SCALING SOLUTIONS OF $\mathcal{N} = 2$ SUPERGRAVITY AND HOLOGRAPHY

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ABSTRACT

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The $4D$ $\mathcal{N} = 2$ SUGRA is the low energy limit of Type II string theory compactified on a Calabi-Yau three-fold. The theory contains BPS black holes with multi-centered configurations consisting of bound states of dyonic single-centered black holes. These stationary solutions have an intrinsic angular momentum originating in electromagnetic interaction between charges. A subset of this family called "*scaling solutions*" for which the relative distances between centers are constrained *up to a scale*. Contrary to the asymptotically flat non-scaling cases, scaling solutions have an AdS_2 factor in their asymptotic geometry and their total angular momentum vanishes.

We showed that the AdS_2 spacetime at far infinity is equipped with an S^2 fiber called a "**twist**". It can be thought of as a rotating two-sphere whose angular velocity is determined by the total charge and the total dipole moment of the black hole. In the absence of an angular momentum, the twist can be considered as a **topological hair** of this family of supersymmetric black holes. It provides a new asymptotic observable to distinguish different black hole solutions.

Moreover, to have a better understanding of the twist and its potential role in a quantum theory of supersymmetric black holes, we S^2 reduced the asymptotic geometry of the scaling solutions. That leads to an effective $2D$ theory with an AdS_2 geometry and a single $U(1)$ gauge field presenting the twist. Consequently, one would expect the AdS/CFT correspondence to clarify the role of the twist as a new hair of these black holes.

ÖZET

$\mathcal{N} = 2$ SÜPER KÜTLEÇEKİMİNİN ÖLÇEKLENEBİLİR ÇÖZÜMLERİ VE HOLOGRAFİ

4 boyutlu $\mathcal{N} = 2$ süperkütleçekim teorisi, Calabi-Yau trifold üzerinde sıkıştırılmış Tip-II sicim teorisinin düşük enerji limitidir ve dyon yükü taşıyan tek-merkezli kara deliklerin çok merkezli bağlı durumlarının konfigürasyonundan oluşan BPS kara deliklerini içermektedir. Hareketsiz olan bu çözümlerin, yükler arasındaki etkileşmeden dolayı içkin bir açısal momentumu vardır. Bu çözüm ailesinin bir alt kümesi olan "ölçeklenebilir çözümler" in merkezleri arasındaki göreceli uzaklık serbestçe ölçeklenebilmektedir. Asimptotik olarak düz olan ölçeklenemeyen durumların tersine, ölçeklenebilir çözümlerin asimptotik geometrisi AdS_2 faktörü içermekte ve açısal momentumu sıfır olmaktadır.

AdS_2 uzayzamanın sonsuz uzaklık limitinde "kıvrılma" (twist) olarak adlandırılan S^2 fiberine sahip olduğunu gösterdik. Bu, kara deliğin toplam yük ve toplam dipol momenti tarafından belirlenen bir açısal hızda dönen küre (S^2) olarak düşünülebilir. Açısal momentumun olmadığı durumda, **kıvrılma**, bu süpersimetrik kara delik ailesi için **topolojik saç** olarak görülebilir ve dolayısıyla farklı kara delik çözümlerini ayırt etmek için yeni bir asimptotik gözlemlenebilir sağlamaktadır.

'Kıvrılma'nın daha iyi anlaşılabilmesi ve süpersimetrik karadeliklerin kuantum teorisindeki potensiyel rolünün anlaşılabilmesi için ölçeklenebilir çözümlerin asimptotik geometrisini S^2 indirgedik. AdS_2 geometrisine sahip 2 boyutlu efektif teoriyi ve kıvrılmayı temsil eden tek bir $U(1)$ ayar alanını elde ettik. Sonuç olarak, kıvrılmanın bu kara deliklerin saçı olma rolünün AdS/CFT benzeşmesi sayesinde açığa çıkarılması öngörülmektedir.

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$a, b, \cdots$	spacetime indices in $2D$
$(D_{\tilde{A}}, D^{\tilde{A}})$	Even cohomology basis of $CY3$
$\mathcal{F}$	Kähler pre-potential
F	Electromagnetic field strength
G	Newton's Gravitational constant
$H^{(r,s)}$	(r, s)-th cohomology group
$h^{(r,s)}$	Hodge number of $H^{(r,s)}$
J	Almost complex structure
$\vec{J}$	Angular momentum
$\mathcal{K}$	Kähler potential
$\mathcal{L}$	Lagrangian
$M, N, \cdots$	spacetime indices in $10D$
$\mathcal{N}$	Determines the number of supercharges
$(p^{\tilde{A}}, q_{\tilde{A}})$	Magnetic and electric charge
$\mathfrak{R}$	Ricci 2-form
$\mathcal{R}$	Ricci scalar
$\mathcal{S}$	Action
S^2	Tow sphere
Z	Central charge
$\vec{\Delta}$	Dipole moment
$\mu, \nu, \cdots$	spacetime indices in $4D$
Γ^a	Dyonic charge
ω	The one-form twist
Ω	Kähler 2-form
$\star$	Hodge star
$\wedge$	Wedge product
∇	Covariant derivative

LIST OF ACRONYMS/ABBREVIATIONS

2D	Two Dimensional
4D	Three Dimensional
10D	Ten Dimensional
AdS	Anti de-Sitter spacetime
CFT	Conformal Field Theory
$CY3$	Calabi-Yau threefold
BH	Bekenstein-Hawking
EH	Einstein-Hilbert
EoM	Equations of Motion
GR	General Relativity
KK	Kaluza-Klein
vev	vacuum expectation value
w.r.t	with respect to
R	Ramond sector
NS	Neveu-Schwarz
BPS solution	Bogomol'nyi–Prasad–Sommerfield solution

1. INTRODUCTION

1.1. Foundations of Modern Physics

Both the theory of general relativity and quantum mechanics are fundamental theories in physics. Their predictions, achievements and accuracy has been amazing not only in physics but even when compared with other revolutionary theories in natural sciences. The former was created by Einstein, who started with questioning the traditional picture of space and time as two absolute and independent notions, a point of view that goes back to Aristotle and later made into a precise theory by Newton. The idea of combining space and time together as a dynamical object called "spacetime" in addition to a geometric interpretation of the gravitational force led to a theory that naturally predicts the existence of black holes as singularities in spacetime, gravitational radiation and lensing effect among other predictions each of which has been observed. In the latest observational proof of the theory, reported in the beginning of 2019, a supermassive black hole and its event horizon was detected at the center of a galaxy located around 55 million light years away from us [2]. The picture captured by a global network of radio telescopes was the first picture containing that much details about the structure of a concrete black hole, a breakthrough that provided strong evidence for the validity of general relativity at the very strong gravitational regime.

On the other side, we have quantum mechanics, another theory that revolutionized some other notions from Newtonian physics such as pointlike particles following a single path. It instead talks about probabilities of existence of a particle at a specific point in spacetime. Later, the theory evolved to its relativistic version, i.e., the theory of quantum fields. In that theory, particles are considered as quanta of fields spread in spacetime. A very successful example of a quantum field theory is *the standard model of particle physics* that categorizes all known matter particles as well as the mediators of all forces, except gravity, in a compatible way. Furthermore, it has enabled us to calculate cross sections of high-energy colliding particles with a stunning accuracy and not a single sign of its violation has been observed so far.

1.1.1. Towards a Unified Theory?

As remarked, each of the two theories works perfectly in its own scale. Still, there are cases in which we need to apply one of these theories in other scales as well. For instance, the physics of black holes or cosmology of the early universe are areas in which one can not ignore quantum effects. On the other hand, there are cases in which gravity may contribute at the quantum scale. For example, one may be interested in possible gravitational corrections to the scattering cross section of colliding particles, which is given by Feynman diagrams including one or more graviton (quantum of the gravitational field). That is why finding a way to unify all fundamental forces has always been an important goal for physicists. There are other motivations like having a "simple unique theory" that describes all of nature, a dream chased by Einstein and others during their whole life. There have been serious obstacles in the way of unification though. Let us have a brief look at some of difficulties that unification has encountered.

1.1.2. Subtleties of Unification

Different scales. The general theory of gravity always concerns large scale structures and long distances and does not care about the way gravity operates at subatomic scales. Still, we know that gravity is a force acting at *all scales*. The scales of activity of all four forces are depicted in Figure 1.1. A simple power counting shows how the gravitational corrections are negligible at particle physics energy levels [3]. Consider two particles propagating with total energy E and its first gravitational correction. The amplitude of the one-graviton exchange is proportional to the square of the gravitational coupling constant G_4, while the amplitudes of these two processes with and without graviton have to be related by a dimensionless factor. The only dimensionless factor one can build out of fundamental constants and characteristic parameters of the process is $\frac{G_4 E^2}{\hbar c^5}$. Setting $\hbar = 1 = c$ and given the Plank mass as $M_P = G_4^{-\frac{1}{2}} \sim 10^{19}$ GeV, one can rewrite the ratio as $(\frac{E}{M_P})^2$, which implies that the gravitational correction is irrelevant for $E \ll 10^{19}$ GeV, whereas the total energy of a typical particle collision at LHC is around 13 Tev.

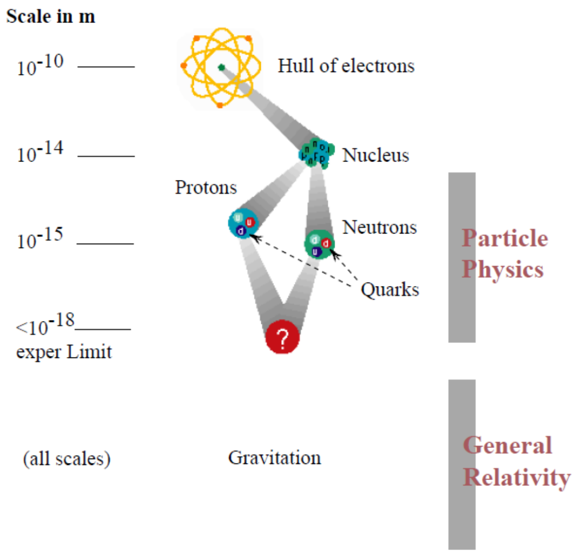

Figure 1.1: Energy scales at which different forces interact [1].

Quantum information in gravitational systems. A paradox shows up when we want to include quantum effects very close to the horizon of a black hole. As showed by Hawking, semi-classic calculations imply that a black hole radiates as if it was a black body. This radiation can be explained by considering quantum vacuum fluctuations very close to the horizon [4]. Black body radiation is thermal and so without much information. This means that most of the information of any quantum state will be lost after being swallowed by the black hole and there is no known way to decode the information from the radiation. This violates one of the initial principles of quantum mechanics which states that in a quantum system, information never gets lost. It has been such a severe contradiction for decades [5, 6], that it caused some prominent physicists such as J. Polchinski to become almost disappointed about making peace between quantum and gravity [7].

Conceptual and computational differences. On the other hand we have quantum field theory in which spacetime can not be considered as a dynamical variable of the theory, and this is the first fundamental difference between general relativity and quantum filed theory. The very first example of such a difficulty shows up when one wants to compute (anti-)commutation relations of two or more quantum fields in spacetime. The commutator has to be calculated at the same time for all fields, meaning that only spacelike separated fields can be considered. However, it is impossible to determine the causal relation between points/fields if we consider the spacetime as an *undetermined* variable [8]. Their conceptual and technical differences go even further and have caused all attempts for their unification to be failed.

Nonrenormalizability of general relativity. The simplest way to see nonrenormalizability of GR is through the gravitational coupling constant. Consider the Einstein-Hilbert action in arbitrary dimensions D given by

$$\mathcal{S}_{\text{EH}} = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} \mathcal{R} ,$$

where G_D denotes the Newton's constant of gravity. Regarding the fact that the Ricci scalar has dimensions $[\mathcal{R}] = L^{-2}$, we need to have $[G_D] = L^{D-2} = M^{2-D}$ which implies negative power of mass for $D > 2$. We then infer that the lower the energy is, the stronger the gravitational interaction will be. This is known as the problem of nonrenormalizability. For instance, Newton's constant in $4D$ is proportional to length squared, i.e., $G_4 \sim (M)^{-2}$ which is not strong enough to suppress ultraviolet divergences.

1.2. Why String Theory?

1.2.1. What Is It About?

The main novelty in string theory was introducing extended dynamical objects which can contribute in any interaction and physical events along with pointlike particles. At first, it only included strings, a line segment that can be either open (topologically a line) or closed (topologically a circle). Strings can oscillate and each specific mode can be assigned to a particular particle. Later, in nonperturbative approach it was realized that there are even higher dimensional objects called *p-branes*. They can be thought of hypersurfaces with p spatial dimensions on which open string can end. However, these objects can be safely neglected in weak coupling regime $g_s \to 0$ since all of them become extremely massive (their tension is proportional to g_s^{-1}) leaving fundamental string to be the only relevant extended object to be considered. So in this point of view, all we have are p-branes in different dimensionality: point particles are 0-branes, strings are 1-branes ans so on. When they move inside spacetime, a $p+1$-dimensional surface is swept out called worldvolume (worldsheet in the case of fundamental strings).

As line segments, strings has a characteristic length l_s that could not be measured experimentally yet. However, one can still make a guess. As we are going to mention as one of most noticeable properties of the theory, string theory propose a relativistic quantum theory which naturally contains gravity. So it can be inferred that l_s should be expressible in terms of corresponding fundamental constants [4], i.e., $\{c, \hbar, G\}$ which are speed of light, Planck's constant divided by 2π and Newton's constant respectively. As we know, there is only one fundamental length scale derivable from these three and it is Planck length given by

$$l_p = \sqrt{\frac{G_4 \hbar}{c^3}} = 1.6 \times 10^{-35} \ (m) \ ,$$

from which one can also read the Planck mass as $M_p = 1.2 \times 10^{19}$ (GeV/c^2). Assuming $l_s \sim l_p$, this implies that for energies less than this scale, point particles are fairly accurate approximation for strings and so conventional quantum field theories containing only point particles have enough accuracy.

In the case of a pointlike particle, the action can be written demanding the worldline followed by the particle to be extremized classically. Analogously, for a generic p-brane we extremize the swept out $(p+1)$-dimensional volume V. In the case of strings, this leads to the *Nambu-Goto action* which is classically equivalent to the following sigma-model on the worldsheet [3, 4, 9]

$$\mathcal{S}_{\text{ws}} = \frac{-T}{2} \int \sqrt{-h} \ h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu d\sigma d\tau \ .$$

Here, $\{\sigma, \tau\}$ are spatial and time coordinates on the worldsheet which is described by an auxiliary metric h. The worldsheet is embedded in flat spacetime via coordinates

X^μ in which vibrations of the string in different directions is encoded. Also, T stands for *tension* (or equivalently, mass) of the string.

For an open string, $X^\mu \in [0, \pi]$ where any of two ends satisfies either Neumann or Dirichlet boundary conditions. It is the second option that requires having D_p-branes as hypersurfaces open strings can stick to. This configuration leads to another crucial property of string theories which will be mentioned later. Furthermore, closed strings can be obtained via identifying the two ends of an open string by imposing $X^\mu(\sigma, \tau) = X^\mu(\sigma+\pi, \tau)$. The graviton is in fact a massless spin-2 mode in closed strings spectrum and this guarantees gravity to be included in all types of string theories.

1.2.2. What Makes It Special?

Having general features of string theory said, let us here focus more on three remarkable properties of the theory that makes it the most powerful candidate for a quantum theory of gravity and unifying all fundamental forces [4, 10].

- Supersymmetry. As mentioned earlier, this is a symmetry between bosonic and fermionic degrees of freedom. Its appearance is one of effective ways to increase the number of symmetries in the theory. Not only is the symmetry admitted naturally by string theory, but also its consistency requires local supersymmetry. In spite of some predictions that expected the symmetry to be observable at the LHC (the Large Hadron Collider), there has not been any experimental evidence of its existence yet. However, there are physicists who still believe that the energy scale of supersymmetry breaking can be higher than available level of energy at the LHC.
- Extra dimensions and extended objects. These two as other options for enlarging the symmetry group of a theory. Interestingly, both of them play a very central role in string theory. First, the theory is essentially about higher dimensional objects such as strings and p-branes. Second, it can be shown that a pure bosonic string theory has to live in 26 dimensional spacetime to be free of anomaly. However, a pure bosonic theory can not be a realistic representation of the nature.

Then one needs to introduce fermionic strings as well via a supersymmetric CFT (SCFT) on the worldsheet. Doing so and requiring an anomaly-free theory reduces the critical dimensions to 10.

- Gravity. This is for sure the most vigorous aspect of the theory which supports it to be the most serious option for the unification. While in any other approach to quantum gravity, the main question is how to introduce gravitational field in a consistent way, it emerges quite naturally in string theory. In fact, graviton is a massless mode in closed string spectrum, a fundamental object that is included in all types of string theories.
- Yang-Mills gauge theories. The unification is achievable via string theory if it gives back the standard model of particle physics at low energies. The existence of D_p-branes are crucial for making this possible. In fact, open strings attached to these branes create gauge fields as their massless oscillation modes. The obtained gauge group then depends to the configurations of D_p-branes and attached open strings. So theoretically we can adjust the configuration in such a way that we get the same gauge group as standard model, which is $SU(3)_C \times SU(2)_L \times U(1)$.

Obviously, in order to obtain an effective theory in four dimensions compatible with our daily life experiences, we need to apply Kaluza-Klein reduction to compactify six spatial dimensions on a compact internal manifold whose volume has to be small enough to stay uncovered in today's available energy levels. As will be argued extensively in the third chapter, the size of the internal manifold is determined by vacuum expectation values of some scalar fields span its moduli space. One point to be noticed is that different internal manifolds give rise to totally different $4D$ theories as the field content of the lower dimensional theory absolutely depends on the topological structure of the internal manifold. This implies that even without observing the $6D$ compact manifold, there is an indirect way to figure its type out from the lower dimensional theory.

There are also some noticeable abilities uniquely observed in string theory. We are going to mention only two among several

- As said before, it requires a specific dimensionality for spacetime to be anomaly-free.

- There is only one undetermined fundamental parameter in the theory which is string length l_s. Note that even string coupling g_s is not a free parameter as the theory is expected to be valid for all values of g_s. As we discuss briefly later, there is a duality that exhibit a relation between specific types of string theories in weak and strong coupling regimes.
- Chiral spectra are allowed in string theory. There are interactions in nature that do not respect parity symmetry. So any theory of nature should provide a way to break this symmetry consistently. While most of other unification theories require parity to be a preserved symmetry, chiral (parity asymmetric) spectra and interactions are naturally allowed in some types of string theory such as Type IIB.

Having all these said as signs of its power, we should also clarify that it is not still a complete theory of physics. Actually, there are still few questions never have been addressed by string theory. Here, we only bring them briefly in the following index

- Why the gauge group of the standard model of particle physics is G_{SM} and not something different?
- Why there are three families of leptons and quarks, not two or four?
- Why the cosmological constant has been set to its very small value $\Lambda \sim 10^{-120} M_P^4$?
- Why the universe is expanding?

1.2.3. Different Types and Dualities

Although it is a strong candidate for unification, it is not a unique theory. There are five known string theories categorized as Type I, Type IIA and IIB, heterotic $SO(32)$ and $E_8 \times E_8$. From all these types, we only discuss Type II spectrum in this thesis and then focus only on Type IIA. This type has a non-chiral spectrum whose massless part describes Type IIA supergravity in $10D$. Being compactified on a Calabi-Yau threefold, it leads to Type IIA supergravity in $4D$ whose black hole solutions will be the main topic of this thesis.

Apparently these are different theories, however there are interesting duality between some of them. For instance, *T-duality* explains how Type IIA and IIB look the same physically if the radius of the circle they are compactified on are related as $R_{IIB} = l_s^2/R_{IIA}$. The radius of the circle is given by vev of a single scalar field. There is the same duality correspondence between two heterotic theories too.

There is another case of duality which relates two theories from different coupling regimes and is called *S-duality*. For example, the Type IIB is self-dual under $g_s \to g_s^{-1}$. Analogously, the string coupling is given by vev of another scalar field, i.e., $g_s \sim e^{\phi}$ where ϕ is dilaton and S-duality can be translated as a field transformation $\phi \to -\phi$. Absorbing this factor takes us from the so-called string frame to the Einstein one.

These duality correspondence as well as the old intention to having a *single* theory for the whole physics made physicists to think of higher dimensions once again. However, if we still require to have supersymmetry, we can not go further than eleven dimensions. The theory is called M-theory and its massless field content is quite simple, a graviton $G_{\mathbb{MN}}$ with $\mathbb{M}, \mathbb{N}, \cdots = 0, \cdots, 10$ and a three-form gauge potential A_3. Being compactified on a circle, it results in Type IIA.

1.3. Supergravity and Black Hole Solutions

It is what this thesis is literally about. However, studying supergravity without knowing about its origin superstring theory will be incomplete. That is why we motivate superstring theory quite a bit before getting to its low energy limit and introducing some interesting black hole solutions.

1.3.1. Supergravity

As just mentioned, in supergravity we are basically focus on massless part of the closed superstring spectrum. That is why it is commonly said that SUGRA is the low energy limit of superstring theory. In $10D$ it contains gravity g_{MN}, an anti-symmetric tensor field B_{MN} usually called the B-field, a dilaton ϕ and two $U(1)$ gauge fields, one is an one-form A_M and the other a three-form A_{MNP} with $M, N, \cdots = 0, \cdots, 9$. One way to classify different supergravity theories is via their number of supercharges. It is denoted by an integer $\mathcal{N}$ which is defined by the number of supercharges divided by the number of real degrees of freedom of the shortest possible spinor in a specific space-time dimensions. Accordingly, the Type IIA SUGRA in $10D$ with 32 supercharges has $\mathcal{N} = 2$. If we are interested in studying some lower dimensional SUGRA via performing compactification, then it is the internal manifold that determines the number of survived supercharges as well as the field content of the lower dimensional effective theory. Here, our main focus is on $\mathcal{N} = 2$ four-dimensional supergravity with 8 unbroken supersymmetries. This restricts the internal manifold to be a Calabi-Yau threefold whose moduli space is a special Kähler manifold. We leave this delicate subject with all of the details to be discussed in the third chapter.

1.3.2. Black Holes in String Theory, Enigmas and Proposals

Now, the main question to be asked is what is new to be learned from string theory about black hole physics? If it is really a theory of quantum gravity and capable of unification of two fundamental theories in physics, then it should be able to provide new perspective and solutions to the questions and obstacles encountered by other unifying approaches. As remarked at the very beginning of this introduction, the physics of black holes is the main arena where general relativity and quantum mechanics face each other and this has led into question some well accepted principles in physics such as unitarity. It turns out that to find a way out of these controversies we first need to search for the microscopic origin of the black hole entropy.

Information loss puzzle

As singular solutions to the general relativity which is a classical theory of gravity, black holes are absolutely stable. They can only swallow everything consisting of mass and energy in their neighborhood and so never throw anything out. However, the Hawking realization of black hole radiation has questioned this stable picture of black holes. In order to explain where this radiation comes from, one needs to invoke quantum effects, and this entails some inconsistencies. There basically arise two subtle questions [5, 11]

- What is the microscopic origin of the black hole entropy? If black holes are some radiating thermodynamic systems, there should be an explanation of their associated entropy, a microscopic quantity that counts the number of quantum microstates of the system. So the question is what are these microstates and how can we count them?
- Why is this radiation thermal? The formation procedure of a black holes starts from a shell of matter collapsing. Assume that this shell is a pure quantum state whose density matrix has only one nonzero entry ($\rho = |\psi\rangle\langle\psi|$). Then it turns into a black hole which is a mixed state ($\rho = \sum_i \rho_i |\psi_i\rangle\langle\psi_i|$ with $\sum_i \rho_i = 1$) and radiates as if it is a black body. This raises several questions like where did the initial information go? How can the whole procedure be explained by quantum mechanics in which evolution of a pure state to a mixed one is excluded?

There are some resolutions to the first problem that cover a large class of supersymmetric black holes but not all of them. In fact, string theory has provided some techniques for counting the number of microstates for black holes with enough supercharges. In this approach first explored by Strominger and Vafa [12], one can continuously transfer from five-dimensional supersymmetric black holes in strong gravitational coupling to the bound states of D-branes in weakly coupled regime. Then the degeneracy of the system of D-branes is interpreted as entropy of the black hole. Regarding the fact that supersymmetric states exist regardless of the values of the parameters in the theory, the number of microstates remains invariant under continuous variations of the gravitational strength. Later, the approach was extended to certain families of four-dimensional supersymmetric black holes as well [13, 14]. Although in this thesis we never discuss these questions directly, but the role of supergravity and its black hole solutions in having a deeper insight into the entropy enigma is definitely important.

The fuzzball proposal

There is another attitude to black hole physics that resolves some of these essential contradictions using the AdS/CFT correspondence. In this picture, the interior of a black hole is considered as a "*fuzzy*" system consisting of regular asymptotically AdS solutions (See, e.g., [15–17]). At far infinity, these AdS geometries look the same as the black hole background and differ from it only up to the horizon scale. The *fuzzball proposal* for black holes was suggested for the first time in [18–20] with the following idea:
on one hand, we have extremal black holes which are asymptotically flat with an AdS factor in their near horizon geometry. So practically one can apply AdS/CFT correspondence at the horizon to interpret microstates of the black hole as supersymmetric states of the dual CFT sitting at the boundary. Since unitarity is guaranteed by the CFT, so is the evolution of the black hole. This argument resolves the information loss problem.
On the other hand, conjectured by AdS/CFT there is an one-to-one correspondence between stable states of any d-dimensional CFT and non-singular $d+1$-dimensional asymptotically AdS solutions where vevs of the gauge invariant CFT operators are all encoded in the boundary of AdS_{d+1}. This implies that for any microstate that contributes in black hole entropy $S_{\rm BH}$, there has to exist a regular asymptotically AdS geometry called *fuzzball*. More importantly, the near horizon of the black hole is essentially described by the same geometry as the asymptotics of the fuzzballs. Consequently, one can replace the interior region of the black hole by these asymptotically AdS solutions so that they attach to the asymptotically flat region of the black hole background at the original location of the horizon. Note that for any black hole with entropy $S_{\rm BH}$ there is an exponential number of such fuzzballs each of which represents a microstate of the corresponding black hole. This proposal for black holes has the following benefits:

- It provides an interpretation of the black hole microstates and its associated entropy.
- Dealing with *horizon-free* geometries guarantees that there is no information loss

any more. A pure state coming from infinity can reach infinity again while it is still a pure state.

- These fuzzballs are regular asymptotically AdS solutions to the string theory and only a subset of them solve the low energy field equations. As such, they generally contain all the field content of the low energy theory (i.e., supergravity) while a Reissner-Nordström black hole only involves gravity and electromagnetic fields.

1.3.3. What Makes Multi-Centered Black Holes Important?

As remarked in the previous part, it was in the case of specific supersymmetric black holes that a statistical interpretation of the Bekenstein-Hawking entropy was achieved for the first time. This shows why these black hole solutions are interesting. At first, only single-centered solutions were known [21, 22]. Later, a significant property of supersymmetric black holes was explored [23, 24]; they can bound together and construct complex configurations. The so-called *multi-centered black holes* have specific properties: in particular, they are stationary solutions with an intrinsic angular momentum, not absolutely stable meaning that they can decay to smaller clusters.

Let us turn back to the problem of black hole entropy and see how it has been explained in the case of supersymmetric black holes. Generally, there is a correspondence between quantum states of a system and its classical phase space so that the entropy is given by the volume of the classical phase space in Planck units. The number of supersymmetric one-particle states is determined by the so-called *Witten index* of the system given by [25]

$$\Omega(\Gamma, t_\infty) = \mathrm{Tr}_\Gamma(-1)^{2J_3}(J_3)^2 e^{-\beta H} \ .$$

Here, Γ is the conserved charge and t_∞ is the value of scalars at spatial infinity [1] . Also, J_3 is the z-component of the angular momentum and H is the energy of the state

[1] As we will explain in chapter 3, these scalars adjust the volume of the internal Calabi-Yau threefold and so called *scalar moduli*. Their asymptotic values play a central role in supergravity solutions as they determine the vacuum energy.

with respect to the supersymmetric lower bound [2] . The subscript Γ shows that the trace has to be taken over all states with charge Γ. It was mentioned before that the entropy of a black hole as a system in the strong gravity regime can be identified with the logarithm of various configurations of D-branes in a weak gravity regime where the D-brane description becomes accurate. These D-branes wrap around cycles of the internal $CY3$ in such a way that they preserve a portion of the supersymmetries. Sending the internal volume to zero, they appear as point-like charges called *BPS particles*. Accordingly, the moduli space of the D-brane configurations is in fact the classical phase space of a supersymmetric black hole from which one can derive the set of ground states. There is a crucial fact that allows us to identify entropy of two physical systems in two different coupling regime and it is the Witten index being independent of any coupling parameter.

Regarding the fact that the trace in $\Omega(\Gamma, t_\infty)$ is over *all* solutions with charge Γ and not necessarily the static spherically symmetric ones, the question is whether $\mathrm{Ln}\,\Omega$ should be attributed only to the single-centered supersymmetric black holes or it contains the corresponding entropy of the multi-centered solutions as well. Intuitively, it was accepted that these are the single-centered solutions which have the dominant entropy among the set of solutions until Denef and Moore showed the reverse [26, 27]. In their argument, they make an example of Type IIA supergravity compactified on $Y = T_1^2 \times T_2^2 \times T_3^2$. Considering a single charge $\Gamma = \lambda(p^0, p, q, q_0)$ they found the entropy as $S_{(1)} \propto \lambda^2$. However, if the area of each two-torus v satisfies a specific lower bound proportional to λ ($v > 2\sqrt{3}\lambda$ in their case), then a two-centered solution with total charge Γ is also possible. Surprisingly, they showed that the entropy of the two-centered case is given as $S_{(2)} \propto \lambda^3$. This implies that for large charge limit $\lambda > 1$ it is actually the entropy of the multi-centered solutions that dominates. In other words, in studying the entropy of highly charged black holes it is the multi-centered solutions that plays the central role.

Ultimately, there is a subset of multi-centered solutions dubbed *scaling solutions*

[2] It will be discussed in the 3rd chapter that supersymmetric solutions satisfy a BPS bound which depends on the minimum value of the scalars.

which are the main focus of this thesis. The relative distances between different centers in these solutions are fixed *up to a scale*, meaning that centers can get arbitrary close to each other so that they look like a single massive charged particle. In addition, in contrary with the typical multi-centered solutions that are not stable, the scaling solutions are absolutely resistant against decay. As such, they show more resemblance to our classic picture of a black hole. They are also interesting solutions to study since there is a correspondence between them and those states in the Higgs branch who can not be mapped to any supergravity solution [28].

1.4. Outlines

We start the more technical part of this thesis in the next chapter and spend quite some time on rather technical discussions each of which are essential to the coming materials and arguments in later chapters. First, we motivate studying black hole thermodynamics and introduce the Reissner-Nordström black holes [4, 29] as non-supersymmetric analogue of the BPS black holes, the main subject of this thesis. We then review the main idea of Kaluza-Klein compactification, a powerful technique that allows reducing a higher dimensional theory to an effective one in lower dimensions [30]. This chapter ends with an introduction to Kähler manifolds and one of its most important subsets, i.e., Calabi-Yau manifolds [31–33].

The third chapter is exclusively devoted to the review of supergravity and a variety of its black hole solutions. We first discuss $\mathcal{N} = 2$ Type IIA supergravity in $10D$ and its Calabi-Yau compactification [34, 35]. We then continue with $\mathcal{N} = 2$ supergravity in $4D$ and its field content classified in different multiplets. Studying self-interacting $U(1)$ gauge fields more carefully reveals a remarkable geometric property of the theory: the scalars target space is a *special Kähler manifold* with a symplectic bundle $V(t^A)$ whose prepotential takes a specific form [29]. The theory in $4D$ contains massive particles carrying electromagnetic charges Γ^a associated with the gauge fields. These pointlike particles have been initially even-dimensional D-branes in $10D$ that get wrapped on non-trivial internal cycles after compactification. Their winding numbers can be interpreted as quantized electromagnetic charges of BPS pointlike particles. We then continue by discussing the static black hole solutions of single massive charges that respect half of the supersymmetries [21, 22]. One way to obtain supersymmetric black holes is by searching for Killing spinors which leads to a set of first order differential equations. However, in this thesis we are going to follow a different but equivalent path called *attractor flow formalism* whose focus is mainly on the radial "flow" of the moduli scalars [23]. As we are going to observe, from any asymptotic values they start at spatial infinity, scalars are always attracted to take their minimum right at the black hole horizon. Moreover, these black holes saturate a BPS bound for the effective Lagrangian, that is the reason for calling them "BPS black holes". This bound leads to attractor flow equations, a set of first order differential equations equivalent to those one obtains from Killing spinor condition. Probing these black holes by another charged particle reveals a possibility of having bound state under certain "stability conditions". As we will review explicitly, the equilibrium distance between two centers depends on scalars asymptotic values. Consequently, the stability conditions divide the "solution space" into two regions where the bound state exists only on one side and will decay by passing through *wall of marginal stability.* Having bound states with more than two centers is also possible and in general such configurations are called "multi-centered black holes" [23, 24]. They are stationary and their rotation shows up via a one-form ω inside the metric. It is important to note that this rotation is absolutely essential, meaning that it originates in electromagnetic charges interacting with each other and so is inevitable. While they have no non-supersymmetric analog, there is a family of multi-centered black holes in which centers can come arbitrary close to each other as the equilibrium distances between different centers can be fixed by stability conditions *up to an overall scale.* These solutions which are called "scaling solutions" exhibit more similarities to non-supersymmetric black holes. They are our main focus for the rest of the thesis. The chapter ends with some original results about asymptotic behavior of the scaling solutions. As we will discuss in chapter 4, the angular momentum plays a decisive role in the asymptotic geometry of these solutions.

Chapter 4 is basically the authors work in which the existence of a "*twist*" in scaling BPS black hole solutions is reported [36]. There we showed that the asymptotic geometry of these solutions is not as simple as what was assumed. i.e., $AdS_2 \times S^2$.

As we argued, at far infinity the one-form ω is promoted to a leading term inside the metric and develops the asymptotic geometry to an AdS_2 fibered by a rotating S^2. It should be noticed that multi-centered solutions typically have a non-vanishing angular momentum. However, one should distinguish the twist from angular momentum $\vec{J}$ since the later vanishes identically for scaling solutions. The angular velocity of the two-sphere is determined by the total amount of charge and dipole moment of the black hole and as such, it can be considered as a new observable that distinguishes between different solutions, i.e., a *hair* for these type of black holes. This suggests the twist as a new label for black hole microstates.

The fifth chapter is devoted to the S^2 reduction of the asymptotic geometry of the scaling solution. This is a original work of the author in collaboration with Dieter Van Den Bleeken and Joris Raeymaekers which is published for the first time in this thesis. Following the common procedure, we encounter some inconsistencies whose resolution needs a redefinition of dynamic variables of the $4D$ action. To examine the new formulation, we first S^2 reduced the Einstein-Maxwell theory and its pure magnetic AdS_2 solution. Then we generalized this procedure so that it contains an arbitrary number of $U(1)$ gauge fields with both electric and magnetic sources. This generalized technique was applied in reducing the asymptotic geometry of scaling multi-centered black holes. Having them successfully reduced, we are left with an AdS_2 geometry whereas the twist appears as a single $U(1)$ gauge field in $2D$ theory. Then the idea is to use holography in order to understand the twist via its realization as a gauge potential with a flat connection. It is through this way that we make some first progress towards a connection between this new hair and microstates of the black hole.

There are three appendices to this thesis. The first one provides some background knowledge of complex manifolds. The second appendix contains a proof for closeness of the asymptotic $d\omega$ we have found. In the last appendix, we shortly mention the origin of an inconsistency that appears in the naive S^2 reduction approach.

2. BACKGROUND DISCUSSION

In the first technical chapter of this thesis, we prepare the scene for materials and discussions to come later in the next chapters and that is why it may seem wandering around different subjects. We start with an introduction to physics of black holes with the main focus on its connection with thermodynamic laws and the way these rules can be translated in terms of quantities characterize black holes. The familiar Schwarzschild black hole is discussed very briefly as the very first example of a black hole solution to Einstein-Hilbert pure gravity. Another case which is discussed is the Reissner-Nordström black hole, the solution to the Einstein-Maxwell theory. Its generalization to the dyonic black hole is also mentioned. The Einstein-Maxwell theory and its black hole solutions are of our special interest since as we are going to remark here and there in chapters 3 and 5, in spite of many subtleties $\mathcal{N} = 2$ supergravity theory and its single and multi-centered black hole solutions exhibit, there are still major similarities between the two theories.

We then digress a bit to study the Clifford algebra in various dimensions. There are few reasons that motivates having this section: First, we learn how to count the number of fermionic degrees of freedom as well as supercharges in any arbitrary dimensions, a crucial primitive knowledge for studying supersymmetric theories. Second, we discuss the commuting subset of Lorentz generators that explains explicitly the Ramond-Ramond spectrum and so filed content of Type IIA/IIB superstring theory. We then continue with conformal symmetry exclusively in two dimensions. In particular, we investigate free bosonic and fermionic CFTs that helps us later in studying string theory on the $2D$ worldsheet with superconformal approach. We also shortly discuss ghost systems in $2D$ CFTs to finally conclude the most stunning claim in string theory: The theory needs to leave in ten dimensions (four large external dimensions and six highly curved internal ones) to be anomaly-free automatically.

Having these materials, we then proceed to talk about Type II superstring theories where our main goal is to obtain bosonic and fermionic spectra of closed strings.

Although we follow the discussion in parallel between Type IIA and IIB, later our attention will get concentrated on Type IIA supergravity which is naturally chiral. In the next step, we digress a bit to generally motivate considering theories with higher dimension via explaining an effective powerful technique established by Kaluza and Klein for the first time, i.e., Kaluza-Klein reduction. There we only focus on the simplest case that is reduction on a circle. However, we are going to apply this technique again in this thesis, once in chapter 3 where we want to obtain effective four dimensional supergravity theory out of the ten-dimensional one through a Calabi-Yau reduction. And the second time is in chapter 5 where we perform an S^2 reduction on multi-centered solutions.

We end this chapter by having an introduction to Kähler manifolds focusing on those geometric properties which are necessary for what come later, i.e., Calabi-Yau manifold. There we explain what reasons motivate to choose a Calabi-Yau threefold as the internal manifold on which we compactify the ten-dimensional superstring theory. We also provide the so called Hodge diamond of $CY3$ as well as our notation for its homology and cohomology basis.

2.1. Black Holes

In this section we are going to have a short look at Schwarzschild black hole as the first and most famous exact solution to the Einstein's equation in vacuum. This introduction let us to briefly enumerate some physical aspects that are common between different black holes. It is also the first step to address one of the most serious controversies in physics community during last decades, i.e., *information paradox* for which we also need to mention thermodynamics laws governing black hole physics such as entropy of a black hole and *Bekenstein-Hawking temperature.*

The Schwarzschild solution found by Karl Schwarzschild in 1916, immediately after field equations of general relativity had been published by Einstein. As an exact solution to the Einstein's equations in vacuum, it describes the gravitational field outside of a static [3] and spherically symmetric massive object. Even for weak gravitational fields like what we have in solar system, Schwarzschild solution reveals some noticeable physical effects such as time delay, bending of light and light redshift non of which are predicted by Newtonian gravity. In the strong gravitational filed limit though, this solution explains the exterior region of a black hole.

Derivation of the Schwarzschild metric is based on staticity and symmetries of the solution and applying *tetrad method* [37]. Here we only bring the familiar final result that is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2 \qquad (2.1)$$

where $d\Omega_{D-2}^2$ is the infinitesimal line element on a $(D-2)$-dimensional sphere S^{D-2}, M is the total mass of the black hole and G denotes the Newoton's constant of gravity that is taken to be one henceforth. This metric is asymptotically flat, meaning that it will reduce to the Minkowskian metric at $r \to \infty$ and the total mass M is what an observer measures at infinity. As one can check easily, the metric components become singular at two points, $r = 0$ and $r = 2M$. However, we should distinguish two types of spacetime singularities

- Coordinate singularities that occur because of the bad choice of coordinates and so are removable.
- Physical singularities that can not be removed from spacetime in any way.

To be able to distinguish these singularities in any spacetime, we need to look at other invariant quantities such as $R_{\mu\nu}R^{\mu\nu}$ or the so-called *Kretschmann scalar* given by $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$. Doing so, we realize that [4] non of these scalars are singular at $r = 2M$ while they become infinite at $r = 0$. In fact, the coordinate singularity at $r = 2M$ can

[3] Let us explain what we mean explicitly by staticity. A spacetime is called *stationary* if it has time translation symmetry which is generated by a timelike Killing vector. Such a spacetime is said to be *static* if there is a (spacelike) hypersurface orthogonal to the orbits of the isometry [37]. To put it simply, a static spacetime is stationary and irrotational .

[4] For instance, Kretschmann scalar equals to $K = \frac{48M^2}{r^6}$ for the Schwarzschild metric.

be removed by choosing Kruskal coordinates. One should also note that for bodies with regular masses, the Schwarzschild radius lies inside the body where the Schwarzschild solution is not valid any more. In other words, this is only for bodies after complete gravitational collapse that $r = 2M$ lies outside of the body and so called *horizon* through which no light ray can pass and reach outside.

2.1.1. Thermodynamics of Black Holes and Bekenstein-Hawking Temperature

If it was not because of those physicists including Hawking, Bekenstein, Gibbons and others who have been shedding lights on dark mysteries of black holes, it had been far from reality to envisage a connection between thermodynamics and physics of the black holes. The first sign of this remarkable connection showed up after Hawking proved *the black hole area theorem* which states in any physically permissible evolution of a black hole, the area of the event horizon A can not decrease [38–40], i.e., we always have $\delta A \geq 0$ [5] . Now, comparing this with one of thermodynamics laws reveals the first analogy between a black hole and a thermodynamic system. The second law of thermodynamics states that during any physical phenomenon, the total amount of entropy in the universe can not be decreased, i.e., $\delta S \geq 0$. At the first glance, it may seem rather a simple resemblance between two physical quantities belonging to two distinct theories. However later, more sophisticated computations and discussions deepened this connection considerably.

Let us first define an important quantity called *surface gravity.* As we will see in a minute, this quantity is actually another piece of puzzle that make the conjectured connection more vigorous and reliable. Take χ^a as the null Killing vector field normal to the horizon. Having $\chi^a \chi_a = 0$ on the horizon means that $\nabla^a(\chi^b \chi_b)$ is also normal

[5]later, Bekenstein [41] showed that this is only true at classical level and becomes violated by considering quantum effects as well. These effects which are responsible for black hole Hawking radiation require a generalization of the second law of black hole thermodynamics saying that this is $S^{'} + \frac{1}{4}A$ that can not decrease in any physical procedure where $S^{'}$ here is the entropy of matter outside the black hole .

to the horizon and so one can write

$$\nabla^a(\chi^b\chi_b) = -2\kappa\chi^a \tag{2.2}$$

where the proportionality factor κ is surface gravity. It can be shown that it is a constant quantity all over the horizon [39, 42], i.e., we have $\mathcal{L}_\chi\kappa = 0$. For a charged rotating black hole called *Kerr-Newman* we have

$$\kappa = \frac{\sqrt{M^2 - a^2 - Q^2}}{2M\big(M + \sqrt{M^2 - a^2 - Q^2}\big) - Q^2} \qquad \text{with} \qquad a := \frac{J}{M} \tag{2.3}$$

where J is the total angular momentum and Q the total electric charge of the black hole. Now, to have a feeling about the important conclusion made by Hawking *et. al.* consider a charged rotating black hole. According to the *no-hair theorem*, any black hole solution that solves Einstein-Maxwell equations can be parameterized by three classical quantities (M, J, Q) where J is the angular momentum and Q denotes the electric charge of the black hole. Then, assume an infinitesimal stationary axisymmetric variation of the metric restricted such that it leave the location of the vent horizon intact. Now the question is how mass of the black hole changes in terms of other quantities. After complicated calculations in [6, 37], it was shown that

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega_H\delta J_H + \Phi\delta Q \tag{2.4}$$

where Ω_H is the angular velocity of the horizon and Φ is the chemical potential. This result exhibits a very clear resemblance to the first law of the thermodynamics that is

$$\delta E = T\delta S - P\delta V + \mu\delta N \tag{2.5}$$

where μ is the chemical potential and N as the total number of particles in the system. To see the analogy better, first note that the left hand side of these two equalities talk about the same quantity: total energy of a physical system. So one can deduce $E \leftrightarrow M$. Now on the right hand side, recall that we have already established a connection between area of the event horizon and entropy, more precisely $S \leftrightarrow \frac{1}{8\pi}A$ which implies that there should be a connection between surface gravity and temperature as well $\kappa \leftrightarrow T$. Surprisingly, the property we just mentioned for the surface gravity to be constant over the horizon makes this guess stronger; according to the zeroth law of the thermodynamics, the temperature is constant all over a body which is in thermal equilibrium with its environment. The second term in (2.4) also looks like the work term $P\delta V$ appears in the first law of thermodynamics. This similarity becomes exact if we write the first law for a rotating body. Finally, the third term is quite familiar in thermodynamics.

The last step that turns this conjecture to a complete analogy is to find a black hole version for the third law. One recalls the third law of thermodynamics states that it is impossible to receive to zero temperature via a physical process. If the whole analogy works, then $T = 0$ is equivalent to $\kappa = 0$, or having $M^2 = a^2 + Q^2$ for a Kerr-Newman black hole using (2.3). Now it has been shown for Kerr black holes [43] as well as extremal Kerr-Newman black holes [44] the closer one gets to the black hole the harder it becomes to take the next step. In other words, $\kappa = 0$ is not achievable in a physical process [6] .

Let us finish this part by explaining how black hole physics started to playing a decisive role in unification of gravity and quantum physics. The thing is if we try to interpret the surface gravity κ as thermodynamic temperature of the black hole then we will encounter a classical contradiction: we use to know black holes as an absolute absorber from which and its ultra strong gravitation, nothing can escape even the fastest physical objects, light rays. However, a non-zero temperature means that black holes are releasing heat out by emission! Here is where quantum physics comes on the scene. It was Hawking in 1974 [45] who explained how an emitting black hole is possible. It happens because of particle pair production, a result of quantum vacuum fluctuations happening everywhere including near event horizon of a black hole. If one particle of the created pair get absorbed by the black hole, the other one has to escape to

[6] There is a tiny difference though between the third law in thermodynamics and black hole physics. In the former $S \to 0$ by sending T to zero while for a black hole the area A remains finite while $\kappa \to 0$.

infinity as a result of momentum conservation law. These escaped particles are those that can be seen as black hole emission. By performing semi-classical calculations in [42], Hawking *et. al* showed that black hole radiation is the same as the one of a black body whose temperature is

$$T = \frac{\hbar\kappa}{2\pi} . \tag{2.6}$$

The appearance of $\hbar$ declares that black hole radiation has a quantum origin. Bring other constants back, the entropy of a four-dimensional black hole becomes [46–48]

$$S = \frac{A}{4\hbar G_4} . \tag{2.7}$$

We refer the reader to [42] where all the four laws of black hole thermodynamics are gathered in a single place.

The natural question that comes to mind immediately is that what are the microstates of a black hole whose logarithm is given by (2.7)? What we know from statistical mechanics is that $S := \mathrm{k}\ \ln\big(\Omega(N, V, E)\big)$ [see, e.g., [49]]. In this definition first suggested by Plank, Ω is the degeneracy of quantum microstates in terms of labeling parameters (N, V, E) for a thermodynamic system and k is the Boltzmann constant. The no-hair theorem tells us that the macroscopic parameters for a Kerr-Newman black holes are (M, J, Q) but it gives no clue about the origin of microstates and the way to calculate the number Ω.

One of the considerable achievements of string theory is that it has provided us a way to calculate entropy of a large class of black holes via counting their microstates. Even though the first invented techniques were only able to count microstates for black holes with large number of preserves supersymmetries, the agreement with the Bekenstein-Hawking entropy formula (2.7) was incredible. Yet, later more advanced techniques enabled physicists to investigate a larger group of black holes and this led to find higher order corrections to the entropy formula. This stunning realization has been a vigorous evidence for many physicists to believe string theory is the best available candidate to resolve the black hole information paradox and unify all forces.

2.1.2. The Reissner-Nordström Black Holes

Here, we are going to briefly introduce a classical theory that describes system of charged massive particles by applying Einstein gravity and Maxwell theory of electromagnetism together. Studying this theory and its black hole solutions (See, e.g., [4,29]) is one of the first essential steps towards understanding supersymmetric black holes in $\mathcal{N}=2$ supergravity. In fact, as we will see later multi-centered black hole solutions are much more complicated, yet they are natural generalizations of the *Reissner-Nordström black holes* which are going to be discussed now.

The so-called *Einstein-Maxwell* action in D-dimension is

$$\mathcal{S} = \int d^D x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{2.8}$$

where $\mu, \nu = 1, ..., D$ and κ here should not be mixed up with the surface gravity. The action describes Einstein gravity coupled to a single electromagnetic field. The equations of motions are

$$R_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{D-2}\, g_{\mu\nu} T_\rho{}^\rho \right) \qquad , \qquad \partial_\mu \left(\sqrt{-g}\; g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \right) = 0 \tag{2.9}$$

where the energy-momentum tensor in the vacuum is just the familiar electromagnetic one, that is

$$T_{\mu\nu} \equiv F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \ . \tag{2.10}$$

The solutions to this action are static and enjoy spherical symmetry. The most general metric ansatz in D-dimension that guarantees these properties is

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2 d\Omega_{D-2}^2 \tag{2.11}$$

where A, B are two functions to be determined that only depend on radial coordinate r. Requiring (2.11) to be an Einstein metric [7] dictates $A' = -B'$ that reduces the metric ansatz to

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2 d\Omega_{D-2}^2 \quad ; \quad F(r) := e^{-2B}. \tag{2.13}$$

Solving (2.9) for components of the Ricci tensor, one obtains

$$F(r) = 1 - \left(\frac{r_0}{r}\right)^{D-3} + \frac{\kappa^2 q^2}{(D-2)(D-3)(\Omega_{D-2} r^{D-3})^2} \,. \tag{2.14}$$

In addition, having only static electric charges and also spherical symmetry dictates the only non-zero components of the strength tensor to be

$$F_{tr} = -F_{rt} := f(r) = \frac{-\,q}{\Omega_{D-2} r^{D-2}} \qquad \text{with} \qquad q = \int d^{D-1}x\ \partial_i\big(\sqrt{-g}\ F^{0i}\big). \tag{2.15}$$

Now, let us focus on four-dimensional case where we have $\kappa^2 = 8\pi G$ with G denotes the Newton's constant and just like Schwarzschild black hole, r_0 is proportional to black hole mass M. Then we read

$$F(r) = 1 - \frac{2MG}{r} + \frac{q^2 G}{4\pi r^2} \,. \tag{2.16}$$

As one can compare, this is the function appears in Schwarzschild metric (2.1) modified by the electric charge. However, in contrast with that case, the Reissner-Nordström

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[7] A metric is called *Einstein metric* if it satisfies

$$R_{\mu\nu} = \kappa(D-1)g_{\mu\nu} \qquad \Longrightarrow \qquad R = \kappa D(D-1) \,. \tag{2.12}$$

black hole may have two horizons in general. They are called inner and outer horizons, although it is only the outer one that is called event horizon since no light ray can pass through it towards out of the black hole. These horizons are located at

$$r_{\pm} = MG\left(1 \pm \sqrt{1 - \frac{q^2}{4\pi GM^2}}\,\right) . \tag{2.17}$$

The same argument we made about singularities for the Schwarzschild solution holds here as well. The only physical singularity is $r = 0$ where the massive charge is located whereas singularities at horizons are of coordinate type [8] . It is worth mentioning that for $q^2 > 4\pi GM^2$, roots of $F(r)$ become imaginary, meaning that the singularity at $r = 0$ is *naked*; not shielded by a horizon any more. This situation is not allowed according to the *cosmic censorship conjecture*, since otherwise arbitrary strong fields are accessible to an external observer. Usually, it is said that these ill-defined singularities in classical solutions can be resolved if one takes quantum effect properly into account [9] .

For $q^2 = 4\pi GM^2$ the inner and outer horizons coincide at $r_0 = MG$ and the black hole solution is called *extremal* whose metric is given by

$$ds^2 = -\big(1 + \frac{r_0}{\tilde{r}}\big)^{-2}dt^2 + \big(1 + \frac{r_0}{\tilde{r}}\big)^2\big(d\tilde{r}^2 + \tilde{r}^2 d\Omega_2^2\big) , \tag{2.18}$$

where we have shifted the radial coordinate by $\tilde{r} := r - r_0$. Consequently, the near horizon geometry is obtainable by taking $\tilde{r} \to 0$ limit

$$ds^2 \to \big(\frac{r_0}{\tilde{\tilde{r}}}\big)^2\big(- dt^2 + d\tilde{\tilde{r}}^2\big) + r_0^2 d\Omega_2^2 \quad ; \qquad \tilde{\tilde{r}} := \frac{r_0^2}{\tilde{r}} . \tag{2.19}$$

This metric is called *Robinson-Bertotti* and describes an $\mathrm{AdS}_2 \times \mathrm{S}^2$ geometry for which AdS scale L and the radius of the two-sphere R_s are the same and equal to $r_0 = MG$. The conclusion is, the extremal Reissner-Nordström black hole admits the same near horizon geometry as all other extremal black hole that we have known of, and it is

[8] Note that at these point, physical invariants like Kretschmann scalar, $R_{\mu\nu}R^{\mu\nu}$ and $F_{\mu\nu}F_{\mu\nu}$ are regular while they are singular at $r = 0$

[9] Another analogous example is the case of naked electron in QED.

AdS $\times$ S^2. Finally, we refer the interested reader to [50] for Penrose diagrams of these black holes and some complementary information about their causal structure.

Temperature and entropy of a Reissner-Nordström black hole

At this point, let us review a very simple way to calculate temperature and entropy of a black hole. We will then apply the same trick to find entropy of a typical Reissner-Nordström black hole. The main clue is a very well-known one which is: consider a physical system behave periodically in Euclideanized time $\tau = it$ and its periodicity is given by β. Then the temperature of the system is $T = \beta^{-1}$. This can be concluded from comparing expressions of the partition function in statistical and quantum mechanics which are given respectively by

$$\mathcal{Z} = \mathrm{Tr}\big(e^{-\beta H}\big) \qquad \longleftrightarrow \qquad \mathcal{Z} = \big(e^{iHt}\big) \tag{2.20}$$

where H is the Hamiltonian of the system. These two expressions match if we first Wick rotate time $t \to \tau = it$ and then translate the trace to periodicity β in Euclidean time τ. Accordingly, to obtain temperature of a black hole, the easiest way is to first analytically continue the Lorentzian time. Then the requirement of having a regular metric near the horizon determines the periodicity in Euclidean time and so the temperature of the black hole. To see how this works, let us take the four-dimensional Reissner-Nordström metric (2.13) with (2.16). To take the near horizon limit, we also define a new radial coordinate ρ via $r = r_+(1+\rho^2)$ and then take the limit $\rho \to 0$. The result will be

$$ds^2 \to \frac{4r_+^3}{r_+ - r_-}\Big[\rho^2\big(\frac{r_+ - r_-}{2r_+^2}\big)^2 d\tau^2 + d\rho^2 + \frac{r_+ - r_-}{4r_+} d\Omega_2^2\Big] \,. \tag{2.21}$$

Defining $\psi := (\frac{r_+ - r_-}{2r_+^2})\tau$, the first two terms describe a $2D$ plane in polar coordinate if for $\tau \to \tau + \beta$ we have $\psi \to \psi + 2\pi$. This requirement is satisfied for

$$\beta = \frac{4\pi r_+^2}{r_+ - r_-} \qquad \Longrightarrow \qquad T = \beta^{-1} = \frac{\sqrt{M^2G^2 - \frac{q^2 G}{4\pi}}}{2\pi r_+^2} \tag{2.22}$$

As one sees, the temperature and so the entropy depends only on the mass and the charge of the black hole. It is also worth mentioning that for an extremal black hole, the temperature vanishes.

More general case

More general family of the Reissner-Nordström black holes are those carrying magnetic as well as electric charges. These solutions are still static and spherically symmetric. The easiest way to study them is applying *electromagnetic duality*. Since this duality plays a crucial role in studying supergravity, we are going to discuss it and its geometrical consequences quite detailed later in subsection 3.3.1. But for now, let us directly introduce a superposition of the electromagnetic strength tensor $F_{\mu\nu}$ and its Hodge dual denoted by $\star_4 F_{\mu\nu}$ [29]

$$\mathcal{F}_{\mu\nu} := qF_{\mu\nu} + p \star F_{\mu\nu} \quad ; \qquad \star_4 F^{\mu\nu} = \frac{1}{2\sqrt{-g}}\, \varepsilon^{\mu\nu\rho\sigma}\, F_{\rho\sigma} \ , \tag{2.23}$$

where we have written F and $\star F$ for the unit electric and magnetic charges. With the same electromagnetic energy-momentum tensor as (2.10), one finds

$$T_{\mu\nu}(\mathcal{F}) = (p^2+q^2)\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \tag{2.24}$$

where the second parenthesis is energy-momentum tensor of a *unit* electric charge whose non-vanishing components are quite familiar, i.e., $F_{tr} = -F_{rt} = -1/4\pi r^2$ and so the only non-zero component of its dual will be $(\star F)_{\theta\phi} = -(\star F)_{\phi\theta} = \sin\theta/4\pi$. What we learn from (2.24) is that the strength tensor F and its dual $\star F$ have the same contribution to the total strength tensor $\mathcal{F}$. Note that it is guaranteed for any strength tensor satisfying Maxwell's equations (2.9) and the Bianchi identity that its dual follows the same properties.

Now, we can read the function $F(r)$ for the general case as well. From the expression (2.24) for $T_{\mu\nu}$ and by a simple comparison with the result for the pure electric case, one can guess that the nominator of the third term in (2.16) should now be replaced by $p^2 + q^2$, i.e., we have

$$F(r) = 1 - \frac{2MG}{r} + \frac{(p^2 + q^2)G}{4\pi r^2} \ . \tag{2.25}$$

We close this part by just saying that surprisingly the upper bound for charges that prevents having a naked singularity matches with the BPS bound of $\mathcal{N} = 2$ supergravity that will be discussed in the third chapter. This upper bound is

$$p^2 + q^2 \leq 4\pi G M^2. \tag{2.26}$$

It is also straightforward to read the temperature of these black holes. Analogous to what we found for pure electric case in (2.22), here the temperature will be given by

$$T = \frac{\sqrt{M^2G^2 - \frac{(p^2+q^2)G}{4\pi}}}{2\pi r_+^2} \ . \tag{2.27}$$

2.2. Kaluza-Klein Reduction

As we mentioned shortly in 1.2, superstring theory is quite restrictive about dimensionality of spacetime, it is anomaly free only in ten dimensions. On the other hand, we know almost without any doubt that the universe we are living in is a $(3+1)$-dimensional spacetime. The facts that supports this idea are not only our daily life experiences, but also all high energy experiments we have done until now. We have not ever observed a direct evidence of extra dimensions in any of these experiments. So what have been motivating theoretical physicists to investigate theories in higher dimensions if there was no way to finally connect them to a four dimensional one? The reason that justifies these attempts is having a powerful and consistent technique that enables us to extract an "effective" four dimensional theory out of any higher dimensional one.

This technique was first innovated by Kaluza and later became accomplished by

Klein and so is called *Kaluza-Klein reduction* method. We are going to encounter this technique twice during this thesis. Once we dimensionally reduce Type IIA supergravity on a Calabi-Yau threefold to obtain a four dimensional supergravity theory and later we S^2 reduce a subclass of $4D$ supersymmetric black holes. Hence it is worthwhile to devote a section to study the idea of dimensional reduction in more details.

2.2.1. S^1 Reduction

One-dimensional reduction on a circle S^1 is basically the simplest case of the Kaluza-Klein reduction. Although we are not going to apply this specific case during this thesis, yet it will be discussed here to clarify the main idea in the most simplest possible way. Moreover, here we explain it in $5D$ to address the initial purpose of its introduction by Kaluza and Klein. There is another case in which S^1 reduction plays an important role. That is S^1 reduction of M-theory which leads to Type IIA superstring theory in ten dimensions [8]. Briefly speaking, M-theory is an eleven-dimensional theory whose low energy limit is supergravity in $11D$ and as such, it is more fundamental than any type of superstring theory that can be obtained via its dimensional reduction [10] .

2.2.1.1. Why Higher Dimensions?. In 1919, Kaluza [51] came up with the idea of adding an extra spatial dimension to $(3+1)$-dimensional spacetime. Inspired by unification of electric and magnetic forces in four dimensions via covariant formulation of special relativity, his main motivation was to unify gravity and electromagnetism this time. Our starting point is pure Einstein gravity in 5D given by

$$\hat{S} = \frac{1}{2\hat{\kappa}} \int d^5\hat{x}\sqrt{-\hat{g}}\ \hat{R} \tag{2.28}$$

with $d\hat{s}^2 = \hat{g}_{\hat{\mu}\hat{\nu}} d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}}$. From now on, all higher dimensional quantities are decorated by a hat " ˆ " that makes our formulation a bit messy. That is why we are going to avoid fully following this notation in the next chapters. Here for instance $\hat{\mu}, \hat{\nu} = 0, ..., 4$

[10] The $E_8 \times E_8$ heterotic superstring can be derived from M-theory as well.

run over five dimensions.

The action (2.28) is invariant under general coordinate transformation $\hat{x}^{\hat{\mu}} \rightarrow \hat{x}'^{\hat{\mu}} = \hat{x}^{\hat{\mu}} + \hat{\xi}^{\hat{\mu}}$ that changes the metric by

$$\delta \hat{g}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}} \hat{\xi}^{\hat{\rho}} \hat{g}_{\hat{\rho}\hat{\nu}} + \partial_{\hat{\nu}} \hat{\xi}^{\hat{\rho}} \hat{g}_{\hat{\rho}\hat{\mu}} + \hat{\xi}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{g}_{\hat{\mu}\hat{\nu}} \ . \tag{2.29}$$

As we will see in a moment, symmetries of the lower dimensional theory is determined by those of the higher dimensional theory [11] .

Now, let us focus on the reduction procedure. As it has been mentioned earlier, Kaluza started from a $5D$ metric and suggested the following 4+1 split of its components

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{\phi} g_{\mu\nu} + e^{-2\phi} A_{\mu} A_{\nu} & e^{-2\phi} A_{\mu} \\ e^{-2\phi} A_{\nu} & e^{-2\phi} \end{pmatrix} . \tag{2.30}$$

We call the fifth dimension y and so have $\hat{x}^{\hat{\mu}} = (x^{\mu}, y)$ where $\mu = 0, ..., 3$. Note that all new (unhatted) fields depend only on x^{μ} and are independent of the fifth coordinate y. But what are these new field really? To answer this question we need to perform the same $4+1$ decomposition of five-dimensional Einstein's equations $\hat{G}_{\hat{\mu}\hat{\nu}} = 0$ as well. Then it reveals that $g_{\mu\nu}(x)$, $A_{\mu}(x)$ and $\phi(x)$ introduced in (2.30) respectively satisfy Einstein's equations, Maxwell's equations and massless Klein-Gordon equations all in $4D$. Consequently, Kaluza's idea to unify gravity and electromagnetism by adding an extra dimension to $3+1$ spacetime really works [12] . Yet, there were two questions had been left unanswered by Kaluza that could weaken his successful idea seriously:

[11] Note that there is also global scale transformation that leaves equations of motion intact (the action (2.28) will be changed by a constant factor, though). It is global scale transformation with a constant parameter λ defined as $\delta \hat{g}_{\hat{\mu}\hat{\nu}} = \lambda \hat{g}_{\hat{\mu}\hat{\nu}}$. The scale transformation belongs to a larger group of conformal transformations under which the metric tensor remains invariant up to a scaling function $\Omega(x)$. But here we only focus on diffeomorphism that is the set of all transformations that keeps the metric invariant.

[12] Actually, by his time there were only these two fundamental forces known, so physicist were only curious about unification of these forces.

- If this idea has something to do with our real four-dimensional universe, it should be able to explain why we haven't ever seen the fifth direction.
- Forcing $4D$ fields to be independent of the fifth direction is a restrictive condition. How we can justify this choice?

As we are about to see, answer to these two questions is hidden in Klein's brilliant idea.

2.2.2. Infinite Towers of States and Symmetries

It seems that both of these ambiguities can be resolved simultaneously by making the fifth dimension *periodic (compact)* by taking $0 \leq y/R \leq 2\pi$ where R is radius of the circle. As proposed by Klein for the first time [52, 53], one can assume the $5D$ topology to be $\mathbb{R}^4 \times S^1$. This assumption allows us to release the restrictive condition on fields to be independent of y. So five-dimensional metric components can depend on the fifth direction but now they are forced to admit the following Fourier expansions as a result of periodicity in fifth direction

$$
\begin{aligned}
\hat{g}_{\mu\nu}(x,y) &\longrightarrow g_{\mu\nu}(x,y) = \sum_{n=-\infty}^{\infty} g_{\mu\nu n}(x) e^{iny/R} \ , \\
\hat{g}_{\mu 4}(x,y) &\longrightarrow A_{\mu}(x,y) = \sum_{n=-\infty}^{\infty} A_{\mu n}(x) e^{iny/R} \ , \\
\hat{g}_{44}(x,y) &\longrightarrow \phi(x,y) = \sum_{n=-\infty}^{\infty} \phi_{n}(x) e^{iny/R} \ ,
\end{aligned}
\tag{2.31}
$$

where for instance $g^*_{\mu\nu(n)}(x) = g_{\mu\nu(-n)}(x)$ and we have the same for other modes. Note that each of modes in expansion (2.31) is a "distinct" field. So one concludes that starting from a single gravitational field $\hat{g}_{\hat{\mu}\hat{\nu}}$ in $5D$ and doing one dimensional reduction on a circle, we obtain an infinite number of gravitational, electromagnetic and scalar fields in $4D$. This infinite spectrum is called *Kaluza-Klein tower*.

The fact is our $4D$ theory would get highly complicated if we had to deal with all of these modes. So we need to look for a reasonable way to truncate the majority part of the spectrum. As we will see soon, the same argument that explains why the higher dimensions have not been observed yet, will fix this problem too. But before going there, let us mention that one can Fourier expand parameter of the general coordinate transformation introduced in (2.29) following exactly the same logic

$$\hat{\xi}^\mu(x,y) = \sum_{n=-\infty}^{\infty} \xi_n^\mu(x) e^{iny/R} \quad , \quad \hat{\xi}^4(x,y) = \sum_{n=-\infty}^{\infty} \xi_n^4(x) e^{iny/R} \ . \tag{2.32}$$

This expansion reveals a delicate point that is: *By dimensional reduction, the number of symmetries in lower dimension becomes infinite too* [30].

For now, let us focus only on zero modes of the four dimensional fields. By substituting only zero modes from (2.31) into (2.30) and plug the result back inside $5D$ action (2.28) we obtain

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2}\, \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-3\phi} F_{\mu\nu} F^{\mu\nu} \right\} \tag{2.33}$$

where $F = dA$ is the field strength for the electromagnetic field and two prefactors are related as $\hat{\kappa}^2 = 2\pi R \kappa^2$ (one can absorb the factor of three by making a simple redefinition $\tilde{\phi} := \sqrt{3}\phi$). The way these fields transform under general coordinate transformation tells us what they are in four dimensions. By splitting indices of (2.29), we find

$$\begin{aligned}
&\mu,\ \nu,\ \hat{\rho}: && \delta g_{\mu\nu} = g_{\mu\hat{\rho}} \partial_\nu \xi^{\hat{\rho}} + g_{\hat{\rho}\nu} \partial_\mu \xi^{\hat{\rho}} + \xi^{\hat{\rho}} \partial_{\hat{\rho}} g_{\mu\nu} + g_{\mu\nu}(\partial_4 \xi^4) \ , \\
&\mu,\ \hat{\nu} = 4,\ \hat{\rho}: && \delta A_\mu = (\partial_\mu \xi^\rho) A_\rho + \xi^\rho \partial_\rho A_\mu + \partial_\mu \xi^4 - A_\mu (\partial_4 \xi^4) \ , \\
&\hat{\mu} = 4 = \hat{\nu},\ \hat{\rho}: && \delta\phi = \xi^\rho \partial_\rho \phi - \partial_4 \xi^4 \ .
\end{aligned} \tag{2.34}$$

Considering only zero modes of (2.32) and setting $\xi^4_{(0)} = cte.$ while keeping $\xi^\rho_{(0)}$ an arbitrary functions of four-dimensional coordinates, we will obtain the transformation laws for a tensor of rank two, a vector and a scalar respectively. Hence it follows that in the quantum level, the $4D$ theory is described by a spin two graviton, a spin one photon and a spin zero dilaton.

As we just saw in an example of Kaluza-Klein reduction, these are symmetries inherited from the higher dimensional theory that dictates the field content of the lower dimensional theory. Now, taking non-constant $\xi^4_{(0)}$ reveals a local gauge symmetry for the vector filed

$$\delta A_\mu = \partial_\mu \xi^4_{(0)} \ . \tag{2.35}$$

In addition to those symmetries inherited from higher dimensional theory, the lower dimensional theory may have its own symmetries. Here for instance, one can check that the lower dimensional Lagrangian (2.33) is invariant under the following global rescaling

$$\delta A_\mu = \lambda A_\mu \ , \qquad \delta\phi = \frac{2}{3}\lambda. \tag{2.36}$$

this symmetry is not obtainable from (2.34), neither it comes from a higher dimensional invariance under a global rescaling. Finally, taking the first mode of ξ^4 into account, i.e., setting $\xi^4 = iy/R\xi^4_{(1)}(x)$ we will get $\delta g_{\mu\nu} = (\partial_4\xi^4_{(1)})g_{\mu\nu}$, $\delta A_\mu = -(\partial_4\xi^4_{(1)})A_\mu$, $\delta\phi = -(\partial_4\xi^4_{(1)})$. After taking $\partial_4\xi^4_{(1)}(x) := \lambda$ we see $\delta\hat{g}_{\mu\nu} = 0$ but here we also varied the 4D metric. Does this rescaling leave the 4D action (2.33) invariant?)

So the global scale invariance in 5D (11) is spontaneously broken to (2.36) which results in to the massless dilaton as a Goldstone boson. Whereas, the graviton and the photon remain massless because of covariance under general coordinate transformations (2.34) and invariance under gauge transformation (2.35) respectively.

2.3. Particular Families of Complex Manifolds

In this section, we will have a short look at a very important family of Hermitian manifolds called Kähler manifolds. A detailed introduction to complex manifolds is provided in Appendix A to which we refer those readers who are not familiar with definitions and some important theorems in the context. Among many references on the topic specified to physicists, [33] covers necessary materials in a brief fashion. In addition, to find some discussions on main approaches to construct CY manifolds as well as computation of physical quantities in these spaces, one can check [54]. The fact that a complete section of this thesis has been devoted to the subject reflects their importance in our later discussions about $4D$ $\mathcal{N}=2$ supergravity. As we will see in the next chapter, the six-dimensional compactification manifold that take us from a $10D$ $\mathcal{N}=2$ supergravity to a four-dimensional one has to be a *Calabi-Yau* threefold ($CY3$). These manifolds make a subset of Kähler manifolds with an $SU(3)$ holonomy group that practically determines the number of survived supercharges after compactification. In addition, later we will discuss that gauge fields and scalar fields are quiet entangled in $4D$ supergravity. We will then explain in more details how this close relation forces the scalars target space to be *special Kähler*, another subset of Kähler manifolds. With this introduction, let us investigate these manifolds in more details.

2.3.1. Kähler Manifolds

As it is explained in Appendix A, *almost complex structure* J is a second ranked tensor field that completely specifies the complex structure of a complex manifold. Only a complex manifold can admit a *globally* constant complex structure. In other words, it is *only* for a complex manifold that J acts independent of chart. It squares to $-I_{2m}$ where m is complex dimensions of the manifold [13] . This suggests that in complex basis J is a diagonal matrix with eigenvalues $\pm i$ where its corresponding eigenvectors called holomorphic and anti-holomorphic vectors, respectively. Being globally constant implies that a vector remains holomorphic (anti-holomorphic) by traveling between patches. Let us now introduce Kähler manifolds

Definition 2.4.1. *A subgroup of Hermitian manifolds whose Kähler forms* (A.30) *are closed,* $d\Omega = 0$ *called* ***Kähler*** *manifolds. The metric* g *of these manifolds called* ***Kähler*** *metric. Obviously not all complex manifolds are Kähler.*

In addition, we have the following theorem that clarifies this definition.

[13]The manifold is called *almost complex* if $J^2 = -I_{2m}$ holds only locally.

Theorem 2.4.1. *A Hermitian manifold (M, g) is Kähler if and only if the almost complex structure J is a **covariantly** constant tensor field*

$$\nabla_\mu J = 0 \ . ^{14} \tag{2.37}$$

To prove this theorem, we first need to note that for any r-form ω

$$d\omega \equiv \nabla\omega = \frac{1}{r!}\nabla_\mu \omega_{\nu_1,\ldots,\nu_r} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_r} \ . \tag{2.38}$$

Then using metric compatibility condition $\nabla g = 0$, one can show that Ω is closed ($\nabla_\mu \Omega = 0$) if and only if $\nabla_\mu J = 0$ [31].

On the other hand, it can be shown that the almost complex structure J is *covariantly* constant with respect to the *Hermitian connection* [A.3.2]. Again, not all Hermitian metrics accept a Hermitian connection [15]. Those which do so are called **Kähler** manifolds. In other words, any metric which admits a (metric compatible) Hermitian connection is Kähler. There is another way to define this family of complex manifolds which will be of more interest in subsection 3.3.1 and that is a Kähler is a complex manifold that admits a symplectic vector bundle. We will pay to this in much more detail in Subsection 3.3.1.

Now recall Ω in terms of the metric (A.30). From $d\Omega = 0$, we obtain

$$\frac{\partial g_{\mu\overline{\nu}}}{\partial z^\lambda} = \frac{\partial g_{\lambda\overline{\nu}}}{\partial z^\mu} \quad , \quad \frac{\partial g_{\mu\overline{\nu}}}{\partial \overline{z}^\lambda} = \frac{\partial g_{\mu\overline{\lambda}}}{\partial \overline{z}^\nu} \ . \tag{2.39}$$

Not only do these relations characterize the Kähler metric, but also they suggest that one can derive it from a function called "**Kähler potential**". Imagine a Hermitian

---

[14] Meaning that its components satisfy $\left(\nabla_\kappa J\right)_\nu^{\ \mu} = 0 = \left(\nabla_{\overline{\kappa}} J\right)_{\overline{\nu}}^{\ \overline{\mu}} = \left(\nabla_{\overline{\kappa}} J\right)_\nu^{\ \mu} = \left(\nabla_\kappa J\right)_{\overline{\nu}}^{\ \overline{\mu}}$.

[15] As explained in [A.3.2], Hermiticity of the connection comes from Hermiticity of the metric plus requiring holomorphic and anti-holomorphic vectors not to get mixed after parallel transportation. It can be shown that this second condition is equivalent to the definition of Kähler manifolds (2.37).

metric g takes the following form on a chart U_i

$$g_{\mu\overline{\nu}} = \partial_\mu \partial_{\overline{\nu}} \mathcal{K}_i \quad ; \quad \mathcal{K}_i \in \mathcal{F}(U_i) \ . \tag{2.40}$$

Clearly, g satisfies (2.39) and so is Kähler. Conversely, it has been proven that any Kähler metric can take this form (2.40) *locally*, and so on U_i we have $\Omega = i\partial\overline{\partial}\mathcal{K}_i$ [16] . On the intersection of two overlapping charts (U_i, φ_i) and (U_j, φ_j) with $z = \varphi_i(p)$ and $\omega = \varphi_j(p)$ one finds

$$\frac{\partial \omega^\alpha}{\partial z^\mu} \frac{\partial \overline{\omega}^\beta}{\partial \overline{z}^\nu} \frac{\partial}{\partial \omega^\alpha} \frac{\partial}{\partial \overline{\omega}^\beta} \mathcal{K}_j = \frac{\partial}{\partial z^\mu} \frac{\partial}{\partial \overline{z}^\nu} \mathcal{K}_i \tag{2.41}$$

which can be satisfied if and only if we have

$$\mathcal{K}_j(\omega, \overline{\omega}) = \mathcal{K}_i(z, \overline{z}) + \phi_{ij}(z) + \psi_{ij}(\overline{z}) \tag{2.42}$$

meaning that Kähler potentials on two different charts can only differ by holomorphic and anti-holomorphic functions. Relation (2.42) is called "*Kähler transformations*" [29, 31]. As it can be deduced from (2.41) the Kähler potential does not transform in a nice way. In fact, there are cases in which the potential can not be defined globally that weaken this approach to the metric. That is why in some cases where other approaches are available we may prefer to avoid focusing on this one. For instance, as we will discuss later in 3.3.1, in the case of special Kähler manifolds it is more helpful to follow their symplectic structure.

It may be clarifying to look at an specific example of a Kähler manifold. Definitely, the very first instant is the complex Euclidean space $\mathbb{C}^m$. Another important example is *complex projective space* $\mathbb{C}P^n$ that appears very frequently. Its Kähler potential on a chart U_α for which $z^\alpha \neq 0$ is given by

$$\mathcal{K}_\alpha(p) \equiv \sum_{\nu=1}^{n+1} \left| \frac{z^\nu}{z^\alpha} \right|^2 \ . \tag{2.43}$$

[16]Having $d = \partial + \overline{\partial}$, it is easy to check that $d\Omega = 0$.

Then $\mathcal{K}_\beta$ at $p \in U_\alpha \bigcap U_\beta$ can be obtained as $\mathcal{K}_\beta(p) = \left|z^\alpha/z^\beta\right|^2 \mathcal{K}_\alpha(p)$ which implies that $\partial\overline{\partial}\ \log\mathcal{K}_\alpha = \partial\overline{\partial}\ \log\mathcal{K}_\beta$ and so we can define the Kähler form locally as $\Omega \equiv i\partial\overline{\partial}\ \log\mathcal{K}_\alpha$ that leads to a positive definite metric on $\mathbb{C}P^m$ called **Fubini-Study** [31].

Let us finish this part by listing some general facts and definitions:

- The only sphere which admits a complex structure is S^2, having $S^2 \simeq \mathbb{C}p^1$ shows it is also a Kähler manifold.
- $S^{2m+1} \times S^{2n+1}$ is always a complex manifold but not Kähler.
- Any orientable one-dimensional complex manifold M is Kähler. The argument follows very simple: Kähler form Ω is a real two-form (A.30), meaning that $d\Omega = 0$ since there can not exist any three-form on M.
- **Betti number** is defined as real dimension of r-th De Rham cohomology of a real manifolds, i.e., $b^r(M) \equiv \dim H^r(M)$. Now, let M be an m-dimensional compact Kähler manifold without any boundary. Then $\Omega^m \equiv \overbrace{\Omega \wedge ... \wedge \Omega}^{\text{m}}$ is closed but not exact, and so we always have $b^{2m} \geq 1$. More generally, $b^{2p} \geq 1$ for $1 \leq p \leq m$.
- M is called a **Riemann surface** if it is an one-dimensional compact orientable complex manifold.

Kähler geometry

Recall that a Kähler metric satisfies (2.39) which guaranties its **torsionlessness**, i.e., we have $T^\lambda{}_{\mu\nu} = 0 = T^{\overline{\lambda}}{}_{\overline{\mu\nu}}$ (A.35). This property makes the connection of a Kähler metric very similar to the Levi-Civita. Also we have an extra symmetry for the Riemannian tensor (A.36)

$$R^\kappa{}_{\lambda\mu\overline{\nu}} = R^\kappa{}_{\mu\lambda\overline{\nu}}\ , \quad R^{\overline{\kappa}}{}_{\overline{\lambda}\overline{\mu}\nu} = R^{\overline{\kappa}}{}_{\overline{\mu}\overline{\lambda}\nu}\ , \quad R^\kappa{}_{\lambda\overline{\mu}\nu} = R^\kappa{}_{\nu\overline{\mu}\lambda}\ , \quad R^{\overline{\kappa}}{}_{\overline{\lambda}\mu\overline{\nu}} = R^{\overline{\kappa}}{}_{\overline{\nu}\mu\overline{\lambda}}\ , \tag{2.44}$$

which cause the Ricci form (A.38) and the Ricci tensor $R_{\mu\overline{\nu}}$ (obtainable from (A.36)) become identical.

Holonomy group

Let p be a point on a manifold M with connection ∇ and c be a loop passing through p. Now, take a vector $V \in T_pM^+$ and parallel transport it along c. Generally, we will end up with another vector $V' \in T_pM^+$ [17] which has to be related to V by an element of $GL(T_pM)$, the group of all *linear invertible transformations*. In other words, there is always a group element attributed to any loop that starts from and ends at p. Considering all possible loops, one can define a map between an elements of the group and its corresponding loop. The group is called *the holonomy group*. As said before, in the most general case the holonomy group is $GL(T_pM)$. However, if the parallel transportation preserve some specific properties of the vector fields (such as their length), then the holonomy group reduces to a subgroup of $GL(T_pM)$. For instance in the Euclidean space the holonomy group is trivial and contains only identity element [32].

Having this definition, let us turn back to the case of an m-dimensional Kähler manifold. In this case, the connection ∇ on M does not mix holomorphic and anti-holomorphic vectors. Hence the parallel transported vector can be written as $V'_\mu = V_\nu h^\nu{}_\mu \in T_pM^+$. Moreover, it preserves length of the vectors, meaning that the holonomy group is restricted to $h^\nu{}_\mu(c) \in U(m) \subset O(2m)$.

2.3.2. Calabi-Yau Manifolds

Calabi-Yau manifolds are a big subset of Kähler manifold that admit a Ricci flat metric h, i.e., we have $R = 0 = \mathfrak{R}$.

Theorem 2.5.1. *The first Chern class vanishes for a Calabi-Yau manifold.*
To prove this theorem, first assume that (M, h) is a Kähler manifold with a flat metric. Then, according to the definition of the first Chern class which is given in section A.3.2,

[17] V' has to lie in T_pM because parallel transportation is a linear isomorphism. Also note that our definition is independent of the vector V we have started with. If we start with any other vector $\tilde{V}$ there is still an element of $GL(T_pM)$ that maps these two vectors.

$\mathfrak{R}(h) = 0$ implies $c_1(M) \equiv [\mathfrak{R}/2\pi]$ vanishes. Moreover, let (M, g) be another Kähler with a non-flat metric g such that $g = h + \delta\tilde{h}$. Moreover, recall the Theorem A.3.2 from the Appendix A which states that c_1 is invariant under a smooth variation in the metric. Consequently, one conclude that the first Chern class of g also vanishes.

Having this property proven for a Calabi-Yau manifold, we have an alternative way to define it as follows

Definition 2.5.1. *A compact Kähler manifold whose first Chern class vanishes is called Calabi-Yau.*

Theorem 2.5.2. *The holonomy group of an m-dimensional Calabi-Yau is $SU(m)$* [31]. To have an understanding of this theorem, let us take $X \in T_pM^+$. Then, the parallel transported vector $X^{'} \in T_pM^+$ is given by

$$X'^{\mu} = X^{\mu} + X^{\nu} R^{\mu}{}_{\nu\kappa\overline{\lambda}}\, \epsilon^{\kappa}\, \overline{\delta}^{\lambda} \tag{2.45}$$

where ϵ and δ are two length of the parallelogram (See Figure 2.1). This shows that the holonomy group becomes trivial if the Riemann tensor vanishes on M. From (2.45) we obtain

$$\Theta_{\nu}{}^{\mu} = \delta_{\nu}{}^{\mu} + R^{\mu}{}_{\nu\kappa\overline{\lambda}}\, \epsilon^{\kappa}\, \overline{\delta}^{\lambda}\ . \tag{2.46}$$

The second term measures deviation of $X^{'}$ from X generated by $R^{\mu}{}_{\nu\kappa\overline{\lambda}}$. Now taking the trace leads to

$$R^{\mu}{}_{\mu\kappa\overline{\lambda}}\epsilon^{\kappa}\overline{\delta}^{\lambda} = \mathfrak{R}_{\kappa\overline{\lambda}}\epsilon^{\kappa}\overline{\delta}^{\lambda} \tag{2.47}$$

that vanishes as a result of Ricci-flatness of the metric. This implies that the generators of the holonomy group (or equivalently, generators of this deviation) have to be traceless. The holonomy group of Kähler manifolds $U(m)$ can be decomposed to $SU(m)\times U(1)$, and so at the level of algebra we have $\mathfrak{u}(m) = \mathfrak{su}(m)\oplus\mathfrak{u}(1)$. The trace is contained in $\mathfrak{u}(1)$ that leaves $\mathfrak{su}(m)$ factor traceless. Consequently, the holonomy group of a Calabi-Yau in which we are interested is generated by $\mathfrak{su}(m)$ and so is $SU(m)$. This conclusion is equivalent to the following statement: *A Calabi-Yau threefold admits a unique covariantly constant spinor*. We will come back to this point later.

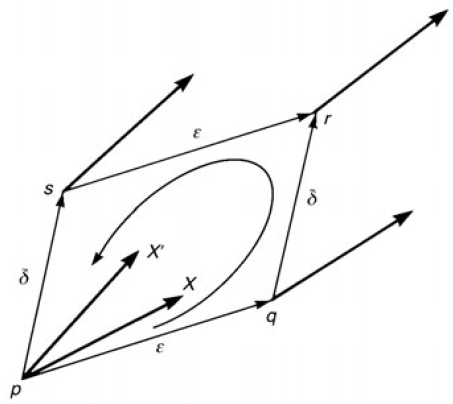

Figure 2.1: Parallelogram of a vector $X \in T_p(M)$ [31]. The second vector $X^{'} \in T_p(M)$ is obtained via parallel transportation along a closed path $pqrsp$. These two vectors are related by a member of the Holonomy group of the manifold M.

As the last step of our introduction of Calabi-Yau manifolds, we want to have a look at its Hodge diamond diagram. To that aim, we first define Hodge numbers as a geometrical quantity of a manifold. We refer those who are not familiar with cohomology classes to the Appendix A where its definition and some helpful details can be found.

The complex dimension of (r,s)-th Dolbeault cohomology $H^{(r,s)}_{\overline{\partial}}$ defined in (A.20) is called **Hodge number** and denoted by $h^{(r,s)}$. These numbers can be listed in the so-called **Hodge diamond diagram**. In general there are $(m+1)^2$ of these numbers, but depending on type of the manifold, the number of independent Hodge numbers can be decreased because of some constraints. This may simplify the Hodge diamond considerably in some cases. For instance, in the case of an m-dimensional Kähler manifold one can show

$$h^{r,s} = h^{s,r} \quad , \quad h^{r,s} = h^{m-r,m-s} \; . \tag{2.48}$$

These two relations make the Hodge diamond of a Kähler manifold symmetric both vertically and horizontally. Generally, for a generic Hermitian manifold, there in no specific relation between Hodge number and Betti numbers (defined on page 39). But they get closely related in the case of a Kähler manifold. The following theorem enumerates some properties for the Betti numbers.

Theorem 2.5.3. *For an m-dimensional Kähler manifold without boundary $\partial M = 0$, the Betti number b^p $(1 \leq p \leq 2m)$ satisfies* [31]:

(a) $b^p = \sum_{r+s=p} h^{r,s}$,

(b) b^{2p-1} is even for $1 \leq p \leq m$,

(c) $b^{2p} \geq 1$ for $1 \leq p \leq m$.

In the case of Calabi-Yau manifolds there is another constraint that relates different Hodge numbers even more, they can be found in [55]. Ultimately, all these information results in the Hodge diagram of a complex manifold.

Till now, our discussion where quite general in dimensionality. However, for the later applications, let us focus on the case of Calabi-Yau threefold and finish this chapter with its rather simple Hodge diamond diagram given in Table 2.1. The Hodge diamond diagram also contains valuable information about *Homology class* of the corresponding manifold via **Poicaré duality**. More precisely, entries of Table 2.1 reveals the number of non-trivial cycles in $CY3$ as well. For instance $h^{1,1}$ shows the number of 2- or 4-cycles while $h^{2,1}$ shows that of 3-cycles. We are going to use these very soon in the next section. To see some concrete examples of Calabi-Yau manifolds, we refer the reader to Chapter 17 of [56].

<u>2.3.2.1. Homology Class and Stokes' Theorem on Complex Manifolds.</u> The theory we are interested in contains electromagnetic charges which correspond to the field strengths of Type IIA supergravity. To be able to define these charges explicitly and also find the decomposition of field strengths in terms of cohomology basis of $CY3$, we first need

Table 2.1: Hodge diamond of Calabi-Yau 3-fold. Because of **Hodge duality**, in any $6D$ manifold $b^2 = b^4$ (the number of 2- and 4-forms in cohomology group).

$H^0(Y)$				1			
$H^1(Y)$			0		0		
$H^2(Y)$		0		$h^{1,1}$		0	
$H^3(Y)$	1		$h^{2,1}$		$h^{2,1}$		1
$H^4(Y)$		0		$h^{1,1}$		0	
$H^5(Y)$			0		0		
$H^6(Y)$				1			

to have a short look at two notions: *the Poincaré duality* and *Stokes' theorem* [31].

Consider X as an m-dimensional compact manifold for which $\alpha \in \Omega^r(X)$ and $b \in \Omega^{m-r}(X)$. Then, one can define the following bilinear *inner product* (ω is the unique volume form on X)

$$(\alpha, \beta) := \int_X \alpha \wedge \beta \ . \tag{2.49}$$

This inner product $(\ ,\ ) : \Omega^r(X) \times \Omega^{m-r}(X) \mapsto \mathbb{R}$ is non-singular as it should be and provides a one-to-one correspondence between cohomology groups $H^r(X)$ and $H^{m-r}(X)$. So we have

$$H^r(X) \approxeq H^{m-r}(X) \ . \tag{2.50}$$

On the other hand, there is another duality between $H^r(X)$ and $H_r(X)$ where the later is the r**th homology** group of the manifold X. Let us first define what is a homology group. In short, there are two nil-potent operators: one is the *boundary operator* ∂

acting on an r-chain [18] as $\partial : C_r(X) \mapsto C_{r-1}(X)$. The other nil-potent operator is the *exterior derivative* d acting on an r-form as $d : \Omega^r(X) \mapsto \Omega^{r+1}(X)$. These two operators define the following complexes on manifold X

$$\begin{aligned}
&\textbf{chain complex } C(X) \;:\quad ... \xleftarrow{\partial_{r-1}} C_{r-1}(X) \xleftarrow{\partial_r} C_r(X) \xleftarrow{\partial_{r+1}} C_{r+1}(X) \xleftarrow{\partial_{r+2}} ...\\
&\textbf{de Rham complex } \Omega(X) \;:\quad ... \xrightarrow{d_{r-1}} \Omega^{r-1}(X) \xrightarrow{d_r} \Omega^r(X) \xrightarrow{d_{r+1}} \Omega^{r+1}(X) \xrightarrow{d_{r+2}} ...
\end{aligned}$$

and so rth homology and cohomology groups defined as follows

$$H_r(X) \equiv Z_r(X)/B_r(X) = \mathrm{ker}\partial_r/\mathrm{im}\partial_{r+1} \tag{2.51}$$

$$H^r(X) \equiv Z^r(X)/B^r(X) = \mathrm{ker}d_{r+1}/\mathrm{im}d_r \;. \tag{2.52}$$

The idea can be generalized to the case of complex manifold as has been discussed in A.2.1.

We also can define an inner product between an r-form and an r-chain. Let us take $c \in C_r(X)$ and $\alpha \in \Omega^r(X)$ and define

$$(c, \alpha) := \int_c \alpha \;. \tag{2.53}$$

This inner product $(\;,\;) : C_r(X) \times \Omega^r(X) \mapsto \mathbb{R}$ is also bilinear and should not be confused with (2.49). This leads us to the **de Rham's theorem**:
Theorem 3.1. Let X be a compact manifold, then $H_r(X)$ and $H^r(X)$ are finite dimensional vector spaces for which we can define a bilinear non-degenerate map

$$\Delta \;:\; H_r(X) \times H^r(X) \mapsto \mathbb{R} \;, \tag{2.54}$$

which reveals that $H^r(X)$ is the dual vector space of $H_r(X)$.

Having (2.50) and (2.54) together, reveals a one-to-one correspondence between

[18] We refer the reader who is not familiar with chains and complexes to [31]

$H^r(X)$ and $H_{m-r}(X)$ which is called "*the Poincaré duality*". More precisely, it states that for an m-dimensional compact manifold, the map $H^r(X) \to H_{m-r}(X)$ is an isomorphism for all r.

Taking these two operators, one can express **Stokes' theorem** by

$$(c, d\alpha) = (\partial c, \alpha) \ . \tag{2.55}$$

If α is a closed form in (2.53), then (c, α) will be called **period** of α over the cycle c.

We finish this chapter by fixing our notation for homology and cohomology basis of $CY3$ in the following table. They satisfy the following orthogonality relations

$$\begin{aligned} \int_{\mathbb{A}^B} D_A &= \int_Y D_A \wedge D^B = \delta_A^B \ , \qquad & \int_{\mathbb{B}_A} D^B &= \int_Y D^B \wedge D_A = -\delta_A^B \\ \int_{\mathbb{C}_K} \beta^L &= \int_Y \beta^L \wedge \alpha_K = -\delta_K^L \ , \qquad & \int_{\mathbb{C}_L} \alpha_K &= \int_Y \alpha_K \wedge \beta^L = \delta_K^L \ . \end{aligned} \tag{2.56}$$

Table 2.2: (Co)homology basis for a $CY3$. One should be careful about lower and upper indices here. Depending on the context, they may show Hodge dual pairs like a p- and a $(6-p)$-form, or they may refer to the Poincaré dual pairs, i.e., a p-form and a $(6-p)$-cycle. Commonly in the literature, $(H_{(2,1)}, H^{(1,2)})$ are denoted by (α_K, β^K).

cohomology group	cohomology basis	homology basis	
$H^{(1,1)}$	D_A	$\mathbb{A}^A$	$A = 1, \cdots, h^{(1,1)}$
$H^{(2,2)}$	D^A	$\mathbb{B}_A$	
$H^{(1,2)}$	χ^K	$\mathbb{C}_K$	$K = 1, \cdots, h^{(2,1)}$
$H^{(2,1)}$	χ_K	$\mathbb{C}^K$	
$H^{(0,3)} \oplus H^{(1,2)}$	$\chi^{\tilde{K}} = (\overline{\Lambda}, \chi^K)$	$\mathbb{C}_{\tilde{K}}$	$\tilde{K} = 0, \cdots, h^{(2,1)}$
$H^{(3,0)} \oplus H^{(2,1)}$	$\chi_{\tilde{K}} = (\Lambda, \chi_K)$	$\mathbb{C}^{\tilde{K}}$	
$H^{(0,0)} \oplus H^{(1,1)}$	$D_{\tilde{A}} = (1, D_A)$		$\tilde{A} = 0, \cdots, h^{(1,1)}$
$H^{(2,2)} \oplus H^{(3,3)}$	$D^{\tilde{A}} = (D^A, \mathcal{V})$		

3. MULTI-CENTERED BLACK HOLE SOLUTIONS TO $\mathcal{N} = 2$ SUPERGRAVITY IN $4D$

In this chapter, we are going to discuss $\mathcal{N} = 2$ supergravity in four dimensions and its black hole solutions. We first review $\mathcal{N} = 2$ supergravity spectrum in $10D$ as the low energy limit of ten-dimensional superstring theory. Then, we are going to investigate how one can effectively obtain $\mathcal{N} = 2$ supergravity in $4D$ via compactifying $10D$ supergravity on a Calabi-Yau threefold which is introduced in the last section of the previous chapter. We also explain how the number of survived supercharges is determined by our choice of internal manifold. More specifically, we will see that only eight out of thirty two initial supercharges survive after CY compactification. Furthermore, the lower dimensional fields and so the corresponding Lagrangian are dictated by the internal manifold as well. It will be also discussed how electromagnetic charges $\{\Gamma\}$ attributed to the gauge fields are obtainable from wrapped D-branes on non-trivial cycles of the CY. Having these discussed, we will investigate the geometry that connects scalars and gauge fields in the theory in a very specific way. We will show that it is given by a Kähler manifold equipped with a symplectic bundle and a special form for its potential, therefor it is called a *special Kähler manifold*.

Then, one can look for specific supersymmetric solutions to the supergravity Lagrangian. We are specially interested in black hole solutions of the theory. As one can follow from sections, we first look at the static black holes with only one center via the well-known BPS formalism. Saturating the BPS bound leads to a set of first order differential equations equivalent to Killing spinor equation called *attractor flow equations*. Solution to these equations is usually called *BPS black holes*. In the next step, we prob such a black hole by another massive charged particle and observe that there can be a bound state between these two in specific conditions. Then, the argument will be generalized to the case with arbitrary number of massive charges. These stationary bound states of BPS black holes are usually called *multi-centered black holes*. As we will discuss with details, these bound states can decay to two or more smaller clusters. There are also cases in which centers go infinitely far away from each other or get arbitrarily close. The later case which is called *scaling solutions* is of main interest for the rest of this thesis. The chapter ends with some original results about asymptotic behavior of this specific case. This reveals some subtleties in the asymptotic geometry of the scaling solutions which is mainly discussed in the next chapter.

3.1. $\mathcal{N} = 2$ SUGRA in $10D$

Supergravity is the low energy limit of the superstring theory in which we only deal with massless fields and states. As such, here we are going to shortly review the closed superstring spectrum first. Having the massless part of this spectrum, we can classify the field content of Type II supergravity and read out the Lagrangian that describes the theory.

3.1.1. Closed Strings, NS and R Spectra

Consider a fermionic string $\psi^\mu(\sigma^1, \sigma^2)$ where (σ^1, σ^2) are parameters on the world-sheet. To study its dynamics on a cylinder, we define $w := \sigma^1 + i\sigma^2$ and then impose a periodicity condition $w \sim w + 2\pi i$. Requiring the Lorentz invariance as well as boundary terms to be vanished leave two choices for periodicity of ψ^μ. This splits the fermionic spectrum into two sectors called *Ramond* (R) and *Neveu-Schwarz* (NS) respectively

$$\psi^\mu(w + 2\pi i) = \exp(2\pi i\nu)\psi^\mu(w) \quad ; \quad \nu = \begin{cases} 0 & \text{for} \quad \text{R sector} \\ \frac{1}{2} & \text{for} \quad \text{NS sector} \end{cases} \tag{3.1}$$

Clearly, there are two options for the right-moving modes as well, i.e., $\overline{\psi}^\mu(\overline{w} - 2\pi i) = \exp(-2\pi i\overline{\nu})\overline{\psi}^\mu(\overline{w})$ with $\overline{\nu} = (0, \frac{1}{2})$.

To obtain spectra for these two sectors of fermionic string, we first need to find the ground state properly. There is no ambiguity for the NS sector. Since there is no zero mode in this case, we can safely take positive modes as annihilation operators that eliminate the ground state

$$\psi_r^\mu \left|0\right\rangle_{NS} = 0 \quad \text{for} \quad r > 0 \ . \tag{3.2}$$

Having this defined, one can build the tower of exited states via successive action of the negative modes (creation operators with $r < 0$) on $\left|0\right\rangle_{NS}$. The NS ground state is a Lorentz singlet annihilated by $\Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ meaning that $\left|0\right\rangle_{NS}$ is a spin-0 state while creation operators are vectors and so change spin of the initial state by one. So all excited states have integer spin and so are bosonic. Also note that because of being Grassmannian (anti-commuting) operators, each mode can be excited only once.

The ground state in the Ramond sector needs more attention. Again, it is defined as the state annihilated by all the positive modes. However, we still need to be careful about the way eight zero modes $\{\psi_0^\mu\}$ act on the ground state. The so-called Dirac representation of $\mathfrak{so}(D-1,1)$ algebra is given by

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = \eta^{\mu\sigma}\Sigma^{\nu\rho} + \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho} \ ; \quad \Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] \ , \tag{3.3}$$

from which one can check that $\{\Sigma^{2\tilde{k},2\tilde{k}+1}\}$ commute among themselves for $\tilde{k} = 0, \cdots, 4$. We define

$$S_{\tilde{k}} := i^{\delta_{\tilde{k},0}}\Sigma^{2\tilde{k},2\tilde{k}+1} \quad \text{with} \quad s_{\tilde{k}} = \pm\frac{1}{2} \ . \tag{3.4}$$

Here, $\{s_{\tilde{k}}\}$ are eigenvalues of these generators and the factor of i is introduced such that it makes S_0 Hermitian and so s_0 is now a good quantum number. Having a set of commuting Hermitian generators means that they have common eigenstates labeled by their corresponding eigenvalues. The half-integer spectrum of these generators reveals that their eigenstates represent spacetime fermions. It also shows that the degeneracy of the Ramond ground state equals $2^5 = 32$. Now, the fact that zero modes anti-commute with all positive modes (i.e., $\{\psi^\mu_r, \psi^\mu_0\} = 0$ for $r > 0$) implies that ψ^μ_0 rotates ground states among themselves. Moreover, one can check that the chirality operator commutes with all Lorentz generators $\Sigma^{\mu\nu}$ which means that the Dirac representation is still reducible to two Weyl representations $\mathrm{R}_\pm$ with ± 1 chirality or equivalently, even and odd numbers of $+1/2$s. So the Weyl decomposition of the Dirac representation is

$$\mathbf{32}_{\text{Dirac}} = \mathbf{16} \times \mathbf{16}^{'} \ , \tag{3.5}$$

each of which can be decomposed further as

$$\mathbf{16} \to \big(+\frac{1}{2}, \mathbf{8} \big) + \big(-\frac{1}{2}, \mathbf{8}^{'} \big) \qquad , \qquad \mathbf{16}^{'} \to \big(+\frac{1}{2}, \mathbf{8}^{'} \big) + \big(-\frac{1}{2}, \mathbf{8} \big). \tag{3.6}$$

where written in boldface is the dimension of the representations.

Closed string spectrum

To construct a closed string, we need to stick two open strings together, one is left-moving and the other is right-moving. The left- and right-moving open strings have to be from the same mass level because of the *level-matching* condition. We have already recognized two sectors for fermionic strings. Hence, potentially we have four different arrangements for closed strings. Given the fact that NS states are all bosonic while the R sector is pure fermionic, it is easy to guess that NS-NS and R-R sectors of closed strings are all with integer spins. Also, one would expect NS-R and R-NS states to have half-integer spins.

The ground state of R-R sector $\big|\mathbf{s}, \mathbf{s}^{'}\big\rangle_R$ has a degeneracy of $(32)^2$ that can be decompose in the following way

$$\mathbf{32}_{Dirac} \times \mathbf{32}_{Dirac} = [0] + [1] + ... + [10] = [0]^2 + [1]^2 + ... + [5] \ , \tag{3.7}$$

where $[n]$ denotes an anti-symmetric tensor of rank n, or equivalently an n-form gauge fields. The second equality is easily obtainable by applying the Hodge duality. The

R-R ground states in terms of the Weyl representations $R_\pm$ are classified inside the Table 3.1. Considering NS-sector as well, level-matching condition excludes NS_- from pairing with any sector other than itself since otherwise the resulting spectrum becomes tachyonic. Hence, we can only allow NS_+ and $R_\pm$ to pair up with themselves and with each other. This leads to six different possibilities listed as $SO(8)$ representations in the Table 3.2.

Table 3.1: Massless R-R states as a representation of $SO(9,1)$. The subscripts $\pm$ in rank five representation shows self-duality and anti-self-duality of the anti-symmetric tensors, respectively.

	$SO(9,1)$ rep.
(R_+, R_+)	$\mathbf{16}\times\mathbf{16} = [1]+[3]+[5]_+$
(R_+, R_-)	$\mathbf{16}\times\mathbf{16}' = [0]+[2]+[4]$
(R_-, R_+)	$\mathbf{16}'\times\mathbf{16} = [0]+[2]+[4]$
(R_-, R_-)	$\mathbf{16}'\times\mathbf{16}' = [1]+[3]+[5]_-$

Now, one can recognize two different possibilities for pairing various sectors of the Table 3.2. They are

$$\begin{aligned}
\text{Type IIA}\ :\quad & \big\{(NS_+, NS_+)\ ,\ (R_+, NS_+)\ ,\ (NS_+, R_-)\ ,\ (R_+, R_-)\big\} \rightarrow \\
& \big\{[0]+[1]+[2]+[3]+(2)+\mathbf{8}+\mathbf{8}'+\mathbf{56}+\mathbf{56}'\big\}\ \ , \\
\text{Type IIB}\ :\quad & \big\{(NS_+, NS_+)\ ,\ (R_+, NS_+)\ ,\ (NS_+, R_+)\ ,\ (R_+, R_+)\big\} \rightarrow \\
& \big\{[0]^2+[2]^2+[4]_+ +(2)+\mathbf{8}'^2+\mathbf{56}^2\big\}
\end{aligned} \tag{3.8}$$

We are interested in these two choices rather than other possibilities because they are tachyon-free and contain both fermionic and bosonic states. As one can see, the first two sectors are same in both types while the last two sectors are different. More precisely, the pairs in NS-R and R-R sectors of Type IIA have opposite chiralities and

Table 3.2: Massless spectrum of the closed superstrings, constructed by production of two copies of open strings $SO(8)$ representations. Again, $\pm$ subscripts shows self-duality and anti-self-duality of the representation.

Sector	$SO(8)$ spin	Tensors	Dimensions
(NS_+, NS_+)	$\mathbf{8}_v \times \mathbf{8}_v$	$[0]+[2]+(2)$	$\mathbf{1}+\mathbf{28}+\mathbf{35}$
(R_+, R_+)	$\mathbf{8}\times\mathbf{8}$	$[0]+[2]+[4]_+$	$\mathbf{1}+\mathbf{28}+\mathbf{35}_+$
(R_+, R_-)	$\mathbf{8}\times\mathbf{8}^{'}$	$[1]+[3]$	$\mathbf{8}_v+\mathbf{56}_t$
(R_-, R_-)	$\mathbf{8}^{'}\times\mathbf{8}^{'}$	$[0]+[2]+[4]_-$	$\mathbf{1}+\mathbf{28}+\mathbf{35}_-$
(NS_+, R_+)	$\mathbf{8}_v\times\mathbf{8}$		$\mathbf{8}^{'}+\mathbf{56}$
(NS_+, R_-)	$\mathbf{8}_v\times\mathbf{8}^{'}$		$\mathbf{8}+\mathbf{56}^{'}$

so this part of the spectrum is non-chiral and invariant under parity [19] . However, these sectors are of the same chirality in Type IIB which makes the spectrum chiral.

3.1.2. Type IIA SUGRA

Returning to the Table 3.2 one realizes that the spectrum we discussed there is exactly that of a ten-dimensional supergravity. There, we classified $SO(8)$ representations of the Lorentz group according to their dimensions, helicities and being self- or anti-self-dual. Now, in Table 3.3 we are going to introduce our notations for these representations as physical fields.

Here, we have followed *democratic formulation* [57] in which all gauge potentials in R-R sector are shown as $A_1, ..., A_9$. Note that generally we can only have odd rank gauge potentials in Type IIA depending on brane types we add to the theory. But in the absence of D-brane fluxes, the only gauge potentials one gets from string theory

[19] Parity changes chirality and so takes $\mathbf{8}\to\mathbf{8}^{'}$ and $\mathbf{56}\to\mathbf{56}^{'}$.

Table 3.3: The spectrum of $\mathcal{N}=2$ Type IIA supergravity in $10D$.

Bosonic		Fermionic	
$NS-NS$	$R-R$	$R-NS$	$NS-R$
$\{\ \phi,\ B_{MN},\ g_{MN}\ \}$	$\{\ A_M,\ A_{MNP}\ \}$	$\{\ \lambda^{(-)},\ \psi_M^{(+)}\ \}$	$\{\ \lambda^{(+)},\ \psi_M^{(-)}\ \}$
$\{\ \mathbf{1},\ \mathbf{28},\ \mathbf{35}\ \}$	$\{\ \mathbf{8}_v,\ \mathbf{56}\ \}$	$\{\ \mathbf{8}',\ \mathbf{56}\ \}$	$\{\ \mathbf{8},\ \mathbf{56}'\ \}$

spectrum are A_1 and A_3 . In the fermionic side, we see two M-W dilatinos $\lambda^{\pm}$ and two M-W gravitino $\psi_M^{\pm}$ whose superscripts show their chirality (+ for left-moving and $-$ for right-moving). They are fermionic superpartners of dilaton ϕ and graviton g_{MN}, respectively. As mentioned before, R-NS and NS-R sectors in Type IIA carry opposite chiralities that makes the whole spectrum non-chiral. In other words, gravitinos in these two sectors appear with opposite chiralities and at each sector, dialtino comes with opposite chirality than the gravitino.

This is not the only option to get a supergravity theory. One can always start from other Types of superstring like Type IIB (3.8) whose low energy limit leads to another version of supergravity with a different field content than what we just explained above. In that case, only even rank gauge potentials $A_0, ..., A_{10}$ are possible and the spectrum will be chiral. Accordingly, the first difference is in the bosonic side where the R-R sector comprises an axion A_0 and two gauge potentials A_{MN} and A_{MNPQ} where the later has a self-dual field strength ($[4]_+$ in Table 3.2) while NS-NS sector has the same field content as its counterpart in Type IIA. It is quite common to combine two scalars ϕ and A_0 into a single one as $\tau := A_0 + ie^{-\phi}$ which parameterizes an $SL(2,\mathbb{R})/U(1)$ space [35]. The second difference between Type IIA and IIB supergravities shows up in the fermionic spectrum where in both R-NS and NS-R sectors of Type IIB we have positive chiralities for gravitino ψ_M while dilatinos λ still carry the opposite chirality than ψ_M [20] . From now on we only focus on Type IIA and avoid studying these two Types in parallel, nevertheless there is always *T-duality* that takes us from one type to

[20]Instead of considering two Majorana-Weyl dilatinos, one can equivalently think of a single Weyl spinor [4]. The same is true for gravitino. Note that doing so is only possible in Type IIB because M-W spinors have the same chiralities.

the other. Without going into details, we just mention that T-duality guaranties that physics described by Type IIA string theory on a circle of radius R_{IIA} is equivalent with the one described by Type IIB on a dual circle of radius $R_{IIB} = \alpha'/R_{IIA}$ [4].

Generally, supergravity theories are classified by an integer number $\mathcal{N}$ which by definition, is the total number of supercharges divided by the dimension of the minimal (the shortest possible) spinor representation in a specific spacetime dimension [56]. For instance, a ten-dimensional Dirac spinor has $2^5 = 32$ complex components, however it is also possible to have Majorana-Weyl (MW) spinors with 16 real components [21] . It is said that at low energy limit, a ten-dimensional Type IIA/IIB string theory reduces to an $\mathcal{N} = 2$ SUGRA, meaning that there are 32 real supercharges in total.

The $\mathcal{N} = 2$ Type IIA SUGRA is described by the following Lagrangian in $10D$

$$\begin{aligned} \mathcal{S}_{IIA} &= \int \Big\{ \frac{-1}{2} \hat{\mathcal{R}} \star \mathbf{1} - \frac{1}{4} d\hat{\phi} \wedge \star d\hat{\phi} - \frac{1}{2} e^{\frac{3}{2}\hat{\phi}} \hat{F}_{(2)} \wedge \star \hat{F}_{(2)} - \frac{1}{2} e^{\frac{1}{2}\hat{\phi}} \hat{F}_{(4)} \wedge \star \hat{F}_{(4)} \\ &- \frac{1}{4} e^{-\hat{\phi}} \hat{H}_{(3)} \wedge \star \hat{H}_{(3)} \Big\} + \mathcal{L}_{top} \ ; \qquad (3.9) \\ \mathcal{L}_{top} &= \frac{-1}{2} \hat{B}_{(2)} \wedge d\hat{A}_{(3)} \wedge d\hat{A}_{(3)} \ , \qquad (3.10) \end{aligned}$$

where NS-NS and R-R field strengths are [22]

$$\hat{H}_{(3)} = d\hat{B}_{(2)} \ , \quad \hat{F}_{(2)} = d\hat{A}_{(1)} \ , \quad \hat{F}_{(4)} = d\hat{A}_{(3)} + \hat{A}_{(1)} \wedge \hat{H}_{(3)} \ . \qquad (3.11)$$

To avoid confusion, from now on all higher dimensional quantities come with a “hat”. All field strengths in (3.11) have to fulfill the Bianchi identity which in NS-NS case simply is

$$d\hat{H}_{(3)} = 0. \qquad (3.12)$$

[21] A factor of 1/2 is because Majorana spinors are real and another one comes from its decomposition to two Weyl representations with different helicities.

[22] Here, the subscripts show rank of the corresponding form. These gauge potentials are usually denoted as $\hat{C}_{(n)}$ in the literature. However to stick to a single notation as much as possible, we are going to show them by $\hat{A}_{(n)}$ all over this thesis.

For R-R gauge potentials, it is common to write all even filed strength as the following formal sum [57]

$$\hat{F}^{(10)} := d\hat{A} + \hat{A} \wedge \hat{H} + me^{\hat{B}} \quad \text{with} \quad \hat{F}^{(10)}_{(n)} = (-1)^{[n/2]} \star_{10} \hat{F}^{(10)}_{(10-n)} , \tag{3.13}$$

where the self-duality condition is imposed to avoid double counting of degrees of freedom. Here, $m \equiv \hat{F}^{(10)}_{(0)}$ is the mass parameter of Type IIA [35, 57]. Then (3.13) obeys the following Bianchi identity

$$d\hat{F}^{(10)} - \hat{H} \wedge \hat{F}^{(10)} = 0 . \tag{3.14}$$

It is also worth mentioning that the action (3.9) has been written in so-called *Einstein-frame* so that we can get familiar terms like Einstein-Hilbert gravitational action, while it is more common in string theory to work in *string-frame*. These two formulations are related via local conformal rescaling of fields [8].

Accordingly, electromagnetic flux for field strength of ten-dimensional SUGRA (3.13) is defined as

$$\Gamma := \frac{1}{\left(2\pi\sqrt{\alpha'}\right)^{p-1}} \int_{\Sigma_p} \hat{\mathbb{F}}_p \in \mathbb{Z} \quad ; \quad \hat{\mathbb{F}}_{(p)} := d\hat{A}_{(p)} + me^{\hat{B}} , \tag{3.15}$$

where Σ_p is a non-trivial p-cycle. More specifically, the associated electric and magnetic charges with each field strength in (3.11) will be

$$\Gamma^{\tilde{K}} := \frac{1}{(2\pi)^2\alpha'} \int_{\mathbb{C}^{\tilde{K}}} \hat{H}_{(3)} \quad , \quad \Gamma_{\tilde{K}} := \frac{1}{(2\pi)^2\alpha'} \int_{\mathbb{C}_{\tilde{K}}} \hat{H}_{(3)} \tag{3.16}$$

$$\Gamma^{A} := \frac{1}{(2\pi)\sqrt{\alpha'}} \int_{\mathbb{A}^{A}} \hat{\mathbb{F}}_{(2)} \quad , \quad \Gamma_{A} := \frac{1}{\left((2\pi)\sqrt{\alpha'}\right)^3} \int_{\mathbb{B}_{A}} \hat{\mathbb{F}}_{(4)} , \tag{3.17}$$

where $\hat{\mathbb{F}}_{(p)}$ has been defined in (3.15). later in the next section we will discuss how Kaluza-Klein reduction dictates specific forms of field strengths in terms of cohomology basis of the internal manifold [35].

Accordingly, one can use orthogonality relations in Table 2.2 to obtain the following decomposition for field strengths in terms of cohomology basis of $CY3$

$$\hat{\mathbb{F}}_{(2)} = \Gamma^A D_A \quad , \quad \hat{\mathbb{F}}_{(4)} = -\Gamma_A D^A$$
$$\hat{\mathbb{F}}_{(0)} = \Gamma^0 \quad , \quad \hat{\mathbb{F}}_{(6)} = -\Gamma_0 d\mathcal{V}$$
$$\hat{H}_{(3)} = \Gamma^K \chi_K - \Gamma_K \chi^K \ , \tag{3.18}$$

where $\mathcal{V}$ is the volume 6-form on the Calabi-Yau. It is common to call Γ with upper and lower indices magnetic and electric charges, respectively. So in summery, there are two types of electromagnetic charges in the theory

$$\Gamma^k = \begin{pmatrix} \Gamma_{\tilde{K}} \\ \\ \Gamma^{\tilde{K}} \end{pmatrix} \quad , \quad \Gamma^a_{R-R} = \begin{pmatrix} \Gamma_{\tilde{A}} \\ \\ \Gamma^{\tilde{A}} \end{pmatrix} \tag{3.19}$$

where we have already defined $(\Gamma_{\tilde{K}}, \Gamma^{\tilde{K}})$ and (Γ_A, Γ^A) in (3.16). These are two symplectic vectors that transform under $Sp(2h^{(2,1)+2}, \mathbb{Z})$ and $Sp(2h^{(1,1)+2}, \mathbb{Z})$ respectively. As we will see later, this symplectic structure originated in the fact that electromagnetic duality also affects the scalars via their interaction with the gauge fields in such a way that the target space of scalar fields is forced be of special Kähler manifold. This structure also enables us to generalize a pure electric $4D$ toy model and its reduction procedure to a more complex theory with electromagnetic charges. This will be discussed in very details in Chapter 5. Let us finish this part by rewrite the R-R field strength in a single expression as

$$\hat{\mathbb{F}}_{R-R} = \mathbb{F}^0 + \mathbb{F}^A D_A - \mathbb{F}_A D^A - \mathbb{F}_0 d\mathcal{V} \ , \tag{3.20}$$

where the various components are given in (3.18). As we will see, self-duality condition restricts $\mathbb{F}_0$ and $\mathbb{F}_A$ to be given in terms of $\mathbb{F}^{\tilde{A}}$ and $\star\mathbb{F}^{\tilde{A}}$ in a very specific way.

3.2. $\mathcal{N} = 2$ SUGRA from $10D$ to $4D$

All we were talking about was a ten-dimensional theory, while here in this thesis, we are more interested in studying a four-dimensional $\mathcal{N} = 2$ supergravity and its black hole solutions. To that aim, we first need to dimensional reduce (3.9,3.10) on a (compact) six-dimensional internal manifold. As discussed before in section 2.2, this is the internal manifold that determines the physical features of the lower dimensional theory such as field content, their form in terms of harmonic basis and the number of supercharges. Note that the smallest possible spinor in four dimensions is a Majorana spinor with four real component. Accordingly, there are only 8 supercharges for $\mathcal{N} = 2$ supergravity in $4D$. In other words, in six-dimensional reduction procedure one need to break 24 supercharges out of 32 which is a rigorous restriction on the type of internal manifold. As explained in [56], this requirement can be fulfilled if the internal manifold admits a unique covariantly constant spinor which is equivalent to saying that the holonomy group of the internal manifold has to be $SU(3)$. Now, we recall from subsection 2.3.2 concluding the same properties for a Calabi-Yau threefold. Considering the fact that these manifolds do not have any isometries, their role is to break redundant symmetries. So in summary:
To obtain $\mathcal{N} = 2$ supergravity in four dimensions, Type IIA superstring should be compactified on a Calabi-Yau three-fold.

In the following sections, we are going to study this reduction procedure in detail as much as possible. This will help us to comprehend the structure of the lower dimensional theory more clear.

3.2.1. Compactification on $CY3$

To obtain the massless spectrum of the $4D$ theory after compactification, we need to arrange indices of $10D$ massless fields in Table 3.3 in a $SU(3)$ covariant way, that is $M = (\mu,\ i,\ \bar{i})$ where $\mu = 0, 1, 2, 3$ runs over spacetime coordinates and $(i,\ \bar{i})$ belong to the internal $CY3$. The obtained fields should be then categorized in *supermultiplets* in such a way that the number of bosonic and fermionic degrees of freedom are the same in each multiplet. These multiplets are labeled by the highest helicity they contain [23] and have to be CPT self-conjugate, otherwise we need to add their conjugate multiplet as well.

The first step is to decompose the ten-dimensional metric as

$$d\hat{s}^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{i\bar{j}}(y)dy^i dy^{\bar{j}} \ , \tag{3.21}$$

where we have separated coordinates as $x^M = \{x^\mu, y^i\}$ with $\{x\}$ to be four external coordinates while $\{y\}$ denotes six internal coordinates of $CY3$. Also, the decomposition of other fields are given as

$$\begin{aligned} \{g_{\mu\nu},\ A_\mu,\ \psi_\mu,\ \tilde{\psi}_\mu\} \quad &, \quad \{g_{i\bar{j}},\ B_{i\bar{j}},\ A_{\mu i\bar{j}},\ \text{fermions}\} \\ \{g_{ij},\ A_{ij\bar{k}},\ \text{fermions}\} \quad &, \quad \{A_{ijk},\ B_{\mu\nu},\ \phi,\ \text{fermions}\} \ . \end{aligned} \tag{3.22}$$

All these fields still leave in ten-dimensions and so depend on x^M. Here, one may encounter other possibilities such as $B_{\mu i}$ and $B_{\mu\bar{i}}$ that come from B_{MN}. The fact is any field other than what we have in (3.22) is not allowed. To see this, we should recall a vital relationship between cohomology group of the internal manifold and field content of the lower dimensional theory. Indeed, as a result of KK-reduction, massless modes of all fields in the lower dimensional theory correspond to the *harmonic forms* of the internal manifold ($CY3$ in our case) [34, 35]. On the other hand, according to the Hodge Theorem, harmonic forms are classified by the cohomology group of the manifold. This implies that the multiplicity of massless spectrum of supergravity in four dimensions is dictated by cohomology of the Calabi-Yau three-fold. For instance, the number of possible (n, m)-form gauge fields is given by the Hodge number $h^{(n,m)}$, i.e., entries of the Hodge diamond diagrams 2.1. This is true for all field originated from the spectrum in $10D$, i.e., all gauge potentials from R-R sector as well as graviton, dilaton and B-field from NS-NS sector (Table 3.3).

[23] Note that helicity is an Lorentz invariant quantity only for massless states.

Now we are going to expand these fields in their possible internal and external components. Having cohomology basis classified in Table 2.2 one obtains the following decomposition for the whole spectrum in(3.22)

$$\text{NS-NS sector: } B_{i\bar{j}}(x,y) = b^A(x) D_A(y) \ , \tag{3.23}$$
$$g_{i\bar{j}}(x,y) = i v^A(x) \big(D_A(y)\big)_{i\bar{j}} \ , \quad g_{ij}(x,y) = i z_K(x) \left(\frac{(\chi^K)_{i\bar{k}\bar{l}} \ \Lambda^{\bar{k}\bar{l}}{}_j}{|\Lambda|^2} \right)(y)$$
$$\text{R-R sector: } \quad A_{\mu i\bar{j}}(x,y) = A^A_\mu(x) \wedge D_A(y) \ , \tag{3.24}$$
$$\left. \begin{aligned} A_{ij\bar{k}}(x,y) &= \xi^K(x) \chi_K(y) \\ A_{ijk}(x,y) &= \xi^0(x) \Lambda(y) \end{aligned} \right\} \to A_3(x,y) = \xi^{\tilde{K}}(x) \chi_{\tilde{K}}(y) + \bar{\xi}_{\tilde{K}}(x) \chi^{\tilde{K}}(y)$$

where in the last line A_3 stands for all gauge potentials with three“internal” indices. Also note that for any of these fields, there exists a partner with conjugate indices that has its own decomposition in terms of proper cohomology basis which we did not mentioned here (except for A_3 that the whole decomposition is given) [24] . Also we have purely external fields which can be expanded in terms of the only zero-form in $CY3$

$$\text{NS-NS sector :} \qquad g_{\mu\nu}(x,y) = g_{\mu\nu}(x) \quad , \quad \phi(x,y) = \phi(x)$$
$$B_{\mu\nu}(x,y) = B_{\mu\nu}(x) \tag{3.25}$$
$$\text{R-R sector :} \qquad A_\mu(x,y) = A_\mu(x) \ . \tag{3.26}$$

[24]For instance there is

$$g_{\bar{i}\bar{j}}(x,y) = i\bar{z}^K(x) \left(\frac{(\chi_K)_{\bar{i}kl} \ \overline{\Lambda}^{kl}{}_{\bar{j}}}{|\Lambda|^2} \right)(y) \quad , \quad A_{\bar{i}\bar{j}k}(x,y) = \bar{\xi}_K(x) \chi^K(y) \quad , \quad A_{\bar{i}\bar{j}\bar{k}}(x,y) = \bar{\xi}_0(x) \overline{\Lambda}(y) \ .$$

Eventually, we can categorize these fields in *supergravity multiplets* as

$$\begin{aligned} \text{gravity multiplet} \quad &: \quad \{g_{\mu\nu},\ A_\mu,\ \psi_\mu,\ \tilde{\psi}_\mu\} \\ h^{(1,1)}\ \text{vector multiplets} \quad &: \quad \{A^A_\mu,\ v^A,\ b^A,\ \text{fermions}\} \\ h^{(2,1)}\ \text{hypermultiplets} \quad &: \quad \{z_K,\ \overline{z}^K,\ \xi_K,\ \xi^K\ \text{fermions}\} \\ \text{tensor multiplets} \quad &: \quad \{B_{\mu\nu},\ \xi^0,\ \overline{\xi}_0\ \phi,\ \text{fermions}\}\ . \end{aligned} \tag{3.27}$$

Here $\{\phi,\ v^A,\ b^A,\ z_K,\ \overline{z}^K,\ \xi^{\tilde{K}},\ \overline{\xi}_{\tilde{K}}\}$ are all real scalars in $4D$, $A^{\tilde{A}}_\mu := (A^0_\mu, A^A_\mu)$ are $h^{(1,1)} + 1$ one-form gauge fields [25] . Also we have $B_{\mu\nu}$ as a two-form and $g_{\mu\nu}$ is the gravitational field. As one can conclude from (3.23), two of these real scalars have specific *geometric* role: there are $h^{(1,1)}$ v^A that deform the Kähler form Ω of the Calabi-Yau via $g_{i\bar{j}}$ while $h^{(2,1)}$ z^K characterize deformations of its complex structure J via g_{ij}. To see this, one may need to recall expressions of Ω and J for a Hermitian manifold given by (A.30) and (A.8) respectively.

3.3. $\mathcal{N} = 2$ SUGRA in $4D$

The $\mathcal{N} = 2$ is the first *extended* [26] supergravity theory with 8 real supercharges in four-dimensions. Studying this theory is of special interest because of two main reasons. First, it admits a family of black hole solutions which are remarkable because of being *multi-centered*. More precisely, these black holes are bound states of a bunch of dyonic black holes. Second, the theory is very rich geometrically. In short, scalar fields of gauge and hyper multiplets of four-dimensional $\mathcal{N} = 2$ supersymmetry leave on specific subclasses of Kähler manifolds called *special geometries* [58–60]. All these features are going to be discussed in detail in the following sections, but first let us have a look at the action of this theory and equations of motion it leads to.

To finally obtain the low energy effective action for $\mathcal{N} = 2$ supergravity in terms of four-dimension multiplets (3.27), we need to plug the expansion of ten-dimensional fields given in (3.23,3.24,3.25,3.26) into the ten-dimensional action (3.9,3.10) and in-

[25] Here $A^0 \equiv A_\mu$ in (3.27). This change of notation enables us to define $A^{\tilde{A}}$.

[26] In the literature, supergravity theories with $\mathcal{N} > 1$ is called *extended*.

tegrate over the internal Calabi-Yau threefold Y. To obtain the final familiar expression [60], [61], [62] we need to redefine some of fields as well. We do so by taking [34]

$$t^A := b^A + iv^A \quad , \quad e^D := \frac{e^\phi}{\sqrt{\mathcal{V}}} \ . \tag{3.28}$$

where $\mathcal{V}$ is the volume of $CY3$ whose definition in terms of Kähler two-form Ω is given by $\Omega^3 := \int_Y \Omega \wedge \Omega \wedge \Omega = 3!\mathcal{V}$. The scalars $\{t^A\}$ live on $CY3$ moduli space and their dynamics is given by a σ-model.

The $4D$ $\mathcal{N} = 2$ supergravity is described by the following action

$$\begin{aligned} \mathcal{S} = \quad & \int \frac{-1}{2}\mathcal{R} \star \mathbf{1} - \tilde{G}_{A\overline{B}} dt^A \wedge \star d\bar{t}^B - h_{uv} dq^u \wedge \star dq^v \\ & -\frac{1}{2} H(t,\bar{t})_{\tilde{A}\tilde{B}} F^{\tilde{A}} \wedge \star F^{\tilde{B}} + \frac{1}{2}\Theta(t,\bar{t})_{\tilde{A}\tilde{B}} F^{\tilde{A}} \wedge F^{\tilde{B}} \ . \end{aligned} \tag{3.29}$$

Let us first give some short explanations about some new quantities in this action. We have already defined complex scalars t^A via (3.28) while new fields q^v are taken as a shorthand notation for all real scalars in various multiplets of $\mathcal{N} = 2$ supergravity (3.27) except (b^A, v^A). In the second line, we have kinetic term of the gauge fields $A^{\tilde{A}}$ with field strengths $F^{\tilde{A}} = dA^{\tilde{A}}$ and a purely topological term $F \wedge F$.

So in summery, the theory we deal with contains gravity, real and complex scalars plus self-interacting $U(1)$ gauge fields. But there are two main differences that makes this action more complicated than a usual one with the same field content: first, we do not have only one of each kind of these fields, instead we have many of each, $\tilde{A} = \{0, 1, ..., h^{(1,1)}\}$. Second and more important difference is that coupling of these fields are given by matrices $\{G, H, \Theta\}$ that can be quite complicated. They are all yet to be explained in detail in the next subsection but for now, let us mention that each of them reveals specific geometric properties of the theory under study. As one sees, there is $\tilde{G}_{A\overline{B}}$ that has been shown to be the metric of special Kähler manifold on which scalars t^A live. There is also a complex matrix $\tau(t,\bar{t})_{\tilde{A}\tilde{B}} := \Theta(t,\bar{t})_{\tilde{A}\tilde{B}} + iH(t,\bar{t})_{\tilde{A}\tilde{B}}$ that depends on scalars t^A and determines coupling between gauge fields. This matrix is delicately related to the underlying geometry of scalars target space, as we will see very soon. Finally, there is h^{uv} that denotes metric of the target space of all other scalars q^v. As we will shortly mention, this one is a quaternionic manifold. Both of these manifolds show up as scalar target spaces belong to a subclass of complex manifolds called *special geometries*, the title of the next subsection.

Henceforth, we ignore scalars $\{q^v\}$ as they can get consistently decoupled from other fields. Let us also have a very short comment on Einstein's equations of the theory and its asymptotic behavior. Varying the action with respect to the metric, we obtain

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \quad & \frac{l_4^2}{16\pi^2}H(t)_{\tilde{A}\tilde{B}}\Big(\frac{1}{8}g_{\mu\nu}F^{\tilde{A}\ \alpha\beta}F^{\tilde{B}}{}_{\alpha\beta} - \frac{1}{2}F^{\tilde{A}}{}_{\mu}{}^{\alpha}F^{\tilde{B}}{}_{\nu\alpha}\Big) + \\
& 32\pi\tilde{G}_{A\overline{B}}\big[(\partial_\mu t^A)(\partial_\nu \bar{t}^B) - \frac{1}{2}g_{\mu\nu}(\partial_\alpha t^A)(\partial^\alpha \bar{t}^B)\big]
\end{aligned}
\tag{3.30}
$$

However, asymptotically scalars t^A take constant values and hence both the action and equations of motion get much simpler

$$
\mathcal{S}^{\text{asymp}} = \frac{2\pi}{\ell_4^2}\int d^4x\sqrt{-g}\mathcal{R} + \frac{1}{16\pi}\int\Big[- H_{\tilde{A}\tilde{B}}F^{\tilde{A}}\wedge\star F^{\tilde{B}} + \Theta_{\tilde{A}\tilde{B}}F^{\tilde{A}}\wedge F^{\tilde{B}}\Big] \tag{3.31}
$$

for which the Einstein's equation can be obtained as

$$
R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \frac{l_4^2}{16\pi^2}H(t)_{\tilde{A}\tilde{B}}\Big(\frac{1}{8}g_{\mu\nu}F^{\tilde{A}\ \alpha\beta}F^{\tilde{B}}{}_{\alpha\beta} - \frac{1}{2}F^{\tilde{A}}{}_{\mu}{}^{\alpha}F^{\tilde{B}}{}_{\nu\alpha}\Big) \tag{3.32}
$$

3.3.1. Rigid and Projective Special Geometries

In this section, we are going to study the geometry of the target space for scalar fields of $\mathcal{N} = 2$ supersymmetry and supergravity multiplets. Our main focus will be on scalars in the gauge multiplet and their underlying geometry. Starting from supersymmetry where we deal with *global* symmetries, we first investigate *rigid/affine* special Kähler. Then, through gauge fixing via projection method, we obtain *local/projective* special Kähler in supergravity [27] . Here, we mostly follow [29,34] but we have changed some notations to keep it as uniform as possible all through this thesis.

Rigid special Kähler

The gauge multiplet of $\mathcal{N}=2$ supersymmetry in $4D$ is given by

$$\{A^A_\mu,\ t^A,\ \vec{Y}^A,\ \lambda^A_i\} \quad , \quad A=1,...,h^{(1,1)}\ . \tag{3.33}$$

Here, λ^A_i denotes gauginos that are $SU(2)$ Majorana doublets whose chirality is determined by the position of its index $i=1,2$. Comparing with its supergravity counterpart in (3.27), we have an extra triplet of auxiliary scalar fields $\vec{Y}^A$ which are added to set the bosonic and fermionic off-shell degrees of freedom equal [28] . The theory is described by the following Lagrangian

$$\mathcal{L}_{\text{SUSY}} = i\mathcal{F}_{I\bar{J}} D_\mu T^I D^\mu \overline{T}^J + \frac{i}{4}\tau_{AB} F^A_{\mu\nu} F^{B\ \mu\nu} + ... + \text{h.c.}\ . \tag{3.34}$$

Here, $\{T^I\}$ are functions of physical scalars $\{t^A\}$. Obviously, (3.34) does not cover the Lagrangian of whole field content of the SUSY gauge multiplets (3.33). The full Lagrangian has been given in detail in [29]. Here our main focus is on kinetic terms of scalars t^A and gauge fields A^A_μ with field strengths $F^A_{\mu\nu}$. Hence, (3.34) seems enough for studying the sigma model of scalars whose target space geometry is determined by interactions between these two group of fields. As we will see, from the kinetic term of scalars, one can conclude that the target space is a *rigid special Kähler manifold*; a specific subset of Kähler manifolds previously defined in section 2.3.1.

Special Kähler manifold: is a Kähler manifold with an extra flat (zero curvature) connection that preserves a *symplectic structure* [29]. In other words, the manifold is equipped at each point with a *holomorphic flat bundle* V that transforms as a vector

[27]The terminology depends on the literature. Here, the first choices are common in Physics literature while the second one is used in Mathematics.

[28]One can check that in gauge multiplet (3.33) there are $5=3+2$ bosonic d.o.f coming from the gauge field and complex scalar, while two Majorana fermions have $8=2\times 4$ d.o.f. So it follows that we need a triplet of auxiliary scalars shown by the vector $\vec{Y}$ to balance the number of bosonic and fermionic d.o.f in the multiplet.

under symplectic transformations [34] given by

$$M = \begin{pmatrix} M_1 & M_2 \\ \\ M_3 & M_4 \end{pmatrix} \in Sp(2m, \mathbb{R}) \quad \Longrightarrow \quad M^T J M = J \quad ; \quad J = \begin{pmatrix} 0 & \mathbf{1} \\ \\ -\mathbf{1} & 0 \end{pmatrix} \ . \tag{3.35}$$

where each of the elements $M_1, ..., M_4$ in the matrix M is an $m \times m$ matrix by itself [29] . The matrix J is symplectic metric that remains invariant under symplectic transformations [30] and so provides a symplectic invariant *inner product* $\langle ., . \rangle$ defined as

$$\left\langle V, \overline{V} \right\rangle := V J \, \overline{V} \ . \tag{3.36}$$

The prefix "special" reflects the fact that Kähler potential of these manifolds is not an arbitrary function [63], but a holomorphic one with a specific form that we will see later.

Now consider the rigid special Kähler $\mathcal{M}$ spanned by holomorphic coordinates $t \equiv t^A$ which are physical scalar fields of the theory and so should not get confused with time. At each point of $\mathcal{M}$ there is a fiber V with $2h^{(1,1)}$ components given by

$$V := \begin{pmatrix} T^I(t) \\ \\ \mathcal{F}_I(T) \end{pmatrix} \quad ; \quad I = 1, ..., h^{(1,1)} \tag{3.37}$$

where T^I and $\mathcal{F}_I$ are vectors under symplectic transformations $Sp(2h^{(1,1)}, \ \mathbb{R})$ and so is V. In other words, the fibers V from different patches are related via symplectic transformations. The reader should be cautious in working with indices all through this section. Although we start from two different types of indices I and A to make discussion general and easier to follow, however it is harmless in most of formula to exchange these two as we will do so whenever it is needed.

[29] In the case of rigid SUSY it is $m = h^{(1,1)}$ while for SUGRA it becomes $m = h^{(1,1)} + 1$.
[30] It is also called *complex structure* in the context of complex manifolds (A.5).

Now, the metric on the scalar target space $\mathcal{M}$ is given by

$$G_{A\overline{B}} = i\left\langle \partial_A V, \partial_{\overline{B}}\overline{V}\right\rangle = i\partial_A\partial_{\overline{B}}\left\langle V, \overline{V}\right\rangle \ , \tag{3.38}$$

from which one can simply read the Kähler potential by using (2.40)

$$\mathcal{K}_{\mathcal{M}} = i\left\langle V, \overline{V}\right\rangle = i\Big(T^I\overline{\mathcal{F}}_I(\overline{T}) - \overline{T}^I\mathcal{F}_I(T)\Big) \ . \tag{3.39}$$

One can easily check that V also satisfies the following condition

$$\langle \partial_A V, \partial_B V\rangle = 0 \ . \tag{3.40}$$

Till now $\mathcal{F}_I(T)$ was an arbitrary holomorphic function that appears as the lower components of the symplectic bundle V. However, provided that the matrix $\partial_A T^I$ is invertible then (3.40) is in fact an integrability condition equivalent to the existence condition of a (local) *prepotential* $\mathcal{F}(T)$. In the case of its existence, $\mathcal{F}(T)$ would be an arbitrary holomorphic function whose first derivative with respect to T^I gives lower half of the symplectic vector V, i.e., we have

$$\mathcal{F}_I(T) \equiv \frac{\partial\mathcal{F}(T)}{\partial T^I} \ . \tag{3.41}$$

Once again we emphasize that our definition of the special geometries should not rely on the prepotential since it may not be a globally well-defined function allover the manifold. We rather prefer to focus on the symplectic formulation in which the transition rules for bundles $V(t)$ from one patch to the other is quite clear and given by action of the symplectic group. Specifically, this approach is much more applicable in supergravity where the matrix $\partial_A T^I$ may not be invertible at all.

As it has been told earlier, to be able to reach supergravity we first demand the supersymmetric theory to have conformal invariance as well. This requirement restricts the prepotential to be a homogeneous function of second order for which the Kähler metric becomes

$$G_{A\overline{B}} = 2\mathrm{Im}[\mathcal{F}_{I\overline{J}}]\ \partial_A T^I\ \partial_{\overline{B}} \overline{T}^J\ , \tag{3.42}$$

and this is what appears in the kinetic terms of the supersymmetric Lagrangian (3.34) [31] . The kinetic term of scalars then becomes

$$\mathcal{L}_{\text{scalar}} = 2G_{A\overline{B}} D_\mu t^A D^\mu \bar{t}^B\ . \tag{3.43}$$

Now, focusing on kinetic term for the gauge field one can read

$$\mathcal{L}_{\text{gauge fields}} = \frac{1}{2}\mathrm{Im}\Big[\tau_{AB}(t) F^A_{\ \mu\nu} F^{B\ \mu\nu}\Big] \quad ; \quad \tau_{AB}(t) \equiv \overline{\mathcal{F}}_{AB}\ . \tag{3.44}$$

Here we switch indices so $\mathcal{F}_{AB}$ denotes the second derivative of prepotential $\mathcal{F}$ with respect to T^A. Under symplectic transformation (3.35) we have

$$\tau \xrightarrow{Sp\left(2h^{(1,1)},\ \mathbb{R}\right)} \tau^{'} = \big(M_3 + M_4\tau\big)\big(M_1 + M_2\tau\big)^{-1}\ . \tag{3.45}$$

One may have realized that the special geometry of the scalars target space in the gauge multiplet originated from their connection with the gauge fields. In the presence of magnetic monopoles, both the equations of motion and Bianchi identities enjoy symplectic invariance. Now from the $4D$ supersymmetry Lagrangian (3.44) we see that the matrix $\tau_{A\overline{B}}$ in the kinetic term depends on the scalars t^A through which the invariance under symplectic transformations is imposed on scalars as well. Hence, the complex manifold of the target space should be able to support symplectic structure as well, which implies that it has to be special Kähler.

[31] Note that in this and also other sections we may have to switch between two notations time to time: whenever there are only complex indices we put a “bar” on top of the index, like the second index in $G_{A\overline{B}}$. However, if the complex variable appears itself, then we move the bar from the index to the variable, like $\overline{T}^J$.

Projective parameterization

The conclusion of the last part of this section was that the target space of scalars in $\mathcal{N}=2$ SUSY gauge multiplet is a rigid special Kähler, while it turns out that its supergravity counterpart is the projective version of special Kähler. To see this, we need to start from the gauge multiplet we had in SUSY spectrum (3.33) and supergravity (3.27) with one more complex scalar, i.e., $h^{(1,1)}+1$ complex scalars. In other words, what we start with is $h^{(1,1)}$ complex scalars t^A and gauge fields A^A from the gauge multiplets (3.33) in addition to an extra complex scalar t^0. The later is an extra degree of freedom that gets killed by gauge fixing in projection procedure. It is just after this gauge fixing that we arrive at supergravity theory with gauge multiplets in (3.27). This means that all results from the previous part are still valid if we replace index A with its tilde version $\tilde{A}=\{0,A\}=0,...,h^{(1,1)}$ for scalar fields and also $\tilde{I}=\{0,I\}=0,...,h^{(1,1)}$ for the set of holomorphic coordinate on the rigid Kähler. As before, $t^{\tilde{A}}$ are complex scalar given by (3.28).

Now let us see how one can achieve a projective Kähler from the rigid one. Generally, gauge fixing of the rescaling symmetry of a given manifold leads to a secondary space called *projective*. In this process, we first find orbits of this symmetry which are generated by rescaling generators of the embedding space. Then, all those points that belong to the same orbit become identified. Interestingly, this is exactly the procedure that takes us from $\mathcal{N}=2$ supersymmetry to its supergravity counterpart. In other words, the rigid special Kähler of SUSY can be thought of as a bundle of "complex" scaling on top of the projective special Kähler of SUGRA as the base manifold. As we will see, the projective Kähler requires the Kähler potential $\mathcal{K}$ to be "homogeneous" and this is exactly what one reads from the coupling matrix which appears in scalars kinetic energy of the supergravity Lagrangian (3.29).

Previously, we saw that complex scalar fields $\tilde{t}^{\tilde{A}}=\{t^0,t^A\}$ can be thought as a holomorphic coordinate system spanning the rigid Kähler manifold of $h^{(1,1)}+1$ dimensions. Now, we define a new set of inhomogeneous coordinates by $\{\tilde{T}^{\tilde{I}}\}:=\{T^{\tilde{I}}/t^0\}$. Generally, $\tilde{T}^{\tilde{I}}$ can be any arbitrary functions of physical scalar fields $\{t^A\}$

with only one condition, $\tilde{T}^{\tilde{I}}$ has to be non-degenerate meaning that the following $(h^{(1,1)}+1)\times(h^{(1,1)}+1)$ matrix should be of rank $h^{(1,1)}+1$

$$\begin{pmatrix} \tilde{T}^{\tilde{I}} \\ \\ \partial_A \tilde{T}^{\tilde{I}} \end{pmatrix} . \tag{3.46}$$

However, among all possible choices for $\tilde{T}^{\tilde{I}}$, we stick to the most natural choice here, that is setting $\tilde{T}^0 = 1$ and $\tilde{T}^I = t^A$ and so one can switch between indices I and A. Then, the line elements in these two sets of coordinates is

$$ds^2 = dT^{\tilde{I}} G_{\tilde{I}\bar{\tilde{J}}}\, d\overline{T}^{\tilde{J}} = \left|t^0\right|^2 d\tilde{T}^{\tilde{I}} \tilde{G}_{\tilde{I}\bar{\tilde{J}}}\, d\overline{\tilde{T}}^{\tilde{J}} . \tag{3.47}$$

Now, imagine a holomorphic vector $h = 2t^0\partial/\partial t^0$ where the factor of two is for more convenience. Its action on the rigid manifold leads to two types of transformations, imaginary part of h generates an isometry while its real part rescales the metric $\tilde{G}_{\tilde{I}\bar{\tilde{J}}}$ as can be seen from (3.47). Let k_T and k_D stand for these two generators, so in terms of h and $\overline{h}$ they are given by

$$k_D := \frac{1}{2}(h+\overline{h}) \quad , \quad k_T := \frac{i}{2}(h-\overline{h}) , \tag{3.48}$$

while λ_T and λ_D denote real parameters of their corresponding infinitesimal transformations, respectively. The variation of other fields under h will be

$$\begin{aligned} &\delta t^0 = \big(\lambda_D + i\lambda_T\big)t^0 \quad , \quad \delta\bar{t}^0 = \big(\lambda_D - i\lambda_T\big)\bar{t}^0 \quad , \quad \delta t^A = 0 = \delta\bar{t}^A , \\ &\delta G_{\tilde{I}\bar{\tilde{J}}} = -\big(\lambda_D \mathcal{L}_{k_D} + \lambda_T \mathcal{L}_{k_T}\big) G_{\tilde{I}\bar{\tilde{J}}} \quad ; \quad \mathcal{L}_{k_D} G_{\tilde{I}\bar{\tilde{J}}} = 2G_{\tilde{I}\bar{\tilde{J}}} , \quad \mathcal{L}_{k_T} G_{\tilde{I}\bar{\tilde{J}}} = 0 , \end{aligned} \tag{3.49}$$

where $\mathcal{L}_k$ denotes the Lie derivative along vector k. From the second line above, one can conclude that generator of T-transformations (simply rotations) k_T is a Killing vector, and so is an isometry that leaves the metric invariant. However, k_D is a *closed homothetic Killing vector* that generate dilatation and rescales the metric [32] . Such a

[32]By definition, a homothetic Killing vector k is the one that satisfies $\mathcal{L}_k g_{ij} = 2w g_{ij}$ with the Weyl

vector satisfies $\nabla_{\tilde{I}} k_D^{\tilde{J}} = \delta_{\tilde{I}}^{\tilde{J}}$ which implies that there is a scalar k whose gradient gives the Killing vector, i.e., $k_{D\tilde{I}} = \partial_{\tilde{I}} k$.

It can be shown [29] that for any Kähler manifold with a homogeneous Kähler potential we have the second line in (3.49) satisfied. More precisely, the condition $\mathcal{L}_{k_D} G_{\tilde{I}\bar{\tilde{J}}} = 2G_{\tilde{I}\bar{\tilde{J}}}$ implies that (under special circumstances) there exist a homogeneous function $\mathcal{K}$ of first degree in $T^{\tilde{I}}$ and in $\overline{T}^{\tilde{J}}$, so that one can write

$$G_{\tilde{I}\bar{\tilde{J}}} = \partial_{\tilde{I}} \partial_{\bar{\tilde{J}}} \mathcal{K} := \frac{\partial}{\partial T^{\tilde{I}}} \frac{\partial}{\partial \overline{T}^{\tilde{J}}} \mathcal{K} \ . \tag{3.50}$$

Homogeneity of the Kähler potential allows us to write

$$T^{\tilde{I}} \frac{\partial}{\partial T^{\tilde{I}}} \mathcal{K} = \mathcal{K} \quad , \quad \overline{T}^{\tilde{I}} \frac{\partial}{\partial \overline{T}^{\tilde{J}}} \mathcal{K} = \mathcal{K} \quad \Longrightarrow \quad \mathcal{K}(t^0 \tilde{T}, \bar{t}^0 \overline{\tilde{T}}) = |t^0|^2 \mathcal{K}(\tilde{T}, \overline{\tilde{T}}) \ . \tag{3.51}$$

It is worth mentioning that to be able to get the final identity, one may need to first, perform a Kähler transformation $\mathcal{K}' \to \mathcal{K}$ defined in (2.42).

Gauge fixing and projective special Kähler

The next step to obtain the projective special Kähler is to first gauge dilatation and T-transformation by localizing transformation parameters $\lambda_D(x)$ and $\lambda_T(x)$ and then fix these redundant gauge symmetries. Clearly, the obtained manifold will be of $h^{(1,1)}$ dimensions. For T-transformation, this is equivalent to setting $t^0 = \bar{t}^0$ that results in a $U(1)$ auxiliary gauge field A_μ whose variation under T-transformation is given by $\delta A_\mu = \partial_\mu \lambda_T$. This is the same gauge field that joins the gravity multiplet (3.27) later as *graviphoton* A_μ.

To fix the dilatation gauge generated by k_D, we note that from (3.51) one reads

$$\delta \mathcal{K} = 2\lambda_D \mathcal{K} \ . \tag{3.52}$$

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weight $w = (D-2)/2$ in D-dimensions. It will be called "closed" if in addition we have $\nabla_i k_j = w g_{ij}$.

So one can impose gauge fixing condition directly on $\mathcal{K}$. As we argued before, we now need to identify all points laid on the same ray, i.e., those points for which there is a non-zero real constant α so that $p^{'} = \alpha p$. This identification projects the $h^{(1,1)} + 1$ dimensional rigid Kähler to a sphere whose radius is a constant depending on the theory under study. It turns out that for $\mathcal{N} = 2$ supergravity, the proper constant is G^{-2}, where G denotes the gravitational coupling constant. So we demand

$$T^{\tilde{I}}\ G_{\tilde{I}\overline{\tilde{J}}}\ \overline{T}^{\tilde{J}} = T^{\tilde{I}}\ \mathcal{K}_{\tilde{I}\overline{\tilde{J}}}\ \overline{T}^{\tilde{J}} \equiv \mathcal{K}(T^{\tilde{I}}, \overline{T}^{\tilde{J}}) = -G^2\ , \tag{3.53}$$

where $\mathcal{K}_{\tilde{I}\overline{\tilde{J}}}$ denotes the second derivative of $\mathcal{K}$ and we have used (3.50). Now, from (3.47), we will get the dilatation gauge fixing condition as

$$\left|t^0\right|^2 = -G^{-2}\left(\tilde{T}^{\tilde{I}}(t)\ \tilde{G}_{\tilde{I}\overline{\tilde{J}}}\ \overline{\tilde{T}}^{\tilde{J}}(\bar{t})\right)^{-1}\ . \tag{3.54}$$

The Kähler potential of the projective special Kähler $\tilde{\mathcal{K}}$ then becomes

$$\tilde{\mathcal{K}}(t, \bar{t}) = G^{-2}\ \ln\left|t^0\right|^2\ . \tag{3.55}$$

Finally, we can write the kinetic term for the set of scalars of gauge multiplet of $\mathcal{N} = 2$ supergravity as a non-linear σ-model whose target space is a projective special Kähler manifold of dimension $h^{(1,1)}$. It is given by

$$\mathcal{L}_{\text{scalars}} = -G_{A\overline{B}} D_\mu t^A D^\mu \bar{t}^B = -\partial_A \partial_{\overline{B}} \tilde{\mathcal{K}}(t, \bar{t}) D_\mu t^A D^\mu \bar{t}^B\ . \tag{3.56}$$

As mentioned before in the case of rigid special Kähler in SUSY, there is another approach to formulate metric and other properties of the manifold which is based on its symplectic structure. Recall the symplectic bundle $V(t)$ given in (3.37). A simple calculation [33] shows that $V(t)$ satisfies

$$k_D{}^A \partial_A V(t) = V(t) \ , \tag{3.57}$$

where k_D is the closed homothetic Killing vector that generates dilatations (3.48). This identity in addition to the way we defined lower half components of $V(t)$ (3.37) by (3.41), will guarantee homogeneity of the prepotential $\mathcal{F}(t)$. Accordingly, our definition of homogeneous coordinates $T^{\tilde{I}}$ in projective parameterization of rigid special Kähler (i.e., factorizing t^0) is now applicable for all $2(h^{(1,1)}+1)$ elements of the symplectic section, so we have

$$V(t) = t^0 \tilde{V}(t) = t^0 \begin{pmatrix} \tilde{T}^{\tilde{I}}(t) \\ \\ \tilde{\mathcal{F}}_{\tilde{I}}(t) \end{pmatrix} \ . \tag{3.58}$$

Here, $\tilde{\mathcal{F}}_{\tilde{I}}$ and $\mathcal{F}_{\tilde{I}}$ have the same functionality of physical scalars t.

Kähler covariant formulation

The symplectic section $\tilde{V}$ in (3.58) and also its complex conjugate $\overline{\tilde{V}}$ are still symplectic vectors with $2(h^{(1,1)}+1)$ components. In other words, the symplectic bundle $\tilde{V}$ of the projective special Kähler is a vector under $Sp(2h^{(1,1)}+2)$. Having Kähler covariant derivatives defined as below, one can show that these two are also holomorphic and anti-holomorphic vectors, i.e.,

$$\begin{aligned} \nabla_A \tilde{V} &:= \partial_A \tilde{V} + \frac{1}{2} G^{-2} \big(\partial_A \tilde{\mathcal{K}}\big) \tilde{V} \quad ; \quad \nabla_{\overline{A}} \tilde{V} \equiv 0 \ , \\ \nabla_{\overline{A}} \overline{\tilde{V}} &:= \partial_{\overline{A}} \overline{\tilde{V}} + \frac{1}{2} G^{-2} \big(\partial_{\overline{A}} \tilde{\mathcal{K}}\big) \overline{\tilde{V}} \quad ; \quad \nabla_A \overline{\tilde{V}} \equiv 0 \ . \end{aligned} \tag{3.59}$$

[33] Take the definition of closed homothetic Killing given in the footnote 32 with $w = 1$. It is easy to conclude that $\partial_A\big(k_D{}^B G_{B\overline{B}}\big) = G_{A\overline{B}}$. Then, by substituting $G_{A\overline{B}}$ from (3.38), we obtain identity (3.57).

Also, the integrability condition (3.40) can be written as

$$\left\langle \tilde{V}, \nabla_A \tilde{V} \right\rangle = 0 = \left\langle \nabla_A \tilde{V}, \nabla_B \tilde{V} \right\rangle \ . \tag{3.60}$$

From (3.55) one can read

$$t^0 = \exp\bigl(G^2 \tilde{\mathcal{K}}/2\bigr). \tag{3.61}$$

Note that we already have fixed the $U(1)$ gauge by setting $t^0 = \bar{t}^0$. This leads to

$$G^2 \ e^{-G^2 \tilde{\mathcal{K}}} = -i \left\langle \tilde{V}, \overline{\tilde{V}} \right\rangle \qquad \Longrightarrow \qquad \tilde{\mathcal{K}} = -G^{-2} \ln \left(-\, iG^{-2} \left\langle \tilde{V}, \overline{\tilde{V}} \right\rangle \right). \tag{3.62}$$

Having $\nabla_A t^0 = 0 = \nabla_A \bar{t}^0$ allows us to write

$$\nabla_A T^{\tilde{I}} = \partial_A T^{\tilde{I}} + \frac{1}{2} G^{-2} \bigl(\partial_A \tilde{\mathcal{K}}\bigr) T^{\tilde{I}} = t^0 \nabla_A \tilde{T}^{\tilde{I}} \qquad \Longrightarrow \qquad \nabla_A V = t^0 \nabla_A \tilde{V} \ . \tag{3.63}$$

This yields to a *Kähler and symplectic covariant* expression for the metric of the embedding Kähler manifold that is (compare with (3.38))

$$G_{A\overline{B}} = i \left\langle \nabla_A V, \nabla_{\overline{B}} \overline{V} \right\rangle = i \left| t^0 \right|^2 \left\langle \nabla_A \tilde{V}, \nabla_{\overline{B}} \overline{\tilde{V}} \right\rangle \ , \tag{3.64}$$

whose invertibility requires the following $2\bigl(h^{(1,1)}+1\bigr) \times \bigl(h^{(1,1)}+1\bigr)$ matrix to be of rank $\bigl(h^{(1,1)}+1\bigr)$

$$\begin{pmatrix} \tilde{V} & \nabla_A \tilde{V} \end{pmatrix} = \begin{pmatrix} \tilde{T}^{\tilde{I}}(t) & \nabla_A \tilde{T}^{\tilde{I}}(t) \\ \\ \tilde{\mathcal{F}}_{\tilde{I}}(t) & \nabla_A \tilde{\mathcal{F}}_{\tilde{I}}(t) \end{pmatrix} \ . \tag{3.65}$$

It is also worth mentioning that the upper $\bigl(h^{(1,1)}+1\bigr) \times \bigl(h^{(1,1)}+1\bigr)$ matrix reveals the existence condition for prepotential. More precisely, it is equivalent to (3.46) in the embedding manifold and its invertibility guarantees the existence of a prepotential.

Invertibility of the metric (3.64) is also pleasant since implies that the matrix $\overline{\tau}_{AB}$ which appears in the kinetic term of the gauge fields is well defined

$$\overline{\tau}_{\tilde{I}\tilde{J}} = \Big(\overline{\tilde{\mathcal{F}}}_{\tilde{I}}(\bar{t}) \qquad \nabla_A \tilde{\mathcal{F}}_{\tilde{I}}(t)\Big)\Big(\overline{\tilde{T}}^{\tilde{J}}(\bar{t}) \qquad \nabla_A \tilde{T}^{\tilde{J}}(t)\Big)^{-1} , \tag{3.66}$$

which is independent of having a prepotential. If it exists however, we can rewrite $\tau_{\tilde{A}\tilde{B}}$ as

$$\tau_{\tilde{A}\tilde{B}} = \overline{\mathcal{F}}_{\tilde{A}\tilde{B}} + 2i\, \frac{\mathrm{Im}[\mathcal{F}_{\tilde{A}\tilde{C}}]T^{\tilde{C}}\mathrm{Im}[\mathcal{F}_{\tilde{B}\tilde{D}}]T^{\tilde{D}}}{\mathrm{Im}[\mathcal{F}_{\tilde{C}\tilde{D}}]T^{\tilde{C}}T^{\tilde{D}}} \ . \tag{3.67}$$

Reformulation of special geometry

Here, we are going to summarize important formula we obtained before but with a new formulation in which the Kähler form Ω plays a crucial role [34] . As a two-form, is has a unique expansion in terms of cohomology basis $\{D_A\}$ of $H^{(1,1)}$ given by $\Omega = v^A D_A$. One may notice that the expansion coefficients v^A are imaginary part of the complex scalars t^A (3.28) from vector multiplets (3.27). In this sense, these scalar fields can be thought of as variation generators of the Kähler two-form, as once mentioned before at the end of section 3.2.1. Also one should note that this expansion of Ω has been given in the string-frame while in the Einstein frame, we need to rescale it as $\Omega_E = e^{-\phi/2}\Omega$ where ϕ is the dilaton field from tensor multiplet (3.27).

As suggested in [64, 65], the Kähler metric of the projective special manifold (the target space of scalars in vector multiplet) can be written as

$$\tilde{G}_{A\overline{B}} = \frac{1}{4\mathcal{V}}\int_Y D_A \wedge \star D_B = \frac{-3}{2}\left(\frac{\Omega_{AB}}{\Omega^3} - \frac{3}{2}\frac{\Omega_A\ \Omega_B}{\Omega^6}\right) = -\partial_A \partial_{\overline{B}}\left(\ln \frac{4}{3}\Omega^3\right) . \tag{3.68}$$

[34]From now on, we set $G = 1$ for more convenience.

Let us also mention the intersection numbers

$$D_{ABC} := \int_Y D_A \wedge D_B \wedge D_C \ , \qquad \Omega_{AB} := \int_Y D_A \wedge D_B \wedge \Omega = D_{ABC} v^C \ , \quad (3.69)$$
$$\Omega_A := \int_Y D_A \wedge \Omega \wedge \Omega = D_{ABC} v^B v^C \ , \quad \Omega^3 := \int_Y \Omega \wedge \Omega \wedge \Omega = D_{ABC} v^A v^B v^C = 3! \mathcal{V} \ .$$

From (3.68), one can read the Kähler potential as

$$\tilde{\mathcal{K}} = -\ln \frac{4}{3} \Omega^3 \ . \quad (3.70)$$

In the case that a prepotential $\tilde{\mathcal{F}}(t)$ exists, the Kähler potential is determined by

$$\tilde{\mathcal{K}} = -\ln i \Big[\overline{T}^{\tilde{I}} \partial_{\tilde{I}} \mathcal{F}(T) - T^{\tilde{I}} \partial_{\bar{\tilde{I}}} \overline{\mathcal{F}}(\overline{T}) \Big] \quad (3.71)$$
$$= -\ln i \left| t^0 \right|^2 \Big[2 \big(\tilde{\mathcal{F}} - \overline{\tilde{\mathcal{F}}} \big) - \big(\tilde{\partial}_{\tilde{I}} \tilde{\mathcal{F}} + \tilde{\partial}_{\bar{\tilde{I}}} \overline{\tilde{\mathcal{F}}} \big) \big(\tilde{T}^{\tilde{I}} - \overline{\tilde{T}}^{\tilde{I}} \big) \Big] \ , \quad (3.72)$$

where $\tilde{\partial}_{\tilde{I}} := \partial / \partial \tilde{T}_{\tilde{I}}$. As one can check it easily, (3.71) is exactly (3.39). It turns out that the prepotential is given by

$$\mathcal{F}(T) = \frac{-1}{6} D_{ABC} \ \frac{T^A T^B T^C}{t^0} \qquad \Longrightarrow \qquad \tilde{\mathcal{F}}(t) = \frac{-1}{6} D_{ABC} \ t^A t^B t^C \ . \quad (3.73)$$

Here, to obtain the last result, we have set $\tilde{T}^I \equiv t^A$. Now, having the form of pre-potential $\tilde{\mathcal{F}}$ explicitly determined, let us also rewrite the coupling matrix $\tau_{\tilde{A}\tilde{B}}$ of the gauge fields of vector multiplet as well. By plugging (3.73) in (3.67) one reads

$$\Theta(t) := \mathrm{Re}\tau = \begin{pmatrix} \frac{-1}{3} D_{ABC} \ b^A b^B b^C & \frac{1}{2} D_{ABC} \ b^B b^C \\ \frac{1}{2} D_{ABC} \ b^B b^C & - D_{ABC} \ b^C \end{pmatrix} \ , \quad (3.74)$$

$$H(t) := \mathrm{Im}\tau = \frac{-\Omega^3}{6} \begin{pmatrix} 1 + 4 G_{AB} \ b^A b^B & - 4 G_{AB} \ b^B \\ -4 G_{AB} \ b^B & 4 G_{AB} \end{pmatrix} \ . \quad (3.75)$$

As an element of even cohomology $H^{2*}(Y)$, it is possible to use the explicit form of the prepotential (3.73) to write the projective symplectic section $\tilde{V}$ in a more abstract way

$$\tilde{V}(t) = \begin{pmatrix} \tilde{T}^{\tilde{I}}(t) \\ \\ \tilde{\mathcal{F}}_{\tilde{I}}(t) \end{pmatrix} \qquad \longleftrightarrow \qquad \tilde{V}(t) = \tilde{T}^{\tilde{A}} D_{\tilde{A}} - \tilde{\mathcal{F}}_{\tilde{A}} D^{\tilde{A}} \equiv -e^{(b^{\tilde{A}}+iv^{\tilde{A}})D_{\tilde{A}}} \ . \tag{3.76}$$

It is common to call $\tilde{V}$ the *period vector*. This expression allows us to rewrite normalized symplectic section $V(t)$ as

$$V(t) = e^{\tilde{\mathcal{K}}/2}\tilde{V}(t) = \frac{-1}{\sqrt{\frac{4}{3}\Omega^3}} e^{t^{\tilde{A}} D_{\tilde{A}}} \ . \tag{3.77}$$

This is just a parallel expression for (3.58) if one recalls (3.70) and (3.61). Moreover, as has been discussed around (3.47), in the last step we set $t^0 = 1$ which is compatible with expansion of (3.77).

We already have mentioned that the projective symplectic section $\tilde{V}(t)$ (and its conjugate $\overline{\tilde{V}}(t)$) is a (anti)-holomorphic with respect to Kähler covariant derivative ∇ defined in (3.59). So in dealing with special Kähler, it is more natural to switch from $\{D_{\tilde{A}}, D^{\tilde{A}}\}$ to another basis for even cohomology $H^{2*}(Y)$ of the internal manifold so that the whole set transform in the same way under Kähler transformation. In fact, the set we are looking for is $\{V, \nabla_A V, \nabla_{\overline{A}}\overline{V}, \overline{V}\}$ with the following *orthonormality* conditions which are the same as (3.60) and (3.62). We put all together for more convenience

$$\langle V, \overline{V} \rangle = -i \ , \qquad \langle \nabla_A V, \nabla_{\overline{B}} \overline{V} \rangle = -iG_{A\overline{B}} \ , \qquad \langle \nabla_A V, V \rangle = 0 \ . \tag{3.78}$$

Hence, in the new basis, there is a unique decomposition for any real element $E \in$

$H^{2*}(Y)$ given by

$$\begin{aligned} E &= i\overline{Z}(E)V - iG^{\overline{A}B}\nabla_{\overline{A}}\overline{Z}(E)\nabla_B V + iG^{A\overline{B}}\nabla_A Z(E)\nabla_{\overline{B}}V - iZ(E)\overline{V} \\ &= 2\text{Im}\big[\overline{Z}(E)V - G^{\overline{A}B}\nabla_{\overline{A}}\overline{Z}(E)\nabla_B\big] \ , \end{aligned} \tag{3.79}$$

where we now introduce the associated *central charge* to any $E \in H^{2*}(Y)$ defined as its intersection product with the symplectic section, i.e., $Z(E) := \langle E, V\rangle$. In $\{D_{\tilde{A}}, D^{\tilde{A}}\}$, one reads

$$Z(E) := \langle E, V\rangle = \frac{-1}{\sqrt{\frac{4}{3}\Omega^3}}\Big(E_0 - E_A t^A + \frac{1}{2}E^A t_A^2 - \frac{1}{3!}E^0 t_0^3\Big) \ . \tag{3.80}$$

The name "central charge" originates from the fact that it appears as a central charge in the supersymmetry algebra in four-dimensions [66]. As we will see in coming sections, the central charge plays an important role in *attractor mechanism* and finding the so called BPS solutions to $\mathcal{N} = 2$ supergravity in $4D$. Work in the new basis has this benefit that one can define a linear real operator $\diamond$ whose action on the basis is given by

$$\diamond V = -iV \ , \quad \diamond \nabla_A V = i\nabla_A V \ , \quad \diamond \nabla_{\overline{A}}\overline{V} = -i\nabla_{\overline{A}}\overline{V} \ , \quad \overline{V} = i\overline{V} \ . \tag{3.81}$$

One can check it is a real operator satisfying $\diamond = \overline{\diamond}$ while $\diamond^2 = -1$. This operator has more complicated action in cohomology basis which we bring here for completeness and also later application. It is

$$\begin{aligned} \diamond D_{\tilde{A}} &= \Big[\text{Re}\tau\big(\text{Im}\tau\big)^{-1}\Big]_{\tilde{A}}^{\ \tilde{B}} D_{\tilde{B}} - \Big[\text{Im}\tau + \text{Re}\tau\big(\text{Im}\tau\big)^{-1}\text{Re}\tau\Big]_{\tilde{A}\tilde{B}} D^{\tilde{B}} \ , \\ \diamond D^{\tilde{A}} &= \Big[\big(\text{Im}\tau\big)^{-1}\Big]^{\tilde{A}\tilde{B}} D_{\tilde{B}} - \Big[\big(\text{Im}\tau\big)^{-1}\text{Re}\tau\Big]^{\tilde{A}}_{\ \tilde{B}} D^{\tilde{B}} \ . \end{aligned} \tag{3.82}$$

As a consequence of τ being dependent on scalar moduli t^A, the action of $\diamond$ is also moduli dependent. Having this defined, one can introduce a bilinear moduli dependent inner product as

$$\big(E_1,E_2\big) := \langle E_1, \diamond E_2\rangle = 2\mathrm{Re}\Big[Z(E_1)\overline{Z}(E_2) + G^{A\overline{B}}\nabla_A Z(E_1)\nabla_{\overline{B}}\overline{Z}(E_2)\Big]\ , \qquad (3.83)$$

where in the last quality, we have used decomposition (3.79) for real E_1 and E_2. Accordingly, we define norm of any real even-form E as $|E| := \sqrt{\big(E,E\big)}$. Moduli dependency of this norm can make it degenerate in the case of having singularities in the moduli space, but we are not going to consider these cases here. It also worth mentioning that for $b^A = 0$ (or equivalently $\Theta(t) \equiv \mathrm{Re}\tau(t) = 0$), the operator $\diamond$ reduces to the Hodge dual, i.e.,

$$\big(E_1,E_2\big)_{b^A=0} = \int_Y E_1 \wedge \star E_2. \qquad (3.84)$$

We finish this part by making a very short remark about the geometry underlies the other set of scalars, i.e., hyperscalars we have denoted the all by q^v in (3.29). In the light of [67], the target space on which these scalar live is a *special quaternionic manifold* with metric h^{uv} given in [34]. These manifolds can be obtained from special Kähler via the so-called c-map [29]. Recall the terminology for rigid(affine)/projective special geometries of gauge multiplet scalars of rigid/local supersymmetries. Analogously, in the case of theory with rigid supersymmetry the underlying geometry of hypermultiplet scalars is called *hyper Kähler* manifold while in the case of localized supersymmetries it is called *quaternionic Kähler*. We are not going to discuss this in detail since hyperscalars q^v do not play an important role in our study of supersymmetric black hole solutions. In fact, they can be consistently set to constants, as we mentioned earlier.

3.3.2. Electromagnetic Fluxes, Wrapped D-Branes and BPS Particles

The $U(1)$ gauge fields $A_\mu^{\tilde{A}}$ in four dimensions classified in (3.27) are sourced by dyons carrying generalized electromagnetic charges $\Gamma^a := (\Gamma^{\tilde{A}}, \Gamma_{\tilde{A}})$ (3.19). These lower dimensional gauge fields by their own have been found from the higher dimensional $R-R$ spectrum. Hence, one may look for a way to extract these charges Γ^a from $R-R$

sources in ten-dimensions [68]. To reveal this relation, we need to study another family of solution to the superstring action generally called *p-branes*. In the following part, we are going to very briefly introduce them and explain how these higher dimensional *extended* objects emerge as charged *point particles* in $4D$ after reduction. We also explicitly write their attributed fluxes to see where Γ^a originates from.

3.3.2.1. D-Branes. The fundamental string are not the only *extended* objects appear in string theory. In fact, there are higher dimensional objects called *p-branes* that become as important as strings in nonperturbative regime. they are hypersurfaces with p spatial dimensions sweeping a $(p+1)$-dimensional *worldvolume* in $10D$ [69]. In this bigger picture of extended objects, fundamental string are nothing but 1-branes. The factor makes strings the most relevant p-branes at weakly coupled regime is that all higher dimensional branes become infinitely heavy as $g_s \to 0$. In Type II string theories, mass (tension) of p-branes is proportional to $(g_s)^{-1}$ that explains why they are only considered in nonperturbative approaches. p-branes may satisfy compound boundary conditions: Dirichlet boundary conditions in $9-p$ directions perpendicular to the hypersurface and Neumann boundary conditions in other directions. Accordingly, they are usually called D_p-*branes*.

The importance of D_p-branes relies on the fact that open strings can end on these hypersurfaces and their quantized fluctuations can be interpreted as scalars and non-abelian gauge fields on the worldvolume. More precisely, their existence enable us to build Yang-Mills type interactions that are crucial for explaining standard model of particle physics (See Figure 3.1). Furthermore, these fields and fluxes on the branes worldvolume, if there is any, can interact with the background fields of supergravity coming from low energy spectrum of closed strings (see Table 3.3). Consequently, from their interactions it can be inferred that D_p-branes configurations depends on type of the corresponding superstring theory. For instance, in Type IIA there are only even-dimensional D_p-branes allowed (stable) while in Type IIB we have just odd-dimensional D_p-branes.

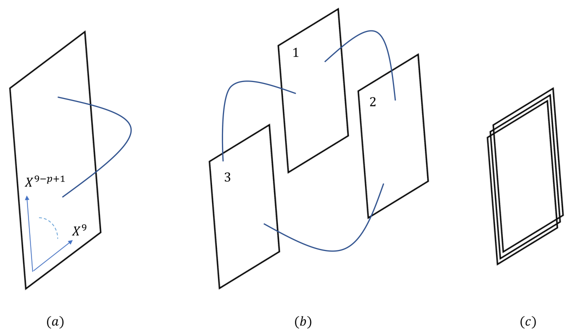

Figure 3.1: (a) A Dp-brane extended in $\{X^{9-p+1}, ..., X^9\}$ directions with an open string ending on it. It sweeps out a $(p+1)$-dimensional worldvolume in $\{X^0, ..., X^{9-p}\}$ directions. The Yang-Mills fields can be thought of massless excitation modes of the attached open strings. (b) A configuration of three separated D-branes that describes $U(1)^3$ gauge fields. (c) A configuration of three D-branes laying on top of each other that describes $U(3)$ gauge fields

Now, consider a D-brane with gauge fields A_α on its worldvolume (should not be confused with $R-R$ gauge fields (A_M, A_{MNP})) whose embedding is given by a scalar field Φ_M. Its dynamics in Type IIA supergravity background is given by [56]

$$\mathcal{S}_{Dp} = \frac{-2\pi}{g_s l_s^{p+1}} \int d^{p+1}\sigma \; e^{\phi} \sqrt{-\det\big(G_{\alpha\beta} + \mathfrak{F}_{\alpha\beta}\big)} + \frac{1}{2} \int e^{\frac{\mathfrak{F}}{2\pi}} \wedge A \; . \tag{3.85}$$

Here $\{\sigma^\alpha\}$ span $(p+1)$-dimensional worldvolume while ϕ and A are background fields. Let us discuss this action term by term. In the first term, $G_{\alpha\beta} := g_{MN}\partial_\alpha\Phi_M\partial_\beta\Phi_N$ is the pullback of the background metric g_{MN} to the worldvolume. Similarly, in $\mathfrak{F}_{\alpha\beta} := B_{MN}\partial_\alpha\Phi_M\partial_\beta\Phi_N + 2\pi\alpha' F_{\alpha\beta}$, the first term is the B-field from $NS-NS$ sector pulled back to the worldvolume and the second term is just field strength of the gauge field A_α living on the worldvolume, i.e., $F = dA$. Clearly, background fields ϕ and A have to be pulled back as well. In the case of vanishing two-form $\mathfrak{F}$, the first term is called *Dirac-Born-Infeld* which is nothing but *area* of the worldvolume. In other words, in the absence of background B-field and worldvolume flux F, the first term simply describes a hypersurface tends to minimize area of its surface. Also note that in the small string coupling regime, this is the first term that is dominant.

The second term describes coupling [35] of the worldvolume field strength $\mathfrak{F}$ to the background $R-R$ gauge fields A. Analogous to a charged point particle (0-brane) that couples to a one-form gauge field, a D_P-brane get coupled to a $(p+1)$-form gauge and that is why we need to introduce generalized electromagnetic charges Γ^a. At the same time note that this term which is called Wess-Zumino (WZ) or Chern-Simons (CS) describes a *self-interacting* theory since B-fields and gauge fields A, both from background, interact via worldvolume two-form $\mathfrak{F}$.

Let us now turn back to our main question of what happens to D-branes of Type IIA SUGRA after performing CY compactification. We have already talked about homology structure of the Calabi-Yau threefold in section 2.3.2.1 and the complete basis for non trivial cycles has been listed in Table 2.2. For simplicity we mention the homology basis here again, that is $\{1, \mathbb{A}^A, \mathbb{B}_A, d\mathcal{V}\}$. By getting ten-dimensional $\mathcal{N}=2$ supergravity compactified on an internal Calabi-Yau, D-branes may wrap on these cycles. Then, by sending the volume of the internal manifold to zero, these wrapped D-branes will seem like charged point particles in $4D$ theory. To read the electromagnetic charges attributed to these wrapped D-branes, consider a D_2-brane wrapped on a 2-cycle $\Gamma = q_A\mathbb{A}^A$. For now we take zero worldvolume two-form $\mathfrak{F}$ and compute the WZ term for the three-form gauge field in R-R sector given in (3.24). It is easy to see that only $A^A_\mu \wedge D_A$ will contribute in the integral

$$\begin{aligned} \int_{\gamma\times\Gamma} A_{(3)} &= \int_{\gamma\times\Gamma} \left(A^A_\mu(x)\wedge D_A + A_3\right) = \int_\gamma A^A_\mu(x) \int_{q_A\mathbb{A}^A} D_A \qquad (3.86) \\ &= q_B \int_\gamma A^A_\mu(x) \int_Y D^B \wedge D_A = q_A \int_\gamma A^A_\mu(x)\ . \end{aligned}$$

Here γ is an integral curve in $4D$ and we have used orthoganility relations from Table 2.2. The last result in (3.86) shows how integration of higher dimensional gauge fields $A_{(3)}$ over a wrapped D_2-brane on a non-trivial 2-cycle turns to a point particle carrying electric charge q_A under four-dimensional gauge field $A^A_\mu(x)$.

[35] To see all allowed possibilities, one should expand the exponential and only keep those terms that fits the integration, i.e., those that are $(p+1)$-forms.

This idea can easily be generalized such that it contains other even dimensional D-branes. More precisely, one can imagine a configuration of $(D6, D4, D2, D0)$ branes with arbitrary number of each type, all coincide at a single point in external $4D$ spacetime after compactification. Accordingly, $D6$-branes get wrapped on full Calabi-Yau three-fold, $D4$ ones are on internal 4-cycles, $D2$-branes are wrapped on 2-cycles and finally, (anti)-$D0$-branes are simple points inside internal CY. Such a configuration ultimately yields a generic electromagnetic charge vector Γ^a previously mentioned in (3.18). So we have

$$\Gamma = (\Gamma^{\tilde{A}}, \Gamma_{\tilde{A}}) = p^0 + p^A D_A + q_A D^A + q_0 d\mathcal{V} := \left(p^0, p^A, q_A, q_0\right) . \tag{3.87}$$

From now on, we mostly refer to Γ as written in the last expression in which we have specified type of charges so that q's are conventionally called electric charges while p's are magnetic ones. The charge Γ above describes p^0 $D6$-, p^A $D4$-,q_A $D2$- and q_0 anti-$D0$-branes. We refer the reader to for having a better imagination of this correspondence between charged point particles in $4D$ supersymmetric theory and wrapped even dimensional branes around non-trivial cycles of internal Calabi-Yau.

<u>3.3.2.2. BPS Particles.</u> Now that we figured out how massive charged point particles in $4D$ originate in even dimensional D-branes wrapped on non-trivial cycles of internal manifold, let us have a look at their dynamics in $\mathcal{N} = 2$ supergravity background. We demand these particle to preserve some of the background supersymmetries. More precisely, these particles correspond to D-branes in higher dimensions whose wrapping around internal manifold via compactification preserve some of supersymmetries. For instance, let us consider a $D2$-brane wrapped on an internal 2-cycle. Effectively, it merges as a point particle in external $4D$ spacetime. Consequently, as discussed in [70] in detail the worldline action of such a particle will be determined uniquely by supersymmetry and it is

$$\mathcal{S}_{\text{wl}} = -\int \big|Z(\Gamma)\big| ds + \frac{1}{2}\int \langle \Gamma, A_\mu \rangle dx^\mu . \tag{3.88}$$

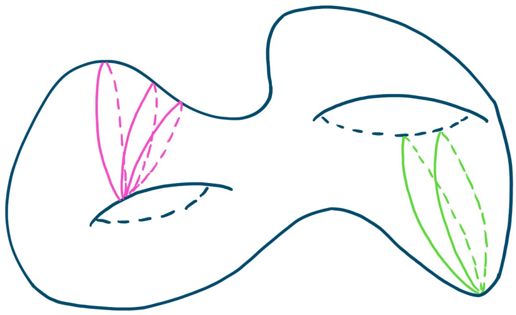

Figure 3.2: D-branes wrapped on non-trivial cycles of the internal manifold ($CY3$). Let the right handle shows a 2-cycle while the left one is a 4-cycle. Hence, in homology basis they are given in terms of $\mathbb{A}^A$ and $\mathbb{B}_A$, respectively. As depicted here, the 2-cycle is wrapped twice by a $D2$-brane (the green lines) while the 4-cycle is wrapped three times by a $D4$-brane (the pink lines). Also there is no pointlike particle ($D0$-brane), neither is any $D6$-brane wrapping on the whole three-fold. Consequently, the corresponding electromagnetic charges of this configuration would be $\Gamma^a = 0 + 2\mathbb{A}^A + 3\mathbb{B}_A + 0d\mathcal{V} = (0, 2, 3, 0)$.

Let us explain this action term by term. Starting from the first term, one first recognizes the associated central charge to Γ while the general definition for $Z(E)$ is given in (3.80). This central charges coupled to the infinitesimal worldline element ds resembles a massive particle moving in a gravitational background. This implies that the central charge $Z(\Gamma)$ can be interpreted as mass of dyons with generalized electromagnetic charge Γ but with a very interesting property: as mentioned before, the central charge depends on scalars t^A via $V(t)$, meaning that what we called "mass" of dyons is not a constant. The second term is the standard electromagnetic charge-field coupling. Considering the fact that total energy is the kinetic energy plus the potential one, one concludes immediately that these charged particles described by (3.88) satisfy BPS bound, an inequality that bounds the mass from below by a function of the (asymptotic) charges of the fields. Here, the lower bound is the central charge that depends on scalar fields t^A. One reads $M \geq Z(\Gamma)$ and so these particles called *BPS particles*.

In the maximally supersymmetric case where we demand to preserve half of supersymmetries (i.e., four out of eight supercharges) [36] , the background metric fixed to be Minkowski. This case is of special interest because of two interesting properties: first, in this background all scalars t^A become constant, meaning that dyons carry constant mass. Although, one should note that this constant is still sensitive to the value of the scalars. In other words it turns out that any specific vacuum is determined by asymptotic values of these scalars in respect to which one measures the mass of the particle. Second, in Minkowski background gauge fields become trivial (i.e., $F = dA = 0$) that makes the second term in (3.88) zero. These properties reduce the worldline action in static frame (i.e., $ds = \sqrt{-g_{tt}}dt$) to

$$\mathcal{S}_{\rm wl} = -\big|Z(\Gamma)\big| \int dt \ . \tag{3.90}$$

This implies that in the case of maximally supersymmetric background, the BPS bound is saturated, i.e., the mass of particles are constant and $M = \big|Z(\Gamma)\big|$, so it is called a BPS state.

3.4. BPS Black Hole Solutions

We are now in a position to start discussing black hole solutions to $\mathcal{N} = 2$ supergravity in four dimensions. As we will see later in this section, the supersymmetric black hole solutions saturate a BPS bound of $\mathcal{N} = 2$ supergravity effective action in 4D. However, these solution only respect half of the supersymmetries already exist for the theory, i.e., only four out of eight supercharges survive. That is why they are usually called $\frac{1}{2}$-BPS black holes. As we will discuss, there are two different but equivalent ways to find these solutions. The first is to solve Killing spinor equations for the only covariantly constant Majorana spinor of these solutions. The second way which we

[36] As discussed in [8], in Calabi-Yau compactification procedure wrapping an even dimensional D-brane on an even dimensional cycle in the internal manifold will preserve *half* of the supersymmetries if the cycle is a holomorphic one. In the same references it is also shown that under this circumstance, the $4D$ Born-Infeld (BI) term (see 3.85) for a D-brane wrapped on a cycle Γ is

$$-\int_\gamma \big|Z(\Gamma)\big| ds \tag{3.89}$$

with $Z(\Gamma) = \int_\Gamma V(t)$ and γ is the worldline followed by the particle in external spacetime.

are going to follow here is called "*the attractor flow mechanism*". The main focus in this approach is on radial flow of the scalars t^A in physical spacetime. We will observe that no matter what is their asymptotic values, they always settle down in a minimum value at the horizon of the BPS black holes. Assuming proper ansatz for fields and then following this approach leads us to first order differential equation called *attractor flow equations*. They are equivalent to those one can find by solving Killing spinor equations and their solution results in BPS black holes.

As mentioned earlier, we can simply ignore hyperscalars $\{q^u\}$ in (3.29) since they are completely decoupled from other fields in the theory. Hence, by taking them to be constant our stating point will be the following action

$$\begin{aligned} \mathcal{S} = \quad & \frac{2\pi}{l_4^2}\int\sqrt{-g}\mathcal{R} - \frac{64\pi^2}{l_4^2}\int G_{A\overline{B}}dt^A\wedge\star d\bar{t}^B + \\ & \frac{1}{16\pi}\int\Big[-H(t,\bar{t})_{\tilde{A}\tilde{B}}F^{\tilde{A}}\wedge\star F^{\tilde{B}} + \Theta(t,\bar{t})_{\tilde{A}\tilde{B}}F^{\tilde{A}}\wedge F^{\tilde{B}}\Big] \end{aligned} \tag{3.91}$$

with $\Theta(t,\bar{t}) \equiv \mathrm{Re}\tau(t,\bar{t})$ and $H(t,\bar{t}) \equiv \mathrm{Im}\tau(t,\bar{t})$. In the beginning of the previous chapter, we motivated studying Einstein-Maxwell theory (EM) by saying that even though there are serious complications in $\mathcal{N}=2$ SUGRA, there are still many analogies between these two theories. We briefly mentioned two of the most significant differences that make (3.91) and its to-be-introduced black hole solutions much more involved than EM (2.8) and its Reissner-Nordström black hole solutions (2.13,2.25). Here we are going to spend more time to clear up these differences and also analogies.

- Both theories contain electromagnetic fields coupled to gravity. Moreover, it is always possible to extend pure electric EM so that magnetic charges get involved as well as electric ones. However, in EM there is only one gauge field while in theory described by (3.91) there are at least two of them [37] as the index runs over $\tilde{A} = 1 + h^{(1,1)}$ and $h^{(1,1)} \geq 1$ for CY threefolds.
- There is no scalar field interacting with the gauge fields in EM. However, as one

[37] The simplest case is when the internal manifold Y is taken to be *quintic Calabi-Yau* for which we have $h^{(1,1)} = 1$. This threefold is embedded in $\mathbb{C}P^4$ by $\sum_{i=1}^5 y_i^5 = 0$ where $\{y_i\}$ are projective coordinate on the embedding space.

sees from (3.91), there are matrices $\tau_{\tilde{A}\tilde{B}}$ that govern coupling between the gauge fields in $\mathcal{N} = 2$ SUGRA and they are explicitly scalar dependent, i.e., $\tau = \tau(t^A)$. In other words, electromagnetic charges "talk" not only to the gauge fields but also to the complex scalars. As mentioned before, this phenomenon is encoded in the geometric structure of the scalar target space and restricts its manifold to be special Kähler. There is another subtlety originates from presence of these complex scalar fields. While gravitational and gauge fields both vanish at infinity, it is asymptotic values of scalars that determines the vacuum state of the theory. Consequently, $\mathcal{N} = 2$ SUGRA can be thought of as a theory with an infinite number of vacua constructing an infinite dimensional Hilbert space, which is a submanifold of scalars target space in general. They also show a very specific behavior called *attractor flow* which we are going to explain in detail very soon.

3.4.1. The Path Towards a Supersymmetric Solution

After getting some insight into the $\mathcal{N} = 2$ SUGRA in $4D$ described by (3.91), the next step is to look for some black hole solutions. It is quite common to start with equation of motion like (3.30) and try to find the solution we are interested in. However it is not an easy task to do in the case of such involved theories as this one. The question is whether there is an alternative way instead of dealing with equations of motion. Luckily the answer is yes: look for *Killing spinors* originally called *super-covariantly constant spinors* by Gibbons and Hull [71].

Generally, supergravity action is supposed to stay invariant under supersymmetric transformations with parameters given by arbitrary spinor functions $\epsilon(x)$. However, a typical solution does not respect all of these symmetries. In fact, the set of supersymmetric transformations that leaves the solution invariant is determined by a finite subset of $\epsilon(x)$. These transformations are generated by Killing spinors. In other words, in a specific supergravity solution, for any Killing spinor one finds there is a one-to-one correspondence between the number of Killing spinors and that of supersymmetry transformation that leaves the solution invariant [38] . Accordingly, the number of super-

[38]In the case of studying fluctuations around Minkowski background, these Killing spinors are

charges is determined by the number of constant parameters in Killing spinors (these parameters are constant for the residual global supersymmetries). Consequently, instead of putting huge attempt on solving equations of motion, one can solve Killing spinor conditions that originate in transformation rules of fields under SUSY transformations. Studying these conditions not only determines the spinors, but also reveals information about spacetime geometry in which these spinors are Killing. So in summary:

The preserved supersymmetries of any classical supergravity solution is determined by Killing spinors. The number of associated supercharges n_Q is as many as the number of constant components of the Killing that parameterize the set of residual rigid (global) supersymmetries respected by the solution. Then the supergravity solution is characterizes by $\mathcal{N}$ which by definition equals to n_Q divided by the number of degrees of freedom of the shortest possible spinor in the specific spacetime dimension [29].

Again making an example of a simpler theory with gauge-gravity coupling such as Einstein-Maxwell theory can be clarifying. Since EM is purely bosonic one need to look for Killing vectors instead. But there is a relation between Killing spinors and vectors in a theory with both bosonic and fermionic degrees of freedom; Killing vectors by which one can determine *bosonic* preserved symmetries are bilinears of Killing spinors [39] . The EM action (2.8) is invariant under general coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x)$. However, its simplest solution which is $g_{\mu\nu} = \eta_{\mu\nu}$ and $F_{\mu\nu} = 0$, only enjoys Poincaré symmetry in D-dimensions generated by Killing vectors of the Minkowski spacetime. It means the number of $\xi^\mu(x)$ leaving this solution invariant decreases from infinity to $\frac{D(D+1)}{2}$. If we demand to have a static and spherically symmetric solution like (2.11,2.15) then the group of isometries become even smaller and reduces to the Lorentz group with $\frac{D(D-1)}{2}$ generators.

On the other hand, as we also mentioned around (3.90) supergravity solutions

simply constant spinors.

[39]For instance, in Minkowski spacetime for any pair $(\epsilon, \epsilon^{'})$ of Killing spinors, the combination $\epsilon^{'} \gamma^\mu \epsilon$ is a constant Killing vector. It generates translation which means global SUSY algebra closes on spacetime translations.

frequently called **BPS** solutions. Here is the reason: in Bogomol'nyi, Prasad and Sommerfield (BPS) formalism [72,73] higher derivatives equations of motion are reduced to first order differential equations. The same happens for Killing conditions: they relate bosonic field components via some first order differential equations.

3.4.2. The Attractor Mechanism

In this subsection we are going to drive the first order differential equations called *attractor flow equations*. To do so, we first set some proper ansatz for present fields in the theory and then plug them in effective supergravity action. Applying BPS formalism leads to the attractor flow equations whose solutions are supersymmetric BPS black holes. In the middle of this way we also explain why this flow of scalars called “attractor”.

The fields ansatz

The solution we are looking for is static and spherically symmetric, so the most general ansatz for the metric has to be in the same form as (2.13). This ansatz also allows half of the supersymmetries to survive [74], [75] and that is why these solutions are called $\frac{1}{2}$-BPS solutions. Here we repeat the ansatz for more convenience

$$ds^2 = -e^{2C(r)}dt^2 + e^{-2C(r)}dx^i dx^i \ , \tag{3.92}$$

where $\{x^i\}$ span three-dimensional flat space and the spherical symmetry restricts the function C to depend only on radial coordinate r. The same is true for the scalars, i.e., we have $t^A = t^A(r)$. Also, demanding (3.92) to be asymptotically Minkowski implies that $C_{r\to\infty} \to 0$.

Now, to find proper ansatz for electromagnetic field strength, we first note that it can be extended to a symplectic vector F^a with $2\tilde{A}$ components, i.e.,

$$\Gamma^a := \begin{pmatrix} \Gamma_{\tilde{A}} \\ \\ \Gamma^{\tilde{A}} \end{pmatrix} \quad \longleftrightarrow \quad F^a := \begin{pmatrix} F_{\tilde{A}} \\ \\ F^{\tilde{A}} \end{pmatrix} \quad \text{with} \quad F_{\tilde{A}} := \mathrm{Re}\Big[\tau_{\tilde{A}\tilde{B}}\big(F^{\tilde{B}} + i \star F^{\tilde{B}}\big)\Big] \ , \quad (3.93)$$

whose magnetic components $F^{\tilde{A}}$ are fixed by spherical symmetry as

$$F^{\tilde{A}} = \Gamma^{\tilde{A}} \ \sin\theta d\theta \wedge d\varphi \ . \qquad (3.94)$$

Then using (3.93), one can easily read the electric part as well. That leads to the following unique ansatz for F^a

$$F^a := \begin{pmatrix} (\diamond\Gamma)_{\tilde{A}} \ e^{2U} dt \wedge d\rho \\ \\ \Gamma^{\tilde{A}} \ \sin\theta d\theta \wedge d\varphi \end{pmatrix} \qquad (3.95)$$

where for later convenience we have redefined the radial coordinate as $\rho \equiv r^{-1}$. Also the operator $\diamond$ and its action on cohomology basis $\{D_{\tilde{A}}, D^{\tilde{A}}\}$ has been defined previously in (3.82). One can compare this ansatz to (2.23) for dyonic Reissner-Nordstöm black hole we discussed before. Notice that the operator $\diamond$ reduces to the normal Hodge star $\star$ (up to a minus sign) if $\Theta(t) \equiv \mathrm{Re}\tau$ vanishes.

By fixing these ansatz for the metric and fields strength, looking for the solution is reduced to finding $h^{(1,1)} + 1$ undetermined functions, i.e., $C(r)$ and the scalars t^A. Plugging these ansatz (3.92,3.95) into equations of motion of $\mathcal{N} = 2$ action (3.30) results in coupled second order differential equations which are very difficult to be solved. However, as mentioned before demanding a specific number of supersymmetries to be preserved is guaranteed only if we can find enough number Killing spinors. This implies that instead of dealing with these second order equations, we can look at Killing conditions obtained from setting gravitino $\psi^{\pm}_M$ and their SUSY variation to zero [75]. This alternative way first was explored in [21], [22] and [76] provides first order differential equations that are much more easier to be dealt with.

The attractor flow formalism

The first order *attractor flow equations* were studied in [23] for the first time and the approach is such that the same techniques can be applied in the case of multi-centered solutions. This is the path we are going to take here.

To obtain the total action (per time) for undetermined fields C and t^A as functions of the inverse radial coordinate ρ, we need to substitute field ansatz (3.92,3.95) we obtained by some symmetry considerations inside the supergravity action (3.91). Then after some simplifications such as neglecting a total derivative term proportional to $\ddot{C}$ we find

$$\mathcal{S}_{\text{eff}} = \frac{\mathcal{S}}{\Delta t} = \frac{-1}{2} \int_0^\infty d\rho \left\{ \dot{C}^2 + G_{A\overline{B}} t^A \dot{\overline{t}}^B + e^{2C} U(t^A) \right\} - \left| e^C Z(\Gamma) \right|_{\rho \to \infty} , \tag{3.96}$$

with

$$U(t^A) = \frac{1}{2} \left| \Gamma \right|^2 = \frac{1}{2} \left| Z(\Gamma) \right|^2 + 4 G^{A\overline{B}} \partial_A \left| Z(\Gamma) \right| \partial_{\overline{B}} \left| Z(\Gamma) \right| . \tag{3.97}$$

Here the dot shows derivation with respect to ρ and we have used $\left|\Gamma\right| = \sqrt{(\Gamma, \Gamma)}$ with (3.83). The last boundary term comes from the action (3.88) describing a wrapped $D2$-brane on a 2-cycle which effectively can be seen as a massive charged point particle. Also note that $e^{2C}U(t^A)$ is proportional to electromagnetic energy density. Accordingly, one can see (3.96) (with the opposite sign) as the effective action of a particle moving in (C, t^A) space experiencing the electromagnetic potential $-e^{2C}U(t^A)$. Now, there are two ways to complete squares in (3.96) and apply Bogomol'nyi trick. These two ways result in two different sets of equations for undetermined functions

(i) The way that is done in [77]

$$\begin{aligned} \mathcal{S}_{\text{eff}} &= \frac{-1}{2}\int_0^\infty d\rho \left\{ \Big(\dot{C} \pm e^C \big|Z(\Gamma)\big| \Big)^2 + \Big| \dot{t}^A \pm 2e^C G^{A\overline{B}} \partial_{\overline{B}} \big|Z(\Gamma)\big| \Big|^2 \right\} \\ &\pm \; e^C \big|Z(\Gamma)\big| \, \Big|_{\rho=0}^{\rho\to\infty} - \Big(e^C \big|Z(\Gamma)\big| \Big)_{\rho\to\infty} \, . \end{aligned} \tag{3.98}$$

From $\pm$ options in each term, only plus sign is compatible with the proper boundary conditions for $\dot{C}$ and $\dot{t}^A$ [23]. By excluding the minus sign option, one reads the following BPS equations

$$\dot{C} = -e^C \big|Z(\Gamma)\big| \quad , \quad \dot{t}^A = -2e^C G^{A\overline{B}} \partial_{\overline{B}} \big|Z(\Gamma)\big| \, . \tag{3.99}$$

(ii) In the second way however, the radial dependent phase of the central charge, $\alpha\big(t^A(\rho)\big)$ plays more important role through its presence in the BPS equations. To complete the square, we need to use orthonormality conditions (3.78) and the definition of inner product (.,.) (3.83). In addition, having Kähler covariant derivative ∇ defined in (3.59), one can easily prove the following identity

$$\Big[\partial_\rho + i\text{Im}\big[\partial_A \mathcal{K} \dot{t}^A\big] + i\dot{\alpha}\Big]\big(e^{-i\alpha}V\big) = e^{-i\alpha}\nabla_A V \dot{t}^A \quad ; \quad \alpha := \arg Z(\Gamma) \; , \tag{3.100}$$

where $\mathcal{K}$ is the Kähler potential of the projective Kähler manifold of the scalars t^A and $V(t^A)$ is its symplectic section. Applying this identity cause a noticeable simplification of the effective action that takes the following form

$$\begin{aligned} \mathcal{S}_{\text{eff}} &= \frac{-1}{4}\int_0^\infty d\rho \; e^{2C} \left| \; 2\text{Im}\Big\{ \Big(\partial_\rho + i\text{Im}\big[\partial_A \mathcal{K} \dot{t}^A\big] + i\dot{\alpha}\Big)\big(e^{-C}e^{-i\alpha}V\big) \Big\} + \Gamma \; \right|^2 \\ &- \; \big|Z(\Gamma)\big|_{\rho=0} \end{aligned} \tag{3.101}$$

where in the last term we put $e^C \, |_{\rho=0} = 1$. This effective action is time-independent so its solutions are static as well. Hence, one can think of (3.101) as the Hamiltonian with an opposite sign, that means any solution to this action admits a BPS bound $M \geq \big|Z(\Gamma)\big|_{\rho=0}$. Another point is the crucial role the central charge $Z(\Gamma)$

plays as its value at the spatial infinity $\left|Z(\Gamma)\right|_{\rho=0}$ determines the vacuum energy of the solution. This action yields the following BPS equation

$$2\mathrm{Im}\Big\{\Big(\partial_\rho + i\mathrm{Im}\big[\partial_A\mathcal{K}\dot{t}^A\big] + i\dot{\alpha}\Big)\big(e^{-C}e^{-i\alpha}V\big)\Big\} = -\Gamma \ . \tag{3.102}$$

Taking intersection product of (3.102) with Γ, one reads $\mathrm{Im}\big[\partial_A\mathcal{K}\dot{t}^A\big] + \dot{\alpha} = 0$ identically. This reduces the BPS equation to

$$2\partial_\rho\Big(e^{-C}\mathrm{Im}\big[e^{-i\alpha}V\big]\Big) = -\Gamma \ , \tag{3.103}$$

which is the first order differential equation called "the attractor flow equations". It is equivalent to the Killing spinor conditions.

Now that we have the BPS equation (3.103), by integrating we get

$$2e^{-C}\mathrm{Im}\big[e^{-i\alpha}V\big] = -\Gamma\rho + 2\mathrm{Im}\big[e^{-i\alpha}V\big]_{\rho=0} \ . \tag{3.104}$$

Regarding the fact that both the charge vector Γ and the projective symplectic section $V(t^A)$ have $2\big(h^{(1,1)}+1\big)$ components, this equation encapsulates $2h^{(1,1)}+1$ independent equations that is enough to find one real function $C(\rho)$ and $h^{(1,1)}$ complex scalars $t^A(\rho)$ (or equivalently $2h^{(1,1)}$ real functions $b^A(\rho)$ and $v^A(\rho)$ (3.28)). The $2h^{(1,1)}+2$nd equation is not an independent one since the intersection product of (3.104) with Γ vanishes identically. Also note that by solving these equations, we are able to express moduli scalars t^A in terms of inverse radial coordinate ρ which can be thought of as a single parameter flow in the moduli space. This is where the term *attractor flow equations* comes from. Let us investigate these equations in more detail.

Why attractor?

There are still more information to be extracted from the attractor flow equations (3.104) by following some simple algebraic calculations. First, taking intersection product of (3.103) with the symplectic vector V we get

$$\dot{C}(\rho) = -e^{C}\big|Z(\Gamma)\big| \ . \tag{3.105}$$

It is the same equation as what we had in the first approach (3.99). Again, by substituting this inside (3.103) and taking the intersection product with $\diamond\Gamma$ [40] we obtain

$$\partial_\rho\big|Z(\Gamma)\big| = -4e^{C}G^{A\overline{B}}\nabla_A Z(\Gamma)\nabla_{\overline{B}}\overline{Z(\Gamma)} \leq 0 \ , \tag{3.106}$$

which implies that the norm of the central charge $Z(\Gamma)$ increases radially coordinate r (note that $\rho = r^{-1}$). In other words, starting from spatial infinity ($\rho = 0$), the closer we get to the origin ($\rho \to \infty$) the smaller becomes the size of central charge until it reaches its minimum right at the origin, so we have $\big|Z(\Gamma)\big|_{\min} = \big|Z(\Gamma)\big|_{\rho\to\infty}$. Now, given the fact that the central charge $Z(\Gamma, t^A)$ is a function of the scalar moduli and it takes a minimum at the origin ($\rho \to \infty$) reveals that at the same place, t^A also settle down to some fixed minimum values which are uniquely determined by the electromagnetic charge Γ. This means that no matter which boundary values $t^A \mid_{\rho=0}$ they start from at radial infinity, these scalars are always *attracted* towards the origin where they take their minimum values $t^A_* \equiv t^A(\infty)$. This behavior is called *the attractor flow mechanism* and first observed in [76], [77]. It might be impossible to understand these solutions and some of their properties (such as their solution space and decay of multi-centered configurations) without studying the one parameter flow of the central charge $Z(\Gamma, t^A)$ in the moduli space. Still, the minimum value of the central charge may not be an absolute one, meaning that depending on the different asymptotic values of the scalars t^A, the flow may fall into different local minima. As discussed in [78], [79], this only happens at possible singularities in the moduli space that we are not going to discuss here.

Let us now have a more concrete look at the infinity limit $\rho \to \infty$. At this limit, the solution to (3.105) simply becomes $e^{-C} \to \rho\big|Z(\Gamma)\big|_{\min}$ which reveals a metric sin-

[40] To do this first act $\diamond$ on (3.79) using (3.81). Then use orthonormality of the basis (3.78) and apply the inner product definition (3.83).

gularity at the origin $r = \rho^{-1} = 0$. So we found:

The metric solution to $\mathcal{N} = 2$ *supergravity in four-dimensions* (3.92) *has a horizon located at the origin* $r = 0$. This implies that the metric (3.92) describes a supersymmetric black hole that is also called $\frac{1}{2}$-BPS black hole. Plugging this result back into the metric, we can also obtain the near horizon geometry of this metric as

$$ds^2 = \frac{r^2}{\left|Z(\Gamma)\right|^2_{\text{min}}} dt^2 + \frac{\left|Z(\Gamma)\right|^2_{\text{min}}}{r^2} dr^2 + \left|Z(\Gamma)\right|^2_{\text{min}} d\Omega_2^2 \,. \tag{3.107}$$

This leads us to some interesting conclusions. The first is the near horizon geometry of such a supersymmetric black hole is $AdS_2 \times S^2$, the very familiar near horizon metric for all known extremal black holes in four dimensions. Second, one can simply read the Bekenstein-Hawking entropy by using "area-entropy" law. It is

$$S(\Gamma) = \pi \left|Z(\Gamma)\right|^2_{\text{min}} \,. \tag{3.108}$$

We emphasis here that the entropy only depends on the electromagnetic charge of the black hole Γ. There is no dependency to the arbitrary values of the scalars at infinity ($t^A(\infty)$). One may recall that the same conclusion is true for entropy of the Reissner-Nordström black hole (2.13) with (2.25), a solution to the Einstein-Maxwell theory (2.8) whose entropy is uniquely determined by charges. As we mentioned before, the main factor that makes $\mathcal{N} = 2$ SUGRA much more involved than the Einstein-Maxwell is the presence of the scalar moduli t^A and their coupling to the gauge fields. However, thank to the attractor mechanism that fixes scalar values at the attractor point, these complications diminishes considerably near the black hole horizon. That is why we get back some familiar results shared with much simpler theories such as EM. So it is true if one says:

At the attractor point ($\rho \to \infty$) *where the moduli scalars take their fixed values, the* $\mathcal{N} = 2$ *supergravity in four dimensions reduces to Einstein-Maxwell theory.*

Finally, let us now find the fixed value of scalar moduli at infinity in terms of the electromagnetic charge Γ. Plugging $e^{-C} \to \rho\left|Z(\Gamma)\right|_{\text{min}}$ back into the integrated BPS equation (3.104) and neglecting the second term in the RHS, we get

$$2\mathrm{Im}\Big[\overline{Z}\big(\Gamma, t^A(\infty)\big)V\big(t^A(\infty)\big)\Big] = -\Gamma \ , \tag{3.109}$$

which can be solved for $t^A(\infty)$. Once again, we see that the charge Γ determines the value of the scalars at the horizon $r = 0$ independently from their arbitrary boundary values at $r \to \infty$.

3.4.3. Solutions to the Attractor Flow Equations

Determined till now are some ansatz and asymptotic behavior of the fields according to our desired symmetries. In addition, we could achieve some valuable information about general properties of the supergravity solutions by studying the attractor flow equations near horizon. However, we still like to find explicit form of all supergravity fields in all regions. To do so we will follow techniques applied in [23] and [80] to first redefine all fields in terms of a single real function $\Sigma(r)$. It is in fact the warp factor appears later in the metric, but it is common to be called "the entropy function". After having all fields rewritten in terms of the entropy function, we will find Σ itself using the attractor flow equations. However, as we will see from the final result, this function is highly dependent on the geometry of the internal manifold on which we compactified ten-dimensional supergravity. In the case of Calabi-Yau three-fold, this function was first found in [81].

Let us go back to the integrated version of the attractor flow equations (3.104) and define a set of master harmonic functions $H(\rho)$ as

$$H(\rho) := \Gamma\rho - 2\mathrm{Im}\big[e^{-i\alpha}V\big]_{\rho=0} \ , \tag{3.110}$$

so that one can write the LHS of the equation as

$$2e^{-C}\mathrm{Im}\big[e^{-i\alpha(H,t^A)}V(t^A)\big] = -H \ . \tag{3.111}$$

We note that $H(\rho)$ also carry a symplectic index as well as Γ and V. Moreover, we treat $\alpha(\Gamma, t^A)$ and $\alpha(H(\rho), t^A)$ the same since ρ dependency of the central charge $Z(\Gamma, t^A)$ can be absorbed in its size that leaves the argument α intact when we switch from Γ to $H(\rho)$. Taking intersection product of (3.111) with $\overline{V}$ we obtain

$$e^{-C(H)} = \rho\mathrm{Re}\big[Z\big(\Gamma, t^A(H)\big)\big] = \big|Z\big(H, t^A(H)\big)\big| \tag{3.112}$$

where we also used (3.78) and expansion of the central charge in that basis (3.79) [41] . Plugging this back in (3.111) reads

$$2\mathrm{Im}\big[\overline{Z}(H, t^A)V(t^A)\big] = -H \ . \tag{3.113}$$

Now, one may recall the symplectic vector $V(t^A)$ in even cohomology basis $\{D_{\tilde{A}}, D^{\tilde{A}}\}$ given by (3.77). So from the first $h^{(1,1)} + 1$ components of (3.113) one reads

$$\begin{aligned} \mathrm{Im}\big[\overline{Z}(H, t^A)\big] &= \sqrt{\Omega/3}\ H^0 \\ \mathrm{Im}\big[\overline{Z}(H, t^A)t^A\big] &= \sqrt{\Omega/3}\ H^A \ . \end{aligned} \tag{3.114}$$

These two relations express imaginary parts of $\overline{Z}(H, t^A)V^{\tilde{A}}$ in terms of their corresponding harmonic functions. If we find a way to do the same for their real part as well, then these equations can be solved for the scalars t^A in terms of H^A. Seeking such relations, we first need to define the entropy function $\Sigma(H)$ by

$$\Sigma(H) := e^{-2C(H)} = \big|Z\big(H, t^A(H)\big)\big|^2 \ . \tag{3.115}$$

Then, by taking intersection product of (3.111) this time with $\nabla_{\tilde{A}}V$ and following some

[41] There are two delicate points in getting the last equality. The first is ρ being absorbed inside $|Z|$ as we mentioned before. The second is $Z\big(\Gamma\rho\big) = Z\big(H(\rho)\big)$ simply because $Z(V) = \langle V, V\rangle = 0$.

tricks from [80], we obtain

$$\mathrm{Re}\big[\overline{Z}(H,t^A)\big] = \frac{-1}{4}\sqrt{\Omega/3}\,\frac{\partial\Sigma(H)}{\partial H_0}$$
$$\mathrm{Re}\big[\overline{Z}(H,t^A t^A)\big] = \frac{1}{4}\sqrt{\Omega/3}\,\frac{\partial\Sigma(H)}{\partial H_A}\ . \tag{3.116}$$

Finally, from (3.113) and (3.116) we can express scalar moduli t^A in terms of harmonic functions and warp factor Σ as

$$t^A = \frac{H^A - i\partial_A\Sigma}{H^0 + i\partial_0\Sigma}\ . \tag{3.117}$$

The next step is to determine gauge fields of the vector multiplet such that the field strengths $F^a = dA^a$ is given by (3.95). We have

$$\diamond\,\Gamma = -2\mathrm{Re}\Big[\partial_\rho\big(e^C e^{-i\alpha}V\big)\Big] \tag{3.118}$$

from which we simply read

$$A^a := \begin{pmatrix} 2e^C\ \mathrm{Re}\big[e^{-i\alpha}V\big]dt \\ \\ -\Gamma^{\tilde{A}}\ \cos\theta d\varphi \end{pmatrix} \tag{3.119}$$

where now one can substitute $\mathrm{Re}\big[e^{-i\alpha}V\big]$ from (3.116) and also $e^C \to \frac{1}{\sqrt{\Sigma}}$ to finally obtain

$$A^a := \begin{pmatrix} -\big(\partial_{H_0}\ln\Sigma\big)dt \\ \\ \big(\partial_{H_A}\ln\Sigma\big)dt \\ \\ -p^A\cos\theta d\varphi \\ \\ -p^0\cos\theta d\varphi \end{pmatrix}\ . \tag{3.120}$$

Having this result, practically we have the explicit form for the metric, scalar moduli and $U(1)$ gauge fields in term of a single real function that is entropy function $\Sigma(H)$ yet to be determined. For more convenience, here we summarize explicit forms of the harmonic function $H(r)$ all the filed content of four-dimensional $\mathcal{N}=2$ supergravity

$$\begin{aligned} H &= \frac{\Gamma}{r} - 2\mathrm{Im}\big[e^{-i\alpha}V\big]_{r\to\infty} \ , \\ ds^2 &= -\Sigma^{-1}dt^2 + \Sigma dx^i dx^i \ , \\ t^A &= \frac{H^A - i\partial_A\Sigma}{H^0 + i\partial_0\Sigma} \ , \\ A^0 &= -\big(\partial_{H_0}\mathrm{ln}\Sigma\big)dt - p^0\mathrm{cos}\theta d\varphi \ , \\ A^A &= \big(\partial_{H_A}\mathrm{ln}\Sigma\big)dt - p^A\mathrm{cos}\theta d\varphi \ , \end{aligned} \tag{3.121}$$

with gauge fields components have been written in a vector notation.

The entropy function

From its definition given by (3.115), we know the near horizon behavior of the entropy function Σ. It is

$$\Sigma(\Gamma\rho)\mid_{\rho\to\infty} = \big|Z\big(\Gamma, t^A(\Gamma)\big)\big|^2_{min}\rho^2 = \frac{1}{\pi}S(\Gamma)\rho^2 \ , \tag{3.122}$$

where to get the second equality we have used (3.108). As showed in [80], the Σ is a homogeneous function of degree 2, i.e., $\Sigma(\lambda H) = \lambda^2\Sigma(H)$ which implies that

$$\Sigma(\Gamma)\mid_{\rho\to\infty} = \frac{1}{\pi}S(\Gamma) \qquad \Longleftrightarrow \qquad \Sigma(H)\mid_{\rho\to\infty} = \frac{1}{\pi}S(H) \ . \tag{3.123}$$

As remarked earlier around (3.104), there are $2h^{(1,1)}+1$ attractor flow equations. However, we only used $h^{(1,1)}+1$ of them in (3.114) and (3.116) to be able to read the scalars t^A as (3.117). So we are left with $h^{(1,1)}$ equation yet unsolved that can be used to determine the entropy function Σ in terms of the Harmonic function $H(r)$ not only at the near horizon, but everywhere. These sophisticated relations that have been found in [81] for the first time exhibit a deep correlation between the warp factor Σ in the metric and properties of the internal manifold. It was in the same reference where the explicit form of Σ was given as following

$$\begin{aligned} \Sigma(H) &= \sqrt{\frac{Q^3(H) - L^2(H)}{\left(H^0\right)^2}} \qquad \text{with} \\ L(H) &= H_0\left(H^0\right)^2 + \frac{1}{3}D_{ABC}H^AH^BH^C - H^AH_AH^0 \ , \\ Q^3(H) &= \left[\frac{1}{3}D_{ABC}y^A(H)y^B(H)y^C(H)\right]^2 \qquad \text{where} \\ D_{ABC}y^Ay^B &= -2H_CH^0 + D_{ABC}H^AH^B \ . \end{aligned} \tag{3.124}$$

So in summary, one first need to find a solution for $y^A(H)$ from the last equation above which can be done easily in some special cases such as having $h^{(1,1)} = 1$ or $H^0 = 0$. Otherwise, there may not be a analytic solution to this equation. Having y^A one can compute two auxiliary functions $L(H)$ and $Q^3(H)$ to be substituted in the first equation that determines Σ in terms of H.

So in practice, what we have obtained here via solving the attractor flow equations (3.104) is the scalar moduli t^A, the gauge fields $A^{\tilde{A}}$ and the entropy function Σ in terms of the master Harmonic function H which by itself is determined by electromagnetic charges $\Gamma = \left(\Gamma^{\tilde{A}}, \Gamma_{\tilde{A}}\right)$ and also the asymptotic values of the scalars $t^A(\infty)$. We are going to present the whole solution inside a single Table 3.4 for more convenience and later applications.

3.5. Multi-Centered Black Holes

In the last section, we reviewed supersymmetric black hole solutions to $\mathcal{N} = 2$ supergravity in $4D$. In summary, we picked proper ansatz for fields and then, by saturating BPS bound of the action we obtained attractor flow equations by solving which we found the solution for all contributing fields. As we remarked before, these BPS black hole solutions are similar to the Reissner-Nordström black holes in many aspects, even though the former is much more involved. In this section however, we

Table 3.4: The BPS black hole solutions of $\mathcal{N}=2$ SUGRA in $4D$.

$$\begin{aligned} ds^2 &= \frac{-1}{\Sigma}dt^2+\Sigma\; dx^i dx_i \;\;, \\ A^0 &= \frac{-L}{\Sigma^2}dt+\mathcal{A}^0_d \;\;, \\ A^A &= \frac{H^A L - Q^{3/2}y^A}{H^0\Sigma^2}dt+\mathcal{A}^A_d \\ t^A &= \frac{H^A}{H^0}+\frac{y^A}{Q^{3/2}}\Big(i\Sigma-\frac{L}{H^0}\Big)\;. \end{aligned} \tag{3.125}$$

$$d\mathcal{A}^{\tilde{A}}_d=\star_3 dH^{\tilde{A}}\;. \tag{3.126}$$

$$H=\frac{\Gamma}{r}-2\mathrm{Im}[e^{-i\alpha}V]_{r\to\infty}\;, \tag{3.127}$$

$$\begin{aligned} &\Sigma=\sqrt{\frac{Q^3-L^2}{\big(H^0\big)^2}}\;\;, \\ &L=H_0\big(H^0\big)^2+\frac{1}{3}D_{ABC}H^AH^BH^C-H^AH_AH^0\;\;, \\ &Q^3=\Big(\frac{1}{3}D_{ABC}y^Ay^By^C\Big)^2\;\;, \\ &D_{ABC}y^Ay^B=-2H_CH^0+D_{ABC}H^AH^B\;. \end{aligned} \tag{3.128}$$

are going to find a very unique solution to the $4D$ supergravity, i.e., *multi-centered black holes*. As we will see with much more details, they are bound states of single-centered BPS black holes that we discussed in the last section. They are also quite unique, in the sense that and also unique in a sense that they do not have any non-supersymmetric analog. Moreover, they are not static but stationary solutions. The rotation has electromagnetic origin and is encoded in an one-form inside the metric which plays a very noticeable role in asymptotic geometry of spacetime. This will be discussed later in chapter 4.

To show that having bound states of BPS black holes is possible, we first probe a typical supersymmetric black hole by a test dyon. We will observe that potential experienced by the test particle becomes minimized for a specific equilibrium distance between the test particle and the probed black hole. There is also a condition to have the bound state and that is the central charge of the test particle and the black hole need to get aligned at the equilibrium distance. This is in fact a condition on the scalars which divides the solution space into two regions whereas the bound state can exist only in one region. The surface that splits these two regions is called "*the wall of marginal stability*" passing through which the bound state will decay. The whole argument can be generalized to the cases with more than two dyons. Again, there is a set of stability conditions that restrict intercenter distances. Decay of these bound states is also possible. The most common case is decay of a multi-centered solution into only two clusters. There are also two special cases of these bound states. In the so-called "*threshold stability*", various centers can get infinitely far away from each other while in another case called "*the scaling solutions*", some centers can get arbitrarily close to each other. Our focus in this thesis will be on the later case whose asymptotic behavior will be studied in detail at the end of this chapter and also chapter 4.

3.5.1. Probing a Single-Centered Solution

In this section, we are going to study the dynamics of a test dyon with a supersymmetric black hole in the back ground. In probing the black hole with another massive charged particle, we calculate the potential experienced by the probe and show that it can have some minimums. This is an evidence showing that the bound states between supersymmetric black holes can exist. Such configurations are called "*multi-centered black holes*" and will be discussed later. First, let us have a closer look at the simplest case of these black holes, a bound state consists of only two BPS black holes [82].

Previously, we wrote the worldvolume action for a BPS particle in an arbitrary $\mathcal{N} = 2$ supergravity background (3.88). Here, we set the background to be of a BPS black hole whose explicit form is given in the Table 3.4. We rewrite the action here for a test particle carrying charge Γ_p

$$\mathcal{S}_{\text{wl}} = -\int \big|Z(\Gamma_p, t^A)\big| ds + \frac{1}{2}\int \langle \Gamma_p, A_\mu \rangle dx^\mu \ , \tag{3.129}$$

with the line element ds is now determined by the metric given in (3.121) pulled back to the worldline of the test particle with coordinate σ. We choose to work in the *static gauge* in which the worldline parameter coincides with the time coordinate of the background, so we have $\sigma = t$. Moreover, we restrict ourselves to a static probe with fixed spatial position so that it describes a bound state of two charged black holes with a fixed relative distance. It allows us to set $\partial_t r = \partial_t \theta = \partial_t \varphi = 0$ which simplifies the line-element to $ds = \sqrt{-g_{\mu\nu}\frac{dx^\mu dx^\nu}{d\sigma^2}}d\sigma = e^C dt$.

For calculating the second term, we need to read the $U(1)$ gauge fields A^a from (3.119) while Γ is the black hole charge and accordingly, α is the argument of its central charge $Z(\Gamma, t^A)$. With the choices we made, the test particle has no kinetic energy, meaning that one can read the potential energy experienced by the test particle right after calculating the action (3.129). The result, after doing some algebra, will be

$$U(r) = \frac{2}{\sqrt{\Sigma}}\big|Z(\Gamma_p, t^A)\big| \sin^2\big(\frac{\alpha - \alpha_p}{2}\big) \quad \text{with} \quad \alpha_p = \arg\big(Z(\Gamma_p, t^A)\big) \ . \tag{3.130}$$

This potential depends only on the radial direction r that comes from radial dependence of the entropy function $\Sigma(r)$ and also background scalar moduli $t^A(r)$ that makes α,

α_p and $Z(\Gamma_p, t^A)$ radial dependent as well.

Now the question is where the possible minima of this potential are located. The first thing to be noticed is that the potential (3.130) is positive everywhere which leave us only zero's of the potential as possible stable minima. Obviously, the potential becomes minimized when the test particle reaches the black hole. To see this, one needs to take the near horizon limit of $U(r)$. Recalling that in this limit we have $\Sigma \mid_{r=0}= \frac{1}{\pi} S(\Gamma) r^2$ and also the fact that the scalars take some fixed values, say t_*^A, we conclude

$$U(r) \sim \frac{2}{\sqrt{S}} \big| Z(\Gamma_p, t_*^A) \big| \ \sin^2 \big(\frac{\alpha_* - \alpha_{p*}}{2} \big) r + \mathcal{O}(r^2) \ , \tag{3.131}$$

for some constant α_* and α_{p*}. As one would expect, in this limit the potential decrease linearly with r until it vanishes at zero relative distance. We can also check the value of this potential at infinity where the metric becomes flat ($\Sigma \sim 1$) and the boundary value of the scalars are given by some constants t_∞^A. We then obtain

$$U(\infty) = 2 \big| Z(\Gamma_p, t_\infty^A) \big| \ \sin^2 \big(\frac{\alpha_\infty - \alpha_{p\infty}}{2} \big) \ , \tag{3.132}$$

which is a positive definite constant. From the form of the potential (3.130) and also its value at these two limits, it becomes clear that $U(r)$ can be minimized at some finite distance r_{ms} if and only if we have

$$\alpha(r_{ms}) = \alpha_p(r_{ms}) \ . \tag{3.133}$$

One can think of this as a constraint defining a hypersurface inside the moduli space. More precisely, this single real equation is satisfied only if the central charge of the black hole and the test particle take the same arguments. Via central charge dependency on the moduli scalars, this relation can be interpreted as a constraint on the attractor flow restricting it to pass through all points in the moduli space at which $\alpha(r_{ms}) = \alpha_p(r_{ms})$. The equality (3.133) defines "*the marginal stability condition*" from which the subscript "ms" originates and accordingly, the hypersurface is called "*the wall of marginal stability*".

To calculate the equilibrium distance r_{ms}, once again we go back to the attractor flow equation (3.104). Taking the intersection product with Γ_p, we get

$$2e^{C}\text{Im}\big[e^{-i\alpha}Z_p\big] = \frac{-\langle\Gamma_p,\Gamma\rangle}{r} + 2\text{Im}\big[e^{-i\alpha}Z_p\big]_{r\to\infty} , \tag{3.134}$$

where $Z_p \equiv Z(\Gamma_p, t^A_\infty)$. Now at $r = r_{ms}$ where we demand the marginal stability condition to be satisfied, the RHS of this equality vanishes [42] . So we conclude

$$r_{ms} = \frac{\langle\Gamma_p,\Gamma\rangle}{2\text{Im}\big[e^{-i\alpha}Z_p\big]_{r\to\infty}} . \tag{3.135}$$

Restricting r_{ms} to be positive and so physically meaningful results in a constraint $\langle\Gamma_p,\Gamma\rangle\,\text{Im}\big[e^{-i\alpha}Z_p\big]_{r\to\infty} > 0$ that splits up the whole Calabi-Yau moduli space into two regions. The marginal stability condition (3.133) is only satisfied in one of these two regions where r_{ms} gets positive values, and fails to be fulfilled in the other region. The barrier between these two regions is nothing but the wall of marginal stability on which we have $\alpha = \alpha_p$ and so a stable bound state between the test dyon and the BPS black hole separated by r_{ms} can exist. In other words, having a bound state between two BPS black holes is possible if and only if the moduli space of the internal manifold (spanned by the scalars) admits having a region in which $\langle\Gamma_p,\Gamma\rangle\,\text{Im}\big[e^{-i\alpha}Z_p\big]_{r\to\infty} > 0$. Note that for given charges Γ and Γ_p, there are still free parameters that alter the division of the moduli space; they are the asymptotic values of the scalars.

In addition, one can imagine the following scenario: we start from some scalars boundary values for which having bound states is possible. Then, we gradually move towards the wall of marginal stability. At the same time, r_{ms} grows gradually until we hit the wall of marginal stability where it becomes infinite [43] . This implies that by passing through the wall, the bound state will decay into two free BPS black holes

[42] In fact we have $e^{-i\alpha_{ms}}Z_p = |Z_p|$ whose imaginary part is zero.

[43] Please note that this argument works because we assumed $\langle\Gamma_p,\Gamma\rangle$ is a non-zero constant and so

sitting infinitely far away from each other.

3.5.2. From Single-Centered to Multi-Centered Solutions

We just saw in the last section that bound states of two supersymmetric black holes can exist. This observation makes us curious about a possible generalization: bound states of a bunch of BPS black holes. Here we will follow the same steps like those that took us to the explicit form of the single-centered supersymmetric black hole solution in section 3.4. So, we are going to search for proper ansatz for all fields according to the desired symmetries of the solution. Then, substitution of these ansatz inside the supergravity action (3.91) yields the BPS equations. Solving these first order differential equations is equivalent to finding Killing spinors.

The fields ansatz

Here we are going to write the most general ansatz for the fields such that they preserve specific symmetries. Like the single-centered case, here we are interested in stable, non evolving solutions supported by a timelike Killing vector. This requirement restricts all fields to be time independent. However, unlike single-centered case, multi-centered solutions are not static but stationary. As discussed in a footnote on page 20, the difference originates from the fact that the multi-centered solution does not enjoy spherical symmetry. More precisely, in the case of two centers we still have an azimuthal symmetry but for more than two centers, no spherical symmetry is preserved.

Breaking the spherical symmetry can be done by adding off-diagonal elements to the metric. As suggested in [74], [75], the most general ansatz for a stationary metric

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at the wall of marginal stability

$$\langle \Gamma_p, \Gamma \rangle \, \mathrm{Im}\big[e^{-i\alpha} Z_p\big]_{r\to\infty} \to 0 \qquad \Longleftrightarrow \qquad \mathrm{Im}\big[e^{-i\alpha} Z_p\big]_{r\to\infty} \to 0 \tag{3.136}$$

or simply $r_{ms} \to \infty$.

is

$$ds^2 = -e^{2C}(dt+\omega)^2 + e^{-2C}dx^i dx^i \ , \tag{3.137}$$

where $\omega = \omega_i dx^i$ is an one-form with only spatial components. We will discuss about it in much more detail as it is quite essential for our argument in the next chapter. Here, because of lack of spherical symmetry, the new function C and also ω_i depend on all spatial coordinates. However, the time independency, that is the only unbroken bosonic symmetry, dictates these functions to stay t-independent.

About the proper ansatz for the $U(1)$ gauge fields, here we just say that the electric part remains the same as the single-centered case. It can be found by looking at the upper half of A^a matrix representation given in (3.119). The magnetic part however needs to be modified as we will see very soon.

The BPS equations

The next step, as mentioned before, is to plug these more general ansatz back into the $\mathcal{N} = 2$ SUGRA action, rewrite it as a complete square term and finally read BPS equations. We did this step-by-step in the case of single-centered solution. However, doing the same here would be much more complicated because of having less symmetries and so more general ansatz. As such, we only bring the final results first were found in [83], [23]. Here they are

$$H: \ = \ -2e^{-C}\mathrm{Im}\big[e^{-i\alpha(H,t^A)}V(t^A)\big] \ , \tag{3.138}$$

$$A \ = \ \mathcal{A}_{\mathrm{d}} + 2e^{C}\ \mathrm{Re}\big[e^{-i\alpha}V\big]dt \quad ; \quad \star_3\, d\mathcal{A}_{\mathrm{d}} = dH \ , \tag{3.139}$$

$$\star_3 d\omega \ = \ \langle dH, H\rangle \ . \tag{3.140}$$

As one may have already noticed, the function H is substantial for other fields to be determined. First, we note that the BPS equation (3.138) for $H(r)$ is quite the same as the single-centered case (3.111), but it should be expressed in its multi-centered version.

Hence, its form can be guessed immediately as a natural multi-charge generalization of the harmonic function given by the first equation in (3.121). Consider a group of charges distributed in $\mathbb{R}^3$. The vector $\vec{r}_n$ denotes the position of nth center carrying charge Γ_n while $\vec{r}$ shows the observer point. Then the total harmonic function of the system is

$$H = \sum_n \frac{\Gamma_n}{r_n} - 2\mathrm{Im}\big[e^{-i\alpha}V\big]_\infty \qquad \text{with} \qquad r_n := |\vec{r} - \vec{r}_n| \ . \tag{3.141}$$

As a consistency check one can quickly show that the BPS equation we have here in (3.138-3.140) will reduce to their single-centered counterparts (3.111),(3.95) and (3.118) by setting $n = 1$ in the multi-centered harmonic function (3.141). More specifically, the RHS of (3.140) identically vanishes in single-centered case.

In order to have a *stable* multi-centered solution, the relative distance between centers has to be tuned in such a way that the whole system reaches an equilibrium state. In other words, stability of the bound state imposes some constraints on mutual distances between centers. One may recall that this equilibrium distance is r_{ms} in the case of bound state of only two centers. Here, these constraints appear as an integrability condition which is derivable from relation (3.140). As we explained in more detail in Appendix B of this thesis, hitting both sides of the last equation by $d\star$ leads to

$$\langle \triangle H, H \rangle = 0 \ . \tag{3.142}$$

Having the explicit form of the harmonic function H, we can go further. We first note that (3.141) results in

$$\triangle\, H = \sum_n \Gamma_n \delta^3(\vec{r} - \vec{r}_n) \ . \tag{3.143}$$

Substituting this inside the integrability condition (3.142) gives

$$\langle \Gamma_n, H \rangle \mid_{\vec{r}=\vec{r}_n} = 0 \qquad \forall n \tag{3.144}$$

that can be written immediately as

$$\sum_{m \neq \underline{n}} \frac{\langle \Gamma_n, \Gamma_m \rangle}{|\vec{r}_n - \vec{r}_m|} = \langle h, \Gamma_n \rangle \qquad \forall n \qquad \text{with} \qquad h := -2\mathrm{Im}[e^{-i\alpha} V]_\infty \tag{3.145}$$

where $\sum_{m \neq \underline{n}}$ is a summation over all $m \neq n$ and $\alpha = \arg Z = \arg \sum_m Z_m$. This result implies that the mutual distance between each pair $|\vec{r}_n - \vec{r}_m|$ is not arbitrary but constrained by (3.145). As such, the system of charges really exhibit a bound state if this set of constraints is satisfied. Once again, one realizes that the existence of these bound states absolutely depend on asymptotic values of the scalars t^A appears here via the constant h. We finish this part by saying that the integrability condition reduces to the equilibrium distance between two centers (3.135) we found earlier by probing a single-centered black hole.

The one-form ω

On the way of solving BPS equations explicitly, let us take our first step by finding the one-form ω whose definition is given by (3.140). In fact, ω is of remarkable importance for us since it is a characteristic property of to-be-determined multi-centered solutions that distinguishes it from more common single-centered black holes. As we will see very soon, there is no explicit expression for ω in general case with more than two centers. The way to think of it is as the summation over contributions of all possible pairs of charges. As such, here we are going to first solve (3.140) for a two-center case explicitly and then write general form of ω as a summation.

Consider two charges Γ_1 and Γ_2 located at $(0, 0, l)$ and $(0, 0, -l)$ respectively [44]

[44] Note that without loss of generality, we can always drag the origin to the middle point of these two centers and also rotate the coordinate system such that the connecting line of the two centers lies on the z-axis.

. From the probe analysis we did in section 3.5.1, we realize that to have these two centers in a stable bound state l has to take a specific value, i.e

$$l = \frac{r_{ms}}{2} = \frac{\langle \Gamma_p, \Gamma \rangle}{4\mathrm{Im}\left[e^{-i\alpha} Z_p\right]_{r\to\infty}} \ . \tag{3.146}$$

Now, writing H for these two centers and from (3.140) we have

$$\begin{aligned} H &= \frac{\Gamma_1}{r_1} + \frac{\Gamma_2}{r_2} + h \qquad \Longrightarrow \\ d\omega &= \langle \Gamma_1, \Gamma_2 \rangle \star_3 \left[\frac{1}{r_2} d\Big(\frac{1}{r_1}\Big) - \frac{1}{r_1} d\Big(\frac{1}{r_2}\Big) + \frac{1}{2l}\Big(\frac{dr_1}{r_1^2} - \frac{dr_2}{r_2^2}\Big)\right] . \end{aligned} \tag{3.147}$$

where we have defined h earlier in (3.145). To obtain the second relation we note that

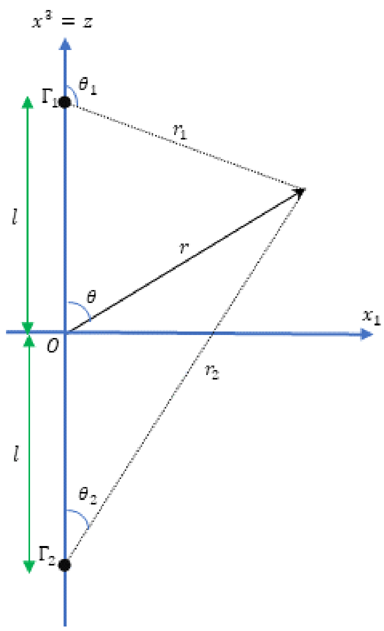

Figure 3.3: A two-charge system consists of Γ_a and Γ_2 located at $(0, 0, l)$ and $(0, 0, -l)$ respectively.

$\star^2 = 1$ for any even form. The subscript of $\star$ denotes that the star operator operates on $\mathbb{R}^3$. In spherical coordinate, we show the observant point by (r, θ, φ) while $(r_1, \theta_1, \varphi_1)$ and $(r_2, \theta_2, \varphi_2)$ are the relative coordinates of Γ_1 and Γ_2 with respect to the observant point, as has been depicted in Figure 3.3. So these three sets of coordinate are related as

$$r_1^2 = r^2 + l^2 - 2rl\cos\theta \quad , \quad r_1\cos\theta_1 = r\cos\theta - l \ , \tag{3.148}$$
$$r_2^2 = r^2 + l^2 + 2rl\cos\theta \quad , \quad r_2\cos\theta_2 = r\cos\theta + l \ .$$

Having these defined and doing some algebra on (3.147), one obtains

$$d\omega = \frac{\langle\Gamma_1,\Gamma_2\rangle}{2l} d\left(\frac{r^2-l^2}{\sqrt{r^4+l^4-2r^2l^2\cos 2\theta}} + 1 - \cos\theta_1 + \cos\theta_2\right) \wedge d\varphi \ , \tag{3.149}$$

from which one can easily read

$$\omega = \frac{\langle\Gamma_1,\Gamma_2\rangle}{2l} \left(\frac{r^2-l^2}{\sqrt{r^4+l^4-2r^2l^2\cos 2\theta}} + 1 - \cos\theta_1 + \cos\theta_2\right) d\varphi \ . \tag{3.150}$$

One can still generalize (3.147) to the cases with more than two centers by replacing $1 \to m$ and $2 \to n$ and then sum over $m < n$ to prevent over counting. Although, doing so does not lead to a compact form for ω in a general multi-centered case, there is an exception; a special configurations in which all charges lie on the z-axis so that the axial symmetry survive. Then for any pair Γ_a and Γ_b we can write

$$\omega_{mn} = \frac{\langle\Gamma_m,\Gamma_n\rangle}{2l} \left(\frac{r^2-l^2}{\sqrt{r^4+l^4-2r^2l^2\cos 2\theta}} + 1 - \cos\theta_m + \cos\theta_n\right) d\varphi \ , \tag{3.151}$$

and the total ω simply is $\omega = \sum_{m<n} \omega_{mn}$. In more general case with charges distributing randomly we need to consider all possible pairs, compute their contribution to the total ω and finally take the summation, i.e.,

$$\omega = \sum_{m<n} \omega_{mn} \ . \tag{3.152}$$

The entropy function and gauge fields

Having the last equation of the set of BPS equations solved for ω, now the last step is to determine the explicit forms of the entropy function Σ, the scalars t^A and also the gauge fields $A^{\tilde{A}}$ in terms of the Harmonic function $H(r)$ given by (3.141). Luckily, we do not have that much to do though; the form of warp factor $C(r)$ in terms of the entropy function Σ (3.115) and also the applied technique to determine scalars t^A as (3.117) are both extensible after replacing the multi-centered version of the Harmonic function H.

Our interpretation of the BPS bound states as "multi-centered" black holes emerges quite manifestly in the near horizon behavior of the entropy function where we demand

$$\lim_{r_n \to 0} \Sigma(H) = \pi S(\Gamma_n) r_n^2 \tag{3.153}$$

which means close to each of these charges, the observer encounters a single-centered black hole whose entropy is given by the minimum value of its corresponding central charge, i.e., $\left|Z_n\right|^2_{\text{min}}$ (see (3.122)).

Finally, as one sees from (3.139) the electric part of the gauge fields $A^{\tilde{A}}$ is the same as their single-centered counterparts given in (3.121). This is a consequence of the fact that the diamond operator $\diamond$ given in (3.118) acts algebraically and so do the electric components of the gauge field. However it is not the case for the magnetic components defined as $\mathcal{A}_{\text{d}}$. We already know the contribution of a single magnetic charge sitting at the origin, it is simply given by $A^{\tilde{A}} = -\Gamma^{\tilde{A}} \cos\theta d\varphi$. This can be extended to the multi-charge case assuming that each charge Γ_n defines its own spherical coordinate (θ_n, φ_n) while the corresponding charge has located at the origin. Then one can write

$$\mathcal{A}^{\tilde{A}}_{\text{d}\ n} = -\Gamma^{\tilde{A}}_n \cos\theta_n d\varphi_n \ , \tag{3.154}$$

that adds up to the total gauge field as

$$\mathcal{A}^{\tilde{A}}_{\text{d}} = \sum_n \mathcal{A}^{\tilde{A}}_{\text{d}\ n} \ . \tag{3.155}$$

like before, it is possible to write the whole expression in a single spherical coordinate, however it is a difficulty that would not bring about any new information. So we just ignore doing so and keep it like (3.155).

$\frac{1}{2}$-BPS solutions to $\mathcal{N} = 2$ supergravity in four dimensions

Here, we bring all necessary relations to determine a specific BPS solution in a single table for more convenience and clearness. It is now easier to compare multi-centered solutions with the simpler single-centered ones given in Table 3.5.

Having the explicit form of the multi-centered black hole solution, one delicate point is still worth mentioning. As one can conclude in the very first glance, the harmonic function $H(r)$ has a substantial role in finding these sophisticated solutions while it is rather a simple function itself. Its first term is quite analogous to the simple Coulomb potential of an electric charge, but it has been generalized in such a way that it now contains both electric and magnetic charges. There is also an additional constant term that clarifies the role of scalars t^A in the theory. Its existence results in an important conclusion: *the vacuum state of the theory is determined by the asymptotic value of the moduli scalars.*

3.5.3. Angular Momentum

As remarked before while discussing the one-form ω, a characteristic property that distinguishes the multi-centered black holes from their single-centered cousins is that they are inevitably rotating and so are stationary solutions. In other words, unlike Kerr black holes whose irrotational version also exists, i.e., Schwarzschild black holes, here there is no irrotational version of these multi-centered solutions. In this short subsection we are going to discuss the angular momentum of these solutions, some of its properties as well as its origin.

Looking at the multi-centered metric (3.156), one would realize that the angular momentum of the bound black holes is obtainable from the one-form ω. In fact, for

Table 3.5: BPS multi-centered black hole solutions of $\mathcal{N} = 2$ SUGRA in $4D$.

$$\begin{aligned} ds^2 &= \frac{-1}{\Sigma}\Big(dt+\omega\Big)^2 + \Sigma\; dx^i dx_i \qquad (3.156) \\ A^0 &= \frac{-L}{\Sigma^2} dt + \mathcal{A}^0_{\rm d} \ , \\ A^A &= \frac{H^A L - Q^{3/2} y^A}{H^0 \Sigma^2}\Big(dt+\omega\Big) + \mathcal{A}^A_{\rm d} \ , \\ t^A &= \frac{H^A}{H^0} + \frac{y^A}{Q^{3/2}}\Big(i\Sigma - \frac{L}{H^0}\Big) \ . \\ d\omega &= \star_3 \Big\langle dH, H \Big\rangle \ , \qquad (3.157) \\ d\mathcal{A}^{\tilde{A}}_{\rm d} &= \star_3 dH^{\tilde{A}} \ . \end{aligned}$$

$$H = \sum_n \frac{\Gamma_n}{r_n} - 2{\rm Im}[e^{-i\alpha} V]_{r\to\infty} \ , \qquad (3.158)$$

$$\sum_{n\neq \underline{m}} \frac{\langle \Gamma_m, \Gamma_n \rangle}{\mid r_m - r_n \mid} = \langle h, \Gamma_m \rangle \quad ; \qquad (3.159)$$

$$\begin{cases} h := -2{\rm Im}[e^{-i\alpha} V]_{r\to\infty} \ , \\ \alpha := {\rm arg} Z_T = {\rm arg} \sum_n \Big\langle \Gamma_n, V(t) \Big\rangle \ . \end{cases}$$

$$\Sigma = \sqrt{\frac{Q^3 - L^2}{\big(H^0\big)^2}} \ , \qquad (3.160)$$

$$L = H_0 \big(H^0\big)^2 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^A H_A H^0 \ ,$$

$$Q^3 = \Big(\frac{1}{3} D_{ABC} y^A y^B y^C\Big)^2 \ ,$$

$$D_{ABC} y^A y^B = -2 H_C H^0 + D_{ABC} H^A H^B \ .$$

any given gravity solution, what an observer measures as the total angular momentum $\vec{J}$ at infinity is given by the asymptotic behavior of the off-diagonal terms of the corresponding metric. At $r \to \infty$ we have

$$\omega = 2\varepsilon_{ijk}\frac{J^i x^j dx^k}{r^3} + \mathcal{O}(r^{-2}) \qquad \text{with} \qquad i,j,k = 1,2,3 \tag{3.161}$$

from which one can extract $\vec{J}$. Obviously, the simplest case is that of two centers for which we have the explicit form of the ω given by (3.150). That leads to

$$\vec{J} = \frac{1}{2}\langle\Gamma_1,\Gamma_2\rangle\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}\ . \tag{3.162}$$

This result can be generalized to the cases with more than two centers as

$$\vec{J} = \frac{1}{2}\sum_{m<n}\langle\Gamma_m,\Gamma_n\rangle\hat{r}_{mn}\ , \tag{3.163}$$

where $\hat{r}_{mn} = \frac{\vec{r}_{mn}}{r_{mn}}$ is the unit vector pointed from mth to the nth center, i.e., $\vec{r}_{mn} = \vec{r}_m - \vec{r}_n$. The angular momentum of the multi-centered supersymmetric black holes posses some interesting properties that we are going to point some of them out here.

- Unlike rotational behavior of other rotating black holes, the rotation of multi-centered dyonic black holes have an electromagnetic origin. To see this more clearly, let us assume for a second that there is only one $U(1)$ gauge field (so we set $h^{(1,1)} = 0$) and two charged particles $\Gamma_1 = (g_1, e_1)$ and $\Gamma_2 = (g_2, e_2)$. Then we have $\langle\Gamma_1,\Gamma_2\rangle = -g_1e_2 + g_2e_1$ which is a quite familiar result from electromagnetic interaction between magnetic monopoles and electric charges.
- The electromagnetic origin of $\vec{J}$ implies that rotation is an intrinsic property of these black holes and so inevitable. This means that there is no static version of these solutions; they have to be stationary.
- As can be deduced from (3.163) immediately, the total angular momentum of the multi-centered black holes is a *topological* conserved charge whose size is given completely in terms of the intersection product of charges. More importantly, its

size is absolutely independent of the relative distance between centers at equilibrium r_{mn} and as such, it is also independent of the internal Calabi-Yau moduli.

- Previously, in 3.3.2.1 we discussed how one can think of these pointlike particles as higher dimensional D-branes wrapped on non-trivial cycles of the internal manifold, and how accordingly these generalized electromagnetic charges Γ^a can be interpreted as the winding number of the D-branes. This interpretation guarantees that Γ^a are all quantized meaning that the total angular momentum of bound black holes are quantized in half integer unites. Once again, to see this easier let us take the same example of a single gauge field. It is a well known fact from Dirac quantization of electromagnetic charges that their multiplication eg is quantized. This observation can be generalized to the case of more $U(1)$ gauge fields and the corresponding charges easily. Having half integer angular momentum becomes of special interest specifically when quantum properties of these black holes comes under investigation [82], [27].

3.5.4. Solution Space

Another characteristic property of the multi-centered cases which distinguishes them from single-centered solutions is that their *solution space* may be quite nontrivial. Firstly, having a solution space originates from the fact that the number of constraints which guarantee stability of a bound state of black holes is not enough to fix the intercenter distances, as we will see in a moment. In this subsection, we are going to first define more precisely what we call the solution space. We also discuss the situation in which a bound state of black holes can decay to smaller clusters. Then we will study two special cases in which different centers can get arbitrary far or come very close together.

The integrability condition, constraint equations and solution space

We recall how the integrability condition (3.142) emerged as a group of constraints restrict the relative distances between centers (3.145) . That is why multi-centered black holes recognized as bound states. For more convenience, we repeat these stability conditions here with more compact notation

$$\sum_{m\neq \underline{n}} \frac{\langle \Gamma_n, \Gamma_m \rangle}{r_{nm}} = 2\mathrm{Im}[e^{-i\alpha} Z_n]_\infty \qquad \forall n \ . \tag{3.164}$$

As defined before $r_{nm} = |\vec{r}_n - \vec{r}_m|$ and $\alpha = \arg Z = \arg \sum_m Z_m$. Having these constraints satisfied means that all centers take their positions in such a way that the total potential energy of the system is minimized and so the bound state is stable. First thing to note is that there is a trivial situation when all centers carry the same electromagnetic charge. Then the relative distances are not constrained at all because for any n and m we have $\langle \Gamma_n, \Gamma_m \rangle = 0$. Putting these trivial cases aside, from now on we consider having nonzero intersection product between charges.

Now, let us have a closer look at these constraint (3.164). Apparently, there are N constraints for N centers. However, they are not all independent, simply because the intersection product is antisymmetric under $m \leftrightarrow n$ and so sum of these constraints vanishes [45]

$$\sum_{m\neq n} \frac{\langle \Gamma_n, \Gamma_m \rangle}{r_{nm}} = 0 \ . \tag{3.166}$$

So practically there are only $N-1$ constraints. For $3N-3$ variables accounting for undetermined positions of N charges in $\mathbb{R}^3$. Note that because of the translational symmetry, three of $3N$ positions are really irrelevant, or equivalently one can get rid of these three coordinates via fixing center of mass position. This amounts to $2N-2$ degrees of freedom, meaning that each multi-centered solution (determined by charge positions and the asymptotic value of the scalar moduli t^A) can be thought of a single point in a $2N-2$ dimensional space which we are going to call "the solution space". As said before, such a manifold may have quite interesting topology. It is very impor-

[45] Note that at the same time, the RHS vanishes as well. According to the definition of α we have

$$\sum_n \mathrm{Im}[e^{-i\alpha} Z_n] = \mathrm{Im}\big[|Z|\big] = 0 \ , \tag{3.165}$$

with Z as the total central charge of the bound state.

tant to be noticed that the "shape" of the solution space depends *implicitly* on the values of scalar moduli at infinity since the equilibrium distances do so via the stability constraints (3.164). One can find a good amount of discussions as well as references in [84]. The existence of extra degrees of freedom here shows that the potential energy of the configuration may have several flat directions. One should note that eventhough the relative positions of centers do not get fixed entirely by the constraints, they are still not free to move arbitrarily in general. There are only two special cases in which centers seem to be able to move much more freely:

- **Threshold stability** in which centers are free to get infinitely far away from each other. As such, these configurations can barely be considered as bound states.
- **Scaling solutions** in which centers can coincide. Although one needs to be careful about the notion of "coincidence" here.

Both of these cases are going to be discussed in some more detail. Specifically, a section is going to be devoted to the scaling solutions in which we will discuss the asymptotic behavior of these solutions as well, a necessary discussion for the next chapter.

Let us finish this part by figuring out what is the solution space in the simplest case made up of only two centers. In this case, the solution space will be two dimensional and the two degrees of freedom are in fact azimuthal and polar angles (θ, φ) determine the orientation of the axis connecting the two centers. Therefor, the solution space is simply S^2.

Marginal stability and decay of bound states

Consider a generic bound state of supersymmetric black holes. As we remarked earlier, multi-centered black holes are $\frac{1}{2}$-BPS solutions to $\mathcal{N} = 2$ supergravity for which the BPS bound is saturated. As such, masses of these states are given by the size of their total central charge Z which by themselves depend on electromagnetic charges Γ^a and the scalars t^A via $Z(\Gamma, t^A) = \langle \Gamma, V(t^A) \rangle$. More precisely, the size of the central charge can be seen as variable mass of the corresponding BPS black hole changing by scalars values at infinity. It can be concluded most easily from (3.129). In addition, phase of the central charge α reveals unbroken supersymmetries. Consequently, one can argue that first, BPS states can not decay to non-BPS states and second, they can not even decay to smaller BPS states. The second possibility is forbidden because of the energy conservation law whose realization for BPS states will be

$$|Z_{1+2}| \leq |Z_1| + |Z_2| \ . \tag{3.167}$$

This simply says that just like any other bound state, some amount of energy will be released during formation of the bound state. This makes energy of the bound state less than the total sum over energies of each component individually. However, there are cases in which the values of scalars are tuned such that the central charges of two smaller BPS states become parallel, i.e., we have $\alpha_1 = \alpha_2$. This implies that the total mass of the initial BPS state $|Z_{1+2}|$ equals the sum of two smaller BPS states. In this situation, our previous argument does not hold anymore, meaning that the energy conservation allows these bound states to decay into smaller BPS states. If this is the case, the BPS state is said to be in *marginal stability.* So in summary:

$$\textbf{Marginal stability:} \quad \mathrm{Im}\big[\overline{Z}_1 Z_2\big] = 0 \quad , \quad \mathrm{Re}\big[\overline{Z}_1 Z_2\big] > 0 \quad \text{and} \quad \langle \Gamma_1, \Gamma_2 \rangle \neq 0, \tag{3.168}$$

where we call it *anti-marginal stability* if $\mathrm{Re}\big[\overline{Z}_1 Z_2\big] < 0$. Furthermore, the necessity of the third condition becomes more clear later when we get to investigate states at threshold stability. This is the only place in the solution space where a bound state of supersymmetric black holes can decay. The marginal stability condition $\alpha_1 = \alpha_2$ is a single equation in terms of scalars t^A (see (3.80)) that defines a surface of codimension one inside the scalar moduli target space. This surface is called *wall of marginal stability* since it splits the solution space into two disjoint subspaces: on one side we have a stable bound state while by passing through this wall, the state decays and hence there is no bound state on the other side of the wall.

It is worth mentioning that on this side of the wall where bound states exist, the solution space is quite regular, meaning that we smoothly move from one stable solution to the another by varying the asymptotic values of the scalars. It only becomes singular when we hit the wall of marginal stability where no bound state can survive the decay.

The fact that by varying the scalars asymptotic values we can move inside the solution space can be seen most easily in the simplest case among the bound states consisting of only two centers. In fact, it was in the same case (probing a single-centered supersymmetric black hole by another test dyon in 3.5.1) that we discussed decay of a bound state for the first time. Here, we just use the main result we have already obtained there to first, achieve a better view from the decay phenomenon and second, to generalize the discussion to the cases with more than two centers. There we found the equilibrium distance between two centers is given by

$$r_{12} = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2\mathrm{Im}\big[e^{-i\alpha_1} Z_2\big]_{r\to\infty}} \ . \tag{3.169}$$

As one can see clearly, the relative distance explicitly depends on $t^A(\infty)$. We have also argued the necessary condition for having a stable bound state by defining its decay as the following: if two centers get infinitely far away from each other, i.e., $r_{12} \to \infty$ the initial bound state is said to be decayed. Assuming $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$ this definition implies that the decay happens when $\mathrm{Im}\big[e^{-i\alpha_1} Z_2\big]_{r\to\infty} = 0$ which is exactly the same condition as the one we have in (3.168). It also explains how one can make a bound state to decay. It is enough to vary the scalars asymptotic values by moving inside the moduli space such that we get closer and closer to the wall of marginal stability inside the solution space. Doing so, we are actually moving towards bound states with larger and larger intercenter distances. This will continue until r_{12} becomes infinite right on the wall so that on the other side, the two smaller clusters are not bound together any more. As we mentioned earlier, these moduli and solution spaces are absolutely entangled.

Decay of a bound state with more than two centers happens exactly in the same fashion. The only difference is that here the decay products are not two single-centereds, but they are really *clusters* each of which can still be considered as a bound state. To make a clear distinction between charges, we label them differently for different cluster. So we define $C_1 = \{\Gamma_i\}$ and $C_2 = \{\Gamma_\alpha\}$ with $Z_1 = \sum_i Z_i$ and $Z_2 = \sum_\alpha Z_\alpha$. Now we recall the stability conditions (3.164) we had for each center. Do a summation on i (or equivalently on α) we read

$$\sum_{\alpha,i} \frac{\langle \Gamma_i, \Gamma_\alpha \rangle}{r_{i\alpha}} = 2\text{Im}[e^{-i\alpha} Z_1]_\infty \ . \tag{3.170}$$

In this case, the marginal stability condition will be again $\alpha_1 = \alpha_2$ but now with $\alpha_1 = \arg Z_1$ and $\alpha_2 = \arg Z_2$. Accordingly, the decay of a bound state with more than two centers is naturally interpreted as the relative distances between centers in different clusters becoming infinite while centers at each cluster are still bound together and stay in some equilibrium distances from each other.

Ultimately, one may ask whether decay to more than two cluster is possible as well. The answer is definitely yes, however they are less likely to happen and here is the reason. Recall that decay of a bound state to two clusters happens on a surface of codimension one in the solution space. Therefor, to have more than two clusters after decay, different walls of marginal stability have to intersect. This leaves a surface of codimension two on which multi cluster decay can happen. This makes these types of decays avoidable since there is always enough “space” to turn around these surfaces without hitting them by taking a proper bypass. We refer the interested reader to [84] where decay procedure has been discussed with lots of details.

Threshold stability

The first special case we want to discuss here is what is called a state at *threshold stability*. As we have emphasized here and there using different words, having stability conditions (3.145) by itself means that the relative distances between different centers is not quite arbitrary. However they are not fixed completely because the number of constraints are not enough. These two together means that there is a relative "freedom" for centers to move around while the intercenter distances have both upper and lower bounds in general. However, in the case of states at threshold stability, it turns out that centers gather in multi-centered groups and stay constrained there while the clusters are completely free to get infinitely far away from each other.

Without telling more about their behavior, let us go directly through the formulation of these special solutions. To do so, we first split index n of centers into two different indices, capital letters $A, B, ...$ label different clusters while Greek indices $\alpha, \beta, ...$ refer to different centers in each cluster. Accordingly, we denote the position of the center α inside the cluster A by $\vec{r}_{A\alpha}$ which is given by

$$\vec{r}_{A\alpha} = \lambda \vec{r}_A + \vec{r}_\alpha + \lambda^{-1} \vec{\mathrm{v}}_\alpha + \mathcal{O}(\lambda^{-2}) \ . \tag{3.171}$$

Here, $\vec{r}_A$ denotes the position of cluster A while $\vec{r}_\alpha$ is that of center α inside the cluster and λ is a scaling parameter. The way we wrote $\vec{r}_{A\alpha}$ and also role of λ becomes more clear if one is familiar with the scaling solutions. However, for our later discussions this would be more convenient to postpone this to the next section. Our definition of a state in threshold stability then would be taking the limit $\lambda \to \infty$. So one can imagine these states as bunch of clusters freely getting further from each other while centers are normally bound together inside each cluster as if there is no other cluster in their neighborhood. In this situation, the stability conditions (3.145) slit into two sets of equation as the following

$$\sum_{\beta \neq \underline{\alpha}} \frac{\langle \Gamma_{A\alpha}, \Gamma_{A\beta} \rangle}{r_{\alpha\beta}} = \langle h, \Gamma_{A\alpha} \rangle \ , \tag{3.172}$$
$$\sum_{A \neq \underline{B}} \frac{\langle \Gamma_A, \Gamma_B \rangle}{r_{AB}} = 0 \quad ; \qquad \Gamma_A = \sum_\alpha \Gamma_{A\alpha} \ ,$$

where as always $r = |\,\vec{r}\,|$. The first equation simply says that centers inside each cluster are bound together in a normal way, just like what we have for a generic multi-centered configuration. Still, performing a summation over α reveals

$$\langle h, \Gamma_A \rangle = \sum_\alpha \langle h, \Gamma_{A\alpha} \rangle = 0 \qquad \Longrightarrow \qquad \mathrm{Im}\big[e^{-i\alpha} Z_A\big] = 0 \ , \tag{3.173}$$

where we have used shorthand introduced in (3.145). This final result implies that all solutions at threshold stability sit on the wall of marginal stability.

Now, the first thing to note about the second equation is that it indeed contains interaction between centers from different clusters, say $\Gamma_{A\alpha}$ and $\Gamma_{B\beta}$. However, such interactions appear only as intercluster interactions in this equation while each cluster behave like a single-centered carrying $\Gamma_A = \sum_\alpha \Gamma_{A\alpha}$. This is a consequence of taking the limit $\lambda \to \infty$. The second point may not seem very clear now but these conditions are just like those one obtains in the case of scaling solutions. So it would be right if we say in the case of threshold stability these are clusters that form a scaling configuration.

The two conclusion from two last paragraphs may seem to contradict each other. On one hand, we saw states in threshold stability laying on the wall of marginal stability which means their relative distances tend to infinity. On the other hand, we said they fulfill scaling conditions which (as we are going to see in the next section) implies that different centers of the solution (clusters in our current case) can get arbitrarily close to each other. Having these two behavior simultaneously clarifies that clusters in threshold stability are absolutely free to move around, get so much close and coincide or go infinitely far away from each other. As such, one can not really interpret this as a decay when intercluster distances tend to infinity. In other words, the main difference between a state in marginal and threshold stability is that in the first case one need to "‘force" the state to decay via hitting the wall of marginal stability whereas in the second case no push is needed. At this moment, our definition of threshold stability is clear enough and also can be compared to that of marginal stability (3.168). Here it is

$$\textbf{Threshold stability:} \quad \mathrm{Im}\big[\overline{Z}_1 Z_2\big] = 0 \ \ , \ \ \mathrm{Re}\big[\overline{Z}_1 Z_2\big] > 0 \ \text{ and } \ \langle \Gamma_1, \Gamma_2 \rangle = 0 \, . \tag{3.174}$$

3.6. Scaling Solution

Finally, we got to the point to introduce a subset of multi-centered black hole solutions which is the main case of our study in this thesis, i.e., "*scaling solutions*". We are going to devote this section and the next one to study these solutions and their asymptotic behavior. As we will see, in this case some centers can get arbitrarily close while the others still stay at some equilibrium distances with respect to these "coincident" centers. In addition, distances between coincident centers are determined *up to a scale*, meaning that solutions in scaling state are invariant under rescaling and so called scaling solutions. This subset of multi-centered black holes is of special importance since they resemble the more common single-centered black holes. Another characteristic property of scaling solutions is that they are absolutely resistant against decay to smaller clusters. It is quite opposite to what we have observed for typical multi-centered black holes for which decaying to smaller clusters are an interesting subtle phenomenon to study. With this motivation, let us now go through details and discus these solutions more concretely.

Just like what we did in the last section to study decay process for states at marginal and threshold stability, here we are going to split centers $\{\Gamma_m\}$ into two groups of charges $\{\Gamma_\alpha\}$ and $\{\Gamma_i\}$. The distinguishing feature between these two sets of charges is that centers labeled by α exhibit a scaling symmetry that allows them to coincide at some point $\vec{r}_0$ while those labeled by i are constrained to stay at equilibrium distances from the scaling centers. The position vectors in such a configuration look like the following

$$\vec{r}_\alpha(\lambda) = \vec{r}_0 + \lambda \vec{\mathrm{w}}_\alpha + \lambda^2 \vec{\mathrm{v}}_\alpha + \mathcal{O}(\lambda^3) \ , \tag{3.175}$$

$$\vec{r}_i(\lambda) = \vec{r}_i + \lambda \vec{\mathrm{z}}_i + \mathcal{O}(\lambda^2) \ . \tag{3.176}$$

Here λ is the scaling factor that parameterizes the solutions. Plugging these positions inside the stability conditions (3.145) and arranging terms according to powers of λ, for $\mathcal{O}(0)$ and $\mathcal{O}(\lambda^{-1})$ one reads

$$\sum_{\alpha\neq\underline{\beta}} \frac{\langle\Gamma_\alpha,\Gamma_\beta\rangle}{|\vec{\mathrm{w}}_\alpha-\vec{\mathrm{w}}_\beta|} = 0 \ , \tag{3.177}$$

$$\sum_{\alpha\neq\underline{\beta}} \frac{\langle\Gamma_\alpha,\Gamma_\beta\rangle}{|\vec{\mathrm{w}}_\alpha-\vec{\mathrm{w}}_\beta|^2}|\vec{\mathrm{v}}_\alpha-\vec{\mathrm{v}}_\beta| + \sum_i \frac{\langle\Gamma_i,\Gamma_\beta\rangle}{|\vec{r}_i-\vec{r}_0|} = \langle\Gamma_\beta,h\rangle \ , \tag{3.178}$$

$$\frac{\langle\Gamma_0,\Gamma_j\rangle}{|\vec{r}_0-\vec{r}_j|} + \sum_{i\neq\underline{j}} \frac{\langle\Gamma_i,\Gamma_\beta\rangle}{|\vec{r}_i-\vec{r}_0|} = \langle\Gamma_j,h\rangle \ . \tag{3.179}$$

Here we defined $\Gamma_0 = \sum_\alpha \Gamma_\alpha$ and as before we have $h = 2\mathrm{Im}\big[e^{-i\alpha}V\big]_\infty$ that allow us to define another shorthand as $h_m := \langle\Gamma_m,h\rangle = 2\mathrm{Im}\big[e^{-i\alpha}Z_m\big]$. Clearly, for later applications we can also define $h_0 = \sum_\alpha h_\alpha$ as a result of linearity of the intersection product.

Starting from the first equation, we immediately realize that a rescaled solution still solves these constraints. In other words, these constraints only determine the equilibrium distances $\mathrm{w}_{\alpha\beta} = |\vec{\mathrm{w}}_\alpha - \vec{\mathrm{w}}_\beta|$ up to an overall rescaling of intercenter distances. That is how this family of solutions gained the name "scaling solutions". Moreover, (3.177) encapsulates a set of constraints only on coinciding centers. As it is clear from the RHS, the requirement of having a bunch of centers coincide is equivalent to setting h_α zero. Another point to note is that having h_α zero means there is no room to play with the asymptotic values of the scalars t^A such that for any set of charges (3.177) can be satisfied. In other words, these set of constraints are independent of the moduli and so depend only on the charges. Therefor, only those charges can coincide for which there is a solution to (3.177). More importantly, independency of the scalar moduli has a very remarkable consequence for these solutions; they can not be made to decay via tuning the scalars asymptotic values. Therefore, beside states at threshold stability, scaling solutions are the second family of multi-centered supersymmetric black holes who are resistant against decay.

The second and third equations can be summarized in a single set of equations if one perform a summation over β in (3.178). The result becomes

$$\sum_{\tilde{i}\neq\tilde{\underline{j}}} \frac{\langle\Gamma_{\tilde{i}},\Gamma_{\tilde{j}}\rangle}{|\vec{r}_{\tilde{i}}-\vec{r}_{\tilde{j}}|} = h_{\tilde{j}} \qquad \text{with} \qquad \tilde{i}=(0,i) \ . \tag{3.180}$$

This very short and clear formula strongly proves our interpretations of the scaling solutions and the way they interact with other normal centers. Comparing to our well-known stability conditions (3.145), it simply describes a bound state between the normal non-collapsing centers $\{\Gamma_i\}$ and the who coincident ones who look like a single center with the total charge Γ_0. This picture also confirm our earlier statement that scaling solutions can be considered as supersymmetric counterparts of familiar single-centered black holes.

There is a very delicate but crucial point about what really happens when a bunch of charges start getting arbitrarily close to each other in the case of scaling solutions. The word "coincidence" we used here may be a bit misleading. As we are going to argue, the relative distance between collapsing points does not become zero in physical space. To make this more concrete, we only focus on collapsing centers $\{\Gamma_\alpha\}$ whose relative distances are given by $\vec{r}_{\alpha\beta}=\lambda\vec{\mathrm{w}}_{\alpha\beta}$. Under rescaling, the harmonic function H behave like

$$H(r,r_{\alpha\beta},h)=\lambda^{-1}H(\mathrm{w},\mathrm{w}_{\alpha\beta},\lambda h) \ . \tag{3.181}$$

As derived in [80] and also one can check from expression of Σ in terms of H and other auxiliary functions L and Q, the warp factor is a homogeneous function of degree 2, i.e.,

$$\Sigma(\lambda^{-1}H)=\lambda^{-2}\Sigma(H) \ . \tag{3.182}$$

Looking at the collapsing limit is equivalent to sending the scaling parameter to zero, i.e., $\lambda\to 0$. Now to calculate the relative distance between these centers we need to take the spatial part of the metric given by 3.5. The behavior of the warp factor Σ

under rescaling (3.182) implies that the relative distance between centers remains *finite* even if they coincide apparently. More precisely with $d\vec{x} = \lambda \vec{\mathrm{w}}_{\alpha\beta} \to 0$ we still have a finite line element $ds^2 = \Sigma(x) d\vec{x}^2 = \Sigma(\mathrm{w}) d\vec{\mathrm{w}}^2$. To see how it is possible one can imagine that these collapsing centers in scaling solution are mutually separated spatially by a "gravitational throat". The closer two centers get the deeper the throat becomes in such a way that the spatial relative distance between centers remains always finite.

It will be instructive to end this part by looking at the first two simple examples of scaling family. The first case would be rather trivial since with only two centers there is only one solution to the scaling constraint equation (3.177) which is $\langle \Gamma_1, \Gamma_2 \rangle = 0$. The next case consists of three centers and can be checked very easily that the following solves the constraints

$$\mathrm{w}_{12} = \langle \Gamma_1, \Gamma_2 \rangle d \quad , \qquad \mathrm{w}_{23} = \langle \Gamma_2, \Gamma_3 \rangle d \quad , \qquad \mathrm{w}_{31} = \langle \Gamma_3, \Gamma_1 \rangle d \ , \tag{3.183}$$

where d is an arbitrary constant whose sign depends on the intersection products of charges such that w_{ij} be positive for all i and j. For instance, for positive $\langle \Gamma_1, \Gamma_2 \rangle$ the other two products have to be positive as well. In fact, this is the first condition imposed on charges by stability constraints (3.177), as we explained before. Furthermore, from the way we wrote the relative distances in a cyclic order one notices that w_{ij} satisfy triangle inequalities and so do the intersection products. These inequalities are in fact constraints on charges coming from stability conditions (3.177)

$$\langle \Gamma_1, \Gamma_2 \rangle + \langle \Gamma_2, \Gamma_3 \rangle \geq \langle \Gamma_3, \Gamma_1 \rangle \quad \text{and two other cyclic permutations} \ . \tag{3.184}$$

One can find discussions on scaling solutions with more than three centers in some specific cases in [85]. Moreover, a systematic way of constructing four-center solutions from three-center ones has been innovated in [86]. We will continue by calculating the one-form ω in the case of scaling solutions with only two centers.

3.6.1. The One-Form ω in Two-Center Scaling Solutions

Like the path we took earlier, here in a short subsection we try to find the one-form ω of scaling solutions for the simple two-center case. We start with the harmonic function H which is

$$H = \frac{\Gamma_1}{r_1} + \frac{\Gamma_2}{r_2} \ , \tag{3.185}$$

where as before we have $r_n := |\vec{r} - \vec{r}_n|$. Substituting this inside the definition (3.140) and taking star, we get

$$d\omega = \langle \Gamma_1 \Gamma_2 \rangle \star \left[\frac{1}{r_2} d\Big(\frac{1}{r_1}\Big) - \frac{1}{r_1} d\Big(\frac{1}{r_2}\Big) \right] \ . \tag{3.186}$$

Comparing to (3.147) we see the last two terms coming from the constant h in the harmonic functions have disappeared. Doing a bit of calculation one finds the following form for the two-form $d\omega$

$$d\omega = \frac{1}{r^4} \varepsilon_{ijk} \Big[r_{12}^i - 2\big(\vec{r}_{12}.\hat{r}\big)\ \hat{r}^i \Big] \big(dr^j \wedge dr^k \big) \quad ; \quad i,j,k = 1,2,3 \tag{3.187}$$

where $\hat{r}$ is the unit vector in radial direction, i.e., $\vec{r} = r\hat{r}$ and we have used an old shorthand $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$. Moreover, dr^j is j component of $d\vec{r}$. Now, using following identity

$$\varepsilon^i{}_{jk} d\hat{r}^j \wedge d\hat{r}^k = \hat{r}^i \varepsilon_{jkl}\ \hat{r}^j \big(d\hat{r}^k \wedge d\hat{r}^l \big) \tag{3.188}$$

one can rewrite (3.187) as

$$d\omega = \frac{-1}{r^2} \varepsilon_{ijk} r_{12}^i \Big(d\hat{r}^j \wedge d\hat{r}^k + \frac{2}{r} d\hat{r}^j \wedge dr\ \hat{r}^k \Big) \tag{3.189}$$

which is an exact form from which one can read ω

$$\omega = \frac{-1}{r^2}\ \varepsilon_{ijk} r_{12}^i\ \hat{r}^j d\hat{r}^k = \frac{-1}{r^4} \varepsilon_{ijk} r_{12}^i\ r^j dr^k\ . \tag{3.190}$$

Working in spherical coordinates and rotating the coordinate system such that $\vec{r}_{12} = 2l\hat{z}$ we get

$$\omega = \frac{-2l}{r^2} \sin^2\theta\ d\varphi\ . \tag{3.191}$$

In the next section we are going to derive ω at spatial infinity for configurations with more than two centers in the case of scaling solutions.

3.7. Asymptotic Behavior of the Supergravity Scaling Solutions

Here in this section we are going to investigate asymptotic behavior of fields and features contribute to the scaling BPS solutions of supergravity in four dimensions discussed in the previous section 3.6. As we emphasized before, the characteristic property of multi-centered solutions is that they are not static but stationary solutions. This property is encoded in the one-form ω introduced in the metric and also time component of the gauge fields $A^{\tilde{A}}$. It is derivable from the harmonic functions H via its definition (3.140). Previously, we found its explicit form in the case with two centers in 3.5.2 and also saw that generalization to multi-centered case is always guaranteed. However, our results there were valid for common multi-centered solutions while here our focus is on scaling case. Due to the fact that for solutions with scaling invariance the constants $h = 2\text{Im}\big[e^{-i\alpha}V\big]_\infty$ vanish, some features and results including ω need to be carefully revised. This is the aim we follow in this section. The results we obtain here are going to be applied in next two chapters.

3.7.1. The One-Form ω

We start with the characteristic one-form since its behavior at spatial infinity is of main importance for our discussion in the next chapter. Usually, to study some solution at spacial infinity the first order expansion of functions and fields seem to be enough. It is not the case here though! As we are going to show in the next chapter, an observer at infinity is able to measure the dipole moment of a charged multi-centered configuration, a quantity that appears in higher order expansions and so negligible at far infinity. The dipole moment of multi-centered solutions emerges surprisingly in asymptotic metric via the one-form ω and that is why we need to go beyond the first order term of the harmonic function H. Here is the familiar multipole expansion for a multi charge configuration up to the forth order in r

$$H^a = \frac{\Gamma^a}{r} + \frac{\vec{\Delta}^a \cdot \hat{r}}{r^2} + \frac{\hat{r} \cdot Q^a \cdot \hat{r}}{2r^3} + \mathcal{O}(r^{-4}) \ , \tag{3.192}$$

where Γ^a is the total charge. The total dipole moment $\vec{\Delta}^a$ and quadrupole moment Q^a of the system in spherical coordinates are given by

$$\vec{\Delta}^a = \sum_n \Gamma^a_n \vec{r}_n \quad , \quad Q^{a\ ij} = \sum_n \Gamma^a_n \big(3 r_n^{\ i}\, r_n^{\ j} - \delta^{ij} r_n^2 \ \big) \tag{3.193}$$

with $i, j = \{1, 2, 3\}$. Also note that the superscript a includes upper and lower indices at the same time, for instance we have $\Gamma^a = (\Gamma_{\tilde{A}}, \Gamma^{\tilde{A}})$. From now on we will drop this index to avoid expressions look messy. Differentiating (3.192) we get

$$\begin{aligned} dH = \quad & \frac{1}{r^2}\Big(-\Gamma dr + \vec{\Delta} \cdot d\hat{r} \Big) + \frac{1}{r^3}\bigg(3 \sum_n \Gamma_n \big(\vec{r}_n \cdot \hat{r}\big)\big(\vec{r}_n \cdot d\hat{r}\big) - 2\big(\vec{\Delta} \cdot \hat{r}\big) dr \bigg) + \\ & \frac{1}{2r^4}\bigg(-9 \sum_n \Gamma_n \big(\vec{r}_n \cdot \hat{r}\big)^2 dr + 3 \sum_n \Gamma_n r_n^2 dr \bigg) + \mathcal{O}\big(r^{-4}\big) \ , \end{aligned} \tag{3.194}$$

which leads to [46]

$$
\begin{aligned}
\star d\omega = \big\langle dH, H\big\rangle = \quad & \frac{1}{r^3}\Big(\big\langle\vec{\Delta}\cdot(\hat{r}\times d\hat{r}),\Gamma\big\rangle - \frac{1}{r}\big\langle\vec{\Delta}\cdot\hat{r},\Gamma\big\rangle dr\Big) + \\
& \frac{1}{r^4}\Big(\big\langle\vec{\Delta}\cdot(\hat{r}\times d\hat{r}),\vec{\Delta}\cdot\hat{r}\big\rangle + 3\Big\langle\sum_n \Gamma_n\big(\vec{r}_n\cdot\hat{r}\big)\big(\vec{r}_n\cdot d\hat{r}\big),\Gamma\Big\rangle\Big) + \\
& \frac{1}{r^5}\Big(- 3\Big\langle\sum_n \Gamma_n(\vec{r}_n\cdot\hat{r})^2,\Gamma\Big\rangle dr + \Big\langle\sum_n \Gamma_n r_n^2,\Gamma\Big\rangle dr\Big) + \mathcal{O}(r^{-6}) \ .
\end{aligned} \tag{3.195}
$$

In taking Hodge star of the RHS, we applied the following identities

$$\star dr = r^2 \sin\theta \ d\theta\wedge d\varphi \tag{3.196}$$

$$(\star d\hat{r})^i = \varepsilon^i{}_{jk}\hat{r}^j dr\wedge d\hat{r}^k \qquad \Longrightarrow \tag{3.197}$$

$$d(\star d\hat{r}) = -dr\wedge d\vec{\Omega} \ \ ; \qquad d\Omega^i \equiv \tfrac{1}{2}\varepsilon^i{}_{jk} d\hat{r}^j\wedge d\hat{r}^k = \hat{r}^i \sin\theta \ d\theta\wedge d\varphi \ .$$

One can show that $d\omega$ is a closed two-form, i.e., $dd\omega = 0$. Note however that this is not trivial. Indeed, the reason enabled us to define $\star\big\langle dH, H\big\rangle$ as $d\omega$ at the first place was that the former is an exact two-form (See appendix B for the proof). It is from the same Appendix that one can follow the calculations step by step. Here, we only bring the last result for second order expansion of the one-form ω which is given by

$$
\begin{aligned}
\omega \ &= \ \frac{-1}{2r^2}\Big\langle\vec{\Delta}\cdot(\hat{r}\times d\hat{r}),\Gamma\Big\rangle \\
&- \ \frac{1}{r^3}\Bigg(\frac{1}{3}\Big\langle\vec{\Delta}\cdot(\hat{r}\times d\hat{r}),\vec{\Delta}\cdot\hat{r}\Big\rangle + \Big\langle\sum_n \Gamma_n(\vec{r}_n\cdot\hat{r})\big(\vec{r}_n\cdot(\hat{r}\times d\hat{r})\big),\Gamma\Big\rangle\Bigg) + \mathcal{O}(r^{-4}) \ ,
\end{aligned} \tag{3.198}
$$

or in more compact notation, it can be written as

$$\omega = \frac{-1}{2r^2}\big\langle\Delta_{\parallel},\Gamma\big\rangle - \frac{1}{3r^3}\Big(\big\langle\Delta_{\parallel},\Delta_{\perp}\big\rangle + \Big\langle Q^{ij}\ \hat{r}_i\ \big(\hat{r}\times d\hat{r}\big)_j,\Gamma\Big\rangle\Big) + \mathcal{O}(r^{-4}) \tag{3.199}$$

[46] Here we take $\vec{r} = r\hat{r}$ and so $d\vec{r} = dr\hat{r} + rd\hat{r}$.

where we have defined

$$\Delta_{\parallel} := \vec{\Delta} \cdot (\hat{r} \times d\hat{r}) \quad , \quad \Delta_{\perp} := \vec{\Delta} \cdot \hat{r} \ , \tag{3.200}$$

and also from vector analysis we can find

$$\hat{r} \times d\hat{r} = \begin{pmatrix} -\sin\varphi \ d\theta - \sin\theta \ \cos\theta \ \cos\varphi \ d\varphi \\ \\ \cos\varphi \ d\theta - \sin\theta \ \cos\theta \ \sin\varphi \ d\varphi \\ \\ \sin^2\theta \ d\varphi \end{pmatrix} . \tag{3.201}$$

As we discussed earlier in footnote 44, it is always possible to rotate the coordinate system in such a way that the vector $\langle \Gamma, \vec{\Delta} \rangle$ lies along z-direction. So we define

$$\vec{K} := \frac{1}{2} \langle \Gamma, \vec{\Delta} \rangle = K \hat{z} \ , \tag{3.202}$$

to get more simple expression for ω

$$\omega = \frac{1}{r^2} K \sin^2\theta \ d\varphi - \frac{1}{3r^3} \Big\langle Q^{ij} \ \hat{r}_i \ \big(\hat{r} \times d\hat{r}\big)_j, \Gamma \Big\rangle + \mathcal{O}(r^{-4}) \ . \tag{3.203}$$

later we will see that the obtained form of ω is crucial to get a S^2 factor in asymptotic geometry of the scaling multi-centered black holes.

3.7.2. Auxiliary Functions and Gauge Fields

We are also interested in the asymptotic behavior of the metric and gauge fields of multi-centered scaling solutions (3.156). To be able to study that, we first need to find higher order expansions of auxiliary function we have introduced in Table 3.5. So we start with y^A which is defined as a solution to the following equation

$$D_{ABC} y^A y^B = -2 H_C H^0 + D_{ABC} H^A H^B \ , \tag{3.204}$$

where D_{ABC} is a symmetric matrix whose entries called *intersection number of Calabi-Yau manifold* first defined in (3.69). It is not possible to analytically solve (3.204) for y^A, except in some special limits like $p^0 \to 0$ or $r \to \infty$ that we can find solutions to this equation. Here, we focus on the second case at which y^A can be written as

$$y^A = \frac{y_1^A}{r} + \frac{y_2^A}{r^2} + \mathcal{O}\big(r^{-3}\big) \ . \tag{3.205}$$

Note that (3.204) dictates the expansion of y^A in terms of r to be the same as expansion of the harmonic functions H^A. Plugging (3.205) back to (3.204) and equating both sides of the equality at the leading order, one can determine y_1^A as a solution to the following equation

$$D_{ABC} y_1^A y_1^B = -2q_C p^0 + D_{ABC} p^A p^B \ . \tag{3.206}$$

Now let us define $(D_{ABC} y_1^A)^{-1} := d^{BC}$, then we get the following expression for y_1^A

$$y_1^A = d^{AB}\big(D_{KMB} p^K p^M - 2q_B p^0\big) \ . \tag{3.207}$$

Doing exactly the same procedure for subleading order, we would get

$$y_2^A = -d^{AB}\Big(q_B \vec{\Delta}^0 + p^0 \vec{\Delta}_B - D_{KMB} p^K \vec{\Delta}^M\Big).\hat{r} \ . \tag{3.208}$$

By substituting this result in Q^3 from (3.160), we read

$$Q^{3/2} = \frac{1}{r^3}\Big(D_{KMC} p^K p^M - 2q_C p^0\Big)\Big(\frac{1}{3} y_1^C + \frac{1}{r} y_2^C\Big) \ . \tag{3.209}$$

Gauge fields

We repeat the expression of the Dirac fields $\mathcal{A}_{\rm d}^{\tilde{A}}$ from (3.156) here for convenience

$$d\mathcal{A}_{\rm d}^{\tilde{A}} = \star dH^{\tilde{A}} \ . \tag{3.210}$$

Using the second order expansion of the harmonic function $H^{\tilde{A}}$ from (3.192) we get

$$d\mathcal{A}_{\rm d}^{\tilde{A}} = -\left[\Gamma^{\tilde{A}} + \frac{2}{r}\big(\vec{\Delta}^{\tilde{A}} \cdot \hat{r}\big)\right] \sin\theta \; d\theta \wedge d\varphi + \frac{1}{r^2}\vec{\Delta}^{\tilde{A}} \cdot \big(dr \wedge (\hat{r} \times d\hat{r})\big) \; , \tag{3.211}$$

from which we can read $\mathcal{A}_d^{\tilde{A}}$ as the following

$$\mathcal{A}_{\rm d}^{\tilde{A}} = \Gamma^{\tilde{A}} \cos\theta \; d\varphi - \frac{1}{r}\Delta_{\parallel}^{\tilde{A}} \; . \tag{3.212}$$

Using (3.196,3.197), it is easy to check that differentiating (3.212) gives back (3.211). Accordingly, the gauge fields get to the following form

$$A^{\tilde{A}} = C^{\tilde{A}}(\theta) r \; dt + \Gamma^{\tilde{A}} \cos\theta \; d\varphi + \frac{1}{r}\left(C^{\tilde{A}(\theta)} K \sin^2\theta - \Delta_{\parallel}^{\tilde{A}}\right) \; , \tag{3.213}$$

where $C^{\tilde{A}}(\theta)$ are some r-independent functions.

4. TWIST ON MULTI-CENTERED AdS_2 SOLUTIONS

This chapter contains original work of the author in collaboration with Dieter Van Den Bleeken that was reported in [36].

4.1. Introduction

The multi-centered black hole solutions of 4D $\mathcal{N} = 2$ supergravity [23, 24, 80] provide an interesting setting to investigate the BPS spectrum of string theory compactified on a Calabi-Yau manifold and the associated physics problem of black hole entropy and microstates [27, 87–90]. Through string/M duality these solutions can be lifted to 5 dimensions [91–94]. It is in this setting that recently a subset of multi-centered solutions, often called 'scaling solutions' [27, 82, 85, 95] have been revisited and their asymptotic AdS_2 nature explored [96], see also [97, 98]. In this short note we point out that somewhat surprisingly the asymptotic geometry typically has a fibered structure, with an S^2 rotating over AdS_2. Interestingly this rotation is not linked to the total angular momentum (which for these scaling solutions vanishes, similar to single-centered black holes) but to a higher moment of the angular momentum. For lack of deeper understanding we call this new feature the 'twist'. This twist provides some hair that distinguishes the asymptotics of AdS_2 multi-centered solutions from the near horizon black hole $AdS_2 \times S^2$ geometry. Since it has been argued that it are precisely the scaling solutions that correspond to the exponential majority of black hole microstates [28, 99, 100] a precise holographic interpretation of the twist would be highly interesting. We leave this last problem for future work. After reviewing some technicalities of the multi-centered solutions in section 4.2 and spelling out some details on both the far and near region of a scaling solution in section 4.3 we come to the point in section 4.4 and derive the asymptotic AdS_2 geometry (4.19) revealing the subtle presence of the twist (4.17). We end with some comments in section 4.5.

4.2. Reminder of $\mathcal{N}=2$ Multi-Centered Black Holes

The multi-centered solutions of $\mathcal{N} = 2$ supergravity are dyonic black holes interacting through electromagnetic and scalar field induced forces in such a way that stable bound states are formed. Although rather intricate, exact explicit solutions are known, for a review see [27, 84]. The theory has $h^{(1,1)}$ complex scalar fields t^A and $h^{(1,1)}+1$ $U(1)$ gauge fields $A^{\tilde{A}} = (A^0, A^A)$. For a generic multi-centered solution these fields and the metric take the following form:

$$
\begin{aligned}
ds^2 &= -\frac{1}{\Sigma}(dt+\omega)^2 + \Sigma\, dx^i dx^i\,, \\
A^0 &= \frac{-L}{\Sigma^2}\,(dt+\omega) + \mathcal{A}^0_{\rm d} \qquad A^A = \frac{H^A L - Q^{3/2} y^A}{H^0 \Sigma^2}\,(dt+\omega) + \mathcal{A}^A_{\rm d}\,, \\
t^A &= \frac{H^A}{H^0} + \frac{y^A}{Q^{\frac{3}{2}}}\left(i\Sigma - \frac{L}{H^0}\right),
\end{aligned}
\tag{4.1}
$$

The whole solution is determined in terms of $2h^{(1,1)}+2$ harmonic functions $H = (H^0, H^A, H_A, H_0)$ which for N dyonic charges $\Gamma_n = (p^0_n, p^A_n, q^n_A, q^n_0)$ at positions $\vec{r}_n$ in the spatial $\mathbb{R}^3$ take the simple form

$$
H = \sum_n \frac{\Gamma_n}{r_n} + h\,,
\tag{4.2}
$$

where we defined $r_n := |\vec{r} - \vec{r}_n|$. These harmonic functions enter the fields above through a set of auxiliary functions. First they define the y^A, obtained by formal solution of the quadratic equations[47]

$$
D_{ABC} y^A y^B = -2 H_C H^0 + D_{ABC} H^A H^B\,,
\tag{4.3}
$$

[47]Here the constant symmetric three tensor D_{ABC} and the symplectic inner product $\langle E_1, E_2\rangle = -E^0_1 E^2_0 + E^A_1 E^2_A - E^1_A E^A_2 + E^1_0 E^0_2$ are those associated to the particular 4D $\mathcal{N}=2$ supergravity under consideration. In the case of Calabi-Yau compactifications these two objects are naturally determined by the internal geometry. The precise value of the constants D_{ABC} will play however no role in the current paper and all of our discussion hence also applies to situations with no known embedding in string theory, as for example some 'magic' $\mathcal{N}=2$ theories [101].

and then

$$Q^3 = (\frac{1}{3}D_{ABC}y^Ay^By^C)^2\,, \qquad L = H_0(H^0)^2 + \frac{1}{3}D_{ABC}H^AH^BH^C - H^AH_AH^0\,,$$
$$\Sigma = \sqrt{\frac{Q^3 - L^2}{(H^0)^2}}\,.$$

Furthermore there are the one-forms[48]

$$d\mathcal{A}_{\rm d}^{\tilde{A}} = \star dH^{\tilde{A}}\;,, \qquad d\omega = \star\langle dH, H\rangle \tag{4.4}$$

At the technical level the bound state nature of these solution appears through a set of equations restricting the (coordinate) distances r_{ab} between the centers:

$$\sum_{m,\,m\neq n} \frac{\langle \Gamma_n, \Gamma_m\rangle}{r_{nm}} = \langle h, \Gamma_n\rangle\,. \tag{4.5}$$

The constants $h = (h^0, h^A, h_A, h_0)$ appearing in the harmonic functions set the asymptotic values of the scalar fields[49] and as such correspond to the choice of vacuum. When these constants are non-zero the multi-centered solutions are easily seen to be asymptotically flat. When one approaches one of the centers, $\vec{r} \to \vec{r}_n$, the geometry becomes (for a generic charge Γ_n) $AdS_2 \times S^2$, which one recognizes as the near-horizon geometry of an extremal Reissner-Nordström black hole, with the scalars taking constant 'attractor' values [21]. Finally it is important to point out that the solutions are stationary with a total angular momentum given by [23]

$$\vec{J} = \frac{1}{2}\sum_{n<m}\langle \Gamma_n, \Gamma_m\rangle \hat{r}_{nm}\,. \tag{4.6}$$

Note that of course the special case of a single-centered reproduces a standard extremal BPS black hole [21] without angular momentum.

[48] The Hodge star is that of flat $\mathbb{R}^3$.

[49] More precisely it is the h being determined by t^A_∞, in such a way that $\langle h, \sum_n \Gamma_n\rangle = 0$ [23].

4.3. A Far and Near Limit for Scaling Solutions

We should point out that although the $3N$ (coordinate) positions of the dyonic black hole centers are constrained by the $N-1$ equations (4.5), $2N-2 = 3N-(N-1)-3_{\text{c.o.m.}}$ remain free, leading to an interesting space of solutions (see [88, 102] for some first explorations of these spaces). For a generic set of charges Γ_n there will be both a minimal and maximal distance between the centers, but in some special case this is not so. In particular the relative coordinate positions of the centers can be made arbitrary small when the charges Γ_n are such that there exist a set of positive numbers $s_{nm} = s_{mn}$, among which each triple satisfies the triangle inequalities and

$$\sum_{m,\, m\neq n} \frac{\langle\Gamma_n, \Gamma_m\rangle}{s_{nm}} = 0\,. \tag{4.7}$$

Indeed, it directly follows that then $r_{nm} = \xi s_{nm}$ solves the constraint equations (4.5) in the limit $\xi \to 0$. The above 'scaling' conditions have not, as far as we are aware, been studied/solved in general (for $N > 3$), but one can find example solutions for any number of centers[50] . Clearly if a set of charges Γ_n satisfies the scaling conditions then so does an arbitrary overall rescaling of these charges, and one can also freely rescale their positions; hence the name.

The supergravity solution degenerates in an interesting way when the coordinate positions of the centers approach each other [27, 95], in particular the physical distance between the centers does not vanish. To understand more clearly what happens it is useful to consider this limit in a slightly different but equivalent way. As on a technical level one is essentially comparing inverse distances to the constants h in the harmonic functions we can study it from that perspective. Let us introduce a parameter λ by redefining $h = \lambda\tilde{h}$, and consider sending $\lambda \to 0$ while keeping $\tilde{h}$ and $\vec{r}_n$ fixed. This procedure produces two different supergravity solutions, depending on how we treat the coordinates (t, x^i) in this limit. We'll refer to these two solutions as the far and

[50] Take for example N ordered points $\vec{r}_k \in \mathbb{R}^3$, $k \in \mathbb{Z}_N$. Defining $l_k = |\vec{r}_{k+1} - \vec{r}_k|$ one can choose $\langle\Gamma_n, \Gamma_m\rangle = (m-n)l_n$ when $|n-m| = 1$ and zero otherwise. In this case $s_{nm} = |\vec{r}_n - \vec{r}_m|$ shows $\langle\Gamma_n, \Gamma_m\rangle$ satisfy the scaling conditions.

near limits respectively.

4.3.1. The Far Limit

Here we rescale the coordinates via $(t, x^i) = \lambda^{-1}(\tilde{t}, \tilde{x}^i)$ and keep the tilded versions fixed in the $\lambda \to 0$ limit. As now the original coordinates $x^i \gg r_n^i$ one sees that the new coordinates $\tilde{x}^i$ parameterize the region far away from the charged centers. Additionally, if we would define the rescaled positions $\tilde{r}_n^i$ they go to zero and so the limit also describes the centers approaching each other. Although the supergravity solution has a rather intricate form involving a number of auxiliary functions a closer look reveals that much of this structure is homogeneous under the above rescaling. If after taking the limit one drops the tildes one finds that the solution has essentially remained intact, the only difference being the replacement

$$H = \sum_n \frac{\Gamma_n}{r_n} + h \quad \rightarrow \quad H = \frac{\sum_n \Gamma_n}{r} + h \tag{4.8}$$

So the far limiting procedure reproduces the single-centered solution of total charge $\Gamma_{\mathrm{t}} = \sum_n \Gamma_n$, in particular at large r the constant in the harmonic function dominates and the solution is asymptotically flat. Physically what happens is that the original centers develop a stronger and stronger gravitational warping deep in the center which for an observer far away becomes indistinguishable from a single extremal black hole carrying the total charge while nothing much happens to the asymptotics of the original solution. Note that for this procedure to make sense as a continuous limit one needs to keep track of the constraint equations (4.5), which reduce to the scaling conditions

$$\sum_{m,\, m\neq n} \frac{\langle \Gamma_n, \Gamma_m \rangle}{r_{nm}} = 0 \tag{4.9}$$

Note that an interesting physical consequence of these conditions is that the angular momentum of the solution vanishes:

$$\sum_m \frac{\langle \Gamma_n, \Gamma_m \rangle}{r_{nm}} = 0 \quad \Rightarrow \quad 0 = \sum_{n,m} \frac{\langle \Gamma_n, \Gamma_m \rangle}{r_{nm}} \vec{r}_n = \sum_{n<m} \frac{\langle \Gamma_n, \Gamma_m \rangle}{r_{nm}} (\vec{r}_n - \vec{r}_m) = 2\vec{J} \tag{4.10}$$

This is of course in agreement with the fact that also the total angular momentum of the corresponding single-centered black hole is zero.

All this might suggest that when the centers of a multi-centered solution approach each other a single-centered black hole is obtained. Although this is true for the far region we'll see in the next subsection this not at all the case in a near region.

4.3.2. The Near Limit

In this case we do not rescale the coordinates at all, rather keeping t, x^i fixed as $\lambda \to 0$. This limit is immediate to perform as it leaves the full solution intact, simply putting h to zero. Apart from imposing the scaling conditions (4.9), it simply amounts to replacing the harmonic functions as

$$H = \sum_n \frac{\Gamma_n}{r_n} + h \quad \to \quad H = \sum_n \frac{\Gamma_n}{r_n} \tag{4.11}$$

Contrary to the far limit, in the near limit the solution retains its multi-centered nature as it does not differ from the original near any of the centers, i.e., when $\vec{r} \to \vec{r}_n$. The large r behavior has however drastically changed as there

$$H = \frac{\sum_n \Gamma_n}{r} + \mathcal{O}(r^{-2}) \tag{4.12}$$

This suggests that the near limit at large distances is no longer asymptotically flat but should behave as the near horizon of a single-centered black hole of total charge $\Gamma_t = \sum_n \Gamma_n$, which is $AdS_2 \times S^2$. It seems a simple picture emerges where the far distance behavior of the near limit matches perfectly with the near behavior of the far limit, both coinciding with the near horizon geometry of a single extremal black hole. Although this is roughly correct there is a small, but we believe important, twist to this intuitive picture which is the main point of this paper. As we will see the large r behavior of the near limit does not exactly reproduce the near horizon geometry but rather some hair remains through a spinning of the 2-sphere. Technically this originates from carefully keeping track of the $\mathcal{O}(r^{-2})$ term in the harmonic functions as we will explain in some more detail the next section.

4.4. Far Asymptotics of the Near Limit

To go beyond the naive analysis of the large distance behavior of the near limit of scaling solutions made at the end of the previous section we will need to keep track of a subleading term in the harmonic functions:

$$H = \frac{\Gamma}{r} + \frac{\Delta^i \hat{r}^i}{r^2} + \mathcal{O}(r^{-3})\,. \tag{4.13}$$

Here we introduced the electro-magnetic dipole

$$\Delta^i = \sum_n \Gamma_n r_n^i\,. \tag{4.14}$$

Let us stress that the unexpected twist we'll uncover does not originate in subleading terms in an asymptotic expansion of the metric. Rather, as we'll see, for certain terms the naive leading part will vanish promoting a 'subleading' part to the dominant contribution.

For the rest of this section we'll focus on the metric as nothing unexpected happens in the gauge fields or scalar expansions, as can be checked by the reader. The main non-triviality of the metric (4.1) is encoded in the warp factor Σ. It is readily calculated by inserting (4.13) in the auxiliary functions, that its leading behavior at large distance is

$$\Sigma = \frac{S}{4\pi\, r^2} + \mathcal{O}(r^{-3})\,, \tag{4.15}$$

where $S(\Gamma_{\mathrm{t}})$ [80, 81] is a constant that has the physical interpretation of entropy (or horizon area).

The key point is now the contribution of ω (4.4) to (4.1). As was pointed out in [23] it generically behaves at large distances as r^{-1}, with the coefficient directly proportional to (and responsible for) the total angular momentum of the solution. But since for scaling solutions the total angular momentum necessarily vanishes (see (4.10)) the term at order r^{-2} becomes the leading contribution. An explicit calculation reveals

$$\omega = \frac{K^i}{r^2}\varepsilon_{ijk}\hat{r}^j d\hat{r}^k + \mathcal{O}(r^{-3}) \tag{4.16}$$

where

$$\vec{K} = \frac{\langle \Gamma, \vec{\Delta} \rangle}{2} = \frac{1}{2}\sum_{n<m}\langle \Gamma_n, \Gamma_m \rangle \vec{r}_{nm} \tag{4.17}$$

Note that we can always choose coordinates such that $\vec{K}$ is oriented along the z-axis, such that in spherical coordinates the expression for ω then becomes

$$\omega = -\frac{K}{r^2}\sin^2\theta\, d\varphi + \mathcal{O}(r^{-3}) \tag{4.18}$$

If the metric were asymptotically flat, where the both the time-like and spatial warp factor would go to a constant, this $\mathcal{O}(r^{-2})$ behavior of ω would remain some subleading angular momentum multipole effect. But the asymptotics have changed by putting $h = 0$. The time-like warp factor now blows up like r^2 enhancing the contribution from ω while at the same time the spatial warp factor falls off like r^{-2} tempering the growth of the spatial sphere, in exactly such a way that both contributions become of the same order. Writing this out produces the far near metric:

$$ds^2 = -\frac{4\pi\, r^2}{S}dt^2 + \frac{S}{4\pi\, r^2}dr^2 + \frac{S}{4\pi}\left(d\theta^2 + \sin^2\theta(d\varphi + A)^2\right) \tag{4.19}$$

here we recognize an S^2 fibered over AdS_2, with a flat connection

$$A = \frac{16\pi^2\, K}{S^2}dt \tag{4.20}$$

So interestingly enough the far near metric is not exactly the near far metric but has the extra twist that the 2-sphere is rotating as time in AdS_2 flows, with the rate of rotation set by the intriguing quantity (4.17), which for lack of better current understanding we might just refer to as the 'twist' vector[51] . Note that the twist can apparently be removed by a coordinate transformation $\tilde{\varphi} = \varphi + \frac{4\pi K}{S^2} t$.

4.5. Comments

Scaling multi-centered solutions provide an interesting source of highly non-trivial asymptotic AdS_2 geometries. Recently [96] argued why they could provide important new insights and directions to 2D holography and they explored some of the first physical properties and consequences. In this note we showed that these solutions might be even richer than naively expected, in that they retain a subtle extra twist (or hair) that is not present in the empty AdS_2 background obtained from the near horizon geometry of an extremal black hole. We end with a number of small comments.

- First a small note of caution. Although it definitely appears as if the twist provides a leading contribution to the asymptotic metric it might be naive to simply treat on equal footing the $dtd\varphi$ and $d\varphi^2$ components. Directly related is the question if the asymptotic coordinate transformation $\tilde{\varphi} = \varphi + \frac{4\pi K}{S^2} t$, that could remove the twist, is indeed large and physically relevant or not. What exactly the correct asymptotic boundary conditions are, and the corresponding asymptotic symmetries, is a subtle issue that needs further careful analysis. This would require a precise 2D bulk theory containing all the relevant fields, which as far as we are aware has not been previously formulated. A simple sphere reduction keeping the connection A as a 2D gauge field seems to have problems with consistency and so a larger framework might be needed. In a lift to 5d terms of the order $r\, dtd\varphi$ are generated [96], but it is unclear if this has any implications for our discussion in 4D.
- If we are more optimistic this result has potentially interesting physical conse-

[51] We refrain from calling $\vec{K}$ spin as it should be clearly distinguished from the angular momentum $\vec{J}$ that vanishes.

quences. It has been assumed that it is exactly the scaling solutions that are key in understanding black hole entropy [28, 100]. On the microscopic side because they seem to be associated with an exponential number of 'pure-Higgs' states and on the gravity side because they closely resemble the black hole. The scaling solutions satisfy the condition [103] that the angular momentum of black hole microstates should vanish and it is interesting that the twist uncovered here provides a new observable that can differentiate, even asymptotically, between different AdS_2 scaling solutions. The natural arena to try and understand the twist better and provide a potential connection to microstates is of course holography. For the moment we have nothing to add to the interesting discussion and list of references in [96], but we hope to investigate this further in the future.

- We should point out that also in the $AdS_3 \times S^2$ limit of multi-centered solutions a spinning sphere made its appearance [104]. We see however no direct relation with the twist here, as the spin there has a direct interpretation as angular momentum, or R-charge in the dual field theory.

5. TOWARDS A HOLOGRAPHIC INTERPRETATION OF THE TWIST

What we are going to discuss in this chapter is based on an original work of the author in collaboration with Dieter Van Den Bleeken and Joris Raeymaekers which has published for the first time in this thesis. We start with considering the S^2 reduction of four-dimensional Einstein-Maxwell theory as a toy model for $\mathcal{N} = 2$ SUGRA. As it will be shown, we first need to find a way to resolve an inconsistency occurs in the naive approach. After generalizing this method such that it becomes applicable to the SUGRA solutions as well, we will successfully S^2 reduce the asymptotic geometry of the scaling solutions. We observe that the twist which appeared in the asymptotic geometry, shows up as a $U(1)$ gauge field with a flat connection. Our ultimate goal is to find a holographic interpretation of the twist as a new "*hair*" for this family of supersymmetric black holes via applying the AdS/CFT correspondence. The last step was in progress at the time of writing this thesis.

5.1. From 4D Einstein-Maxwell Theory to 2D Dilaton-Gravity

The ultimate aim is to S^2 reduce the asymptotic geometry of the scaling solutions (4.19) and obtain a two-dimensional theory with AdS spacetime. Doing so, the twist will emerge as a $U(1)$ gauge field in the lower dimensional theory. Although we do not discuss this point very concretely in this thesis, but having this two dimensional theory one can argue AdS_2/CFT_1 correspondence to explore a field theory origin for the twist we have reported in the gravity side. In the path, we can warm up by doing S^2 reduction procedure for the Einstein-Maxwell theory first as a toy model that has many similarities to $\mathcal{N} = 2$ supergravity. Although, as we will see the procedure will not be that much straight forward even in this simpler case.

Einstein-Maxwell theory as a toy model

As previously remarked, the $\mathcal{N}=2$ supergravity in $4D$ reduces to the Einstein-Maxwell if we set scalar moduli $\{t^A\}$ to constants and also reduce the number of $U(1)$ gauge fields $\{A^{\tilde{A}}\}$ to one. Yet, doing so leaves a tiny difference between $AdS_2\times S^2$ solutions in these two theories. As it will be discussed later, this solution comes with pure magnetic charges in Einstein-Maxwell theory. In the contrary, even the simplified version of $\mathcal{N}=2$ supergravity contains electromagnetic charges. We will see later that this difference can be removed by applying electromagnetic duality to "*rotate*" from pure magnetic charges towards mixed electromagnetic ones. For some technical reasons, it is easier to first dimensionally reduce the pure magnetic $AdS_2\times S^2$ solution of Einstein-Maxwell theory with and then go to the more general case we are mainly interested in, which is supergravity with multi gauge fields and electromagnetic charges. That is why once again here we mention the Einstein-Maxwell Lagrangian and its equations of motion for more convenience. Here they are

$$\mathcal{S}_{4D}=\frac{1}{16\pi}\int d^4x\sqrt{-\hat{g}}\,\left(\hat{R}-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right)\,, \tag{5.1}$$

where $\mu,\nu=1,...,4$ and all quantities with hat correspond to the four-dimensional theory. The equations of motion are

$$\hat{R}_{\mu\nu}-\frac{1}{2}\,\hat{R}\,\hat{g}_{\mu\nu}=\hat{T}_{\mu\nu}\quad;\quad\hat{T}_{\mu\nu}=\frac{1}{2}\left[\hat{F}_{\mu}{}^{\lambda}\hat{F}_{\nu\lambda}-\frac{1}{4}\,\hat{g}_{\mu\nu}\hat{F}_{\lambda\rho}\hat{F}^{\lambda\rho}\right]\,. \tag{5.2}$$

Now, let us for a second go all the way back to the very beginning of this thesis, i.e., subsection 2.1.2 where we talked about this theory in arbitrary dimensions. Taking trace of (2.9) one reads

$$R=\frac{-2\kappa^2}{D-2}T_{\rho}{}^{\rho}\,. \tag{5.3}$$

First, it shows (5.2) and (2.9,2.10) are the same but differently expressed equations (to see this easily take $\kappa^2=\frac{1}{2}$ and plug (5.3) into (5.2)). Second, for $D>2$ the Ricci scalar is always negative. However, exceptionally in $D = 4$ the energy-momentum tensor (5.2) becomes traceless and so the Ricci scalar vanishes. Trivially, the Minkowski metric is possible only for zero electromagnetic tensor $\hat{F}_{\mu\nu}$. Yet, there is another vacuum solution in which we are interested here, i.e., $AdS_2 \times S^2$. This is a metric with a compact $2D$ factor (the two-sphere) that can be dimensionally reduced to an effective $2D$ theory by applying Kaluza-Klein reduction technique.

5.1.1. No Gauge Potential

We are interested in an $AdS_2 \times S^2$ solution to $4D$ Einstein-Maxwell theory. Accordingly, the reduction ansatz we take are

$$ds^2 = \ell^2 \Big(e^{2\alpha\phi} ds^2_{2D} + e^{2\beta\phi} d\Omega^2 \Big) \quad , \quad \hat{F} = 2\ell \ \sin\theta \ d\theta \wedge d\varphi \ . \tag{5.4}$$

where $d\Omega^2$ is the surface element of a two-sphere. We have not specified the two dimensional manifold here. However, it is not an arbitrary one. More precisely, it only can have negative Ricci scalar so that the total Ricci scalar of the $4D$ manifold vanishes. Furthermore, we have not turned on any Kaluza-Klein gauge potential yet. We postpone this case to the next subsection where we will show there are some subtleties to achieve a consistent reduction.

What we are going to do now is to check consistency of our reduction ansatz (5.4) at the level of Lagrangian and equations of motion. Figure 5.1 depict the consistency check of a typical reduction procedure.

To read the Lagrangian for $2D$ theory, we plug reduction ansatz into $\mathcal{L}_{4D}$ and integrate over S^2 which leads to

$$\begin{aligned} \mathcal{L}_{2D} &= \frac{\ell^2}{4}\sqrt{-g_{2D}} \left\{ e^{2\beta\phi} \Big[R_{2D} - 2(\alpha + 2\beta)\Box\phi - 6\beta^2 \partial_a\phi\partial^a\phi \Big] - 2e^{2(\alpha-\beta)\phi} \Big(1 - e^{2\beta\phi} \Big) \right\} \\ &= \frac{\ell^2}{4}\sqrt{-g_{2D}} \left\{ e^{2\beta\phi} \Big[R_{2D} + 2\beta(2\alpha + \beta)\partial_a\phi\partial^a\phi \Big] - 2e^{2(\alpha-\beta)\phi} \Big(1 - e^{2\beta\phi} \Big) \right\} \end{aligned} \tag{5.5}$$

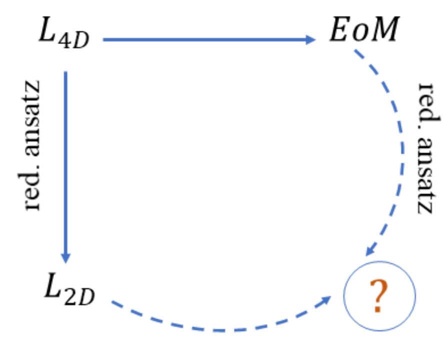

Figure 5.1: Consistency check of a reduction procedure. Given a set of reduction ansatz, we can obtain two families of EoM in lower dimensions. The first family is given by plugging the ansatz into higher dimensional EoM while the second one is found from the lower dimensional Lagrangian. A reduction ansatz is called consistent if the lower dimensional EoM solves the higher dimensional ones.

where ϕ depends only on $2D$ coordinates (t, r) and all differential operators are defined in 2D. In addition, to get the second equality one needs to perform a partial integration. The theory described by $\mathcal{L}_{2D}$ is a Dilaton-Gravity and the last term is the potential of the Dilaton which results in a non-zero vacuum energy, similar to cosmological constant.

To obtain equations of motion in lower dimension, we start from (5.5). In order to do variation with respect to the metric, we need some useful identities [105]

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab} \ , \tag{5.6}$$

$$\nabla_c g_{ab} = 0 = \nabla_c \mathfrak{g}_{ab} \quad ; \quad \mathfrak{g}_{ab} := \sqrt{-g}\ g_{ab} \ , \tag{5.7}$$

$$\delta\Gamma^a_{bc} = \frac{1}{2}g^{ad}\Big(\nabla_b\delta g_{dc} + \nabla_c\delta g_{bd} - \nabla_d\delta g_{bc}\Big) \ , \tag{5.8}$$

$$\delta R^a_{bcd} = \nabla_c\delta\Gamma^a_{bd} - \nabla_d\delta\Gamma^a_{bc} \ , \tag{5.9}$$

$$\delta R_{ab} = \nabla_c\delta\Gamma^c_{ab} - \nabla_b\delta\Gamma^c_{ac} \ . \tag{5.10}$$

Here $\mathfrak{g}_{ab}$ is the *metric density of weight* $+1$. [52] Having these, let us now vary the first term of the Lagrangian $\delta\mathcal{L}^{(1)}_{2D}$ with respect to the metric and follow the calculation step

[52]Generally $\mathfrak{M}^{a\cdots}_{b\cdots}$ is called a *tensor density of weight* m if its transformation law under general

by step

$$\frac{4}{\ell^2}\delta\mathcal{L}^{(1)}_{2D}\big|_{\delta g^{ab}} = \tag{5.13}$$
$$\sqrt{-g_2}\ e^{2\beta\phi}\Big[\Big(R_{ab}-\frac{1}{2}Rg_{ab}\Big)\delta g^{ab}+g^{ab}\Big(\nabla_c\delta\Gamma^c_{ab}-\nabla_b\delta\Gamma^c_{ac}\Big)\Big] =$$
$$-\beta\sqrt{-g_2}\ e^{2\beta\phi}\Big(\partial_c\phi\Big)\Big[g^{ab}g^{cd}\Big(\nabla_a\delta g_{db}+\nabla_b\delta g_{ad}-\nabla_d\delta g_{ab}\Big)-g^{ac}g^{bd}\Big(\nabla_a\delta g_{db}\Big)\Big] =$$
$$-2\beta\sqrt{-g_2}\ e^{2\beta\phi}g^{ab}g^{cd}\Big(\partial_c\phi\Big)\Big(\nabla_a\delta g_{db}-\nabla_d\delta g_{ab}\Big) =$$
$$-2\beta\sqrt{-g_2}\ e^{2\beta\phi}\Big\{2\beta\Big[\partial_a\phi\partial_b\phi-g_{ab}\partial_c\phi\partial^c\phi\Big]+\nabla_b\partial_a\phi-g_{ab}\Box\phi\Big\}\delta g^{ab}\ .$$

First of all, we notice that two last terms in $\delta\Gamma^c_{ac}$ cancel each other [53] . Then, to obtain the second equality we note that the Einstein tensor G_{ab} is identically zero in $2D$. We also need to do partial integration. To get the third equality one needs to play with contracted indices, and finally in the last step, partially integrate once again (note that there is a minus sign coming from partial integration, but it will be canceled because of replacing δg_{ab} by δg^{ab}). The variation of the two other terms in the Lagrangian leads to

$$\begin{aligned}\frac{4}{\ell^2}\delta\mathcal{L}^{(2)}_{2D}\big|_{\delta g^{ab}} &= 2\beta(2\alpha+\beta)\sqrt{-g_2}\ e^{2\beta\phi}\Big[\frac{-1}{2}\ g_{ab}\partial^c\phi\partial_c\phi+\partial_a\phi\partial_b\phi\Big]\delta g^{ab}\ ,\\ \frac{4}{\ell^2}\delta\mathcal{L}^{(3)}_{2D}\big|_{\delta g^{ab}} &= \sqrt{-g_2}\ e^{2(\alpha-\beta)}\Big(1-e^{2\beta\phi}\Big)g_{ab}\delta g^{ab}\ .\end{aligned} \tag{5.14}$$

coordinate transformations $x^a \to x^{'a}$ is given by

$$\mathfrak{M}'^{a\ldots}_{b\ldots} = J^m\frac{\partial x^{'a}}{\partial x^c}\cdots\frac{\partial x^d}{\partial x'^b}\cdots\mathfrak{M}^{c\ldots}_{d\ldots}\ , \tag{5.11}$$

where $J=\left|\frac{\partial x^a}{\partial x'^b}\right|$ is the Jacobian of the corresponding transformation [105]. The covariant derivative of a tensor density contains all normal terms as if it is a tensor plus all possible terms like $-m\Gamma^d_{dc}\mathfrak{M}^{a\ldots}_{b\ldots}$. In particular, for a vector density $\mathfrak{M}^a$ of weight $m=+1$ we have $\nabla_a\mathfrak{M}^a=\partial_a\mathfrak{M}^a$ because all extra terms on the RHS cancel out. That is why vector densities are of special interest. Luckily, any tensor can turn into a tensor density very easily via multiplying it by $\sqrt{-g}$ which is a scalar density of weight $m=+1$ by itself, i.e., $\sqrt{-g'}=J\sqrt{-g}$. Then we can use the following result

$$\sqrt{-g}\ \nabla_a M^a = \nabla_a\mathfrak{M}^a = \partial_a\mathfrak{M}^a = \partial_a\big(\sqrt{-g}\ M^a\big)\ . \tag{5.12}$$

[53](5.8) is anti-symmetric under exchanging a and d and so getting contracted by g^{ad} kills the last two terms.

Putting all these together, we obtain the first EOM

$$\begin{aligned} \frac{\delta \mathcal{L}_{2D}}{\delta g^{ab}} \sim \quad & 2\beta\Big[(2\alpha-\beta)\partial_a\phi\partial_b\phi - \nabla_a\partial_b\phi\Big] + \\ & g_{ab}\Big[2\beta\Box\phi - \beta(2\alpha-3\beta)\partial_c\phi\partial^c\phi + e^{2(\alpha-2\beta)\phi}\big(1-e^{2\beta\phi}\big)\Big] = 0 \end{aligned} \tag{5.15}$$

whose trace is

$$2\beta\Box\phi + 4\beta^2\partial_a\phi\partial^a\phi + 2e^{2(\alpha-2\beta)\phi}\big(1-e^{2\beta\phi}\big) = 0\ . \tag{5.16}$$

This equality will help us later to match two families of EOMs.

The variation of $\mathcal{L}_{2D}$ w.r.t to the scalar field ϕ results in the second EOM

$$\begin{aligned} \frac{4}{\ell^2}\delta\mathcal{L}^{(1)}_{2D}\big|_{\delta\phi} &= 2\beta\sqrt{-g_2}\ e^{2\beta\phi}R_2\delta\phi\ , \\ \frac{4}{\ell^2}\delta\mathcal{L}^{(2)}_{2D}\big|_{\delta\phi} &= -4\beta(2\alpha+\beta)\sqrt{-g_2}\ e^{2\beta\phi}\Big[\beta\partial_a\phi\partial^a\phi + \Box\phi\Big]\delta\phi\ , \\ \frac{4}{\ell^2}\delta\mathcal{L}^{(3)}_{2D}\big|_{\delta\phi} &= -2\sqrt{-g_2}\ e^{2\beta\phi}e^{2(\alpha-2\beta)\phi}\Big[2(\alpha-\beta) - 2\alpha e^{2\beta\phi}\Big]\delta\phi\ . \end{aligned} \tag{5.17}$$

Be gathered in a single equation it becomes

$$\frac{\delta\mathcal{L}_{2D}}{\delta\phi} \sim 2\beta R_2 - 4\beta(2\alpha+\beta)\Big[\beta\partial_a\phi\partial^a\phi + \Box\phi\Big] - 2e^{2(\alpha-2\beta)\phi}\Big[2(\alpha-\beta) - 2\alpha\ e^{2\beta\phi}\Big] = 0\ . \tag{5.18}$$

On the other hand, one can find a set of $2D$ EOMs by substituting reduction ansatz (5.4) in $4D$ EOMs. Doing so we get

$$\begin{aligned} EE_{ab} = \quad & 2\beta\Big[(2\alpha-\beta)\partial_a\phi\partial_b\phi - \nabla_a\partial_b\phi\Big] + \\ & g_{ab}\Big[2\beta\Box\phi - \beta(2\alpha-3\beta)\partial_c\phi\partial^c\phi + e^{2(\alpha-2\beta)\phi}\big(1-e^{2\beta\phi}\big)\Big] = 0 \end{aligned} \tag{5.19}$$

$$EE_{\varphi\varphi} = \quad -1/2R_2 + \beta^2\partial_a\phi\partial^a\phi + (\alpha+\beta)\Box\phi - e^{2(\alpha-2\beta)\phi} = 0\ , \tag{5.20}$$

and trace of the first equation is

$$2\beta\Box\phi + 4\beta^2\partial_a\phi\partial^a\phi + 2e^{2(\alpha-2\beta)\phi}\Big(1 - e^{2\beta\phi}\Big) = 0 \ . \tag{5.21}$$

The first equation is the same as the first EOM we found from the Lagrangian (5.15). To get the second equation, one needs to add 2α times (5.21) to (5.18)

$$2\beta\Big[R_2 - 2\beta^2\partial_a\phi\partial^a\phi - 2(\alpha+\beta)\Box\Phi + 2e^{2(\alpha-2\beta)}\Big] = 0 \tag{5.22}$$

which is -4β times (5.20). This shows that the reduction ansatz (5.4) we started with yields a consistent S^2 reduction.

5.1.2. Turning the KK Gauge Potential On

As mentioned in the last subsection, we also can allow having Kaluza-Klein gauge modes. More precisely, there are three $SO(3)$ gauge potentials $\{A^i\}$ that can be turned on [54] . Then we need to modify the reduction ansatz accordingly and check consistency of reduction procedure. However, as we will see the normal reduction procedure leads to some inconsistencies to resolve which we suggest a slightly different formulation.

Now following [106] for our case, we suggest the following reduction ansatz (in which S^2 is now fibered on top of $2D$ external manifold)

$$ds^2 = \ell^2\Big(e^{2\alpha\phi}ds^2_{2D} + e^{2\beta\phi}\mathcal{D}y^i\mathcal{D}y_i\Big) \tag{5.23}$$

$$\hat{F} = \ell\varepsilon_{ijk}\Big(\mathcal{D}y^i\wedge\mathcal{D}y^j - F^{ij}\Big)y^k \quad ; \quad F^{ij} = dA^{ij} - A^i{}_k\wedge A^k{}_j \ . ^{55} \tag{5.24}$$

[54] Generally in any dimensional reduction case, the internal isometries emerge as gauge potentials in lower dimensional theory. The Calabi-Yau threefold has no isometry which make the reduction procedure much simpler.

where from now on we assume the following ranges for different indices

$$i,j,k,... = \{1,2,3\} \quad , \quad a,b,c,... = \{1,2\} \quad , \quad \mu,\nu,... = \{1,...,4\} \ . \tag{5.25}$$

We also take $\{y_i\}$ as internal coordinates via which the unit S^2 is embedded in $\mathbb{R}^3$, so they satisfy $y_i y^i = 1$ and their covariant derivatives are given by

$$\mathcal{D}y^i = dy^i - A^i{}_j y^j \quad ; \quad A^i{}_j = \varepsilon^i{}_{jk} A^k \ . \tag{5.26}$$

So A^i are some $SO(3)$-valued one-form whose curvature two-form F^{ij} can be read from (5.24). For later applications, we also mention the following identity that is easily obtainable from (5.26)

$$F^k = \varepsilon^k{}_{ij} F^{ij} \ . \tag{5.27}$$

For now, let us reduce the $SU(2)$ (or equivalently $SO(3)$) gauge group to the simplest case by setting $A^1 = 0 = A^2$. So the only non-vanishing gauge potential is A^3, a $U(1)$ valued one-form whose components depend only on the external coordinates (t,r)

$$A^3 = A(r)dt + A(t)dr \ . \tag{5.28}$$

This choice simplifies the curvature two-form to that of an abelian gauge field by killing the second term of F^{ij} (5.24). However, the reason behind this choice is not making things simpler. In fact, comparing (5.23) to the asymptotic metric of the scaling solutions (4.19) one can explicitly deduce the same gauge potential. Having these ansatz, once again we start from the higher dimensional Lagrangian (5.1) and

[55] Commonly in the literature, the two-form $\hat{F}$ is called *global angular two-form* and denoted by e_2 (See, e.g. [107]).

substitute (5.23,5.24). We then get

$$\begin{aligned}
\mathcal{L}_{2D} &= \frac{\ell^2}{4}\sqrt{-g_{2D}}\times \\
&\Big\{e^{2\beta\phi}\Big[R_{2D}-2(\alpha+2\beta)\Box\phi-6\beta^2\partial_a\phi\partial^a\phi-\frac{1}{3}e^{-2\alpha\phi}\Big(1+\frac{1}{2}e^{2\beta\phi}\Big)F_{(2D)ab}F^{ab}_{(2D)}\Big] \\
&-2e^{2(\alpha-\beta)\phi}\Big(1-e^{2\beta\phi}\Big)\Big\} = \\
&\frac{\ell^2}{4}\sqrt{-g_{2D}}\times \\
&\Big\{e^{2\beta\phi}\Big[R_{2D}+2\beta(2\alpha+\beta)\partial_a\phi\partial^a\phi-\frac{1}{3}e^{-2\alpha\phi}\Big(1+\frac{1}{2}e^{2\beta\phi}\Big)F_{(2D)ab}F^{ab}_{(2D)}\Big] \\
&-2e^{2(\alpha-\beta)\phi}\Big(1-e^{2\beta\phi}\Big)\Big\}\ ,
\end{aligned} \tag{5.29}$$

where $F_{(2D)} \equiv F^3 = dA^3$ given by (5.27). Moreover, to obtain the second equality we partially integrated the second term. Comparing to (5.5) we had before, one finds an extra term coming from the $U(1)$ gauge field. We call it $\mathcal{L}^{(new)}_{\rm 2D}$ which is

$$\mathcal{L}^{(new)}_{\rm 2D} = \frac{-\ell^2}{12}\sqrt{-g_{2D}}\ e^{2(\beta-\alpha)\phi}\Big(1+\frac{1}{2}e^{2\beta\phi}\Big)F_{(2D)ab}F^{ab}_{(2D)}\ . \tag{5.30}$$

So the new terms in EOMs are originated from this term whose variation w.r.t the lower dimensional fields are given by

$$\begin{aligned}
&\delta\mathcal{L}^{(new)}_{2D}\big|_{\delta A^b} = \\
&\frac{\ell^2}{3}\sqrt{-g}\ e^{2(\beta-\alpha)\phi}\Big\{\Big(1+\frac{1}{2}e^{2\beta\phi}\Big)\nabla^aF_{ab}+\Big[2(\beta-\alpha)+(2\beta-\alpha)e^{2\beta\phi}\Big](\partial^a\phi)F_{ab}\Big\}\delta A^b = \\
&\frac{\ell^2}{3}\sqrt{-g}\ e^{2(\beta-\alpha)\phi}\Big\{\nabla^aF_{ab}+2(\beta-\alpha)(\partial^a\phi)F_{ab}+ \\
&\qquad\frac{1}{2}e^{2\beta\phi}\Big[\nabla^aF_{ab}+2(2\beta-\alpha)(\partial^a\phi)F_{ab}\Big]\Big\}\delta A^b\ ,
\end{aligned} \tag{5.31}$$

$$\delta\mathcal{L}^{(new)}_{2D}\big|_{\delta g^{ab}} = \frac{-\ell^2}{12}\sqrt{-g}e^{2(\beta-\alpha)\phi}\Big(1+\frac{1}{2}e^{2\beta\phi}\Big)\Big[-\frac{1}{2}\ g_{ab}F_{cd}F^{cd}+2F_a{}^cF_{bc}\Big]\delta g^{ab}\ , \tag{5.32}$$

$$\delta\mathcal{L}_{2D}^{(new)}\big|_{\delta\phi} = \frac{-\ell^2}{12}\sqrt{-g}\ F_{ab}F^{ab}e^{2(\beta-\alpha)\phi}\Big[2\beta\big(1+e^{2\beta\phi}\big) - \alpha\big(2+e^{2\beta\phi}\big)\Big]\delta\phi =$$

$$\frac{-\ell^2}{12}\sqrt{-g}\ F_{ab}F^{ab}e^{2(\beta-\alpha)\phi}\Big[2(\beta-\alpha)+(2\beta-\alpha)e^{2\beta\phi}\Big]\delta\phi\ . \tag{5.33}$$

Put all these together with our previous results (5.13,5.14) and (5.17) we can read three EOMs

$$\begin{aligned}\frac{\delta\mathcal{L}_{2D}}{\delta g^{ab}} \sim\quad & 2\beta\Big[(2\alpha-\beta)\partial_a\phi\partial_b\phi - \nabla_a\partial_b\phi\Big] + \\ & g_{ab}\Big[2\beta\Box\phi - \beta(2\alpha-3\beta)\partial_c\phi\partial^c\phi + e^{2(\alpha-2\beta)\phi}\big(1-e^{2\beta\phi}\big)\Big] - \\ & 1/3e^{-2\alpha\phi}\big(1+1/2e^{2\beta\phi}\big)\Big[-1/2g_{ab}F_{cd}F^{cd}+2F_a{}^cF_{bc}\Big] = 0\ ,\end{aligned} \tag{5.34}$$

$$\begin{aligned}\frac{\delta\mathcal{L}_{2D}}{\delta\phi} \sim\quad & 2\beta R_2 - 4\beta(2\alpha+\beta)\Big[\beta\partial_a\phi\partial^a\phi + \Box\phi\Big] - \\ & 2e^{2(\alpha-2\beta)\phi}\Big[2(\alpha-\beta) - 2\alpha\ e^{2\beta\phi}\Big] \\ & -1/3\ F_{ab}F^{ab}e^{-2\alpha\phi}\Big[2(\beta-\alpha)+(2\beta-\alpha)e^{2\beta\phi}\Big] = 0\ ,\end{aligned} \tag{5.35}$$

$$\begin{aligned}\frac{\delta\mathcal{L}_{2D}}{\delta A^a} \sim\quad & \big(1+1/2e^{2\beta\phi}\big)\Big[\nabla^aF_{ab}+2(\beta-\alpha)(\partial^a\phi)F_{ab}\Big] + \\ & \beta e^{2\beta\phi}(\partial^a\phi)F_{ab} = 0\ .\end{aligned} \tag{5.36}$$

On the other hand, by plugging reduction ansatz (5.23,5.24) in $4D$ equations (5.2), one reads

$$\begin{aligned}EE_{ab} =\quad & 2\beta\Big[(2\alpha-\beta)\partial_a\phi\partial_b\phi - \nabla_a\partial_b\phi\Big] + \\ & g_{ab}\Big[2\beta\Box\phi + \beta(3\beta-2\alpha)\partial_c\phi\partial^c\phi + e^{2(\alpha-2\beta)\phi}\big(1-e^{2\beta\phi}\big)\Big] = 0\ ,\end{aligned} \tag{5.37}$$

$$EE_{\varphi\varphi} = \quad -1/2\ R + (\alpha+\beta)\Box\phi + \beta^2\partial_a\phi\partial^a\phi - e^{2(\alpha-2\beta)\phi} = 0\ , \tag{5.38}$$

and the following constraint

$$F_{ab} = F_{(2D)} = 0 \qquad \Longleftrightarrow \qquad d\hat{F} = 0\ . \tag{5.39}$$

This restricts A^3 to be a flat $U(1)$ gauge potential, i.e., with zero curvature which guarantees that the Bianchi identity is satisfied by $\hat{F}$. However, there is no condition in $4D$ theory that confines the gauge field to have zero curvature. In other words, we need to modify the $4D$ theory such that this requirement shows up as an equation of motion and so become manifested. As we will show in the next section, this happens in the so-called "*first order formalism*" in which $\hat{F}$ is considered as the true variable instead of the gauge field A, i.e., the action $\mathcal{S}_{4D} = \mathcal{S}_{4D}[\hat{g}, \hat{F}, \hat{a}]$ will be a functional of the field strength and *dual gauge field* $\hat{a}$.

5.2. Consistent Reduction via the First Order Formalism

As was explained, the common reduction ansatz will stop working properly as soon as we turn on KK gauge potentials. This inconsistency appears via the constraint (5.39) that only shows up in the lower dimensional theory and does not have any higher dimensional manifestation. Consequently, the way to resolve this inconsistency seems to be finding a way to make this constraint manifested in $4D$ equations of motion. It then turns out to that aim, we need to take $\hat{F}$ as a dynamical variable of the four-dimensional theory. We also introduce a one-form a as a Lagrange multiplier via which we can introduce $d\hat{F} = 0$ as a constraint [55] . This results in the following action

$$\mathcal{S}_{4D}[\hat{g}, \hat{F}, \hat{a}] = \int d^4x\sqrt{-\hat{g}}\Big(\hat{R} - \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\Big) + \int \hat{a} \wedge d\hat{F} \ , \tag{5.40}$$

from which one can read EOMs in $4D$

$$\begin{aligned}
\frac{\delta \mathcal{S}_{4D}}{\delta \hat{g}^{\mu\nu}} = 0 \quad &\Rightarrow \quad \hat{G}_{\mu\nu} - \frac{1}{2}\hat{F}_{\mu\rho}\hat{F}_{\nu}{}^{\rho} + \frac{1}{8}\hat{g}_{\mu\nu}\hat{F}_{\rho\gamma}\hat{F}^{\rho\gamma} = 0 \ , \\
\frac{\delta \mathcal{S}_{4D}}{\delta \hat{F}} = 0 \quad &\Rightarrow \quad \star\hat{F} = d\hat{a} \ , \\
\frac{\delta \mathcal{S}_{4D}}{\delta a} = 0 \quad &\Rightarrow \quad d\hat{F} = 0 \ .
\end{aligned} \tag{5.41}$$

[55] Later on, it becomes clear that this one-form is actually the dual gauge field satisfying $\star\hat{F} = d\hat{a}$.

The third equation is the Bianchi identity for $\hat{F}$ which we were seeking. Furthermore, from the second equation we deduce that a is actually the dual gauge field. The same equation leads to the Maxwell's equation $d\star\hat{F}=0$.

To do the S^2 reduction, we are going to take the same ansatz as (5.23) for the metric, while the proper one for the field strength is now given by

$$\hat{F} = 2\ell\ p\ \varepsilon_{ijk}\big(\mathcal{D}y^i \wedge \mathcal{D}y^j\big)y^k\ . \tag{5.42}$$

Here p is the magnetic charge which is needed to have $AdS_2 \times S^2$ [56] as a solution of (5.40). One can check that this choice solves Maxwell's equations as well. Notably, the requirement of $\hat{F}$ satisfying the Bianchi identity leads naturally to the constraint (5.39), i.e., $F_{(2D)} = dA = 0$. In other words, by adding the Lagrange multiplier and using consistent reduction ansatz (5.42), the constraint we were looking for appears as a solution to one of EOMs. Also, by substituting the second equation from (5.41) in (5.40) one gets back the original Einstein-Maxwell theory.

We are interested in the following $4D$ solution

$$ds^2 = \ell^2\Big(e^{2\alpha\phi}ds^2_{2D} + e^{2\beta\phi}\mathcal{D}y^i\mathcal{D}y_i\Big)\ , \tag{5.43}$$

$$\hat{F} = 2\ell\ p\sin\theta d\theta\wedge\big(d\varphi + A\big)\ \ ; \qquad dA = 0\ , \tag{5.44}$$

$$\hat{a} = \hat{a}_\mu(t,r)dx^\mu\ \ ; \qquad \hat{a}_\theta, \hat{a}_\phi : const., \qquad d\hat{a} = \ell e^{-2\beta\phi} vol_{AdS_2}\ . \tag{5.45}$$

The 2D action can be obtained by plugging these reduction ansatz (5.43,5.44,5.45). But before going there, let us have a more careful look at the last term in the Lagrangian. So for a moment, we are not restricted any more to the $AdS_2\times S^2$ solution and also the aforementioned reduction ansatz it requires. Releasing this restriction, now we digress a bit from (5.45) by taking $\hat{a}_\varphi = B$ with a non-constant B (clearly it was already allowed to have a constant $\hat{a}_\varphi$ in (5.45)). One should note that even though (5.46) does

[56]It is not really a direct product of AdS_2 and S^2. Indeed, the two-sphere appears as a fiber bundle on top of AdS_2 spacetime as the base manifold.

not satisfy $4D$ EOMs (5.41) exactly, but we are allowed to make these choices because as we are just about to see, these ansatz leads to a $2D$ theory whose EOMs solves the $4D$ ones which guarantees "*consistency*" of the reduction process. So the one-form $\hat{a}$ which we are going to take is

$$\hat{a} = \xi + B d\varphi \ , \tag{5.46}$$

where ξ is any one-form such that $d\xi \sim vol_{AdS_2}$ (this relation only determines $d\xi$ up to a sign which will be fixed later). We are also going to release the constraint (5.39), i.e., we consider $F_{(2D)} = dA \neq 0$. So the only term that contributes to the last term of the Lagrangian will be

$$\int \hat{a} \wedge d\hat{F} = 2\ell\, p \int B F_{(2D)} \wedge \big(\sin\theta d\theta \wedge d\varphi \big) \ . \tag{5.47}$$

Integrating this over S^2 and plugging ansatz given in (5.43,5.44,5.46) (with $dA \neq 0$) we obtain the following $2D$ Lagrangian

$$\begin{aligned} \mathcal{S}_{2D}[g, \phi, B, A] = 4\pi\ell^2 \int d^2x \sqrt{-g} e^{2\beta\phi} \times \\ \Big[R + 2\beta(2\alpha+\beta)\partial_a\phi\partial^a\phi - \frac{1}{6} e^{-2(\alpha-\beta)\phi} F_{ab}F^{ab} + 2e^{2(\alpha-\beta)\phi}(1-e^{-2\beta\phi}) \Big] \\ + 8\pi\ell\, p \int BF \ . \end{aligned} \tag{5.48}$$

Varying this action w.r.t the dynamical variables results in the following EOMs

$$\begin{aligned} \frac{\delta \mathcal{S}_{2D}}{\delta g^{ab}} \sim \quad & 2\beta\Big[(2\alpha-\beta)\partial_a\phi\partial_b\phi - \nabla_a\partial_b\phi \Big] + \\ & g_{ab}\Big[2\beta\Box\phi - \beta(2\alpha-3\beta)\partial_c\phi\partial^c\phi + e^{2(\alpha-2\beta)\phi}\big(1-e^{2\beta\phi}\big) \Big] + \\ & \frac{1}{3} e^{-2(\alpha-\beta)\phi} \left[\frac{1}{4} g_{ab}F_{cd}F^{cd} + 2\beta F_a{}^c F_{bc} \right] = 0 \ , \end{aligned} \tag{5.49}$$

$$\frac{\delta \mathcal{S}_{2D}}{\delta \phi} \sim 2\beta R_2 - 4\beta(2\alpha+\beta)\Big[\beta\partial_a\phi\partial^a\phi + \Box\phi\Big] - \tag{5.50}$$
$$2e^{2(\alpha-2\beta)\phi}\Big[2(\alpha-\beta) - 2\alpha\, e^{2\beta\phi}\Big] + \frac{1}{3}\,(\alpha-2\beta)e^{-2(\alpha-2\beta)\phi}F_{ab}F^{ab} = 0\ ,$$

$$\frac{\delta \mathcal{S}_{2d}}{\delta B} \sim F_{(2D)} = 0\ , \tag{5.51}$$

$$\frac{\delta \mathcal{S}_{2D}}{\delta A} \sim dB = 0\ . \tag{5.52}$$

Finally, after imposing the third equation $F_{(2D)} = 0$ to the first two we obtain

$$2\beta\Big[(2\alpha-\beta)\partial_a\phi\partial_b\phi - \nabla_a\partial_b\phi\Big] + \tag{5.53}$$
$$g_{ab}\Big[2\beta\Box\phi - \beta(2\alpha-3\beta)\partial_c\phi\partial^c\phi + e^{2(\alpha-2\beta)\phi}\big(1-e^{2\beta\phi}\big)\Big] = 0\ ,$$

$$2\beta R_2 - 4\beta(2\alpha+\beta)\Big[\beta\partial_a\phi\partial^a\phi + \Box\phi\Big] - 2e^{2(\alpha-2\beta)\phi}\Big[2(\alpha-\beta) - 2\alpha\, e^{2\beta\phi}\Big] = 0\ . \tag{5.54}$$

Like before, it is useful to take trace of (5.53)

$$2\beta\Box\phi + 4\beta^2\partial_c\phi\partial^c\phi + 2e^{2(\alpha-2\beta)\phi}\big(1-e^{2\beta\phi}\big) = 0\ . \tag{5.55}$$

Now, after fixing two free parameters as $\alpha = -\frac{1}{4}, \beta = \frac{1}{2}$, making a redefinition $\Phi = e^{\phi}$ and also applying (5.55) we will find

$$\nabla_a\partial_b\Phi - g_{ab}\Big[\Box\Phi - \Phi^{-\frac{1}{2}}\big(1-\Phi^{-1}\big)\Big] = 0\ , \tag{5.56}$$
$$R - \Phi^{-\frac{3}{2}}\big(1-3\Phi^{-1}\big) = 0\ ,$$
$$F_{(2D)} = dA = 0\ ,$$
$$dB = 0\ .$$

As one may realize from the last two relations, to obtain these we have imposed those two constraints as well, i.e., F satisfying Bianchi identity and $\hat{a}_\varphi$ being a constant. This proves that any solution to $2D$ EOMs also solves $4D$ ones.

5.3. From Einstein-Maxwell to $\mathcal{N}=2$ SUGRA

By construction, the $\mathcal{N}=2$ SUGRA respects electromagnetic duality (at the level of equations of motion). Hence, to be able to generalize our successful reduction ansatz (5.43,5.44,5.46) to the scaling solutions via making analogy, we first need to rewrite our results in a duality invariant formulation.

5.3.1. Electromagnetic Duality

The toy model described by (5.40) is a pure magnetic model and so not invariant under electromagnetic duality, while our main purpose is to find the proper S^2 reduction ansatz for scaling solution to $\mathcal{N}=2$ supergravity. The theory contains massive dyons whose electromagnetic charges are gathered inside a symplectic vector $\Gamma^a = (\Gamma_{\tilde{A}}, \Gamma^{\tilde{A}})$ given by (3.18) or equivalently by second line of (3.16). The same equations reveal form of the corresponding $U(1)$ field strengths. We repeat relation (3.93) here for more convenience

$$\Gamma^a := \begin{pmatrix} \Gamma_{\tilde{A}} \\ \\ \Gamma^{\tilde{A}} \end{pmatrix} \quad \longleftrightarrow \quad F^a := \begin{pmatrix} F_{\tilde{A}} \\ \\ F^{\tilde{A}} \end{pmatrix} \quad \text{with} \quad F_{\tilde{A}} := \mathrm{Re}\Big[\tau_{\tilde{A}\tilde{B}}\big(F^{\tilde{B}} + i\star F^{\tilde{B}}\big)\Big] \ . \tag{5.57}$$

The first thing to note is that the lower half of F^a are not new quantities, they are rather given in terms of $F^{\tilde{A}}$ and its dual, as shown above. It has been defined such that it satisfies the Bianchi identity too, so we can write a single identity for all components of F^a that is $dF^a = 0$ [108]. Notably, here $\tau_{\tilde{A}\tilde{B}}$ is a scalar dependent complex matrix with $\Theta_{\tilde{A}\tilde{B}} \equiv \mathrm{Re}[\tau_{\tilde{A}\tilde{B}}]$, $H_{\tilde{A}\tilde{B}} \equiv \mathrm{Im}[\tau_{\tilde{A}\tilde{B}}]$ and the index $a = 2\tilde{A} = 1, ..., 2h^{(1,1)} + 2$. More precisely, not only does it explain the way gauge fields are coupled together but also reveals a coupling between scalars and gauge fields. As discussed in very detail in subsection 3.3.1, this gauge-scalar coupling emerges in a symplectic vector bundle fibered on top of scalars target space which as shown in the same subsection is a special Kähler manifold. Luckily, in the case of asymptotic theory things get considerably easier since scalars take constant values and so do $\tau_{\tilde{A}\tilde{B}}$.

The electromagnetic duality transformation between two system of charges $\{\Gamma^a\}$ and $\{\Gamma'^a\}$ is given by a $2\tilde{A} \times 2\tilde{A}$ matrix M

$$\Gamma'^a = M^a{}_b\,\Gamma^b \quad ; \qquad M = \begin{pmatrix} (M_1)_{\tilde{A}}{}^{\tilde{B}} & (M2)_{\tilde{A}\tilde{B}} \\ \\ (M3)^{\tilde{A}\tilde{B}} & (M4)^{\tilde{A}}{}_{\tilde{B}} \end{pmatrix} . \tag{5.58}$$

For now, M belongs to the largest group of linear matrix transformations, i.e., $M \in GL(2\tilde{A},\mathbb{R})$ and field strength F^a transforms in its fundamental representation. Also, one may recall that the Hodge dual operator in $\{D_{\tilde{A}}, D^{\tilde{A}}\}$ is given by $\diamond$ whose action has been defined in (3.81). Here, this operator takes the following matrix form

$$\star_4\, F^a = \diamond F^a \equiv \mathcal{I}^a{}_b F^b \quad ; \qquad \mathcal{I}^a_{\ b} = \begin{pmatrix} -\big[\Theta H^{-1}\big]_{\tilde{A}}{}^{\tilde{B}} & \big[H + \Theta H^{-1}\Theta\big]_{\tilde{A}\tilde{B}} \\ -\big[H^{-1}\big]^{\tilde{A}\tilde{B}} & \big[H^{-1}\Theta\big]^{\tilde{A}}{}_{\tilde{B}} \end{pmatrix} \tag{5.59}$$

According to the fact that $\mathcal{I}^2 = -1$, one can consider $\mathcal{I}$ as a complex structure (or equivalently a symplectic metric like J in (3.35)), and so (5.59) implies imaginary self-duality of F^a. Requiring this property to remain intact under duality transformation, i.e., having $\star F'^a = \mathcal{I}'^a{}_b F'^b$ demands $\mathcal{I}' = M\mathcal{I}M^{-1}$ that restricts M to the smaller group $Sp(2\tilde{A},\mathbb{R}) \subset GL(2\tilde{A},\mathbb{R})$ and so its four $\tilde{A}\times\tilde{A}$ blocks satisfy the following conditions

$$M_1{}^T\, M_4 - M_3{}^T\, M_2 = 1 \quad , \quad M_1{}^T\, M_3 = M_3{}^T\, M_1 \quad , \quad M_2{}^T\, M_4 = M_4{}^T\, M_2 \, . \tag{5.60}$$

Then the transformed coupling matrix τ'_{AB} is given by natural action of the symplectic matrix M

$$\tau'_{\tilde{A}\tilde{B}} = \Big[\big(M_1\tau + M_2\big)\big(M_3\tau + M_4\big)^{-1}\Big]_{\tilde{A}\tilde{B}} \, . \tag{5.61}$$

Now consider the following form of the Lagrangian

$$\mathcal{S} = \int F^{\tilde{A}} \wedge F_{\tilde{A}} = -\int \mathrm{Im}\Big[\tau_{\tilde{A}\tilde{B}}\, F^{\tilde{A}} \wedge \big(\star F^{\tilde{B}} - iF^{\tilde{B}}\big)\Big] \ . \tag{5.62}$$

It can be shown that this action is not invariant under duality transformations. However, like said before, it leads to duality invariant equations of motion.

5.3.2. Generalized Einstein-Maxwell Theory

Here, as a short discussion and for our later purposes, we generalize the toy model (5.40) so that it contains arbitrary numbers of $U(1)$ gauge fields. The following action describes such a theory in $4D$

$$\mathcal{S}_{4D} = \int d^4x\sqrt{-\hat{g}}\Big(\hat{R} - \frac{1}{4}\hat{F}^{\tilde{A}}{}_{\mu\nu}\hat{F}_{\tilde{A}}{}^{\mu\nu}\Big) + \int \hat{a}_{\tilde{A}} \wedge d\hat{F}^{\tilde{A}} \tag{5.63}$$

where index $\tilde{A}$ runs over gauge fields. Clearly a model described by (5.40) is just the simplest case where $\tilde{A} = 1$ and so can be ignored there. To avoid any confusion, let's rewrite the action as

$$\mathcal{S}_{4D} = \int \Big(d^4x\sqrt{-\hat{g}}\hat{R} - \frac{1}{2}\hat{F}^{\tilde{A}} \wedge \hat{F}_{\tilde{A}}\Big) + \int \hat{a}_{\tilde{A}} \wedge d\hat{F}^{\tilde{A}} \ , \tag{5.64}$$

that leads to the following equations of motion

$$\begin{aligned}
&\hat{G}_{\mu\nu} - \frac{1}{2}\hat{F}^{\tilde{A}}{}_{\mu\rho}\hat{F}_{\tilde{A}\nu}{}^{\rho} + \frac{1}{8}\hat{g}_{\mu\nu}\hat{F}^{\tilde{A}}{}_{\rho\gamma}\hat{F}_{\tilde{A}}{}^{\rho\gamma} = 0 \ ,\\
&\hat{F}_{\tilde{A}} = d\hat{a}_{\tilde{A}} \ ,\\
&d\hat{F}^{\tilde{A}} = 0 \ .
\end{aligned} \tag{5.65}$$

Now, let's have a closer look at the second equation since it does not seem to be a trivial generalization of its simpler case in (5.41). Comparing the second term of the Lagrangian (5.64) to (5.62), one can read $\tau_{\tilde{A}\tilde{B}} = i\delta_{\tilde{A}\tilde{B}}$ which means $\hat{F}_{\tilde{A}} = -\delta_{\tilde{A}\tilde{B}} \star \hat{F}^{\tilde{B}}$. Consequently, the main difference between second equations in (5.65) and (5.41) is a minus sign which is quite crucial. In fact without this extra minus sign we would get wrong sign for electromagnetic kinetic energy. More precisely, substituting $d\hat{a}_{\tilde{A}}$ from (5.65) and after partially integrating the last term of (5.64), one obtains

$$\int \hat{a}_{\tilde{A}} \wedge d\hat{F}^{\tilde{A}} = \int d\hat{a}_{\tilde{A}} \wedge \hat{F}^{\tilde{A}} = \int \hat{F}_{\tilde{A}} \wedge \hat{F}^{\tilde{A}} = -\delta_{\tilde{A}\tilde{B}} \int F^{\tilde{A}} \wedge \star F^{\tilde{B}} \ , \tag{5.66}$$

while from the second term of the Lagrangian (5.64) we get

$$-\frac{1}{2}\int \hat{F}^{\tilde{A}} \wedge \hat{F}_{\tilde{A}} = \frac{1}{2}\delta_{\tilde{A}\tilde{B}} \int F^{\tilde{A}} \wedge \star F^{\tilde{B}} \tag{5.67}$$

These two terms add up and give rise to

$$\int \Big(-\frac{1}{2}\hat{F}^{\tilde{A}} \wedge \hat{F}_{\tilde{A}} + \hat{a}_{\tilde{A}} \wedge d\hat{F}^{\tilde{A}} \Big) = -\frac{1}{2}\delta_{\tilde{A}\tilde{B}} \int F^{\tilde{A}} \wedge \star F^{\tilde{B}} = -\frac{1}{4}\delta_{\tilde{A}\tilde{B}} \int \hat{F}^{\tilde{A}}_{\mu\nu} \hat{F}^{\tilde{B}\ \mu\nu} \tag{5.68}$$

which is the electromagnetic kinetic energy with the correct sign. Furthermore, given $\hat{F}_{\tilde{A}} \equiv -\delta_{\tilde{A}\tilde{B}} \star \hat{F}_{\tilde{B}}$ plus the second line in (5.65), now it becomes clear why we called the Lagrange multiplier $\hat{a}_{\tilde{A}}$ *the dual gauge field* at the beginning.

The reduction ansatz (5.44) should be generalized as well. It happens quite naturally

$$\hat{F}^{\tilde{A}} = 2\ell\ p^{\tilde{A}} \sin\theta d\theta \wedge \big(d\varphi + A\big)\ . \tag{5.69}$$

Given $\tau_{\tilde{A}\tilde{B}} = i\delta_{\tilde{A}\tilde{B}}$ one can read F^a as

$$\hat{F}^a = \begin{pmatrix} -p_{\tilde{A}} e^{-\frac{3}{2}\phi} vol_{AdS_2} \\ \\ \\ p^{\tilde{A}} vol_{S^2} \end{pmatrix} \qquad \text{with} \qquad \begin{cases} vol_{AdS_2} = dt \wedge dr\ , \\ vol_{S^2} = \sin\theta\ d\theta \wedge d\tilde{\varphi}\ . \end{cases} \tag{5.70}$$

In the case of one-form a in (5.46), it is enough to add a lower index $\tilde{A}$ to the scalar

B. Having this generalization done, now the last step towards finding proper reduction ansatz for asymptotic geometry of scaling solutions is to apply a duality transformation that take us from a pure magnetic theory to the one that contains dyons.

5.3.3. Reduction of $\mathcal{N}=2$ SUGRA and Its Multi-Centered Solutions

Now, starting from $\mathcal{N}=2$ SUGRA action in 4D (3.29) and considering the fact that scalars take some constant values at infinity, one can deduce asymptotic form of the action as

$$\begin{aligned} \mathcal{S}^{(\text{asym})}_{\mathcal{N}=2} &= \frac{-1}{4}\int d^4x \left\{\sqrt{-\hat{g}}\hat{R} + \text{Im}\left[\tau_{\tilde{A}\tilde{B}}\left(\hat{F}^{\tilde{A}}_{\mu\nu}\left(\hat{F}^{\tilde{B}\ \mu\nu} + i\star\hat{F}^{\tilde{B}\ \mu\nu}\right)\right)\right]\right\} \\ &= \int\left\{\frac{-1}{2}\hat{\mathcal{R}}\star\mathbf{1} - H_{\tilde{A}\tilde{B}}\hat{F}^{\tilde{A}}\wedge\star\hat{F}^{\tilde{B}} + \Theta_{\tilde{A}\tilde{B}}\hat{F}^{\tilde{A}}\wedge\hat{F}^{\tilde{B}}\right\}, \end{aligned} \tag{5.71}$$

whose equivalence to (5.62) can easily be shown. On the other hand, we have the generalized toy model with pure magnetic charges $\Gamma^a = (0, p^{\tilde{A}})$ described by (5.63) which is not invariant under electromagnetic duality. Consequently, to figure out the consistent reduction ansatz for $\mathcal{N}=2$ scaling solutions, we first need to transfer to a frame with $\Gamma'^a = (q'_{\tilde{A}}\ ,\ p'^{\tilde{A}})$ by making a duality transformation given by

$$M = \begin{pmatrix} \alpha_{\tilde{A}}{}^{\tilde{B}} & \dfrac{q'_{\tilde{A}}}{p^{\tilde{B}}} \\ \\ \dfrac{(-p+\alpha p')^{\tilde{A}}}{q'_{\tilde{B}}} & \dfrac{p'^{\tilde{A}}}{p^{\tilde{B}}} \end{pmatrix} \implies M^{-1} = \begin{pmatrix} \dfrac{p'_{\tilde{A}}}{p_{\tilde{B}}} & \dfrac{-q'_{\tilde{A}}}{p^{\tilde{B}}} \\ \\ \dfrac{(p-\alpha p')^{\tilde{A}}}{q'_{\tilde{B}}} & \alpha^{\tilde{A}}{}_{\tilde{B}} \end{pmatrix} \tag{5.72}$$

where α is an arbitrary $\tilde{A}\times\tilde{A}$ matrix. Now, using (5.70) we can determine the transformed field strength $\hat{F}'^a$

$$F'^a = \begin{pmatrix} -\alpha_{\tilde{A}}{}^{\tilde{B}} p_{\tilde{B}}\ e^{-\frac{3}{2}\phi} vol_{AdS_2} + q'_{\tilde{A}} vol_{S^2} \\ \\ \frac{p_{\tilde{B}}}{q'_{\tilde{B}}}\left(p - \alpha p'\right)^{\tilde{A}}\ e^{-\frac{3}{2}\phi} vol_{AdS_2} + p'^{\tilde{A}} vol_{S^2}\ . \end{pmatrix} \tag{5.73}$$

This result can be written in a more elegant way if we take one more step and transform matrix $\mathcal{I}$ as well. From $\mathcal{I}' = M\mathcal{I}M^{-1}$ we obtain

$$\mathcal{I}' = \begin{pmatrix} \frac{\alpha_{\tilde{A}\tilde{C}}}{q'_{\tilde{B}}}\left(p - \alpha\ p'\right)^{\tilde{C}} - \frac{q'_{\tilde{A}}\ p'_{\tilde{C}}}{p_{\tilde{C}}\ p_{\tilde{B}}} & \frac{q'_{\tilde{A}}\ q'_{\tilde{C}}}{p_{\tilde{C}}\ p^{\tilde{B}}} + \alpha_{\tilde{A}\tilde{C}}\alpha^{\tilde{C}}{}_{\tilde{B}} \\ \\ \frac{-p'^{\tilde{A}}\ p'_{\tilde{C}}}{p^{\tilde{C}}\ p_{\tilde{B}}} - \frac{\left(p - \alpha\ p'\right)^{\tilde{A}}\left(p - \alpha\ p'\right)^{\tilde{C}}}{q'^{\tilde{C}}\ q'_{\tilde{B}}} & \frac{p'^{\tilde{A}}\ q'_{\tilde{C}}}{p_{\tilde{C}}\ p^{\tilde{B}}} - \frac{\alpha^{\tilde{C}}{}_{\tilde{B}}}{q'^{\tilde{C}}}\left(p - \alpha\ p'\right)^{\tilde{A}} \end{pmatrix} \tag{5.74}$$

and so we read

$$\diamond'\ \Gamma' \equiv \mathcal{I}'^a{}_b \Gamma'^b = \begin{pmatrix} \alpha_{\tilde{A}}{}^{\tilde{B}} p_{\tilde{B}} \\ \\ \frac{-p_{\tilde{B}}}{q'_{\tilde{B}}}\left(p^{\tilde{A}} - \alpha_{\tilde{C}}{}^{\tilde{A}} p'^{\tilde{C}}\right) \end{pmatrix} \tag{5.75}$$

from which we can extract the following compact form for F'^a (5.73)

$$\hat{F}'^a = \Gamma'^a \sin\theta d\theta \wedge d\tilde{\varphi} - (\diamond\Gamma)'^a e^{-\frac{3}{2}\phi} dt \wedge dr\ . \tag{5.76}$$

Now recall the following result from asymptotic scaling black hole solutions 3.7.2 where we found

$$\hat{F}^{\tilde{A}} = \Gamma^{\tilde{A}} \sin\theta d\theta \wedge d\tilde{\varphi} - \frac{(\diamond\Gamma)^{\tilde{A}}}{R^2} dt \wedge dr\ . \tag{5.77}$$

To avoid any confusion, let us emphasis that here $\hat{F}^{\tilde{A}}$ denotes the field strength of the scaling configuration consisting of dyonic charges and so should not get mixed with its pure magnetic counterpart given by (5.69). Even though (5.77) has been written with upper indices, using (5.57) one can easily check that $\hat{F}_{\tilde{A}}$ is nothing but the same expression with lower index, and hence we can safely replace index $\tilde{A}$ with a in (5.77). If we do so and compare the result to (5.76) we see they match.

Having the field strengths $\hat{F}^a$ for the pure magnetic case (5.70), we now can find their corresponding gauge fields $\hat{a}^a$ as well. Considering (5.46) we can write

$$\hat{F}^a = d\hat{a}^a \quad ; \qquad \hat{a}^a = \begin{pmatrix} -p_{\tilde{A}} e^{-\frac{3}{2}\phi}\ \xi + B_{\tilde{A}}\ d\tilde{\varphi} \\ \\ p^{\tilde{A}}\ \chi \end{pmatrix} \tag{5.78}$$

where ξ and χ are two one-forms such that $d\xi = vol_{AdS_2}$ and $d\chi = vol_{S^2}$ and so we can choose $\chi = -\cos\theta d\tilde{\varphi}$ (or $\chi = (1-\cos\theta)d\tilde{\varphi}$). Given the fact that under duality transformations, gauge fields have to transform in the same way as field strengths do, we can extract the dyonic gauge fields from their pure magnetic counterparts

$$\hat{a}'^a = M^a_{\ b}\hat{a}^b = \begin{pmatrix} -(\alpha p)_{\tilde{A}}\ e^{-\frac{3}{2}\phi}\xi + q'_{\tilde{A}}\ \chi + (\alpha B)_{\tilde{A}}\ d\tilde{\varphi} \\ \\ \frac{p_{\tilde{B}}}{q'_{\tilde{B}}}(p-\alpha p')^{\tilde{A}}\ e^{-\frac{3}{2}\phi}\xi + p'^{\tilde{A}}\ \chi - \frac{B_{\tilde{B}}}{q'_{\tilde{B}}}(p-\alpha p')^{\tilde{A}}\ d\tilde{\varphi} \end{pmatrix}. \tag{5.79}$$

Or, using (5.72,5.75) we read

$$\hat{a}'^a = -(\diamond'\Gamma')^a\ e^{-\frac{3}{2}\phi}\xi + \Gamma'^a\ \chi + B'^a\ d\tilde{\varphi} \tag{5.80}$$

where $B'^a = M^a_{\ b}B^b$.

Furthermore, applying duality transformation one can also fix $\tau'_{\tilde{A}\tilde{B}}$ which describes coupling between gauge fields in $4D$ $\mathcal{N}=2$ SUGRA

$$\tau'_{\tilde{A}\tilde{B}} = \Big(-i\alpha_{\tilde{A}\tilde{C}} + \frac{q'_{\tilde{A}}}{p^{\tilde{C}}} \Big) \Big(\frac{i\big(p-\alpha\ p'\big)}{q'} + \frac{p'}{p} \Big)^{-1}_{\tilde{C}\tilde{B}} \ . \tag{5.81}$$

Finally and as a consistency check, one can examine imaginary self-duality condition for both F^a and F'^a from (5.59).

$$\mathcal{I}^a{}_b F^b = \begin{pmatrix} p_{\tilde{A}} \sin\theta d\theta \wedge d\tilde{\varphi} \\ \\ \dfrac{p^{\tilde{A}}}{R^2} dt \wedge dr \end{pmatrix} = \begin{pmatrix} \star F_{\tilde{A}} \\ \\ \star F^{\tilde{A}} \end{pmatrix} \quad \Longrightarrow \quad \mathcal{I}^a{}_b F^b = \star F^a \ , \tag{5.82}$$

and also

$$\mathcal{I}'^a{}_b F'^b = \begin{pmatrix} \dfrac{q'_{\tilde{A}}}{R^2} dt \wedge dr + \big(\alpha\ p\big)_{\tilde{A}}\ \sin\theta d\theta \wedge d\tilde{\varphi} \\ \\ \dfrac{p'^{\tilde{A}}}{R^2} dt \wedge dr - \dfrac{p^{\tilde{B}}}{q'^{\tilde{B}}} \big(p - \alpha\ p'\big)^{\tilde{A}}\ \sin\theta d\theta \wedge d\tilde{\varphi} \end{pmatrix} = \begin{pmatrix} \star F'_{\tilde{A}} \\ \\ \star F'^{\tilde{A}} \end{pmatrix} \tag{5.83}$$

which implies $\mathcal{I}'^a{}_b F'^b = \star F'^a$ and so F'^a is imaginary self-dual as well.

The last step is to determine reduction ansatz in a consistent way inspired by the lesson we have learned from studying reduction procedure of our toy model. Here, we remind the reader the asymptotic form of the field content of $\mathcal{N}=2$ supergravity theory in $4D$ (3.31). They are given as

$$ds^2 = e^{-\frac{1}{2}\phi} ds_2^2 + e^{\phi}\big(d\theta^2 + \sin^2\theta d\tilde{\varphi}\big) \quad ; \qquad d\tilde{\varphi} = d\varphi + A \tag{5.84}$$

$$\hat{F}^{\tilde{A}} = \Gamma^{\tilde{A}} \sin\theta d\theta \wedge d\tilde{\varphi} - (\diamond\Gamma)^{\tilde{A}} e^{-\frac{3}{2}\phi} dt \wedge dr \ ,$$

$$\hat{a}_{\tilde{A}} = \Gamma_{\tilde{A}}\big(1 - \cos\theta\big) d\tilde{\varphi} + (\diamond\Gamma)_{\tilde{A}} e^{-\frac{3}{2}\phi}\ r\ dt \ , \tag{5.85}$$

where $2D$ external spacetime is AdS_2 which requires scalar field ϕ to be constant. So to S^2 reduce this solution consistently, we naively follow the procedure by adding to the Lagrangian exactly the same term as what we added in (5.63). Consequently, the Lagrangian we start with is the following

$$\begin{aligned}\mathcal{S}^{\rm asym}[\hat{g},\hat{F}^{\tilde{A}},\hat{a}_{\tilde{A}}] = \quad & \frac{-\pi}{\ell_4^2}\int \hat{\mathcal{R}}\star\mathbf{1}+ \\ & \frac{1}{16\pi}\int\Big[H_{\tilde{A}\tilde{B}}\hat{F}^{\tilde{A}}\wedge\star\hat{F}^{\tilde{B}}-\Theta_{\tilde{A}\tilde{B}}\hat{F}^{\tilde{A}}\wedge\hat{F}^{\tilde{B}}+\hat{a}_{\tilde{A}}\wedge d\hat{F}^{\tilde{A}}\Big]\ .\end{aligned} \tag{5.86}$$

Now let compare (5.85) to what we have found from duality transformation in (5.80). Specifically, the first two terms in (5.80) are the same as (5.85). However, one should not confuse the term $\Gamma_{\tilde{A}}d\tilde{\varphi}$ in (5.85) with the term containing the B-field in (5.80) as the former is just a closed one-form we are allowed to add to the gauge field (a pure gauge) whereas the later is not necessarily closed (dB can be non-zero off-shell). So our consistent ansatz to take would be

$$\hat{a}'_{\tilde{A}} = \Gamma_{\tilde{A}}\big(1-\cos\theta\big)d\tilde{\varphi}+(\diamond\Gamma)_{\tilde{A}}e^{-\frac{3}{2}\phi}\ r\ dt+B'_{\tilde{A}}d\tilde{\varphi}\ . \tag{5.87}$$

Then we are going to substitute these ansatz in (5.86). The term-by-term result would be

$$\begin{aligned}\mathcal{S}_{2D}^{(1)} &= \frac{8\pi^2}{\ell_4^2}\int dtdr\sqrt{-g}e^{\phi}\{\mathcal{R}-\frac{3}{2}\Box\phi-\frac{3}{2}\partial_a\phi\partial^a\phi+2e^{-\frac{3}{2}\phi}-1/6e^{\frac{3}{2}\phi}F_{(2D)ab}F^{ab}_{(2D)}\} \\ &= \frac{8\pi^2}{\ell_4^2}\int dtdr\sqrt{-g}\ \{e^{\phi}\mathcal{R}+2e^{-\frac{1}{2}\phi}-1/6e^{\frac{5}{2}\phi}F_{(2D)ab}F^{ab}_{(2D)}\}\ ,\end{aligned} \tag{5.88}$$

$$\mathcal{S}_{2D}^{(2)} = \frac{1}{8}\int dtdr\ e^{-\frac{3}{2}\phi}H_{\tilde{A}\tilde{B}}\Big[\big(\diamond'\,\Gamma'\big)^{\tilde{A}}\big(\diamond'\,\Gamma'\big)^{\tilde{B}}-\Gamma'^{\tilde{A}}\Gamma'^{\tilde{B}}\Big]\ ,$$

$$\mathcal{S}_{2D}^{(3)} = -\frac{1}{4}\int dtdr\ e^{-\frac{3}{2}\phi}\Theta_{\tilde{A}\tilde{B}}\Gamma'^{\tilde{A}}\big(\diamond'\,\Gamma'^{\tilde{B}}\big)\ ,$$

and finally, for the last term in the action (5.86) we obtain

$$\mathcal{S}_{2D}^{(4)} = \frac{1}{16\pi}\int\hat{a}'_{\tilde{A}}\wedge d\hat{F}'^{\tilde{A}} = \frac{1}{4}\int dtdr\Big(\Gamma'_{\tilde{A}}\Gamma'^{\tilde{A}}+\Gamma'^{\tilde{A}}B'_{\tilde{A}}\Big)\ F_{(2D)}\ , \tag{5.89}$$

or after partial integration

$$\begin{aligned}\mathcal{S}_{2D}^{(4)} &= \frac{1}{16\pi}\int d\hat{a}'_{\tilde{A}}\wedge\hat{F}'^{\tilde{A}} \\ &= \frac{1}{4}\int dtdr\Big[-2e^{-\frac{3}{2}\phi}\Gamma'_{\tilde{A}}\big(\diamond'\,\Gamma'^{\tilde{A}}\big)+\Big(\Gamma'_{\tilde{A}}\Gamma'^{\tilde{A}}+\Gamma'^{\tilde{A}}B'_{\tilde{A}}\Big)\;F_{(2D)}\Big]\,. \end{aligned}\qquad(5.90)$$

Here $F_{(2D)}\equiv F=dA$ with A is the $U(1)$ gauge field describing the twist (5.84).

5.4. The Twist and Its Holographic Interpretation

In this short section, we first review the idea of holography and then mention how gauge/gravity duality may help us in finding a holographic interpretation of the twist we observed by investigating the asymptotic geometry of the scaling solutions [36].

5.4.1. Holography

As remarked at the beginning of this thesis, the Bekenstein-Hawking law relates the black hole entropy to the *area* of its event horizon [46, 48], while as an extensive quantity one would expect the entropy to be proportional to the *volume* enclosed by the horizon. It turns out that in order to investigate quantum properties of any gravitational system such as a black hole, we need a *holographic* picture, meaning that any gravitational system in $D+1$ dimensions should be described by a quantum field theory in D dimensions without gravity [109, 110]. The AdS/CFT correspondence is one example of such dualities which states that there is a regular asymptotically AdS_{D+1} geometry corresponding to every stable state of a CFT$_D$ whose asymptotics contain the information about vevs of gauge invariant operators of that state.

The idea of holography and its principles first formulated in [111–113]. In particular, the required maps that relate various quantities from gravity theory to its dual QFT were introduced there. To have a rough idea of this correspondence, let us consider an $AdS_{D+1}\times X_q$ geometry in the bulk with compact space X_q. Then its dual theory is a CFT_D in a sense that there is a one-to-one correspondence between the Kaluza-Klein spectrum in the supergravity side and the gauge invariant primary operators of CFT_D. In addition, the string theory partition function is mapped to the generating function of the dual CFT. While the first is a function of the boundary conditions of the gravity solution, the second is a functional of the correlation functions of the fields parameterizing those boundary conditions that are mapped to sources of the dual operators.

In a CFT, the first non-trivial correlation function is a 2-point function which is holographically obtainable from the linearized fluctuations around $AdS_{D+1} \times X_q$. Those solutions which are asymptotically $AdS_{D+1} \times X_q$ correspond either to a deformed CFT_D or CFT in a specific non-trivial state. More precisely, for any state of the deformed CFT_D there is an asymptotically AdS_{D+1} while the deformation parameters and vevs of gauge invariant operators are encoded in non-trivial matter fields in the supergravity side.

As mentioned once in the introduction, the number of microstates of specific supersymmetric black holes can be obtained from the degeneracy of bound states of its corresponding D-brane system. This was first explored in the case of the so-called *Strominger-Vafa black hole*, a $5D$ supersymmetric black hole whose microstates were counted via this approach [12]. In the low coupling regime where the brane description is accurate, this black hole corresponds to a 3-charge $D1 - D5 - P$ system [57] . The near-horizon geometry of these black holes is $AdS_3 \times S^3 \times X_4$ where X_4 is either T^4 or $K3$. In fact, later it became clear that what was counted as the black hole microstates is indeed the number of supersymmetric states of a dual CFT, and so it was the first example of an AdS/CFT correspondence.

[57]Supersymmetry allows us to conclude the same entropy for the same systems in different coupling regime.

5.4.2. Twist As a New Hair

From the effective $2D$ action we found in (5.88,5.90), the only term that matters for our purpose is the last term in (5.90) which we are going to call "*the BF term*". We repeat this part of the $2D$ action here

$$\mathcal{S}_{\rm BF} = \frac{1}{4}\Gamma'^{\tilde{A}} \int dt dr B'_{\tilde{A}}\ F_{(2D)} + \mathcal{S}_{\rm bdy}\ , \tag{5.91}$$

with $F_{(2D)} = dA$. Here we introduced $\mathcal{S}_{\rm bdy}$ in order to have a well-defined variation principle. Then the on-shell variation of the action on the boundary reads

$$\delta\mathcal{S}^{\text{on-shell}}_{\rm BF} = \frac{1}{4}\Gamma'^{\tilde{A}} \int dt dr B'_{\tilde{A}}\ \delta A + \delta\mathcal{S}_{\rm bdy}\ . \tag{5.92}$$

The AdS/CFT dictionary [111] tells us there are two boundary conditions to be considered

- A is fixed and so there is no need to have $\mathcal{S}_{\rm bdy}$. Then A can be interpreted as a source for a (weight-zero) operator Q in the dual CFT whose vev is then given by

  $$\langle Q\rangle = \frac{\delta\mathcal{S}^{\text{on-shell}}_{\rm BF}}{\delta A}\ |_{A=0} = \frac{1}{4}\Gamma'^{\tilde{A}} B'_{\tilde{A}}\ . \tag{5.93}$$

  Regarding the fact that A represents a $U(1)$ gauge field and the equations of motion $F_{(2D)} = dA = 0$ and $dB = 0$ remain invariant under the gauge transformation $A \to A + d\Lambda$, we realize that B is in fact the associated conserved charge to the global $U(1)$ invariance.

- A is allowed to fluctuate while $B'_{\tilde{A}}$ remain fixed. In this case we need to consider having a boundary term as

  $$\mathcal{S}_{\rm bdy} = \frac{-1}{4}\Gamma'^{\tilde{A}} \int_{\rm bdy} B'_{\tilde{A}} A\ , \tag{5.94}$$

and we have

$$\langle \mathcal{O}^{\tilde{A}} \rangle = \frac{\delta \mathcal{S}_{\text{BF}}^{\text{on-shell}}}{\delta B'_{\tilde{A}}} = \frac{-1}{4} \Gamma'^{\tilde{A}} A \ . \tag{5.95}$$

It is in this picture that the twist can be interpreted as a " *topological hair*" distinguishing different ground states of the scaling solutions.

6. CONCLUSION

Here, we are going to remark main conclusions of this thesis. Furthermore, we address some questions that we plan to discuss in future based on our results.

- Non-scaling BPS black holes asymptotes to the four-dimensional Minkowski space. It also has been known that scaling solutions have an AdS_2 factor in their asymptotic geometry and so are more interesting to be investigated. What we showed reveals even more involved geometry that arises exciting questions considering AdS/CFT correspondence. We basically showed that the AdS_2 geometry at far infinity is equipped with a S^2 fiber we have called a "*twist*". This fiber can be thought of as a rotating two-sphere whose angular velocity is determined by the intersection product of the total charge of the solution Γ and its total dipole moment $\vec{\Delta}$, i.e., $\vec{K} = \frac{1}{2}\langle\Gamma, \vec{\Delta}\rangle$.
- It has been shown that the total angular momentum of the scaling solutions vanishes. However, interestingly the twist we found can be interpreted as a new *hair* for these black holes solutions. It provides a new observable to distinguish different black holes. In other words, this twist may give us a clue on counting microstates of scaling black holes by labeling these states. To have a better understanding of its role, we S^2 reduced the asymptotic geometry that led to an effective $2D$ theory with AdS_2 geometry and a single $U(1)$ gauge field presenting the twist.
- A holographic dictionary via AdS_2/CFT$_1$ correspondence can clarify consequences of such a twist for the dual quantum theory. This will not be very straightforward though, since it is not clear yet what AdS_2/CFT$_1$ really is. The ambiguities in the gravity side is originated from a characteristic properties of AdS_2 spacetime. Given the fact that it has two causally disconnected boundaries, the question is that how many copies of CFT are needed to have this correspondence working properly. If there are two copies needed, then it is the first example of AdS/CFT correspondence in which we need to consider more than one CFT. If only one copy is enough, then the question is on which of these two boundaries it should

be lied. There are ambiguities in the field theory side as well. In fact, dealing with a one-dimensional CFT is still vague. Consequently, there is still a lot to be explored about AdS_2/CFT$_1$ correspondence

APPENDIX A: COMPLEX MANIFOLDS

In this appendix, we briefly study complex manifolds and the necessary corresponding calculus. To do so, we mainly follow [31] to introduce some of necessary and related concepts including the complex structure and differential forms, Hermitian manifolds and $\overline{\partial}$-cohomology group, etc. which are essential for our introduction of Kähler and Calabi-Yau manifolds. There are some other nice references for studying this subject such as [33,54] as well as a short note that contains a majority of necessary materials in the topic [114].

A.1. Complex Manifolds

By definition, M is a complex manifold if the following axioms hold:

(i) M is a topological space.

(ii) M is provided with a family of pairs $\{(U_i, \varphi_i)\}$.

(iii) $\{U_i\}$ is a family of open sets which covers M. The map φ_i is a homeomorphism from $\{U_i\}$ to an open subset U of $\mathbb{C}^m$ [58] .

(iv) Given U_i and U_j such that $U_i \bigcap U_j \neq 0$, the map $\psi_{ji} = \varphi_j \circ \varphi_i^{-1}$ from $\varphi_i\big(U_i \bigcap U_j\big)$ to $\varphi_j\big(U_i \bigcap U_j\big)$ is *holomorphic.*

To make this definition more clear, let us define what holomorphicity means. A complex valued function $f : \mathbb{C}^m \to \mathbb{C}$ is *holomorphoic (analytic)* if $f = f_1 + if_2$ satisfies the Cauchy-Riemann relation for any $z^\mu = x^\mu + iy^\mu$, i.e.,

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu} \quad , \quad \frac{\partial f_1}{\partial y^\mu} = -\frac{\partial f_2}{\partial x^\mu} \ . \tag{A.1}$$

Holomorphic map: Let M and N being two complex manifolds with $\dim_{\mathbb{C}} M = m$ and $\dim_{\mathbb{C}} N = n$ [59] . Consider a map $f : M \to N$ such that for any $p \in U$ one has

[58] This implies that M is even-dimensional where $m = dim_{\mathbb{C}} M$.

[59] From now on, m (or n) denotes complex dimensions of the manifold, otherwise it will be mentioned explicitly.

$f(p) \in V$ where (U,ϕ) and (V,ψ) are two charts in M and N, respectively. Given $\{z^\mu\} = \phi(p)$ and $\{w^\mu\} = \psi\big(f(p)\big)$ as coordinates on M and N, we call the map $\psi \circ f \circ \phi^{-1} : \mathbb{C}^m \to \mathbb{C}^n$ a *holomorphic map* if w^ν is a holomorphic function of z^μ for $1 \leq \nu \leq n$. This definition is independent of our choice for the coordinate. Moreover, M is called *biholomorphic* to N if the holomorphic map $f : M \to N$ is a diffeomorphism too (then $f^{-1} : N \to M$ is holomorphic automatically).

Let $\{(U_i,\varphi_i)\}$ and $\{(V_j,\psi_j)\}$ be two atlases for M. If their union is again an atlas which satisfies the axioms of the definition (A.1), then these two atlas are said to define the same *complex structure*. A generic complex manifold may admit a number of different complex structures.

Examples

The two-sphere S^2 is the very first example of a complex manifold which can be identified with the Riemannain sphere $\mathbb{C}\bigcup\{\infty\}$.
The second well-known example is a two-dimensional torus T^2 which can be obtained by identifying any two points $z_1, z_2 \in \mathbb{C}$ if they are separated as $z_1 - z_2 = kw_1 + lw_2$ for some integer k, l and two non-vanishing complex numbers w_1 and w_2 with $w_2/w_1 \notin \mathbb{R}$. Then, two pairs (w_1, w_2) and (w_1', w_2') define the same complex structure if there exists a matrix A such that [60]

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \equiv SL(2,\mathbb{Z})/\mathbb{Z}_2 \; .^{61} \tag{A.2}$$

There is another famous example of complex manifolds called **complex projective space** $\mathbb{C}P^n$. As we discuss in subsection 3.3.1, projective manifolds are very crucial

[60]It is usually said that $L(w_1, w_2) \equiv \{kw_1 + lw_2 \; ; \; k, l \in \mathbb{Z}\}$ defines a *lattice* on $\mathbb{C}$, so a torus is actually nothing but $\mathbb{C}/L(w_1, w_2)$.

[61]$PSL(2,\mathbb{Z})$ is the same group as $SL(2,\mathbb{Z})$ for which A and $-A$ are identified. Also note that $L(w_1, w_2)$ and $L(\lambda w_1, \lambda w_2)$ define the same complex structure for $\lambda \in \mathbb{C}$. To get rid of this redundency, one can define $\tau \equiv w_2/w_1 \in H = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$ that is called *the modular parameter*. Accordingly, any lattic is defined by a pair $(1, \tau)$ meaning that different complex structures are generated by different $\tau \in H/PSL(2,\mathbb{Z})$.

in supergravity where they appear as target space of the scalar fields. Although in those cases we need to consider projective special geometries, but studying $\mathbb{C}P^n$ can convey the idea in its simplest possible way. To obtain $\mathbb{C}P^n$, we need to introduce an equivalence relation $z \sim w$ if there is a non-zero complex number a such that $w = az$. Then $\mathbb{C}P^n \equiv (\mathbb{C}^{n+1} - \{0\}/\sim)$. The (n+1)-tuple denoted by $[z_0, ..., z_n]$ are called *homogeneous coordinates* while one can define *inhomogeneous coordinate* as $\xi^\nu_{(\mu)} = z^\nu / z^\mu$ for $\mu \neq \nu$ in a chart U_μ in which $z^\mu \neq 0$. It is easy to show that for $U_\mu \bigcap U_\nu \neq 0$ the coordinate transformation $\psi_{\mu\nu} : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic.

A.1.1. Almost Complex Structure

Let M be an m-dimensional complex manifold with coordinates $z^\mu = x^\mu + iy^\mu$. Then the tangent space T_pM and its dual T^*_pM are spanned by $2m$ basis respectively given by

$$\left\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^m}\right\} \quad , \quad \left\{dx^1, ..., dx^m, dy^1, ..., dy^m\right\} . \tag{A.3}$$

Now, let us define a linear map $J_p : T_pM \to T_pM$ as

$$J_p\Big(\frac{\partial}{\partial x^\mu}\Big) = \frac{\partial}{\partial y^\mu} \quad , \quad J_p\Big(\frac{\partial}{\partial y^\mu}\Big) = -\frac{\partial}{\partial x^\mu} . \tag{A.4}$$

J_p is a real tensor of type $(1,1)$ which squares to minus identity, i.e., $J_p^2 = -\mathrm{Id}_{T_pM}$. One can easily check that the action of J_p is independent of the chart. So in the basis (A.3) J_p takes the following form

$$J_p = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} , \tag{A.5}$$

where I_m is $m \times m$ unit matrix. So J_p defines a smooth tensor field whose form at the point p is given by (A.5). Such a tensor field is called *almost complex structure* and it determines the complex structure of the complex manifold M completely. Also note that any $2m$-dimensional real manifold *locally* admits a tensor field that squares to $-I_{2m}$. However, it is only on a complex manifold that the tensor field J which we have defined in (A.4) is independent of the chart and hence can be defined *globally* via getting patched across charts. In other words, this is only on a complex manifold that two sets of coordinates $\{x^\mu\}$ and $\{y^\mu\}$ do not get mixed.

Taking $z^\mu = x^\mu + iy^\mu$ as the coordinate of a point $p \in (U, \phi)$, one can introduce $\{\partial/\partial z^\mu, \partial/\partial \overline{z}^\mu\}$ [62] and $\{dz^\mu, d\overline{z}^\mu\}$ which are sets of basis for $T_pM^{\mathbb{C}}$ and $T_p^*M^{\mathbb{C}}$ respectively [63] . These sets satisfying duality conditions

$$\langle dz^\mu, \partial/\partial \overline{z}^\nu\rangle = 0 = \langle d\overline{z}^\mu, \partial/\partial z^\nu\rangle \ ,$$
$$\langle dz^\mu, \partial/\partial z^\nu\rangle = \langle d\overline{z}^\mu, \partial/\partial \overline{z}^\nu\rangle = \delta^\mu_{\ \nu} \ . \tag{A.6}$$

Working in these basis has this advantage that J_p becomes diagonal with eigenvalues $\pm i$

$$J_p\Big(\frac{\partial}{\partial z^\mu}\Big) = i\,\frac{\partial}{\partial z^\mu} \quad , \quad J_p\Big(\frac{\partial}{\partial \overline{z}^\mu}\Big) = -i\,\frac{\partial}{\partial \overline{z}^\mu} \qquad \Longrightarrow \tag{A.7}$$

$$J_p = i dz^\mu \otimes \frac{\partial}{\partial z^\mu} - i d\overline{z}^\mu \otimes \frac{\partial}{\partial \overline{z}^\mu} \qquad \Longleftrightarrow \qquad J_p = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix} \ , \tag{A.8}$$

where $\partial/\partial z^\mu(\partial/\partial \overline{z}^\mu)$ is called *holomorphic (anti-holomorphic) basis*. Given the block-diagonal form of almost complex structure, one can define a projection operator $\mathcal{P}^\pm :$ $T_pM^{\mathbb{C}} \to T_pM^\pm$ given by

$$\mathcal{P}^\pm \equiv \frac{1}{2}\big(I_{2m} \mp iJ_p\big) \ . \tag{A.9}$$

Specifically, $\mathcal{P}^\pm$ decompose any vector Z to $Z^\pm \equiv \mathcal{P}^\pm Z \in T_pM^\pm$ where the two disjoint

[62] Clearly and $\overline{z}^\mu = x^\mu - iy^\mu$ is the complex conjugate of z^μ and $\partial/\partial \overline{z}^\mu = \overline{\partial/\partial z^\mu}$.

[63] These are spaces of complexified vectors and one-forms, meaning that any element of these spaces can be decomposed to some real and imaginary part.

subspaces $T_pM^{\pm}$ are defined as

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^- \quad ; \quad T_pM^{\pm} = \{Z \in T_pM^{\mathbb{C}} \ ; \ \ J_pZ = \pm iZ\} \ . \tag{A.10}$$

Accordingly, $T_pM^{\pm}$ are spanned by holomorphic and anti-holomorphic basis respectively and so called *holomorphic (anti-holomorphic) vector spaces.* The separation (A.10) implies that for any $Z \in T_pM^{\mathbb{C}}$, there exists a **unique** decomposition $Z = Z^+ + Z^-$ where $Z^{\pm} \in T_pM^{\pm}$. One can check following properties as well

- Just like J, the separation (A.10) also holds independent of charts which means a (anti-)holomorphic vector still remains (anti-)holomorphic in any other chart.
- $T_pM^- = \overline{T_pM^+} = \{\overline{Z} \ ; \ \ Z \in T_pM^+\}$.
- $\dim_{\mathbb{C}} T_pM^+ = \dim_{\mathbb{C}} T_pM^- = \frac{1}{2}\dim_{\mathbb{C}} T_pM^{\mathbb{C}} = \frac{1}{2}\dim_{\mathbb{C}} M$.
- A vector field $Z \in \mathcal{X}(M)^{\mathbb{C}}$ is real if and only if $Z^+ = \overline{Z^-}\big(\overline{Z}\big)^-$.
- For $X, Y \in \mathcal{X}(M)^{\pm}$, one can show that $[X, Y] \in \mathcal{X}(M)^{\pm}$ as well.

A.2. Complex Differential Forms and Cohomology

To be able to discuss about topological properties of complex manifolds, we need to know about their cohomology group for which an accurate definition of differential forms is necessary. Having $T_pM^{\pm}$ defined, we are now going to first explain how to construct a complex (r, s)-form by complexification. Then, we investigate the action of the exterior derivative d on a complex form by decomposing it into two *Dolbeault operators* ∂ and $\overline{\partial}$. This allows us to ultimately define $\overline{\partial}$-cohomology group of a complex manifold.

Complexification

Consider two real q-forms ω and η both belonging to $\Omega^q_p(M)$. Then $\zeta = \omega + i\eta$ is called a complex q-form, i.e., $\zeta \in \Omega^q_p(M)^{\mathbb{C}}$. It is called real if it equals its complex conjugate, i.e., $\zeta = \overline{\zeta}$. In addition, for any $V_i \in T_p(M)^{\mathbb{C}}$ and $\lambda \in \mathbb{C}$, the following properties hold

$$\overline{\omega}\big(V_1, ..., V_q\big) = \overline{\omega\big(\overline{V}_1, ..., \overline{V}_q\big)} \ , \tag{A.11}$$
$$\overline{\omega + \eta} = \overline{\omega} + \overline{\eta} \quad , \quad \overline{\lambda\omega} = \overline{\lambda}\overline{\omega} \quad , \quad \overline{\overline{\omega}} = \omega \ .$$

Let M be an m-dimensional complex manifold for which we have $T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-$. Take $\omega \in \Omega_p^q(M)^{\mathbb{C}}$, two positive integers r, s such that $r + s = q \leq 2m$ and vectors V_i with $1 \leq i \leq q$ to be either holomorphic or anti-holomorphic. If $\omega\big(V_1, ..., V_q\big) = 0$ unless r vectors among V_i's belong to T_pM^+ while s of them are in T_pM^-, then ω said to be a *bidegree* (r, s) or simply an (r, s)-form. The set of these (r, s)-forms at the point p is denoted by $\Omega_p^{r,s}(M)^{\mathbb{C}}$ which is spanned by the set of basis $\{dz^{\mu_1} \wedge ... \wedge dz^{\mu_r} \wedge d\overline{z}^{\nu_1} \wedge \cdots \wedge d\overline{z}^{\nu_s}\}$. Any (r, s)-form $\omega \in \Omega_p^{(r,s)}(M)^{\mathbb{C}}$ in the chart (U, ϕ) with coordinates $\phi(p) = z^\mu$ can be uniquely written as

$$\omega = \frac{1}{r!\ s!}\ \omega_{\mu_1\cdots\mu_r\overline{\nu}_1\ldots\overline{\nu}_s} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\overline{z}^{\nu_1} \wedge \cdots \wedge d\overline{z}^{\nu_s} \ . \tag{A.12}$$

It can be shown that ω remains a (r, s)-form in passing through charts. Furthermore, we have the following proposition for any differential form on M

- If $\omega \in \Omega^{(r,s)}(M)$ then $\overline{\omega} \in \Omega^{(s,r)}(M)$.
- For $\omega \in \Omega^{(r,s)}(M)$ and $\xi \in \Omega^{(r',s')}(M)$, we have $\omega \wedge \xi \in \Omega^{(r+r',s+s')}(M)$.
- We have the following decomposition for the set of q-form on M

$$\Omega^q(M)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega^{(r,s)}(M) \ . \tag{A.13}$$

Specifically, for any q-form ω, we have the following expansion in terms of all (r, s)-forms

$$\omega = \sum_{r+s=q} \omega^{(r,s)} = \sum_{r+s=q} \frac{1}{r!\ s!}\ \omega_{\mu_1\cdots\mu_r\overline{\nu}_1\cdots\overline{\nu}_s} dz^{\mu_1} \wedge\cdots\wedge dz^{\mu_r} \wedge d\overline{z}^{\nu_1} \wedge\cdots\wedge d\overline{z}^{\nu_s} \ , \tag{A.14}$$

where the expansion coefficients are obtainable as

$$\omega_{\mu_1\cdots\mu_r\overline{\nu}_1\cdots\overline{\nu}_s} = \omega\Big(\frac{\partial}{\partial z^{\mu_1}},\cdots,\frac{\partial}{\partial z^{\mu_r}},\frac{\partial}{\partial \overline{z}^{\nu_1}},\cdots,\frac{\partial}{\partial \overline{z}^{\nu_s}}\Big) \ . \tag{A.15}$$

- We have

$$\dim_{\mathbb{R}}\Omega_p^{(r,s)}(M) = \begin{cases} \begin{pmatrix} m \\ r \end{pmatrix}\begin{pmatrix} m \\ s \end{pmatrix} & \text{for } \ 0 \leq r, s \leq m \ , \\ 0 & \text{othewise .} \end{cases} \tag{A.16}$$

Consequently, as we may expect $\dim_{\mathbb{R}}\Omega_p^q(M)^{\mathbb{C}} = \sum_{r+s=q}\dim_{\mathbb{R}}\Omega_p^{(r,s)}(M) = \begin{pmatrix} 2m \\ q \end{pmatrix}$.

Dolbeault operators

Consider an (r,s)-form like ω. Knowing the fact that $d\omega$ is a combination of $(r+1,s)$- and $(r,s+1)$-forms, it is useful to decompose exterior derivative operator d as

$$d = \partial + \overline{\partial} \qquad \text{with} \qquad \begin{cases} \partial : \Omega^{(r,s)}(M) \to \Omega^{(r+1,s)}(M) \ , \\ \overline{\partial} : \Omega^{(r,s)}(M) \to \Omega^{(r,s+1)}(M) \ . \end{cases} \tag{A.17}$$

where ∂ and $\overline{\partial}$ are called **Dolbeault operators**. Given a q-form ω and a p-form ξ on a complex manifold M, one can check following properties for Dolbeault operators

$$\begin{aligned} &\partial\partial\omega = \overline{\partial}\overline{\partial}\omega = (\partial\overline{\partial} + \overline{\partial}\partial)\omega = 0 \ , \\ &\partial\overline{\omega} = \overline{\overline{\partial}\omega} \quad , \quad \overline{\partial}\overline{\omega} = \overline{\partial\omega} \ , \\ &\partial(\omega\wedge\xi) = \partial\omega\wedge\xi + (-1)^q\omega\wedge\partial\xi \ , \\ &\overline{\partial}(\omega\wedge\xi) = \overline{\partial}\omega\wedge\xi + (-1)^q\omega\wedge\overline{\partial}\xi \ . \end{aligned} \tag{A.18}$$

Definition A.1. *$\omega \in \Omega^{(r,0)}(M)$ is called a **holomorphic** r**-form** if $\overline{\partial}\omega = 0$ which is equivalent to the requirement of $\omega_{\mu_1 \ldots \mu_r}$ being holomorphic functions.*

A.2.1. $\overline{\partial}$-Cohomology Group

Starting from any $(r, 0)$-form, $\overline{\partial}$ constructs the following sequence of $\mathbb{C}$-linear maps which is called **Dolbeault complex**

$$\Omega^{(r,0)}(M) \xrightarrow{\overline{\partial}} \Omega^{(r,1)}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{(r,m-1)}(M) \xrightarrow{\overline{\partial}} \Omega^{(r,m)}(M) \ . \tag{A.19}$$

Considering the fact that $\overline{\partial}^2 = 0$, one can distinguish two sets of (r, s)-forms as the following

- $Z^{(r,s)}_{\overline{\partial}}$: Set of $\overline{\partial}$-closed (r, s)-forms, i.e., those $\omega \in \Omega^{(r,s)}(M)$ for which $\overline{\partial}\omega = 0$. They are called (r, s)-*cocycles.*
- $B^{(r,s)}_{\overline{\partial}}$: Set of $\overline{\partial}$-exact (r, s)-forms, i.e., those $\omega \in \Omega^{(r,s)}(M)$ that can be written as $\omega = \overline{\partial}\eta$ for some $\eta \in \Omega^{(r,s-1)}(M)$. They are called (r, s)-*coboundary.*

Then the following complex vector space is called (r, s)**th $\overline{\partial}$-cohomology group** or (r, s)**th Dolbeault cohomology group**

$$H^{(r,s)}_{\overline{\partial}} \equiv Z^{(r,s)}_{\overline{\partial}} / B^{(r,s)}_{\overline{\partial}} \ . \tag{A.20}$$

It is clear without mentioning that one can talk about (r, s)**th ∂-cohomology group** following the same logic for ∂ instead.

A.3. Hermitian Manifolds and Hermitian Differential Geometry

Let M be an m-dimensional complex manifold and g be a Riemannian metric on M [64] . Taking two complex vectors $Z = X + iY,\ W = U + iV \in T_pM^{\mathbb{C}}$, their inner

[64] It briefly means that g is a symmetric positive-definite bilinear form at each point on M.

product is

$$g_p(Z,W) = g_p(X,U) - g_p(Y,V) + i\Big(g_p(X,V) + g_p(Y,U)\Big) \ . \tag{A.21}$$

In $\{\partial/\partial z^\mu, \partial/\partial \overline{z}^\mu\}$ basis, metric components obey the following equalities

$$g_{\mu\nu} = g_{\nu\mu} \ , \ g_{\overline{\mu}\overline{\nu}} = g_{\overline{\nu}\overline{\mu}} \ , \ g_{\overline{\mu}\nu} = g_{\nu\overline{\mu}} \ , \ \overline{g_{\mu\overline{\nu}}} = g_{\overline{\mu}\nu} \ , \ \overline{g_{\mu\nu}} = g_{\overline{\mu}\overline{\nu}} \ . \tag{A.22}$$

The Hermitian metric

A Riemannian metric g of a complex manifold M is called **Hermitian** if at each point $p \in M$ and for any $X, Y \in T_pM$ it satisfies

$$g_p(J_pX, J_pY) = g_p(X,Y) \ , \tag{A.23}$$

where J is the complex structure (A.8). The pair (g, M) is then called a *Hermitian manifold*. To see the reason for this terminology, let us define $h_p(X,Y) \equiv g_p(X, \overline{Y})$ for $X, Y \in T_p^+M^{\mathbb{C}}$. By taking complex conjugate we have

$$\overline{h_p(X,Y)} = \overline{g_p(X,\overline{Y})} = g_p(\overline{X},Y) = g_p(J\overline{X},JY) = g_p(-i\overline{X},iY) = g_p(Y,\overline{X}) = h(Y,X) \ , \tag{A.24}$$

where to get the second equality, we used (A.22) and for the third one we applied (A.23). Also note that $\overline{X} \in T_p^-M$ and so it is an eigenvector of J with eigenvalue $-i$. Finally note that both g and h are symmetric. From (A.24), we conclude h is a Hermitian inner product and so is g. We can also verify that h is positive definite. Taking $X = X_1 + iX_2$, we have $h(X,X) = g(X,\overline{X}) = g(X_1,X_1) + g(X_2,X_2) \geq 0$.

Due to the fact $J_p^2 = -I_{2m}$, two vectors X and J_pX are orthogonal with respect to a Hermitian metric. Moreover, the condition (A.23) and the way J acts on $\{\partial/\partial z^\mu, \partial/\partial \overline{z}^\mu\}$ (see (A.7)) tell us that for a Hermitian metric we have $g_{\mu\nu} = 0 = g_{\overline{\mu}\overline{\nu}}$, and so it takes the following form

$$g = g_{\mu\overline{\nu}} dz^{\mu} \otimes d\overline{z}^{\nu} + g_{\overline{\mu}\nu} d\overline{z}^{\mu} \otimes dz^{\nu} \ . \tag{A.25}$$

Theorem A.1. *A complex manifold always admits a Hermitian metric.*
To prove this theorem, we note that by taking any positive definite Riemannian metric g on a complex manifold M, one can define a metric $\hat{g}_p$ as

$$\hat{g}_p(X,Y) \equiv \frac{1}{2}\Big\{ g_p(X,Y) + g_p(J_pX, J_pY) \Big\} \quad \Longrightarrow \quad \hat{g}_p(J_pX, J_pY) = \hat{g}_p(X,Y) \ . \tag{A.26}$$

Also $\hat{g}_p$ is positive definite and hence a Hermitian metric.

A.3.1. Kähler Form

Let (M, g) be a Hermitian manifold and $X, Y \in T_pM$. Define a tensor filed Ω of rank $(2, 0)$ with the following action on T_pM

$$\Omega_p(X,Y) := g_p(J_pX, Y) \ , \tag{A.27}$$

for which one can verify the following properties

- Ω is anti-symmetric and so a two-form called **Kähler form**

$$\Omega(X,Y) = g(JX,Y) = -g(JY,X) = -\Omega(Y,X). \tag{A.28}$$

- Ω is invariant under the action of J

$$\Omega(JX,JY) = g(J^3X, J^2Y) = \Omega(X,Y). \tag{A.29}$$

- As a two-form of bidegree $(1,1)$, Ω takes the following form in complex basis

$$\Omega = -J_{\mu\overline{\nu}} dz^{\mu} \wedge d\overline{z}^{\nu} \qquad ; \qquad J_{\mu\overline{\nu}} = g_{\mu\overline{\lambda}} J^{\overline{\lambda}}_{\ \overline{\nu}} = -ig_{\mu\overline{\nu}} \ . \tag{A.30}$$

- Ω is real, i.e., $\overline{\Omega} = -i\overline{g_{\mu\overline{\nu}}} d\overline{z}^{\mu} \wedge dz^{\nu} = ig_{\nu\overline{\mu}} dz^{\nu} \wedge d\overline{z}^{\mu} = \Omega$.
- Using Kähler form, one can show that any Hermitian manifold (and so any complex manifold) is orientable. We refer the reader to [31] for a more detailed discussion, but the main idea is simple. Consider the $2m$-form $\underbrace{\Omega \wedge \cdots \wedge \Omega}^{\text{m}}$ on an m-dimensional complex manifold M which is real and nowhere vanishing. These properties suggests that $\Omega \wedge \cdots \wedge \Omega$ can provide a **volume form** for M and so the manifold is orientable.

A.3.2. Covariant Derivatives, Connection, Torsion and Curvature

Let (M, g) be a Hermitian manifold. Hermiticity of the metric implies that $\Gamma^{\lambda}_{\overline{\mu\nu}} = 0 = \Gamma^{\overline{\lambda}}_{\mu\nu}$. In addition, it is reasonable to require a holomorphic vector $V \in T_pM^+$ to keep its holomorphicity after parallel transportation from point p to q, i.e., $\tilde{V} \in T_qM^+$. Certainly, there is the same requirement for anti-holomorphic vectors as well. Now, let $\{e^{\lambda}\}$ to be a holomorphic vector basis. Then we have

$$\nabla_{\mu} e^{\lambda} = \Gamma^{\lambda}_{\mu\nu} e^{\nu} + \Gamma^{\lambda}_{\mu\overline{\nu}} e^{\overline{\nu}}, \tag{A.31}$$

and the same calculation for a anti-holomorphic vector basis. So these requirements enforce the connection to satisfies $\Gamma^{\lambda}_{\mu\overline{\nu}} = 0 = \Gamma^{\overline{\lambda}}_{\overline{\mu}\nu}$. This restricts non-zero components to $\Gamma^{\lambda}_{\mu\nu}$ and $\Gamma^{\overline{\lambda}}_{\overline{\mu\nu}}$ where $\Gamma^{\overline{\lambda}}_{\overline{\mu\nu}} = \overline{\Gamma^{\lambda}_{\mu\nu}}$. So for instance, the covariant derivative of a holomorphic vector $X^+ = X^{\mu}\partial/\partial z^{\mu}$ is

$$\nabla_{\mu} X^+ = \left(\partial_{\mu} X^{\lambda} + X^{\nu}\Gamma^{\lambda}_{\mu\nu}\right)\frac{\partial}{\partial z^{\lambda}} \qquad , \qquad \nabla_{\overline{\mu}} X^+ = (\partial_{\overline{\mu}} X^{\lambda})\frac{\partial}{\partial z^{\lambda}} \ . \tag{A.32}$$

One can easily generalize this to any arbitrary tensor field. Requiring **metric compatibility** condition to be hold, i.e., $\nabla_k g_{\mu\overline{\nu}} = 0 = \nabla_{\overline{k}} g_{\mu\overline{\nu}}$ help us to read connection coefficients as the following

$$\Gamma^{\lambda}_{\kappa\mu} = g^{\overline{\nu}\lambda}\ \partial_\kappa g_{\mu\overline{\nu}} \qquad , \qquad \Gamma^{\overline{\lambda}}_{\overline{\kappa}\overline{\nu}} = g^{\overline{\lambda}\mu}\ \partial_{\overline{\kappa}} g_{\mu\overline{\nu}}\ . \tag{A.33}$$

A metric compatible connection for which Γ with mixed indices vanishes is called the **Hermitian connection** whose components are **uniquely** determined by(A.33).

To find non-vanishing components of the torsion T and the Riemann curvature tensor R, it is enough to have a look at their action on a vector field

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y]\ , \\ R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\ , \end{aligned} \tag{A.34}$$

from which we find out [31]

$$\begin{aligned} T^{\lambda}_{\mu\nu} &= \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = g^{\bar{\xi}\lambda}\big(\partial_\mu g^{\nu\bar{\xi}} - \partial_\nu g^{\mu\bar{\xi}}\big)\ , \\ T^{\overline{\lambda}}_{\overline{\mu}\overline{\nu}} &= \Gamma^{\overline{\lambda}}_{\overline{\mu}\overline{\nu}} - \Gamma^{\overline{\lambda}}_{\overline{\nu}\overline{\mu}} = g^{\overline{\lambda}\xi}\big(\partial_{\overline{\mu}} g^{\overline{\nu}\xi} - \partial_{\overline{\nu}} g^{\overline{\mu}\xi}\big)\ , \\ T^{A}_{\mu\bar{\nu}} &= T^{A}_{\bar{\mu}\nu} = 0\ . \end{aligned} \tag{A.35}$$

Also for the curvature tensor we have

$$\begin{aligned} R^{\kappa}{}_{\lambda\overline{\mu}\nu} &= \partial_{\overline{\mu}}\Gamma^{\kappa}{}_{\nu\lambda} = \partial_{\overline{\mu}}\big(g^{\bar{\xi}\kappa}\ \partial_\nu g_{\lambda\bar{\xi}}\big), \\ R^{\overline{\kappa}}{}_{\overline{\lambda}\mu\overline{\nu}} &= \partial_\mu \Gamma^{\overline{\kappa}}{}_{\overline{\nu}\overline{\lambda}} = \partial_\mu\big(g^{\overline{\kappa}\xi}\ \partial_{\overline{\nu}} g_{\xi\overline{\lambda}}\big), \\ R^{\kappa}{}_{\overline{\lambda}AB} &= R^{\overline{\kappa}}{}_{\lambda AB} = R^{A}{}_{B\kappa\lambda} = R^{A}{}_{B\overline{\kappa}\overline{\lambda}} = 0\ , \end{aligned} \tag{A.36}$$

where in the last set of equalities, A and B take any holomorphic and anti-holomorphic indices.

Ricci form

Note that the Riemann tensor (A.36) is anti-symmetric in its last two indices and hence by contracting its first two indices, we will obtain a two-form

$$R^{\kappa}{}_{\kappa\mu\overline{\nu}} = -\partial_{\overline{\nu}}\big(g^{\kappa\bar{\xi}}\partial_\mu g_{\kappa\bar{\xi}}\big) = -\partial_{\overline{\nu}}\partial\mu \ \log\mathcal{G} \ \ ; \qquad \mathcal{G} \equiv \det(g_{\mu\overline{\nu}}) = \sqrt{g} \ , \tag{A.37}$$

in light of which we can introduce **Ricci two-form** as

$$\mathfrak{R} := iR^{\kappa}{}_{\kappa\mu\overline{\nu}}dz^\mu \wedge d\overline{z}^\nu = i\partial\overline{\partial} \ \log\mathcal{G} \ . \tag{A.38}$$

One can verify the following properties for $\mathfrak{R}$

- $\mathfrak{R}$ is real, $\overline{\mathfrak{R}} = \mathfrak{R}$.
- $d\mathfrak{R} = 0$. Making use of the identity $\partial\overline{\partial} = -\frac{1}{2}d(\partial - \overline{\partial})$ and also considering the fact that d is a nilpotent operator, we conclude that $\mathfrak{R}$ is closed. However, it is not an exact two-form necessarily.

Let us discuss a bit more about the second property. The fact that $\mathcal{G}$ is not a scalar and $(\partial - \overline{\partial})\log\mathcal{G}$ is not a globally defined quantity, suggests that one can define $c_1(M) \equiv [\mathfrak{R}/2\pi]$ that is rather a non-trivial element of the second cohomology class of M, i.e., $[\mathfrak{R}/2\pi] \in H^2(M;\mathbb{R})$. $c_1(M)$ is called the **first Chern class** for which we have an important theorem.

Theorem A.2. *A smooth variation in the metric $g \to g + \delta g$ keeps the first Chern class invariant.*

To prove this, let us calculate $\delta\mathfrak{R}$ from (A.38)

$$\delta\log\mathcal{G} = g^{\mu\overline{\nu}}\delta g_{\mu\overline{\nu}} \qquad \Longrightarrow \qquad \delta\mathfrak{R} = i\partial\overline{\partial}g^{\mu\overline{\nu}}\delta g_{\mu\overline{\nu}} \ . \tag{A.39}$$

Now note that $g^{\mu\overline{\nu}}\delta g_{\mu\overline{\nu}}$ is a scalar by using which one can define a well-defined one-form ξ as $\xi \equiv \frac{-1}{2}(\partial - \overline{\partial})g^{\mu\overline{\nu}}\delta g_{\mu\overline{\nu}}$. Taking this one-form, one can easily rewrite $\delta\mathfrak{R} = d\xi =$

$\frac{-i}{2}\, d(\partial-\overline{\partial})g^{\mu\overline{\nu}}\delta g_{\mu\overline{\nu}}$ which implies that $\delta\mathfrak{R}$ is actually an **exact** two-form, meaning that $\mathfrak{R}$ and $\delta\mathfrak{R}$ lie in the same class, $[\mathfrak{R}] = [\mathfrak{R}+\delta\mathfrak{R}]$.

A.4. $\overline{\partial}$-Cohomology Group

Generally speaking, $\overline{\partial}$-cohomology group measures the topological non-triviality of the manifold M. Its (r,s)th subgroup is given by

$$H^{(r,s)}_{\overline{\partial}}(M) \equiv Z^{(r,s)}_{\overline{\partial}}(M)/B^{(r,s)}_{\overline{\partial}}(M). \tag{A.40}$$

$\overline{\partial}$-cohomology group of $\mathbb{C}$ is trivial, just like its de Rham cohomology group.

Adjoint operators

In order to define the Laplacian operator $\triangle$ for our later applications in (3.142) and Also Appendix B, we need first to introduce **adjoint operators** $\partial^\dagger$ and $\overline{\partial^\dagger}$. Let M be an m-dimensional Hermitian manifold. The inner product between $\alpha, \beta \in \Omega^{(r,s)}(M)$ is defined by

$$(\alpha,\beta) \equiv \int_M \alpha\wedge\overline{\star}\beta\ . \tag{A.41}$$

Note that Hodge star operator maps an (r,s)-form to an $(m-s, m-r)$-form, so the operator we really need is $\overline{\star} : \Omega^{(r,s)}(M) \to \Omega^{(m-r,m-s)}(M)$ which satisfies $\overline{\star}\beta \equiv \overline{\star\beta} = \star\overline{\beta}$. We denote the adjoint operators of ∂ and $\overline{\partial}$ respectively by $\partial^\dagger$ and $\overline{\partial^\dagger}$ defined as

$$(\alpha,\ \partial\beta) = (\partial^\dagger\alpha,\ \beta) \quad , \quad (\alpha,\ \overline{\partial}\beta) = (\overline{\partial}^\dagger\alpha,\ \beta) \quad ; \quad d^\dagger = \partial^\dagger + \overline{\partial}^\dagger\ . \tag{A.42}$$

Accordingly, we have $\partial^\dagger : \Omega^{(r,s)}(M) \to \Omega^{(r-1,s)}(M)$ and $\overline{\partial}^\dagger : \Omega^{(r,s)}(M) \to \Omega^{(r,s-1)}(M)$. Moreover, for an even-dimensional manifold $d^\dagger = -\star d\star$, so we find

$$\partial^\dagger = -\star\overline{\partial}\star \quad , \quad \overline{\partial}^\dagger = -\star\partial\star \implies \left(\partial^\dagger\right)^2 = 0 = \left(\overline{\partial}^\dagger\right)^2. \tag{A.43}$$

Now recall that on a differentiable manifold, the Laplacian operator is defined as $\triangle \equiv d^2 = dd^\dagger + d^\dagger d$. Consequently, in the case of a Hermitian manifold, $\triangle$ can be decomposed as

$$\triangle_\partial = \partial\partial^\dagger + \partial^\dagger\partial \quad , \quad \triangle_{\overline{\partial}} = \overline{\partial}\overline{\partial}^\dagger + \overline{\partial}^\dagger\overline{\partial} \, . \tag{A.44}$$

An (r,s)-form ω is called *∂-harmonic* (*$\overline{\partial}$-harmonic*) if it satisfies $\triangle_\partial\omega = 0$ ($\triangle_{\overline{\partial}}\omega = 0$). For instance being ∂-harmonic means that $\partial\omega = 0 = \partial^\dagger\omega$ and the same is true for $\overline{\partial}$-harmonic forms. So we can have complex version of the *Hodge decomposition theorem.*

Theorem A.3. (**Hodge's theorem**) $\Omega^{(r,s)}(M)$ *has the following unique orthogonal decomposition* [115]

$$\begin{aligned} \Omega^{(r,s)}(M) &= \overline{\partial}\Omega^{(r,s-1)}(M) \oplus \overline{\partial}^\dagger\Omega^{(r,s+1)}(M) \oplus \mathrm{Harm}^{(r,s)}_{\overline{\partial}}(M) \; ; \\ \mathrm{Harm}^{(r,s)}_{\overline{\partial}}(M) &\equiv \Big\{\omega \in \Omega^{(r,s)}(M) \; ; \; \triangle_{\overline{\partial}}\,\omega = 0\Big\} \, . \end{aligned} \tag{A.45}$$

We have defined Kähler manifolds with details in the section 2.3.1, but let us mention a very special property of these manifolds right here: Three different Laplacian operators $\triangle_\partial$, $\triangle_{\overline{\partial}}$ and consequently $\triangle$ all coincide for the Kähler manifolds [115], [116]. More precisely, one can show that

$$\triangle = 2\triangle_\partial = 2\,\triangle_{\overline{\partial}} \; . \tag{A.46}$$

This property has an immediate consequence which is "*any holomorphic form is harmonic and conversely, being a $(p,0)$ harmonic form means it is holomorphic as well*". To see this, take a holomorphic $(p,0)$-form ω for which $\overline{\partial}\omega = 0$ and so (A.43) implies that $\overline{\partial}^\dagger\omega = 0$ too (simply because $\star\omega \in \Omega^{(m,m-p)}(M)$) and hence ω is a harmonic $(p,0)$-form, $\triangle_{\overline{\partial}}\omega = \triangle\omega = 0$. In the reverse direction, we already discussed that vanishing $\triangle_{\overline{\partial}}\omega$ implies $\overline{\partial}\omega = 0$ which means the $(p,0)$-form is holomorphic.

APPENDIX B: CLOSENESS OF $d\omega$

As we discussed in the vicinity of (3.142), $\langle dH, H\rangle$ is closed. Here, we are going to first prove this using some definitions and identities which we found at the end of previous appendix. After doing so, we check closeness of asymptotic $d\omega$ order by order.

B.1. Closeness of $\langle dH, H\rangle$

The one-form ω first introduced in (3.137) is responsible for angular symmetry breaking and staticity of multi-centered solutions. As defined earlier in (3.140), it is given by

$$\star_3 \, d\omega := \langle dH, H\rangle \ , \tag{B.1}$$

where $\star_3$ acts on three dimensional Euclidean space $\mathbb{R}^3$. It is said before that H is a harmonic function satisfying $\triangle H = 0$. Using this property, we show that $\langle dH, H\rangle$ is a closed one-form. It then implies that what we defined as ω has to be an one-form and this observation justifies our definition in (B.1). To see this, let first hit $\langle dH, H\rangle$ by $d\star_3$

$$d \star \langle dH, H\rangle = \langle d \star dH, H\rangle + \langle \star dH, dH\rangle \ . \tag{B.2}$$

The second term vanishes by using $\star\alpha \wedge \beta = \alpha \wedge \star \ \hat{\eta}^{n+1}\beta$ where $\hat{\eta}\alpha = (-1)^{deg\ (\alpha)}\alpha$ while the intersection product $\langle \cdot, \cdot \rangle$ is antisymmetric. For the first term, one can use the definition of the codifferential (adjoint) operator on an m-dimensional Riemannian manifold $d^\dagger = (-1)^m \star_3^{-1} \ d \ \star_3 \ \hat{\eta}$ and write

$$\langle d \star_3 dH, H\rangle = -\langle \star_3 \, d^\dagger dH, H\rangle \ . \tag{B.3}$$

Now using the Laplacian operator $\triangle = dd^{\dagger} + d^{\dagger}d$ and the fact that $d^{\dagger}$ eliminates any zero-form (function), one can write

$$\langle \star_3 d^{\dagger} dH, H\rangle = -\langle \star_3 \triangle H, H\rangle \ , \tag{B.4}$$

which also vanishes, meaning that $\star_3\langle dH, H\rangle$ is a closed two-form. This result plus taking this into account that for a p-form in n-dimensional space $\star^2 = sgn(g)(-1)^{p(n-p)}$ allow us to define a one-form ω whose exterior derivative satisfies (B.1).

B.2. Closeness of Asymptotic $d\omega$

The second order expansion of one-form ω in the case of scaling solution has been found in (3.198). Here, to double-check the validity of that result, we examine our calculations in two different directions: first, we test if what we found from $\star_3\langle dH, H\rangle$ is closed. Then, we suggest an one-form and show that its exterior derivative exactly equals $\star_3\langle dH, H\rangle$. We are going to do these tests at each order separately. For instance, $dd\omega$ has to vanish at each order, so if $\omega^{(1)}$ stands for the leading terms and $\omega^{(2)}$ for subleadings, one need to show that $dd\omega^{(1)} = 0$ as well as $dd\omega^{(2)} = 0$. To do so, we need to apply identities and make some definitions. First, let us prove (3.197) which is obtainable from the following identity

$$(\star d\vec{r}\ )^i = \frac{1}{2}\varepsilon^i{}_{jk} d\vec{r}^{\ j} dr \wedge d\vec{r}^{\ k} \ , \tag{B.5}$$

where $d\vec{r} = (dr)\hat{r} + rd\hat{r}$. We already know that $\star dr$ is proportional to $d\theta \wedge d\varphi$, so to specify what is $\star d\hat{r}$ it is enough to first expand (B.5) and then ignore all $d\theta \wedge d\varphi$ components. The result will be what is given in (3.197).

In the second step and in order to simplify our calculations, we define four matrices

$$\mathcal{K}^{ij} d\theta \wedge d\varphi \equiv \varepsilon^{i}{}_{kl}\hat{r}^k d\hat{r}^l \wedge d\hat{r}^j \quad , \quad \mathcal{Q}^{ij} d\theta \wedge d\varphi \equiv \frac{1}{2}\varepsilon^{i}{}_{kl} d\hat{r}^k \wedge d\hat{r}^l \hat{r}^j \ , \qquad \text{(B.6)}$$
$$\mathcal{P}^{ij} d\theta \wedge d\varphi \equiv \varepsilon^{j}{}_{kl}\hat{r}^k d\hat{r}^i \wedge d\hat{r}^l \quad , \quad \mathcal{L}^{ij} d\theta \wedge d\varphi \equiv \varepsilon^{j}{}_{kl}\hat{r}^i d\hat{r}^k \wedge d\hat{r}^l \ .$$

For instance, it can be shown that all these matrices are symmetric, a crucial property that we are going to use later [65] . Having these defined, let us first take $d\omega^{(1)}$ from (3.195) and calculate

$$d\omega^{(1)} = \star\left[\frac{1}{r^3}\left\langle\Delta_\parallel,\Gamma\right\rangle - \frac{1}{r^4}\left\langle\Delta_\perp,\Gamma\right\rangle dr\right] \qquad \Longrightarrow$$

$$d\left(\frac{1}{r^3}\left\langle\vec{\Delta}\cdot(\star d\hat{r}),\Gamma\right\rangle\right) = \frac{1}{r^3}\left\langle\vec{\Delta}\cdot(d\star d\hat{r}),\Gamma\right\rangle = \frac{-1}{r^3}\left\langle\vec{\Delta}\cdot(dr\wedge d\vec{\Omega}),\Gamma\right\rangle \ , \qquad \text{(B.7)}$$

$$d\left(\frac{1}{r^4}\left\langle\vec{\Delta}\cdot\hat{r},\Gamma\right\rangle(\star dr)\right) = \frac{-2}{r^3}\left\langle\vec{\Delta}\cdot\hat{r},\Gamma\right\rangle \sin\theta \ dr\wedge d\theta\wedge d\varphi \ . \qquad \text{(B.8)}$$

Applying (3.196,3.197), we conclude $dd\omega^{(1)} = 0$. Then, in the second order we have

$$\star d\omega^{(2)} = \frac{1}{r^4}\left[\langle\Delta_\parallel,\Delta_\perp\rangle + 3\Big\langle\sum_a \Gamma_a(\vec{r}_a\cdot\hat{r})(\vec{r}_a\cdot d\hat{r}),\Gamma\Big\rangle\right] \qquad \text{(B.9)}$$
$$+ \frac{1}{r^5}\left[-3\Big\langle\sum_a \Gamma_a(\vec{r}_a\cdot\hat{r})^2,\Gamma\Big\rangle dr + \Big\langle\sum_a \Gamma_a r_a^2,\Gamma\Big\rangle dr\right] .$$

Differentiating the first term, we get

$$d\left(\frac{1}{r^4}\left\langle\vec{\Delta}\cdot(\star d\hat{r}),\vec{\Delta}\cdot\hat{r}\right\rangle\right) = \frac{-4}{r^5}dr\wedge\left\langle\vec{\Delta}\cdot(\star d\hat{r}),\vec{\Delta}\cdot\hat{r}\right\rangle \qquad \text{(B.10)}$$
$$- \frac{1}{r^4}dr\wedge\left\langle\vec{\Delta}\cdot d\vec{\Omega},\vec{\Delta}\cdot\hat{r}\right\rangle + \frac{1}{r^4}\left\langle\vec{\Delta}\cdot(\star d\hat{r}),\wedge\vec{\Delta}\cdot d\hat{r}\right\rangle \ .$$

In the light of (3.197), the first term vanishes because $dr \wedge \star d\hat{r} = 0$. Looking at the definition of $d\vec{\Omega}$, one realizes that the second term also vanishes because $\langle\cdot,\cdot\rangle$ is

[65] To see this, it is enough to multiply them by ε_{ijk} and then take this into account that $\hat{r}_j d\hat{r}^j = \frac{1}{2}d(\hat{r}^2) = 0$.

anti-symmetric. So we are left with the last term which can be rewritten as

$$\frac{1}{r^4}\left\langle\vec{\Delta}\cdot(\star d\hat{r}),\wedge\vec{\Delta}\cdot d\hat{r}\right\rangle=\frac{1}{r^4}\left\langle\Delta_i,\Delta_j\right\rangle\mathcal{K}^{ij}d\theta\wedge d\varphi\ , \tag{B.11}$$

where (B.6) has been used. This term also vanishes as $\mathcal{K}^{ij}$ is a symmetric matrix. Differentiating the second term of (B.9), we find

$$d\left(\frac{3}{r^4}\Big\langle\sum_n\Gamma_n(\vec{r}_n\cdot\hat{r})(\vec{r}_n\cdot\star d\hat{r}),\Gamma\Big\rangle\right)=\frac{-12}{r^5}dr\wedge\Big\langle\sum_n\Gamma_n(\vec{r}_n\cdot\hat{r})(\vec{r}_n\cdot\star d\hat{r}),\Gamma\Big\rangle+$$
$$\frac{3}{r^4}\Big\langle\sum_n\Gamma_n(\vec{r}_n\cdot d\hat{r})\wedge(\vec{r}_n\cdot\star d\hat{r}),\Gamma\Big\rangle-\frac{3}{r^4}dr\wedge\Big\langle\sum_n\Gamma_n(\vec{r}_n\cdot\hat{r})(\vec{r}_n\cdot d\vec{\Omega}),\Gamma\Big\rangle\ . \tag{B.12}$$

With the same argument we had about the first term in (B.10), the first term of (B.12) vanishes. The two remaining terms of (B.12) can be rewritten as

$$\left(\frac{3}{r^4}\Big\langle\sum_n\Gamma_n r_{ni}r_{nj}\mathcal{K}^{ij},\Gamma\Big\rangle-\frac{6}{r^4}\sin\theta\Big\langle\sum_n\Gamma_n(\vec{r}_n\cdot\hat{r})^2,\Gamma\Big\rangle\right)dr\wedge d\theta\wedge d\varphi\ . \tag{B.13}$$

Differentiating the third term of (B.9) results in

$$d\left(\frac{-3}{r^5}\Big\langle\sum_n\Gamma_n(\vec{r}_n\cdot\hat{r})^2,\Gamma\Big\rangle\star dr\right)=\frac{9}{r^4}\sin\theta\Big\langle\sum_n\Gamma_n(\vec{r}_n\cdot\hat{r})^2,\Gamma\Big\rangle dr\wedge d\theta\wedge d\varphi\ , \tag{B.14}$$

where we use $d\hat{r}\wedge d\theta\wedge d\varphi=0$. Finally, from differentiating the last term of (B.9), we obtain

$$d\left(\frac{1}{r^5}\Big\langle\sum_n\Gamma_n r_n^2,\Gamma\Big\rangle\star dr\right)=\frac{-3}{r^4}\sin\theta\Big\langle\sum_n\Gamma_n r_n^2,\Gamma\Big\rangle dr\wedge d\theta\wedge d\varphi\ . \tag{B.15}$$

Now if we decompose the position vector $\vec{r}_n$ into its normal and tangent components, i.e., replacing $\vec{r}_n=\vec{r}_{n\perp}+\vec{r}_{n\parallel}$ (with $\vec{r}_{n\perp}=\vec{r}_n.\hat{r}$) in (B.15), then we see that its perpendicular component cancels the summation of the second term of (B.13) and (B.14). Furthermore, its tangent component and the first term of (B.13) cancel out each other. Consequently, $d\omega^{(2)}$ is closed as well. Until now and along with closeness of $d\omega^{(1)}$, this implies that our calculations for $\star_3\langle dH,H\rangle$ passed the first test.

Now, to run the second test let us suggest the following form for the leading order of ω

$$\omega^{(1)} = \frac{-1}{2r^2}\left\langle \vec{\Delta}\cdot(\hat{r}\times d\hat{r}),\Gamma\right\rangle \ , \tag{B.16}$$

then from its exterior derivative we get

$$d\left(\omega^{(1)}\right) = \frac{1}{r^3}dr\wedge\left\langle \vec{\Delta}\cdot(\hat{r}\times d\hat{r}),\Gamma\right\rangle - \frac{1}{2r^2}\left\langle \varepsilon^{i}{}_{jk}\Delta_i d\hat{r}^j\wedge d\hat{r}^k,\Gamma\right\rangle \ , \tag{B.17}$$

and using (3.196,3.197) leads us to (B.7). Then, we consider the following expression for $\omega^{(2)}$

$$\omega^{(2)} = -\frac{1}{r^3}\left(\frac{1}{3}\left\langle \vec{\Delta}\cdot(\hat{r}\times d\hat{r}),\vec{\Delta}\cdot\hat{r}\right\rangle + \left\langle \sum_n \Gamma_n(\vec{r}_n\cdot\hat{r})\big(\vec{r}_n\cdot(\hat{r}\times d\hat{r})\big),\Gamma\right\rangle\right) \ . \tag{B.18}$$

To check the validity of this suggestion, we start from differentiating the first term

$$d\left(\frac{-1}{3r^3}\left\langle \vec{\Delta}\cdot(\hat{r}\times d\hat{r}),\vec{\Delta}\cdot\hat{r}\right\rangle\right) = \frac{1}{r^4}\left\langle \vec{\Delta}\cdot(\star d\hat{r}),\vec{\Delta}\cdot\hat{r}\right\rangle - \frac{1}{3r^3}\langle\Delta_i,\Delta_j\rangle\left(\mathcal{K}^{ij}-\mathcal{Q}^{ij}\right)d\theta\wedge d\varphi \ . \tag{B.19}$$

The last term vanishes recalling that both $\mathcal{K}^{ij}$ and $\mathcal{Q}^{ij}$ are symmetric matrices. So we are left with the first term which is exactly the same as first term in (B.9). Taking the second term of (B.18), we obtain

$$d\left(\left\langle \sum_n \Gamma_n(\vec{r}_n\cdot\hat{r})\big(\vec{r}_n\cdot(\hat{r}\times d\hat{r})\big),\Gamma\right\rangle\right) = \tag{B.20}$$
$$\frac{3}{r^4}\left\langle \sum_n \Gamma_n(\vec{r}_n\cdot\hat{r})\big(\vec{r}_n\cdot(\star d\hat{r})\big),\Gamma\right\rangle - \frac{1}{r^3}\left\langle \sum_n \Gamma_n r_{ni}r_{nj},\Gamma\right\rangle\left(\mathcal{P}^{ij}+\mathcal{L}^{ij}\right)d\theta\wedge d\varphi \ .$$

Now, it can be shown that

$$r_{ni}r_{nj}\left(\mathcal{P}^{ij}+\mathcal{L}^{ij}\right) = \sin\theta\big(3(\vec{r}_n\cdot\hat{r})^2 - r_n^2\big) \ . \tag{B.21}$$

Plugging which back into (B.20) and adding the result to (B.19), we will get (B.9) back. This, in addition to our previous observation, guarantee that our guesses for $\omega^{(1)}$ and $\omega^{(2)}$ which are given respectively by (B.16) and (B.18), works perfectly. Hence, the final conclusion for the second order expansion of the one-form ω will be

$$\omega = \frac{-1}{2r^2}\left\langle \vec{\Delta}\cdot(\hat{r}\times d\hat{r}), \Gamma\right\rangle \tag{B.22}$$

$$-\frac{1}{r^3}\left(\frac{1}{3}\left\langle \vec{\Delta}\cdot(\hat{r}\times d\hat{r}), \vec{\Delta}\cdot\hat{r}\right\rangle + \left\langle \sum_n \Gamma_n(\vec{r}_n\cdot\hat{r})(\vec{r}_n\cdot(\hat{r}\times d\hat{r})), \Gamma\right\rangle\right) + \mathcal{O}(r^{-4})\ .$$

APPENDIX C: MORE ABOUT INITIAL INCONSISTENCY IN EINSTEIN-MAXWELL S^2 REDUCTION

In this very short appendix, we are going to provide more details about the constraint (5.39) as well as some middle steps in derivation of equations of motion we have reported in (5.37,5.38). First, by subtracting the $\varphi\varphi$-component of the Einstein's equation from its $\theta\theta$-component, we read

$$EE_{\varphi\varphi} - \sin^2\theta EE_{\theta\theta} = F_{ab}F^{ab} \ , \tag{C.1}$$

which we demand to vanish to keep the spherical symmetry intact. It is helpful to also have a look at what one really gets as Einstein's equations in $4D$ before imposing the constraint and equations of motion. Initially, they are given by

$$\begin{aligned} EE_{ab} = \quad & -2\beta\nabla_a\partial_b\phi + 2\beta(2\alpha-\beta)\partial_a\phi\partial_b\phi + \\ & g_{ab}\Big[2\beta\Box\phi + \beta(3\beta-2\alpha)\partial_c\phi\partial^c\phi + e^{2(\alpha-2\beta)\phi}(1-e^{2\beta\phi}) \\ & -\frac{1}{4\sqrt{g}}e^{-2\alpha\phi}\big(4\cos^2\theta + e^{2\beta\phi}\sin^2\theta\big)F_{cd}F^{cd}\Big] + \\ & A_a EE_{b\varphi} + A_b EE_{a\varphi} - A_a A_b EE_{\varphi\varphi} = 0 \ , \end{aligned} \tag{C.2}$$

$$\begin{aligned} EE_{\varphi\varphi} = \quad & -1/2R + (\alpha+\beta)\Box\phi + \beta^2\partial_a\phi\partial^a\phi - e^{2(\alpha-2\beta)\phi} \\ & +\frac{1}{4\sqrt{g}}e^{-2\alpha\phi}\big(4\cos^2\theta + 3e^{2\beta\phi}\sin^2\theta\big)F_{ab}F^{ab} = 0 \end{aligned} \tag{C.3}$$

$$EE_{a\varphi} = \quad e^{2(\beta-\alpha)\phi}\sin^2\theta\Big\{A_a \ EE_{\varphi\varphi} - \frac{1}{2}\nabla^b F_{ab} - (2\beta-\alpha)(\partial^b\phi)F_{ba}\Big\} = 0 \ , \tag{C.4}$$

which will reduce to (5.37,5.38) after imposing equations of motion and the constraint

(5.39). Finally, there are some identities applied

$$A \wedge F = A_b F_{ca} + A_c F_{ab} - A_a F_{cb} = 0 \ , \tag{C.5}$$

$$A^c \nabla_{[a} F_{bc]} = 0 \qquad , \qquad dF = 0 \ , \tag{C.6}$$

where the first condition can be understood by recalling that no three-form is allowed in $2D$.

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