m \equiv n - \frac{N-1}{2} \quad j \equiv \frac{N-1}{2}. \quad (4.51)$$

We can describe $n \rightarrow j+m$ and $N \rightarrow 2j+1$ plug these into the equations (4.36 - 4.38) as

$$a_{j+m} = \frac{(-Q)^{j+m+1}}{1+Q}A, \quad (4.52)$$

$$d_{j+m} = \frac{(-Q)^{2j+1} - (-Q)^{j+m} + 1}{1+Q}A, \quad (4.53)$$

$$c_{j+m} = b_{j+m} = \sqrt{\left(\frac{(-Q)^{j+m} - Q^{2(j+m)}}{1+Q}\right) \left(\frac{1 - (-Q)^{j-m+1}}{1+Q}\right)}A. \quad (4.54)$$

4.2. Q-Fermionic Oscillator Representation

We take two commuting q-fermionic oscillators which are generated by $c_1, \tilde{c}_1, N_1, c_2, \tilde{c}_2$ and N_2 verifying

$$c_1 \tilde{c}_1 + Q \tilde{c}_1 c_1 = 1, \quad (4.55)$$

$$c_2 \tilde{c}_2 - Q^2 \tilde{c}_2 c_2 = (-Q)^{N_2+1}, \quad (4.56)$$

$$[c_1, c_2] = [c_1, \tilde{c}_2] = 0, \quad (4.57)$$

$$N_1 |n_1\rangle = n_1 |n_1\rangle, \quad (4.58)$$

$$N_2 |n_2\rangle = n_2 |n_2\rangle. \quad (4.59)$$

Here \tilde{c}_1 can be chosen to be Hermitian conjugate of c_1 but \tilde{c}_2 cannot be chosen to be Hermitian conjugate of c_2 . This is related to the fact that the $BM_q(2)$ matrix we are considering cannot be made Hermitian for finite dimensional representation which these oscillators generate.

Then we demonstrate the action of the operators on a complex vector space as follows

$$c_1 |n_1\rangle = \sqrt{\left(\frac{1 - (-Q)^{n_1}}{1 + Q}\right)} |n_1 - 1\rangle, \quad (4.60)$$

$$\tilde{c}_1 |n_1\rangle = \sqrt{\left(\frac{1 - (-Q)^{n_1+1}}{1 + Q}\right)} |n_1 + 1\rangle, \quad (4.61)$$

$$c_2 |n_2\rangle = \sqrt{\left(\frac{(-Q)^{n_2} - Q^{2n_2}}{1 + Q}\right)} |n_2 - 1\rangle, \quad (4.62)$$

$$\tilde{c}_2 |n_2\rangle = \sqrt{\left(\frac{(-Q)^{n_2+1} - Q^{2(n_2+1)}}{1 + Q}\right)} |n_2 + 1\rangle. \quad (4.63)$$

Describing j and m as follows

$$m \equiv \frac{n_2 - n_1}{2} \quad \text{and} \quad j \equiv \frac{n_2 + n_1}{2}. \quad (4.64)$$

It can be obtained as

$$\begin{aligned} \tilde{c}_1 c_2 |j - m\rangle |j + m\rangle &= \sqrt{\left(\frac{1 - (-Q)^{j-m+1}}{1 + Q}\right) \left(\frac{(-Q)^{j+m} - Q^{2(j+m)}}{1 + Q}\right)} |j, m - 1\rangle, \\ \tilde{c}_2 c_1 |j - m\rangle |j + m\rangle &= \sqrt{\left(\frac{1 - (-Q)^{j-m}}{1 + Q}\right) \left(\frac{(-Q)^{j+m+1} - Q^{2(j+m+1)}}{1 + Q}\right)} |j, m + 1\rangle, \end{aligned}$$

$$\begin{aligned} \left(\frac{N_1 + N_2}{2}\right) |j - m\rangle |j + m\rangle &= j |j, m\rangle, \\ \left(\frac{N_1 - N_2}{2}\right) |j - m\rangle |j + m\rangle &= m |j, m\rangle. \end{aligned}$$

By identifying

$$a \equiv \frac{(-Q)^{N_2+1}}{1+Q}A, \quad (4.65)$$

$$d \equiv \frac{(-Q)^{N_1+N_2+1} - (-Q)^{N_2} + 1}{1+Q}A, \quad (4.66)$$

$$b \equiv \tilde{c}_1 c_2 A, \quad (4.67)$$

$$c \equiv \tilde{c}_2 c_1 A, \quad (4.68)$$

from equations (4.55 - 4.59), there is possible to state the braided algebra in $n_1+n_2+1 = N$ by means of the q-fermionic oscillators. We can write Q^2 instead of Q and thus is a realization of braided matrices $BM_q(2)$ which can be defined as

$$u = A \begin{pmatrix} \frac{(-Q)^{2(N_2+1)}}{1+Q^2}A & \tilde{c}_1 c_2 \\ \tilde{c}_2 c_1 & \frac{(-Q)^{2(N_1+N_2+1)} - (-Q)^{2N_2+1}}{1+Q^2}A \end{pmatrix} \quad (4.69)$$

We define $A' = (-Q)^{(N_1+N_2)}A$. From equations (4.55 - 4.59) describing as

$$g_1 \equiv (-Q)^{-N_1}c_1, \quad (4.70)$$

$$\tilde{g}_1 \equiv (-Q)^{-N_1+1}\tilde{c}_1, \quad (4.71)$$

$$g_2 \equiv (-Q)^{-(N_2+1)}c_2, \quad (4.72)$$

$$\tilde{g}_2 \equiv (-Q)^{-N_2}\tilde{c}_2. \quad (4.73)$$

We get

$$\tilde{g}_1 g_2 \equiv (-Q)^{-(N_1+N_2)}c_1^\dagger c_2, \quad (4.74)$$

$$\tilde{g}_2 g_1 \equiv (-Q)^{-(N_1+N_2)}c_2^\dagger c_1, \quad (4.75)$$

$$\tilde{g}_1 g_1 \equiv \frac{1 - (-Q^2)^{-N_1}}{1 + Q^{-2}}, \quad (4.76)$$

$$\tilde{g}_2 g_2 \equiv \frac{1 - (-Q^2)^{N_2}}{1 + Q^2}, \quad (4.77)$$

and

$$g_1 \tilde{g}_1 + Q^{-2} \tilde{g}_1 g_1 = 1, \quad (4.78)$$

$$g_2 \tilde{g}_2 + Q^2 \tilde{g}_2 g_2 = 1. \quad (4.79)$$

It is important to note that there is the $Q \rightarrow Q^{-1}$ symmetry between these two independent q-fermionic oscillators. The u matrix which can be defined as

$$u = A' \begin{pmatrix} \frac{(-Q)^{N_2 - N_1 + 2}}{1 + Q^2} & \tilde{g}_1 g_2 \\ \tilde{g}_2 g_1 & \frac{(-Q)^{N_1 + N_2 + 2} - (-Q)^{N_2 - N_1} + (-Q)^{-(N_2 + N_1)}}{1 + Q^2} \end{pmatrix} \quad (4.80)$$

constructed in terms of two q-fermionic oscillators is a braided matrix $BM_q(2)$.

4.3. The $Q \rightarrow 1$ limit

To renormalize u we divide out the central element A' subtract a Q dependent multiple of the identity matrix. Then, in the $Q \rightarrow 1$ limit we obtain

$$\lim_{Q \rightarrow 1} \left(\frac{u}{A'} - \frac{Q^2}{1 + Q^2} I \right) = \begin{pmatrix} -\frac{(N_1 - N_2)}{2} & c_1^\dagger c_2 \\ c_2^\dagger c_1 & \frac{(N_1 - N_2)}{2} \end{pmatrix} \quad (4.81)$$

where c_1^\dagger , c_1 , c_2^\dagger and c_2 are the creation and annihilation operators of a pair of independent ordinary fermionic oscillators and N_1 and N_2 are the corresponding number operators. By means of the commutation relations between these operators we can determine

$$J^- = c_1^\dagger c_2, \quad (4.82)$$

$$J^+ = c_2^\dagger c_1, \quad (4.83)$$

$$J^0 = \frac{1}{2} (N_1 - N_2), \quad (4.84)$$

which verify the commutation relation in the following way

$$[J^+, J^-] = 2j^0, \quad (4.85)$$

$$[J^0, J^\pm] = \pm J^\pm. \quad (4.86)$$

It is important to note that in the $Q \rightarrow 1$ limit, the allowed eigenvalues of N_1 and N_2 take only 0 and 1. In other words, the Fock space reduces one fermion and vacuum state, thus obeying the Pauli Exclusion principle in this limit. It is also observed that the q-deformed fermions reduce the ordinary fermions in the $Q \rightarrow 1$ limit.

5. CONCLUSION

Firstly, we have reviewed the standard bosonic and fermionic Jordan-Schwinger construction of $SU(2)$ in this thesis. It is seen that one can obtain the all unitary representations of $SU(2)$ by means of the Jordan-Schwinger construction for the $SU(2)$ algebra from a pair of bosonic oscillators due to the fact that j can take arbitrary positive integer or half integer values. In contrast to the bosonic case, in the fermionic case one can get the full set of unitary representations of the $SU(2)$ algebra only by combining the spin 0 and spin 1/2 representations. We have focused that as obtained by Macfarlane and Biedernharn, the Jordan-Schwinger construction can be generalized to realize the quantum group $SU_q(2)$, with q as deformation parameter, by using two independent q -bosonic oscillators. We have also explained a realization of the $SU_q(2)$ based on the Jordan-Schwinger construction by using a pair of commuting q -fermionic oscillators. It is demonstrated that q -deformed fermionic oscillator with $c^2 = 0$ and $(c^\dagger)^2 = 0$ does not admit a q -deformation and obey the Pauli exclusion principle, with the Fock space restricted only the vacuum and one fermion state [15, 21, 22]. However, we have proposed a non-trivial q -deformation of the fermion oscillator algebra which has relaxed the Pauli exclusion principle as investigated by Parthasarathy [23].

In the next chapter of this thesis, we have expressed the braided algebra of Hermitian $BM_q(2)$ in terms of a pair of independent q -bosonic oscillators in the Fock space as achieved by Arık and Yıldız. It is seen that the u matrix which can be defined by constituted in terms of these oscillators is a Hermitian braided matrix $BM_q(2)$. We have observed that there is the $q \rightarrow q^{-1}$ symmetry between two independent linear q -deformed bosonic oscillators. In the fourth section, using a similar approach we have found the representations of the braided matrix group $BM_q(2)$ in terms of a pair of q -deformed fermionic oscillators which cannot satisfy the Hermiticity conditions. So we have shown a representation of u of $BM_q(2)$ in a complex vector space. We have obtained that \tilde{c}_1 can be taken to be Hermitian conjugate of c_1 but \tilde{c}_2 cannot be taken to be Hermitian conjugate of c_2 . This depends on the fact that these oscillators generate the $BM_q(2)$ matrix which cannot satisfy Hermitian matrix for finite dimensional repre-

sentations . Finally, we have seen that the Pauli exclusion principle is valid only in the $Q \rightarrow 1$ limit owing to the fact that the Fock space has only the allowed eigenvalues of N_1 and N_2 which take 0 and 1. We would like to note that in this limit the Hermiticity conditions are satisfied.

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