CONNECTIONS BETWEEN ADJOINT FUNCTORS AND LIMITS

by

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ABSTRACT

CONNECTIONS BETWEEN ADJOINT FUNCTORS AND LIMITS

The concept of an adjoint functor is one of the most important concepts in category theory. Their close relation with universal arrows and limits makes them indispensable. In this thesis connections between adjoint functors and limits are explored. Firstly the general theory of adjoint functors is presented. In this respect characterization of adjunctions by universal arrows and also by units and counits are given. Secondly the notion of a limit and construction of limits by products and equalizers are presented. As the final step, general and special adjoint functor theorems are proven. These important theorems characterize the existence of a left adjoint to a functor in terms of limits and illuminate the adjoint functor-limit relation most. Also specific examples of adjunctions and applications of adjoint functor theorems in different fields of mathematics are presented.

ÖZET

EK FUNKTORLAR VE LİMİTLER ARASINDAKİ İLİŞKİLER

Ek funktor kavramı kategori teorisinin en önemli kavramlarından biridir. Özellikle üniversal morfizmler ve limitlerle olan yakın ilişkileri onları vazgeçilmez kılar. Bu tezde ek funktorlar ve limitlerin aralarındaki ilişkiler araştırılmıştır. İlk olarak ek funktorların genel teorisi sunulmuştur. Bu bağlamda ek funktorların üniversal morfizmler tarafından ve ayrıca birim ve kobirimler tarafından karakterize edilmesi verilmiştir. Daha sonra limit kavramı ve limitlerin çarpım ve eşitleyiciler tarafından inşa edilmesi sunulmuştur. Son olarak genel ve özel ek funktor teoremleri ispat edilmiştir. Bu önemli teoremler bir funktorun sol ek funktorunun olmasını limitler cinsinden karakterize ederek limitler ve ek funktorlar arasındaki ilişkiyi en iyi şekilde aydınlatırlar. Ayrıca ek funktorlara özel örnekler verilmiş ve ek funktor teoremlerinin matematiğin değişik alanlarındaki uygulamaları sunulmuştur.

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LIST OF SYMBOLS

$\mathbb{A}, \mathbb{B}, \mathbb{C} \ldots$	Categories
$A, B, C \ldots$	Objects in the category
$f, g, h \ldots$	Arrows in the category
$F, G, H \ldots$	Functors
$\varepsilon, \eta, \tau \ldots$	Natural transformations

1. INTRODUCTION

In contrast to the classical set theory based mathematics, category theory provides a different perspective to view and shape the world of mathematics. As nothing is meaningful in isolation, in this new setting every mathematical object should be considered in the web of relations with other objects. Indeed what makes a thing that thing is its relation with others. So in category theory, absolute existence and objects pass their dominant position to relational existence and morphisms. Distinguished objects are determined by the way of universal mapping properties. Universals of category theory not only have some property but also give machinery to determine others which have that property (namely every object having that property should factor uniquely through the universal object). In short category theory involves determination through morphisms and particularly determination through universals via the universal mapping properties. In the setting where universals gain the dominance, adjoints have a special place. This is because most of the universals come as a part of adjunctions.

The following remark by Awodey emphasizes the importance of adjoint functors best.

> The notion of adjoint functor applies everything we've learned up to now to unify and subsume all different universal mapping properties that we have encountered, from free groups to limits and exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics. [1]

Adjoint functors is perhaps the most successful concept of mathematics that is captured by category theory. They arise everywhere in mathematics and adjoint functor theorems have a wide range applications. Just to mention two of them, the proof of Fermat's Last Theorem by Wiles make use of the adjoints, and Grothendieck's formulation of Serre's duality turns on the existence of a right adjoint to a particular functor.

The aim of this thesis is to present the theory of adjoint functors and their relation to limits. As an end goal we will prove the general and special adjoint functor theorems in terms of limits.

2. PRELIMINARIES

This is an introductory chapter to give the reader a flavor of category theory and category theoretical methods. Also some of the material in the chapter will be useful for our later work. We first start from the axioms.

2.1. Basics

A *category* $\mathbb{A}$ consists of *objects* $A, B, C, \ldots$; and *arrows* $f, g, h, \ldots$ going from an object to an object. We denote an arrow f from A to B as $A \xrightarrow{f} B$ or $f : A \to B$. In this context we call A as the *domain* of f and B as the *codomain*.

Besides these elements in the category, there is a partial operation on arrows, called *composition*. Two arrows f and g are said to be *composable* if codomain$f$=domain$g$. And as a result of the composition they have a composite, an arrow denoted by $g \circ f$. Composition operation is associative when defined. Also for any object A in $\mathbb{A}$, there exists an arrow $A \xrightarrow{1_A} A$ such that for any arrow $B \xrightarrow{g} A$ and $A \xrightarrow{h} C$ we have $1_A \circ g = g$ and $h \circ 1_A = h$. By simple computation it can be seen that this arrow, 1_A, is unique and it is called the identity arrow of A.

For objects A and B, we will denote the collection of all arrows from A to B by $hom(A, B)$ and call it as *hom-set*. We call a category as *small* if the collection of its objects and the collection of its arrows are sets. In general we will call a collection as small if it constitutes a set.

Examples 2.1.1. (1) *Category of sets,* **Set**, *is the primary example of a category. Its objects are sets and its arrows are functions.*

(2) *A preordered set P can be seen as a category. Its objects are elements of P. And for $A, B \in P$, there exists a unique arrow $A \longrightarrow B$ if and only if $A \leq B$. Then reflexivity provides identity arrows and transitivity provides associativity of the composition.*

(3) *Any type of structures and structure preserving maps between them make a cat-*

egory. As an example monoids with monoid homomorphisms, **Mon**, *constitute the category of monoids.*

(4) *A set can be viewed as a category. Its objects are elements of the set and its arrows are the identity arrows. A category which only has objects and identity arrows is called a* discrete category.

In category theory diagrams are extensively used. A diagram is said to be commutative if after fixing any two objects in the diagram, say A and B, taking composition of arrows along any path starting from A and ending at B gives us the same result. If a diagram is commutative we put a circle in the middle of the diagram to indicate that. In the simplest case if for $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, $g \circ f = h$, then we can draw the following commutative diagram.

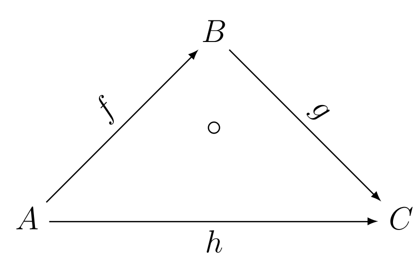


Figure 2.1.

According to the category theoretical philosophy the important thing is not the object itself but its relation with other objects. So they are the arrows which are important. Indeed there is an equivalent formulation of category theory without any mention to objects. In this respect objects are just the things which help us in understanding and visualizing the situation.

Having mentioned the importance of arrows, we now define three types of special arrows.

Definition 2.1.2. *An arrow* $A \xrightarrow{f} B$ *is called an* isomorphism, *or* iso *for short, if there exists an arrow* $B \xrightarrow{g} A$ *such that* $f \circ g = 1_B$ *and* $g \circ f = 1_A$.

In a case as above f and g are said to be *inverses* of each other. Since inverse of an arrow is unique (found by a simple computation) we denote g by f^{-1} and call it *the inverse* of f. If $A \xrightarrow{f} B$ is an iso, then A and B are said to be *isomorphic*. This is denoted as $A \simeq B$. Isomorphic objects have the same categorical properties so they are usually treated as same.

Definition 2.1.3. *An arrow* $A \xrightarrow{f} B$ *is called* monic *if for any two arrows* $C \underset{k}{\overset{h}{\rightrightarrows}} A$, $f \circ h = f \circ k$ *implies* $h = k$.

Definition 2.1.4. *An arrow* $B \xrightarrow{f} A$ *is called* epic *if for any two arrows* $A \underset{k}{\overset{h}{\rightrightarrows}} C$, $h \circ f = k \circ f$ *implies* $h = k$.

By the above two definitions it can be seen that monics are cancellable from the left and epics are cancellable from the right. It is easy to observe that an iso is both a monic and an epic. For the converse we have the following proposition.

Proposition 2.1.5. *A monic* $A \xrightarrow{f} B$ *which has a right inverse is iso.*

Proof. Let f be a monic and have a right inverse $B \xrightarrow{g} A$. Then $f \circ g = 1_B$. If we compose both sides of the equation by f, we get

$$\begin{aligned}(f \circ g) \circ f &= 1_B \circ f \\ f \circ (g \circ f) &= f \\ f \circ (g \circ f) &= f \circ 1_A\end{aligned}$$

Then since f is a monic, $g \circ f = 1_A$. Hence f is an iso with its inverse g. □

Proposition 2.1.6. *Composition of two monics is also a monic.*

Proof. Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be monics. Suppose there exists arrows $T \underset{k}{\overset{h}{\rightrightarrows}} A$ such that $(g \circ f) \circ h = (g \circ f) \circ k$. Then we have $g \circ (f \circ h) = g \circ (f \circ k)$. Since g is monic $f \circ h = f \circ k$. And since f is monic, $h = k$. Therefore $g \circ f$ is a monic. □

Proposition 2.1.7. *For two arrows* $A \xrightarrow{f} B$ *and* $B \xrightarrow{g} C$*, if* $g \circ f$ *is monic then so is* f*.*

Proof. Suppose there exists arrows $T \underset{k}{\overset{h}{\rightrightarrows}} A$ such that $f \circ h = f \circ k$. Then $g \circ f \circ h = g \circ f \circ k$. Since $g \circ f$ is monic, $h = k$. So f is monic. □

We now consider some special objects of a category.

Definition 2.1.8. *Let* $\mathbb{A}$ *be a category. An object* I *of* $\mathbb{A}$ *is called an* initial object *of* $\mathbb{A}$ *if for any object* A *in* $\mathbb{A}$ *there exists a unique arrow from* I *to* A*. Dually an object* T *is called* terminal object *if for every* A *in* $\mathbb{A}$ *there exists a unique arrow from* A *to* T*.*

Proposition 2.1.9. *Initial objects are unique up to isomorphism. In other words if* I *is an initial object of category* $\mathbb{A}$ *and there is an iso* $I' \xrightarrow{f} I$ *then* I' *is also an initial object. Furthermore if* I *and* I' *are both initial objects then they are isomorphic.*

Proof. Let I be an initial object and $I' \xrightarrow{f} I$ be an iso. Then for any A in $\mathbb{A}$ there exists a unique arrow $I \xrightarrow{k} A$. Composing with f, $k \circ f$ is an arrow from I' to A. For the uniqueness part, let h be another arrow from I' to A. Then $h \circ f^{-1}$ is an arrow from I to A. But since I is initial there exists a unique arrow from I to A. Hence $h \circ f^{-1} = k$, $h = k \circ f$. So there is a unique arrow from I' to A and I' is an initial object.
Now suppose that I and I' are both initial objects. Then by definition there exists unique arrows $I \xrightarrow{m} I'$, $I' \xrightarrow{n} I$. $n \circ m$ is an arrow from I to I. But we know that 1_I is also such an arrow. Since I is initial, $n \circ m = 1_I$. Similarly $m \circ n = 1_{I'}$ and m and n are iso. Therefore I and I' are isomorphic. □

Similarly terminal objects are unique up to isomorphism.

Definition 2.1.10. *Let* A, B *be two objects in a category* $\mathbb{A}$*. A* product *of* A *and* B *consists of an object* P *and arrows* $P \xrightarrow{p_1} A$*,* $P \xrightarrow{p_2} B$ *which satisfy the following. For* *any object T and arrows $T \xrightarrow{f} A$, $T \xrightarrow{g} B$, there exists a unique arrow $T \xrightarrow{k} P$ such that $p_1 \circ k = f$ and $p_2 \circ k = g$.*

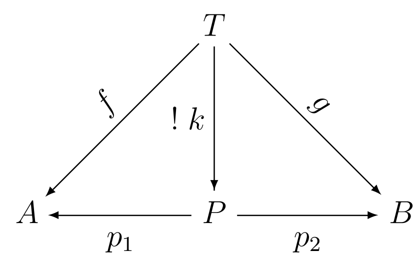


Figure 2.2.

We denote P by $A \times B$ and call p_1 and p_2 as projection arrows.

Product are unique up to isomorphism. It is useful to observe that for any two arrows i, j, if i and j have the same composites with the projection arrows, then $i = j$. For $(A \times B, p_1, p_2)$ and $(C \times D, q_1, q_2)$ if we have two arrows $A \xrightarrow{f} C$, $B \xrightarrow{g} D$, we denote the unique arrow $A \times B \longrightarrow C \times D$ induced by $f \circ p_1$ and $g \circ p_2$ by $f \times g$.

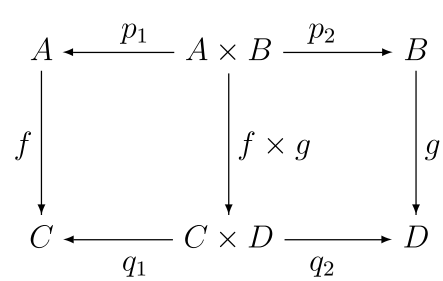


Figure 2.3.

In the case where $g = 1_B$ we write $f \times 1_B$ or $f \times B$.

In the obvious way we can extend the definition of a product for a collection $\{A_i\}_{i \in I}$ of objects in $\mathbb{A}$, their product denoted by $\left(\prod_{i \in I} A_i, \{p_i\}_{i \in I}\right)$. Then for any object

T and a collection of arrows $\{T \xrightarrow{f_i} A_i\}$, there exists a unique arrow $T \xrightarrow{k} \prod_{i \in I} A_i$ such that $p_i \circ k = f_i$ for every $i \in I$.

Definition 2.1.11. *Let $A \overset{f}{\underset{g}{\rightrightarrows}} B$ be a parallel pair of arrows. An* equalizer *of f and g is an arrow $E \xrightarrow{e} A$ such that $f \circ e = g \circ e$ and for any other arrow $T \xrightarrow{h} A$ such that $f \circ h = g \circ h$, there exists a unique arrow $T \xrightarrow{k} E$ for which $e \circ k = h$.*

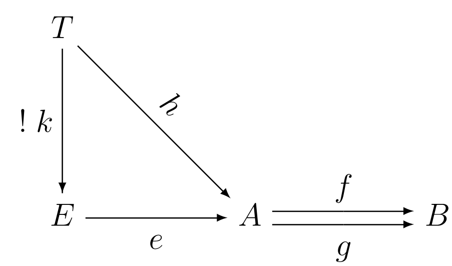


Figure 2.4.

If for an arrow e we have $f \circ e = g \circ e$, we say that e equalizes f and g. So an equalizer e of f and g is an arrow which equalizes f and g. Furthermore any other arrow which equalizes them should factor through e uniquely. As it is the case with the products, equalizers are unique up to isomorphism.

Proposition 2.1.12. *An equalizer is a monic.*

Proof. Let $E \xrightarrow{e} A$ be an equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$. Suppose that there exists arrows $C \overset{i}{\underset{j}{\rightrightarrows}} E$ such that $e \circ i = e \circ j$. Then from $f \circ e = g \circ e$ we obtain $f \circ e \circ i = g \circ e \circ i$ and $f \circ e \circ j = g \circ e \circ j$. So $e \circ i$ equalizes f and g. Since e is an equalizer, there exists a unique arrow $C \xrightarrow{k} E$ such that $e \circ k = e \circ i$. But we already know that i satisfies the equation, so the unique arrow k is equal to i. Similarly $e \circ j$ factors through e by a unique arrow l which is indeed equal to j. But since $e \circ i = e \circ j$, $k = l$. Hence we obtain $i = j$. Therefore e is monic. $\square$

Definition 2.1.13. *Let $B \overset{f}{\underset{g}{\rightrightarrows}} A$ be a parallel pair of arrows. A* coequalizer *of f and g is an arrow $A \xrightarrow{e} E$ such that $e \circ f = e \circ g$ and for any other arrow $A \xrightarrow{h} T$ such that $h \circ f = h \circ g$, there exists a unique arrow $E \xrightarrow{k} T$ for which $k \circ e = h$.*

It is easy to see the similarity of definitions of equalizer and coequalizer. If we reverse the direction of all arrows in one of them we obtain the other. Because of this, these definitions are said to be *duals* of each other. Similarly if we reverse all the arrows in a proof about an arbitrary category, we again obtain a proof about that category. This is because dualizing the set of the axioms gives us the original set of axioms. From the dual axioms (which are axioms themselves) and dual assumptions, we get the proof of the dual statement. In this respect the result every equalizer is monic gives us at the same time that every coequalizer is epic.

Now let A be an object of $\mathbb{A}$, on the family of all monics with codomain A we define a relation $\leq$ as follows: $h \leq k$ if and only if h factors through k, i.e. there exists an arrow u such that $h = k \circ u$.

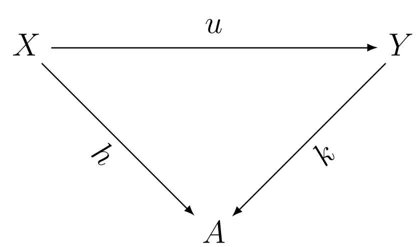


Figure 2.5.

Observe that if $h \leq k$, u is also monic by Proposition 2.1.7 and h factors through k uniquely since k is monic.

If $h \leq k$ and $k \leq h$, we write $h \equiv k$. $\equiv$ is an equivalence relation on the family of all monics with the fixed codomain A. Equivalence classes with respect to $\equiv$ are called subobjects of A. But we will often identify a monic h with its equivalence class $[h]$ and say that h is a subobject of A.

Definition 2.1.14. *Let* $A \xrightarrow{f} B$ *and* $C \xrightarrow{g} B$ *be two arrows with the same codomain. A* pullback *of* f *and* g *consists of an object* P *and arrows* $P \xrightarrow{p_1} A$, $P \xrightarrow{p_2} B$ *for which the following conditions are satisfied.*

(1) $f \circ p_1 = g \circ p_2$

(2) *If for any object T and arrows $T \xrightarrow{h} A$, $T \xrightarrow{k} C$, $f \circ h = g \circ k$, then there exists a unique arrow $T \xrightarrow{w} P$ such that $p_1 \circ w = h$ and $p_2 \circ w = k$.*

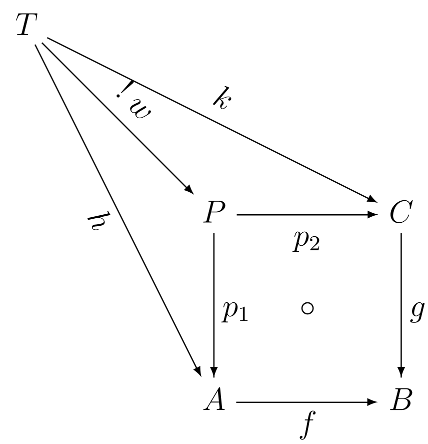


Figure 2.6.

As it is the case with the products, arrows p_1 and p_2 are called projection arrows. And for any two arrows u and w if they have the same composites with projection arrows then they are equal. As usual pullbacks are unique up to isomorphism.

Proposition 2.1.15. *Let P, p_1, p_2 be a pullback of $A \xrightarrow{f} B$ and $C \xrightarrow{g} B$ where f is a monic. Then p_2 is a monic.*

Proof. Let there be two arrows $T \underset{w}{\overset{u}{\rightrightarrows}} P$ such that $p_2 \circ u = p_2 \circ w$. Then $g \circ p_2 \circ u = g \circ p_2 \circ w$. Since P, p_1, p_2 is a pullback of f and g, $g \circ p_2 = f \circ p_1$. So $f \circ p_1 \circ u = f \circ p_1 \circ w$. But f is a monic, hence $p_1 \circ u = p_1 \circ w$. We see that u and w have the same composites with projections p_1 and p_2. So $u = w$ and p_2 is a monic. □

By a similar reasoning whenever g is a monic, so is p_1.

If $A \xrightarrow{f} B$ and $C \xrightarrow{g} B$ are both monics, then their pullback is called intersection of f and g, denoted by $A \cap C$. In such a case, by the proposition above, both projection arrows are also monic.

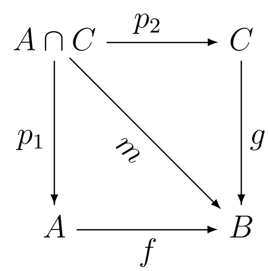


Figure 2.7.

We can see that, there is an arrow $A \cap C \xrightarrow{m} B$ such that $m = f \circ p_1 = g \circ p_2$. m is also a monic and $m \leq f$, $m \leq g$. Since m factors through f and g uniquely, it is also possible to give the pullback of f and g by $A \cap C$ and the monic $A \cap C \xrightarrow{m} B$.

Proposition 2.1.16. *$A \xrightarrow{f} B$ is monic if and only if pullback of f with itself is $A, 1_A, 1_A$.*

Proof. Let $A \xrightarrow{f} B$ be a monic. Clearly $f \circ 1_A = f \circ 1_A$ and the first condition of being a pullback is satisfied. Now let $T \xrightarrow{h} A$ and $T \xrightarrow{k} A$ be two arrows such that $f \circ h = f \circ k$. Then since f is monic, $h = k$. And this is the unique arrow w from T to A which satisfies $1_A \circ w = h$ and $1_A \circ w = k$. So $A, 1_A, 1_A$ is a pullback of f with itself. Conversely let $A, 1_A, 1_A$ be a pullback of f with itself. And let there exist two arrows $T \underset{k}{\overset{h}{\rightrightarrows}} A$ such that $f \circ h = f \circ k$. Then since $A, 1_A, 1_A$ is a pullback there exists a unique arrow $T \xrightarrow{w} A$ for which $1_A \circ w = h$ and $1_A \circ w = k$. So $h = w = k$. Hence f is monic. $\square$

When we take the pullback of an arrow $A \xrightarrow{f} B$ with itself, the pair of projection arrows is called the *kernel pair* of f. Rephrasing the above proposition, f is a monic if and only if it has the kernel pair $(1_A, 1_A)$.

Definition 2.1.17. *Let A, B be two objects in $\mathbb{A}$. An* exponential *of B by A consists of an object B^A and an arrow $B^A \times A \xrightarrow{ev_{A,B}} B$ which satisfies the following condition.*

For any T in $\mathbb{A}$ and an arrow $T \times A \xrightarrow{h} B$, there exists a unique arrow $T \xrightarrow{\overline{h}} B^A$ such that $ev_{A,B} \circ (\overline{h} \times A) = h$.

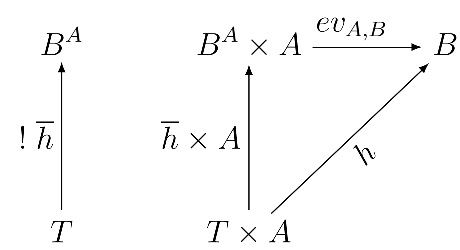


Figure 2.8.

Exponentials are unique up to isomorphism. A category which has finite products and exponentials for any pair of objects is called *Cartesian closed*.

$T \xrightarrow{\overline{h}} B^A$ is called the transpose of $T \times A \xrightarrow{h} B$. There is a one to one correspondence between arrows from T to B^A and arrows from $T \times A$ to B in terms of taking transposes. By definition an arrow $T \times A \xrightarrow{h} B$ uniquely determines its transpose, i.e. for h there is a unique $T \xrightarrow{\overline{h}} B^A$ such that $ev_{A,B} \circ (\overline{h} \times A) = h$. Conversely for $T \xrightarrow{g} B^A$, $ev_{A,B} \circ (g \times A)$ is the unique arrow from $T \times A$ to B whose transpose is g. In this respect we consider taking transposes in both directions and for $T \xrightarrow{g} B^A$ call $T \times A \xrightarrow{ev_{A,B} \circ (g \times A)} B$ as the transpose of g, denoted by $\overline{g}$. So transpose of a transpose gives us the original arrow.

For $C \xrightarrow{f} B$, we define the arrow f^A as the transpose of $f \circ ev_{A,C}$.

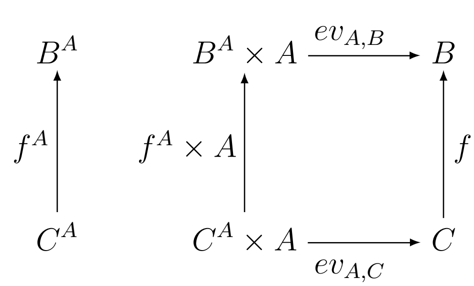


Figure 2.9.

From this definition it is not hard to see that $(1_B)^A = 1_{B^A}$ and $(g \circ f)^A = g^A \circ f^A$. So exponentiation preserves identity arrows and composition.

2.2. Functors and Constructions on Categories

Definition 2.2.1. *Let $\mathbb{A}, \mathbb{B}$ be two categories. A functor $F : \mathbb{A} \to \mathbb{B}$ is a map which sends every object of $\mathbb{A}$ to an object of $\mathbb{B}$ and every arrow of $\mathbb{A}$ to an arrow of $\mathbb{B}$ such that,*

(1) *If $A \xrightarrow{f} A'$ in $\mathbb{A}$, then $FA \xrightarrow{Ff} FA'$ in $\mathbb{B}$;*
(2) *For any A in $\mathbb{A}$, $F(1_A) = 1_{FA}$;*
(3) *For any f, g in $\mathbb{A}$, $F(g \circ f) = Fg \circ Ff$.*

We define the composition of two functors $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{C}$ by $(G \circ F)(A) = G(F(A))$ and $(G \circ F)(f) = G(F(f))$. From this definition we see that composition is associative. Also for any category $\mathbb{A}$, we have an identity functor $1_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ which acts as identity map on objects and arrows. Hence **Cat**, which takes all small categories as objects and functors between them as arrows, is a category. It is the category of categories.

Examples 2.2.2. (1) *Let $\mathbb{A}$ have all finite products and A be an object in it. Then $\_ \times A : \mathbb{A} \to \mathbb{A}$ defined as $\_ \times A(B) := B \times A$ on objects and $\_ \times A(f) := f \times A$ on arrows gives us a functor.*
(2) *For $\mathbb{A}$, cartesian closed, and A in $\mathbb{A}$, $\_^A$ defined as $\_^A(B) := B^A$ on objects and $\_^A(f) := f^A$ on arrows is a functor from $\mathbb{A}$ to $\mathbb{A}$.*
(3) *Given a specific category, a functor from this category which omits some of the structure on the objects is called a forgetful functor. As an example consider the forgetful functor U : **Mon** $\to$ **Set**. Any M in **Mon** is mapped under U to its underlying set in **Set**.*

For every category $\mathbb{A}$, we can construct another category, called the opposite category of $\mathbb{A}$ and denoted by $\mathbb{A}^{op}$, by reversing all the arrows in $\mathbb{A}$.

Definition 2.2.3. *Let* $\mathbb{A}$ *be a category. The* opposite category *of* $\mathbb{A}$, $\mathbb{A}^{op}$ *is defined as follows:*

(1) *Objects of* $\mathbb{A}^{op}$ *are objects of* $\mathbb{A}$;

(2) *If* $A \xrightarrow{h} B$ *in* $\mathbb{A}$, *then* $B \xrightarrow{h} A$ *in* $\mathbb{A}^{op}$;

(3) *Whenever* $h \circ k = g$ *in* $\mathbb{A}$, $k \circ h = g$ *in* $\mathbb{A}^{op}$.

Looking at the definition we see that $\mathbb{A}^{op}$ is just $\mathbb{A}$ with the reversed order of composition. Regarding on our argument on duality , it is the category where dual theorems of $\mathbb{A}$ hold. $\mathbb{A}^{op}$ will particularly help us in the case that follows.

Definition 2.2.4. *Let* $\mathbb{A}, \mathbb{B}$ *be two categories and* $F : \mathbb{A} \to \mathbb{B}$ *a map which satisfies*

(1) *If* $A \xrightarrow{f} A'$ *in* $\mathbb{A}$ *then* $FA' \xrightarrow{Ff} FA$ *in* $\mathbb{B}$;

(2) *For any* A *in* $\mathbb{A}$, $F(1_A) = 1_{FA}$;

(3) *For any* f *and* g *in* $\mathbb{A}$, $F(g \circ f) = Ff \circ Fg$.

Then F *is called a* contravariant functor *from* $\mathbb{A}$ *to* $\mathbb{B}$.

A functor in the original sense which satisfies the Definition 2.2.1 is called a *covariant functor*. Now we will consider an example about covariant and contravariant functors which will be useful to us later.

Let $\mathbb{A}$ be a category and A a fixed object in $\mathbb{A}$. We define the functor $hom(A, \_) : \mathbb{A} \to \mathbf{Set}$ as follows:

(2.1) For any B in $\mathbb{A}$, $hom(A, \_)(B) = hom(A, B)$;

(2.2) For any $B \xrightarrow{g} C$, $hom(A, \_)(g) = hom(A, B) \xrightarrow{hom(A,g)} hom(A, C)$.

The function $hom(A, g)$ takes any $A \xrightarrow{f} B$ in $hom(A, B)$ to $A \xrightarrow{g \circ f} C$ in $hom(A, C)$ by composing with g on the left. It is easy to verify that $hom(A, \_)$ is a covariant functor from $\mathbb{A}$ to $\mathbf{Set}$. Now for a fixed B in $\mathbb{A}$, we similarly define the functor

$hom(\_\,, B) : \mathbb{A} \to \mathbf{Set}$ as,

(2.3) For any A in $\mathbb{A}$, $hom(\_\,, B)(A) = hom(A, B)$;
(2.4) For any $C \xrightarrow{h} A$, $hom(\_\,, B)(h) = hom(A, B) \xrightarrow{hom(h,B)} hom(C, B)$.

The function $hom(h, B)$ takes any $A \xrightarrow{f} B$ in $hom(A, B)$ to $C \xrightarrow{f \circ h} B$ in $hom(C, B)$ by composing with h on the right. We see that $hom(\_\,, B)$ is a contravariant functor from $\mathbb{A}$ to **Set**, but we can write it covariantly as a functor from $\mathbb{A}^{op}$ to **Set**. Then for any $C \xrightarrow{h} A$, $hom(\_\,, B)(h) = hom(C, B) \xrightarrow{hom(h,B)} hom(A, B)$, since $C \xrightarrow{h} A$ in $\mathbb{A}^{op}$ indicates the existence of the arrow $A \xrightarrow{h} C$ in $\mathbb{A}$. We will continue to use the notation $hom(\_\,, B)$ for the covariant functor from $\mathbb{A}^{op}$ to **Set**.

Any contravariant functor can be made covariant and vice versa by the help of opposite categories. Also from a functor $F : \mathbb{A} \to \mathbb{B}$, we can create the functor $F^{op} : \mathbb{A}^{op} \to \mathbb{B}^{op}$ in the obvious way. An arrow $A \xrightarrow{f} B$ in $\mathbb{A}^{op}$ is mapped to $FA \xrightarrow{Ff} FB$ in $\mathbb{B}^{op}$. F^{op} is a covariant functor which is just the same as F. Only it maps mirror image of $\mathbb{A}$, $\mathbb{A}^{op}$ to the mirror image of $\mathbb{B}$, $\mathbb{B}^{op}$.

Definition 2.2.5. *Let $\mathbb{A}, \mathbb{B}$ be two categories. The* product category *of $\mathbb{A}$ and $\mathbb{B}$ has pairs $\langle A, B\rangle$, where A in $\mathbb{A}$, B in $\mathbb{B}$, as objects; and pairs $\langle f, g\rangle : \langle A, B\rangle \longrightarrow \langle A', B'\rangle$, where $A \xrightarrow{f} A'$ in $\mathbb{A}$, $B \xrightarrow{g} B'$ in $\mathbb{B}$, as arrows. Composition is defined componentwise as $\langle f, g\rangle \circ \langle f' \circ g'\rangle = \langle f \circ f', g \circ g'\rangle$ with the identity arrows $\langle 1_A, 1_B\rangle$ for the objects $\langle A, B\rangle$. We denote the product category as $\mathbb{A} \times \mathbb{B}$.*

From $\mathbb{A} \times \mathbb{B}$ to $\mathbb{A}$ and $\mathbb{B}$, we have the natural projection functors $P_1 : \mathbb{A} \times \mathbb{B} \to \mathbb{A}$ and $P_2 : \mathbb{A} \times \mathbb{B} \to \mathbb{B}$ for which $P_1(\langle A, B\rangle) = A$, $P_1(\langle f, g\rangle) = f$, $P_2(\langle A, B\rangle) = B$, $P_2(\langle f, g\rangle) = g$. The product category with functors P_1 and P_2 is indeed the product of $\mathbb{A}$ and $\mathbb{B}$ in **Cat**. Hence the notation $\mathbb{A} \times \mathbb{B}$ justifies itself.

A functor $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ from a product category is called a *bifunctor*, a functor of two variables. Going further on our example on *hom* functors we now define a bifunctor $hom(\_\,, \_) : \mathbb{A}^{op} \times \mathbb{A} \to \mathbf{Set}$ as follows:

(2.5) For any A in $\mathbb{A}^{op}$ and B in $\mathbb{A}$, $hom(\_\,,\_)(\langle A, B\rangle) = hom(A, B)$.

(2.6) For any $A \xrightarrow{f} A'$ in $\mathbb{A}^{op}$, $B \xrightarrow{g} B'$ in $\mathbb{A}$,

$$hom(\_\,,\_)(\langle f, g\rangle) = hom(A, B) \xrightarrow{hom(f,g)} hom(A', B')$$

where for $A \xrightarrow{h} B$ in $hom(A, B)$, $hom(f, g)(h) = g \circ h \circ f$, $A' \xrightarrow{g \circ h \circ f} B'$.

Restricting $hom(\_\,,\_)$ to A in the first variable we obtain $hom(A, \_) : \mathbb{A} \to$ **Set** and to B in the second variable we get $hom(\_\,, B) : \mathbb{A}^{op} \to$ **Set**. This is not a surprise. For any bifunctor fixing a variable in one component gives us a functor from the other component category to the codomain category. For example if $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ then for any B in $\mathbb{B}$, $F(\_\,, B) : \mathbb{A} \to \mathbb{C}$ is a functor with $F(\_\,, B)(A) = F(\langle A, B\rangle)$ and $F(\_\,, B)(f) = F(\langle A, B\rangle) \xrightarrow{F(\langle f, 1_B\rangle)} F(\langle A', B\rangle)$ for $A \xrightarrow{f} A'$.

2.3. Natural Transformations

Definition 2.3.1. *Let $F, G : \mathbb{A} \to \mathbb{B}$ be two functors.* A natural transformation *$\eta : F \to G$ from F to G is a collection $\{FA \xrightarrow{\eta_A} GA\}_{A \text{ in } \mathbb{A}}$ of arrows in $\mathbb{B}$ such that for any $A \xrightarrow{f} A'$ in $\mathbb{A}$ we have $Gf \circ \eta_A = \eta_{A'} \circ Ff$ in $\mathbb{B}$.*

$$\begin{array}{ccccc} A & \qquad & FA & \xrightarrow{\eta_A} & GA \\ \Big\downarrow f & & Ff \Big\downarrow & \circ & \Big\downarrow Gf \\ A' & & FA' & \xrightarrow[\eta_{A'}]{} & GA' \end{array}$$

Figure 2.10.

Commutativity of the square above is called naturality condition. When $\eta : F \to G$ is a natural transformation, for any A we call η_A a component of η and say that $\eta_A : FA \to GA$ is natural in A.

A natural transformation with each component iso is called a natural isomorphism.

Example 2.3.2. *Let* $\mathbb{A}$ *be a cartesian closed category. Consider the composition of two functors* $\_^A$ *and* $\_ \times A$, $(\_^A) \times A : \mathbb{A} \to \mathbb{A}$ *and the identity functor* $1_{\mathbb{A}}$ *on* $\mathbb{A}$. *Then for every* B *in* $\mathbb{A}$, $ev_{A,B}$ *is an arrow from* $B^A \times A$ *to* B. *And the collection* $\{ev_{A,B}\}_{B \text{ in } \mathbb{A}}$ *is a natural transformation from* $(\_^A) \times A$ *to* $1_{\mathbb{A}}$.

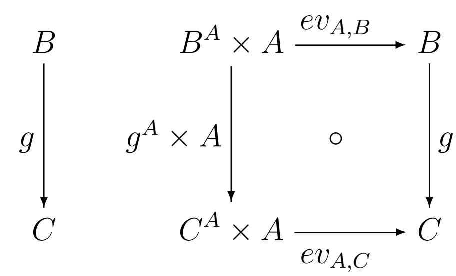


Figure 2.11.

Natural transformations compose. For $\eta : F \to G$ and $\tau : G \to H$, $\tau \circ \eta$ is defined as $(\tau \circ \eta)_A = \tau_A \circ \eta_A$ for any A in $\mathbb{A}$. This composition is associative. Also for any functor F, there is an identity natural transformation $1_F : F \to F$. It has the components 1_{FA} for any A in $\mathbb{A}$. By the above remarks, we can assert the existence of functor categories which takes functors as objects and natural transformations as arrows.

By combining functors and natural transformations we can obtain new natural transformations. Let $G : \mathbb{D} \to \mathbb{A}$, $H, K : \mathbb{A} \to \mathbb{B}$, $F : \mathbb{B} \to \mathbb{C}$ be functors and $\eta : H \to K$ be a natural transformation. Then we can define the natural transformation $F\eta : F \circ H \to F \circ K$ with components $F\eta_A$ for each A in $\mathbb{A}$.

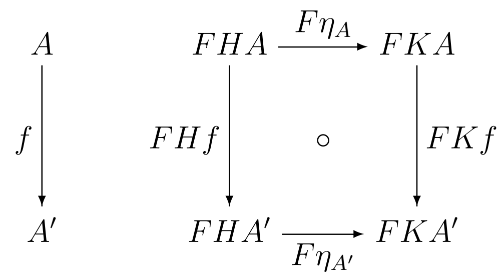


Figure 2.12.

The square above commutes since η is a natural transformation and it is the image of the commutative square below under F.

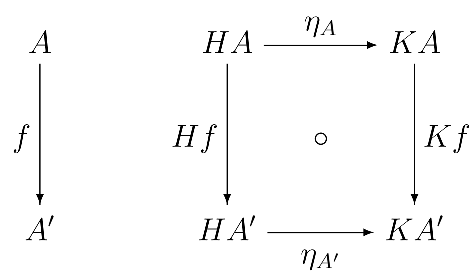


Figure 2.13.

We also define natural transformation $\eta_G : H \circ G \to K \circ G$ with components η_{GD} for each D in $\mathbb{D}$.

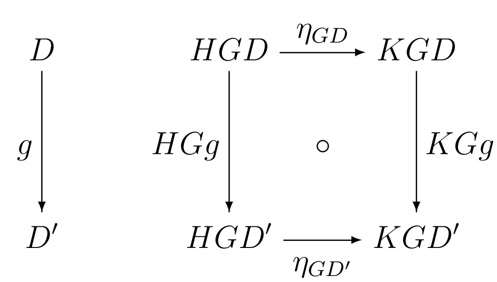


Figure 2.14.

The square above commutes since η is a natural transformation from H to K and we are considering it as restricted to objects in the form GD.

Now we consider natural transformations between bifunctors. Let $F, G : \mathbb{X} \times \mathbb{A} \to \mathbb{B}$ be two bifunctors and $\eta = \left\{ F(\langle X, A \rangle) \xrightarrow{\eta_{X,A}} G(\langle X, A \rangle) \right\}_{X \text{ in } \mathbb{X},\ A \text{ in } \mathbb{A}}$ be a collection of arrows. For any X in $\mathbb{X}$ we say that $\eta_{X,A}$ is natural in X, or η is natural in X, if for any A in $\mathbb{A}$ $\eta_{\_\,,A} : F(\langle \_\,, A \rangle) \to G(\langle \_\,, A \rangle)$ is natural transformation of functors from $\mathbb{X}$ to $\mathbb{B}$, i.e. we have the following commutative square for any $X \xrightarrow{f} X'$ and A in $\mathbb{A}$.

$$
\begin{array}{ccccc}
X & \qquad & F(\langle X, A \rangle) & \xrightarrow{\eta_{X,A}} & G(\langle X, A \rangle) \\
{\scriptstyle f}\downarrow & & {\scriptstyle F(\langle f, 1_A \rangle)}\downarrow & \circ & \downarrow{\scriptstyle G(\langle f, 1_A \rangle)} \\
X' & & F(\langle X', A \rangle) & \xrightarrow[\eta_{X',A}]{} & G(\langle X', A \rangle)
\end{array}
$$

Figure 2.15.

Proposition 2.3.3. *Let $F, G : \mathbb{X} \times \mathbb{A} \to \mathbb{B}$ be two bifunctors. Then $\eta = \left\{ F(\langle X, A \rangle) \xrightarrow{\eta_{X,A}} G(\langle X, A \rangle) \right\}_{X \text{ in } \mathbb{X},\ A \text{ in } \mathbb{A}}$ is a natural transformation between functors F and G if and only if $\eta_{X,A}$ is natural in X and natural in A for each X in $\mathbb{X}$, A in $\mathbb{A}$.*

Proof. Suppose η is a natural transformation from F to G. Then for each A in $\mathbb{A}$, we have the above commutative diagram for any $X \xrightarrow{f} X'$. So $\eta_{X,A}$ is natural in X. And for each X in $\mathbb{X}$, $A \xrightarrow{g} A'$ we have

$$
\begin{array}{ccccc}
A & \qquad & F(\langle X,A\rangle) & \xrightarrow{\eta_{X,A}} & G(\langle X,A\rangle) \\
\Big\downarrow g & & F(\langle 1_X, g\rangle)\Big\downarrow & \circ & \Big\downarrow G(\langle 1_X, g\rangle) \\
A' & & F(\langle X,A'\rangle) & \xrightarrow[\eta_{X,A'}]{} & G(\langle X,A'\rangle)
\end{array}
$$

Figure 2.16.

So $\eta_{X,A}$ is natural in A.

Conversely suppose that $\eta_{X,A}$ is natural in X and natural in A for each X in $\mathbb{X}$, A in $\mathbb{A}$. Let $\langle X, A\rangle \xrightarrow{\langle f,g\rangle} \langle X', A'\rangle$ be an arrow in $\mathbb{X} \times \mathbb{A}$. Then we have $X \xrightarrow{f} X'$ in $\mathbb{X}$ and $A \xrightarrow{g} A'$ in $\mathbb{A}$. By naturality conditions we get the two commutative squares below.

$$
\begin{array}{cccccc}
X & & \qquad & F(\langle X,A\rangle) & \xrightarrow{\eta_{X,A}} & G(\langle X,A\rangle) \\
\Big\downarrow f & & & F(\langle f, 1_A\rangle)\Big\downarrow & \circ & \Big\downarrow G(\langle f, 1_A\rangle) \\
X' & A & & F(\langle X',A\rangle) & \xrightarrow{\eta_{X',A}} & G(\langle X',A\rangle) \\
 & \Big\downarrow g & & F(\langle 1_{X'}, g\rangle)\Big\downarrow & \circ & \Big\downarrow G(\langle 1_{X'}, g\rangle) \\
 & A' & & F(\langle X',A'\rangle) & \xrightarrow[\eta_{X',A'}]{} & G(\langle X',A'\rangle)
\end{array}
$$

Figure 2.17.

By composing the vertical arrows we get,

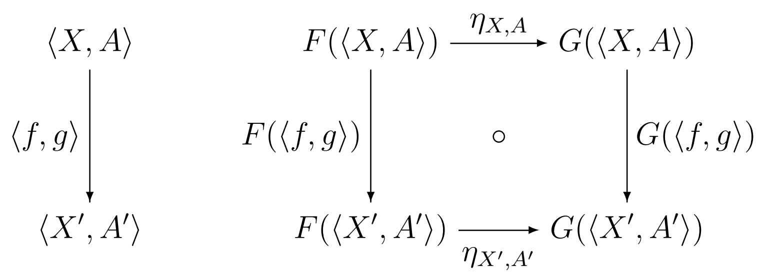


Figure 2.18.

So $\eta : F \to G$ is a natural transformation. □

We will consider natural transformations between bifunctors when we define adjunctions. An adjunction will be a natural isomorphism between bifunctors $hom(F\_\,,\_)$ and $hom(\_\,,G\_)$ where $F : \mathbb{X} \to \mathbb{A}$ and $G : \mathbb{A} \to \mathbb{X}$ are functors. In this setting $hom(F\_\,,\_)$ is the composition of functors $\mathbb{X}^{op} \times \mathbb{A} \xrightarrow{F^{op}\times 1_{\mathbb{A}}} \mathbb{A}^{op} \times \mathbb{A} \xrightarrow{hom} \mathbf{Set}$ and $hom(\_\,,G\_)$ is the composition $\mathbb{X}^{op} \times \mathbb{A} \xrightarrow{1_{\mathbb{X}^{op}}\times G} \mathbb{X}^{op} \times \mathbb{X} \xrightarrow{hom} \mathbf{Set}$.

2.4. Universal Arrows

Definition 2.4.1. *Let $G : \mathbb{A} \to \mathbb{X}$ be a functor and X be an object in $\mathbb{X}$. A* universal arrow *from X to G consists of an object A in $\mathbb{A}$ and an arrow $X \xrightarrow{\eta_X} GA$ such that for any B in $\mathbb{A}$ and an arrow $X \xrightarrow{f} GB$ in $\mathbb{X}$ there exists a unique arrow $A \xrightarrow{g} B$ for which $Gg \circ \eta_X = f$. In such a case we may say that η_X is universal from X to G.*

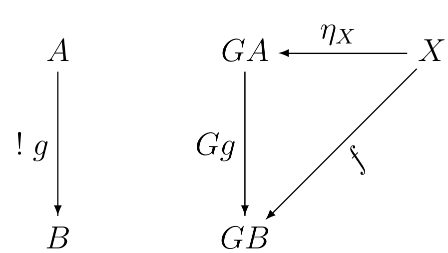


Figure 2.19.

Dually for $F : \mathbb{X} \to \mathbb{A}$ *and an object* A *in* $\mathbb{A}$*, a universal arrow from* F *to* A *consists of an object* X *in* $\mathbb{X}$ *and an arrow* $FX \xrightarrow{\varepsilon_A} A$ *such that for any* Y *in* $\mathbb{X}$ *and any arrow* $FY \xrightarrow{h} A$ *there exists a unique arrow* $Y \xrightarrow{k} X$ *for which* $\varepsilon_A \circ Fk = h$*. In this case we say that* ε_A *is universal from* F *to* A*.*

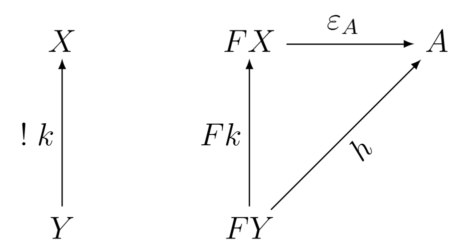


Figure 2.20.

Examples 2.4.2. (1) *Take a cartesian closed category* $\mathbb{A}$*, then for any* A *and* B *in* $\mathbb{A}$*,* $(B^A, ev_{A,B})$ *is a universal arrow from* $\_ \times A$ *to* B*.*

(2) *For a category* $\mathbb{A}$ *define a functor* $\Delta : \mathbb{A} \to \mathbb{A} \times \mathbb{A}$ *as* $\Delta(A) = \langle A, A \rangle$ *on objects and* $\Delta(f) = \langle f, f \rangle$ *on arrows.* Δ *is called the diagonal functor. Suppose further that* $\mathbb{A}$ *has finite products. Then for each* $\langle A, B \rangle$ *in* $\mathbb{A} \times \mathbb{A}$ *there is a universal arrow* $(A \times B, \langle p_1, p_2 \rangle)$ *from* Δ *to* $\langle A, B \rangle$ *where* $A \times B \xrightarrow{p_1} A$ *and* $A \times B \xrightarrow{p_2} B$ *are projection arrows. To verify this let* C *be an object in* $\mathbb{A}$ *and* $\langle C, C \rangle \xrightarrow{\langle f,g \rangle} \langle A, B \rangle$ *be an arrow in* $\mathbb{A} \times \mathbb{A}$*. Then in* $\mathbb{A}$ *we will have two arrows* $C \xrightarrow{f} A$ *and* $C \xrightarrow{g} B$*. Since* $A \times B$ *is a product, there exists a unique arrow* $C \xrightarrow{k} A \times B$ *such that* $p_1 \circ k = f$ *and* $p_2 \circ k = g$*. Taking the image of* k *under* Δ*, we see that*

$$\langle p_1, p_2 \rangle \circ \langle k, k \rangle = \langle p_1 \circ k, p_2 \circ k \rangle = \langle f, g \rangle$$

So for an arbitrary object C *in* $\mathbb{A}$ *and an arrow* $\langle C, C \rangle \xrightarrow{\langle f,g \rangle} \langle A, B \rangle$ *there exists a unique arrow* $C \xrightarrow{k} A \times B$ *such that* $\langle p_1, p_2 \rangle \circ \Delta(k) = \langle f, g \rangle$*.*

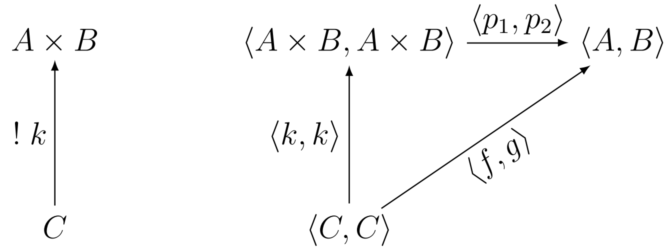


Figure 2.21.

Therefore $\langle p_1, p_2 \rangle$ *is universal from* Δ *to* $\langle A, B \rangle$.

(3) *For any set* A*, the set of finite strings of elements of* A *with concatenation as the operation is called the free monoid on* A*. In this monoid empty string* () *serves as the identity element. We will denote the free monoid on* A *by* FA *(this notation will justify itself in a few lines). Also for any function* $A \xrightarrow{f} B$*, there exists a corresponding monoid homomorphism* $FA \xrightarrow{\overline{f}} FB$*. This homomorphism takes the identity element of* FA *to the identity element of* FB*, and an arbitrary element* $(a_1, a_2, \ldots, a_n)$ *of* FA *to* $(f(a_1), f(a_2), \ldots, f(a_n))$ *of* FB*.*

By the above remarks it can be seen that there is a functor from **Set** *to* **Mon** *which takes any set to its free monoid and any function between sets to corresponding monoid homomorphism between free monoids. We will denote this functor by* $F : \mathbf{Set} \to \mathbf{Mon}$ *and call it the free monoid functor. Remember that we also have the forgetful functor,* $U : \mathbf{Mon} \to \mathbf{Set}$*. For a set* A*,* UFA *will be the underlying set of the free monoid* FA*. And we will denote the function from* A *to* UFA *which sends each* $a \in A$ *to* $\langle a \rangle \in UFA$ *by* η_A*.*

Now take any set X *in* **Set***. We claim that* $(FX, X \xrightarrow{\eta_X} UFX)$ *is a universal arrow from* X *to* U*. For any monoid* M *and a set function* $X \xrightarrow{f} UM$*, there exists a monoid homomorphism* $FX \xrightarrow{g} M$ *which sends the identity element of* FX *to the identity element of* M *and each* $\langle x \rangle$ *to* $f(x) \in M$*. By this construction,* $Ug \circ \eta_X = f$*.*

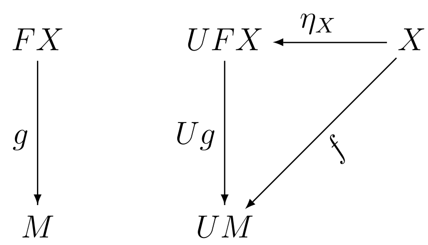


Figure 2.22.

To see that g is the unique monoid homomorphism which satisfies this condition, let $FX \xrightarrow{h} M$ be another monoid homomorphism with $Uh \circ \eta_X = f$. Then for any $x \in X$, $Uh(\langle x \rangle) = f(x)$ as a set function and hence $h(\langle x \rangle) = f(x)$ as a monoid homomorphism. For an arbitrary $\langle x_1, x_2, \ldots, x_n \rangle$ in FX,

$$\begin{aligned} h(\langle x_1, x_2, \ldots, x_n \rangle) &= h(\langle x_1 \rangle \langle x_2 \rangle \ldots \langle x_n \rangle) \\ &= h(\langle x_1 \rangle) h(\langle x_2 \rangle) \ldots h(\langle x_n \rangle) \\ &= f(x_1) f(x_2) \ldots f(x_n) \\ &= g(\langle x_1 \rangle) g(\langle x_2 \rangle) \ldots g(\langle x_n \rangle) \\ &= g(\langle x_1 \rangle \langle x_2 \rangle \ldots \langle x_n \rangle) \\ &= g(\langle x_1, x_2, \ldots, x_n \rangle) \end{aligned}$$

So $h = g$, and g is the unique monoid homomorphism satisfying $Ug \circ \eta_X = f$. This shows that η_X is universal from X to U.

Universal arrows are unique up to isomorphism, i.e. if $(A, X \xrightarrow{\eta_X} GA)$ is a universal arrow from X to $G : \mathbb{A} \to \mathbb{X}$ and there is an isomorphism $A \xrightarrow{f} B$ in $\mathbb{A}$, then $(B, X \xrightarrow{Gf \circ \eta_X} GB)$ is also a universal arrow from X to G. Also if $(A, X \xrightarrow{\eta_X} GA)$ and $(B, X \xrightarrow{\eta'_X} GB)$ are two universal arrows from X to G, then there is an isomorphism $A \xrightarrow{f} B$ in $\mathbb{A}$.

2.5. Comma Categories

Definition 2.5.1. *Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ be categories and $F : \mathbb{A} \to \mathbb{B}$, $G : \mathbb{C} \to \mathbb{B}$ be functors. The* comma category $(F \downarrow G)$ *has as objects the triples $\langle A, C, f \rangle$; where A is an object in $\mathbb{A}$, C is an object in $\mathbb{C}$, $FA \xrightarrow{f} GC$ is an arrow in $\mathbb{B}$; and as arrows $\langle A, C, f \rangle \longrightarrow \langle A', C', f' \rangle$ the pairs $\langle h, k \rangle$; where $A \xrightarrow{h} A'$, $C \xrightarrow{k} C'$ which satisfy $Gk \circ f = f' \circ Fk$.*

$$\begin{array}{ccc} FA & \xrightarrow{\quad f \quad} & GC \\ {\scriptstyle Fh}\downarrow & \circ & \downarrow{\scriptstyle Gk} \\ FA' & \xrightarrow[\quad f' \quad]{} & GC' \end{array}$$

Figure 2.23.

In particular we will be interested in comma categories where F is a constant functor.

Definition 2.5.2. *Let B be an object in $\mathbb{B}$. By abuse of notation denote the constant functor which sends every object of $\mathbb{A}$ to B and every arrow to 1_B as B. Then the comma category $(B \downarrow G)$ is called* objects G-under B.

The objects of $(B \downarrow G)$ are the pairs $\langle f, C \rangle$ where f is an arrow from B to GC in $\mathbb{B}$. And arrows $\langle f, C \rangle \xrightarrow{h} \langle f', C' \rangle$ are just arrows $C \xrightarrow{h} C'$ in $\mathbb{C}$ for which we have $Gh \circ f = f'$.

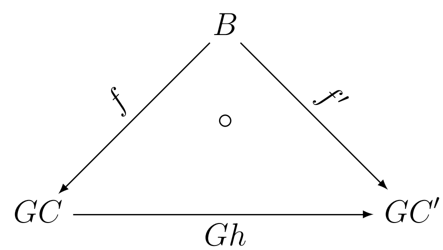


Figure 2.24.

In other words we can identify objects of $(B \downarrow G)$ with the arrows $B \longrightarrow GC$ in $\mathbb{B}$ and the arrows $(B \longrightarrow GC) \xrightarrow{h} (B \longrightarrow GC')$ with the arrows $C \xrightarrow{h} C'$ of $\mathbb{C}$ for which the triangle above commutes.

We can also define a functor $P : (B \downarrow G) \rightarrow \mathbb{C}$, called the *projection functor* as follows:

(2.7) For any object $\langle f, C \rangle$ of $(B \downarrow G)$, $P(\langle f, C \rangle) = C$

(2.8) For any arrow $\langle f, C \rangle \xrightarrow{h} \langle f', C' \rangle$, $P(h) = h$ as an arrow in $\mathbb{C}$.

In diagrams,

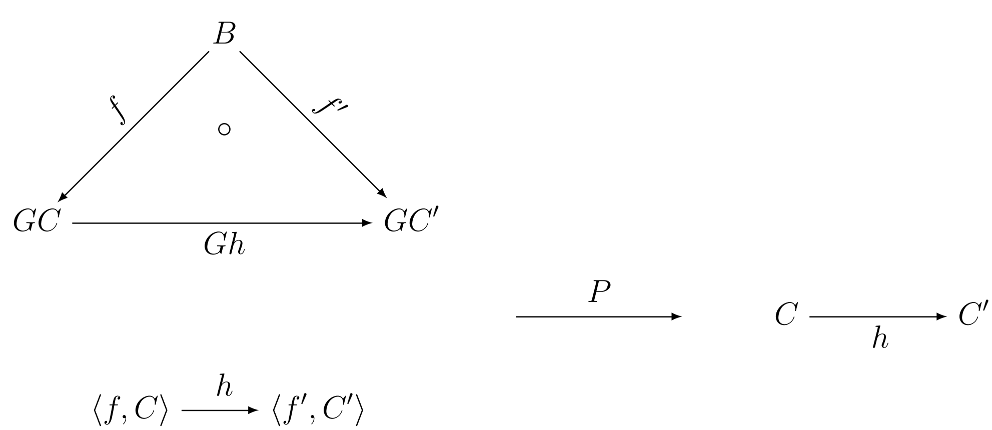


Figure 2.25.

Example 2.5.3. *A non-empty set together with a selected element of it is called a* pointed set. *And this distinguished element is called the* base point *of the set. We can form the category of pointed sets by taking pointed set as objects. Arrows of this category are functions between pointed sets which send the base point of the domain to the base point of the codomain.*

There is also an alternative presentation of category of pointed sets in terms of comma categories. By abuse of notation let's denote the functor from **Set** *to* **Set** *which takes any set to the singleton* $\{*\}$ *by* $*$. *Also denote the identity functor from* **Set** *to* **Set** *by* **Set**. *Then the comma category* $(* \downarrow \mathbf{Set})$ *is just the same with the category of pointed sets. Every pointed set* (C, c) *can be identified with the object* $\{*\} \xrightarrow{c} C$ *of the comma category (Here we denote the function which takes* $*$ *to* c *also by* c*). And an arrow* $(C, c) \xrightarrow{f} (D, d)$ *in the category of pointed sets corresponds to the arrow* $c \xrightarrow{f} d$ *in* $(* \downarrow \mathbf{Set})$.

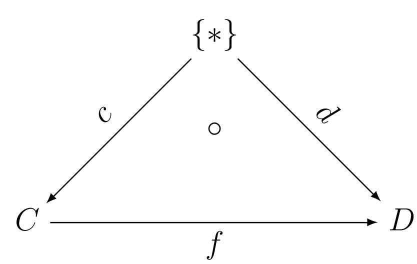


Figure 2.26.

3. ADJOINTS

In this chapter we introduce the theory of adjoint functors. Theorems presented here give all the important properties of adjunctions in a rather compact way. We end the chapter with examples of adjoints occurring in different areas of mathematics.

Historically, the notion of an adjoint functor was introduced by Kan [2] for the needs of homological algebra. There he presented the natural equivalence between $hom(M \otimes N, A)$ and $hom(M, hom(N, A))$, where M, N, A are R-modules and R is a commutative ring, as an adjunction between functors $\_ \otimes N$ and $hom(N, \_)$. He called the functor $\_ \otimes N$ as the left adjoint of $hom(N, \_)$ and $hom(N, \_)$ as the right adjoint of $\_ \otimes N$. This terminology comes from adjoint operators in functional analysis. There for any linear operator $S : H \to G$ from one Hilbert space to another there exists an operator $T : G \to H$ such that for every $h \in H$ and $g \in G$, $\langle h, Tg\rangle = \langle Sh, g\rangle$. In this context we say that S is left adjoint of T as T is right adjoint of S.

3.1. Characterization of Adjunctions

We now formally define an adjunction.

Definition 3.1.1. *Let $\mathbb{X}$ and $\mathbb{A}$ be two categories. An* adjunction from $\mathbb{X}$ to $\mathbb{A}$ *consists of two functors $F : \mathbb{X} \to \mathbb{A}$, $G : \mathbb{A} \to \mathbb{X}$ and a natural isomorphism ϕ between bifunctors $hom(F\_\,, \_)$ and $hom(\_\,, G\_)$. We denote such an adjunction by $\langle F, G, \phi\rangle$ and call F as the left adjoint of G and G as the right adjoint of F.*

By Proposition 2.3.3, ϕ is a natural transformation between $hom(F\_\,, \_)$ and $hom(\_\,, G\_)$ if and only if $\phi_{X,A} : hom(FX, A) \to hom(X, GA)$ is natural in X and natural in A for each X in $\mathbb{X}$, A in $\mathbb{A}$. So an adjunction $\langle F, G, \phi\rangle$ can be taken as two functors and a collection of bijections $\phi_{X,A} : hom(FX, A) \to hom(X, GA)$ which is natural in X and A for X in $\mathbb{X}$ and A in $\mathbb{A}$.

$\phi_{X,A}$ being natural in X implies that we have the following commutative square for each A in $\mathbb{A}$.

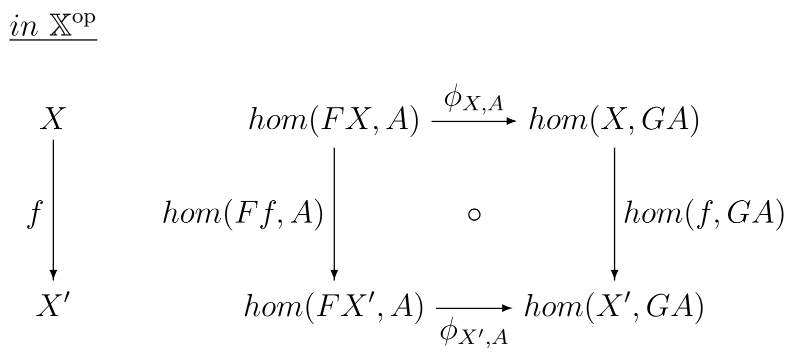


Figure 3.1.

In equational form, $\phi_{X,A}(h) \circ f = \phi_{X',A}(h \circ Ff)$ for any h in $hom(FX, A)$.

And $\phi_{X,A}$ being natural in A implies the following commutative square for each X in $\mathbb{X}$.

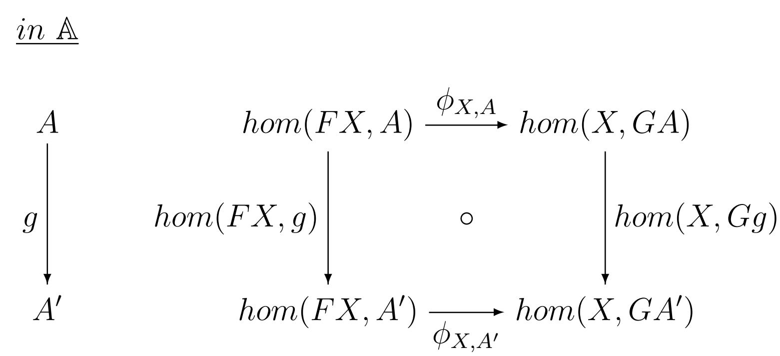


Figure 3.2.

In equations, we have $Gg \circ \phi_{X,A}(h) = \phi_{X,A'}(g \circ h)$ for any h in $hom(FX, A)$.

Since each $\phi_{X,A}$ is an iso, replacing them in the diagrams above by their inverses preserves the commutativity of the diagrams. Hence also $\phi^{-1}_{X,A} : hom(X, GA) \to$

$hom(FX, A)$ is natural in X and A for X in $\mathbb{X}$ and A in $\mathbb{A}$. This means that for an arrow k in $hom(X, GA)$, we have $\phi^{-1}_{X,A}(k) \circ Ff = \phi^{-1}_{X',A}(k \circ f)$ and $g \circ \phi^{-1}_{X,A}(k) = \phi^{-1}_{X,A'}(Gg \circ k)$.

Definition 3.1.1 is a compact way of defining an adjunction. But this definition carries more information than being just a certain natural isomorphism. The following theorem shows us the relation between adjunctions and universal arrows. It follows that every adjunction give rise to specific universal arrows and these universal arrows are the basic determinants of an adjunction.

Theorem 3.1.2. *Let $\langle F, G, \phi \rangle$ be an adjunction. Then*

(1) *For each X in $\mathbb{X}$, there exists an arrow $\eta_X : X \to GFX$ which is universal from X to G. The collection $\{\eta_X\}_{X \text{ in } \mathbb{X}} = \eta$ is a natural transformation from $1_{\mathbb{X}}$ to GF. And for any $FX \xrightarrow{f} A$ in $\mathbb{A}$, $\phi_{X,A}(f) = Gf \circ \eta_X$;*

(2) *For each A in $\mathbb{A}$, there exists an arrow $\varepsilon_A : FGA \to A$ which is universal from F to A. The collection $\{\varepsilon_A\}_{A \text{ in } \mathbb{A}} = \varepsilon$ is a natural transformation from FG to $1_{\mathbb{A}}$. And for any $X \xrightarrow{g} GA$ in $\mathbb{X}$, $\phi^{-1}_{X,A}(g) = \varepsilon_A \circ Fg$;*

(3) *$\varepsilon_F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta_G = 1_G$.*

Proof. (1) For any X in $\mathbb{X}$, FX is an object in $\mathbb{A}$. Consider $\phi_{X,FX} : hom(FX, FX) \to hom(X, GFX)$. We set $\eta_X = \phi_{X,FX}(1_{FX})$.

Then for any $FX \xrightarrow{f} A$, we have the following commutative square.

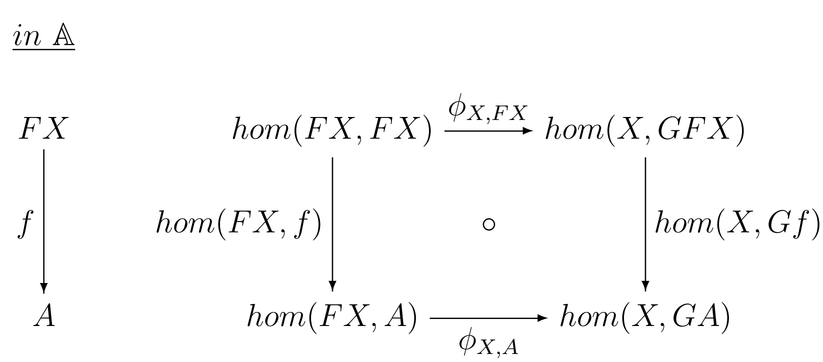


Figure 3.3.

If we chase 1_{FX} around the commutative square via the upper route we obtain

$Gf \circ \phi_{X,FX}(1_{FX}) = Gf \circ \eta_X$, while via the lower route we obtain $\phi_{X,A}(f \circ 1_{FX}) = \phi_{X,A}(f)$. Since the square is commutative we get $\phi_{X,A}(f) = Gf \circ \eta_X$.

This determines ϕ completely since $FX \xrightarrow{f} A$ was arbitrary.

We can picture the action of $\phi_{X,A}$ as,

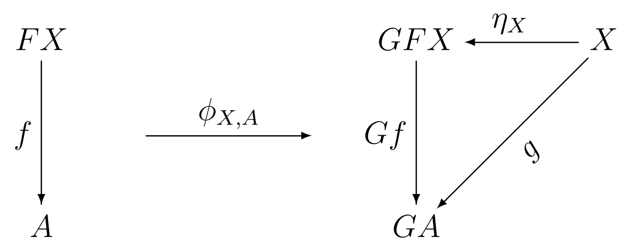


Figure 3.4.

Since $\phi_{X,A}$ is a bijection, for any $X \xrightarrow{g} GA$ there exists a unique $FX \xrightarrow{f} A$ such that $\phi_{X,A}(f) = Gf \circ \eta_X = g$. This implies that η_X is universal from X to G.

Lastly we show that $\eta : 1_{\mathbb{X}} \to GF$ is a natural transformation, i.e. the square below commutes for any $X \xrightarrow{f} X'$ in $\mathbb{X}$.

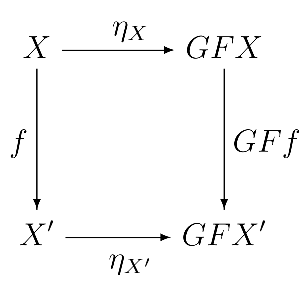


Figure 3.5.

In symbols, $\eta_{X'} \circ f = GFf \circ \eta_X$.

We can establish this identity by the help of the following commutative diagram.

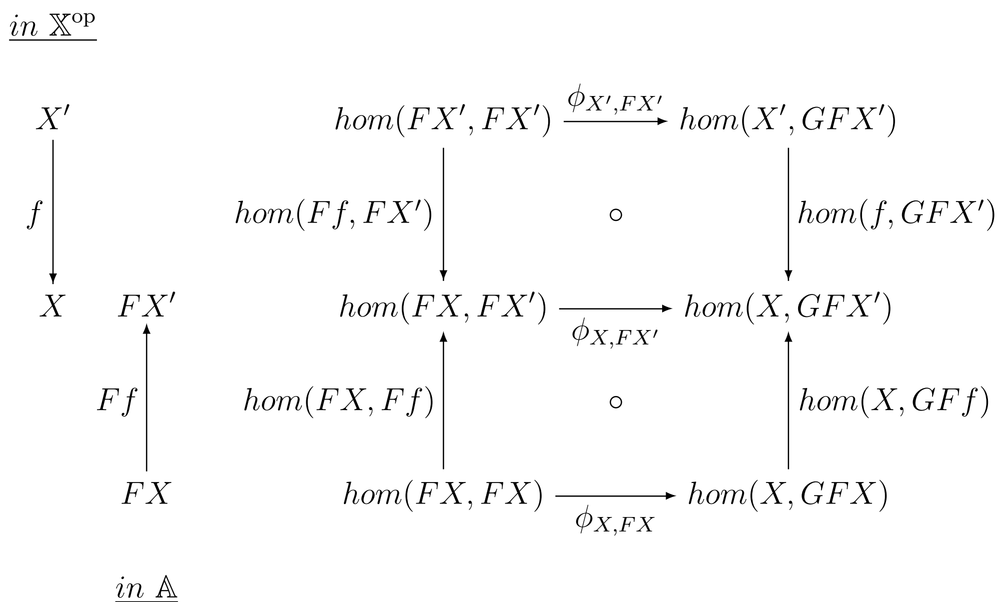


Figure 3.6.

If we chase $1_{FX'}$ around the upper commutative square we obtain

$$\phi_{X,FX'}(Ff) = \eta_{X'} \circ f$$

and if we chase 1_{FX} around the lower commutative square we obtain

$$\phi_{X,FX'}(Ff) = GFf \circ \eta_X.$$

Combining these two equations we get the desired result

$$\eta_{X'} \circ f = GFf \circ \eta_X.$$

(2) Similar to the proof of (1). Now we take $\varepsilon_A = \phi^{-1}_{GA,A}(1_{GA})$. Then we find that $\phi^{-1}_{X,A}(g) = \varepsilon_A \circ Fg$ for any $X \xrightarrow{g} GA$, ε_A is universal from F to A and $\varepsilon : FG \to 1_{\mathbb{A}}$ is a natural transformation.

(3) $\phi^{-1}_{X,A}(g) = \varepsilon_A \circ Fg$ for any $X \xrightarrow{g} GA$. If we take $A = FX$ and $g = X \xrightarrow{\eta_X} GFX$,

then we obtain

$$\begin{aligned}\phi^{-1}_{X,FX}(\eta_X) &= \varepsilon_{FX} \circ F\eta_X \\ \phi^{-1}_{X,FX}(\phi_{X,FX}(1_{FX})) &= \varepsilon_{FX} \circ F\eta_X \\ 1_{FX} &= \varepsilon_{FX} \circ F\eta_X.\end{aligned}$$

This is for any X in $\mathbb{X}$, so $1_F = \varepsilon_F \circ F\eta$.

Similarly $\phi(\varepsilon_A)_{GA,A} = G\varepsilon_A \circ \eta_{GA}$ for any A in $\mathbb{A}$. Hence $1_G = G\varepsilon \circ \eta_G$.

□

In the case we have an adjunction $\langle F, G, \phi\rangle$ and we define η and ε as in the theorem above, we call η as the unit of the adjunction while ε as the counit.

An adjunction can be determined by less data. The following theorem shows equivalent ways of defining an adjunction.

Theorem 3.1.3. *An adjunction $\langle F, G, \phi\rangle$ can be determined by any one of the following.*

(1) *Functors F, G and a natural transformation $\eta : 1_{\mathbb{X}} \to GF$ such that each $\eta_X : X \to GFX$ is universal from X to G;*

(2) *Functors F, G and a natural transformation $\varepsilon : FG \to 1_{\mathbb{A}}$ such that each $\varepsilon_A : FGA \to A$ is universal from F to A;*

(3) *A functor G, for each X in $\mathbb{X}$ an object A_X in $\mathbb{A}$ and $\eta_X : X \to GA_X$ such that η_X is universal from X to G;*

(4) *A functor F, for each A in $\mathbb{A}$ an object X_A in $\mathbb{X}$ and $\varepsilon_A : FX_A \to A$ such that ε_A is universal from F to A;*

(5) *Two functors F, G, two natural transformations $\eta : 1_{\mathbb{X}} \to GF$, $\varepsilon : FG \to 1_{\mathbb{A}}$ such that $\varepsilon_F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta_G = 1_G$.*

Proof. (1) First we show that the data given is enough to determine an adjunction.

For F, G and η, define $\widehat{\phi}_{X,A} : hom(FX, A) \to hom(X, GA)$ by $\widehat{\phi}_{X,A}(f) = Gf \circ \eta_X$ for any $FX \xrightarrow{f} A$.

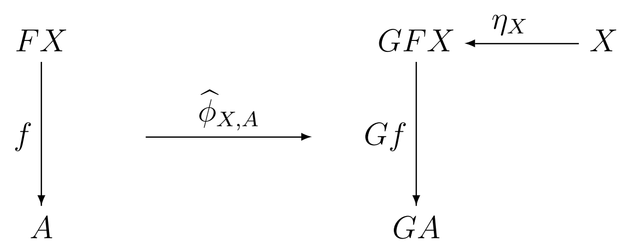


Figure 3.7.

Since η_X is universal from X to G, for any $X \xrightarrow{g} GA$ there exists a unique $FX \xrightarrow{f} A$ such that $g = Gf \circ \eta_X$. So $\widehat{\phi}_{X,A}$ is surjective. And for any $FX \overset{f_1}{\underset{f_2}{\rightrightarrows}} A$ such that $Gf_1 \circ \eta_X = Gf_2 \circ \eta_X$ universality of η_X implies that $f_1 = f_2$. So $\widehat{\phi}_{X,A}$ is injective. Hence $\widehat{\phi}_{X,A} : hom(FX, A) \to hom(X, GA)$ is a bijection.
To see that $\widehat{\phi}_{X,A}$ is natural in X, we will should prove that

$$\widehat{\phi}_{X,A}(h) \circ f = \widehat{\phi}_{X',A}(h \circ Ff)$$

for any $X \xrightarrow{f} X'$ in $\mathbb{X}^{op}$ and $FX \xrightarrow{h} A$ in $\mathbb{A}$, i.e. we should prove that

$$(Gh \circ \eta_X) \circ f = G(h \circ Ff) \circ \eta_{X'}.$$

This can be seen easily from the following diagram since η is a natural transformation.

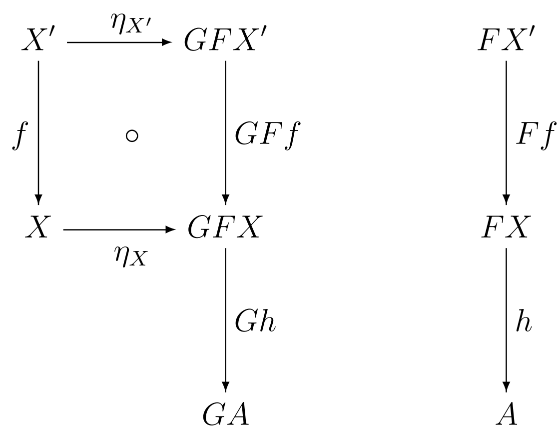


Figure 3.8.

And for the naturality in A, we try to find

$$Gg \circ \widehat{\phi}_{X,A}(h) = \widehat{\phi}_{X,A'}(g \circ h)$$

for any $A \xrightarrow{g} A'$ and $FX \xrightarrow{h} A$ in $\mathbb{A}$, i.e.

$$Gg \circ (Gh \circ \eta_X) = G(g \circ h) \circ \eta_X.$$

Equality follows since G is a functor.

So with the given data we construct a natural isomorphism $\widehat{\phi}$ by defining $\widehat{\phi}_{X,A}(f) = Gf \circ \eta_X$ for any $FX \xrightarrow{f} A$; hence the adjunction $\langle F, G, \widehat{\phi} \rangle$.

Conversely given an adjunction $\langle F, G, \phi \rangle$, if we forget about ϕ but only recall $\eta_X = \phi_{X,FX}(1_{FX})$ and hence η; what we construct as $\widehat{\phi}$ above is equal to ϕ. This is because $\phi_{X,A}(f) = Gf \circ \eta_X = \widehat{\phi}_{X,A}(f)$. (Note that ϕ is determined by η by the way $\widehat{\phi}$ is defined by it.)

(2) Dual proof of part (1).

(3) To show that these data are enough to construct an adjunction, we will try to satisfy the conditions of part (1). The first step is to construct a functor $F' : \mathbb{X} \to \mathbb{A}$. For this we set $F'(X) = A_X$ for any X in $\mathbb{X}$. And for any arrow $X \xrightarrow{f} Y$, we define $F'(f)$ as the unique arrow from A_X to A_Y which satisfies $\eta_Y \circ f = G(F'f) \circ \eta_X$. Since η_X is universal from X to G this is well defined.

Mimicking the notation of transposes in cartesian closed categories we will denote $F'f$ by $\overline{\eta_Y \circ f}$.

We need to show that defined in this manner F' becomes a functor. Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be two arrows, then we have the diagram below.

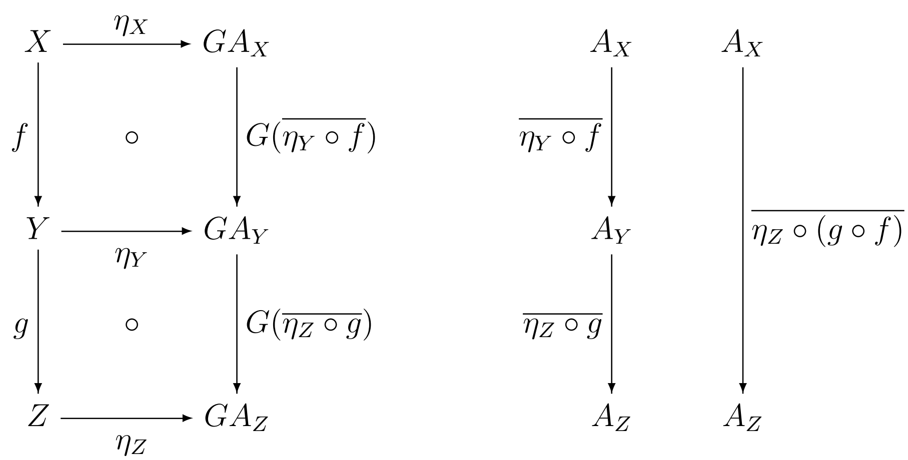


Figure 3.9.

By definition $\overline{\eta_Y \circ f}$ and $\overline{\eta_Z \circ g}$ are the unique arrows whose images under G make upper and lower squares commute. But then $\overline{\eta_Z \circ g} \circ \overline{\eta_Y \circ f}$ becomes an arrow whose image under G makes the whole diagram on the left commute. By uniqueness of such arrow from A_X to A_Z,

$$\overline{\eta_Z \circ g} \circ \overline{\eta_X \circ f} = \overline{\eta_Z \circ (g \circ f)}.$$

Hence,

$$F'g \circ F'f = F'(g \circ f)$$

So F' preserves composition.

The case for $X \xrightarrow{1_X} X$ is trivial and shown by the diagram below.

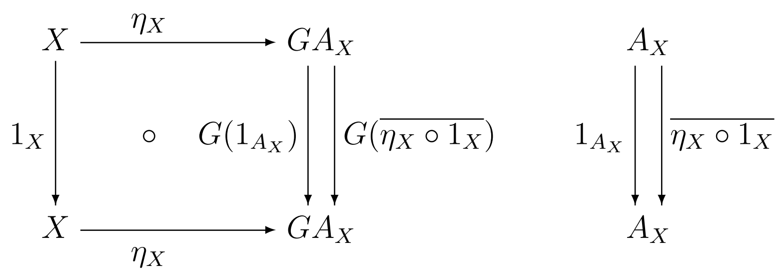


Figure 3.10.

Since images of both 1_{A_X} and $\overline{\eta_X \circ 1_X}$ make the diagram commute, $1_{A_X} = \overline{\eta_X \circ 1_X} = F'(1_X)$ i.e. $1_{F'X} = F'(1_X)$.

Hence $F' : \mathbb{X} \to \mathbb{A}$ is a functor. Moreover by this definition $\eta : 1_{\mathbb{X}} \to GF'$ becomes a natural transformation. For any $X \xrightarrow{f} Y$ we have the following commutative square,

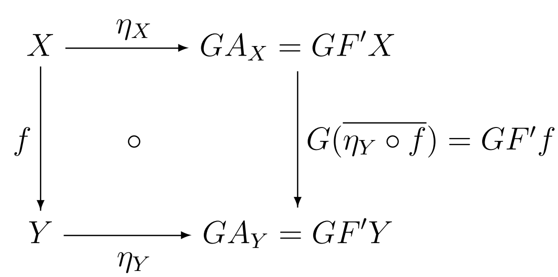


Figure 3.11.

Therefore conditions of part (1) is satisfied and given data is enough to determine an adjunction.

Conversely given an adjunction $\langle F, G, \phi \rangle$ if we forget about F and ϕ but only recall the action of F on objects and the universality of $\eta_X : X \to GFX$ for each X; the functor F' we construct as above by the data given will be equal to the original functor F.

After the construction of F' we will have $\eta : 1_{\mathbb{X}} \to GF'$. And originally we know that $\eta : 1_{\mathbb{X}} \to GF$. So we will have,

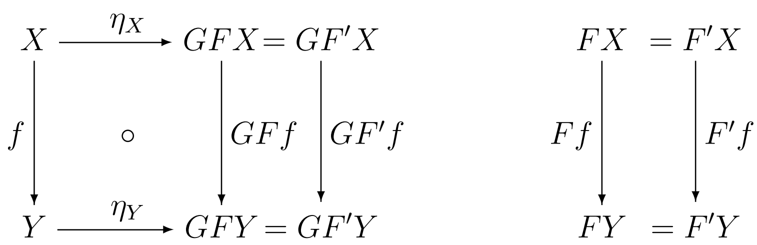


Figure 3.12.

Since η_X is universal from X to G, there is exists a unique arrow $FX \longrightarrow FY$ which makes the square on the left commute. Hence $Ff = F'f$. So F and F' agree on arrows as well as on objects. Hence $F = F'$.

(4) Dual proof of part (3).

(5) For any $FX \xrightarrow{f} A$, define $\widehat{\phi} : hom(F\_\, , \_) \to hom(\_\, , G\_)$ componentwise, by putting $\widehat{\phi}_{X,A}(f) = Gf \circ \eta_X$; and for any $X \xrightarrow{g} GA$, define $\psi : hom(\_\, , G\_) \to hom(F\_\, , \_)$ componentwise, by putting $\psi_{X,A}(g) = \varepsilon_A \circ Fg$. Then,

$$\begin{aligned} \psi_{X,A}(\widehat{\phi}_{X,A}(f)) = \psi_{X,A}(Gf \circ \eta_X) &= \varepsilon_A \circ F(Gf \circ \eta_X) \\ &= \varepsilon_A \circ FGf \circ F\eta_X \\ &= f \circ \varepsilon_{FX} \circ F\eta_X \quad (\varepsilon \text{ is nat. trans.}) \\ &= f \circ 1_{FX} \\ &= f. \end{aligned}$$

Also,

$$\begin{aligned} \widehat{\phi}_{X,A}(\psi_{X,A}(g)) = \widehat{\phi}_{X,A}(\varepsilon_A \circ Fg) &= G(\varepsilon_A \circ Fg) \circ \eta_X \\ &= G\varepsilon_A \circ GFg \circ \eta_X \\ &= G\varepsilon_A \circ \eta_{GA} \circ g \quad (\eta \text{ is nat. trans.}) \\ &= 1_{GA} \circ g \\ &= g. \end{aligned}$$

So $\widehat{\phi}$ and ψ are inverses of each other. Hence they are bijections.

$\widehat{\phi}$ is natural in X and A by the same argument in part (1). Hence $\langle F, G, \widehat{\phi}\rangle$ is an adjunction.

Conversely if we had started with an adjunction $\langle F, G, \phi\rangle$ and constructed $\widehat{\phi}$ as above from η and ε of the original adjunction; we will again obtain ϕ. This is because $\phi_{X,A}(f) = Gf \circ \eta_X = \widehat{\phi}_{X,A}(f)$ for each X in $\mathbb{X}$ and A in $\mathbb{A}$.

□

Examples 3.1.4. (1) *Let $\mathbb{A}$ be a cartesian closed category and A be an object in it. By Example 2.4.2-(1), for any B in $\mathbb{A}$, $(B^A, ev_{A,B})$ is a universal arrow from $\_ \times A$ to B. Then by Theorem 3.1.3-(4), $\_^A$ is the right adjoint of $\_ \times A$ with counit $\{ev_{A,B}\}_{B in\, \mathbb{A}}$. For any object A, exponentiation by A and taking product with A are adjoints of each other.*

(2) *By Example 2.4.2-(2) for any $\langle A, B\rangle$ in $\mathbb{A}\times\mathbb{A}$, $(A\times B, \langle p_1, p_2\rangle)$ is a universal arrow from Δ to $\langle A, B\rangle$. Then by Theorem 3.1.3-(4), product functor $P : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ which takes $\langle A, B\rangle$ to $A \times B$ is the right adjoint of $\Delta : \mathbb{A} \to \mathbb{A} \times \mathbb{A}$. Counit of the adjunction is the collection of pairs of projection arrows. Diagonal functor and product functor are adjoints of each other.*

(3) *By Example 2.4.2-(3) for any X in* **Set**, *$(FX, X \xrightarrow{\eta_X} UFX)$ is a universal arrow from X to F. By Theorem 3.1.3-(3), free monoid functor F :* **Set** → **Mon** *is the left adjoint of the forgetful functor U :* **Mon** → **Set**. *This example can be generalized to other free constructions such as free groups, free abelian groups and free modules. Taking F as the relevant free object functor, we see that F and the forgetful functor are adjoints of each other.*

(4) *A functor may have both a left and a right adjoint. As an example take the forgetful functor U :* **Top** → **Set** *from the category of topological spaces and continuous functions to* ***Set***. *U has a right adjoint T :* **Set** → **Top** *which sends every set A to the topological space A with the trivial topology. For any set A and topological space X the bijection $\phi_{X,A} : hom(UX, A) \to hom(X, TA)$ sends the function $UX \xrightarrow{f} A$ between sets to $X \xrightarrow{f} TA$ between topological spaces. f is in* **Top** *since every function with a trivial topology as codomain is continuous. It can be easily seen that $\phi_{X,A}$ is natural in X and A.*

On the other hand discrete topology functor $D : \mathbf{Top} \to \mathbf{Set}$ *is left adjoint of* U*. Similar to above, one-to-one correspondence between* $hom(DA, X)$ *and* $hom(A, UX)$ *is established by sending* $DA \xrightarrow{g} X$ *in* **Top** *to* $A \xrightarrow{g} UX$ *in* **Set**.

(5) *Let* $G : \mathbf{CpctHaus} \to \mathbf{Top}$ *be the inclusion functor from the category of compact Hausdorff topological spaces to the category of topological spaces. Then* G *has a left adjoint* $F : \mathbf{Top} \to \mathbf{CpctHaus}$ *which takes any topological space to its Stone-Čech compactification. More on this will be said after we prove Special Adjoint Functor Theorem.*

(6) *Let* P *and* Q *be two preordered sets and* $F : P \to Q$, $G : Q \to P$ *be order preserving functions. We can see* P, Q *as categories and* F, G *as functors between them. Then* F *is left adjoint to* G *if and only if for every* $a \in P$ *and* $b \in Q$, $Fa \leq b$ *if and only if* $a \leq Gb$*. Observe that this is just the statement of the adjunction in terms of a natural isomorphism. Naturality condition is trivially satisfied since every hom-set contains at most one arrow. Universal arrows induced by this adjunction are* $a \leq GFa$ *and* $FGb \leq b$ *for every* $a \in P$, $b \in Q$*. If we take image of* $a \leq GFa$ *under* F*, we obtain* $Fa \leq FGFa$*. Now considering* Fa *as element of* Q *we get* $Fa \leq FGFa \leq Fa$*. This corresponds to the triangular identity* $\varepsilon_{FA} \circ F\eta_A = 1_{FA}$ *in terms of our preorder category. Similarly we have* $Gb \leq GFGb \leq Gb$ *for the other triangular equation. In the case where* P *and* Q *are posets;* $G = GFG$ *and* $F = FGF$.

As a specific example take any two sets A, B *and consider their power sets* $\mathcal{P}(A)$ *and* $\mathcal{P}(B)$*.* $\mathcal{P}(A)$ *and* $\mathcal{P}(B)$ *are posets with* $\subseteq$ *as the relation, hence we can consider them as categories. For any* $A \xrightarrow{f} B$*, there exists* $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ *such that* $f^{-1}(B_0) = \{a \in A \mid f(a) \in B_0\}$*.* $B_0 \subseteq B_1$ *implies* $f^{-1}(B_0) \subseteq f^{-1}(B_1)$*. So* f^{-1} *is a functor from* $\mathcal{P}(B)$ *to* $\mathcal{P}(A)$*, called the inverse image functor. On the other hand we have the direct image functor* $f_* : \mathcal{P}(A) \to \mathcal{P}(B)$*, which takes* A_0 *to* $f(A_0)$*. Then* f^{-1} *is right adjoint of* f_* *since* $f_*(A_0) \subseteq B_0$ *if and only if* $A_0 \subseteq f^{-1}(B_0)$*.*

(7) *Let* $\mathbb{X}$ *be the category of abelian groups,* **Ab***, and* $\mathbb{A}$ *be the category of homomorphisms between abelian groups. Objects of* $\mathbb{A}$ *are homomorphisms* $A \xrightarrow{f} B$*. And for any two objects* $A \xrightarrow{f} B$ *and* $A' \xrightarrow{f'} B'$ *in* $\mathbb{A}$ *an arrow from* f *to* f' *is*

a pair $\langle g, h \rangle : f \longrightarrow f'$ *consisting of two functions* $A \xrightarrow{g} A'$ *and* $B \xrightarrow{h} B'$ *which satisfy* $h \circ f = f' \circ g$.

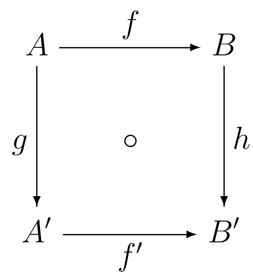


Figure 3.13.

Now let $G : \mathbb{A} \to \mathbb{X}$ *be the kernel functor which takes any* $\langle g, h \rangle : f \longrightarrow f'$ *to* $Kerf \xrightarrow{g} Kerf'$. *And let* $F : \mathbb{X} \to \mathbb{A}$ *be the functor which maps any abelian group* X *to the homomorphism* $X \xrightarrow{0_X} 0$ *in* $\mathbb{A}$. *In diagrams,*

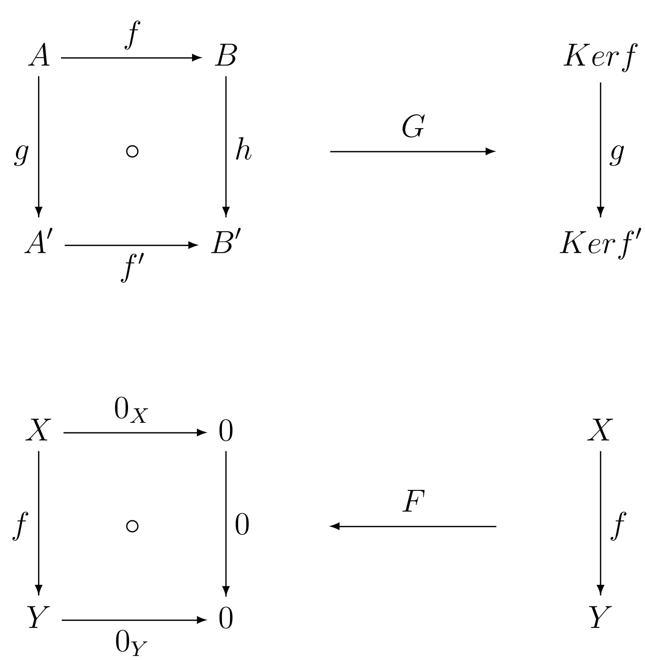


Figure 3.14.

Then F *is left adjoint of* G *with unit* $\eta = \{1_X\}_{X \, in \, \mathbb{X}}$ *from* $1_{\mathbb{X}}$ *to* GF. *Clearly* η *is a natural transformation and for each abelian group* X, η_X *is universal from* X *to* G. *This is shown by the diagram below.*

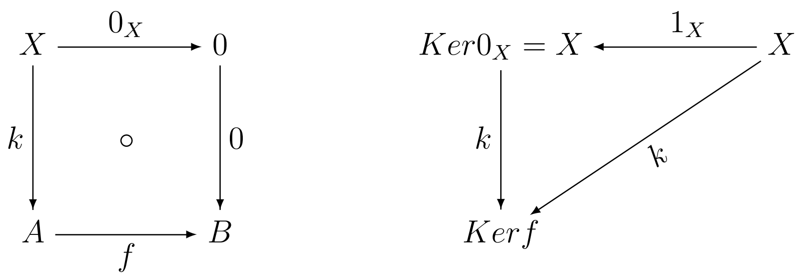


Figure 3.15.

Then Theorem 3.1.3-(1) ensures the existence of the claimed adjunction. While unit of the adjunction is the collection of identity homomorphisms, counit is the collection of embeddings of kernels into their domains.

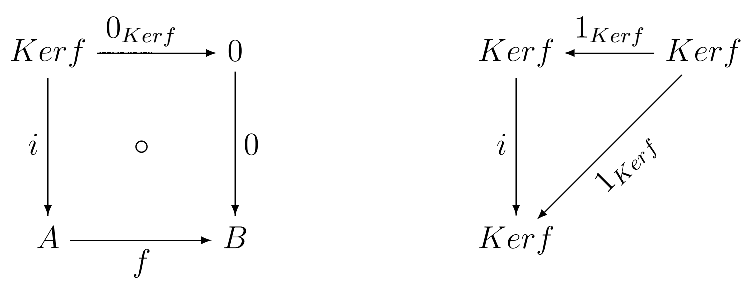


Figure 3.16.

Proposition 3.1.5. *If $F : \mathbb{X} \to \mathbb{A}$ and $F' : \mathbb{X} \to \mathbb{A}$ are both left adjoints of $G : \mathbb{A} \to \mathbb{X}$ then there is a natural isomorphism $\tau : F \to F'$.*

Proof. Let $\langle F, G, \eta, \varepsilon \rangle$ and $\langle F', G, \eta', \varepsilon' \rangle$ be the adjunctions mentioned above. Then for any X in $\mathbb{X}$ we will have two universal arrows η_X and ${\eta'}_X$ from X to G. And since universal arrows are unique up to isomorphism, there exists an iso $FX \xrightarrow{\tau_X} F'X$ such that $G\tau_X \circ \eta_X = {\eta'}_X$.

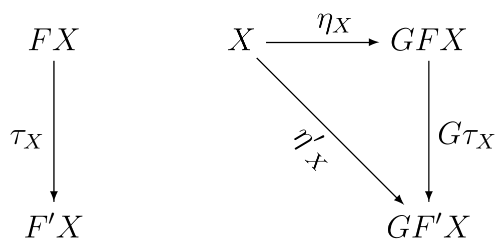


Figure 3.17.

Our claim is that $\{\tau_X\}_{X \text{ in } \mathbb{X}}$ is a natural transformation from F to F'. For this we need to show that the following square commutes for any $Y \xrightarrow{f} Z$, i.e. $\tau_Z \circ Ff = F'f \circ \tau_Y$.

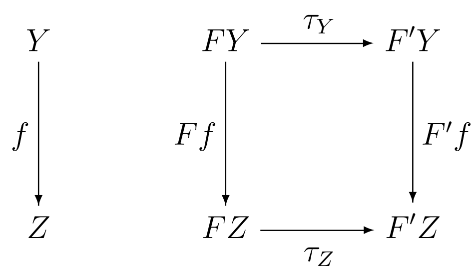


Figure 3.18.

Observe that if $G(\tau_Z \circ Ff) \circ \eta_Y = G(F'f \circ \tau_Y) \circ \eta_Y$ then $\tau_Z \circ Ff = F'f \circ \tau_Y$. This is because in such a case we will have $\phi_{Y,F'Z}(\tau_Z \circ Ff) = \phi_{Y,F'Z}(F'f \circ \tau_Y)$. We show that $G(\tau_Z \circ Ff) \circ \eta_Y = G(F'f \circ \tau_Y) \circ \eta_Y$ by the commutative diagrams below.

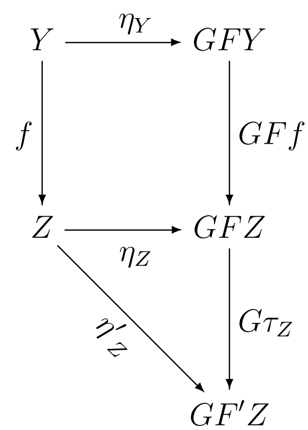


Figure 3.19.

The upper square commutes since η is a natural transformation and the lower triangle commutes by the fact represented in Figure 3.17. So $G(\tau_Z \circ Ff) \circ \eta_Y = \eta'_Z \circ f$. By a similar reasoning the following is a commutative diagram.

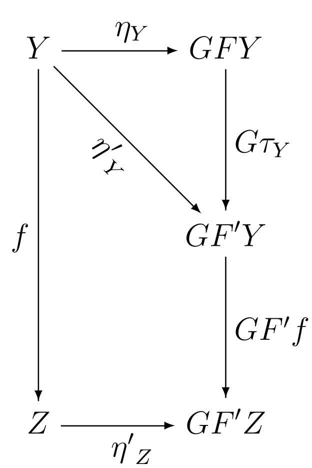


Figure 3.20.

Hence $G(F'f \circ \tau_Y) \circ \eta_Y = \eta'_Z \circ f$. So $\tau_Z \circ Ff = F'f \circ \tau_Y$ and $\tau : F \to F'$ is a natural isomorphism. $\square$

Dually if G and G' are right adjoints of F then G and G' are naturally isomorphic.

4. LIMITS AND ADJOINTS

This is the final chapter where our target theorems Freyd's Adjoint Functor Theorem and Special Adjoint Functor Theorem are proved. These theorems characterize the functors with left adjoints and in doing so they reveal a connection between limits and adjoints. But in order to understand these important theorems we first need to observe some facts about limits.

4.1. Limits

Definition 4.1.1. *Let* $F : \mathbb{I} \to \mathbb{A}$ *be a functor.* A cone on F *consists of an object* A *of* $\mathbb{A}$ *and a family of arrows* $\{f_X : A \to FX\}_{X \, in \, \mathbb{I}}$ *such that for any* $X \xrightarrow{g} Y$ *in* $\mathbb{I}$, $Fg \circ f_X = f_Y$.
In diagram,

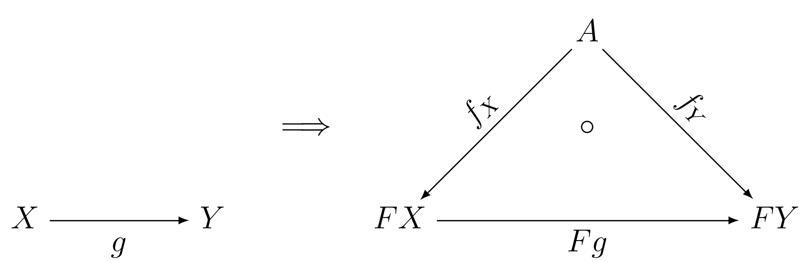


Figure 4.1.

Definition 4.1.2. *Let* $F : \mathbb{I} \to \mathbb{A}$ *be a functor. A cone* $(A, \{f_X\}_{X \, in \, \mathbb{I}})$ *on* F *is called a* limit of F *if for any cone* $(B, \{h_X\}_{X \, in \, \mathbb{I}})$ *on* F *there exists a unique arrow* $B \xrightarrow{k} A$ *such that* $f_x \circ k = h_X$ *for any* X *in* $\mathbb{X}$.

Remark 4.1.3. *Limits are unique up to isomorphism.*

Some of the notions previously introduced are well known examples of limits. For example, taking $\mathbb{I}$ as a two object discrete category gives us a product of two objects in $\mathbb{A}$. Taking $\mathbb{I}$ as $\cdot \rightrightarrows \cdot$ with two objects and two nonidentity arrows induces equalizers, and taking it as $\cdot \longrightarrow \cdot \longleftarrow \cdot$ with two arrows with common codomain induces pullbacks.

Definition 4.1.4. *A category* $\mathbb{A}$ *is called* small complete *if every functor* $F : \mathbb{I} \to \mathbb{A}$ *from a small category* $\mathbb{I}$ *has a limit in* $\mathbb{A}$*.*

The next theorem describes the creation of limits by products and equalizers.

Theorem 4.1.5. *If* $\mathbb{A}$ *has all small products and equalizers for every parallel pair of arrows then* $\mathbb{A}$ *is small complete.*

Proof. Let $\mathbb{A}$ have all small products and equalizers for every parallel pairs of arrows. And suppose that $F : \mathbb{I} \to \mathbb{A}$ is a functor from a small category $\mathbb{I}$. We need to show that F has a limit in $\mathbb{A}$.

Since $\mathbb{I}$ is small, the collections $\{FX\}_{X \text{ in } \mathbb{I}}$ and $\{FX \xrightarrow{Fg} FY\}_{g \text{ in } \mathbb{I}}$ are sets in $\mathbb{A}$.
By our assumption we can form the products $\left(\prod_{X \text{ in } \mathbb{I}} FX, \{p_X\}_{X \text{ in } \mathbb{I}}\right)$ and $\left(\prod_{g \text{ in } \mathbb{I}} F(codg), \{q_g\}_{g \text{ in } \mathbb{I}}\right)$ respectively.
For each $X \xrightarrow{g} Y$ in $\mathbb{I}$, there is an arrow $p_Y : \prod_{X \text{ in } \mathbb{I}} FX \to FY = F(codg)$. Hence by definition of products there is a unique arrow $r : \prod_{X \text{ in } \mathbb{I}} FX \to \prod_{g \text{ in } \mathbb{I}} F(codg)$ such that $q_g \circ r = p_Y$ for any g in $\mathbb{I}$. Also $X \xrightarrow{g} Y$ implies the existence of the arrow $Fg \circ p_X : \prod_{X \text{ in } \mathbb{I}} FX \to FY$. Similarly there exists a unique arrow $s : \prod_{X \text{ in } \mathbb{I}} FX \to \prod_{g \text{ in } \mathbb{I}} F(codg)$ such that $Fg \circ p_X = q_g \circ s$. These arrows are shown in the diagram below.

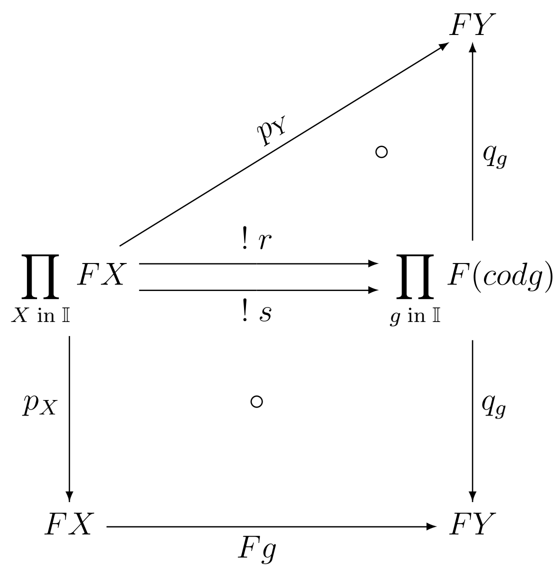


Figure 4.2.

Now r and s constitute a pair of parallel arrows. By our assumption they have an equalizer l. Our claim is that $(L, \{p_X \circ l\}_{X \text{in } \mathbb{I}})$ is a cone on F. If we denote $p_X \circ l$ by π_X what we have to show is that $Fg \circ \pi_X = \pi_Y$ for any $X \xrightarrow{g} Y$ in $\mathbb{I}$.

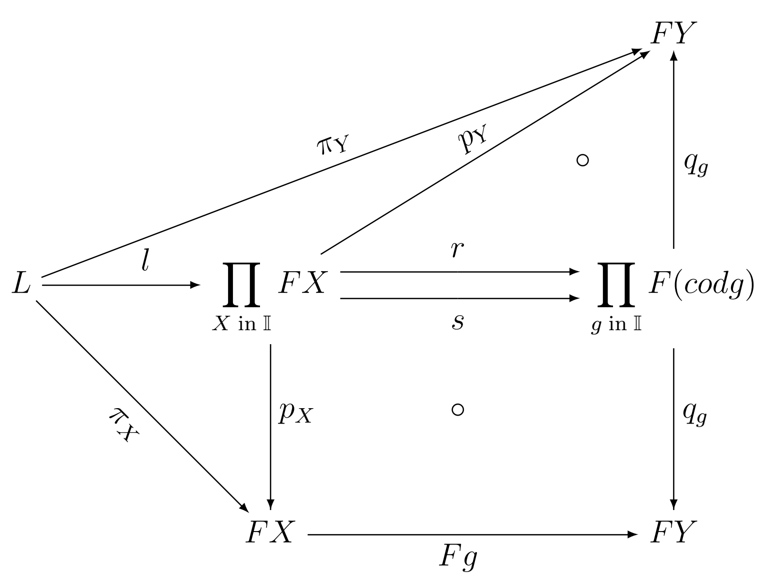


Figure 4.3.

This can be shown as follows:

$$\begin{aligned} Fg \circ \pi_X = Fg \circ (p_X \circ l) &= (Fg \circ p_X) \circ l \\ &= (q_g \circ s) \circ l \\ &= q_g \circ (s \circ l). \end{aligned}$$

Since l equalizes r and s, we continue as,

$$\begin{aligned} q_g \circ (s \circ l) &= q_g \circ (r \circ l) \\ &= (q_g \circ r) \circ l \\ &= p_Y \circ l \\ &= \pi_Y. \end{aligned}$$

Hence $Fg \circ \pi_X = \pi_Y$ and $(L, \{\pi_X\}_{X \text{in } \mathbb{I}})$ is a cone on F.
Now suppose that $(L', \{\pi'_X\}_{X \text{in } \mathbb{I}})$ is another cone on F. Then since $\prod_{X \text{ in } \mathbb{I}} FX$ is a product, there exists a unique $L' \xrightarrow{u} \prod_{X \text{ in } \mathbb{I}} FX$ such that $p_X \circ u = \pi'_X$ for every X in $\mathbb{I}$.

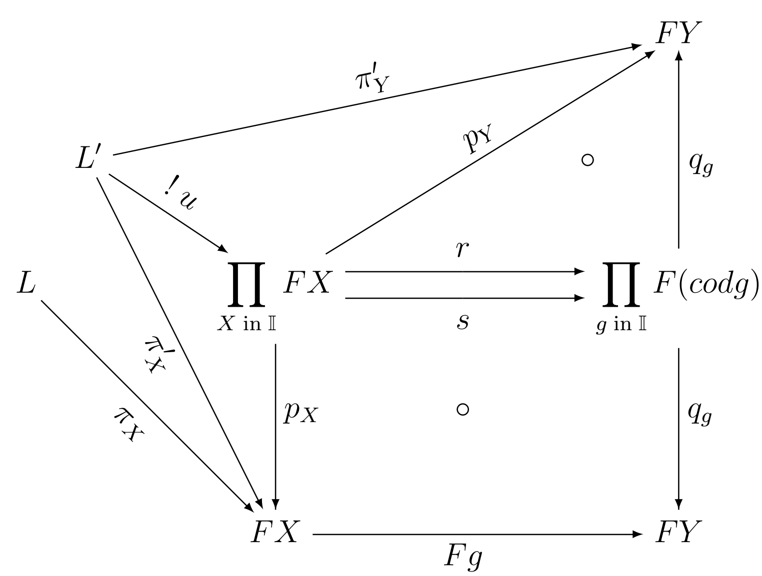


Figure 4.4.

Since $(L', \{\pi'_X\}_{X \text{in } \mathbb{I}})$ is a cone on F, $Fg \circ \pi'_X = \pi'_Y$. Hence

$$\begin{aligned} \pi'_Y = Fg \circ \pi'_X = Fg \circ (p_X \circ u) &= Fg \circ (p_X \circ u) \\ &= (q_g \circ s) \circ u \\ &= q_g \circ (s \circ u). \end{aligned}$$

And $\pi'_Y = p_Y \circ u = (q_g \circ r) \circ u = q_g \circ (r \circ u)$.

So $s \circ u$ and $r \circ u$ have the same composites with all projection arrows q_g of $\prod_{g \text{ in } \mathbb{I}} F(codg)$.

Then $s \circ u = r \circ u$; u equalizes r and s.

Since l is an equalizer of r and s, there exists a unique $L' \xrightarrow{k} L$ such that $l \circ k = u$.

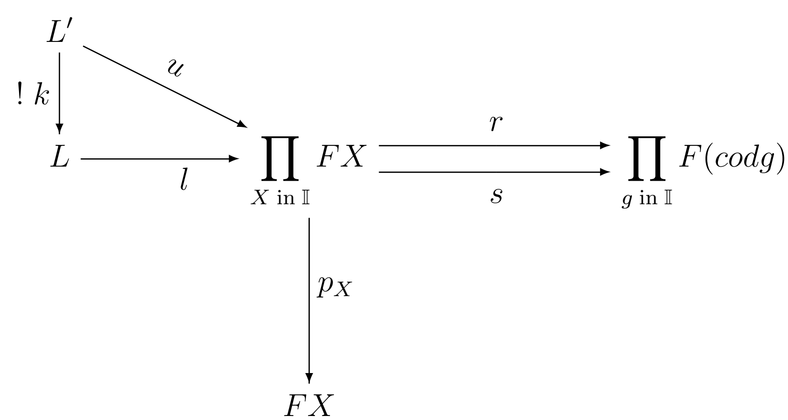


Figure 4.5.

Hence $\pi'_X = p_X \circ u = p_X \circ (l \circ k) = (p_X \circ l) \circ k = \pi_X \circ k$ for any X in $\mathbb{I}$. So for any other cone $(L', \{\pi'_X\}_{X \text{in } \mathbb{I}})$ on F there exists an arrow $L' \xrightarrow{k} L$ such that $\pi'_X = \pi_X \circ k$ for every X in $\mathbb{I}$.

For $(L, \{\pi_X\}_{X \text{in } \mathbb{I}})$ to be a limit of F, this arrow k should be unique. For any other $L' \xrightarrow{w} L$ such that $\pi'_X = \pi_X \circ w$ we have

$$\pi'_X = \pi_X \circ w = (p_X \circ l) \circ w = p_X \circ (l \circ w).$$

Also $\pi'_X = p_X \circ u$. Since $l \circ w$ and u have the same composites with the projection arrows p_X, $l \circ w = u$. Hence $l \circ w = l \circ k$.

But l is an equalizer, so u factors through l uniquely. This implies that $k = w$. Therefore k is unique. □

Definition 4.1.6. *A functor* $F : \mathbb{Y} \to \mathbb{A}$ preserves limits of $G : \mathbb{I} \to \mathbb{Y}$ *if whenever* $(Y, \{g_X\}_{X in\, \mathbb{I}})$ *is a limit of* G *in* $\mathbb{Y}$, $(FY, \{Fg_X\}_{X in\, \mathbb{I}})$ *is a limit of* $F \circ G$ *in* $\mathbb{A}$.

Definition 4.1.7. *A functor* $F : \mathbb{Y} \to \mathbb{A}$ creates limits for $G : \mathbb{I} \to \mathbb{Y}$ *if it satisfies the following condition.*
For any $(A, \{f_X\}_{X in\, \mathbb{I}})$*, a limit of* $F \circ G$ *in* $\mathbb{A}$*, there exists a unique cone* $(Y, \{g_X\}_{X in\, \mathbb{I}})$ *on* G *in* $\mathbb{X}$ *such that* $F(Y) = A$ *and* $F(g_X) = f_X$ *for* X *in* $\mathbb{X}$*. Moreover this* $(Y, \{g_X\}_{X in\, \mathbb{I}})$ *is a limit of* G *in* $\mathbb{X}$*.*

Theorem 4.1.8. *If* $F : \mathbb{Y} \to \mathbb{A}$ *creates limits for* $G : \mathbb{I} \to \mathbb{Y}$ *and* $F \circ G : \mathbb{I} \to \mathbb{A}$ *has a limit in* $\mathbb{A}$ *then* F *preserves limits of* G*.*

Proof. Assume that $F : \mathbb{Y} \to \mathbb{A}$ creates limits for $G : \mathbb{I} \to \mathbb{Y}$ and $F \circ G : \mathbb{I} \to \mathbb{A}$ has a limit $(A, \{f_X\}_{X \text{ in } \mathbb{I}})$ in $\mathbb{A}$. Then since F creates limits for G, there exists a unique cone $(Y, \{g_X\}_{X \text{ in } \mathbb{I}})$ in $\mathbb{Y}$ whose image under F is $(A, \{f_X\}_{X \text{ in } \mathbb{I}})$ in $\mathbb{A}$. Also this $(Y, \{g_X\}_{X \text{ in } \mathbb{I}})$ is a limit of G.
Let $(Y', \{g'_X\}_{X \text{ in } \mathbb{I}})$ be also a limit of G. We have to show that the image of $(Y', \{g'_X\}_{X \text{ in } \mathbb{I}})$ under F, $(FY', \{Fg'_X\}_{X \text{ in } \mathbb{I}})$, is a limit of $F \circ G$ in $\mathbb{A}$. Since limits are unique up to isomorphism there exists a unique isomorphism $Y' \xrightarrow{u} Y$ in $\mathbb{Y}$ such that $g'_X = g_X \circ u$ for every X in $\mathbb{I}$. And since functors preserve composition $(FY', \{Fg'_X\}_{X \text{ in } \mathbb{I}})$ is a cone on $F \circ G$. Also $FY' \xrightarrow{Fu} FY$ is an isomorphism satisfying $Fg'_X = Fg_X \circ Fu = f_X \circ Fu$ for every X in $\mathbb{I}$. Since limits are unique up to isomorphism $(FY', \{Fg'_X\}_{X \text{ in } \mathbb{I}})$ is a limit of $F \circ G$. □

A functor that has a left adjoint preserves all limits in its domain category. This is a useful criterion to check existence of left adjoints. If a functor does not preserve some limit then surely it can not have a left adjoint.

Theorem 4.1.9. *Let* $G : \mathbb{A} \to \mathbb{X}$ *be a functor having a left adjoint. Then for any* $S : \mathbb{I} \to \mathbb{A}$ *and* $(A, \{f_X\}_{X in\, \mathbb{I}})$*, a limit of* S *in* $\mathbb{A}$*, its image under* G*,* $(GA, \{Gf_X\}_{X in\, \mathbb{I}})$*, is a limit of* $G \circ S$ *in* $\mathbb{X}$*.*

Proof. Let $\langle F, G, \phi\rangle$ be the adjunction mentioned in the theorem. And let $S : \mathbb{I} \to \mathbb{A}$ be any functor from $\mathbb{I}$ to $\mathbb{A}$ with its limit $(A, \{f_X\}_{X \text{ in } \mathbb{I}})$ in $\mathbb{A}$. Then clearly $(GA, \{Gf_X\}_{X \text{ in } \mathbb{I}})$ is a cone on $G \circ S$ in $\mathbb{X}$.

To see that it is a limit of $G \circ S$, let $(Z, \{h_X\}_{X \text{ in } \mathbb{I}})$ be another cone on $G \circ S$. Then $(FZ, \{\phi^{-1}(h_X)\}_{X \text{ in } \mathbb{I}})$ is a cone on S by the adjunction isomorphism ϕ^{-1}.

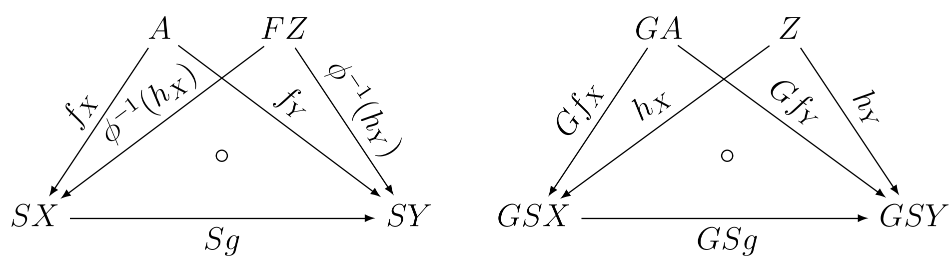


Figure 4.6.

This can be shown as follows:

Since $(Z, \{h_X\}_{X \text{ in } \mathbb{I}})$ is a cone on $G \circ S$, we have $GSg \circ h_X = h_Y$ for any $X \xrightarrow{g} Y$ in $\mathbb{I}$. Hence $\phi^{-1}(GSg \circ h_X) = \phi^{-1}(h_Y)$. By naturality of ϕ^{-1},

$$\phi^{-1}(GSg \circ h_X) = Sg \circ \phi^{-1}(h_X).$$

So $Sg \circ \phi^{-1}(h_X) = \phi^{-1}(h_Y)$ for any $X \xrightarrow{g} Y$ in $\mathbb{I}$, and $(FZ, \{\phi^{-1}(h_X)\}_{X \text{ in } \mathbb{I}})$ is a cone on S.

Now since $(A, \{f_X\}_{X \text{ in } \mathbb{I}})$ in $\mathbb{A}$ is a limit of S, there exists a unique $FZ \xrightarrow{k} A$ such that $f_X \circ k = \phi^{-1}(h_X)$ for any X in $\mathbb{I}$. By adjunction isomorphism ϕ, $\phi(k)$ is an arrow from Z to GA and we claim that $\phi(k)$ is the unique arrow for which $Gf_X \circ \phi(k) = h_X$ holds for every X in $\mathbb{I}$.

Observe that proof of this claim will give us that $(GA, \{Gf_X\}_{X \text{ in } \mathbb{I}})$, is a limit of $G \circ S$.

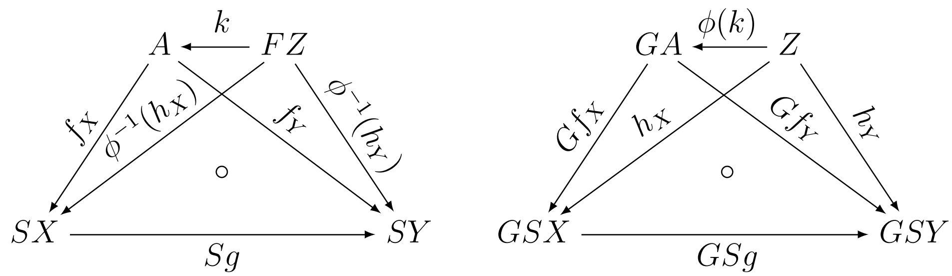


Figure 4.7.

$f_X \circ k = \phi^{-1}(h_X)$ for any X in $\mathbb{I}$, taking their image under ϕ we have $\phi(f_X \circ k) = \phi(\phi^{-1}(h_X)) = h_X$. By the naturality of ϕ, $\phi(f_X \circ k) = Gf_X \circ \phi(k)$. So $Gf_X \circ \phi(k) = h_X$ for any X in $\mathbb{I}$.

For the uniqueness part assume that there exists an arrow $Z \xrightarrow{w} GA$ such that $Gf_X \circ w = h_X$ for every X in $\mathbb{I}$. Then by a similar argument $FZ \xrightarrow{\phi^{-1}(w)} A$ satisfies $f_X \circ \phi^{-1}(w) = \phi^{-1}(h_X)$ for every X in $\mathbb{I}$. But since $(A, \{f_X\}_{X \text{ in } \mathbb{I}})$ is a limit of S in $\mathbb{A}$, k is the unique such arrow. So $\phi^{-1}(w) = k$ and hence $w = \phi(\phi^{-1}(w)) = \phi(k)$. Therefore $\phi(k)$ is unique. $\square$

4.2. Freyd's Adjoint Functor Theorem

In this section we will prove Freyd's Adjoint Functor Theorem. We access to the proof by the help of some preliminary theorems.

Theorem 4.2.1. *Let $\mathbb{B}$ be a category with small hom-sets which is small complete. Then $\mathbb{B}$ has an initial object if and only if it satisfies the*
Solution Set Condition: There exists a small collection of objects, $\{B_i\}_{i\in I}$ of $\mathbb{B}$ such that for any C in $\mathbb{B}$ there exists an arrow $B_j \xrightarrow{k} C$ in $\mathbb{B}$ for some $B_j \in \{B_i\}_{i\in I}$.

Proof. If $\mathbb{B}$ has an initial object say D, then clearly $\{D\}$ satisfies the solution set condition.

Conversely assume that the solution set condition holds for $\mathbb{B}$. Since $\{B_i\}_{i\in I}$ is a small collection and $\mathbb{B}$ is small complete, we can form the product $\prod_{i\in I} B_i$ in $\mathbb{B}$. Observe that for any C in $\mathbb{B}$ there exists an arrow from $\prod_{i\in I} B_i$ to C by the composition with projection arrows.

$$\prod_{i\in I} B_i \longrightarrow B_j \longrightarrow C$$

Figure 4.8.

Since $\mathbb{B}$ is a category with small hom-sets, $hom(\prod_{i\in I} B_i, \prod_{i\in I} B_i)$ is small. Hence there is an equalizer $E \xrightarrow{e} \prod_{i\in I} B_i$ of all arrows in $hom(\prod_{i\in I} B_i, \prod_{i\in I} B_i)$.

Our claim is that E is an initial object of $\mathbb{B}$.

For any C in $\mathbb{B}$ there is an arrow from E to C by the composition of arrows

$$E \xrightarrow{e} \prod_{i\in I} B_i \longrightarrow B_j \longrightarrow C$$

Figure 4.9.

What we need to show is that there is a unique arrow from E to C. To show this, let $E \underset{n}{\overset{m}{\rightrightarrows}} C$ be two arrows from E to C. Since $\mathbb{B}$ is small complete, they have an equalizer $F \xrightarrow{f} E \underset{n}{\overset{m}{\rightrightarrows}} C$. And we know that there is an arrow $\prod_{i\in I} B_i \xrightarrow{u} F$. Now, the composite $e \circ f \circ u$ is an arrow from $\prod_{i\in I} B_i$ to $\prod_{i\in I} B_i$.

$$\prod_{i\in I} B_i \xrightarrow{u} F \xrightarrow{f} E \xrightarrow{e} \prod_{i\in I} B_i$$

Figure 4.10.

Since $1_{(\prod_{i\in I} B_i)}$ is an arrow in $hom\big(\prod_{i\in I} B_i, \prod_{i\in I} B_i\big)$ and e is their equalizer, we can write

$$\begin{aligned}(e\circ f\circ u)\circ e &= 1_{(\prod_{i\in I} B_i)}\circ e\\ (e\circ f\circ u)\circ e &= e\\ e\circ(f\circ u\circ e) &= e\circ 1_E\end{aligned}$$

Since e is an equalizer it is a monic, hence left cancelable. So we get $f\circ(u\circ e) = 1_E$. But f is an equalizer, therefore a monic, which has a right inverse $u\circ e$. This implies that f is an iso, and $m\circ f = n\circ f$ yields $m = n$. So there exists a unique arrow from E to C. Therefore E is an initial object of $\mathbb{B}$. □

Lemma 4.2.2. *Let $G : \mathbb{A} \to \mathbb{X}$ preserve all small limits. Then for any X in $\mathbb{X}$, $P : (X \downarrow G) \to \mathbb{A}$ creates all small products and equalizers for any parallel pair of arrows.*

Proof. Let $\mathbb{I}$ be a small, discrete category and $H : \mathbb{I} \to (X \downarrow G)$ be a functor. Then the image of $\mathbb{I}$ under H will be a collection of objects $\{X \xrightarrow{f_i} GA_i\}_{i \text{ in } \mathbb{I}}$ in $(X \downarrow G)$. Since $P\circ H : \mathbb{I} \to \mathbb{A}$, where $\mathbb{I}$ is discrete, the limit of $P\circ H$ is a small product in $\mathbb{A}$. Let's denote this product by $\prod_{i \text{ in } \mathbb{I}} A_i$ with projection arrows $\prod_{i \text{ in } \mathbb{I}} A_i \xrightarrow{q_j} A_j$. Since G preserves all small limits, in particular small products, $\big(G\prod_{i \text{ in } \mathbb{I}} A_i, \{G\prod_{i \text{ in } \mathbb{I}} A_i \xrightarrow{Gq_j} GA_j\}_{j \text{ in } \mathbb{I}}\big)$ is a product of GA_i's in $\mathbb{X}$.

Now, for each A_j, j in $\mathbb{I}$, there exists an arrow $X \xrightarrow{f_j} GA_j$ in $\mathbb{X}$. But since $G\prod_{i \text{ in } \mathbb{I}} A_i$ is the product of GA_i's in $\mathbb{X}$, there exists a unique arrow $X \xrightarrow{f} G\prod_{i \text{ in } \mathbb{I}} A_i$ such that $Gq_j \circ f = f_j$ for any j in $\mathbb{I}$, i.e. in $\mathbb{X}$ we have the following diagram for every j in $\mathbb{I}$.

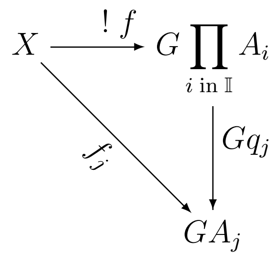


Figure 4.11.

Considering $X \xrightarrow{f} G\prod_{i \text{ in } \mathbb{I}} A_i$ as an object of $(X \downarrow G)$, we see that $(f, \{f \xrightarrow{q_j} f_j\}_{j \text{ in } \mathbb{I}})$ is a cone on H. Furthermore it is the unique cone whose image under P is $\big(\prod_{i \text{ in } \mathbb{I}} A_i, \{\prod_{i \text{ in } \mathbb{I}} A_i \xrightarrow{q_j} A_j\}_{j \text{ in } \mathbb{I}}\big)$.
To show that it is a limit, let $(h, \{h \xrightarrow{r_j} f_j\}_{j \text{ in } \mathbb{I}})$ be another cone on H, i.e. in $\mathbb{X}$ we have the following diagram with $Gr_j \circ h = f_j$ for all j in $\mathbb{I}$.

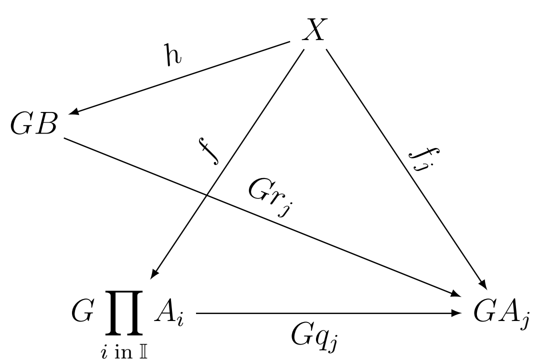


Figure 4.12.

Then in $\mathbb{A}$ there exists $B \xrightarrow{r_j} A_j$ for any j in $\mathbb{I}$. Since $\prod_{i \text{ in } \mathbb{I}} A_i$ is a product, there exists a unique $B \xrightarrow{s} \prod_{i \text{ in } \mathbb{I}} A_i$ such that $q_j \circ s = r_j$ for every j in $\mathbb{I}$. Taking the image of this under G, we get $Gq_j \circ Gs = Gr_j$.

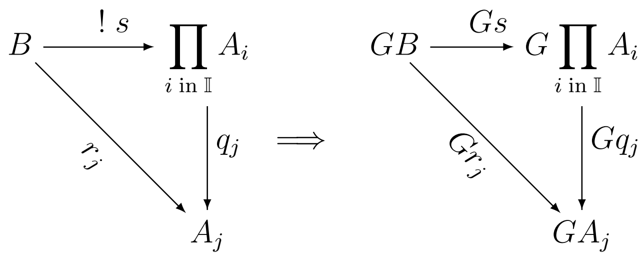


Figure 4.13.

Taking composite of $(Gs \circ h)$ with Gq_j's we see that

$$\begin{aligned} Gq_j \circ (Gs \circ h) &= (Gq_j \circ Gs) \circ h \\ &= Gr_j \circ h \\ &= f_j \\ &= Gq_j \circ f. \end{aligned}$$

So $Gs \circ h$ and f give the same arrows when composed with the projection arrows q_j of the product $G \prod_{i \text{ in } \mathbb{I}} A_i$. Hence $Gs \circ h = f$ and s is an arrow from h to f in $(X \downarrow G)$. This implies that for $(h, \{h \xrightarrow{r_j} f_j\}_{j \text{ in } \mathbb{I}})$ on H, there exists a unique $h \xrightarrow{s} f$ such that $q_j \circ s = r_j$ for every j in $\mathbb{I}$.

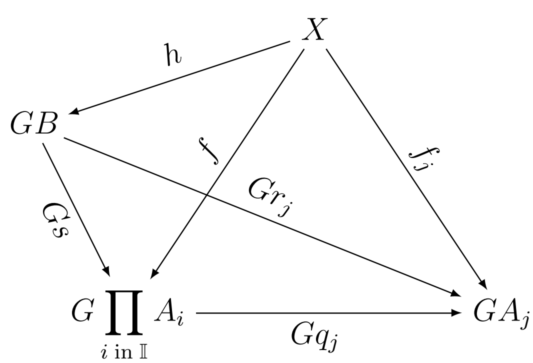


Figure 4.14.

So $(f, \{q_j\}_{j \text{ in } \mathbb{I}})$ is a limit of H in $(X \downarrow G)$. Therefore $P : (X \downarrow G) \to \mathbb{A}$ creates small products.

Now we consider the case with equalizers. Let $\mathbb{J}$ be the category consisting of two objects and two nonidentity arrows between them. Also let $K : \mathbb{J} \to (X \downarrow G)$ be a functor. Then the image of $\mathbb{J}$ under K consists of arrows $u \underset{g}{\overset{f}{\rightrightarrows}} w$ in $(X \downarrow G)$. In diagrams we have,

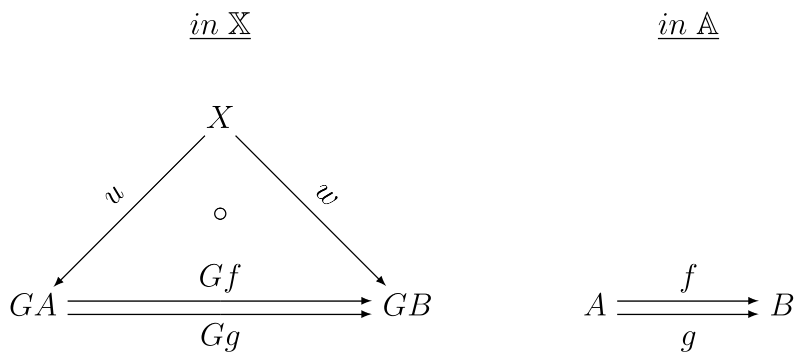


Figure 4.15.

To show that P creates equalizers, suppose that f and g in $\mathbb{A}$ has an equalizer, let's call it e.

$$E \xrightarrow{\;e\;} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

Figure 4.16.

Then since G preserves all small limits and hence equalizers, Ge is an equalizer of Gf and Gg.

$$GE \xrightarrow{\;Ge\;} GA \underset{Gg}{\overset{Gf}{\rightrightarrows}} GB$$

Figure 4.17.

In $\mathbb{X}$ we have the following picture,

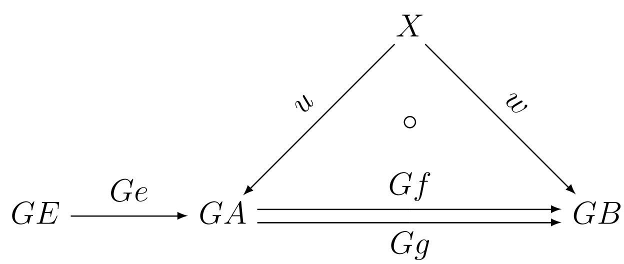


Figure 4.18.

Since $Gf \circ u = Gg \circ u = w$, also u equalizes Gf and Gg. But Ge is an equalizer, hence there exists a unique arrow $X \xrightarrow{z} GE$ such that $Ge \circ z = u$.

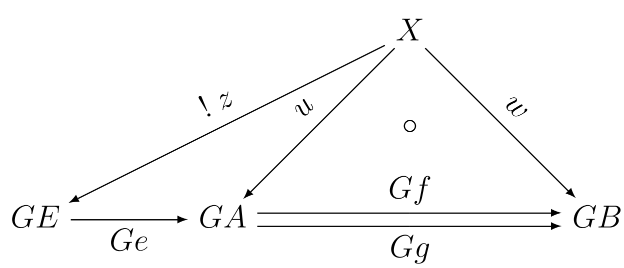


Figure 4.19.

This means that in $(X \downarrow G)$, there exists an object z and an arrow $z \xrightarrow{e} u$ such that $f \circ e = g \circ e$. Hence e equalizes f and g in $(X \downarrow G)$.

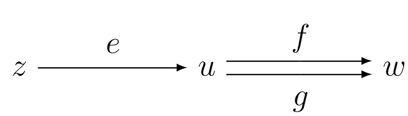
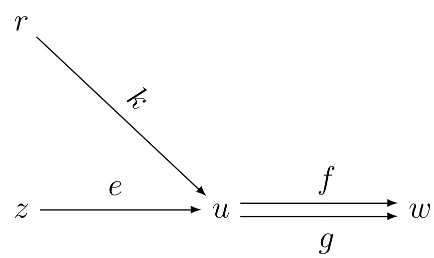


Figure 4.20.

Now let $r \xrightarrow{k} u$ also equalize f and g.

$\underline{in\ (X \downarrow G)}$



$\underline{in\ \mathbb{X}}$

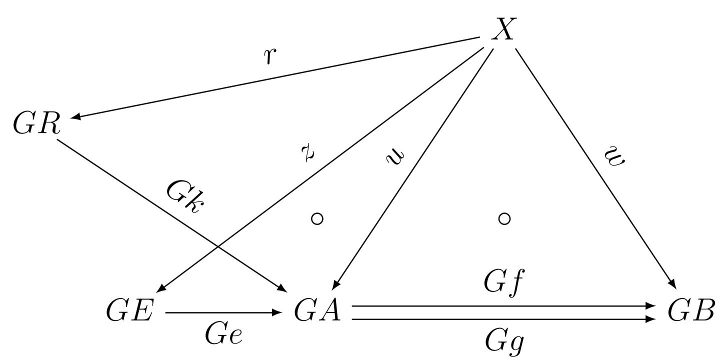


Figure 4.21.

So there exists an arrow $R \xrightarrow{k} A$ in $\mathbb{A}$ such that $f \circ k = g \circ k$ and also $Gk \circ r = u$. Since k equalizes f and g and e is an equalizer of them there exists a unique $R \xrightarrow{s} E$ such that $e \circ s = k$.

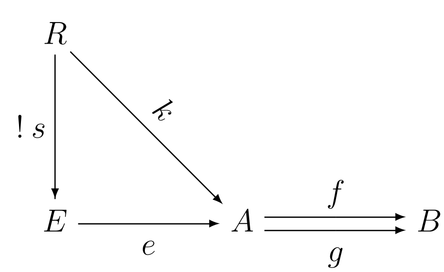


Figure 4.22.

Taking its image under G, we obtain $Ge \circ Gs = Gk$.

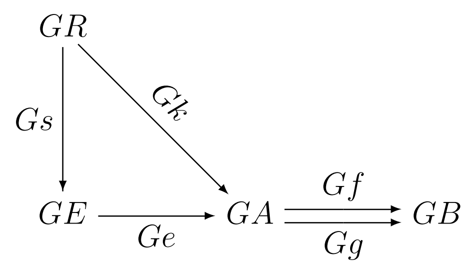


Figure 4.23.

So there is an arrow $GR \xrightarrow{Gs} GE$ for which $e \circ s = k$. In $\mathbb{X}$ we have,

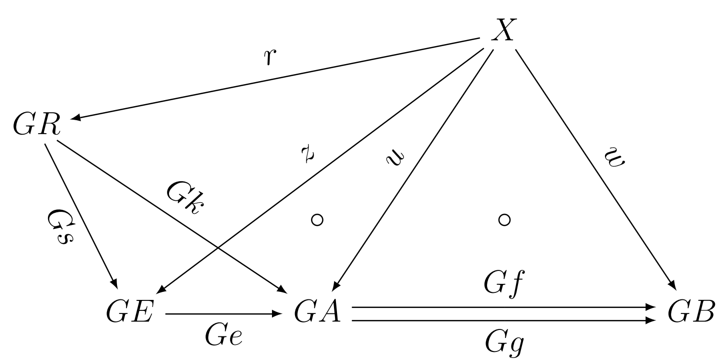


Figure 4.24.

If we show that s is an arrow from r to z in $(X \downarrow G)$, i.e. $Gs \circ r = z$, then we will be done. In such a situation s will be the unique arrow from r to z for which $e \circ s = k$.

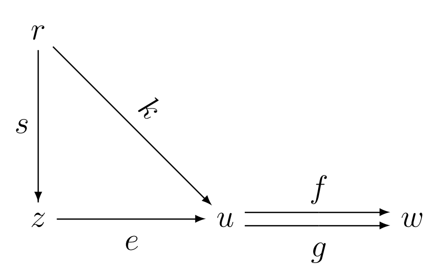


Figure 4.25.

Now,

$$\begin{aligned} Ge \circ (Gs \circ r) &= (Ge \circ Gs) \circ r \\ &= Gk \circ r \\ &= u. \end{aligned}$$

But also $Ge \circ z = u$. So we have the following commutative diagram in $\mathbb{X}$.

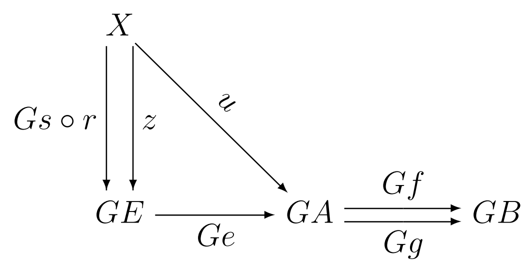


Figure 4.26.

Since Ge is an equalizer, $Gs \circ r = z$. Therefore s is an arrow from r to z and $z \xrightarrow{e} u$ is an equalizer of $u \underset{g}{\overset{f}{\rightrightarrows}} w$ in $(X \downarrow G)$ whose image under P is $E \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} B$.
So P creates equalizers for any parallel pair of arrows. □

By a similar reasoning as in the above proof $P : (X \downarrow G) \to \mathbb{A}$ creates all small limits.

Corollary 4.2.3. *Let $\mathbb{A}$ be a small complete category and $G : \mathbb{A} \to \mathbb{X}$ preserve all small limits. Then $(X \downarrow G)$ is small complete.*

Proof. By the Theorem 4.1.5, if $(X \downarrow G)$ has all small products and equalizers for any parallel pair of arrows, then $(X \downarrow G)$ is small complete.
Now, for any small collection $\{X \longrightarrow GA_i\}$ of objects in $(X \downarrow G)$, their image under P is a small collection A_i of objects in $\mathbb{A}$. Since $\mathbb{A}$ is small complete they have a product in $\mathbb{A}$. By the above lemma P creates small products in $(X \downarrow G)$. So the collection $\{X \longrightarrow GA_i\}$ has a product in $(X \downarrow G)$. Hence $(X \downarrow G)$ has all small products.

By the similar reasoning every parallel pair of arrows in $(X \downarrow G)$ has an equalizer. Hence $(X \downarrow G)$ is small complete. □

Theorem 4.2.4. (Freyd's Adjoint Functor Theorem) *Let $\mathbb{A}$ be a small complete category with small hom-sets. Then a functor $G : \mathbb{A} \to \mathbb{X}$ has a left adjoint if and only if it preserves all small limits and satisfies the following condition.*
For each X in $\mathbb{X}$ there exists a small collection of arrows $\{X \xrightarrow{f_i} GA_i\}_{i \ in\ \mathbb{I}}$ such that for any $X \xrightarrow{h} GB$ in $\mathbb{X}$, there exists a j in $\mathbb{I}$ and an arrow $A_j \xrightarrow{k} B$ in $\mathbb{A}$ such that $Gk \circ f_j = h$

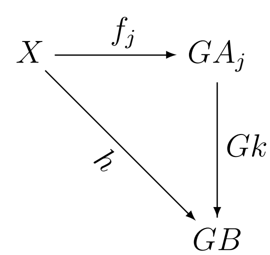


Figure 4.27.

Proof. Suppose that $G : \mathbb{A} \to \mathbb{X}$ has a left adjoint. Then it preserves all limits by Theorem 4.1.9. And for each X in $\mathbb{X}$, $\{X \xrightarrow{\eta_X} GFX\}$ satisfies the condition stated since it is universal from X to G.
Conversely by Theorem 3.1.3 $G : \mathbb{A} \to \mathbb{X}$ has a left adjoint if for each X in $\mathbb{X}$ there exists a universal arrow from X to G. In other words for each X in $\mathbb{X}$ we should look for an arrow $X \xrightarrow{f} GA$ such that whenever there exists $X \xrightarrow{g} GB$ there is a unique arrow $A \xrightarrow{h} B$ for which $Gh \circ f = g$

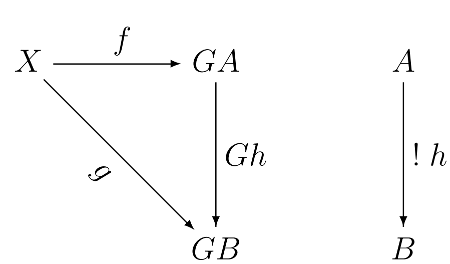


Figure 4.28.

Observe that in the terminology of the comma category $(X \downarrow G)$ this is just an initial object of $(X \downarrow G)$. So $G : \mathbb{A} \to \mathbb{X}$ has a left adjoint if for each X in $\mathbb{X}$ $(X \downarrow G)$ has an initial object.
To show that $(X \downarrow G)$ has an initial object we will use the Theorem 4.2.1. For this we have to ensure that $(X \downarrow G)$ is a small complete category with small hom-sets. Also it should satisfy the solution set condition.
Starting from the last one, the condition stated in the theorem is just the solution set condition given in terms of $(X \downarrow G)$. Secondly $(X \downarrow G)$ has small hom-sets since arrows of $(X \downarrow G)$ are arrows of $\mathbb{A}$ satisfying some commutativity condition. So $\mathbb{A}$ has small hom-sets implies that $(X \downarrow G)$ has small hom-sets.
Lastly $(X \downarrow G)$ is small complete by the preceding corollary. By Theorem 4.1.9, $G : \mathbb{A} \to \mathbb{X}$ has a left adjoint implies that G preserves all limits. Also it is given that $\mathbb{A}$ is small complete. Hence Corollary 4.2.3 is available for our use and $(X \downarrow G)$ is small complete. □

Adjoint functor theorems are important since they give us a tool for deciding whether a functor has a left adjoint without knowing anything about this potential left adjoint. For example by Freyd's Adjoint Functor Theorem we can deduce that every forgetful functor from a suitable algebraic category to **Set** has a left adjoint [3].

It can be insightful to compare Theorem 3.1.3-(3) with Freyd's Adjoint Functor Theorem. There it was stated that a functor $G : \mathbb{A} \to \mathbb{X}$ has a left adjoint if there exists a universal arrow $X \xrightarrow{\eta_X} GA_X$ from X to G for each X in $\mathbb{X}$. Freyd's Adjoint Functor Theorem relaxes this condition. It says that under suitable conditions it is not necessary to find a unique object A_X and an arrow $X \xrightarrow{\eta_X} GA_X$ that satisfies unique factorization constraint. Now it is enough to find a small collection of objects and arrows which will ensure (not necessarily unique) factorization.

4.3. Special Adjoint Functor Theorem

As Freyd's Adjoint Functor Theorem, Special Adjoint Functor Theorem also asserts the existence of a left adjoint to a functor. But different from the former it uses a small cogenerating set instead of the solution set condition.

Definition 4.3.1. *Let $\mathbb{B}$ be a category. A collection S of objects of $\mathbb{B}$ is said to cogenerate $\mathbb{B}$ if for any two arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ such that $f \neq g$, there exists an object R in S and an arrow $B \xrightarrow{h} R$ for which $h \circ f \neq h \circ g$.*

Examples 4.3.2. (1) *In* **Set**, *any set A with two elements induces a cogenerating set $\{A\}$.*

(2) *Let P be a preordered set. Then any subset of P is a cogenerating set. This is because there is at most one arrow between any two objects.*

(3) *In* **CpctHaus**, *$\{[0,1]\}$ is cogenerating. To see this let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be two continuous functions such that $f \neq g$. Then there exists $a \in A$ such that $f(a) = b_0$, $g(a) = b_1$ and $b_0 \neq b_1$. From topology, a compact Hausdorff space is normal. And by Urysohn's lemma for any two closed disjoint subspaces X, Y of a normal space B there exists a continuous map $B \xrightarrow{h}$ such that $h(X) = 0$ and $h(Y) = 1$. In our case $\{b_0\}$ and $\{b_1\}$ are closed subspaces of B. So there exists a continuous map $B \xrightarrow{h}$ such that $h(b_0) = 0$ and $h(b_1) = 1$. Then $h \circ f \neq h \circ g$, and hence $\{[0,1]\}$ is cogenerating.*

Theorem 4.3.3. *Let $\mathbb{B}$ be a small complete category with small hom-sets and a small cogenerating set S. Then $\mathbb{B}$ has an initial object if for every B in $\mathbb{B}$ any collection of subobjects of B has an intersection (pullback).*

Proof. Since S is small and $\mathbb{B}$ is small complete we can form the product of objects in S, call it $\prod_{R \in S} R$ with projections $\prod_{R \in S} R \xrightarrow{p_R} R$.

Now consider the collection of all subobjects of $\prod_{R \in S} R$ and take their intersection, say T with its monic $T \xrightarrow{m} \prod_{R \in S} R$. We claim that T is an initial object of S. For this we have to ensure that for any B in $\mathbb{B}$ there exists a unique arrow from T to B.

We first consider the existence part. For a fixed B in $\mathbb{B}$, consider the collection $K = \{B \xrightarrow{u} R \mid R \in S\}$ of arrows. This is a small collection since $\mathbb{B}$ has small hom-sets and also S is small. Then we form the product $\prod_{u \in K} R$ $(= codu)$ with projections $\prod_{u \in K} R \xrightarrow{q_u} R$, by taking the codomains of arrows in K into consideration. It is clear that we have an arrow $\prod_{R \in S} R \xrightarrow{h} \prod_{u \in K} R$ which satisfies $q_u \circ h = q_R$ for any $B \xrightarrow{u} R$ by the property of products.

Similarly we have an arrow $B \xrightarrow{k} \prod_{u \in K} R$ such that $q_u \circ k = u$ for any $B \xrightarrow{u} R$. We claim that k is monic. To see this let $A \overset{f}{\underset{g}{\rightrightarrows}} B$ be two arrows such that $f \neq g$. Then since S is a cogenerating set there exists an object $R_0 \in S$ and an arrow $B \xrightarrow{u_0} R_0$ such that $u_0 \circ f \neq u_0 \circ g$.

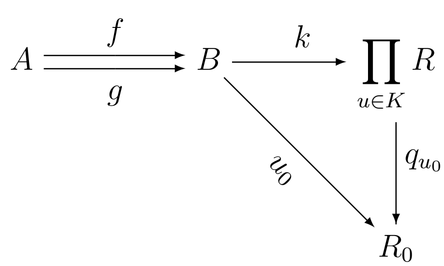


Figure 4.29.

By the commutative triangle above,

$$\begin{aligned} (q_{u_0} \circ k) \circ f &\neq (q_{u_0} \circ k) \circ g \\ q_{u_0} \circ (k \circ f) &\neq q_{u_0} \circ (k \circ g) \end{aligned}$$

This implies that $k \circ f \neq k \circ g$, since otherwise it would be the case that $q_{u_0} \circ (k \circ f) = q_{u_0} \circ (k \circ g)$. So $f \neq g$ implies that $k \circ f \neq k \circ g$. Hence k is monic.

Now for the arrows $\prod_{R \in S} R \xrightarrow{h} \prod_{u \in K} R$ and $B \xrightarrow{k} \prod_{u \in K} R$, form the pullback

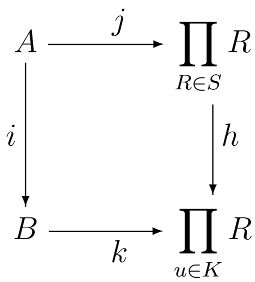


Figure 4.30.

Since k is monic, so is j. And since T is the intersection of all subobjects of $\prod_{R \in S} R$, m factors through j by an arrow $T \xrightarrow{n} A$.

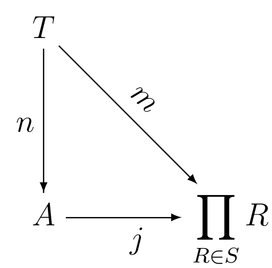


Figure 4.31.

Then $i \circ n$ is an arrow from T to B. This completes the existence part.
Also we have to ensure that an arrow from T to B is unique. To see this let $T \underset{g}{\overset{f}{\rightrightarrows}} B$ be two arrows. Since $\mathbb{B}$ is small complete, they have an equalizer $E \xrightarrow{e} T \underset{g}{\overset{f}{\rightrightarrows}} B$. e is an equalizer implies that it is a monic. Then so is the arrow $E \xrightarrow{m \circ e} \prod_{R \in S} R$. Since T is the intersection of all subobjects of $\prod_{R \in S} R$, m factors through $m \circ e$ by an arrow $T \xrightarrow{l} E$. So

$$\begin{aligned} (m \circ e) \circ l &= m \\ m \circ (e \circ l) &= m \circ 1_T \end{aligned}$$

Since m is monic, we have $e \circ l = 1_T$. Hence e has a right inverse l. But we also know that e is monic. So e is an iso. Then we get,

$$\begin{aligned} f \circ e &= g \circ e \\ f \circ e \circ e^{-1} &= g \circ e \circ e^{-1} \\ f &= g \end{aligned}$$

So an arrow from T to B is unique.

By the existence and uniqueness parts, we conclude that T is an initial object in $\mathbb{B}$. □

Proposition 4.3.4. *Let $\mathbb{A}$ be a small complete category and $G : \mathbb{A} \to \mathbb{X}$ preserve all small limits. Then an arrow $\langle f, A\rangle \xrightarrow{h} \langle f', A'\rangle$ is monic in $(X \downarrow G)$ if and only if $A \xrightarrow{h} A'$ is monic in $\mathbb{A}$.*

Proof. Let $A \xrightarrow{h} A'$ be monic in $\mathbb{A}$. To show that $\langle f, A\rangle \xrightarrow{h} \langle f', A'\rangle$ is monic assume there exists arrows $\langle k, D\rangle \overset{i}{\underset{j}{\rightrightarrows}} \langle f, A\rangle$ in $(X \downarrow G)$ such that $h \circ i = h \circ j$. This means that there are arrows $D \overset{i}{\underset{j}{\rightrightarrows}} A$ in $\mathbb{A}$ for which $h \circ i = h \circ j$. But h is monic. Therefore $i = j$ and h is a monic in $(X \downarrow G)$.

Conversely let $\langle f, A\rangle \xrightarrow{h} \langle f', A'\rangle$ be monic in $(X \downarrow G)$. Recall that by Proposition 2.1.16, $A \xrightarrow{h} A'$ is monic if and only if its kernel pair is $(1_A, 1_A)$. We know that $\mathbb{A}$ is small complete so h has a kernel pair in $\mathbb{A}$. Also $P : (X \downarrow G) \to \mathbb{A}$ creates all kernel pairs by a similar reasoning indicated in Lemma 4.2.2. Then by Theorem 4.1.8, $P : (X \downarrow G) \to \mathbb{A}$ preserves all kernel pairs. $\langle f, A\rangle \xrightarrow{h} \langle f', A'\rangle$ is monic in $(X \downarrow G)$ so it has the kernel pair $(1_A, 1_A)$ in $(X \downarrow G)$. Its image under P, $(1_A, 1_A)$ in $\mathbb{A}$ is a kernel pair of h in $\mathbb{A}$ since P preserves kernel pairs.

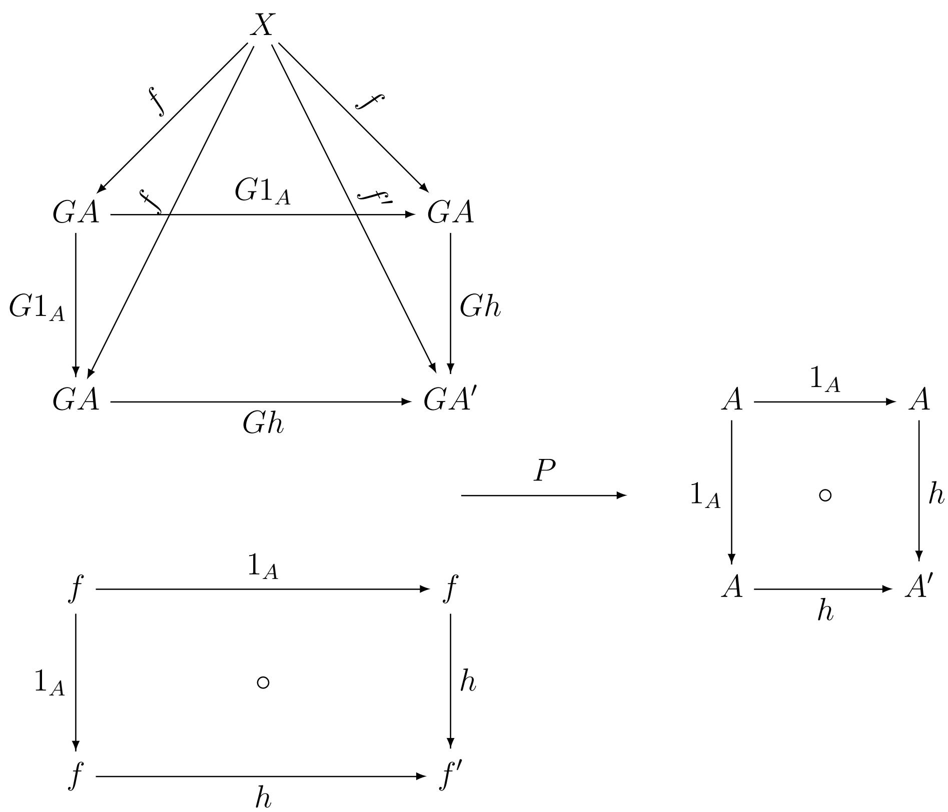


Figure 4.32.

Therefore h is monic in $(X \downarrow G)$. □

Theorem 4.3.5. (Special Adjoint Functor Theorem) *Let $\mathbb{A}$ be a small complete category with small hom-sets and a small cogenerating set S. Also suppose that for any A in $\mathbb{A}$ any collection of subobjects of A has an intersection and $\mathbb{X}$ is a category with small hom-sets.*

Then $G : \mathbb{A} \to \mathbb{X}$ has a left adjoint if and only if G preserves all small limits and intersection of any collection of monics.

Proof. Suppose that G has a left adjoint. Then by Theorem 4.1.9, G preserves all limits. In particular it preserves all small limits and intersection of any collection of monics.

Conversely suppose that G preserves all small limits and intersection of any collection of monics. Similarly to Freyd's Adjoint Functor Theorem, for each X in $\mathbb{X}$, finding an initial object in $(X \downarrow G)$ will suffice. And for this we will make use of the Theorem 4.3.3.

$(X \downarrow G)$ is small complete by Corollary 4.2.3 and it has small hom-sets since $\mathbb{A}$ has small hom-sets.

To see that $(X \downarrow G)$ has a small cogenerating set, let $h \overset{f}{\underset{g}{\rightrightarrows}} k$ be two arrows in $(X \downarrow G)$ such that $f \neq g$. So there are arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathbb{A}$ such that $f \neq g$ but $Gf \circ h = k$ and $Gg \circ h = k$.

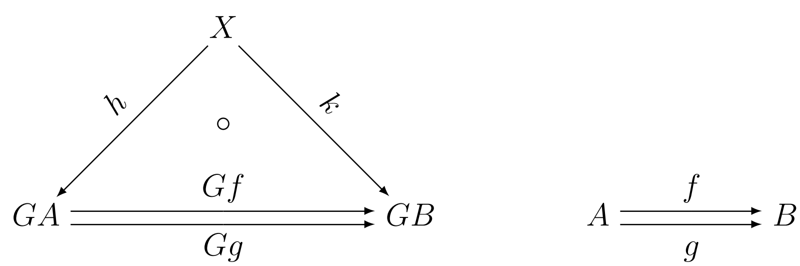


Figure 4.33.

Since S is cogenerating in $\mathbb{A}$, there exists $R \in S$ and an arrow $B \xrightarrow{s} R$ such that $s \circ f \neq s \circ g$. Then Gs is an arrow from GB to GR and by composing with k we obtain a new object $Gs \circ k$ in $(X \downarrow G)$ for which $k \xrightarrow{s} Gs \circ k$.

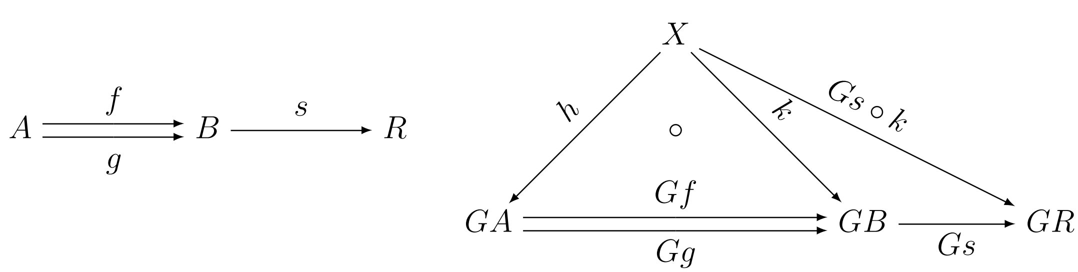


Figure 4.34.

So in $(X \downarrow G)$ we have obtained an object $Gs \circ k$ and an arrow s such that $s \circ f \neq s \circ g$.

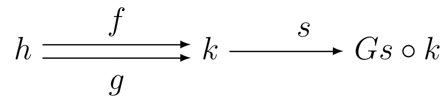


Figure 4.35.

Clearly the collection $\{hom(X, GR)\}$ for $R \in S$ is a cogenerating set for $(X \downarrow G)$. Also it is small since $\mathbb{X}$ has small hom-sets and S is small.

Lastly we have to show that any collection of subobjects in $(X \downarrow G)$ has an intersection. Let $\{\langle h_i, A_i\rangle \xrightarrow{f_i} \langle k, B\rangle\}_{i\in I}$ be a collection of monics in $(X \downarrow G)$. By the Proposition 4.3.4, $\{A_i \xrightarrow{f_i} B\}_{i\in I}$ is a collection of monics in $\mathbb{A}$. And by our assumption they have an intersection in $\mathbb{A}$, call it A with its monic m. So m factors through every monic f_i by an arrow g_i.

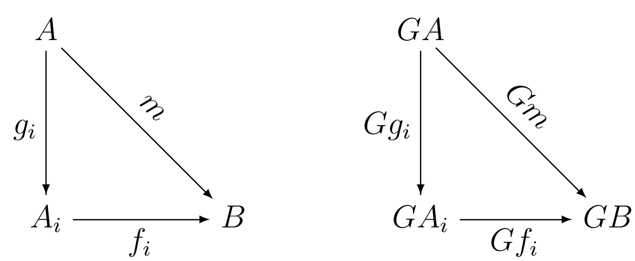


Figure 4.36.

Since G preserves intersection of monics, i.e. pullbacks of f_i's, GA together with projections Gg_i's constitute a pullback of Gf_i's in $\mathbb{X}$.

We know that $\{h_i \xrightarrow{f_i} k\}_{i\in I}$ are arrows in $(X \downarrow G)$, so $Gf_i \circ h_i = k = Gf_j \circ h_j$ for any $i, j \in I$. Then by definition of pullbacks, there exists a unique arrow $X \xrightarrow{u} GA$ such that $Gg_i \circ u = h_i$ for all $i \in I$.

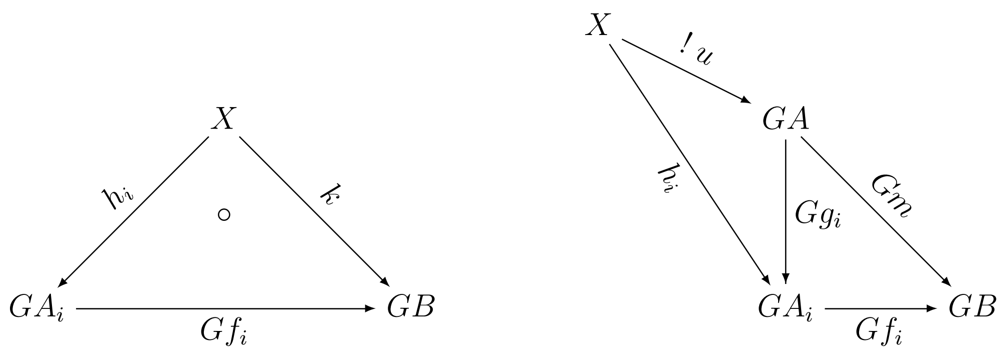


Figure 4.37.

So there exist arrows $\{u \xrightarrow{g_i} h_i\}_{i \in I}$ in $(X \downarrow G)$.

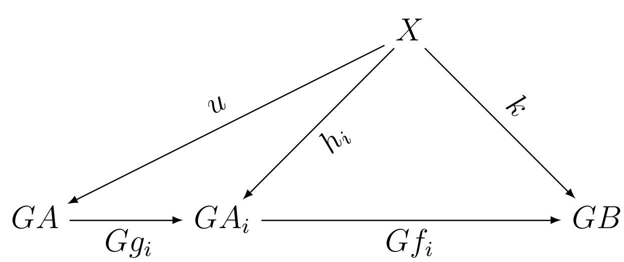


Figure 4.38.

$(u, \{u \xrightarrow{g_i} h_i\}_{i \in I})$ is a cone in $(X \downarrow G)$ whose image under $P : (X \downarrow G) \to \mathbb{A}$ is the pullback of f_i's.

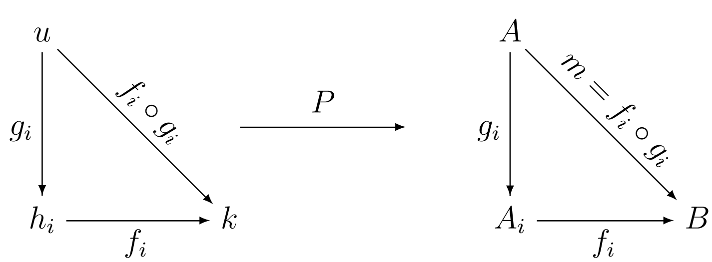


Figure 4.39.

Since P creates pullbacks, u together with projection arrows, g_i's, form a pullback of

f_i's in $(X \downarrow G)$. □

In the following examples Special Adjoint Functor Theorem helps us to assert the existence of a left adjoint to the given functor.

Examples 4.3.6. (1) *Let P, Q be two small preordered collections and $P \xrightarrow{F} Q$ be an order preserving function. Suppose that any subcollection of P has an infimum and F preserves infimums. Then by Special Adjoint Functor Theorem F has a left adjoint. To verify this, observe that any limit in P is an infimum of a certain subcollection of P. So our assumption implies that P is complete. Since P, Q are preorders they have small hom-sets. We also know that P has a cogenerating set by Example 4.3.2-(2). And lastly if F preserves infimums then it preserves all limits. So we can safely use Special Adjoint Functor Theorem.*

(2) *Inclusion functor G : **CpctHaus** $\to$ **Top** has a left adjoint called Stone-Čech compactification functor by Special Adjoint Functor Theorem. This can be shown as follows. **Top** and and its subcategory **CpctHaus** has small hom-sets.*

***CpctHaus** is small complete since it has products for any small collection of objects and an equalizer for any parallel pair of arrows. It has small products since product of Hausdorff spaces is also Hausdorff and by Tychonoff's Theorem product of compact spaces is compact. To see that it has an equalizer for any parallel pair of arrows, let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be any two continuous functions. Then the subset $Z = \{x \in X \mid f(x) = g(x)\}$ of X is closed since X is Hausdorff. And being closed in a Hausdorff space, Z is compact. As a subspace of X, Z is also Hausdorff. So Z together with the inclusion function $Z \longrightarrow X$ is an equalizer of f and g in **CpctHaus**.*

*Monics in **CpctHaus** are injections (this is easy to prove since one point space is compact Hausdorff). So for any X in **CpctHaus** we can identify subobjects of X with its subspaces, and these constitute a small collection. Hence for any X, any collection of subobjects of X is small. And since **CpctHaus** is small complete every collection of subobjects of X has an intersection.*

*Also $\{[0, 1]\}$ is a cogenerating set for **CpctHaus** by Example 4.3.2-(3) and inclusion functor G : **CpctHaus** $\to$ **Top** preserves limits.*

*Therefore all the conditions of Special Adjoint Functor Theorem is satisfied and G has a left adjoint F : **Top** $\to$ **CpctHaus**, the Stone-Čech compactification functor.*

5. CONCLUSIONS

In this thesis connections between adjoint functors and limits are explored. As the first step we presented the theory of adjoint functors in the third chapter. After having experience about characterizations and properties of adjunctions, we presented basic facts about limits. Connections between adjoint functors and limits are examined in the fourth chapter starting by the theorem which states that a functor having a left adjoint preserves all limits in its domain category. Adjoint functor-limit relation is best revealed by the adjoint functor theorems. So as the ultimate goal of the thesis we proved Freyd's Adjoint Functor Theorem and Special Adjoint Functor Theorem. These theorems characterize the existence of a left adjoint to a functor in terms of limits. They are important since they give us a way to assert the existence of the left adjoint without having any knowledge about it. We presented the proof of Freyd's Adjoint Functor Theorem by making use of the solution set condition. And in the proof of Special Adjoint Functor Theorem we replaced this condition by the existence of a small cogenerating set. We also presented examples from different areas of mathematics to help the reader to visualize these abstract concepts and see the applications of adjoint functor theorems.

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