CONSTRUCTION OF METRICS ON SASAKI-EINSTEIN MANIFOLDS

by

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ABSTRACT

CONSTRUCTION OF METRICS ON SASAKI-EINSTEIN MANIFOLDS

Sasaki-Einstein manifolds are Einstein manifolds occurring in odd dimensions, which have positive curvature and Calabi-Yau metric cones. We start with an introduction of Sasakian manifolds, following the historical development of the subject. For this part, we define almost contact and contact structures and normality. Then, we define Killing spinors and examine their relationship with Sasaki-Einstein manifolds. Afterwards we study two different explicit nonsingular metric constructions of Sasaki-Einstein manifolds, in the form of principle U(1) bundles over Kähler-Einstein base manifolds. We see that even though the base manifolds are singular, we can obtain a nonsingular total space. In the second construction, base manifolds are themselves S^2 bundles over a Kähler-Einstein manifold, whereas in the first one they are S^2 bundles over a product of Kähler-Einstein manifolds. For the first one, we give a detailed analysis in dimension 7 and then a generalization to arbitrary odd dimensions ≥7. The second construction applies to all odd dimensions.

ÖZET

SASAKI-EINSTEIN MANİFOLDLARI ÜZERİNDE METRİK YAPILANDIRMALARI

Sasaki-Einstein manifoldları, pozitif eğriliğe ve Calabi-Yau metrik konilerine sahip, tek boyutlu manifoldlardır. Bu tezde konunun tarihsel gelişimi izlenerek neredeyse kontak ve kontak yapılar ile normallik ve Sasaki manifoldları tanımlanmıştır. Daha sonra Killing spinörleri tanımlanıp Sasaki-Einstein manifoldları ile ilişkileri incelenmiştir. Ardından Kähler-Einstein taban manifoldları üzerine asal U(1)-demeti formunda iki farklı açık sorunsuz Sasaki-Einstein metrik yapılandırması çalışılmıştır. Taban manifoldları sorunlu olsa da, manifoldun kendisinin sorunsuz olduğu görülmüştür. Yapılandırmalarda kullanılan taban manifoldlarının kendileri de küre demetleridir. İkinci yapılandırmada kullanılan demetin taban manifoldunun taban manifoldu bir Kähler-Einstein manifold iken, ilk yapılandırmada çeşitli Kähler-Einstein manifoldlarının çarpımıyla elde edilmiştir. Bu tezde, ilk yapılandırma 7 boyutta detaylı incelendikten sonra, yediden büyük tek boyutlara genellenmiştir. İkinci yapılandırma tüm tek boyutlarda geçerlidir.

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LIST OF SYMBOLS

C(M)	Metric cone of M
d	Exterior derivative
D	Dirac operator
g_{ab}	Metric tensor
Ker	Kernel
$\mathcal{L}$	Lie derivative
LM	Frame bundle on M
N_J	Nijenhuis tensor
R	Scalar curvature
R_{ab} or Ric	Ricci tensor
$R^{\mu}_{\nu\alpha\beta}$ or R(.,.).	Riemann curvature tensor
S^n	n dimensional sphere
S(M)	Spinor bundle on M
TM	Tangent bundle of M
$\Gamma(S(M))$	Section of S(M)
δ_{ij}	Kronecker delta
η	One form of the (almost) contact structure
ξ	Reeb vector field
ϕ	(1,1)-tensor of the (almost) contact structure
∇	Covariant derivative
$\lrcorner$	Interior product
[.,.]	Lie bracket
$\wedge$	Wedge product

1. INTRODUCTION

"A Sasaki-Einstein Manifold is a Riemannian manifold (M,g) that is both Sasakian and Einstein" [2]. More explicitly, a Sasaki-Einstein manifold is a Riemannian manifold, whose metric cone is Kähler and whose Ricci curvature is proportional to its metric.

In 1960, Shigeo Sasaki introduced a geometric structure related to an almost contact structure in [3]. This geometry became known as the Sasakian geometry. Similar to the relation between contact and symplectic manifolds, Sasakian manifolds are closely related to Kähler manifolds. After mid-1970's, Sasakian geometry was almost forgotten until early 1990's. In 1990's, manifolds admitting real Killing spinors became important in physics. In fact, Sasaki-Einstein manifolds are compact Einstein manifolds of positive scalar curvature occurring in odd dimensions. Moreover, when they are simply connected, they admit real Killing spinors, which make them important for Supergravity, Superstring, and M-Theory. They are a candidate for the 7-dimensional manifold, which complements the 4-dimensional "universe", in the 11-dimensional M-Theory, or a candidate for the 5-dimensional manifold in the 10-dimensional Superstring Theory.

Einstein manifolds have always been important in physics, since their definition stems from the well-known Einstein's equation. A manifold is Einstein if $R_{ab} = \lambda g_{ab}$, where λ is a constant. In General Relativity, Einstein's equation with a cosmological constant Λ is $R_{ab} - \frac{1}{2} g_{ab} R + g_{ab}\Lambda = 8\pi T_{ab}$, where R_{ab} is the Ricci curvature and speed of light c and gravitational constant G are both taken to be 1. The stress-energy tensor T_{ab} gives the matter and energy content of the underlying spacetime. In vacuum, where $T_{ab} = 0$, one can rewrite Einstein's equation in a form, which is actually equivalent to the basic condition of an Einstein manifold. The equation becomes $R_{ab} = \frac{2\Lambda}{n-2} g_{ab}$. A good source for Einstein's manifolds is [4].

Until 2004, the known explicit Sasaki-Einstein metrics were relatively sparse. Well-known examples were the metrics on round spheres in any odd dimensions and the sphere with the non-standard "squashed" Einstein metric in dimensions 4n-1, described as a coset Sp(n+1)/Sp(n). Other examples included the five dimensional $T^{1,1}$ space (topologically $S^2 \times S^3$) and higher dimensional analogues. These isolated examples are all homogeneous. There were also various existence theorems for further inhomogeneous Sasaki-Einstein metrics (for example on $S^2 \times S^3$ [5]). The collection of examples increased dramatically in 2004 with the explicit construction of infinitely many inhomogeneous non-singular Sasaki-Einstein metrics in all odd dimensions $d = 2n + 3 \geq 5$ [6, 7]. The same year, in [8], a generalization of the constructions in [6, 7] was given. After these works, we now have a countably infinite class of explicit Sasaki-Einstein metrics.

After these developments, in [9], another class of metrics of Sasaki-Einstein manifolds was given, which is not going to be investigated in this thesis.

In the second chapter, we give various equivalent definitions of Sasakian manifolds. Then adding the condition for being Einstein, we define Sasaki-Einstein manifolds. Along the way, we study almost contact and contact structures, nearly Sasakian manifolds and normality.

In the third chapter, we discuss Killing spinors briefly, since complete simply connected Sasaki-Einstein manifolds admit real Killing spinors, which make them important in physics due to the role of the real Killing spinors in supersymmetry. We finish the second chapter with the classification of complete, simply connected Riemannian spin manifolds admitting nontrivial real Killing spinors. A helpful reference is [10].

In the fourth chapter, we start with the local metric construction of a Sasaki-Einstein manifold as a principle bundle over a Kähler-Einstein base. Then we study two explicit constructions of nonsingular Sasaki-Einstein metrics. Both have cohomogenity 1 according to the classification in [11], i.e. there is a compact Lie group G of isometries, preserving the Sasakian structure, which acts such that its orbit has codimension 1. In both examples, the Kähler-Einstein base manifold is actually a S^2-bundle over a Kähler-Einstein manifold. Although the metrics on the Kähler-Einstein base of the principle bundle is singular, we will obtain nonsingular Sasaki-Einstein metrics as total space. We mainly follow [8] and [7].

In this thesis a basic knowledge of Riemannian and complex geometry is assumed. [12] is a good reference for Riemannian geometry and [13] is a good source for complex geometry.

Good references for the Sasaki-Einstein manifolds are the monograph [14] and the reviews [15] and [2].

2. FROM ALMOST CONTACT STRUCTURE TO SASAKIAN STRUCTURE

In this chapter, we follow the historical development of Sasakian manifolds. We start with the definition of almost contact manifolds, on which Sasakian manifolds are based. Then we give definitions and some basic properties of contactness and normality. Having the tools we need, we define Sasaki-Einstein manifolds and study some facts about their curvature properties. Finally imposing the condition to be Einstein, we end this chapter with the definition of Sasaki-Einstein manifolds.

2.1. Almost Contact and Contact Manifolds

Definition. *An **almost contact structure** on a differentiable manifold M is a triple (ξ, η, ϕ), where ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form satisfying*

$$\eta(\xi) = 1 \qquad \phi \circ \phi = -\mathrm{I} + \xi \otimes \eta$$

where I *is the identity on TM.*
*A smooth manifold with such a structure is called an **almost contact manifold**.*

An almost contact structure is not unique; an almost contact structure (ξ, η, ϕ) has three other almost contact structures canonically associated to it, namely (ξ, η, -ϕ), (-ξ, -η, ϕ),(-ξ, -η, -ϕ).

Theorem 2.1. *For an almost contact structure (ξ, η, ϕ)*

(i) $\phi(\xi)$=0
(ii) $\eta(\phi X)$=0

Proof. (i) $\phi^2\xi = -\xi + \eta(\xi)\xi = -\xi + \xi = 0$

$\phi(\phi\xi) = 0 \Rightarrow$ Either $\phi\xi = 0$ or $\phi\xi$ is an eigenvector of ϕ corresponding to the eigenvalue 0.

Suppose $\phi\xi \neq 0$.

$0 = \phi(\phi^2\xi) = \phi^2(\phi\xi) = -\phi\xi + \eta(\phi\xi)\xi$

Hence $\eta(\phi\xi)\xi = \phi\xi \neq 0$

$0 = \phi^2\xi = \eta(\phi\xi)(\phi\xi) \Rightarrow$ Either $\phi\xi = 0$ or $\eta(\phi\xi) = 0$, which is a contradiction with the equation on the previous line and our assumption.

Thus $\phi\xi = 0$

(ii) $\phi^2 X = -X + \eta(X)\xi$

$\phi^3 X = -\phi X + \eta(X)\phi\xi = -\phi X \Rightarrow \phi^2(\phi X) = -\phi X + \eta(\phi X)\xi = -\phi X$

$\eta(\phi X)\xi = 0 \Rightarrow \eta(\phi X) = 0$ since $\xi \neq 0$

□

Theorem 2.2. *Every almost contact manifold M admits a Riemannian metric such that for every vector field X, Y on M,*

$g(X,\xi) = \eta(X)$

$g(X,Y) = g(\phi X,\phi Y) + \eta(X)\eta(Y)$

$g(\phi X,Y) + g(X,\phi Y) = 0$

Proof. Assume that M admits a Riemannian metric tensor f (for its existence, see Proposition 11.26 in [16]). Then set

$$\tilde{g}(X,Y) = f(X - \eta(X)\xi, Y - \eta(Y)\xi) + \eta(X)\eta(Y)$$
$$\tilde{g}(X,\xi) = f(X - \eta(X)\xi, \xi - \eta(\xi)\xi) + \eta(X)\eta(\xi) = \eta(X)$$

Let $g(X,Y) = \frac{1}{2}[\tilde{g}(X,Y) + \tilde{g}(\phi X,\phi Y) + \eta(X)\eta(Y)]$. It is easy to check that g defines a Riemannian metric. We start with showing that it is symmetric:

$$g(X,Y) = \frac{1}{2}[f(X - \eta(X)\xi, Y - \eta(Y)\xi) + \eta(X)\eta(Y) + f(\phi X - \eta(\phi X)\xi, \phi Y - \eta(\phi Y)\xi) + \eta(\phi X)\eta(\phi Y) + \eta(X)\eta(Y)]$$

$$
\begin{aligned}
&= \frac{1}{2}[f(Y-\eta(Y)\xi, X-\eta(X)\xi)+\eta(Y)\eta(X)+f(\phi Y-\eta(\phi Y)\xi, \phi X-\eta(\phi X)\xi) \\
&\qquad +\eta(\phi Y)\eta(\phi X)+\eta(Y)\eta(X)] \\
&= g(Y,X) \quad \text{since f is symmetric.}
\end{aligned}
$$

Next we check whether g is positive definite:

$$
\begin{aligned}
g(X,X) &= \frac{1}{2}[\tilde{g}(X,X)+\tilde{g}(\phi X,\phi X)+\eta(X)\eta(X)] \\
&= \frac{1}{2}[f(X-\eta(X)\xi, X-\eta(X)\xi)+\eta(X)\eta(X)+f(\phi X-\eta(\phi X)\xi, \phi X-\eta(\phi X)\xi) \\
&\qquad +\eta(\phi X)\eta(\phi X)+\eta(X)\eta(X)] \\
&= \frac{1}{2}[f(X-\eta(X)\xi, X-\eta(X)\xi)+\eta(X)\eta(X)+f(\phi X,\phi X)+\eta(X)\eta(X)]
\end{aligned}
$$

We see that g is positive definite since f is a Riemannian metric. It is straightforward to show that g is linear. Hence, g is a Riemannian metric.

Now we show the identities stated in the theorem are true. We start with the first one:

$$
\begin{aligned}
g(X,\xi) &= \frac{1}{2}[\tilde{g}(X,\xi)+\tilde{g}(\phi X,\phi\xi)+\eta(X)\eta(\xi)] \\
&= \frac{1}{2}[\eta(X)+\tilde{g}(\phi X,0)+\eta(X)] = \eta(X)
\end{aligned}
$$

Now we check the second identity:

$$
\begin{aligned}
g(\phi X,\phi Y) &= \frac{1}{2}[\tilde{g}(\phi X,\phi Y)+\tilde{g}(\phi^2 X,\phi^2 Y)+\eta(\phi X)\eta(\phi Y)] \\
&= \frac{1}{2}[\tilde{g}(\phi X,\phi Y)+\tilde{g}(-X+\eta(X)\xi, -Y+\eta(Y)\xi)] \\
&= \frac{1}{2}[\tilde{g}(\phi X,\phi Y)+\tilde{g}(X,Y)-\tilde{g}(X,\eta(Y)\xi)-\tilde{g}(\eta(X)\xi,Y)+\eta(X)\eta(Y)\tilde{g}(\xi,\xi)] \\
&= \frac{1}{2}[\tilde{g}(\phi X,\phi Y)+\tilde{g}(X,Y)-\eta(Y)\eta(X)-\eta(X)\eta(Y)+\eta(X)\eta(Y)] \\
&= \frac{1}{2}[\tilde{g}(\phi X,\phi Y)+\tilde{g}(X,Y)+\eta(X)\eta(Y)]-\eta(X)\eta(Y) \\
&= g(X,Y)-\eta(X)\eta(Y)
\end{aligned}
$$

We verify the last identity:

$$\begin{aligned}
g(\phi X, \phi^2 Y) &= g(X, \phi Y) - \eta(X)\eta(\phi Y) = g(X, \phi Y) \\
g(\phi X, \phi^2 Y) &= g(\phi X, -Y + \eta(Y)\xi) = -g(\phi X, Y) + \eta(Y)g(\phi X, \xi) \\
&= = -g(\phi X, Y) + \eta(Y)\eta(\phi X) = -g(\phi X, Y) \\
\Rightarrow -g(\phi X, Y) &= g(X, \phi Y) \\
\Rightarrow g(\phi X, Y) &+ g(X, \phi Y) = 0
\end{aligned}$$

□

Now, we define contact manifolds:

Definition. *A (2n+1)-dimensional manifold M is a* ***contact manifold*** *if there exists a 1-form η, called the* ***contact 1-form on M*** *such that*

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M.

On a contact manifold (M, η), there is a unique vector field ξ, called the *Reeb vector field*, satisfying

$$\xi \lrcorner \eta = 1 \qquad \xi \lrcorner d\eta = 0$$

where $\lrcorner$ stands for interior product.

ξ generates a 1-dimensional subbundle L_ξ of the tangent bundle TM; hence, an almost contact manifold M has an associated 1-dimensional foliation F_ξ, called the *characteristic foliation.* The 1-form η is called the *characteristic 1-form*, and the Reeb vector field is also called as the *characteristic vector field.*

Let (M,ξ) be a contact manifold with a contact 1-form η and consider $D = Ker\eta \subset TM$. The pair (D,ω), where ω is the restriction of $d\eta$ to D gives the structure of a symplectic vector bundle, on which ϕ acts as an almost complex structure.

Theorem 2.3. *Let (M,g) be a contact manifold and ξ be its characteristic vector field. Then there exists an almost contact metric structure (ξ,η,ϕ,g) such that $g(X,\phi Y) = d\eta(X,Y)$.*

Proof. In Theorem 2.2, we had $\tilde{g}(X,\xi) = \eta(X)$. Let $D = Ker\eta$. On D, $d\eta$ leads to a metric $\bar{g}$ and a complex structure J such that $\bar{g}(X,JY) = d\eta(X,Y)$. Extending $\bar{g}$ to a metric g agreeing with $\tilde{g}$ in the direction ξ and extending J by requiring J=ϕ, we have an almost contact metric structure such that $g(X,\phi Y) = d\eta(X,Y)$. □

2.2. Nearly Sasakian Manifolds

Definition. *An almost contact metric structure (ξ,η,ϕ,g) on a manifold M is said to be **nearly Sasakian** if the following holds for all vector fields X,Y on M:*

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X \tag{2.1}$$

Theorem 2.4. *A vector field X on a Riemannian manifold M is a Killing vector field if and only if the following holds for all vector fields X,Y,Z on M:*

$$g(\nabla_Y X,Z) + g(\nabla_Z X,Y) = 0$$

Proof.

$$\begin{aligned}
(\mathcal{L}_X g)(Y,Z) &= \mathcal{L}_X(g(Y,Z)) - g(\mathcal{L}_X Y,Z) - g(Y,\mathcal{L}_X Z)\\
&= Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z])\\
&= Xg(Y,Z) - g(\nabla_X Y - \nabla_Y X,Z) - g(Y,\nabla_X Z - \nabla_Z X)\\
&= (\nabla_X g)(Y,Z) + g(\nabla_Y X,Z) + g(Y,\nabla_Z X) = g(\nabla_Y X,Z) + g(Y,\nabla_Z X)
\end{aligned}$$

Therefore $(\mathcal{L}_X g)(Y,Z) = 0$ if and only if $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ □

Theorem 2.5. *On a nearly Sasakian manifold, the vector field ξ is Killing.*

Proof. Let $X = \xi$, $Y = \xi$ in (2.1).

$$(\nabla_\xi \phi)\xi + (\nabla_\xi \phi)\xi = 2g(\xi,\xi)\xi - \eta(\xi)\xi - \eta(\xi)\xi = 0$$

$$(\nabla_\xi \phi)\xi = 0 \Rightarrow \phi \nabla_\xi \xi = 0$$

$$\phi^2 \nabla_\xi \xi = -\nabla_\xi \xi + \eta(\nabla_\xi \xi)\xi = -\nabla_\xi \xi = 0$$

$$\nabla_\xi(\eta(\xi)) = 0 = (\nabla_\xi \eta)\xi + \eta(\nabla_\xi \xi) = (\nabla_\xi \eta)\xi$$

$$\Rightarrow \nabla_\xi \eta = 0$$

Taking the covariant derivative of the second identity proved in Theorem 2.2, we get

$$\nabla_\xi[g(\phi X, \phi Y)] = \nabla_\xi[g(X,Y) - \eta(X)\eta(Y)] \tag{2.2}$$

$$= g(\nabla_\xi(\phi X), \phi Y) + g(\phi X, \nabla_\xi(\phi Y)) = g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y) - \eta(\nabla_\xi X)\eta(Y) - \eta(\nabla_\xi Y)\eta(X)$$

$$= g((\nabla_\xi \phi)X, \phi Y) + g(\phi(\nabla_\xi X), \phi Y) + g(\phi X, (\nabla_\xi \phi)Y) + g(\phi X, \phi(\nabla_\xi Y))$$

$$= g((\nabla_\xi \phi)X, \phi Y) + g(\phi X, (\nabla_\xi \phi)Y) - g(\nabla_\xi X, \phi^2 Y) - g(\phi^2 X, \nabla_\xi Y)$$

Cancelling with the right hand side, we get

$$g((\nabla_\xi \phi)X, \phi Y) + g(\phi X, (\nabla_\xi \phi)Y) = 0 \tag{2.3}$$

Using (2.1), equation (2.3) becomes

$$\begin{aligned}
&= g(-(\nabla_X \phi)\xi + 2g(X,\xi)\xi - \eta(X)\xi - \eta(\xi)X, \phi Y) \\
&\qquad + g(\phi X, -(\nabla_Y \phi)\xi + 2g(\xi, Y)\xi - \eta(\xi)Y - \eta(Y)\xi) \\
&= g(\phi(\nabla_X \xi) + \eta(X)\xi - X, \phi Y) + g(\phi X, \phi(\nabla_Y \xi) + \eta(Y)\xi - Y) \\
&= -g(\nabla_X \xi, \phi^2 Y) + \eta(X)g(\xi, \phi Y) - g(X, \phi Y) - (g(\phi^2 X, \nabla_Y \xi) + \eta(Y)g(\phi X, \xi) - g(\phi X, Y) \\
&= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)
\end{aligned}$$

since $g(X,\phi Y) = -g(Y,-\phi X)$, we have $g(\phi X,\xi) = -g(X,\phi\xi) = 0$ and $g(\nabla_X\xi,\xi) = 0$. Hence

$$g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi) = 0$$

So, ξ is Killing. □

2.3. Normality

Definition. *For any Riemannian metric g on a smooth manifold M, $(C(M),\bar{g}) = (\mathbb{R}^+ \times M, dr^2 + r^2 g)$ is called the **Riemannian cone** or **metric cone** on M, where $r \in \mathbb{R}^+$.*

Proposition 2.6. *There is a one-to-one correspondence between almost contact structures (ξ,η,ϕ,g) on M and almost complex structures J on C(M).*

Proof. Let the metric on C(M) be $\bar{g} = dr^2 + r^2 g$, where g is the metric on M. Define $\Psi := r\dfrac{\partial}{\partial r}$. Define J=$\phi$+$\Psi \otimes \eta$ and JΨ=-ξ. Then for any $Y \in TM$

$$\begin{aligned} JY &= \phi Y + \eta(Y)\Psi \\ J(JY) &= \phi(\phi Y + \eta(Y)\Psi) + \eta(\phi Y + \eta(Y)\Psi)\Psi \\ &= -Y + \eta(Y)\xi + \eta(Y)\phi\Psi + \eta(\phi Y)\Psi + \eta(Y)\eta(\Psi)\Psi \\ &= -Y + \eta(Y)\xi + \eta(Y)J\Psi - \eta(Y)\eta(\Psi)\Psi + \eta(Y)\eta(\Psi)\Psi = -Y \end{aligned}$$

For Ψ, $J^2(\Psi) = J(J(\Psi)) = J(-\xi) = -[\phi(\xi) + \eta(\xi)\Psi] = -\Psi$.
Hence $J^2 = -id$. □

Definition. *An almost contact structure (ξ,η,ϕ,g) is said to be **normal** if the corresponding almost complex structure J on C(M) is integrable, i.e. if its Nijenhuis tensor $N_J(X,Y) = J^2[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY]$ vanishes.*

Theorem 2.7. *An almost contact structure on M is normal if and only if $N_\phi = -\xi \otimes d\eta$.*

Proof. It is enough to check for the vector fields X,Y on M and Ψ on C(M).

$$\begin{aligned}
N_J(X,Y) &= J^2[X,Y]+[JX,JY]-J[JX,Y]-J[X,JY]\\
&= -[X,Y]+[\phi X+\eta(X)\Psi,\phi Y+\eta(Y)\Psi]-J[X,\phi Y+\eta(Y)\Psi]\\
&\qquad -J[\phi X+\eta(X)\Psi,Y]\\
&= [\phi X,\phi Y]+[\phi X,\eta(Y)\Psi]+[\eta(X)\Psi,\phi Y]+\phi^2[X,Y]-\eta([X,Y])\xi-\phi[X,\phi Y]\\
&\quad -\eta([X,\phi Y])\Psi-J[X,\eta(Y)\Psi]-\phi[\phi X,Y]-\eta([\phi X,Y])\Psi-J[\eta(X)\Psi,Y]\\
&= N_\phi(X,Y)-[\phi Y\eta(X)-\phi X\eta(Y)+\eta([X,\phi Y])+\eta([\phi X,Y])]\Psi \qquad (2.4)\\
&\qquad +[X\eta(Y)-Y\eta(X)-\eta([X,Y])]\xi\\
&= N_\phi(X,Y)+d\eta(X,Y)\xi+d\eta(\phi X,Y)\Psi+d\eta(X,\phi Y)\Psi\\
&= N_\phi(X,Y)+d\eta(X,Y)\xi-(\mathcal{L}_{\phi Y}\eta)(X)\Psi+(\mathcal{L}_{\phi X}\eta)(Y)\Psi
\end{aligned}$$

since by Cartan's equation

$$\mathcal{L}_{\phi Y}\eta(X)=((\phi Y)\lrcorner d+d\phi Y\lrcorner)\eta(X)=((\phi Y)\lrcorner d\eta(X)=d\eta(\phi Y,X)$$

and

$$d\eta(X,Y)=X\eta(Y)-Y\eta(X)-\eta([X,Y]) \qquad (2.5)$$

For a vector field X on M and Ψ, we have

$$\begin{aligned}
N_J(X,\Psi) &= J^2[X,\Psi]+[JX,J\Psi]-J[JX,\Psi]-J[X,J\Psi]\\
&= -[\phi X+\eta(X)\Psi,\xi]+\phi[X,\xi]+\eta([X,\xi])\Psi-\phi[\phi X+\eta(X)\Psi,\Psi]\\
&\qquad -\eta([\phi X+\eta(X)\Psi,\Psi])\\
&= -[\phi X,\xi]+\xi\eta(X)\Psi+\phi[X,\xi]+\eta([X,\xi])\Psi\\
&= [X\eta(\xi)-d\eta(X,\xi)]\Psi-[\phi X,\xi]+X\phi(\xi)-\xi\phi(X)+d\phi(\xi,X)\\
&= (\mathcal{L}_\xi\eta)(X)\Psi+(\mathcal{L}_\xi\phi)(X)
\end{aligned}$$

If we define

$$\begin{aligned} N^{(1)}(X,Y) &= N_\phi(X,Y) + d\eta(X,Y)\xi & (2.6)\\ N^{(2)}(X,Y) &= (\mathcal{L}_{\phi X}\eta)(Y) - (\mathcal{L}_{\phi Y}\eta)(X) & (2.7)\\ N^{(3)}(X) &= (\mathcal{L}_\xi \phi)(X) & (2.8)\\ N^{(4)}(X) &= (\mathcal{L}_\xi \eta)(X) & (2.9) \end{aligned}$$

we see that an almost contact structure on M is normal if $N^{(i)} = 0$ for i=1,2,3,4. In the next lemma, we'll prove that this is equivalent to $N^{(1)} = 0$, i.e. $N_\phi = -\xi \otimes d\eta$. □

Lemma 2.8. *If $N^{(1)}$ vanishes, so does $N^{(2)}$,$N^{(3)}$,$N^{(4)}$.*

Proof. For any vector field X on M, consider

$$\begin{aligned} N^{(1)}(X,\xi) &= N_\phi(X,\xi) + d\eta(X,\xi)\xi\\ &= \phi^2[X,\xi] + [\phi X, \phi\xi] - \phi[X,\phi\xi] - \phi[\phi X,\xi] + [X\eta(\xi) - \xi\eta(X) - \eta([X,\xi])]\xi\\ &= -[X,\xi] + \eta[X,\xi]\xi - \phi[\phi X,\xi] - \xi\eta(X)\xi - \eta([X,\xi])\xi\\ &= -[X,\xi] - \phi[\phi X,\xi] - \xi\eta(X)\xi & (2.10) \end{aligned}$$

If $N^{(1)}$ vanishes, then

$$[X,\xi] + \phi[\phi X,\xi] + \xi\eta(X)\xi = 0$$

$$\eta([X,\xi]) + \eta\phi[\phi X,\xi] + \xi\eta(X)\eta(\xi) = 0$$

$$\eta([X,\xi]) + \xi\eta(X) = 0 = X\eta(\xi) - d\eta(X,\xi) = -d\eta(X,\xi) = -(\mathcal{L}_\xi\eta)(X) = -N^{(4)}(X)$$

$$\Rightarrow N^{(4)} = 0$$

Replacing X by ϕX in (2.10), we get

$$\begin{aligned} N^{(1)}(\phi(X),\xi) &= -[\phi(X),\xi] - \phi[\phi^2 X,\xi] - \xi\eta(\phi(X))\xi\\ &= -[\phi(X),\xi] + \phi[X,\xi] - \phi[\eta(X)\xi,\xi] = -[\phi(X),\xi] + \phi[X,\xi] = N^{(3)}(X) \end{aligned}$$

Applying η to $N^{(1)}(\phi X, Y)$, we get

$$\begin{aligned}
\eta(N^{(1)}(\phi X,Y)) &= \eta(N_\phi(\phi X,Y)) + d\eta(\phi X,Y)\eta(\xi) \\
&= \eta([\phi^2 X,\phi Y] + \phi^2[\phi X,Y] - \phi[\phi^2 X,Y] - \phi[\phi X,\phi Y]) + d\eta(\phi X,Y)\eta(\xi) \\
&= -\eta([X,\phi Y]) + \eta[\eta(X)\xi,\phi Y] + d\eta(\phi X,Y) \\
&= -\eta([X,\phi Y]) - \phi Y\eta(X) - \eta(X)d\eta(\xi,\phi Y) + d\eta(\phi X,Y) \\
&= d\eta(X,\phi Y) - \eta(X)d\eta(\xi,\phi Y) + d\eta(\phi X,Y) \\
&= N^{(2)}(X,Y) - \eta(X)d\eta(\xi,\phi Y)
\end{aligned}$$

where the second term vanishes if $N^{(4)}$=0. □

Proposition 2.9. *Let (ξ,η,φ,g) be a contact metric structure. Then* $N^{(2)}$*=*$N^{(4)}$*=0 and* $N^{(3)}$*=0 if and only if ξ is Killing.*

Proof.

$$\begin{aligned}
N^{(4)} &= (\mathcal{L}_\xi \eta) = d\eta(\xi) + \xi \lrcorner\, d\eta = 0 \\
N^{(2)}(X,Y) &= d\eta(\phi X,Y) + d\eta(X,\phi Y) = -g(Y,X) + g(X,Y) = 0
\end{aligned}$$

$N^{(4)} = \mathcal{L}_\xi \eta = 0$, and since d$\eta$ is also invariant under ξ, $\mathcal{L}_\xi d\eta$=0,

$$\begin{aligned}
0 = (\mathcal{L}_\xi d\eta)(X,Y) &= \mathcal{L}_\xi d\eta(X,Y) - d\eta(\mathcal{L}_\xi X,Y) - d\eta(X,\mathcal{L}_\xi Y) \\
&= \mathcal{L}_\xi g(X,\phi Y) - g(\mathcal{L}_\xi X,\phi Y) - g(X,\phi\mathcal{L}_\xi Y) \\
&= \xi g(X,\phi Y) - g([\xi,X],\phi Y) - g(X,\phi[\xi,Y]) \qquad (2.11)
\end{aligned}$$

Looking at the Lie derivative of the metric with respect to ξ, we get

$$\begin{aligned}
(\mathcal{L}_\xi g)(X,\phi Y) &= \mathcal{L}_\xi g(X,\phi Y) - g(\mathcal{L}_\xi X,\phi Y) - g(X,\mathcal{L}_\xi \phi Y) \\
&= \xi g(X,\phi Y) - g([\xi,X],\phi Y) - g(X,[\xi,\phi Y]) \qquad (2.12)
\end{aligned}$$

Combining equation (2.12) with (2.11), we get

$$\begin{aligned} 0 &= (\mathcal{L}_\xi g)(X,\phi Y) + g(X,[\xi,\phi Y]) - g(X,\phi[\xi,Y]) \\ &= (\mathcal{L}_\xi g)(X,\phi Y) + g(X,\mathcal{L}_\xi(\phi Y) - \phi\mathcal{L}_\xi Y) \\ &= (\mathcal{L}_\xi g)(X,\phi Y) + g(X,(\mathcal{L}_\xi \phi)Y) \end{aligned}$$

ξ is Killing if and only if $\mathcal{L}_\xi g=0 \Leftrightarrow \mathcal{L}_\xi\phi=0 \Rightarrow N^{(3)}=0$. □

2.4. Sasakian Manifolds

Definition. *Let (M,g) be a Riemannian manifold of real dimension m. We say that (M,g) is* ***Sasakian*** *if m=2n+1 and $(C(M),\bar{g})$ is Kähler . In particular, the holonomy group of the metric cone on M reduces to a subgroup of $U(\frac{m+1}{2})$.*

Historically Sasakian structures were defined via additional conditions on almost contact structures as follows:

Theorem 2.10. *[17] For a nearly Sasakian structure normality is equivalent to contact metric structure. In particular, a normal nearly Sasakian structure is Sasakian.*

Theorem 2.11. *An almost contact metric structure (ξ,η,φ,g) on M is a Sasakian structure if and only if*

$$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.13}$$

Proof. Let (M,g) be embedded in its cone $(C(M),dr^2+r^2g)$. Then the almost contact metric structure (ξ,η,ϕ,g) is Sasakian if and only if the metric of the cone is Kähler with respect to the corresponding complex structure. The complex structure J is Kähler with respect to the metric on the cone if and only if it is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of the metric on the cone (Thm. 8.5, [18])

Recalling the Gauss' formula $\bar{\nabla}_X Y = \nabla_X Y + g(sX,Y)\Psi$ and Weingarten's equation

$\bar{\nabla}_X \Psi = -sX$ for hypersurfaces where $\Psi := r\dfrac{\partial}{\partial r}$ and s is the shape operator (which is -1 in this case) [19], we get

$$\begin{aligned}
0 &= (\bar{\nabla}_X I)(Y) = \bar{\nabla}_X(IY) - I\bar{\nabla}_X Y \\
&= \bar{\nabla}_X(\phi Y + \eta(Y)\Psi) - (\phi + \Psi \otimes \eta)\bar{\nabla}_X \\
&= \nabla_X(\phi Y) + g(sX, \phi Y)\Psi + X(\eta(Y))\Psi - \eta(Y)sX - \psi\nabla_X Y - \eta(\nabla_X Y)\Psi - g(sX, Y)I\Psi \\
&= (\nabla_X \phi)Y + \eta(Y)X - g(X,Y)\xi + [X\eta(Y) - g(X, \phi Y) - \eta(\nabla_X Y)]\Psi
\end{aligned}$$

The term on M gives us the result $(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$, where the Ψ-term gives us a known identity. □

Theorem 2.12. *In a Sasakian manifold* (ξ, η, ϕ, g)*, the following relations hold:*

(i) $\nabla_X \xi = -\phi X$

(ii) $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$

(iii) $R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi$

(iv) $\eta(R(X,Y)Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z)$

Proof. (i) Set Y=ξ in (2.13)

$$\begin{aligned}
(\nabla_X \phi)\xi = g(X,\xi)\xi - \eta(\xi)X &= -\phi(\nabla_X \xi) \\
-\phi^2(\nabla_X \xi) = \eta(X)\phi(\xi) - \phi X &= -\phi X \\
\nabla_X \xi - \eta(\nabla_X \xi)\xi &= -\phi X \\
\nabla_X \xi &= -\phi X
\end{aligned}$$

(ii)

$$\begin{aligned}
R(X,Y)\xi &= \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi \\
&= \nabla_X(-\phi Y) + \nabla_Y(-\phi X) + \phi([X,Y]) \\
&= -(\nabla_X \phi)Y - \phi\nabla_X Y + (\nabla_Y \phi)X - \phi\nabla_Y X + \phi(\nabla_X Y) - \phi(\nabla_Y X) \\
&= -g(X,Y)\xi + \eta(Y)X + g(Y,X)\xi - \eta(X)Y = \eta(Y)X - \eta(X)Y
\end{aligned}$$

(iii)

$$\begin{aligned} g(R(X,Y)\xi,V) &= \eta(Y)g(X,V)-\eta(X)g(Y,V) = g(Y,\xi)g(X,V)-g(X,\xi)g(Y,V) \\ &= g(R(\xi,V)X,Y) = -g(R(V,\xi)X,Y) \\ &\Rightarrow R(V,\xi)X = -g(X,V)\xi + g(X,\xi)V \end{aligned}$$

(iv) $\eta(R(X,Y)Z) = g(R(X,Y)Z,\xi) = -g(R(X,Y)\xi,Z) = -(\eta(Y)g(X,Z)+\eta(X)g(Y,Z)$

□

Theorem 2.13. *A Sasakian manifold cannot be flat.*

Proof. Suppose that a Sasakian manifold is flat, i.e. R(X,Y)Z=0. Choose X=ξ. By (iv) in Theorem 2.12

$$0 = \eta(R(\xi,Y)Z) = \eta(\xi)g(Y,Z)-\eta(Y)g(\xi,Z) = g(Y,Z)-\eta(Y)\eta(Z) = g(\phi Y,\phi Z)$$

which gives us a contradiction if Y=Z, since a Riemannian metric is positive definite.

□

Theorem 2.14. *If a (2n+1)-dimensional Sasakian manifold is of constant curvature k, then k must be unity and its scalar curvature is equal to 2n(2n+1).*

Proof. If a Sasakian manifold is of constant curvature k, then R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y] [12]. But (ii) in Theorem 2.12 implies that k=1.
(iv) in Theorem 2.12 implies that R(X,Y)Z=g(Y,Z)X-g(X,Z)Y. Let e_i be the local orthonormal coordinates on the manifold for i=1,..,2n+1. Then

$$\begin{aligned} Ric(e_j,e_k) &= g(R(e_i,e_j)e_k,e_i) = g(e_j,e_k)g(e_i,e_i)-g(e_i,e_k)g(e_i,e_j) \\ &= (2n+1)g(e_j,e_k)-g(e_j,e_k) \\ &= 2ng(e_j,e_k) \end{aligned}$$

Contracting once more, we get scalar curvature equals to 2n(2n+1). □

Example: An example of a Sasakian manifold is the odd dimensional sphere

$$S^{2n+1} = \{\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n} | z_i |^2 = \sum_{i=1}^{n}((x_i)^2 + (y_i)^2) = 1\}$$

where x_i, y_i denote the real and imaginary components of $z_i \in \mathbb{C}$.

Define $\eta = \sum_{i=0}^{n}(y_i dx_i - x_i dy_i)$ $\quad 0 \le i \le n$

It's easy to see that η is the characteristic 1-form. Then the Reeb vector field is

$$\xi = \sum_{i=0}^{n}(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i})$$

Our standard complex structure is

$$J = \sum_{i=0}^{n}(\frac{\partial}{\partial y_i} \otimes dx_i - \frac{\partial}{\partial x_i} \otimes dy_i)$$

Extending J to the real line bundle generated by ξ:

$$\phi = \sum_{i,j}((x_i x_j - \delta_{ij})\frac{\partial}{\partial x_i} \otimes dy_j - (y_i y_j - \delta_{ij})\frac{\partial}{\partial y_i} \otimes dx_j + x_j y_i \frac{\partial}{\partial y_i} \otimes dy_j - x_i y_j \frac{\partial}{\partial x_i} \otimes dx_i)$$

The extended metric is

$$g = d\eta \circ (\phi \otimes \mathrm{I}) + \eta \otimes \eta$$

With the metric defined above, using Cartan's equation ($\mathcal{L}_X \omega = X \lrcorner\, d\omega + d(X \lrcorner\, \omega)$ for any smooth vector field X and any smooth differential form ω [16]), one can show that ξ is Killing:

$$\mathcal{L}_\xi g = \xi \lrcorner\, (dg) + d(\xi \lrcorner\, g) = \xi \lrcorner\, (0 + d\eta \otimes \eta + \eta \otimes d\eta) + d(0 + 1 \otimes \eta) = 1 \otimes d\eta - 1 \otimes d\eta = 0$$

2.5. Sasaki-Einstein Manifolds

Definition. *A Sasakian manifold (M,g) is* ***Sasaki-Einstein*** *if the metric g is also Einstein, i.e. its Ricci curvature is proportional to its metric by a constant. To formalize it in other form, g is Einstein if* $R_{\mu\nu} = \lambda g_{\mu\nu}$*, where* λ *is a constant called* ***Einstein constant****.*

Lemma 2.15. *Let (M,g) be a Riemannian manifold of dimension n, and consider* $(C(M) = \mathbb{R}^+ \times M, \bar{g})$ *the cone on M with metric* $\bar{g} = dr^2 + r^2 g$*. Then if* $\bar{g}$ *is Ricci-flat if and only if g is Einstein with Einstein constant n-1.*

Proof. Let $\{\theta^i\}$ where $1 \le i \le n$ be a local orthonormal coframe for (M,g), then we obtain a local orthonormal coframe $\{\phi^\mu\}$ for the cone metric $\bar{g}$ on $M \times \mathbb{R}^+$ by setting

$$\phi^i = r\theta^i \qquad \phi^0 = dr$$

where $0 \le \mu \le n$. So let the Greek indices range from 0 to n, the Latin indices range from 1 to n.

Consider the Cartan structure equations:

$$d\phi^\mu + w^\mu_\nu \wedge \phi^\nu = 0 \tag{2.14}$$

$$dw^\mu_\nu + w^\mu_\lambda \wedge w^\lambda_\nu = \frac{1}{2}\bar{R}^\mu_{\upsilon\lambda\rho}\phi^\lambda \wedge \phi^\rho \tag{2.15}$$

where w^μ_ν is the connection 1-form with respect to the Levi-Civita connection on $M \times \mathbb{R}^+$ which is antisymmetric, and $\bar{R}^\mu_{\upsilon\lambda\rho}$ denotes the Riemannian curvature tensor.

Using equation (2.14), one can show that $w^i_0 = \theta^i$:

$$d\phi^0 + w^0_i \wedge \phi^i = d(dr) + w^0_i \wedge r\theta^i = w^0_i \wedge r\theta^i = 0$$

w^j_i are connection 1-forms for the Levi-Civita connection with respect to g on M. Then using equation (2.15), one sees that $\bar{R}^{\mu}_{0\lambda\rho} = 0$:

$$\frac{1}{2}\bar{R}^{i}_{0\lambda\rho}\phi^{\lambda} \wedge \phi^{\rho} = dw^{i}_{\ 0} + w^{i}_{\ \nu} \wedge w^{\nu}_{\ 0} = d\theta^{i} + w^{i}_{\ j} \wedge \theta^{j} = 0$$

where the last equation follows from the Cartan's first structure equation for M, and

$$\bar{R}^{i}_{\ jkl}\phi^{k} \wedge \phi^{l} = 2dw^{i}_{\ j} + 2w^{i}_{\ \lambda} \wedge w^{\lambda}_{\ j} = 2dw^{i}_{\ j} + 2w^{i}_{\ k} \wedge w^{k}_{\ j} + 2w^{i}_{\ 0} \wedge w^{0}_{\ j} = R^{i}_{\ jkl}\phi^{k} \wedge \phi^{l} + \theta^{i} \wedge \theta_{j} - \theta_{j} \wedge \theta^{i}$$

Contracting this equation, one gets the result. □

Proposition 2.16. *Let (M,g) be a Sasakian manifold of dimension 2n+1. Then the metric g is Einstein if and only if the metric of its metric cone (C(M),* $\bar{g}$*) is Ricci-flat. In other words, (C(M),* $\bar{g}$*) is Kähler Ricci-flat, i.e. Calabi-Yau. It follows that the restricted holonomy group is contained in SU(n+1) and that the Einstein constant of g is positive and equals 2n.*

Proof. Follows from Lemma 2.15. □

Definition. *Let (M,g) be a Riemannian manifold of dimension m. (M,g) is said to be* ***3-Sasakian*** *if the metric cone (C(M),*$\bar{g}$*) = (*$\boldsymbol{R}^{+} \times M$*,* $dr^2 + r^2 g$*) on M is hyperkähler.*

Corollary 2.17. *Every 3-Sasakian manifold (M,g) of dimension 4n+3 is Einstein with Einstein constant* $\lambda = 2(2n+1)$*.*

Proof. (M,g) 3-Sasakian ⇒ (C(M),$\bar{g}$) hyperkähler ⇒ (C(M),$\bar{g}$) Ricci-flat ⇒ (M,g) Einstein (by Lemma 2.15). □

3. DIRAC OPERATOR AND KILLING SPINORS

Motivated by quantum mechanics, it is desirable to define a first order operator whose square gives the Laplacian. This first order operator is called the Dirac operator. An educated guess for the Dirac operator would be

$$D = \sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i}$$

Then one sees that, in order to have $D^2 = \Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, γ_i's must satisfy:

$$\gamma_i^2 = -1 \tag{3.1}$$

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 0 \qquad (i \neq j) \tag{3.2}$$

i.e. γ_i's must satisfy Clifford algebra, the algebra multiplicatively generated by n elements $\gamma_1, ..., \gamma_n$ satisfying (3.1) and (3.2) [10].

A *principal bundle* is a fibre bundle, whose fibres are identical to the structure group, which acts on the fibre on the left. *Frame bundle LM* is a principal bundle, associated with the tangent bundle over the manifold M of dimension n. $LM = \bigcup_{p \in M} L_p M$, where $L_p M$ is the set of frames at p and a frame $u = X_1, ..., X_n$ at p is given by $X_\alpha = X_\alpha^\mu \frac{\partial}{\partial x^\mu} |_p$ where $X_\alpha^\mu \in GL(n, \mathbb{R})$ and $\frac{\partial}{\partial x^\mu}$ denotes the coordinate basis on M. One can show that local trivialization of the frame bundle is $GL(n, \mathbb{R})$, any point of this frame bundle can be seen as (p, X_α^μ). For a detailed construction of frame bundles, see [13].

If the frame bundle of a manifold M allows a reduction to the double cover Spin(n) of the structure group $SO(n, \mathbb{R})$, we call M a *spin manifold*. Then, we can define on M the vector bundle S associated with this reduction, which is called the *spinor bundle*. *Spinor fields* over M are sections of the bundle S.

Therefore, spinor fields and Dirac operators cannot be introduced on every Riemannian space, but, nevertheless, they exist for a large class. The existence of a Spin(n)-reduction of the frame bundle on M translates into a topological condition on the manifold, namely, the first two Stiefel-Whitney classes have to vanish [13].

We are interested mainly in the case when (M,g) is a Riemannian spin manifold. The Levi-Civita connection ∇ on TM induces a connection, also denoted by ∇, on any of the spinor bundles S(M), or more appropriately on the sections Γ(S(M)).

Definition. *Let* (M^n, g) *be a Riemannian spin manifold and let S(M) be any spinor bundle. The* ***Dirac operator*** *is the first order differential operator* $D : \Gamma(S(M)) \to \Gamma(S(M))$ *defined by*

$$D\Psi = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} \Psi$$

where $\{e_j\}_{j=1}^n$ *is a local orthonormal frame and* $\cdot$ *denotes the Clifford multiplication, i.e. the operation on the Clifford algebra, satisfying (3.1) and (3.2).*

An interesting question on any spin manifold is the following: What are the eigenvectors of the Dirac operator? Now, we are going to focus on the special sections of certain spinor bundles called *Killing spinor fields* or just *Killing spinors* for short. They are used in supergravity and superstring theory, in particular to find solutions which preserve supersymmetry.

Definition. *Let* (M^n, g) *be a complete Riemannian spin manifold, and let S(M) be a spinor bundle on M and* Ψ *be a smooth section of S(M).* Ψ *is a* ***Killing spinor*** *if for every vector field X, there is an* $\alpha \in \mathbb{C}$*, called* ***Killing number****, such that*

$$\nabla_X \Psi = \alpha X \cdot \Psi$$

Here $X \cdot \Psi$ *denotes the Clifford multiplication of X and* Ψ*.*

We say that Ψ is *imaginary* if $\alpha \in Im(\mathbb{C}^*)$, *parallel* if $\alpha = 0$ and *real* if $\alpha \in Re(\mathbb{C}^*)$.

The standard terminology can be misleading. The Killing spinor Ψ is usually a section of a complex spinor bundle. So a real Killing spinor just means that α is real.

Proposition 3.1. *[10] In a vector space of n-spinors, there exists a positive definite Hermitian scalar product $<,>$ with the invariance property*

$$< x \cdot \Psi, \varphi > + < \Psi, x \cdot \varphi >= 0$$

where $x \in \mathbb{R}^n$, φ, Ψ are spinors.

The name Killing spinor comes from the fact that for any real Killing spinor, there is an associated Killing vector field:

Proposition 3.2. *If Ψ is a nontrivial Killing spinor and α is real, the vector field*

$$X_\Psi = \sum_{j=1}^{n} g(\Psi, e_j \cdot \Psi) e_j$$

is a Killing vector field for the metric g.

Proof. For a local orthonormal frame $e_1, ..., e_n$

$$\begin{aligned}
\nabla_i X_k &= g(\nabla_i \Psi, e_k \cdot \Psi) e_k + g(\Psi, e_k \cdot \nabla_i \Psi) e_k \\
&= g(\alpha e_i \cdot \Psi, e_k \cdot \Psi) e_k + g(\Psi, e_k \cdot \alpha e_i \cdot \Psi) e_k \\
&= \alpha [g(e_i \cdot \Psi, e_k \cdot \Psi) + g(\Psi, e_k \cdot e_i \cdot \Psi)] e_k \\
&= \alpha [-g(\Psi, e_i \cdot e_k \cdot \Psi) + g(\Psi, e_k \cdot e_i \cdot \Psi)] e_k \\
&= \alpha g(\Psi, e_k \cdot e_i \cdot \Psi - e_i \cdot e_k \cdot \Psi) e_k
\end{aligned}$$

Since Clifford multiplication is antisymmetric in i and k, $\nabla_i X_k + \nabla_k X_i = 0$, i.e. X_Ψ is a Killing vector. □

Remark: If Ψ is a Killing spinor on a spin manifold, then

$$D\Psi = \sum_{n}^{j=1} e_j \cdot \nabla_{e_j}\Psi = \sum_{n}^{j=1} e_j \cdot \alpha e_j \cdot \Psi = \alpha \sum_{n}^{j=1} e_j \cdot e_j \cdot \Psi = \alpha(-n)\Psi = -n\alpha\Psi$$

So Killing spinors are eigenspinors of the Dirac operator with eigenvalue $-n\alpha$.

In 1980, Friedrich [20] proved the following theorem:

Theorem 3.3. *Let (M^n, g) be a Riemannian spin manifold which admits a nontrivial Killing spinor with Killing number α. Then (M^n, g) is Einstein with scalar curvature R=4n(n-1)α^2.*

From this theorem, one can see that α^2 must be real, since R is real. So α is either real or pure imaginary.

Proof. (In this proof, we mainly follow the conventions and definitions of [10]).

$$\begin{aligned}
\nabla_X \nabla_Y \Psi &= \nabla_X(\alpha Y \cdot \Psi) = \alpha \nabla_X Y \cdot \Psi + \alpha Y \cdot \nabla_X \Psi = \alpha \nabla_X Y \cdot \Psi + \alpha^2 Y \cdot X \cdot \Psi \\
\mathcal{R}(X,Y)\Psi &= \nabla_X \nabla_Y \Psi - \nabla_Y \nabla_X \Psi - \nabla_{[X,Y]}\Psi \\
&= \alpha \nabla_X Y \cdot \Psi + \alpha^2 Y \cdot X \cdot \Psi - \alpha \nabla_Y X \cdot \Psi - \alpha^2 X \cdot Y \cdot \Psi - \alpha[X,Y] \cdot \Psi \\
&= \alpha^2 (Y \cdot X - X \cdot Y) \cdot \Psi \\
&= \alpha^2 (Y \cdot X + Y \cdot X + 2g(X,Y)) \cdot \Psi \\
&= 2\alpha^2 (Y \cdot X + g(X,Y)) \cdot \Psi \\
\sum_{i=1}^{n} e_i \cdot \mathcal{R}(X, e_i)\Psi &= \sum_{i=1}^{n} e_i \alpha^2 (e_i \cdot X - X \cdot e_i)\Psi \\
Ric(X)\Psi &= -2\sum_{i=1}^{n} e_i \cdot \mathcal{R}(X, e_i)\Psi = -4\alpha^2 \sum_{i=1}^{n} e_i \cdot (e_i \cdot X + g(X, e_i))\Psi \\
&= -4\alpha^2(-nX)\Psi - 4\alpha^2 \sum_{i=1}^{n} g(X, e_i)\Psi = 4\alpha^2(n-1)X \cdot \Psi
\end{aligned}$$

implying that

$$Ric(X) = 4\alpha^2(n-1)X$$
$$Ric(X,Y) = 4\alpha^2(n-1)g(X,Y)$$

Contracting, we get the result $R = 4n(n-1)\alpha^2$. □

So, since $R = 4n(n-1)\alpha^2$, if the Killing number is real, then (M, g) must be an Einstein manifold with positive scalar curvature. In particular, if M is complete, then it is compact by Bonnet-Myer's theorem [12].
On the other hand, if the Killing number is pure imaginary, Friedrich shows that M must be noncompact [21].

Theorem 3.4. *[20] Let (M^n, g) be a compact Riemannian spin manifold and R_0 be the minimum scalar curvature. Then, for any eigenvalue λ of the Dirac operator, we have the inequality*

$$|\lambda|^2 \geq \frac{1}{4}\frac{n}{n-1}R_0$$

Moreover, if the lower bound is an eigenvalue and Ψ is an eigenspinor, then

$$\nabla_X\Psi = \pm\frac{1}{2}\sqrt{\frac{R_0}{n(n-1)}}X\cdot\Psi$$

i.e. Ψ is a real Killing spinor with $\alpha = \pm\frac{1}{2}\sqrt{\frac{R_0}{n(n-1)}}$.

Notice that if a Sasakian manifold M of dimension 2n+1 admits a Killing spinor, then it must also be Einstein with scalar curvature 2n(2n+1) (remember that the Einstein constant was 2n by Lemma 2.15), and Theorem 3.4 also implies that $\alpha = \pm\frac{1}{2}$.

Theorem 3.5. *[21] Every simply connected Sasaki-Einstein manifold admits non-trivial real Killing spinors. Furthermore,* *(i) If M has dimension 4m+1, then (M,g) admits exactly one Killing spinor for each of the values* $\alpha = \pm\frac{1}{2}$.

(ii) If M has dimension 4m+3, then (M,g) admits at least two Killing spinors for one of the values $\alpha = \pm\frac{1}{2}$.

By 1990, as a result of the research by many people, classification of such manifolds in terms of their underlying geometric structures had begun. Friedrich and Kath [22] began their investigation in dimension 5 and showed that a simply-connected compact 5 manifold, which admits a Killing spinor must be Sasaki-Einstein. Then they continued with dimension 7 [23], where they found that the manifolds having the above propoerties must be either weak G_2-manifolds, or 3-Sasakian manifolds, or Sasaki-Einstein manifolds which are not 3-Sasakian. Later, Grünewald [24] investigated 6-manifolds admitting Killing spinors and Hijazi showed that only 8-dimensional manifolds with Killing spinors are the round spheres. Later it was found that round spheres in any dimension and Sasaki-Einstein manifolds in odd dimensions have the desired properties. The problem was reduced to showing that the only possible choices with the given properties in dimensions greater than or equal to 7 are the round spheres in even dimensions and Sasaki-Einstein manifolds in odd dimensions.

Although the relation between parallel spinors and reduced holonomy was anticipated by Hitchin [25] and Bonan [26] first, Wang [27] was the one who showed the correspondence of reduced holonomy and existence of parallel spinors on simply connected Riemannian spin manifolds. This led Bär [28] to a description of the geometry of manifolds admitting real Killing spinors in terms of special holonomies of the associated cones. The correspondence between real Killing spinors on M and parallel spinors on the cone C(M) (equivalently reduced holonomy) is called *Bär's correspondence*:

Theorem 3.6. *Let* (M^n, g) *be a complete Riemannian spin manifold and* $(C(M), \bar{g})$ *is the Riemannian cone. Then there is a 1-1 correspondence between real Killing spinors on* (M, g) *with* $\alpha = \pm\frac{1}{2}$ *and parallel spinors on* $(C(M), \bar{g})$.

Proof. If a parallel spinor exists on $(C(M), \bar{g})$, then $\alpha = 0$, so $\bar{g}$ is Ricci-flat. Then,

(M,g) is Einstein with scalar curvature R=n(n-1) by Lemma 2.15. So

$$\alpha = \pm\frac{1}{2}\sqrt{\frac{R_0}{n(n-1)}} = \pm\frac{1}{2}\sqrt{\frac{n(n-1)}{n(n-1)}} = \pm\frac{1}{2}$$

$\nabla^{\pm}_X = \nabla_X \pm \frac{1}{2}X$ defines a connection in the spin bundle S(M). The connection 1-forms $\omega^{\pm} of \nabla^{\pm}$ are related to the connection 1-form ω of the Levi-Civita connection by $\omega^{\pm} = \omega \pm \frac{1}{2}\beta$, where β is a 1-form called the soldering form. So, parallel spinors on the cone correspond to parallel spinors on (M,g) with respect to the connection $\nabla^{\pm}$, which correspond precisely to real Killing spinors with respect to the Levi-Civita connection.

□

Definition. *A Riemannian spin manifold (M,g) is* ***of type (p,q)*** *if it carries exactly p linearly independent real Killing spinors with* $\alpha > 0$ *and exactly q linearly independent real Killing spinors with* $\alpha < 0$.

Theorem 3.7. *Let* (M,g) *be a complete simply connected Riemannian spin manifold, and let* $Hol(\bar{g})$ *be the holonomy group of the Riemannian cone* $(C(M),\bar{g})$. *Then* (M,g) *admits a nontrivial real Killing spinor with* (M,g) *of type (p,q) if and only if (dim*$(M), Hol(\bar{g})$*, Cone, Geometry, (p,q)) is one of the 6 possible choices listed in table below* $(m \geq 1, n > 1)$*:*

Table 3.1. Classification of complete, simply connected Riemannian spin manifolds admitting nontrivial real Killing spinors.

dim(M)	**Hol(ḡ)**	**Cone**	**Geometry**	**(p,q)**
n	id	flat	round sphere	$(2^{[\frac{n}{2}]}, 2^{[\frac{n}{2}]})$
4m+1	SU(2m+1)	Calabi-Yau	Sasaki-Einstein	(1,1)
4m+3	SU(2m+1)	Calabi-Yau	Sasaki-Einstein	(2,0)
4m+3	Sp(m+1)	Hyperkähler	3-Sasakian	(m+2,0)
7	Spin(7)	Spin(7)	weak G_2	(1,0)
6	G_2	G_2	nearly Kähler	(1,1)

Proof. (Sketch) Assume (M,g) is complete and has a nontrivial real Killing spinor. Then it has positive curvature by Theorem 3.3 and is compact by Bonnet-Myer's theorem. Then, if (M,g) is in addition simply connected, the cone $(C(M), \bar{g})$ is either irreducible or flat [29]. If irreducible, Berger's classification [30] can be used. Wang first showed that the groups listed on Berger's classification but not on the above list admit no parallel spinors. Then upon decomposing the spin representation of the group in question into irreducible pieces, the number of parallel spinors corresponds to the multiplicity of the trivial representation. The flat case was already known [14]. The number of constant spinors on $\mathbb{R}^{n+1}$ with the flat metric was already well-known to be $2^{[\frac{n}{2}]}$ for each of the values $\alpha = \pm\frac{1}{2}$. Knowing that, one can apply Theorem 3.6 and get the result. □

Corollary 3.8. *Let (M^{2n}, g) be a complete Riemannian spin manifold of dimension 2n with $n \neq 3$, admitting a nontrivial real Killing spinor. Then M is isometric to the round sphere.*

Proof. If M carries a Killing spinor, so does its universal cover $\widetilde{M}$. In particular, $\widetilde{M}$ is compact. The only nontrivial holonomy group for 2n-dimensional manifolds in the above list is G_2 for n=3. So, $\widetilde{M}$ is isomorphic to the standart sphere S^n. For n even, the only quotients of S^n are non-orientable (and hence non-spin) projective spaces $\mathbb{RP}^n$. Thus $M = \widetilde{M} = S^n$. □

For complete non-simply connected Riemannian spin manifolds, the classification was done by Wang, Moroianu and Semmelmann [31].

4. CONSTRUCTION OF SASAKI-EINSTEIN METRICS

In this chapter we first show that a metric on a principal U(1)-bundle with a Kähler-Einstein base manifold is Sasaki-Einstein . Then we study two types of explicit constructions of Sasaki-Einstein metrics as U(1)-bundles with Kähler-Einstein base manifolds. Our Kähler-Einstein base manifolds are themselves S^2 bundles. We will see that although the metric on the base manifold has singularities, the metric on the total space, i.e. on the Sasaki-Einstein manifold, can avoid metric singularities. Choosing our parameters appropriately, we can have countably infinite nonsingular Sasaki-Einstein metrics on the total space. First, we work on a 7-dimensional manifold and then generalize the construction to arbitrary odd dimensions following [8]. Then we study the construction given in [7], which is a generalization of [6], in which the first example of a countably infinite family of Sasaki-Einstein metrics was given.

4.1. Sasaki-Einstein Manifolds as U(1)-bundles over Kähler-Einstein Manifolds

In this section we will try to verify the idea presented in [1].

Theorem 4.1. *Let M be a Kähler-Einstein manifold with metric ds^2_{KE}. Then, its U(1) bundle is a Sasaki-Einstein manifold with metric ds^2_{SE}, which can be locally written as*

$$ds^2_{SE} = (d\Psi + 2\mathcal{A})^2 + ds^2_{KE}$$

where $\mathcal{A}$ is a local one form such that $d\mathcal{A}$ is the Kähler form on M.

Proof. Suppose we have a Kähler-Einstein metric g_{ab} on a manifold M_n of dimension n=2m. By the standard formulae of Kaluza-Klein dimensional reduction, where one reduces an (n+1)-dimensional manifold to a n-dimensional manifold by assuming that one of the coordinates define a circle of small radius [32], the (n+1)-dimensional metric

$$d\hat{s}^2 = (d\Psi + 2\mathcal{A})^2 + ds^2$$

has Ricci curvature $\hat{R}_{AB}$ where $A, B = 0, 1, ..., n$ whose components are given in an orthonormal basis by

$$\hat{R}_{ab} = R_{ab} - 2F^c_{\ a}F_{bc}$$
$$\hat{R}_{0a} = \nabla^b F_{ab}$$
$$\hat{R}_{00} = F_{ab}F^{ab}$$

where $F = d\mathcal{A}$, $a, b = 1, ..., n$ and $\hat{e}^0 = d\Psi + 2\mathcal{A}$, $\hat{e}^a = e^a$ constitute an orthonormal basis for the (n+1)-dimensional manifold. (e^a is an orthonormal basis for the Kähler-Einstein manifold.)

Taking $F_{ab} = J_{ab}$, where J_{ab} is the Kähler form on M_n, we have

$$\hat{R}_{ab} = \Lambda g_{ab} - 2J^c_{\ a}J_{bc} = \Lambda g_{ab} - 2g_{ab} = (\Lambda - 2)g_{ab}$$
$$\hat{R}_{0a} = \nabla^b \mu J_{ab} = 0$$
$$\hat{R}_{00} = J_{ab}J^{ab} = n$$

where $R_{ab} = \Lambda g_{ab}$ and the Kähler form is covariantly constant, since M_n is Kähler-Einstein . Choosing $\Lambda = n + 2$ and $\hat{\Lambda} = n$, the (n+1)-dimensional metric $d\hat{s}^2$ will be Einstein.

Now consider the metric cone on this (n+1)-dimensional Einstein metric:

$$d\bar{s}^2 = dt^2 + t^2 d\hat{s}^2 = dt^2 + t^2(d\psi + 2\mathcal{A})^2 + t^2 ds^2 \tag{4.1}$$

Let $B = t^2(d\psi + 2\mathcal{A})$ such that $dB = \Omega = 2tdt \wedge (d\psi + 2\mathcal{A}) + t^2 2d\mathcal{A} = 2tdt \wedge (d\psi + 2\mathcal{A}) + 2t^2 J$.

One sees that Ω defines a Kähler form on the metric cone of dimension n+2=2m+2:

$$d\Omega = -2tdt \wedge 2d\mathcal{A} + 4tdt \wedge J = 0$$
$$\Omega^{m+1} = (2t^2)^m J^m \wedge 2tdt \wedge (d\psi + 2\mathcal{A}) \neq 0$$

The second equation follows from the fact that $J^m \neq 0$ since J is a Kähler form on M_n and Ω^{m+1} becomes a volume form on the cone. Hence the cone is Kähler and the (n+1)-dimensional manifold with the metric $d\hat{s}^2$ is Sasakian. Combining with the above result that it is also Einstein, we obtain an (n+1)-dimensional Sasaki-Einstein metric. □

The relationship between 2m and 2m+2 dimensional Kähler manifolds and 2m+1 dimensional Sasaki-Einstein manifolds was obtained in [1] using spinors, which we introduced in Chapter 3. We can summarize Theorem 4.1 together with the results of [1] as follows:

Table 4.1. Summary of Theorem 4.1 and the results of [1]

(2m+2)-dimensions	**(2m+1)-dimensions**	**2m-dimensions**
Calabi-Yau Cone	Sasaki-Einstein	Kähler-Einstein
covariantly-constant spinor	Killing spinor	gauge covariantly-constant spinor

4.2. A Construction of Sasaki-Einstein Metrics on Seven Dimensional Sasaki-Einstein Manifolds

As mentioned in Theorem 4.1, a Sasaki-Einstein metric can always be viewed as a circle bundle over a Kähler-Einstein base space, with metric (4.1). The Kähler-Einstein base metrics might be singular. This need not necessarily imply that the Sasaki-Einstein metrics on circle bundles over them are going to be singular. For suitable choices of parameters the Sasaki-Einstein metrics can be extended smoothly.

In 2004, in [8], a countably infinite family of explicit Sasaki-Einstein metrics were constructed. To investigate the construction, let us start with 6-dimensional Kähler-Einstein metrics of the form

$$ds_6^2 = dt^2 + c^2(d\tau' + B_{(1)})^2 + a^2 d\Omega_2^2 + b^2 d\tilde{\Omega}_2^2 \tag{4.2}$$

where a, b and c are functions of t and

$$\begin{aligned} d\Omega_2^2 &= d\theta^2 + sin^2\theta d\phi^2 \\ d\tilde{\Omega}_2^2 &= d\tilde{\theta}^2 + sin^2\tilde{\theta} d\tilde{\phi}^2 \end{aligned} \tag{4.3}$$

and the one form $B_{(1)}$ is such that

$$dB_{(1)} = pV_{(2)} + q\tilde{V}_{(2)} \equiv -pdA_{(1)} - qd\tilde{A}_{(1)}$$

where $V_{(2)}$ and $\tilde{V}_{(2)}$ are the volume forms of the two unit 2-spheres, p and q are constants. Without loss of generality, we can take the simplest 1-forms, which gives us volume forms after exterior differentiation:

$$\begin{aligned} A_{(1)} &= cos\theta d\phi \\ \tilde{A}_{(1)} &= cos\tilde{\theta} d\tilde{\phi} \end{aligned} \tag{4.4}$$

By a rescaling, one can choose p and q to be relatively prime integers, which characterize the so called winding numbers of the circle bundle over the two spheres of the base. Without loss of generality, p and q can be taken as positive. In order that the circle bundle to be well-defined, the ratio $\frac{p}{q}$ must be rational, so that the periods dictated for τ' by the consideration of the bundle over each S^2 factor commensurate.

As we proceed to construct Sasaki-Einstein 7-metrics as circle bundles over these 6-metrics, we will see that $\frac{p}{q}$ will no longer need to be rational.

4.2.1. Conditions for the 6-dimensional Base Manifold to be Kähler-Einstein

For the 6-dimensional Kähler-Einstein manifold, choose a complex structure for which the Kähler 2-form is

$$J = cdt \wedge \left(d\tau' + B_{(1)}\right) + a^2\Omega_{(2)} + b^2\tilde{\Omega}_{(2)} \tag{4.5}$$

Since this is a Kähler manifold, dJ should be zero:

$$dJ = -cdt \wedge \left(p\Omega_{(2)} + q\tilde{\Omega}_{(2)}\right) + 2a\dot{a}dt \wedge \Omega_{(2)} + 2b\dot{b}dt \wedge \tilde{\Omega}_{(2)} = 0$$

Thus we have

$$\frac{\dot{a}}{a} = \frac{pc}{2a^2} \tag{4.6}$$

$$\frac{\dot{b}}{b} = \frac{qc}{2b^2} \tag{4.7}$$

where $\dot{a}$ stands for the derivative of a with respect to t. We also want our metric (4.2) to be Einstein. We choose the normalization for the Einstein constant as $R_{\mu\nu} = 8g_{\mu\nu}$. If we calculate the Ricci tensor for the metric, we find that the Einstein condition $R_{\mu\nu} = \lambda g_{\mu\nu}$ implies:

$$\begin{aligned}
-\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{2\dot{a}\dot{b}}{ab} - \frac{\dot{a}\dot{c}}{ac} - \frac{p^2c^2}{2a^4} + \frac{1}{a^2} &= 8 \\
-\frac{\ddot{b}}{b} - \frac{\dot{b}^2}{b^2} - \frac{2\dot{a}\dot{b}}{ab} - \frac{\dot{b}\dot{c}}{bc} - \frac{q^2c^2}{2b^4} + \frac{1}{b^2} &= 8 \\
-\frac{\ddot{c}}{c} - \frac{2\dot{a}\dot{c}}{ac} - \frac{2\dot{b}\dot{c}}{bc} - \frac{p^2c^2}{2a^4} - \frac{q^2c^2}{2b^4} &= 8 \\
-\frac{2\ddot{a}}{a} - \frac{2\ddot{b}}{b} - \frac{\ddot{c}}{c} &= 8
\end{aligned} \tag{4.8}$$

Substituting (4.6) and (4.7) into these equations, we get

$$\frac{\dot{c}}{c} = -\frac{pc}{2a^2} - \frac{qc}{2b^2} + \frac{1 - 8a^2}{pc} \tag{4.9}$$

together with an algebraic constraint

$$p - q + 8(qa^2 - pb^2) = 0 \tag{4.10}$$

So, if (4.6), (4.7), (4.9) and (4.10) are satisfied, then we have ds_6^2 given in 4.2 as a Kähler-Einstein metric.

To solve (4.6), (4.7) and (4.9), introduce a new radial variable r such that dr=cdt. Then we have

$$\begin{aligned} a^2 &= pr + l_1 \\ b^2 &= qr + l_2 \\ c^2 &= \frac{2}{a^2b^2} \int_0^r a^2b^2\left(\frac{1-8a^2}{p}\right)dr' \\ &= -\frac{2r}{12p(pr+l_1)(qr+l_2)}\big[24p^2qr^3 + 4p[8(pl_2 + 2ql_1 - q)]r^2 \\ &+ 6[8l_1(ql_1 + 2pl_2) - (ql_1 + pl_2)]r + 96l_1^2l_2 - 12l_1l_2\big] \end{aligned} \tag{4.11}$$

Without loss of generality, we choose the lower limit of the integration to be $r = 0$. For $r = 0$, the constraint (4.10) becomes $p - q + 8(ql_1 - pl_2) = 0$. It follows that the two integration constants l_1 and l_2 are non-trivial parameters.

4.2.2. Construction of 7-dimensional Sasaki-Einstein Metric

Now introduce a potential $\mathcal{A}_{(1)}$ for J given in (4.5): $\mathcal{A}_{(1)} = rd\tau' - a^2A_{(1)} - b^2\tilde{A}_{(1)}$ Note that,

$$\begin{aligned} d\mathcal{A}_{(1)} &= dr \wedge d\tau' - 2a\dot{a}dt \wedge A_{(1)} - a^2dA_{(1)} - 2b\dot{b}dt \wedge \tilde{A}_{(1)} - b^2d\tilde{A}_{(1)} \\ &= cdt \wedge d\tau' - pcdt \wedge A_{(1)} - qcdt \wedge \tilde{A}_{(1)} + a^2\Omega_{(2)} + b^2\tilde{\Omega}_{(2)} \\ &= cdt \wedge (d\tau' + B_{(1)}) + a^2\Omega_{(2)} + b^2\tilde{\Omega}_{(2)} \\ &= J \end{aligned}$$

Using Theorem 4.1, the 7-dimensional Sasaki-Einstein metric is given by

$$d\hat{s}_7 = (d\psi' + 2\mathcal{A}_{(1)})^2 + ds_6^2 \tag{4.12}$$

whose Ricci tensor of ds_7^2 satisfies $\hat{R}_{\mu\nu} = 6\hat{g}_{\mu\nu}$.

In order to avoid the metric singularities caused by the zeros of a, b and c, we take the radial coordinate r to range between two zeros of the metric function $c(r)$, $r_- \leq r \leq r_+$, for which $a(r)$ and $b(r)$ remain nonvanishing. To have a smooth extension of the metric, $c(r)$ must approach zero at the two endpoints at an appropriate rate. This rate determines the period required for τ' so that the metric on the (r, τ') plane extend smoothly at $r = r_-$ and $r = r_+$. For the metric to extend globally onto a smooth manifold, the periods for τ' at the two endpoints need to be identical and must be consistent with that allowed by the requirement of well-definedness of the 1-form $(d\tau' + B_{(1)})$. Actually, although these criteria are not fulfilled for the 6-dimensional Kähler-Einstein metrics ds_6^2, this does not necessarily imply that the 7-dimensional Sasaki-Einstein metric $d\hat{s}_7^2$ on the circle bundle over ds_6^2 is singular. We therefore need to study the global structure of $d\hat{s}_7^2$.

Define new fiber coordinates τ and ψ, related to τ' and ψ' by

$$\begin{aligned} \psi' &= 2\tau \\ \tau' &= \frac{8\tau - \psi}{8\beta} \end{aligned}$$

In terms of these, we can re-express the Sasaki-Einstein metric (4.12) as

$$\begin{aligned} d\hat{s}_7^2 &= \frac{dr^2}{c^2} + \frac{c^2}{16(c^2 + 4(\beta + r)^2)}(d\psi - A_{(1)} - \tilde{A}_{(1)})^2 + a^2 d\Omega_{(2)}^2 + b^2 d\tilde{\Omega}_{(2)}^2 \\ &+ \frac{c^2 + 4(\beta + r)^2}{\beta^2}\Big(d\tau - l_1 A_{(1)} - l_2 \tilde{A}_{(1)} - \frac{c^2 + 4r(\beta + r)}{8(c^2 + 4(\beta + r)^2)}(d\psi - A_{(1)} - \tilde{A}_{(1)})\Big)^2 \end{aligned} \tag{4.13}$$

where

$$\beta = \frac{8l_1 - 1}{8p} = \frac{8l_2 - 1}{8q} \tag{4.14}$$

The metric has a rescaling symmetry under which $p \to \lambda p$, $q \to \lambda q$ and $r \to \frac{r}{\lambda}$ ($c^2 \to \frac{c^2}{\lambda^2}$, $a^2 \to a^2$, $b^2 \to b^2$). Thus only the ratio of $\frac{p}{q}$ of the parameters p and q is nontrivial. This ratio is determined in terms of l_1 and l_2 by $\frac{p}{q} = \frac{8l_1 - 1}{8l_2 - 1}$.

As we discussed in the beginning of this section, $\frac{p}{q}$ had to be rational in order that the circle bundle over $S^2 \times S^2$ were non-singular, but this requirement is no longer necessary when considering the regularity of the Sasaki-Einstein space. Now the parameters p and q need only satisfy $\frac{p}{q} = \frac{8l_1 - 1}{8l_2 - 1}$. Thus as far as local considerations are concerned, we have a family of Sasaki-Einstein metrics described by the two nontrivial real parameters l_1 and l_2.

In the solutions for the functions a,b,c given by equations (4.11), we chose an integration constant for c^2 so that $r = 0$ is one of the zeroes of the function $c(r)$. Thus we take r to lie in the range $0 \le r \le r_+$, where r_+ is the smallest positive zero of $c(r)$. The functions $a(r)$ and $b(r)$ should remain non-zero in the entire range $0 \le r \le r_+$. Therefore we require the following conditions:

$$\begin{aligned} l_1 > 0, l_2 > 0 \\ r_+ > 0, c(r_+) = 0 \\ (c^2)'(0) > 0 \end{aligned}$$

Together with the condition $c(0) = 0$, we see that $c^2(r) > 0$ for $0 < r < r_+$ and that $(c^2)'(r_+) < 0$.

4.2.3. Structure of the 6-dimensional Base Manifold

Now, we consider the base manifold, whose metric can be obtained from (4.13) by ignoring the last summand. We will see that, it can be viewed as an S^2 bundle over $S^2 \times S^2$, where the S^2 bundle is coordinated by r and ψ. Without loss of generality, assume $0 \le p \le q$, in which case $l_1 > 0$, $l_2 > 0$ and $(c^2)'(0) > 0$ implies that

$$
\begin{aligned}
0 < l_2 < \frac{1}{8} \\
\frac{1}{8}(1 - \frac{p}{q}) < l_1 < \frac{1}{8}
\end{aligned}
$$

The vector $\frac{\partial}{\partial \psi}$ degenerates at the points $r = 0$ and $r = r_+$ where $c(r)$ vanishes. If $c^2(r)$ has slope $(c^2)'(r_0) = \kappa(r_0)$ at one of these endpoints, then writing $c^2(r) \sim \kappa(r_0)(r - r_0)$ nearby, and defining $\rho^2 = r - r_0$, we see that in the (r, ψ) frame we have

$$
\begin{aligned}
\frac{dr^2}{c^2} + \frac{c^2}{16\big(c^2 + 4(\beta + r)^2\big)} d\psi^2 &= \frac{4\rho^2 d\rho^2}{\kappa(r_0)\rho^2} + \frac{\kappa\rho^2}{16\big(c^2(r_0) + 4(\beta + r_0)^2\big)} d\psi^2 \\
&\sim \frac{4}{\kappa}\Big(d\rho^2 + \frac{\kappa^2\rho^2}{256(\beta + r_0)^2} d\psi^2\Big)
\end{aligned}
$$

Looking at the coefficient of $d\psi^2$, we see that the period of ψ is

$$
\Delta\psi = \left| \frac{2\pi \cdot 16(\beta + r_0)}{\kappa(r_0)} \right|
$$

The periods determined by these conditions at $r_0 = 0$ and $r_0 = r_+$ must agree:

$$
\frac{\beta}{\frac{8l_1 - 1}{8p}} = \frac{\beta + r_+}{r_+ + \frac{8l_1 - 1}{8p}} \Rightarrow \beta\Big(r_+ + \frac{8l_1 - 1}{8p}\Big) = (\beta + r_+)\Big(\frac{8l_1 - 1}{8p}\Big)
$$

This is satisfied if $\beta = \dfrac{8l_1 - 1}{8p}$.
Then, together with the previous condition $p - q + 8(ql_1 - pl_2) = 0$, this gives the condition $\beta = \dfrac{8l_1 - 1}{8p} = \dfrac{8l_2 - 1}{8q}$, that we already have. Checking $\Delta\psi$, we see that ψ has period 2π.

4.2.4. Conditions on the U(1)-bundle

The U(1)-fibre parametrized by the coordinate τ never collapses, so it follows that the period of τ is governed only by the connection 1-form on the fibre, given by

$$d\tau - l_1 A_{(1)} - l_2 \tilde{A}_{(1)} - \frac{c^2 + 4r(\beta + r)}{8(c^2 + 4(\beta + r)^2)}(d\psi - A_{(1)} - \tilde{A}_{(1)}) \tag{4.15}$$

The global structure can be examined by looking at the periods of the 1-form over the cycles at $r = 0$ and $r = r_+$ where c^2 vanishes. At $r = 0$, (4.15) becomes $d\tau - l_1 A_{(1)} - l_2 \tilde{A}_{(1)}$, so the periods for $A_{(1)}$ and $\tilde{A}_{(1)}$ given by (4.4) are anti-proportional to l_1 and l_2, respectively. At $r = r_+$, (4.15) becomes $d\tau - l_1 A_{(1)} - l_2 \tilde{A}_{(1)} - \frac{r_+}{8(\beta + r_+)}(d\psi - A_{(1)} - \tilde{A}_{(1)})$, thus the periods are proportional to:

$$\begin{aligned} A_{(1)} &: \left(\frac{r_+}{8(\beta + r_+)} - l_1\right)^{-1} \\ \tilde{A}_{(1)} &: \left(\frac{r_+}{8(\beta + r_+)} - l_2\right)^{-1} \\ d\psi &: \left(\frac{r_+}{8(\beta + r_+)}\right)^{-1} \end{aligned}$$

For (4.15) to be globally extendible, the ratios of the above quantities must all be rational. There are two independent requirements:

$$\frac{l_1}{l_2} = \alpha \in \mathbb{Q} \quad \text{and} \quad \frac{r_+}{(\beta + r_+) l_1} = \gamma \in \mathbb{Q}$$

Then we solve the cubic polynomial that comes from $c(r_+)^2 = 0$, using the above conditions and $p - q + 8(q l_1 - p l_2)) = 0$. We see from (4.11) that l_2 can be expressed purely in terms of α and γ:

$$\begin{aligned} 0 &= 1536\alpha^2\gamma^3 l_2^4 + 64\alpha\gamma^2\big(\alpha(\gamma - 96) - 30\gamma\big) l_2^3 + 8\gamma\big(32\alpha^2(36 - \gamma) + 27\gamma^2 + 4\alpha\gamma(72 + 7\gamma)\big) l_2^2 \\ &+ \big(384\alpha^2(\gamma - 16) - 32\alpha\gamma(24 + 7\gamma) - \gamma^2(192 + 29\gamma)\big) l_2 + 48\alpha(16 - \gamma) + 16\gamma(2\gamma - 3) \end{aligned}$$

Appropriate choices of rational values for α and γ lead to a countably infinite number of solutions for l_2, which is in general real but not necessarily rational, satisfying the condition $0 < l_2 < \frac{1}{8}$.

4.2.5. Some Special Cases

Although in general l_2 does not need to be, and is not, rational; special cases can arise when l_2 is rational. (By a quick observation this immediately implies that $\frac{p}{q}$, β and r_+ are also rational.) Using scaling symmetry, one can choose p and q to be relatively prime integers. In the special case with (p,q)=(1,2), the polynomial for c^2 factorises, giving

$$c^2 = -\frac{r(r+l_2)(128r^2 + 128rl_2 + 64l_2^2 - 1)}{(2r+l_2)(16r + 8l_2 + 1)}$$

and hence

$$r_+ = \frac{\sqrt{2 - 64l_2^2} - 8l_2}{16}$$

Define L such that $64L^2 + 64l_2^2 = 2$. In order r_+ to be rational, it is necessary that L should be rational. If that is the case, then

$$r_+ = \frac{1}{2}(L - l_2)$$

So, the existence of a rational solution becomes equivalent to finding rational solutions for $64L^2 + 64_2^2 = 2$, in which one of L and l_2 must be less than $\frac{1}{8}$ (a condition that we have already required for l_2), and the other greater than $\frac{1}{8}$. Having a rational solution for $64L^2 + 64l_2^2 = 2$ is equivalent to having integer valued solutions to $x^2 + y^2 = 2z^2$. By using a computer enumeration, many, and presumably infinitely many solutions were found [8]. For (p,q)=(1,3), any rational solutions for l_1 and l_2 cannot be found, but it is not clear for them whether such solutions are intrinsically absent, or their search was insufficient. Also for some other values for (p,q), isolated rational solutions have been found.

4.3. A General Class of Sasaki-Einstein Metrics in D≥7

Now we generalize the work done in the previous section for dimensions greater than 7.

Recall that for N Riemannian manifolds $M_1, M_2, ..., M_N$ with metrics $g_1, g_2, ..., g_N$ respectively, $M_1 \times M_2 \times ... \times M_N$ has a Riemannian metric $g_1 \oplus g_2 \oplus ... \oplus g_N$ [19]. Let the d-dimensional Kähler-Einstein space be constructed as a complex line bundle over a product of N Kähler-Einstein spaces, with dimensions n_i and metrics $d\Sigma_{n_i}^2$. Then $d = 2 + \sum_{i=1}^{N} n_i$, and the metric is

$$ds_d^2 = dt^2 + c^2\Big(d\tau' - \sum_{i=1}^{N} p_i A_{(1)}\Big)^2 + \sum_{i=1}^{N} a_i^2 d\Sigma_{n_i}^2$$

where $J_{(2)} = dA_{(1)}^i$ is the Kähler form for the Kähler-Einstein metric $d\Sigma_{n_i}^2$, with Einstein constant λ_i, p_i are constants and a_i and c are functions of t. ds_d^2 is itself a Kähler-Einstein with Einstein constant Λ, provided that the functions c and a_i satisfy the following first-order equations

$$\begin{aligned} \frac{\dot{a}_i}{a_i} &= \frac{p_i c}{2a_i^2} \\ \frac{\dot{c}}{c} &= \frac{\lambda_1 - \Lambda a_1^2}{p_1 c} - \frac{1}{2}\sum_{i=1}^{N}\frac{n_i \dot{a}_i}{a_i} \end{aligned}$$

together with algebraic constraints

$$\lambda_j p_i - \lambda_i p_j + \Lambda(p_j a_i^2 - p_i a_j^2) = 0$$

Note that there are (N-1) independent constraints. The solutions are

$$\begin{aligned} a_i^2 &= p_i r + l_i \\ c^2 &= \frac{2}{\prod_{i=1}^{N} a_i^{n_i}} \int_0^r \frac{\lambda_1 - \Lambda a_1^2}{p_1} \prod_{i=1}^{N} a_i^{n_i} \end{aligned}$$

where r is defined by $dr = cdt$. As in the 7-dimensional case, we choose the lower limit of integration to be r=0.

The integration constants l_i satisfy the constraints

$$\beta = \frac{\Lambda l_i - \lambda_i}{\Lambda p_i}$$

where β is a constant for all i.

The $D = d + 1 = 3 + \sum_{i=1}^{N} n_i$ dimensional Sasaki-Einstein metric is given by

$$ds_D^2 = (d\psi' + 2\mathcal{A}_{(1)})^2 + ds_d^2 \tag{4.16}$$

where

$$\mathcal{A}_{(1)} = r d\tau' - \sum_{i=1}^{N} a_i^2 A_{(1)}^i$$

For the solution to be Einstein, we must have $\Lambda = 4 + \sum_{i=1}^{N} n_i$ (after choosing, without loss of generality, $\lambda_i = 1$).

Next, apply the coordinate transformations:

$$\psi' = 2\tau \qquad\qquad \tau' = \beta^{-1}(\tau - \Lambda^{-1}\psi)$$

The metric (4.16) becomes

$$\begin{aligned} ds_D^2 &= \frac{dr^2}{c^2} + \frac{4c^2}{\Lambda^2(c^2 + 4(\beta + r)^2)}\Big(d\psi - \sum_{i=1}^{N} \lambda_i A_{(1)}^i\Big)^2 + \sum_{i=1}^{N} a_i^2 d\Sigma_{n_i}^2 \\ &+ \frac{c^2 + 4(\beta + r)^2}{\beta^2}\left(d\tau - \sum_{i=1}^{N} l_i A_{(1)}^i - \frac{c^2 + 4r(\beta + r)}{\Lambda(c^2 + 4(\beta + r)^2)}\big(d\psi - \sum_{i=1}^{N} \lambda_i A_{(1)}^i\big)\right)^2 \end{aligned}$$

As before, the rate of the collapsing of the circle parametrized by ψ should be the same at all the roots of $c^2(r) = 0$. That implies that the period of ψ is 2π. Moreover, consideration of the connection on fibres parametrized by τ (which never shrinks to zero) implies

$$\frac{l_i}{l_N} = \alpha_i \in \mathbb{Q} \quad i = 1, 2, ..., N-1$$
$$\frac{r_+}{\Lambda(\beta + r_+)l_N} = \gamma \in \mathbb{Q}$$

Substituting the above conditions and $\beta = \dfrac{\Lambda l_i - \lambda_i}{\Lambda p_i}$ is a constant into the equation $c^2 = 0$, one gets a polynomial equation in l_N of order $1 + \frac{1}{2}\sum_{i=1}^{N} n_i$, with rational coefficients that are polynomials in α_i and γ. Without loss of generality, we can choose $0 \leq p_1 \leq p_2 \leq ... \leq p_N$. The constant l_N must lie in the range $0 < l_N < \Lambda^{-1}$. Then the corresponding set of rational numbers (α_i, γ) gives a non-singular Sasaki-Einstein metric.

In [8], the case when l_N is rational is investigated as a special case. In this case, after rescaling, the parameters p_i can be chosen as relatively prime integers. Such solutions seems to occur sporadically, and in [8] no explanation can be found.

4.4. A Different Construction of Metrics on Sasaki-Einstein Manifolds

In this section, we follow first [33] and then [7].

In [6], a countably infinite class of metrics on 5-dimensional Sasaki-Einstein manifolds were given for the first time. In [7], the authors generalized this construction to arbitrary odd dimensions.

This time we start with a metric $d\hat{s}^2$ on a (2n+2)-dimensional manifold, having the form

$$d\hat{s}^2 = \alpha^2 dr^2 + \beta^2(d\tau - 2A)^2 + \gamma^2 ds^2 \qquad (4.17)$$

where ds^2 is a Kähler-Einstein metric on a 2n-dimensional positive curvature Kähler manifold with Einstein constant λ and Kähler from J, r and τ are two additional coordinates on the (2n+2)-dimensional manifold, A is a 1-form such that $dA = J$ and α, β and γ are functions of r.

4.4.1. Conditions for the (2n+2)-dimensional base manifold to be Kähler-Einstein

Let $\{e^i\}_{i=2,...,2n+1}$ be an orthonormal dual basis for the 2n-dimensional manifold. We choose an orthonormal dual basis for (4.17) as

$$\hat{e}^0 = \alpha dr \qquad \hat{e}^1 = \beta(d\tau - 2A) \qquad \hat{e}^i = \gamma e^i \tag{4.18}$$

Using Cartan's structure equations (2.14) and (2.15), one can calculate the curvature 2-forms $\hat{\Theta}^a_{\ b} = \frac{1}{2}\hat{R}^a_{bcd}\hat{e}^c\hat{e}^d = d\hat{w}^a_{\ b} + \hat{w}^a_{\ c} \wedge \hat{w}^c_{\ b}$ where $a = 0, 1, ..., 2n+1$:

$$\hat{\Theta}^0_{\ 1} = \left(-\frac{\beta''}{\alpha^2\beta} + \frac{\alpha'\beta'}{\alpha^3\beta} \right)\hat{e}^0 \wedge \hat{e}^1 + 2\left(\frac{\beta'}{\alpha} - \frac{\beta\gamma'}{\alpha\gamma} \right)F \tag{4.19}$$

$$\hat{\Theta}^0_{\ i} = \left(-\frac{\gamma''}{\alpha^2\gamma} + \frac{\alpha'\gamma'}{\alpha^3\gamma} \right)\hat{e}^0 \wedge \hat{e}^i + \left(\frac{\beta'}{\alpha\gamma^2} - \frac{\beta\gamma'}{\alpha\gamma^3} \right)F_{ij}\hat{e}^1 \wedge \hat{e}^j \tag{4.20}$$

$$\hat{\Theta}^1_{\ i} = \left(-\frac{\beta'\gamma'}{\alpha^2\beta\gamma}\delta_{ij} - \frac{\beta^2}{\gamma^4}F_{jk}F^k_{\ i} \right)\hat{e}^1 \wedge \hat{e}^j - \frac{\beta}{\gamma^3}\nabla_k F_{ij}\hat{e}^k \wedge \hat{e}^j + \left(-\frac{\beta'}{\alpha\gamma^2} + \frac{\beta\gamma'}{\alpha\gamma^3} \right)F_{ij}\hat{e}^0 \wedge \hat{e}^j \tag{4.21}$$

$$\hat{\Theta}^i_{\ j} = \Theta^i_{\ j} - \left(\frac{\gamma'}{\alpha\gamma} \right)^2 \hat{e}^i \wedge \hat{e}^j - \frac{\beta^2}{\gamma^4}(F_{ij}F_{km} + F_{ik}F_{jm})\hat{e}^k \wedge \hat{e}^m + 2\left(\frac{\beta'}{\alpha\gamma^2} - \frac{\beta\gamma'}{\alpha\gamma^3} \right)F_{ij}\hat{e}^0 \wedge \hat{e}^1 - \frac{\beta}{\gamma^3}\nabla_k F_{ij}\hat{e}^1 \wedge \hat{e}^k \tag{4.22}$$

where prime denotes differentiation with respect to r, $F = \frac{1}{2}F_{ij}e^i \wedge e^j = dA$. One can then calculate the components of the Ricci tensor $\hat{R}_{ab} = \hat{R}^c_{\ acb}$ requiring $\hat{R}_{ab} = \Lambda\delta_{ab}$, since we are looking for a Einstein metric in an orthonormal basis. Subtracting the equations of $\hat{R}_{00}$ and $\hat{R}_{11}$ from each other and choosing α and β such that $\alpha\beta$ equals to a constant, which we denote by c, we are left with an equation of γ only. $\gamma^2 = c(1 - r^2)$

satisfies this equation, where $c = \alpha\beta$, a constant. Then solving $\hat{R}_{ij} = \Lambda\delta_{ij}$ gives

$$\beta^2 = c^2\alpha^{-2} = c^2(1-r^2)^{-n}P(r) \tag{4.23}$$

where $P(r)$ is a polynomial of degree (2n+2) satisfying

$$\frac{d}{dr}(r^{-1}P) = r^{-2}[\Lambda(1-r^2)^{n+1} - \lambda c^{-1}(1-r^2)^n] \tag{4.24}$$

Changing α, β and γ as $\alpha\beta = c$ and $\gamma^2 = c(1-r^2)$, we can rewrite our metric (4.17) as

$$d\hat{s}^2 = (1-r^2)^n P(r)^{-1}dr^2 + c^2(1-r^2)^{-n}P(d\tau - 2A)^2 + c(1-r^2)ds^2 \tag{4.25}$$

which is Einstein with Einstein constant Λ, provided that ds^2 is Kähler-Einstein with Einstein constant λ and Kähler form $J = dA$. If we further introduce ρ and U, defined by,

$$\rho^2 = c(1-r^2) \qquad U = c(1-r^2)^{-n-1}P \tag{4.26}$$

We can rewrite (4.24) in terms of ρ and U and get

$$\frac{1}{\sqrt{1-\frac{\rho^2}{c}}}\frac{d}{d\rho}\left(\frac{1}{\sqrt{1-\frac{\rho^2}{c}}}U\rho^{2n+2}\right) = -\frac{1}{1-\frac{\rho^2}{c}}\left[\Lambda\rho^{2n+3} - \lambda\rho^{2n+1}\right] \tag{4.27}$$

We can solve this differential equation in the limit $|c| \to \infty$ while holding ρ and U finite, where (4.27) reduces to

$$\frac{d}{d\rho}(\rho^{2n+2}U) = \lambda\rho^{2n+1} - \Lambda\rho^{2n+3} \tag{4.28}$$

and the (2n+2)-dimensional Kähler-Einstein metric (4.25) becomes

$$d\hat{s}^2 = U^{-1}d\rho^2 + U\rho^2(d\tau - 2A)^2 + \rho^2 ds^2 \tag{4.29}$$

When one solves U from (4.28), we see that the general solution for U is

$$U = \frac{\lambda}{2n+2} - \frac{\Lambda\rho^2}{2n+4} + \frac{\Lambda}{2(n+1)(n+2)}\left(\frac{\lambda}{\Lambda}\right)^{n+2}\frac{\kappa}{\rho^{2n+2}} \tag{4.30}$$

where $\frac{\Lambda}{2(n+1)(n+2)}\left(\frac{\lambda}{\Lambda}\right)^{n+2}\kappa$ is an integration constant. Now, we want our (2n+2)-dimensional manifold with the metric (4.25) to be also Kähler . The Kähler form of the manifold can be given by

$$\hat{J} = \rho^2 J + \rho(d\tau - 2A) \wedge d\rho \tag{4.31}$$

since

$$d\hat{J} = 2\rho d\rho \wedge J + \rho d\rho \wedge (-2dA) = 2\rho d\rho \wedge J - \rho d\rho \wedge 2J = 0$$
$$(\hat{J})^{n+1} = \rho^{2n} J^n \wedge \rho(d\tau - 2A) \wedge d\rho \neq 0$$

In [33], it is shown that the (2n+2)-dimensional metric (4.25) describes a complete metric on a manifold if and only if $\kappa = 0$, which is the case of $\mathbb{C}P^n$ as the base manifold and $\mathbb{C}P^{n+1}$ as the total space.

4.4.2. Construction of (2n+3)-dimensional Sasaki-Einstein Metric

Now we construct a local Sasaki-Einstein metric on a (2n+3)-dimensional manifold as in Section 4.1. We write the metric as

$$d\tilde{s}^2 = d\hat{s}^2 + (d\psi' + 2\sigma)^2 \tag{4.32}$$

where $d\sigma = 2\hat{J}$. As in section 4.25, if Λ=2(n+2), then this (2n+3)-dimensional metric has Einstein constant 2n+2. σ can be chosen as the following:

$$\sigma = \frac{\lambda}{\Lambda}A + \frac{1}{2}(\frac{\lambda}{\Lambda} - \rho^2)(d\tau - 2A) \tag{4.33}$$

From which we get

$$d\sigma = \frac{\lambda}{\Lambda}dA + (\frac{\lambda}{\Lambda} - \rho^2)(-dA) + (-\rho)d\rho \wedge (d\tau - 2A) = \rho^2 J + \rho(d\tau - 2A) \wedge d\rho = \hat{J}$$

Next, we define the new coordinates α and ψ as

$$\theta = -\tau - \frac{\Lambda}{\lambda}\psi' \qquad \frac{\Lambda}{\lambda}\psi' = \psi \tag{4.34}$$

In these coordinates the metric (4.32) becomes

$$d\tilde{s}^2 = U(\rho)^{-1}d\rho^2 + \rho^2 ds^2 + q(\rho)(d\psi + 2A)^2 + w(\rho)[d\theta + f(\rho)(d\psi + 2A)]^2 \tag{4.35}$$

where

$$w(\rho) = \rho^2 U(\rho) + (\rho^2 - \frac{\lambda}{\Lambda})^2 \tag{4.36}$$

$$q(\rho) = \frac{\lambda^2}{\Lambda^2}\frac{\rho^2 U(\rho)}{w(\rho)} \tag{4.37}$$

$$f(\rho) = \frac{\rho^2(U(\rho) + \rho^2 - \frac{\lambda}{\Lambda})}{w(\rho)} \tag{4.38}$$

To have a Riemannian metric, we should have $U \geq 0$, $w \geq 0$ and $q \geq 0$ in order to preserve positive definiteness. From (4.36), (4.37) and (4.38), one can deduce that it is enough to have $U \geq 0$. To guarantee that, we choose the range of ρ as $\rho_1 \leq \rho \leq \rho_2$, where ρ_i are two appropriate roots of the equation $U(\rho) = 0$ such that $U(\rho) \geq 0$ in that interval. We also want to exclude $\rho = 0$, where we have a metric singularity. Therefore, without loss of generality, we take both of the roots positive. Another metric singularity occurs at $w = 0$. To prevent this situation, we choose $\rho_i^2 \neq \frac{\lambda}{\Lambda}$, so that w never vanishes on the interval allowed for ρ.

We define $x = \frac{\Lambda}{\lambda}\rho^2$ and introduce the polynomial

$$P(x,\kappa) = -(n+1)x^{n+2} + (n+2)x^{n+1} + \kappa \tag{4.39}$$

and write $U(\rho)$ as

$$U(\rho) = \frac{\lambda}{2(n+1)(n+2)}\frac{1}{x^{n+1}}P(x,\kappa) \tag{4.40}$$

For fixed κ, a zero of $U(\rho)$ corresponds to a root of $P(x,\kappa) = 0$. $P(x,\kappa)$ has critical points at $x = 0$ and $x = 1$. At $x = 1$, it has a maximum. For any $\frac{\Lambda}{\lambda}\rho^2 \geq 0$, if κ is positive, then there is only a single real root. Similarly, if $\kappa < -1$, there are no real roots and if $\kappa = -1$, there is only one real root. Therefore these cases are excluded, and we take the range of κ as $-1 < \kappa \leq 0$. Since $\kappa = 0$ corresponds to a "known" case as mentioned above, we also exclude 0 from the range of κ.

4.4.3. Structure of the (2n+2)-dimensional Base Manifold

Now, we first look at the (2n+2)-dimensional manifold, transverse to the θ-direction. We'll see that it is the total space of an S^2-bundle over the 2n-dimensional Kähler-Einstein manifold. Taking the range of ρ as $[\rho_1, \rho_2]$, if ψ is periodically identified, the $\rho - \psi$ part of the metric at a fixed point on the base is a circle fibred over the line segment with coordinate ρ, where the size of the circle goes to zero at the endpoints ρ_1 and ρ_2.

Near a root ρ_i, we have $U(\rho) \approx U'(\rho_i)(\rho - \rho_i)$. We define a new coordinate

$$R^2 = \frac{4(\rho - \rho_i)}{U'(\rho_i)} \tag{4.41}$$

Then $2RdR = \frac{4d\rho}{U'(\rho_i)}$ and at any fixed point on the base near the root ρ_i, the $\rho - \psi$ part of the metric becomes

$$\begin{aligned} U(\rho)^{-1}d\rho^2 + \frac{\lambda^2}{\Lambda^2}\frac{\rho_i^2 U(\rho_i)}{\rho_i^2 U(\rho_i) + (\rho_i^2 - \frac{\lambda}{\Lambda})^2}d\psi^2 &= \frac{1}{4}\frac{R^2 dR^2 (U'(\rho_i))^2}{U'(\rho_i)(\rho - \rho_i)} + \frac{\lambda^2}{\Lambda^2}\frac{\rho_i^2 U'(\rho_i)(\rho - \rho_i)}{(\rho_i^2 - \frac{\lambda}{\Lambda})^2}d\psi^2 \\ &= \frac{1}{4}\frac{R^2 dR^2 (U'(\rho_i))^2}{U'(\rho_i) R^2 U'(\rho_i)\frac{1}{4}} + \frac{\lambda^2}{\Lambda^2}\frac{\rho_i^2 (U'(\rho_i))^2 R^2}{4(\rho_i^2 - \frac{\lambda}{\Lambda})^2}d\psi^2 \\ &= dR^2 + \frac{\lambda^2}{4\Lambda^2}\frac{\rho_i^2 (U'(\rho_i))^2}{(\rho_i^2 - \frac{\lambda}{\Lambda})^2}R^2 d\psi^2 \end{aligned} \tag{4.42}$$

Using (4.30), one can directly calculate that $\dfrac{\rho_i^2(U'(\rho_i))^2}{(\rho_i^2 - \frac{\lambda}{\Lambda})^2} = \Lambda^2$. Hence (4.42) becomes $dR^2 + \frac{\lambda^2}{4}R^2 d\psi^2$.

Remember that λ was the Einstein constant of the 2n-dimensional manifold. The metric can be rescaled so that $\lambda = 2$. Then we will have the period of ψ as 2π and $\rho - \psi$ part of the metric as $dR^2 + R^2 d\psi^2$. Hence the $\rho - \psi$ fibre is topologically S^2.

4.4.4. Conditions on the U(1)-bundle

Now we want to check the θ-coordinate in order to obtain a principle U(1) bundle over the (2n+2)-dimensional space for appropriate values for κ. Since $w(\rho)$ is greater that 0 for $\rho_1 \le \rho \le \rho_2$, $[d\theta + f(\rho)(d\psi + 2A)]$ determines the period of the U(1)-bundle parametrized by θ. As was done in Section 4.2, we should check the periods of θ at the endpoints $\rho = \rho_1$ and $\rho = \rho_2$.

For the periods to be rationally related, we should have $\dfrac{f(\rho_1)}{f(\rho_2)} \in \mathbb{Q}$. Define

$$R(\kappa) \equiv \frac{f(\rho_1)}{f(\rho_2)} = \frac{\rho_1^2(\rho_2^2 - \frac{\lambda}{\Lambda})}{\rho_2^2(\rho_1^2 - \frac{\lambda}{\Lambda})} = \frac{x_1(x_2 - 1)}{x_2(x_1 - 1)} \tag{4.43}$$

where x_1, x_2 are roots of $P(x,\kappa)$ corresponding to ρ_1 and ρ_2 accordingly. R is a continuous function of $\kappa \in (-1, 0]$. Moreover, $R(0) = 0$ and in the limit $\kappa \to -1$, $R(-1) = -1$, since at $\kappa = 0$, the smaller root is 0 and at $\kappa = -1$, there is only one real root, hence in the limit, x_1 and x_2 approach each other from opposite sides. According to the definition of R in (4.43), R must be rational in the interval $(-1, 0]$, and there are countably infinite number of values of κ for which $R(\kappa)$ is rational. For these values of κ, we have a principle U(1)-bundle over M_{2n+2}, and hence a complete Sasaki-Einstein metric on a (2n+3)-dimensional manifold.

In Chapter 2, we saw that every Sasakian manifold has a contact structure on it. Here, the contact structure consists of the vector field $\xi = \dfrac{\partial}{\partial\psi'}$, its dual η and the (1,1)-

tensor constructed by raising an index of $d\eta$ with the metric. $\xi = \frac{\partial}{\partial\psi'} = \frac{\Lambda}{\lambda}\left(\frac{\partial}{\partial\psi} - \frac{\partial}{\partial\theta}\right)$ is globally defined on the manifold. The 1-form η may be written as

$$\eta = (\rho^2 - \frac{\lambda}{\Lambda})[d\theta + f(\rho)(d\psi + 2A)] + \frac{\Lambda}{\lambda}q(\rho)(d\psi + 2A) \tag{4.44}$$

One can directly calculate that $\eta(\xi) = 1$. Moreover, calculating $d\eta$, one sees that $(d\eta)^{2n+2}$ has a nonvanishing $d\rho \wedge [d\theta + f(\rho)(d\psi + 2A)] \wedge (dA)^n$ term, where $dA = J$. Hence $\eta \wedge d\eta$ is nonvanishing due to the $d\rho \wedge d\theta \wedge (d\psi + 2A) \wedge (J)^n$ term. Hence we have a contact manifold.

5. CONCLUSION

In this thesis, we started with an introduction of Sasaki-Einstein manifolds. We gave some of the equivalent definitions of a Sasaki-Einstein manifold, along its most commonly used description, namely to have a Calabi-Yau metric cone. Then we discussed some of their general properties, such as having positive scalar curvature and admitting real Killing spinors if they are simply-connected. Since their explicit metrics other than some specific examples were not known until 2004, we studied two examples of their metric constructions given in [8] and [7]. In these constructions, we described the Sasaki-Einstein manifold as a U(1)-bundle over a Kähler-Einstein base manifold, and the Kähler-Einstein base was seen as an S^2 bundle. Choosing the parameters appropriately, we obtained countably infinite families of Sasaki-Einstein metrics. In Section 4.2, we started with 7-dimensional Sasaki-Einstein manifolds, where we had 6-dimensional basis as S^2 bundle over $S^2 \times S^2$. In Section 4.3 the basis was an S^2 bundle over a product of Kähler-Einstein manifolds. This construction was indeed a generalization of the construction in Section 4.2 to arbitrary odd dimensions greater than or equal to 7. In Section 4.4, we dealt with metrics on U(1) bundles over a base, which is itself an S^2 bundle over a Kähler-Einstein manifold. This last construction is a generalization of [6], where the first countably infinite family of Sasaki-Einstein metrics in dimension 5 were given in 2004, to arbitrary odd dimensions.

As we mentioned in the introduction, metrics that we obtained are of cohomogenity one. In 2006, in [9], another family of Sasaki-Einstein metrics of cohomogenity two were constructed. A classification of metrics on Sasaki-Einstein manifolds according their (co)homogenity can be found in [11]. The subject is still actively studied by physicists, due to its importance for the supersymmetric theories. Also the classification of Sasaki-Einstein manifolds is an open problem for mathematicians. Even in 5-dimensions, only simply connected Sasaki-Einstein manifolds, which are homogeneous and cohomogenity one are classified. A general classification of Sasaki-Einstein manifolds is still not found. Especially for 5-dimensional simply-connected Sasaki-Einstein manifolds, this is an active field of research [2].

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