To My Family

OPTIMAL CONTROL OF GENERALIZED STORAGE MODELS

by

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ABSTRACT

The purpose of this dissertation is to study to optimal control problem of the generalized storage processes over an infinite planning horizon. The generalization of the controlled storage process allows for both positive and negative jumps by the stochastic input process as well as controlled inputs and outputs. The extension of the theory for the optimal control of generalized storage processes mainly consists of studying various aspects of the uncontrolled storage model, deriving the sufficient condition of optimality, verifying the existence of a unique solution and studying its properties. The approach is to specify the stochastic structure of the processes involved in the model and monotonicity properties of the controls so as to guarantee the existence of a unique solution to the storage equation, to construct the Markov process model for the content level of the store and then to apply Markov decision theory in order to characterize the expected infinite time horizon discounted return. Consequently the sufficient condition of optimality is established as a functional differential equation in terms of the generator of the storage process and shown to possess a unique and continuously differentiable solution. In the process of verifying the

existence and uniqueness of the optimal return and optimal controls, the deterministic version is considered first so as to shed light upon the nature of the solution methodology and then the results obtained are extended as to include the stochastic processes inherent in the generalized storage model.

GENEL BİRİKİM MODELLERİNİN ENİYİ KONTROLU

ÖZET

Bu çalışmanın amacı, genel birikim süreçlerinin eniyi kontrol problemini sonsuz planlama çevreni içinde incelemektir. Kontrol altındaki birikim sürecinin genelleştirilmesi, kontrol edilebilir girdi ve çıktıların yanısıra rassal girdi ve çıktı süreçlerinin sıçramalarına izin vermektedir. Birikim modellerinin eniyi kontrol kuramının geliştirilmesi, birikim modelinin çeşitli yönlerinin incelenmesi, eniyilik yeterli koşulunun türetilmesi, tek bir çözüm varlığının doğrulanması ve bu çözümün özelliklerinin ayrıntılı değerlendirilmesi ile gerçekleştirilmiştir. Yaklaşım, modeldeki rassal süreçlerin yapılarını ve birikim denklemine tek bir çözümün varolmasını sağlayacak girdi ve çıktı kontrollerinin monoton özelliklerini saptamak, depo içerik düzeyi için Markof süreci modelini kurmak ve beklenen indirilmiş kazancı belirlemek için Markof karar kuramını uygulamaktır. Böylece, eniyilik yeterli koşulu birikim sürecinin üreteci çerçevesinde işlevsel türevsel bir denklem olarak ifade edilmekte ve sürekli türevlenebilir tek bir çözümünün varolduğu gösterilmektedir. Eniyi kazanç ve eniyi kontrollerin, varlık ve teklik koşulunu sağladıklarının doğrulanmasında, çözüm yönteminin yapısını anlayabilmek için önce gerekirci model ele alınıp çözülmüş, elde edilen sonuçlar daha sonra genel birikim modelindeki rassal süreçleri kapsayacak şekilde geliştirilmiştir.

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I. INTRODUCTION

The objective of this dissertation is to study the optimal control problem of a special class of Markov processes, namely storage processes. The generalized storage process is described by the following stochastic integral equation:

$$X_t = X_o + A_t \dot{-} B_t + \int_o^t p(X_s)ds - \int_o^t r(X_s)ds \quad , t \geq 0 \qquad (1.1)$$

where X_t is the content level of the store at time t, A_t is the cumulative uncontrolled input to the store during the time interval [0,t], B_t is the cumulative output from the store during [0,t] and X_o is the initial content level of the store. The processes $X = (X_t)_{t \geq 0}$, $A = (A_t)_{t \geq 0}$ and $B = (B_t)_{t \geq 0}$ will be referred to as the content or storage process, the input process, and the output process, respectively. The first integral gives the total controlled input to the store up to time t where p(x) is the rate of input when the content level of the store is x. Similarly, the second integral gives the total controlled output from the store up to time t where r(x) is the rate of output when the content level of the store is x. Hence the equation under consideration expresses the simple observation that the content level at time t is equal to the sum of the initial content and the total input during [0,t] diminished by the total output during the same time period.

The main concern in this study is to control the content level of the store by choosing within an admissible class a proper output rate function r and a proper input rate function p which will be referred to as the output control and the input control, respectively. The content level of the store will be observed continuously by the controller who possesses an objective function yielding utility at a rate given by

$$L(X_t,p(X_t),r(X_t)) = L_1(X_t) + L_2(p(X_t)) + L_3(r(X_t)) \qquad (1.2)$$

at any time t. Thus at each instant of time, based upon his own observation that the current content level of the store is x, the controller should decide upon a proper input rate p and a proper output rate r so as to optimize some measure of utility given by (1.2), in which case his net rate of earnings is $L(x,p(x),r(x))$.

The objective underlying in this model is to determine (r,p) which achieves the maximization of the expected infinite time horizon discounted earnings. This is accomplished by developing a Markov process model for the content level of the store and then applying Markov decision theory to characterize the optimal controls as functions of the content level. Specifically DYNKIN's [1] theory of weak infinitesimal generators of Markov processes is employed to characterize the expected infinite time horizon discounted return in terms of a functional differential equation. Consequently the balance of this dissertation is concerned with the study of the existence and uniqueness of a return function and the associated controls satisfying this equation. In the process of showing this, the dissertation will have mainly fulfilled two tasks: the first will be to analyze the generalized storage process described by (1.1) which includes two random processes and two control possibilities for increasing and decreasing the content of the store. In connection with this, the generator of the storage process, which is the main tool in the Markov decision theoretic approach employed in the optimal control problem, is studied extensively. The second task will be to control the content level of the store in an optimal manner where the reward and cost structure is specified by (1.2)

The storage process X is the core of the optimal control problem, so it is crucial to specify its stochastic structure which is basically determined by the stochastic structure of the input and output processes. Throughout this study the input and output processes are assumed to be two independent compound Poisson processes which will be described in detail in a later chapter. As far as the restrictions on the physical properties of the store are concerned, it is assumed that the store has infinite physical capacity and that there does exist no backlogging. So although there is no upper bound imposed upon the physical capacity of the store, the content level is not permitted to fall below zero. Random jumps of the output process B decrease the content level at random times, but any jump that will drop the content level below zero is lumped at the critical point of emptiness. Accordingly the construction of the storage process X as described by (1.1) is done so as to incorporate this requirement. Furthermore the controls which specify the input and output rates, at each instant of time, as functions of the content level are of vital importance in our analysis, and their structures should be as general as possible. So we will try to impose minimal restrictions on the input and output rate functions just to guarantee the existence of a unique solution to the generalized storage equation given by (1.1).

A review of the research carried out so far on storage theory is presented in Chapter II. Most of the studies are basically concerned with the theoretical analysis of the uncontrolled storage models, and the optimal control problem constitutes a rather new area of research where Markov decision theory or the diffusion approximation may be employed to characterize the optimal controls. The innovations leading to the originality of our model become apparent in connection with previous studies. Chapter III provides an insight into application areas of the generalized storage process, verifying the novelty of the model under consideration.

The analysis of the uncontrolled storage model is accomphished in Chapter IV. The stochastic structures of the input and output processes are set forth in Section 1. The concept of admissibility is introduced in Section 2, and the restrictions imposed on the controls in order to meet the model requirements and to ensure the existence and uniqueness of a solution to the storage equation are shown to compose the admissible class. The storage process is constructed in Section 3 and proven to be strong Markov. In Section 4, the expression for the generator of the storage process is obtained, and its domain and range are explicitly characterized.

Chapter V is devoted to the formulation of the optimal control problem. In Section 1 the control problem is introduced, and necessary restrictions are imposed upon the reward and cost structure. In Section 2, the sufficient condition of optimality is derived in terms of a functional differential equation which is obtained by employing DYNKIN's [1] theory of generator of Markov processes.

The balance of Chapter VI is primarily concerned with the study of the deterministic version of the generalized storage model in which case stochastic input and output processes are not taken into account. The sufficient condition of optimality is restated for the deterministic problem in Section 1. In Section 2, it is shown that there exists a unique return function and an associated control pair satisfying the optimality condition over a certain subset of the admissible class. This way a sequence of locally optimal return functions is created and shown to be convergent. In Section 3, the limit of the locally optimal return functions is shown to be the function we are seeking for only if some monotonicity assumptions are further made about the structure of L_1.

The results of the deterministic problem are extended in Chapter VII so as to include the stochastic processes inherent in the generalized storage model. In Section 1, as it is done in the deterministic case, locally optimal return function and the associated control pair is constructed in M_n, and their properties are studied. In Section 2, the monotonicity assumption imposed on L_1 enables us to demonstrate that the functional differential sufficiency condition has a unique and continuously differentiable solution in M only when the existence of the stochastic output process is excluded in our analysis of the generalized storage model.

In Chapter VIII some possible generalizations are provided by relaxing the restrictions on the model features, and some suggestions are made to readapt the solution procedure proposed by our model. Section 1 drops the assumption of infinite physical capacity and discusses the applicability of the model to stores with finite capacity. In Section 2 no backlogging assumption is relaxed, and it is shown that our solution procedure can be immediately employed for finite backlogging. In Section 3 more general cost and reward structures are considered to illustrate that results similar to those of Chapters VI and VII are readily obtained. Chapter IX mainly dwells upon the conditions under which the optimal controls turn out to possess a bang-bang structure. The theory is discussed in Section 1 and used to solve some simple problems in Section 2. Finally Chapter X concludes this dissertation by providing a summary of results.

Our notation and terminology will follow those of BLUMENTHAL and GETOOR [2]. We will let $\mathbb{N}_+ = \{1,2,\ldots\}$, $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_0 = (0,\infty)$, $\mathbb{R}_+ = [0,\infty)$ and let $\mathcal{R}$, $\mathcal{R}_0$ and $\mathcal{R}_+$ denote the set of subsets of $\mathbb{R}$, $\mathbb{R}_0$ and $\mathbb{R}_+$, respectively. If $(\mathbf{E},\varepsilon)$ and $(\mathbb{F},\mathcal{F})$ are measurable spaces and $f; \mathbf{E} \to \mathbb{F}$ is measurable relative to and $\mathcal{F}$, then we write $f \in \varepsilon/\mathcal{F}$. In particular if $(\mathbb{F},\mathcal{F}) = (\mathbb{R},\mathcal{R})$ we simply write $f \in \varepsilon$. If in addition f is bounded, we write $f \in b\varepsilon$. The σ-algebra generated by (.) will be denoted by $\sigma(.)$. A history $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ on a probability space $(\Omega,\mathcal{H},P)$ is an increasing family of sub σ-algebras of $\mathcal{H}$, and the set of all stopping times of $\mathcal{F}$ will be denoted by $s(\mathcal{F}_t)$.

For any real-valued function f defined on a set F we will let $||f|| = \sup|f(x)|$, $\overline{f} = \sup_{x \in F} f(x)$ and $\underline{f} = \inf_{x \in F} f(x)$. Finally we will let $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$ for any $a,b \in \mathbb{R}$.

II. LITERATURE SURVEY

Storage theory was introduced in 1950's with MORAN's [3] pioneering investigations. Later various studies with different assumptions concerning the model variables-time parameter, state space, release function, stochastic structure of the input process-succeeded his papers and shed light upon different aspects of storage theory.

Most of the studies carried out so far focused attention mainly on storage systems which described the stochastic evolution of the water level in dams. In such a storage model the input process is random in nature, and there does not exist any form of uncontrollable random output out of the store. Furthermore, the control of the store is possible via a proper choice of the release rule which prescribes how and when the water is to be released while an input control is not taken into account. The model developed under these assumptions is described by the following equation

$$X_t = X_o + A_t - \int_o^t r(X_s)ds \quad , \qquad t \geq 0 \tag{2.1}$$

where X_t is the content of the store at time t, A is the input process and r(.) is the associated release rule.

The originality of this dissertation arises from the fact that the random uncontrolled output process and the input control are also incorporated into the above-mentioned storage model. Thus it takes into account the existence of a random process which diminishes the content level of the store and the possibility of an input control rule which, depending on the content level of the store, prescribes when and how much the content level is to be increased by some means varying according to the nature of the particular model under consideration.

Most of the studies on storage theory are basically concerned with the theoretical analysis of the uncontrolled storage model in a way to construct the process and to derive the expressions for its limiting distribution, its generator and the local times. Studies dwelling upon the optimal control problem are less in number and can be classified into two distinct groups as far as the formulation of the optimal control problem is concerned. One group employs Markov decision theory to maximize the expected infinite time horizon discounted reward while the other uses a diffusion approximation to achieve the maximization of the long-run average reward.

MORAN [3] first studies the discrete time problem where the system is observed at discrete time points. He assumes that the system has a finite capacity K and the inputs to the store at different time points $\{A_t : t \in \mathbb{N}_+\}$ are independent and identically distributed random variables. The content level is given by

$$X_{t+1} = X_t + A_t - \min(M, X_t + A_t) \tag{2.2}$$

for some constant $M < K$ where X_t is the content level of the dam just before the input A_t occurs. He proposes several numerical techniques to find the stationary probability distribution of the imbedded Markov chain, but indicates the difficulty of obtaining explicit solution for the finite dam. He then simplifies the problem by dropping the finite capacity assumption and identifies finite capacity dams with queues in two distinct respects: one is known as queueing with bulk service and the other is known as Smith's queueing model. Thus he refers to BAILEY's [4] and LINDLEY's [5] methods for the solution of stationary distribution equations.

MORAN [6] later studies the continuous case in which time varies continuously so that the input is a continuous flow and the release occurs at a continuous rate so long as there is any water in the dam. The input process into the dam is taken to be a right continuous additive homogeneous process with nonnegative independent increments whose means are finite for finite time intervals, i.e. $\mathbb{E}[A_{t+h} - A_t] = mh > 0$ for $h > 0$. The release rule r is chosen so that in any interval of time $(t, t+dt)$ the amount of water released is $r(X_t)dt + O(dt)$ for any bounded realization of the content process. A heuristic description of the resulting process is given by

$$dX_t = -r(X_t)dt + dA_t \tag{2.3}$$

where dX_t and dA_t are the increments of X_t and A_t over the time interval of length dt. First r is taken to satisfy $r(0) = 0$ and $r(u) = \rho u$ $(\rho > 0)$ for $u > 0$, and X_t is written as a linear functional of A_t and shown to be ergodic if $\rho > m$ under these assumptions. Later more restrictive conditions are imposed upon r and given by

i. $r(u)$ is a continuous function of u for $u \geq 0$;

ii. $r(0) = 0$, $r(u) > 0$ for $u > 0$;

iii. $r(u)$ is non-decreasing;

iv. for any finite interval I, there exists a constant K such that $|r(u_1) - r(u_2)| \leq K|u_1 - u_2|$ for $u_1, u_2 \in I$.

Under these assumptions, he derives the following integral equation as a solution to equation (2.3)

$$X_t - X_o = A_t - A_o - \int_0^t r(X_s)ds \quad , \quad t \geq 0$$

and shows that for any realization of A, X_t is a nondecreasing function of X_o. Furthermore he considers two distinct processes X_t^a and X_t^b with the same input process but different release rules $r_a(u)$ and $r_b(u)$ such that $r_a(u) \leq r_b(u)$ for $u \geq 0$ while the initial conditions are equal, i.e. $X_o^a = X_o^b$. He then proves that for all $s \geq 0$ and $t \geq 0$

$$P\{X_t^a \leq s\} \leq P\{X_t^b \leq s\} .$$

The first attempt to consider correlations among inputs is made by LLOYD and ODOOM [7] where the sequence of inflows $\{A_t\}$ during consecutive time intervals constitutes a simple Markov chain with a finite number of states. The release rule is similar to MORAN [6], and the content level of the infinite dam under consideration is as given by (2.2). What distinguishes this study from the previous studies is due to the fact that the input A_t is dependent on the previous values of both the content and the input processes. They show that (X,A) forms a bivariate Markov chain under these considerations.

KHAN and GANI [8] extend the study of correlated inputs by considering a similar model where the release rate M is taken to be unity. Using the moment-generating function and matrix theory, they determine the probability of first emptiness of the dam, the joint probability distribution for (X,A) and the limiting distribution for X.

ÇINLAR and PINSKY [9] improve the dam model by imposing more refined assumptions on the input process and the release rule. The control r is taken to be a Lipschitz continuous strictly increasing function of the content level and the input process A is assumed to possess stationary independent increments with a finite jump rate. From the general theory of processes with stationary independent increments, it is obvious that

$$A_t = a.t - \tilde{A}_t \tag{2.4}$$

where $a \geq 0$ is a constant and $\tilde{A}_t$ is a compound Poisson process with a finite jump rate. So the content process X satisfies the differential equation

$$dX_t = -r(X_t)dt + a.dt \tag{2.5}$$

between the jumps of A. They verify that equation (2.5) has a unique solution q(x,t) which is monotonically non-decreasing, continuous, and satisfies

$$-\frac{\partial q(x,t)}{\partial t} = [r(X) - a]\,\frac{\partial q(x,t)}{\partial t} \quad . \tag{2.6}$$

Then X_t is recursively defined in terms of q(x,t), so that the storage equation is shown to possess a unique solution. They also show that the content process X is a normal standard Markov process and obtain the sufficiency condition for the existence of the limiting distribution for X.

HARRISON and RESNICK [10] obtain the expression for the generator of the content process under the assumptions of [9]. They study the limiting behaviour and the recurrence properties of the content process X and provide a necessary and sufficient condition for the existence of the stationary distribution. They derive some useful results explicitly in terms of a positive kernel, concerning the first hitting time of zero and the transition behaviour among the states of the content level.

ÇINLAR [11] changes the structure of the input process A by assuming it to be semi-Markovian which implies that although the magnitudes of successive inputs form a Markov chain, the sojourn times between successive inputs are not independent and identically distributed. He contructs the content process X and shows that (X,A) is a Markov renewal process to further characterize its transition function.

In a subsequent study, ÇINLAR and PINSKY [12] analyze the situation where the input process has infinitely many jumps in any finite time interval. They further drop the restriction on the Lipschitz continuity of the input rate control r. In the case of infinite jump rate, the input process A is considered as the limit of an increasing sequence of compound Poisson processes plus a drift term, so

$$A_t^n = a.t + \sum_{s \leq t} (A_s - A_{s-}).I_{\{A_s - A_{s-} \geq 1/n\}} \quad (2.7)$$

for all $n \geq 1$ and $t \geq 0$, and

$$A_t = \lim_{n\to\infty} A_t^n \quad . \quad (2.8)$$

Then it is obvious that for each $n \geq 1$ $A^n = \{A_t^n: t \geq 0\}$ is an increasing compound Poisson process with a finite jump rate. Hence the results of previous studies as applied to the storage equation

$$X_t^n = X_o + A_t^n - \int_o^t r(X_s^n)ds \tag{2.9}$$

reveal that there exists a unique solution to (2.9). They then show that X^n converges to X almost surely and this convergence is uniform in t over any finite interval.

Considering the dependence of the input process on environmental factors, ÇINLAR [13] includes the environmental factors as a stochastic process in his model. He lets the environment progress as a standard Markov process Z on an abstract state space and defines the input process A as a non-stationary additive process on the environment process. A Levy-Khinchin type decomposition is provided for $A = \{A_t: t \geq 0\}$

$$A_t = C_t + A_t^f + A_t^d \quad , \qquad t \geq 0 \tag{2.10}$$

where

a) $C = \{C_t: t \geq 0\}$ is a continuous additive functional of the Markov process Z;

b) $A^f = \{A_t^f: t \geq 0\}$ is a pure jump process of the form

$$A_t^f = \sum_j W_j . I_{\{\tau_j \leq t\}}$$

where each τ_j is a stopping time and the corresponding jump magnitude W_j is a random variable whose distribution depends on the values of Z near τ_j;

c) $A^d = \{A_t^d : t \geq 0\}$ is a pure jump process which is stochastically continuous.

He furthermore lets the release function be an arbitrary continuous non-decreasing function vanishing at the origin. Then the content process is constructed by solving the integral equation (2.1) for X. The solution for X is obtained first for input processes which are continuous, then for those which have only finitely many jumps in any finite interval, and finally for those which have infinitely many jumps in any open interval as the limit of processes converging to the given input process. As an important result it is shown that the resulting two dimensional process (Z,X) is a Hunt process.

ÇINLAR [14] investigates the behaviour of the storage process X at zero through a study of the hitting time of zero, local time at zero and the inverse local time. He first computes the Laplace transform for the "time to emptiness", namely the hitting time $S = \inf\{t > 0 : X_t = 0\}$. He then considers the problem of constructing a local time at zero, which is a continuous additive functional whose support is the singleton {0}. So the local time at zero $L = \{L_t : t \geq 0\}$ is defined as

$$L_t = \int_0^t I_{\{0\}}(X_s)ds \quad , \quad t \geq 0 \tag{2.11}$$

and its λ-potential is computed when zero is regular for {0}. The inverse of local times on the other hand is defined as

$$Z_t = \inf\{s > 0 : L_s > t\} \tag{2.12}$$

and shown to be an increasing Levy process. BROCKWELL and CHUNG [15] also study the local time of the content process X at zero under the same assumptions.

COHEN and RUBINOVITCH [16] deal with the stochastic properties of level crossings in a classical dam. They show that the sequence of successive up- and down-crossings of level x forms a renewal process and consequently compute the expected total time spend below x, expected total time spent above x, expected total time spent at level zero and the expected number of down-crossings of level x. Furthermore they pose a simple problem of determining the optimal value of the capacity of a finite dam so as to achieve the maximization of the expected revenue per unit time. So their study lays the ground for cost optimization studies of the dam process.

The above mentioned studies basically deal with constructing various theoretical aspects of the storage theory; however the optimal control problem of the storage processes also receives attention from researchers nowadays, and noteworthy studies contribute greatly to the characterization of the optimal control aspects. The Markov decision theory and the diffusion approximation are the two main tools used in handling with this problem.

The vital importance of the Markov property of processes in control theory is first pointed out by BELLMAN [17]. As a consequence of this perception he and many other researchers succeeding him endeavor to obtain necessary and sufficient conditions for the optimality of controls in Markov decision processes. A Markov decision process is a stochastic process defined on a state space which is controlled by choosing an admissible action from an action space based on the state of the process. The actions interact with chance environment in determining the evolution of the process, but given the present state and the action the evolution of the process until the next decision is made is stochastically independent of the past. A policy which is defined as a Lebesgue measurable, memoryless, deterministic rule prescribes the actions to be chosen, and for each policy and initial state an economic efectiveness is defined as the infinite horizon total expected discounted return. Studies in this area are mainly based upon deriving the conditions under which an optimal policy exists in the sense that it maximizes the total expected discounted return. In fact the necessary and sufficient condition of optimality is derived in terms of the infinitesimal generator of the content process with different model characteristics.

Much of the earlier work in this area is done by BLACKWELL [18] and STRAUCH [19] who restrict themselves to discrete time parameter case. HINDERER [20] gives an extensive account of Markov decision processes with discrete time parameter. MILLER [21] considers Markov decision processes with continuous time parameter, but restricting his attention to the finite state space case. KAKUMANU [22] studies the continuous time Markov decision process in which both the state space and the action space are countable. He proves the existence of a unique optimal return function which satisfies the dynamic optimality condition given in terms of the generator. He furthermore provides a policy space iterative procedure which yields a convergent sequence of stationary policies. VERMES [23] uses the functional analytic theory of Markov processes to prove a sufficient optimality condition for the control of general discrete or continuous-time Markov processes. DOSHI [24] deals with continuous time Markov decision processes on a fairly general state space. In his model no restrictive assumptions are made about the specific nature of the controlled process, and the existence and uniqueness of a solution to the dynamic programming functional equation which expresses the sufficient optimality condition in terms of the generator is proven. It is then shown that the existence of an optimal policy in the special case of time independent reward function implies the existence of a stationary optimal policy. For the problems with finite action space a refined algorithm is presented to generate successively improving stationary policies. Similar results are obtained by PLISKA [25, 26] as well where consideration is focused on the transient, discounted, positive and negative cases all with an infinite time horizon. It is shown that the maximum expected total reward is the limit of a fixed point of an operator on the space of upper semicontinuous functions defined on the state space.

A functional differential equation that arises frequently in the Markov decision problems, specifically in the optimal control of storage models, is studied by PLISKA [27]. Letting S be an interval of the real line and A denote a compact subset of n-dimensional Euclidean space, he defines a nonnegative measure $\beta(x,a,.)$ on the Borel subsets of S for each pair $(x,a) \in S\times A$ and considers two continuous real-valued functions on $S\times A$ p and r with r nonnegative. He then shows that for $S = [0,\infty)$ and each $x > 0$ there exists a unique continuous real-valued function v that satisfies (with $v' \equiv dv/dx$).

$$v'(x) = \sup_{a\in A}\{r^{-1}(x,a)[\int[v(y) - v(x)]\beta(x,a,dy) - \lambda v(x) + p(x,a)]\}, \quad x > 0 \qquad (2.13)$$

and the boundary condition

$$\sup_{a \in A}\{\int[v(y) - v(0)]\beta(0,a,dy) - \lambda v(0) + p\,(0,a)]\} = 0 \qquad (2.14)$$

where λ is a positive constant. Moreover it is shown that v is continuously differentiable on $(0,\infty)$. The proof of this important result is constructed in a way to overcome the difficulty created by the possibility that $r(x,a) = 0$ at $x = 0$.

The results of the studies on the general theory of Markov decision processes are fully utilized by researchers for the optimal control of various storage models; thus besides abstract versions of the Markov decision processes, outstanding applications of the controlled storage processes are involved. MORAIS [28] and MORAIS and PLISKA [29] use the Markov decision theory and the generator of HARRISON and RESNICK [30] to analyze the optimal control problem of the storage model that assumes a pure jump input with a non-stationary content-dependent jump rate and jump size distribution.

DESHMUKH and PLISKA [31] present a controlled storage process model of the problem of optimally consuming a natural resource and exploring for new sources of supply of that resource. Their objective is to choose an optimal consumption and exploration policy so as to maximize the expected discounted utility of consumption diminished by the exploration cost over an infinite planning horizon given the amount of proven reserves of the resource. Their approach is to develop a Markov process model for the level of the proven reserves, to derive the dynamic programming functional equation and to show that it has a unique, nonnegative, increasing, concave and differentiable solution which turns out to be the maximum expected discounted return. In the process of proving the existence of an optimal consumption and exploration policy, they show that this policy is admissible, its associated return is in fact the solution of the functional equation and the optimal consumption rate is strictly positive and nondecreasing while the optimal exploration rate is nonicreasing. In the deterministic models, the shadow price of the resource is shown to rise at the social rate of discount by HOTILLING [32], DASGUPTA and HEAL [33], SOLOW [34]. Deshmukh and Pliska prove the stochastic analog of this result in their model so that the expected rate of increase of the shadow price equals the discount rate; thus the expected scarcity rent on proven reserves rises exponentially in time at the discount rate, but the actual rents may decrease by random amounts at random times whenever new resource deposits are discovered.

The optimal control problem of storage models with Markov additive inputs introduced by ÇINLAR [13] is extensively investigated by ÖZEKİCİ [35]. The environmental process Z is taken to be a regular Hunt process with an infinite lifetime and the input process is taken to be a regular Markov process with a Levy-Khinchin type decomposition as given by (2.10). His aim is to control the content level of the infinite dam by determining the release rate r defined as a function of both the environment and content processes so as to maximize the total expected infinite time horizon discounted earnings under fairly general assumptions imposed upon his admissible set of controls. His reward and cost structure is specified by

$$L(Z_t,X_t,r(Z_t,X_t)) = L_1(X_t) + L_2(r(Z_t,X_t)) \ , \quad t \geq 0 \ . \tag{2.15}$$

In the process of showing the existence and uniqueness of an optimal return function and the associated optimal release rate satisfying the sufficient optimality condition, he considers different properties satisfied by L_1 and studies the conditions which result in bang-bang controls. His approach mainly consists of analyzing the corresponding deterministic model and generalizing the results obtained to the stochastic model.

In addition to Markov decision theory, there exists another method of dealing with the optimal control problem of a storage model, which is known as the diffusion approximation approach. The fact that the input occurs according to a jump process does not allow the input process to have continuous path functions. Consequently the optimality condition is expressed in terms of a functional differential equation. However the relaxation of the jump-process condition will allow the input to possess normal increments and continuous path functions, yielding a considerably easier differential equation. This amounts to assuming that the input process occurs according to a Brownian motion. Although this is a rather crude representation of the true input process-since it allows for negative inputs as well - its advantages are evident from a computational point of view. In all of the studies carried out so far to determine the optimal release rule in diffusion approximated processes the input process into the storage model is assumed to be a Wiener process with positive drift; that is the input flow during the time interval $(t,t+dt]$ is distributed by $N(\mu dt,\sigma^2 dt)$ where the parameter μ is taken to be strictly positive to guarantee positive drift. Another assumption encountered in all is that the store has finite capacity.

BATHER [36] is concerned with determining the optimal release rule for a finite capacity dam where a Wiener process is used to describe the random input flow into the reservoir. His utility function measures the gain per unit time when water is released from the dam at a certain rate and is taken to be strictly concave with continuous second-order derivatives.

The content process in his model does not involve boundary conditions, so that he assumes the water level to be actually zero when the content level falls down below zero, and he allows any excess of input over the capacity of the dam to be simply wasted and not to enter the controlled output. This makes pure reflection impossible. The optimal policy is obtained by solving a second-order differential equation of the potential utility function which represents the current state of the system with regard to the total expectation of utility over an infinite future.

FADDY [37] refined the model developed by BATHER [35] by allowing the content at any time to be negative, effectively assuming a reflecting boundary for the process at the top. He assumes that water may be released at 0 or M units per unit time. The objective is to control this output in a way as to minimize the long term average cost of operating the system. At any time the cost of increasing the output rate from 0 to M is KM, K being a nonnegative constant; likewise the rate may be decreased from M to zero with zero cost. Finally if the output rate is M during a time interval of length dt, then a running cost of -aMdt is incurred where a is a nonnegative constant. He proposes a bang-bang form for the optimal release policy given by

$$\mathbb{P}_{\lambda}^{M} : \begin{cases} \text{Increase Release Rate to M if } x \geq \lambda \\ \text{Decrease Release Rate to 0 if } x = 0. \end{cases} \quad (2.16)$$

He furthermore proves the optimality of the $\mathbb{P}_{\lambda}^{M}$ policy and determines the value of λ explicitly through a renewal argument.

PLISKA [38] further improves the model in BATHER [36] by considering the boundary conditions both at the bottom and at the top of the reservoir as pure reflections. Moreover, the drift and diffusion coefficients are assumed to be continuous functions of the content level and the release rate, while the set of admissible controls consists of all piecewise continuous real-valued functions. He shows that in this considerably general set-up, the minimization of both the expected long run average cost and the expected infinite time discounted cost can be accomplished.

The optimal policy for a two-stage release policy of a finite fam as introduced by FADDY [37] is further analyzed by ZUCKERMANN [39] who allows for a reflecting boundary at the top of the reservoir, and no boundary at zero level. He examines particularly two extreme cases:

i. He assumes that $K = 0$, which means that the release rate may be increased at zero cost. The associated optimal policy turns out to be releasing water at the maximum possible rate as long as the storage level is positive.

ii. He assumes that $a = 0$, which implies that the release of water does not yield any earnings. Then the optimal policy results in switching off the output rate permanently.

He establishes a different version of the proof for the optimality of monotone policies $\mathbb{P}_\lambda^M$ as given by (2.16) for the cases mentioned above.

ATIA and BROCKWELL [40] study the same model, restricting themselves to monotone optimal policies $\mathbb{P}_\lambda^M$ as given by (2.16), but assuming two reflecting boundaries at the top and bottom of the reservoir. They obtain similar results in the optimization of the long-run average cost per unit time.

This review of the studies on the theory of storage models reveals the novelty of this dissertaion. In addition to the random input process and the output control encountered in all the studies mentioned above, our primary emphasis is on presenting and analyzing a model which explicitly incorporates the existence of a random output process and the possibility of an input control. Our approach is to develop a Markov process model for the content level of the store and then to apply Markov decision theory to construct the storage process and to characterize the optimal return and optimal controls.

III. APPLICATIONS OF THE GENERALIZED STORAGE MODEL

In this chapter prominent applications of the generalized storage model will be described briefly. Particularly emphasis will be given to demonstrate how the generalization accomplished in this study can be applied to problems and situations which can't be handled adequately by the currently available storage models. The more general features of the model developed here will allow for the analysis of new problems which constitute brand new application areas.

DAMS. A dam is a store where there is a random input flow of water to be stored in the reservoir and to be discharged for purposes of flood control, power generation, processing drinking water, irrigation and recreation. In such a store, evaporation, seepage and overflow conditions inevitably prevailing in the environment require the consideration of a random output process as well. As far as the control problem is concerned, besides the possibility of decreasing the content level at a controlled rate, there exists another control mechanism especially in a network of dams. This enables one to increase the water level in any one of the dams in the network through an inlet permitting water flow at a particular rate. So the control problem of the content level of a dam in a network of dams can be modelled with the generalized storage equation (I.1).

We let X_t be the content or water level of a dam at time t and assume that the random input to the dam is given by a process A and the random output out of the dam is given by a process B. Here A is the process describing the jump inputs to the dam which arise from climatic conditions. Thus A_t denotes the total amount of water randomly flowing into the dam during [0,t]. B, on the other hand, is a process which describes the random outputs out of the dam due to evaporation and overflow. B_t then is the total amount of water randomly evaporating, overflowing or seeping out of the dam during [0,t].

The controller who observes the content level of the dam continuously decides upon an output rate for discharging water out of the dam and an input rate for letting water into the dam from some other dam in series with it or another source according to the current content level. If the output rate is given by a function r of the water level and if the input rate is given by a function p of the water level, then the storage equation (I.1) clearly describes the model under consideration.

NATURAL RESOURCES. This is the case where in a socially managed economy there is an exhaustible resource such as oil, mineral deposits, energy which is essential and can be stored indefinitely over the planning horizon. Although the resources cannot be produced, the amount on hand may be increased by exploring and searching for new sources of supply of the resource. The exploration process involves uncertainty regarding the time until a successful discovery as well as the magnitude of supply gained upon discovery. This randomness arising from the exploration process is conveyed in the model by the input process. The stock of this resource in turn is depleted through consumption for the sake of social and economic utility. In fact, it is a well-known observation that high consumption of the resource yields high returns. Moreover in some cases the exploration activity directly depletes a portion of the extracted reserves of the resource. Exploring for new oil, for example, depletes the existing stock of oil. It is a decision maker's problem to decide upon the appropriate rate of consumption. Besides the random input process, deterioration decreasing the level of extracted reserves held in storage occurs randomly in time, and misdetermination of the proven reserves may be realized upon the extraction and less than what is estimated may be extracted; so these constitute a random output process. Furthermore there exists some controlled means of increasing the level of extracted reserves. In macro level, the country suffering from the scarcity of the resource may be obliged to import it. In micro level especially when the exploration itself consumes a portion of the stock, the firm may require the procurement of the resource. Thus the availability of that resource at a controlled rate from exogenerous sources should be taken into account and incorporated into the model. At each instant of time given the amount of proven reserves, the problem is reduced to determining the consumption rate which includes the amount consumed for social welfare and the amount depleted for further exploration activities, if such a situation exists, the exploration rate and the procurement rate which directly increases the amount of extracted reserves in the stock.

Here X_t denotes the level of proven reserves at time t without distinguishing between known reserves in the ground and extracted reserves held in inventory. The exploration increases the stock in a random manner as described by the process A while the process B shows the effect of deterioration and the impact of errors made in forecasts. The control r corresponds to the consumption rate of the resource, regardless of the purpose it is being used, while the control p corresponds to the controlled input rate. So the process X is clearly explained by the generalized storage equation (I.1).

QUEUING MODELS. A queue is a single or multiple server system at which customers arrive, demanding a random amount of service, experience delays before they are served, and leave at the completion of their service demand. Moreover customers may decide to leave the system before they are served. This random departure process is due to either balking, reneging or jockeying conditions. In the presence of these random processes the virtual waiting time of the queue may be controlled in two respects. The output rate which is the service rate in queuing theory can be adjusted by changing the number of servers, the rate at which servers work etc. On the other hand, it is possible to increase the work load of a particular queue in a network of queues. The controller may decide to feed customers at a proper rate to a particular queue under consideration from some other queue in the network with higher virtual waiting time.

Here A_t is the total amount of service demand that enters the queue during the time interval [0,t], and B_t is the total amount of service demand that leaves the queue during [0,t] because of balking, reneging and jockeying. Consequently X_t denotes the outstanding demand for service at time t, i.e. A_t plus the controlled input up to time t diminished by B_t

and the total amount of service delivered during [0,t]. It is obvious that X_t becomes the virtual waiting time of a customer in a "first-come-first-served" priority rule if he arrived at time t. The rate at which service is delivered is specified through the output rate r, and the rate at which service load is increased is specified by the input control p. Thus the process X imbedded in the queueing model is described by the generalized storage equation (I.1).

INSURANCE MODELS. An insurance company receives premiums from customers and in return pays for claims made in random amounts at random time points. The company's current fund position increases due to premiums, arriving randomly through time, which constitute the stochastic input process, and decreases by random magnitudes due to claims, also arising randomly through time, which constitute the stochastic output process. It can be controlled by altering the premiums charged, by advertising and promotion campaigns, and by re-specifying the customer selection policy based on the risks involved. The company's fund position can be increased by raising the periodic premiums charged from the customers or by relaxing the customer selection criterion; it can be decreased by imposing reinvestment opportunities of any kind. At any instant of time the fund level is controlled through proper choices of an output rate, which is to evaluate the reinvestment possibilities of the money received from premiums, and an input rate, which is to determine the premium and customer policies.

Thus, X_t is the company's fund position at time t. The premium arrival process is described by A, and the claim arrivals are given by B. If the rate at which money is being expanded for various investment purposes is given by the output rate control r and the rate at which money flows through the premiums received from potential customers is given by the input rate control p, then the fund process X is described by the generalized storage equation (I.1).

BANKING MODELS. A bank is a store where people deposit money that earns interest at a certain rate and can withdraw the unpaid principal plus the interest accrued. The stochastic input process arises from the arrival process of customer deposits which are random both in magnitude and timing wherein the stochastic output process is due to the arrival process of customer withdrawals which also occur randomly through time at random quantities. The bank's current fund position can be controlled by interest rates, promotional efforts, credits, bonds, shares and various service and investment decisions. The bank's managers observing the current financial situation may decide to increase the credits provided for industrial, agricultural and other socio-economic purposes, to invest money on any business venture or to employ further promotional activities. On the other hand, the bank itself may issue and sell bonds and shares, borrow money from financial organizations or borrow cash from the State Bank or some other bank. Interest rates as applied to customer deposits and credits, and repayment plans of the loans provided by the bank are other tools of controlling the financial situation.

In this application, X_t represents the bank's fund position at time t. The input process A describes the arrival pattern of customers depositing money at the bank while the withdrawal process is given by the process B. The output rate control r gives the rate at which money is expanded, and the input rate control p gives the rate at which money is gained by any one of the means mentioned above. So the bank's fund position X can be described by the generalized storage equation (I.1).

INVENTORY MODELS. An inventory model may be considered as a storage node where a commodity either purchased or manufactured is accumulated to be used to satisfy some future demand. As far as the occurrence time and the magnitude of the demand are concerned, the demand process conveys uncertainty and constitutes a stochastic process. Especially in case of in-process inventories, the demand process can be taken to possess a controllable component in the sense that the controller may decide upon an increase in the production level of some stage which requires the depletion of the in-process inventory. The output process is partially under control such that one may decide when and at which rate to order or to produce while there exist some uncontrolled factors which occur randomly and which cause reductions in the outstanding demand.

In an inventory model, X_t denotes the outstanding demand, i.e. the total demand that has occurred minus the demand that has been met during [0,t]. The random demand is given by the process A which depicts the total demand for the product under consideration. The process B describes all possible random demand withdrawals which decrease the existing demand requirement. If the product is being manufactured, the output rate is controlled by the production rate r specified by the number of workers, number of machines, rates at which machines are operating etc. If the product is being purchased, the output rate is controlled by the procurement rate r. The demand rate is controlled by the input control p in case of in-process inventories. So within this set-up the inventory process X can be explained by the storage equation (I.1).

As the above mentioned applications show, the generalized storage model can be used for a variety of applications in financial management, industrial engineering and micro-economics all of which involve inflow, storage and outflow under uncertainty. Furthermore the models encountered in health services - such as blood banks - and computer memories can be described by the generalized storage equation with slight modifications.

In our model, assumptions on the stochastic structure of the input and output processes are derived so as to meet the conditions required by the arrival processes inherent in the applications. The input and output processes are taken to be two independent and increasing Compound Poisson processes which are employed frequently to explain the arrival patterns. Furthermore it is assumed that they have finitely many jumps in any finite time interval. The admissible output and input controls are assumed not to affect the stochastic behaviour of the input and output processes in any way although the structure of the controls are made as general as possible. Our model will be clarified by our assumptions which will be described in detail in Chapter IV.

IV. THE UNCONTROLLED STORAGE MODEL

This chapter is devoted to a detailed description of the generalized storage model introduced in previous chapters. The input process A and the output process B are assumed to be two independent compound Poisson processes with only finitely many jumps in any finite time interval while the input rate control p and the output rate control r are defined as functions of the content level. In Section 1, the stochastic structures of the input process A and the output process B are described in detail. In Section 2, the restrictions to be imposed upon the controls are discussed, and consequently the admissibility conditions which should be satisfied by p and r are formally put forward. In the meantime special attention is given to make the controls as general as possible by imposing minimal restrictions upon them. In Section 3, the storage process X is constructed and shown to be strong Markov for any given pair of admissible controls (r,p). In Section 4, the generator of the storage process, which turns out to be the main tool in the optimal control problem, is to be analyzed, and its expression together with its domain and range is determined explicitly.

4.1 THE INPUT PROCESS AND THE OUTPUT PROCESS

As we proceed further, it will become clear how the stochastic structure of the storage process X is uniquely determined by the stochastic structure of the input process A and the output process B; thus it is crucially necessary to comprehend the stochastic properties of A and B. In this section we will specify the input and output processes and try to provide an insight into their quantitative properties. The main assumption concerning the stochastic structure of A and B is stated below. Let (Ω,μ, P) be a complete probability space.

(1.1) ASSUMPTION. The input process A and the output process B are two independent and increasing compound Poisson processes defined on (Ω,μ , P) with finitely many jumps in any finite time interval. $\square$

This assumption implies that for any $\omega \in \Omega$, the mapping $t \to A_t(\omega)$ is non-decreasing, right continuous, increases by jumps only, and $A_0(\omega) = 0$. Similarly for any $\omega \in \Omega$ the mapping $t \to B_t(\omega)$ is non-decreasing, right continuous, increases by jumps only, and $B_0(\omega) = 0$.

Throughout this study we define $\{T_n\}$ as the jump times of the input process A recursively by:

$$T_0 = 0, \quad T_{n+1} = \inf\{t > T_n\colon A_t \neq A_{t-}\}, \quad n \in \mathbb{N}_+ . \tag{1.1}$$

Similarly we define $\{\tau_n\}$ as the jump times of the output process B recursively by:

$$\tau_0 = 0, \quad \tau_{n+1} = \inf\{t > \tau_n\colon B_t \neq B_{t-}\} , \; n \in \mathbb{N}_+ . \tag{1.2}$$

Furthermore if the magnitudes of the successive jumps of the input process are denoted by $\{Y_n\}$, it is possible to represent the input process in the form

$$A_t = \sum_{T_n \leq t} Y_n \quad , \qquad t \geq 0$$

where $\{T_n\}$ is as defined by (1.1).

If the magnitudes of the successive jumps of the output process are denoted by $\{Z_n\}$, it is possible to represent the output process in the form

$$B_t = \sum_{\tau_n \leq t} Z_n \quad , \qquad t \geq 0$$

where $\{\tau_n\}$ is as defined by (1.2).

By the fact that the magnitudes of the successive jumps in a compound Poisson process are independent and identically distributed random variables independent of the jump times, $\{Y_n\}$, namely the jump sizes of the input process, are independent, and

$$P\{Y_n \in D\} = G_a(D) \quad , \quad D \in R_+ \quad , \quad n \geq 0$$

for some distribution function $G_a(.)$ on R_+. Similarly the jump sizes of the output process $\{Z_n\}$ are independent, and

$$P\{Z_n \in D\} = G_b(D) \quad , D \in R_+ \quad , n \geq 0$$

for some distribution function $G_b(.)$ on R_+.

Moreover it is another well-known result that the interarrival times in a compound Poisson process are independent and identically distributed exponential random variables; thus the distributions of the interarrival times of the input and output processes are given by

$$P\{T_{n+1} - T_n > t\} = e^{-\lambda_a t} \quad , \quad t \geq 0$$

and

$$P\{\tau_{n+1} - \tau_n > t\} = e^{-\lambda_b t} \quad , \quad t \geq 0$$

for some finite parameters $\lambda_a \geq 0$, $\lambda_b \geq 0$ respectively.

The important point to note here is that the storage process X jumps whenever A or B jumps and increases or decreases accordingly. So the jump times of A and B completely determine the jump times of X. Therefore we define the jump times of X denoted as $\{S_n\}$ by

$$S_o = 0 \ , \quad S_{n+1} = \inf\{t > S_n: A_t \neq A_{t-} \quad \text{or} \quad B_t \neq B_{t-}\}, \quad n \in \mathbb{N}_+ \ . \tag{1.3}$$

It is clear that for any $\omega \in \Omega$, $S_1(\omega) = \min(T_1(\omega), \tau_1(\omega))$, $\{T_n(\omega)\} \subset \{S_n(\omega)\}$ and $\{\tau_n(\omega)\} \subset \{S_n(\omega)\}$. Furthermore without loss of generality we assume that for all

$$\lim_{n\to\infty} T_n(\omega) = \infty, \quad \lim_{n\to\infty} \tau_n(\omega) = \infty \ , \quad \lim_{n\to\infty} S_n(\omega) = \infty \ .$$

The stochastic processes involved in the storage equation, namely the output and input processes, which are taken to be two compound Poisson processes independent of each other, their jump times and jump magnitudes are specified by the above argument to clarify the stochastic behaviour of the storage process X.

4.2 ADMISSIBLE CONTROLS

In this section we will dwell upon the input and output rate controls p and r, respectively, and specify the conditions that should be met by p and r so as to guarantee the uniqueness and existence of a solution to the storage equation. Since the controls p and r provide the only means of controlling the generalized storage model in the optimal control problem, they play a crucial role which necessitates a thorough analysis. Note that the optimal control problem requires the admissibility conditions which are to be imposed upon p and r to be minimal, so we will define the set of admissible controls by imposing a set of minimal restrictions upon p and r.

We define the set of admissible controls M to be the set of all pairs of functions (r,p) both defined on R_+ satisfying the admissibility conditions which we discuss next.

<u>ADMISSIBILITY CONDITION 1</u>. The controls p and r are both bounded; this implies that

$$0 \leq r(x) \leq \bar{r} \quad , \quad x \in \mathbb{R}_+$$
$$0 \leq p(x) \leq \bar{p} \quad , \quad x \in \mathbb{R}_+$$

for some $\bar{r} \in \mathbb{R}_+$ and $\bar{p} \in \mathbb{R}_+$. Δ

<u>ADMISSIBILITY CONDITION 2</u>. The content level of the store is restricted not to fall below the zero level. In other words, there can not be any output from the store when it is empty. This is guaranteed by assuming that

$0 \leq r(0) \leq p(0)$. Δ

<u>ADMISSIBILITY CONDITION 3.</u> The optimal control problem requires (r,p) to satisfy:

i. r(.) and p(.) are both left-continuous on $D_1 = \{r(.) \geq p(.)\}$, and if $\{x_n\} \subset D_1$ with $x_n \uparrow \bar{x}$, then $\lim_{n\to\infty}[r(x_n) - p(x_n)] = r(\bar{x}) - p(\bar{x})$;

ii. r(.) and p(.) are both right-continuous on $D_2 = \{r(.) \leq p(.)\}$, and if $\{x_n\} \subset D_1$ with $x_n \downarrow \bar{x}$, then $\lim_{n\to\infty}[r(x_n) - p(x_n)] = r(\bar{x}) - p(\bar{x})$. Δ

This condition will enable us to characterize the generator by Proposition (4.1) and Theorem (4.1) and to use it in the optimal control of the generalized storage model through Theorem (V.2.5).

<u>ADMISSIBILITY CONDITION 4.</u> The controls (r,p) should be chosen in a way so that for every $\omega \in \Omega$ the equation

$$f(t) = x + A_t - B_t + \int_0^t (p - r)(f(s))ds \quad , \quad t \geq 0 \qquad (2.1)$$

possesses a unique solution for every $x \in \mathbb{R}_+$. Δ

A typical control pair (r(.), p(.)) satisfying Admissibility Conditions 1-3 is depicted in Figure (2.1). Note that $p(x) \geq 0$ and $r(x) \geq 0$, and $p(x) \leq \bar{p}$ and $r(x) \leq \bar{r}$ for all $x \in \mathbb{R}_+$ by Admissibility Condition 1. At zero content level $r(0) \leq p(0)$ by Admissibility Condition 2. Also note that by Admissibility Condition 3 r(.) and p(.) are right-continuous on $[0,x_1]$ and r(.) and p(.) are left-continuous on $[x_2,\infty)$.

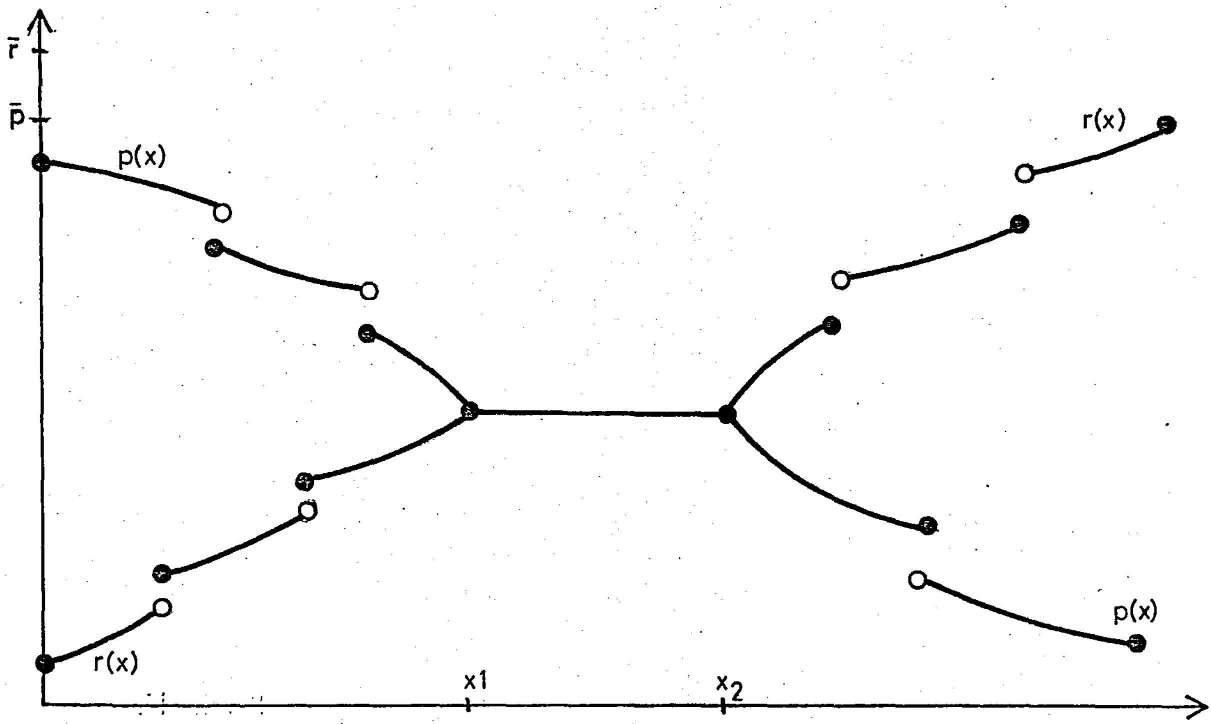

FIGURE 2.1 - A typical control pair (r(.),p(.)) satisfying Admissibility Conditions 1-3.

An important consequence of Admissibility Conditions 3 is that the set $\{r(.) = p(.)\}$ is closed. To see this, note that if $\{x_n\} \subset \{r(.) = p(.)\}$ with $x_n \uparrow \bar{x}$, then $r(\bar{x}) = p(\bar{x})$ by the left continuity of r and p on $\{r(.) \geq p(.)\}$. On the other hand if $\{x_n\} \subset \{r(.) = p(.)\}$ with $x_n \downarrow \bar{x}$, then $r(\bar{x}) = p(\bar{x})$ by the right continuity of both r and p on $\{r(.) \leq p(.)\}$; this argument implies that the set $\{r(.) \neq p(.)\}$ is open.

The final admissibility condition is the most important restriction imposed upon the controls (r,p), so it deserves more emphasis. In fact, Admissibility Condition 4 can be simplified by the fact that there are only finitely many jumps in any finite time interval. As it will be verified by Lemma (2.1), the jumps of the input and output processes do not affect the structure of the controls as far as the existence and uniqueness of a solution to equation (2.1) is concerned. Thus it suffices to consider the associated deterministic problem in evaluating Admissibility Condition 4.

(2.1) LEMMA. Admissibility Condition 4 is satisfied if and only if there exists some $t_1 > 0$ such that the equation

$$f(t) = x + \int_0^t (p - r)(f(s))ds \quad , \quad t \geq 0 \tag{2.2}$$

has a unique solution on $[0,t_1)$ for every $x \in \mathbb{R}_+$.

Proof. Note that any function f satisfying (2.1) also satisfies

$$f(t+s) = f(s) + A_{t+s} - A_s - (B_{t+s} - B_s) + \int_0^t (p - r)(f(s + u))du, \quad t \geq 0$$

for any fixed $s \geq 0$. Therefore, Admissibility Condition 4 is satisfied if and only if for any $\omega \varepsilon \Omega$, equation (2.1) has a unique solution on some finite interval $[0,t(\omega))$ for some $t(\omega) > 0$ and for every $x \varepsilon R_+$. To show necessity take $t_1 = \min(t(\omega), S_1(\omega)) > 0$ for any $\omega \varepsilon \Omega$. This implies that t_1 is less than any input or output jump time; thus $A_t(\omega) = 0$ and $B_t(\omega) = 0$ on $[0,t_1)$ and equation (2.1) reduces to equation (2.2). So the desired result follows immediately. Sufficiency follows in a similar manner by simply taking $t(\omega) = \min(t_1,S_1(\omega)) > 0$ for any $\omega \varepsilon \Omega$. Δ

Lemma (2.1) states that equation (2.1) has a unique solution for all t if and only if it has a unique solution for all t smaller than the first jump times of both the input and output processes. The presence of the two stochastic processes does not influence the existence and uniqueness of a solution of equation (2.1). So it follows from Lemma (2.1) that we can treat the generalized storage model as if it is simply a deterministic model with an input control p and an output control r. It consequently becomes easier to study the admissibility of control pairs (r,p). The existence or the uniqueness of a solution to equation (2.2) may fail to some $x \in \mathbb{R}_+$. It therefore still remains to show under what conditions there exists a unique solution for the admissibility condition introduced by Lemma (2.1). We now analyze some explicit conditions to be imposed upon p and r, which will insure the existence and uniqueness of a solution to equation (2.2)

ÇINLAR [9] showed that in the storage model with no controlled input and no random output the storage equation has a unique solution if the mapping $x \to r(x)$ is continuous and increasing. He further showed that the same result holds true when r is Lipschitz continuous or continuously differentiable. ÖZEKİCİ [35] later extended these results to the case where the output rate control r is dependent upon his environmental process. Similar results are obtained in the generalized storage model, and the Lipschitz or increasing property of r(.) - p(.) is shown to insure the existence and uniqueness of a solution.

(2.1) PROPOSITION. Let M_i be the set of all pairs of functions (r,p) both defined on R_+ satisfying Admissibility Conditions 1-3 and

i) r(.) and p(.) have only finitely many discontinuities in any finite interval;

ii) if either $r(x^-) \leq p(x)$, $r(x^+) \geq p(x)$ or $r(x^-) \geq p(x)$, $r(x^+) \leq p(x)$ for some $x \in \mathbb{R}_+$, then $r(x) = p(x)$;

iii) $r(.) - p(.)$ is increasing;

then $M_i \subset M$.

Proof. The proof follows ÖZEKİCİ [35], replacing his control r by r - p. Δ

(2.1) REMARK. An important consequence of Proposition (2.1) is that the control pair $(r,p) \in M_i$ turns out to be admissible if r is taken to be increasing and p is taken to be decreasing. Δ

A similar result is obtained when the controls are both piecewise Lipschitz functions.

(2.2) PROPOSITION. Let M_L be the set of all pairs of functions (r,p) both defined on R_+ satisfying Admissibility Conditions 1-3 and

i) $r(.)$ and $p(.)$ have only finitely many discontinuities in any finite interval;

ii) if either $r(x^-) \leq p(x)$, $r(x^+) \geq p(x)$ or $r(x^-) \geq p(x)$, $r(x^+) \leq p(x)$ for some $x \in \mathbb{R}_+$, then $r(x) = p(x)$;

iii) $r(.)$ and $p(.)$ are piecewise Lipschitz, i.e. $|r(x_1) - r(x_2)| \leq m_1|x_1-x_2|$ for some $m_1 < \infty$ whenever $x_1, x_2 \in I$ for some interval I on which $r(.)$ is continuous. Similarly, $|p(x_1) - p(x_2)| \leq m_2|x_1 - x_2|$ for some $m_2 < \infty$ whenever $x_1, x_2 \in I$ for some interval I on which $p(.)$ is continuous;

then $M_L \subset M$.

Proof. The proof follows ÖZEKİCİ [35], replacing his control r by r - p. Δ

(2.2) REMARK. Propositions (2.1) and (2.2) reveal that if either p is decreasing and r is increasing or both p and r piecewise Lipschitz, there exists a unique solution to equation (2.2). Although these do not constitute the complete set of admissible controls, these subsets of M are general enough for our purposes. Δ

Actually we are now in a position to define for any $(r,p) \in M$ and $x \in \mathbb{R}_+$ a solution $q(x,.)$ of (2.2) on R_+ by:

$$q(x,t) = \begin{cases} \sup\{L(x) \le y \le x: \int_y^x dz/(r(z)-p(z)) \ge t\} \vee L(x) & \text{if } r(x) > p(x) \\ x & \text{if } r(x) = p(x) \\ \inf\{x \le y \le U(x): \int_x^y dz/(p(z)-r(z)) \ge t\} \wedge U(x) & \text{if } r(x) < p(x) \end{cases} \quad (2.3)$$

where

$$L(x) = \sup\{z \le x: r(z) = p(z)\}$$

$$U(x) = \inf\{z \ge x: r(z) = p(z)\}.$$

Note that $\int_a^b dz/|r(z)-p(z)|$ gives us the total time required to change the content level from a to b when there exist only controlled input and output; thus the function $q(x,t)$ as defined by (2.3) is the content level of the store at time t if the initial content is x. In fact $q(x,t)$ is the unique solution of equation (2.2) where the input process is deterministic with no jump inputs, and the output process is deterministic with no jump outputs, i.e. $A_t = B_t = 0$ identically for all t.

(2.3) REMARK. Some properties of q can be listed as:

i) $q(x,0) = x$ for every $x \in \mathbb{R}_+$;

ii) $q(x,t)$ is decreasing in t if $r(x) \geq p(x)$ for $x \in \mathbb{R}_+$;

iii) $q(x,t)$ is increasing in t if $r(x) \leq p(x)$ for $x \in \mathbb{R}_+$;

iv) the mapping $t \to q(x,t)$ is continuous for fixed $x \in \mathbb{R}_+$. $\square$

These observations are pictorially summarized in Figure (2.2). If the output rate exceeds the input rate at the initial level x, the content level continuously decreases until the output rate equals the current input rate and then remains at that particular level forever. The reverse holds true when the initial output rate is less than the initial input rate: the content level increases until the input and output rates become equal and remains there forever.

In this section attention is primarily focused upon the admissibility of the control pair (r,p) as far as the existence and uniqueness of a solution to the storage equation is concerned. We characterized some subsets of admissible controls in M by considering the deterministic problem and consequently were able to identify the unique solution q which satisfies equation (2.2). An important point to note here is that results we have obtained in this section are in accordance with the results of all the work to date on storage theory.

4.3 CONSTRUCTION OF THE GENERALIZED STORAGE PROCESS

In this section we construct the storage process X and show that it is a Hunt process. In doing so, an important aspect to be incorporated into the construction is the no-backlogging condition. Recall that the content level of the store increases due to an input jump and decreases due to an output jump where the input and output processes are as defined

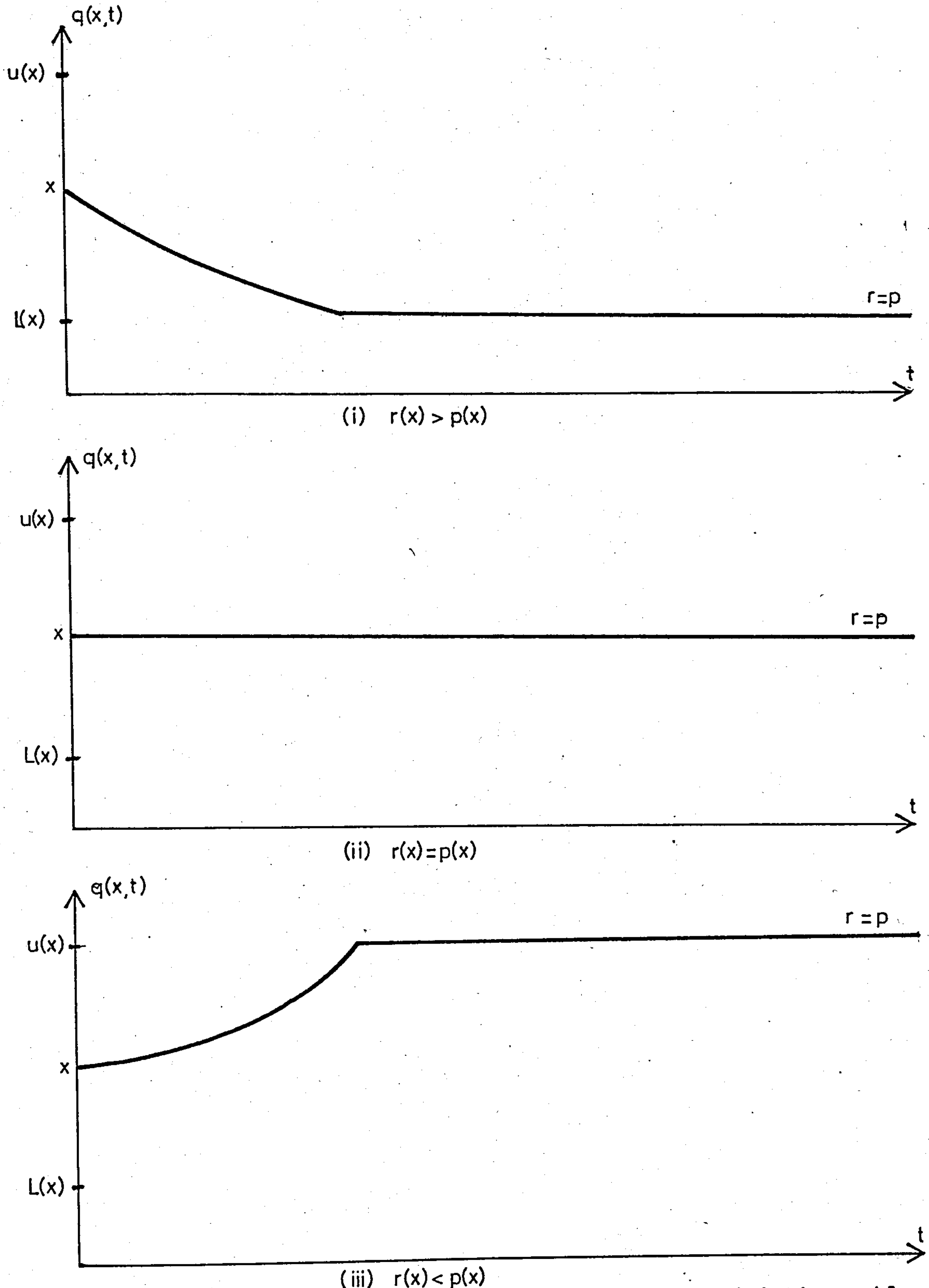

FIGURE 2.2 - The content level q(x,.) of the deterministic problem for any $(r,p) \in M$ and a given $x \in \mathbb{R}_+$.

in Section IV.1. When a jump which brings the current content level below zero occurs, the content level should be prevented from falling below zero to avoid backlogging, so it is lumped into zero at the critical point of emptiness. To achieve this scheme, we introduce for fixed $\omega \varepsilon \Omega$

$$Z_n(\omega) = \{X_{S_n^-}(\omega) + A_{S_n}(\omega) - A_{S_n^-}(\omega) - (B_{S_n}(\omega) - B_{S_n^-}(\omega))\} \vee 0 \quad , n \varepsilon \mathbb{N}_+.$$

Now we are in a position to define X in terms of Z and q as defined by (2.3). We assume that $(r,p) \varepsilon M$ is fixed. For every fixed $\omega \varepsilon \Omega$ and $x \varepsilon \mathbb{R}_+$, define $X_t(\omega)$ recursively by:

$$X_t(\omega) = \begin{cases} q(X_0(\omega), t) & 0 \leq t < S_1(\omega) \\ q(Z_n(\omega), t - S_n(\omega)) & S_n(\omega) \leq t < S_{n+1}(\omega). \end{cases} \quad (3.1)$$

This is the unique solution of the generalized storage equation (I.1), which follows from Lemma (2.1) and the fact that Lim $S_n(\omega) = \infty$. Obviously for all $\omega \varepsilon \Omega$, $X_t(\omega)$ jumps only when $A_t(\omega)$ or $B_t(\omega)$ jumps. Therefore the jump times of X coincide with those of A or B. The evolution of X in between the jumps is deterministic and described by $q(x,t-t_0)$ if at time t_0 there occurred a jump which brought the content level to x. Note that if the jump which took place at t_0 was due to an output jump and its magnitude was large enough to bring the content level below zero, then the evolution of x is described by $q(0,t-t_0)$ until the next jump occurs.

Assume that $\hat{A}$ and $\hat{B}$ are the input and output processes defined on $(\hat{\Omega}, \hat{F}, \hat{p})$, respectively, as given by Assumption (1.1). Now define the shift operators $\{\hat{\theta}_t\}$ on $\hat{\Omega}$ such that for each $t,s \geq 0$, $\hat{A}_{t+s}(\hat{\omega}) = \hat{A}_t(\hat{\omega}) + \hat{A}_s \circ \hat{\theta}_t(\hat{\omega})$ and $\hat{B}_{t+s}(\hat{\omega}) = \hat{B}_t(\hat{\omega}) + \hat{B}_s \circ \hat{\theta}_t(\hat{\omega})$.

Let $\hat{\hat{F}}$ be the completion of $\sigma(\hat{A}_s, \hat{B}_s : s \geq 0)$ with respect to the family of measures $\{P_\mu : \mu$ a finite measure on $\hat{F}\}$. Furthermore let $\hat{F}_t$ be the completion of $\sigma(\hat{A}_s, \hat{B}_s : s \leq t)$ in $\hat{F}$ with respect to the same family $\{P_\mu\}$.

We now let

$$\Omega = \mathbb{R}_+ \times \hat{\Omega}, \quad F^0 = R_+ \times \hat{F}, \quad F_t^0 = R_+ \times \hat{F}_t$$

and for each $\omega = (x, \hat{\omega}) \in \Omega$ define

$$A_t(\omega) = \hat{A}_t(\hat{\omega}), \quad B_t(\omega) = \hat{B}_t(\hat{\omega}), \quad \theta_t(\omega) = (X_t(\omega), \hat{\theta}_t(\hat{\omega}))$$

where $X_t(\omega)$ is the unique solution of the storage equation as given by (3.1) such that $X_0(\omega) = x$.

For $x \in \mathbb{R}_+$, define a probability measure on F^0 by

$$\mathbb{P}_x = \delta_x \times \hat{P}$$

where δ_x is the Dirac measure concentrating its unit mass at x. It is clear that the mapping $x \to \mathbb{P}_x(\Lambda)$ is in R_+ for any $\Lambda \in F^0$ and thus

$$\mathbb{P}_\mu(\Lambda) = \int_{R_+} \mu(dx)\mathbb{P}_x(\Lambda)$$

is well-defined for any finite measure μ on R_+.

The construction of the storage process X will now be completed by letting F be the completion of F^0 with respect to the family of measures $\{P_\mu\}$ and F_t be the completion of F_t^0 in F with respect to the same family $\{P_\mu\}$. Thus it is obvious that F_t conveys all the information contained in the input, output and storage processes during [0,t).

Theorem (3.1) based upon this construction states an important result which enables us to employ Markov decision theory in the optimal control problem.

(3.1) THEOREM. The storage process

$$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}_x) \quad .$$

is a Hunt process taking values in $\mathbb{R}_+$.

Proof. In view of Definition (9.2) in BLUMENTAL and GETOOR [2], we need to check the following properties:

a) *Normality*. From the definition of X_0 and the probability measure $\mathbb{P}_x$, $\mathbb{P}_x\{X_0 = x\} = 1$ for any $x \in \mathbb{R}_+$.

b) *Right Continuity and the Existence of Left-Hand-Limits*. The way $q(x,t)$ is defined by (2.3) and the storage process X is constructed in terms of q by (3.1) ensures us that the mapping $t \to X_t$ is right continuous and has left-hand-limits.

c) *Homogeneity*. It suffices to check that $X_{t+s} = X_s \circ \theta_t$. Note that $\hat{A}_{t+s}(\hat{\omega}) = \hat{A}_t(\hat{\omega}) + \hat{A}_s \circ \hat{\theta}_t(\hat{\omega})$ and $\hat{B}_{t+s}(\hat{\omega}) = \hat{B}_t(\hat{\omega}) + \hat{B}_s \circ \hat{\theta}_t(\hat{\omega})$ for fixed $\omega = (x,\hat{\omega})$. From the definition of X, $X_{t+s}(\omega)$ satisfies (2.1) for all $s \geq 0$; so we can write

$$X_{t+s}(\omega) = x + \hat{A}_{t+s}(\hat{\omega}) - \hat{B}_{t+s}(\hat{\omega}) + \int_0^{t+s} (p-r)(X_u(\omega))du, \quad t \geq 0.$$

Writing the integral as the sum of the two integrals, we obtain for $s \geq 0$

$$X_{t+s}(\omega) = x + \hat{A}_t(\hat{\omega}) + \hat{A}_s \circ \hat{\theta}_t(\hat{\omega}) - \hat{B}_t(\hat{\omega}) - \hat{B}_s \circ \hat{\theta}_t(\hat{\omega})$$

$$+ \int_0^t (p - r)(X_u(\omega))du + \int_0^s (p - r)(X_{t+u}(\omega))du$$

$$= X_t(\omega) + \hat{A}_s \circ \hat{\theta}_t(\hat{\omega}) - \hat{B}_s \circ \hat{\theta}_t(\hat{\omega}) + \int_0^s (p - r)(X_{t+u}(\omega))du \ .$$

Hence $s \to X_{t+s}(\omega)$ satisfies equation (2.1) with (x,ω) replaced by $(X_t(\omega),\hat{\theta}_t(\hat{\omega}))$, respectively. From the uniqueness of the solution this implies

$$X_{t+s}(\omega) = X_s(X_t(\omega),\hat{\theta}_t(\hat{\omega})) = X_s \circ \theta_t(\omega)$$

since $\theta_t(\omega) = (X_t(\omega),\hat{\theta}_t(\omega))$.

d) *Quasi-Left Continuity.* Let $\{T_n\}$ be an increasing sequence of $\{\mathcal{F}_t\}$ stopping times with Limit T. By the fact that $t \to X_t$ has left-hand limits everywhere, $\lim_n X_{T_n}$ exists. Since X_t is continuous on (S_n,S_{n+1}), $X_{T_n} \to X_T$ everywhere on Ω except on the set

$$\{T_n < T \text{ for all } n: \ \lim T_n = T; \ T < \infty\}$$

and then only if T is a point of discontinuity for X_t. If $\lim_n X_{T_n} \neq X_T$, then either $\lim_n A_{T_n} \neq A_T$ or $\lim_n B_{T_n} \neq B_T$ since X_t has the same points of discontinuity as either A_t or B_t. But this is not possible since A and B are both obviously quasi-left-continuous, which follows from the fact that A and B are both compound Poisson processes with finite jump rates. Hence X is also quasi-left-continuous.

e) *Regularity Conditions.* Conditions on the state space $(\mathbb{R}_+,\mathcal{R}_+)$ are met automatically, and our construction further implies that

$X_t \in F_t/R_+$

which in turn implies that X is progressively measurable with respect to $\{F_t\}$ by the right continuity. So if T is any $\{F_t\}$ stopping time,

$X_T \in F_T/R^*$.

Moreover it follows from our construction of F_t^o and completion of F_t that $\{F_t\}$ is right continuous.

f) *Strong Markov Property*. Since A and B are processes with stationary independent increments, $(\Omega, F, F_{t+}, A_t, \theta_t, \mathbb{P}_x)$ and $(\Omega, F, F_{t+}, B_t, \theta_t, \mathbb{P}_x)$ are strong Markov processes and for any $\{F_{t+}^o\}$ stopping time T

$$\mathbb{P}_x\{A_{T+t} - A_T \in D_1/F_{T+}^o\} = \mathbb{P}_x\{A_t \in D_1\} \tag{3.2}$$

and

$$\mathbb{P}_x\{B_{T+t} - B_T \in D_2/F_{T+}^o\} = \mathbb{P}_x\{B_t \in D_2\}$$

for all $t \geq 0$ and $D_1, D_2 \in R_+$ independent of x. Define

$$A_t^+ = A_{T+t} - A_T, \quad B_t^+ \doteq B_{T+t} - B_T, \quad X_t^+ = X_{T+t}, \quad t \geq 0$$

for any stopping time T in $\{F_{t+}^o\}$. Then it follows from (I.1) that

$$X_t^+ = X_o^+ + A_t^+ - B_t^+ + \int_o^t (p - r)(X_u^+)du \quad , \quad t \geq 0$$

where $X_o^+ \doteq A_t^+ - B_T^o + \int_o^T (p - r)(X_u)du$. Since $\sigma(A_t^+ : t \geq 0)$ and $\sigma(B_t^+ : t \geq 0)$ are both independent of F_T^o by (3.2),

$$\mathbb{P}_x\{X_t^+ \in D_3/F_{T+}^o\} = \mathbb{P}_y\{X_t \in D_3\} \quad \text{on} \quad \{X_o^+ = y\} . \tag{3.3}$$

That is,

$$\mathbf{P}_x\{X_{T+t} \in D_3/\mathcal{F}^o_{T+}\} = \mathbb{P}_{X_T}\{X_t \in D_3\}$$

for all $x \in \mathbb{R}_+$, $t \geq 0$ and $D_3 \in \mathcal{R}_+$. This implies that $(\Omega,\mathcal{F},\mathcal{F}^o_{t+},X_t,\theta_t, \mathbb{P}_x)$ is a strong Markov process. We know that if X is Markov relative to $\{\mathcal{F}^o_{t+}\}$, then

$$\mathcal{F}_t = \mathcal{F}_{t+} \qquad \text{for each } t \geq 0 \tag{3.4}$$

by Proposition (8.12) in BLUMENTAL and GETOOR [2]. Thus expression (3.3) reduces to

$$\mathbb{P}_x\{X_{T+t} \in D_3/\mathcal{F}_T\} = \mathbf{P}_{X_T}\{X_t \in D_3\}$$

for all $t \geq 0$, $x \in \mathbb{R}_+$, $D_3 \in \mathcal{R}_+$ by Theorem (7.3) in BLUMENTAL and GETOOR [2].

Note that if $\overline{\mathcal{F}}_t$ is the completion of $\mathcal{F}_t$ in $\mathcal{F}$ with respect to $\{P_\mu\}$, then obviously $\overline{\mathcal{F}}_t = \mathcal{F}_t$ and together with (3.4) we have

$$\mathcal{F}_t = \mathcal{F}_{t+} = \overline{\mathcal{F}}_t \quad . \square$$

Theorem (3.1) states that the storage process X is in fact a standard normal strong Markov process with the termination time being infinite almost surely. So X is a Markov decision process where (r,p) are the associated controls.

4.4 THE GENERATOR OF THE GENERALIZED STORAGE PROCESS

The generator possesses vital importance in the optimal control of strong Markov processes. In this section we obtain the expression for the generator and characterize its domain and range. Our definition of the generator is equivalent to the weak infinitesimal generator given in DYNKIN [1] and BREIMAN [41] while our procedure follows ÖZEKİCİ [35].

For every $f \in b(R_+)$ and $x \in \mathbb{R}_+$, let

$$D^+f(x) = \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}, \qquad D^-f(x) = \lim_{h \downarrow 0} \frac{f(x) - f(x-h)}{h}$$

and for fixed $(r,p) \in M$

$$D_{rp}f(x) = \begin{cases} D^+f(x) & \text{if} \quad r(x) < p(x) \\ D^-f(x) & \text{if} \quad r(x) > p(x) . \end{cases}$$

If $f(.)$ is absolutely continuous, then

$$f'(.) \equiv D^+f(.) \equiv D^-f(.) \qquad \text{almost everywhere.}$$

(4.1) <u>DEFINITION</u>. A sequence of functions $\{f_t\} \subset b(R_+)$ converges boundedly pointwise to a function $f \in b(R_+)$ as $t \downarrow 0$ if

i) $\lim_{t \downarrow 0} f_t(x) = f(x)$ for every $x \in \mathbb{R}_+$;

ii) there exists some constant $M < \infty$ such that

$$||f_t|| = \sup_{x \in R_+} |f_t(x)| \leq M$$

for all t sufficiently small. Δ

The generator G_{rp} of the process X, the range $R(G_{rp})$ of G_{rp}, and the domain $D(G_{rp})$ of G_{rp} for any $(r,p) \in M$ are defined in Definition (4.2).

(4.2) DEFINITION. For any $(r,p) \in M$,

i) The range $R(G_{rp})$ of the generator G_{rp} is the set of all $f \in b(R_+)$ such that

$$\mathbb{E}_x[f(X_t)] \to f(x) \quad \text{as} \quad t \downarrow 0 \quad \text{for all } x \in \mathbb{R}_+;$$

ii) The domain $D(G_{rp})$ of the generator G_{rp} is the set of all $f \in R(G_{rp})$ such that

$\mathbb{E}_x[f(X_t) - f(x)]/t$ converges boundedly pointwise on R_+ as $t \downarrow 0$ to a function in $R(G_{rp})$;

iii) for any $f \in D(G_{rp})$, $G_{rp}f$ is defined to be the limiting function in (ii). Δ

The stopping times T_1, τ_1 and S_1 of F as defined by (1.1), (1.2), and (1.3) are the first jump times of the input process A, the output process B and the storage process X, respectively. We know that for any $\omega \in \Omega$ $S_1(\omega) = \min(T_1(\omega), \tau_1(\omega))$, so it is clear hat

$$\begin{aligned} &\mathbb{P}_x\{S_1 > u\} = e^{-(\lambda_a+\lambda_b)u}, \\ &\mathbb{P}_x\{S_1 > u,\ S_1 = \tau_1\} = \frac{\lambda_b}{\lambda_a + \lambda_b} e^{-(\lambda_a+\lambda_b)u}, \\ &\mathbf{P}_x\{S_1 > u,\ S_1 = T_1\} = \frac{\lambda_a}{\lambda_a + \lambda_b} e^{-(\lambda_a+\lambda_b)u} \end{aligned} \tag{4.1}$$

for every $x \in \mathbb{R}_+$, $u \geq 0$. Finally let $(P_t)_{t \geq 0}$ be the contraction semigroup on $b(R_+)$ defined by

$$P_t f(x) = \mathbb{E}_x[f(X_t)], \qquad t \geq 0.$$

(4.1) PROPOSITION. For $(r,p) \in \mathcal{M}$, $\mathcal{R}(G_{rp})$ consists of all $f \in b(\mathcal{R}_+)$ such that

i) $f(.)$ is right continuous on $r(.) < p(.)$;

ii) $f(.)$ is left continuous on $r(.) > p(.)$.

Proof. For every $x \in \mathbb{R}_+$ and $t \geq 0$,

$$P_t f(x) = \mathbb{E}_x[f(X_t):S_1 > t] + \mathbb{E}_x[f(X_t):S_1 \leq t].$$

On $\{S_1 > t\}$, $f(X_t) = f(q(X_o,t))$; thus

$$\mathbb{E}_x[f(X_t):S_1 > t] = e^{-(\lambda_a+\lambda_b)t} \cdot f(q(x,t)). \tag{4.2}$$

Note that

$$\mathbb{E}_x[f(X_t):S_1 \leq t] = \mathbb{E}_x[f(X_t):S_1 \leq t,\ S_1 = T_1] + \mathbb{E}_x[f(X_t):S_1 \leq t,\ S_1 = \tau_1]$$

$$= k_1(x,t) + k_2(x,t) .$$

Using the strong Markov property of X at S_1, we obtain

$$k_1(x,t) = \mathbb{E}_x[I_{\{S_1 \leq t\}} \cdot I_{\{S_1=T_1\}} \cdot \mathbb{E}_x[f(X_{t-S_1}) \circ \theta_{S_1}/\mathcal{F}_{S_1}]]$$

$$= \mathbb{E}_x[I_{\{S_1 \leq t\}} \cdot I_{\{S_1=T_1\}} \cdot P_{t-S_1} f(X_{S_1})].$$

On $\{S_1 = T_1\}$ we have

$$X_{S_1} = q(X_o,S_1) + A_{S_1} - A_{S_1^-} .$$

Since $\mathbb{P}_x\{A_{S_1} - A_{S_1^-} \in D/S_1 = T_1\} = G_a(D)$ for $D \in \mathcal{R}_+$,

$$k_1(x,t) = \lambda_a \int_0^t e^{-(\lambda_a+\lambda_b)s}\, ds \int_0^\infty G_a(dy) P_{t-s} f(q(x,s) + y).$$

By a change of variables setting u = t - s, we get

$$k_1(x,t) = \lambda_a \int_0^t e^{-(\lambda_a+\lambda_b)(t-u)}du \int_0^\infty G_a(dy)P_u f(q(x,t-u) + y). \quad (4.3)$$

Similarly we have

$$k_2(x,t) = \mathbb{E}_x[I_{\{S_1\le t\}} \cdot I_{\{S_1=\tau_1\}} \cdot \mathbb{E}_x[f(X_{t-S_1}) \circ \theta_{S_1}/\mathcal{F}_{S_1}]]$$

$$= \mathbb{E}_x[I_{\{S_1\le t\}} \cdot I_{\{S_1=\tau_1\}} \cdot P_{t-S_1} f(X_{S_1})] .$$

On $\{S_1 = \tau_1\}$ we have

$$X_{S_1} = \{q(X_o,S_1) - (B_{S_1} - B_{S_1^-})\} \vee 0$$

by the definition of X as given in (3.1). Since we have

$$\mathbb{P}_x\{B_{S_1} - B_{S_1^-} \in D/S_1 = \tau_1\} = G_b(D) \quad \text{for} \quad D \in \mathcal{R}_+ ,$$

$$k_2(x,t) = \lambda_b \int_0^t e^{-(\lambda_a+\lambda_b)s} ds \int_0^{q(x,s)} G_b(dy)P_{t-s} f(q(x,s) - y)$$

$$+ \lambda_b \int_0^t e^{-(\lambda_a+\lambda_b)s} ds \int_{q(x,s)}^\infty G_b(dy)P_{t-s} f(0) .$$

By another change of variables setting u = t-s,

$$k_2(x,t) = \lambda_b \int_0^t e^{-(\lambda_a+\lambda_b)(t-u)}du \int_0^{q(x,t-u)} G_b(dy)P_u f(q(x,t-u) - y)$$

$$+ \lambda_b \int_0^t e^{-(\lambda_a+\lambda_b)(t-u)}du \int_{q(x,t-u)}^\infty G_b(dy)P_u f(0) . \quad (4.4)$$

From (4.3) and (4.4) it is obvious that for all $x \in R_+$

$$\lim_{t \downarrow 0} k_1(x,t) = 0 , \qquad \lim_{t \downarrow 0} k_2(x,t) = 0.$$

So by (4.2) $f \in b(R_+)$ is in $R(G_{rp})$ if and only if for all $x \in R_+$

$$\lim_{t \downarrow 0}[e^{-(\lambda_a+\lambda_b)t} f(q(x,t))] = f(x)$$

or if and only if

$$\lim_{t \downarrow 0} f(q(x,t)) = f(x) .$$

Now the desired result follows immediately by recalling from Remark (2.3) that

a) $q(x,t) \downarrow x$ as $t \downarrow 0$ on $\{r(.) < p(.)\}$

b) $q(x,t) = x$ for $t \geq 0$ on $\{r(.) = p(.)\}$

c) $q(x,t) \uparrow x$ as $t \downarrow 0$ on $\{r(.) > p(.)\}$. Δ

Proposition (4.1) characterizes the range of the generator by specifying the conditions to be satisfied by any function in the range. Now there remains to find an expression for the generator itself and to characterize its domain.

For $0 \leq x_1 \leq x_2$, we define

$$t^+(x_1,x_2) = \inf\{t \geq 0 : q(x_1,t) = x_2\} \tag{4.5}$$
$$t^-(x_1,x_2) = \inf\{t \geq 0 : q(x_2,t) = x_1\}.$$

So $t^+(x_1,x_2)$ is the total amount of time it takes for the process to increase from x_1 to x_2, and $t^-(x_1,x_2)$ is the total amount of time it takes for the process to decrease from x_2 to x_1 when there are no random input and output jumps. At least one of these two quantities is obviously infinite.

If $t^-(x_1,x_2) < \infty$, then for $0 \leq x_1 \leq u \leq v \leq x_2$, $r(u) > p(u)$, $r(v) > p(v)$, $t^-(x_1,u) \leq t^-(x_1,v) < \infty$ and

$$t^-(u,v) = \int_u^v \frac{ds}{r(s) - p(s)} \quad . \tag{4.6}$$

Similarly if $t^+(x_1,x_2) < \infty$, then for $0 \leq x_1 \leq u \leq v \leq x_2$, $p(u) > r(u)$, $p(v) > r(v)$, $t^+(v,x_2) \leq t^+(u,x_2) < \infty$ and

$$t^+(u,v) = \int_u^v \frac{ds}{p(s) - r(s)} \quad . \tag{4.7}$$

Keeping these definitions in mind, we now state the conditions that should be met by $\mathcal{D}(G_{rp})$ and G_{rp} in the following theorem.

<u>THEOREM (4.1)</u>. For $(r,p) \in \mathcal{M}$, $\mathcal{D}(G_{rp})$ consists of all $f \in \mathcal{R}(G_{rp})$ such that

i) $D_{rp}f(x)$ exists for all $x \in \mathbb{R}_+$;

ii) The function $f(.)$ is absolutely continuous on every interval $I \subset \{x \in \mathbb{R}_+ : r(x) \neq p(x)\}$ with $\min_{x\in I} |r(x) - p(x)| > 0$;

iii) If for some $\bar{x} < x$. $t^-(\bar{x},x) < \infty$, then $f(.)$ is right continuous at $\bar{x}$. Similarly if for some $\bar{x} > x$ $t^+(x,\bar{x}) < \infty$, then $f(.)$ is left-continuous at $\bar{x}$;

iv) The function

$$D_{rp}f(x)[p(x) - r(x)] - (\lambda_a + \lambda_b)f(x) + \lambda_a \int_0^\infty G_a(dy)f(x + y)$$

$$+ \lambda_b \int_0^x G_b(dy)f(x - y) + \lambda_b f(0)(1 - G_b(x)), \quad x \in \mathbb{R}_+$$

is in $R(G_{rp})$.

Furthermore $G_{rp}f(x)$ is equal to the function given in (iv) for any $f \varepsilon R(G_{rp})$.

Proof. From the definition given by (4.1)

$$G_{rp}f(x) = \lim_{t \downarrow 0} \frac{1}{t} [P_t f(x) - f(x)] , \quad f \varepsilon b(R_+) \tag{4.8}$$

where the domain $D(G_{rp})$ is the set of all $f \varepsilon b(R_+)$ for which this limit exists boundedly pointwise and belongs to $R(G_{rp})$.

Combining (4.2)-(4.4) together, we obtain

$$\frac{1}{t} [P_t f(x) - f(x)] = \frac{1}{t} k_1(x,t) + \frac{1}{t} k_2(x,t) + \frac{1}{3} k_3(x,t) \tag{4.9}$$

where $k_3(x,t) = e^{-(\lambda_a+\lambda_b)t} f(q(x,t)) - f(x)$. To find the generator, it suffices to find the limit of each of the terms involved in expression (4.9). It is obvious that for $f \varepsilon R(G_{rp})$

$$\lim_{t \downarrow 0} \frac{1}{t} k_1(x,t) = \lambda_a \int_0^\infty G_a(dy)f(x + y) \tag{4.10}$$

and

$$\lim_{t \downarrow 0} \frac{1}{t} k_2(x,t) = \lambda_b \int_0^x G_b(dy)f(x - y) + \lambda_b f(0)(1 - G_b(x)). \tag{4.11}$$

It follows from (4.3) and (4.4) that for all $f \varepsilon R(G_{rp})$ and $f \varepsilon b(R_+)$

$$\frac{1}{t}|k_1(x,t)| \leq \lambda_a ||f|| \frac{1}{t} \int_0^t e^{-(\lambda_a+\lambda_b)s} ds \leq \lambda_a ||f||, \tag{4.12}$$

$$\frac{1}{t}|k_2(x,t)| \leq \lambda_b ||f|| \frac{1}{t} \int_0^t e^{-(\lambda_a+\lambda_b)s} ds \leq \lambda_b ||f||$$

which in turn implies that limits (4.10) and (4.11) exist boundedly pointwise. Now there remains to show that $(1/t)k_3(x,t)$ converges boundedly pointwise. In evaluating $\lim_{t \downarrow 0} (1/t)k_3(x,t)$, note that Taylor's series expansion yields

$$e^{-(\lambda_a+\lambda_b)t} = 1 - (\lambda_a + \lambda_b)t + O(t)$$

where $o(t)/t \to 0$ as $t \downarrow 0$. Consequently,

$$\frac{1}{t} k_3(x,t) = \frac{1}{t}[f(q(x,t) - f(x)] + [-(\lambda_a + \lambda_b) + O(t)/t]f(q(x,t)). \tag{4.13}$$

Furthermore,

$$\lim_{t \downarrow 0}[-(\lambda_a + \lambda_b) + O(t)/t]f(q(x,t)) = -(\lambda_a + \lambda_b)f(x) \tag{4.14}$$

boundedly pointwise for all $f \in R(G_{rp})$ by the boundedness of $O(t)/t$. To complete the proof, there remains to show that

$$\frac{1}{t}[f(q(x,t)) - f(x)] \tag{4.15}$$

converges boundedly pointwise as $t \downarrow 0$ and $G_{rp}f \in R(G_{rp})$ if and only if f satisfies conditions (i)-(iv) of the Theorem.

We will first show necessity by assuming that the function given by (4.15) converges boundedly pointwise as $t \downarrow 0$ for $f \in R(G_{rp})$ and $G_{rp}f \in R(G_{rp})$. Then it is necessary to prove that f satisfies the condition of the Theorem.

i) Note that for $x \in \{r(.) \neq p(.)\}$

$$\frac{1}{t}[f(q(x,t)) - f(x)] = \frac{f(q(x,t)) - f(x)}{q(x,t)-x} \cdot \frac{q(x,t) - x}{t} \tag{4.16}$$

Now we know that q satisfies

$$q(x,t) = x + \int_0^t (p - r)q(x,s)ds, \quad t \geq 0 .$$

So,

$$\lim_{t \downarrow 0} \frac{q(x,t) - x}{t} = \lim_{t \downarrow 0} \frac{\int_0^t (p - r)q(x,s)ds}{t} \tag{4.17}$$

$$= \lim_{t \downarrow 0}[p(q(x,t)) - r(q(x,t))]$$

$$= p(x) - r(x) \neq 0$$

boundedly pointwise since p and r are both bounded and they satisfy Admissibility Condition 3. This in turn implies that the function

$$\frac{f(q(x,t)) - f(x)}{q(x,t) - x} \tag{4.18}$$

converges pointwise as $t \downarrow 0$ for every $x \in \{r(.) \neq p(.)\}$. If $r(x) = p(x)$, then $q(x,t) = x$ for all $t \geq 0$ and the left hand side of (4.16) is trivially zero and the derivative $D_{rp}f(x)$ is not defined. Then the first term of the expression in (iv) is set to be identically equal to zero. This together with the definition of $D_{rp}f$ amounts to saying that $D_{rp}f(x)$ exists for all $x \in R_+$;

ii) Let I be an interval of $\{r(.) \neq p(.)\}$ and assume that $r(.) > p(.)$ on I such that $c = \min_{x \varepsilon I}\{r(x) - p(x)\} > 0$.

If $(1/t)[f(q(x,t)) - f(x)]$ converges boundedly pointwise, by Definition (4.1) there exists $M < \infty$ and $t_M > 0$ such that for every $x \varepsilon \mathbb{R}_+$ and $t \leq t_M$

$$|f(q(x,t)) - f(x)| \leq M.t \ . \tag{4.19}$$

For arbitrary $\varepsilon > 0$, let $\delta = \min\{\bar{r}\ t_M, (\bar{r}/M)\varepsilon\}$. For any finite collection $\{(x_i,x_i')\}_{i\leq n}$ of nonoverlapping intervals of I with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$, we have

$$t^-(x_i,x_i') = \int_{x_i}^{x_i'} \frac{du}{r(u) - p(u)} \leq \frac{x_i' - x_i}{\bar{r}} \leq t_M \ . \tag{4.20}$$

Note that for all i = 1,...n

$$x_i = q(x_i', t^-(x_i,x_i')).$$

So,

$$\sum_{i=1}^{n} |f(x_i) - f(x_i')| = \sum_{i=1}^{n} f(q(x_i',t^-(x_i,x_i'))) - f(x_i')| \ .$$

By (4.19),

$$\sum_{i=1}^{n} |f(x_i) - f(x_i')| \leq M \sum_{i=1}^{n} t^-(x_i,x_i') \ .$$

Furthermore by (4.20),

$$\sum_{i=1}^{n} |f(x_i) - f(x_i')| \leq \frac{M}{\overline{r}} \sum_{i=1}^{n} |x_i' - x_i| \leq \frac{M}{\overline{r}} \delta \leq \varepsilon .$$

This implies that f(.) is absolutely continuous when r(.) > p(.) on I. Note that the same argument can be repeated when r(.) < p(.) using $t^+(.,.)$.

iii) We will show the desired result only for the case where $t^-(\overline{x},x_1) < \infty$ for some $\overline{x} < x_1$, leaving the other to the reader. Now assume that f(.) is not right continuous at $\overline{x}$. For every $x \in [\overline{x},x_1]$ we have

$$\frac{|f(q(x,t^-(\overline{x},x))) - f(x)|}{t^-(\overline{x},x)} = \frac{|f(\overline{x}) - f(x)|}{t^-(\overline{x},x)} .$$

Since f(·) is assumed not to be right continuous at $\overline{x}$, $\lim_{x \downarrow \overline{x}} f(x) \neq f(\overline{x})$. This together with the fact that $\lim_{x \downarrow \overline{x}} t^-(\overline{x},x) = 0$ implies that

$$\sup_{x \in \mathbb{R}_+} \frac{|f(\overline{x}) - f(x)|}{t^-(\overline{x},x)} = \infty$$

which on the other hand contradicts the boundedly pointwise convergence of $(1/t)[f(q(x,t)) - f(x)]$ by (4.21); hence, f(.) should be right continuous at $\overline{x}$.

Putting statements (4.9)-(4.18) together, the expression for the generator $G_{rp}f$ is explicitly obtained, which is in fact given by Condition (iv) of the Theorem. The proof of necessity condition is completed by noting that $f \in \mathcal{D}(G_{rp})$ satisfies Condition (i)-(iii) of the Theorem if expression (4.15) converges boundedly pointwise.

To show sufficiency assume that $f \in R(G_{rp})$ satisfies conditions (i)-(iv) of the Theorem. Then it is necessary to show that the function given by (4.15) converges boundedly pointwise.

Since $D_{rp}f$ exists for all $x \in \mathbb{R}_+$ by Condition (i) it follows from (4.16) that the pointwise limit of (4.15) exists for all $x \in \mathbb{R}_+$. By (iv), $G_{rp}f \in R(G_{rp})$ and all we need to show is that this convergence is bounded. Note that if $r(x) = p(x)$ for some $x \in \mathbb{R}_+$, $(q(x,t) - x)/t$ becomes zero and thus (4.15) is trivially zero for all $t \geq 0$.

We will first prove that the convergence of (4.15) is bounded on the set $J = \{x \in \mathbb{R}_+ : r(x) > p(x)\}$. To do this it suffices to show that for all $t \geq 0$ and $x \in J$

$$[f(q(x,t)) - f(x)] = \int_x^{q(x,t)} D_{rp}f(u)du \tag{4.21}$$

since a change of variables $s = t^-(u,x)$ and the fact that

$$u = q(x,s), \qquad t^-(u,x) = \int_u^x \frac{dv}{r(v) - p(v)}$$

give

$$[f(q(x,t)) - f(x)] = \int_0^t D_{rp}f(q(x,s))[p(q(x,s)) - r(q(x,s))]ds. \tag{4.22}$$

The fact that $f \in R(G_{rp})$ and $G_{rp}f \in R(G_{rp})$ are bounded by Proposition (4.1) implies that the right hand side of (4.22) is bounded by Condition (iv), which in turn implies the bounded pointwise convergence of (4.15).

To show (4.21) note that $q(x,t) \leq x$ and $D_{rp}f(.) = D^-f(.)$ on $[q(x,t),x]$ for all $t \geq 0$ since $r(x) > p(x)$ on J; thus the right-hand side of (4.21) is well-defined. Define $y(t) = q(x,t)$, so $y(0) = x$ and $y(t)$ is decreasing with respect to t. Let $\bar{y} = \lim_{t\to\infty} y(t)$. Since $t^-(y,x) < \infty$ for all $y \in (\bar{y},x)$, f(.) is right-continuous on $(\bar{y},x)$ by Condition (iii). So f(.) is continuous and left-differentiable on $(\bar{y},x]$ by Proposition (4.1) since $f \in R(G_{rp})$ and $r(y) > p(y)$ for all $y \in (\bar{y},x]$. This and (ii) imply that (4.21) is true for all $t < t^-(\bar{y},x)$. If $t^-(\bar{y},x) = \infty$, then we are done. If $t^-(\bar{y},x) < \infty$, then obviously $y(t) = \bar{y}$ for all $t \geq t^-(y,x)$ and f(.) is right continuous at $\bar{y}$ by Condition (iii). So f(.) is continuous and left-differentiable at $\bar{y}$ by Proposition (4.1) since $f \in R(G_{rp})$ and $r(\bar{y}) > p(\bar{y})$. So (4.21) is still true for all $t \geq t^-(\bar{y},x)$. Now it is proven that (4.21) holds true for all $t \geq 0$ and $x \in J$. The same argument can be repeated here to prove the bounded convergence of (4.15) on $\{r(.) < p(.)\}$ by using $t^+(.,.)$.

Although it is quite difficult to check whether a given function satisfies the conditions imposed by Theorem (4.1), fortunately we are able to find a set of functions which readily meet those requirements.

<u>COROLLARY (4.1)</u>. Let f be a bounded function on R_+ so that f(.) is absolutely continuous, $D^+f(.)$ and $D^-f(.)$ exist and are bounded on R_+. Then $f \in D(G_{rp})$ for any $(r,p) \in M$ and

$$G_{rp}f(x) = f'(x)[p(x) - r(x)] - (\lambda_a + \lambda_b)f(x) + \lambda_a \int_0^\infty G_a(dy)f(x + y)$$

$$+ \lambda_b \int_0^x G_b(dy)f(x - y) + \lambda_b f(0)(1 - G_b(x)).$$

<u>Proof</u>. This follows directly from Theorem (4.1). Note that the absolute continuity of f(.) implies that f'(.) exists and is equal to $D_{rp}f(.)$ almost everywhere independent of (r,p). $\square$

REMARK (4.1). It is clear from Corollary (4.1) that every bounded function f on $\mathcal{R}_+$ with a bounded continuous derivative f'(.) on $\mathcal{R}_+$ is in $\mathcal{D}(G_{rp})$ for every (r,p) ε $\mathcal{M}$. Furthermore $D_{rp}f(.) = f'(.)$ independent of (r,p). Δ

An important point to be emphasized is that the dependence of the derivative on (r,p) is highly undesirable in the optimal control problem; however the set of functions introduced by Remark (4.1) overcomes this difficulty and will be employed throughout this paper.

V. A GENERAL FRAMEWORK FOR OPTIMAL CONTROL OF THE GENERALIZED STORAGE MODEL

In this chapter the problem of optimally controlling the content level of the generalized storage model is analyzed in detail. In Section 1 the control problem is formulated in the formal procedure of Markov decision theory, and the main assumptions on the reward and cost structure of the model are stated. In Section 2 sufficient optimality conditions are derived in terms of functional differential equations for both local and global purposes.

5.1 THE OPTIMAL CONTROL PROBLEM

The main purpose imbedded in the optimal control problem is to control the storage process X of the generalized storage model given by (I.1). The controller will observe the content level and accordingly decide upon appropriate input and output rates continuously in time. If at time t the content level is observed to be x, he is to choose an input rate p(x) and an output rate r(x) within the admissible class. The set of conditions that should be satisfied by his controls (r,p) constitutes his set of admissible controls; in fact, the set of all pairwise functions (r,p) both defined on $\mathbb{R}_+$ satisfying Admissibility Conditions 1-4 is defined to be the set of admissible controls $\mathcal{M}$, and is studied extensively in Section IV.2. We have furthermore characterized explicitly some subsets of $\mathcal{M}$ to be used frequently in this paper. In particular, recall that $\mathcal{M}_i$ is the set of all pairs $(r,p) \in \mathcal{M}$ such that r(.)-p(.) is increasing, and $\mathcal{M}_\ell$ is the set of $(r,p) \in \mathcal{M}$ such that both are piecewise Lipschitz.

In selecting the input and output controls based upon his observation of the content level, the controller should optimize a return function specified by

$$L(X_t,p(X_t),r(X_t)) = L_1(X_t) + L_2(p(X_t)) + L_3(r(X_t)) \tag{1.1}$$

where at any time t L(x,p(x), r(x)) is the rate of earnings given that the content level is x, the input rate chosen is p(x) and the output rate chosen is r(x). Although from a theoretical point of view no sign restriction is required on L_1, L_2 and L_3, we will assume that L_1 and L_2 have negative contributions while L_3 has positive contribution. So $L_1(x)$ can be interpreted as the holding cost rate when the content level of the store is x. Moreover $L_2(p(x))$ can be interpreted as the rate of expense incurred by procurement when there is a controlled input to the store at a rate p(x) while $L_3(r(x))$ can be interpreted as the rate of earnings obtained from sales when there is a controlled output from the store at a rate r(x).

We now define the return function v_{rp} as the expected infinite time horizon discounted earnings given by

$$v_{rp}(x) = E_x[\int_0^\infty e^{-\alpha t} L(X_t, p(X_t), r(X_t))dt] \quad , \quad x \in \mathbb{R}_+ \tag{1.2}$$

for any $(r,p) \in \mathcal{M}$ and $\alpha > 0$. The control problem aims at choosing controls (r,p) in the admissible class $\mathcal{M}$ so as to maximize the expected infinite time horizon discounted earnings.

Throughout this paper, $(r^*,p^*) \in \mathcal{M}$ are said to be optimal controls, and v^* is said to be an optimal return function if

$$v_{r^*p^*}(x) = v^*(x) \geq v_{rp}(x)$$

for all $(r,p) \in \mathcal{M}$ and $x \in \mathbb{R}_+$. Similarly for an arbitrary subset $\hat{\mathcal{M}} \subset \mathcal{M}$, $(\hat{r},\hat{p}) \in \hat{\mathcal{M}}$ is optimal in $\hat{\mathcal{M}}$ and $\hat{v} = v_{\hat{r}\hat{p}}$ is the optimal return function in $\hat{\mathcal{M}}$ if

$$v_{\hat{r}\hat{p}}(x) = \hat{v}(x) \geq v_{rp}(x)$$

for all $(r,p) \in \hat{\mathcal{M}}$ and $x \in \mathbb{R}_+$.

We now state our basic assumptions imposed upon the reward and cost structure.

ASSUMPTION (1.1).

i) L_1 is a bounded Lipschitz continuous function on R_+ and $L_1(\infty) \equiv \lim_{x\to\infty} L_1(x)$ exists;

ii) $L_2 \in C^2([0,\overline{p}])$ is concave decreasing and $|L_2''| > \varepsilon$ for some $\varepsilon > 0$;

iii) $L_3 \in C^2([0,\overline{r}])$ is concave increasing and $|L_3''| > \varepsilon$ for some $\varepsilon > 0$;

iv) $\overline{r} > \overline{p}$. $\Box$

Although the assumptions imposed on L_2 and L_3 seem quite restrictive, resulting in the elimination of some interesting cases, it should be pointed out that they are made for the sake of simplicity. In fact, similar results will be obtained by dropping these assumptions and studying the problem with less restrictive conditions in Chapter VIII. Furthermore, we will show that nicer results will be obtained if L_1 satisfies some monotonicity properties. The assumption $\overline{r} > \overline{p}$ is crucial, but it is still a prominent assumption which implies that it is always possible to decrease the content level whatever the input rate is.

It follows from Admissibility Condition 3 that $L_2(p(.))$ and $L_3(r(.))$ are left continuous on $\{r(.) \geq p(.)\}$, and $L_2(p(.))$ and $L_3(r(.))$ are right continuous on $\{r(.) \leq p(.)\}$. So by Proposition (IV.4.1), $L_2(p(.)) \in \mathcal{R}(G_{rp})$ and $L_3(r(.)) \in \mathcal{R}(G_{rp})$. Furthermore $L_1 \in \mathcal{R}(G_{rp})$ which follows from Assumption (1.1) and Proposition (IV.4.1). So $L \in \mathcal{R}(G_{rp})$ for all $(r,p) \in \mathcal{M}$.

Now we are in a position to provide a characterization of the expected infinite time horizon discounted earnings v. To do this we refer to a well-known result due to DYNKIN [2] and BREIMAN [41], which states that v_{rp} is the unique solution in $\mathcal{D}(G_{rp})$ of a functional differential equation.

<u>THEOREM (1.1)</u>. For $\alpha > 0$ and $(r,p) \in \mathcal{M}$, the expected infinite time horizon discounted earnings v_{rp} given by (1.2) is the unique solution in $\mathcal{D}(G_{rp})$ of the equation

$$(\alpha I - G_{rp})v_{rp} = \mathbb{L} .$$

That is,

$$\alpha v_{rp} = L + G_{rp} v_{rp} \ . \ \Delta$$

Theorem (1.1) relates the expected infinite time horizon discounted earnings to the storage process X through its generator. It bears great importance in the optimal control problem since this result is used to derive the sufficient condition of optimality.

We conclude this section by providing the full expression for the functional differential equation given in Theorem (1.1). It follows from Corollary (IV.4.1) that v_{rp} satisfies for $(r,p) \varepsilon \mathcal{M}$ and $x \varepsilon \mathbb{R}_+$.

$$\begin{aligned}\alpha v_{rp}(x) &= L_1(x) + L_2(p(x)) + L_3(r(x)) + v'_{rp}(x)[p(x) - r(x)] \\ &\quad - (\lambda_a + \lambda_b) v_{rp}(x) + \lambda_a \int_0^\infty v_{rp}(x + y) G_a(dy) \\ &\quad + \lambda_b \int_0^x v_{rp}(x - y) G_b(dy) + \lambda_b v(0)[1 - G_b(x)] \qquad (1.3)\end{aligned}$$

where $v'_{rp}(.) \equiv D_{rp} v_{rp}(.)$ is well-defined on $\{r(.) \neq p(.)\}$ since $v_{rp} \varepsilon \mathcal{D}(G_{rp})$. Note that in case $r(x) = p(x)$ for some $x \varepsilon \mathbb{R}_+$ the fourth term on the right hand side of equation (1.3) is trivially zero.

5.2 A SUFFICIENT CONDITION OF LOCAL AND GLOBAL OPTIMALITY

In this section we establish a sufficient optimality condition for v_{r*p*} to be optimal in $\mathcal{D}(G_{rp})$. Our most important tool in this respect is the generator of the storage process X with its domain and range as specified in Chapter IV. So the results of Proposition (IV.4.1), Theorem (IV.4.1) and Theorem (1.1) are fully utilized to accomplish our aim.

Continuous time control of Markov decision processes have been studied by many researchers who stated the sufficient conditions of optimality in various forms. We refer the reader to DOSHI [24] and VERMES [23] for detailed treatment of these processes. We follow the approach of ÖZEKİCİ [35] to obtain the existence and uniqueness results on optimal controls as well as the sufficient optimality conditions.

For simplicity of notation for every $x \in \mathbb{R}_+$, let

$$\hat{\alpha} = \alpha + \lambda_a + \lambda_b \tag{2.1}$$

and

$$Kv(x) = \lambda_a \int_0^\infty v(x+y)G_a(dy) + \lambda_b \int_0^x v(x-y)G_b(dy) + \lambda_b v(0)[1 - G_b(x)] \; . \tag{2.2}$$

Then it follows from (1.3) that for $x \in \mathbb{R}_+$ $\dot{v}_{rp}$ satisfies

$$\hat{\alpha} v_{rp}(x) = L_1(x) + L_2(p(x)) + L_3(r(x)) + v'_{rp}(x)[p(x) - r(x)] + Kv_{rp}(x) \; . \tag{2.3}$$

Since $Kv(x)$ will be encountered frequently in our analysis of the optimal control problem, it is necessary to dwell upon some of the properties it possesses.

<u>LEMMA (2.1)</u> For any $f \in b(R_+)$,

i) $Kf(.)$ is continuous if $f(.)$ is continuous;

ii) $Kf(.)$ is Lipschitz continuous if $f(.)$ is Lipschitz continuous;

iii) $Kf(.)$ is decreasing if $f(.)$ is decreasing on $[0,\infty)$.

Proof. Note that for $x \in \mathbb{R}_+$

$$Kf(x) = \lambda_a c_1(x) + \lambda_b c_2(x)$$

where

$$c_1(x) = \int_0^\infty f(x + y)G_a(dy)$$

$$c_2(x) = \int_0^x f(x - y)G_b(dy) + f(0)[1 - G_b(x)].$$

It is convenient to introduce for $x \in \mathbb{R}_+$

$$g(x) = \begin{cases} f(x) & , \quad x \geq 0 \\ f(0) & , \quad x < 0 \end{cases}$$

and to express c_1 and c_2 in terms of expectations by

$$c_1(x) = \mathbb{E}[f(x + Y)]$$

and

$$c_2(x) = \mathbb{E}[g(x - Z)]$$

where Y and Z have the probability distributions $G_a(.)$ and $G_b(.)$, respectively.

i) The continuity of f implies that $f(x_n) \to f(x)$ as $x_n \to x$. So $x_n + Y \to x + Y$ and $f(x_n + Y) \to f(x + Y)$. Then by the bounded convergence theorem $\mathbb{E}[f(x_n + Y)] \to \mathbb{E}[f(x + Y)]$, and consequently c_1 is continuous. The continuity of f further implies the continuity of g by the way g is defined. Similarly then $g(x_n - Z) \to g(x - Z)$, and $\mathbb{E}[g(x_n - Z)] \to \mathbb{E}[g(x - Z)]$. Thus c_2 is also continuous;

ii) If f is Lipschitz continuous, then

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$$

which implies

$$|f(x_1 + Y) - f(x_2 + Y)| \leq M|x_1 - x_2|.$$

Then note that

$$|\mathbb{E}[f(x_1 + Y)] - \mathbb{E}[f(x_2 + Y)]| = |\mathbb{E}[f(x_1 + Y) - f(x_2 + Y)]|$$

$$\leq |\mathbb{E}[M\ x_1 - x_2\ ]| = M|x_1 - x_2|.$$

So c_1 is continuous. The Lipschitz property of c_2 can be shown in a similar manner by using the same argument.

iii) If f is decreasing on $[0,\infty)$, i.e. for $x_1 \leq x_2$ $f(x_1) \geq f(x_2)$, then $x_1 + Y \leq x_2 + Y$ and $f(x_1 + Y) \geq f(x_2 + Y)$ which implies that c_1 is decreasing since $\mathbb{E}[f(x_1 + Y)] \geq \mathbb{E}[f(x_2 + Y)]$. Likewise the fact that f is decreasing on $[0,\infty)$ implies that g is decreasing on $(-\infty,\infty)$, so $x_1 - Z \leq x_2 - Z$ and $g(x_1 - Z) \geq g(x_2 - Z)$ for $x_1 \leq x_2$. Thus $\mathbb{E}[g(x_1 - Z)] \geq \mathbb{E}[g(x_2 - Z)]$, and accordingly c_2 is also decreasing. Δ

The fact that the properties of f are inherent in Kf as stated by Lemma (2.1) will be useful in the proof of the following theorem.

THEOREM (2.1) If there is a bounded function v on R_+ which satisfies:

i) $v(.)$ is absolutely continuous, $D^+v(.)$ (resp, $D^-v(.)$) exists and it is bounded, right-continuous (resp, left-continuous) on R_+;

ii)
$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)D_{rp}v(x)\} + L_1(x) + Kv(x) - \hat{\alpha}v(x) = 0 , \quad x > 0$$

$$\sup_{\substack{P\ \varepsilon\ [0,\overline{p}] \\ R\ \varepsilon\ [0,P]}}\{L_2(P) + L_3(R) + (P - R)D_{rp}v(0)\} + L_1(0) + Kv(0) - \hat{\alpha}v(0) = 0.$$

Then $v(x) \geq v_{rp}(x)$ for every $x \in \mathbb{R}_+$ and $(r,p) \in M$.

Proof. Let $v_{rp}(x)$ be the return function in $D(G_{rp})$ satisfying (2.3) for every $x \in \mathbb{R}_+$ and assume that v satisfies Condition (i)-(ii) of the Theorem. Then by Corollary (IV.4.1) $v \in D(G_{rp})$. It is obvious that for every $x \in \mathbb{R}_0$

$$\hat{\alpha}v_{rp}(x) - \hat{\alpha}v(x) = L_2(p(x)) + L_3(r(x)) + [p(x) - r(x)]D_{rp}v_{rp}(x)$$

$$-\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)D_{rp}v(x)\} + Kv_{rp}(x) - Kv(x). \quad (2.4)$$

Adding and subtracting $[p(x) - r(x)]D_{rp}v(x)$ on the right hand side of (2.4), we obtain

$$\hat{\alpha}[v_{rp}(x) - v(x)] = L_2(p(x)) + L_3(r(x)) + [p(x) - r(x)]D_{rp}(v_{rp} - v)(x)$$

$$-\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)D_{rp}v(x)\} + K(v_{rp} - v)(x) + [p(x) - r(x)]D_{rp}v(x) \quad (2.5)$$

which reduces to

$$\alpha u(x) = L_2(p(x)) + L_3(r(x)) + [p(x) - r(x)]D_{rp}v(x) \quad (2.6)$$

$$-\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)D_{rp}v(x)\} + [p(x) - r(x)]D_{rp}u(x) + Ku(x) - (\lambda_a + \lambda_b)u(x)$$

by letting $u = v_{rp} - v$. A similar argument also yields

$$\alpha u(0) = L_2(p(0)) + L_3(r(0)) + [p(0) - r(0)]D_{rp}v(0)$$

$$-\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,P]}}\{L_2(P) + L_3(R) + (P - R)D_{rp}v(0)\} + [p(0) - r(0)]D_{rp}u(0) + Ku(0) - (\lambda_a + \lambda_b)u(0) .$$

So define for every $x \in \mathbb{R}_0$

$$g(x) = L_2(p(x)) + L_3(r(x)) + [p(x) + r(x)]D_{rp}v(x)$$
$$-\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)D_{rp}v(x)\}. \quad (2.7)$$

and

$$g(0) = L_2(p(0)) + L_3(r(0)) + [p(0) - r(0)]D_{rp}v(0) - \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[0,P]}}\{L_2(P) + L_3(R) + (P - R)D_{rp}v(0)\}.$$

Note by Theorem (IV.4.1) that for every $x \varepsilon \mathbb{R}_+$

$$G_{rp}u(x) = [p(x) - r(x)]D_{rp}u(x) - (\lambda_a + \lambda_b)u(x) + Ku(x). \qquad (2.8)$$

Thus rewriting (2.6) in terms of (2.7) and (2.8), we obtain

$$\alpha u(x) = g(x) + G_{rp}u(x) \qquad (2.9)$$

for every $x \varepsilon \mathbb{R}_+$. By the fact that $(r,p) \varepsilon M$, the first three terms on the right hand side of (2.7) are in $R(G_{rp})$. The continuity and boundedness of the fourth term on the right hand side of (2.7) follows from the boundedness and continuity of $Kv(.)$ on R_+ by Lemma (2.1) and Condition (ii) of the Theorem. So it is also in $R(G_{rp})$, which implies that $g \varepsilon R(G_{rp})$.

Since $g \varepsilon R(G_{rp})$, $u \varepsilon D(G_{rp})$, it follows from Theorem (1.1) that

$$u(x) = \mathbb{E}_x[\int_0^\infty e^{-\alpha t} g(X_t)dt]$$

for $x \varepsilon \mathbb{R}_+$ and $(r,p) \varepsilon M$. The way $g(.)$ is defined by (2.7) implies that $g(x) \leq 0$ for all $x \varepsilon \mathbb{R}_+$ and $(r,p) \varepsilon M$. So

$$u(x) \leq 0 \quad , \quad x \varepsilon \mathbb{R}_+$$

which in turn implies that for every $x \varepsilon \mathbb{R}_+$

$$v_{rp}(x) \leq v(x). \quad \Delta$$

Condition (ii) of Theorem (2.1) is hard to verify because of the dependence of $D_{rp}v$ upon (r,p), so a more practical sufficient optimality condition is provided by the following corollary.

COROLLARY (2.1). Assume there is a bounded function v* on R_+ which satisfies

i) v*(.) is differentiable with a bounded continuous derivative v*'(.) on R_+;

ii) $$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)v^{*\prime}(x)\} + L_1(x) + Kv^*(x) - \hat{\alpha}v^*(x) = 0, \quad x > 0$$

$$\sup_{\substack{P\,\varepsilon\,[0,\overline{p}] \\ R\,\varepsilon\,[0,P]}}\{L_2(P) + L_3(R) + (P - R)v^{*\prime}(0)\} + L_1(0) + Kv^*(0) - \hat{\alpha}v^*(0) = 0.$$

Then,

a) $v^*(x) \geq v_{rp}(x)$ for all $(r,p) \in M$, $x \in \mathbb{R}_+$;

b) if there are controls $(r^*,p^*) \in M$ such that $v^* = v_{r^*p^*}$ then (r^*,p^*) are the unique optimal controls.

Proof. Note that (a) follows directly from Theorem (2.1) since $v^{*\prime} = D_{rp}v^*$ for every $(r,p) \in M$. To show (b) assume that $(\hat{r},\hat{p})$ is another optimal control pair, so it satisfies $v_{\hat{r}\hat{p}} = v_{r^*p^*} = v^*$. Then (2.3) implies that

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)v^*(x)\} = L_2(p^*(x)) + L_3(r^*(x)) + [p^*(x) - r^*(x)]v^{*\prime}(x)$$

and

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)v^{*\prime}(x)\} = L_2(\hat{p}(x)) + L_3(\hat{r}(x)) + [\hat{p}(x) - \hat{r}(x)]v^{*\prime}(x)$$

which contradicts the strict concavity of L_2 on $[0,\overline{p}]$ and L_3 on $[0,\overline{r}]$, so $(r^*,p^*) = (\hat{r},\hat{p})$. Δ

In the following chapters we will illustrate the proof for the existence of a function v^* and controls $(r^*,p^*) \varepsilon M$ satisfying Conditions (i)-(ii) of Corollary (2.1) such that $v^* = v_{r^*p^*}$. This will be achieved by constructing v^* in a step by step procedure using the suboptimal results on subsets of M using a similar proof of Theorem (2.1) and Corollary (2.1). These subsets of M can be defined by

$$M_n = \{(r,p) \varepsilon M\text{: } r(0) = p(0),\ r(x) \geq p(x) + (1/n) \text{ for all } x \varepsilon \mathbb{R}_o\} \tag{2.10}$$

for every $n \varepsilon \mathbb{N}_+$. We will show that it is possible to obtain a sufficient condition of optimality for M_n. Assume without loss of generality that $\overline{r} > \overline{p} + (1/n)$ for every $n \varepsilon \mathbb{N}_+$, which can be interpreted in the same manner as the assumption $\overline{r} > \overline{p}$ given by Assumption (1.1.iv).

<u>COROLLARY (2.2)</u>. Let $n \varepsilon \mathbb{N}_+$ be fixed and assume there is a bounded function v_n on R_+ which satisfies:

i) $v_n(.)$ is differentiable with a bounded continuous derivative $v_n'(.)$ on R_+;

ii) $\sup_{\substack{P\varepsilon[0,\overline{p}] \\ P\varepsilon[P+(1/n),\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)v_n'(x)\} + L_1(x) + Kv_n(x) - \hat{\alpha}v_n(x) = 0$ $\quad x > 0$

$$\sup_{\substack{P \varepsilon [0,\overline{p}] \\ R = P}}\{L_2(P) + L_3(R) + (P - R)v_n'(0)\} + L_1(0) + Kv_n(0) - \hat{\alpha}v_n(0) = 0 .$$

Then

a) $v_n(x) \geq v_{rp}(x)$ for all $(r,p) \varepsilon M_n$, $x \varepsilon \mathbb{R}_+$;

b) if there are controls $(r_n,p_n) \varepsilon M_n$ such that $v_n = v_{r_n p_n}$, then (r_n,p_n) are the unique optimal controls in M_n.

Proof. This can be proved by using the procedure of Theorem (2.1) and Corollary (2.1). The point to be emphasized here is that in (b) (r_n,p_n) are unique only in M_n, i.e. there may exist controls $(\hat{r},\hat{p}) \varepsilon M$ such that $v_n = v_{\hat{r}\hat{p}}$ but $(\hat{r},\hat{p}) \notin M_n$. Δ

REMARK (2.1). Note that the conditions on L_1, L_2 and L_3 could be less restrictive in constructing the locally optimal return function and controls in M_n. Since $r_n(.) \geq p_n(.)$ in M_n, L_1 could be taken to be bounded and left continuous only, and L_2 and L_3 could be assumed to be strictly concave simply without twice continuous differentiability in M_n. Furthermore, absolute continuity of v_n and left-continuity of D^-v_n would be sufficient to be able to use the theory developed. Δ

Corollary (2.2) is very important in the sense that it will be cited repeatedly in proving the existence of a unique function v_n which satisfies the sufficient condition of optimality in $\mathcal{M}_n$ and controls $(r_n, p_n) \in \mathcal{M}_n$ such that $v_n = v_{r_n p_n}$. This way we will construct an improving sequence of locally optimal controls (i.e. optimal in $\mathcal{M}_n$) which will be shown to converge to the function we are seeking.

VI. THE DETERMINISTIC OPTIMAL CONTROL PROBLEM

The aim of this chapter is to analyze the associated deterministic optimal control problem in order to set forth the basic features of the solution procedure. In Section 1, the deterministic storage model will be described shortly, and the sufficient condition of optimality will be restated for the resulting optimal control problem. In Section 2, a sequence of suboptimal return functions which are optimal only in some subset of the admissible class, namely M_n, will be constructed and shown to converge to a limiting function. In Section 3, this limiting function will be shown to be the global optimal return function only if L_1 is taken to satisfy some monotonicity properties. In particular, the optimal return function and the corresponding optimal controls will be constructed explicitly under the additional asumption that L_1 is decreasing.

6.1 A SUFFICIENT OPTIMALITY CONDITION FOR THE DETERMINISTIC STORAGE MODEL

By the deterministic model we mean the special case where the sto - chastic input process A and the stochastic output process B are excluded, so that there are no jump inputs and outputs. The only input to the store is through the controlled input rate p and the only output of the store is through the controlled release rate r. As it is seen in Figure (IV.2.2), starting at an initial content level x, the content level of the store is either increased or decreased by using controls (r(.),p(.)) until it reaches some level $\bar{x} \in \mathbb{R}_+$ such that $p(\bar{x}) = r(\bar{x})$ and is kept there forever. The simplicity of the deterministic structure enables us to obtain significant, results which can be easily extended so as to include the original stochastic control model.

In the deterministic model for every $(r,p) \in M$ and $x \in \mathbb{R}_+$,

$$\lambda_a = 0, \quad \lambda_b = 0 \; ; \tag{1.1}$$
$$Kv_{rp} = 0;$$
$$G_{rp}v_{rp}(x) = [p(x) - r(x)]Dv_{rp}(x) \; .$$

The content level of the store at any time t is given by

$$x(t) = x + \int_0^t (p - r)(x(s))ds \quad , \quad t \geq 0 \tag{1.2}$$

for any $(r,p) \in M$ while the return function is defined to be

$$v_{rp}(x) = \int_0^\infty e^{-\alpha t}[L_1(x(t)) + L_2(p(x(t))) + L_3(r(x(t)))]dt \tag{1.3}$$

for any $(r,p) \in M$ and $x \in \mathbb{R}_+$. Now the control problem can be restated as finding control $(r^*,p^*) \in M$ such that

$$v_{r^*p^*}(x) \geq v_{rp}(x)$$

for every $(r,p) \in M$ and $x \in \mathbb{R}_+$.

This set-up reveals that the deterministic model is only a special case of the general problem described in previous chapters. So the results obtained so far still hold true. Note that the return function v satisfies

$$\alpha v_{rp}(x) = L_2(p(x)) + L_3(r(x)) + [p(x) - r(x)]Dv_{rp}(x) + L_1(x) \quad (1.4)$$

for any $(r,p) \in M$ and $x \in \mathbb{R}_+$. The dependence of Dv_{rp} upon (r,p) will be eliminated to acquire practicality in handling with the results encountered in our analysis.

Hence sufficient conditions of optimality can be expressed in terms of an ordinary differential equation. Below we state the deterministic version of Corollary (V.2.1).

<u>COROLLARY (1.1)</u>. Assume there is a bounded function v on R_+ which satisfies:

i. v(.) is differentiable with a bounded continuous derivative v'(.) on R_+;

ii. $$\sup_{\substack{P \in [0,\bar{p}] \\ R \in [0,\bar{r}]}} \{L_2(P) + L_3(R) + (P - R)v'(x)\} + L_1(x) - \alpha v(x) = 0 \qquad x > 0$$

$$\sup_{\substack{P \in [0,\bar{p}] \\ R \in [0,\bar{r}]}} \{L_2(P) + L_3(R) + (P - R)v'(0)\} + L_1(0) - \alpha v(0) = 0 .$$

Then,

a) $v(x) \geq v_{rp}(x)$ for all $(r,p) \in M$, $x \in \mathbb{R}_+$;

b) if there are controls $(r,p) \in M$ such that $v = v_{rp}$, then (r,p) are the unique optimal controls.

<u>Proof.</u> The proof follows directly from Corollary (V.2.1). $\square$

In a similar fashion a deterministic version can be provided for Corollary (V.2.2) as well.

<u>COROLLARY (1.2).</u> Let $n \in \mathbb{N}_+$ and assume there is a bounded function v_n on R_+ which satisfies:

i. $v_n(.)$ is differentiable with a bounded continuous derivative $v_n'(.)$ on R_+;

ii. $$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P - R)v_n'(x)\} + L_1(x) - \alpha v_n(x) = 0, \quad x >> 0$$

$$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R=P}} \{L_2(P) + L_3(R) + (P - R)v_n'(0)\} + L_1(0) - \alpha v_n(0) = 0 .$$

Then,

a) $v_n(x) \geq v_{rp}(x)$ for all $(r,p) \in M_n$, $x \in \mathbb{R}_+$;

b) if there are controls $(r_n,p_n) \in M_n$ such that $v_n = v_{r_n p_n}$, then (r_n,p_n) are the unique optimal controls in M_n.

<u>Proof.</u> The proof follows directly from Corollary (V.2.2). □

The simplicity of the deterministic problem mainly arises from the fact that the sufficient optimality condition is expressed in terms of an ordinary differential equation rather than a functional differential equation which is considerably difficult to deal with; thus we start our analysis by showing the existence and uniqueness of a function satisfying the sufficiency condition of Corollary (1.1) for the deterministic model. In Chapter VII, the results to be obtained will be fully utilized in characterizing the optimal return function and the optimal controls for the stochastic problem.

6.2 CONSTRUCTION OF SUBOPTIMAL CONTROLS

The first step in our procedure of showing the existence and uniqueness of a function satisfying the sufficiency condition of Corollary (1.1) is to obtain a sequence of suboptimal return functions, each satisfying the sufficiency condition of Corollary (1.2). We are in fact interested in the limit of this sequence of suboptimal return functions $\{v_n\}$, each of which is optimal in $\mathcal{M}_n$. This limiting function and the corresponding controls can be expected to be optimal in $\mathcal{M}_\infty$ defined by

$$\mathcal{M}_\infty = \bigcup_{n\geq 1} \mathcal{M}_n = \{(r,p) \varepsilon \mathcal{M};\ r(0) = p(0),\ \ r(x) \geq p(x)\ \ \text{for}\ \ x \varepsilon \mathbb{R}_0\}.$$

However we cannot be assured of the optimality of the limiting function in $\mathcal{M}$. In fact in some cases the limiting function will not be optimal in $\mathcal{M}$. However, in the next section, we will be able to obtain some explicit conditions which guarantee that the limiting function turns out to be the one we are seeking.

In this section we show that there exists a unique return function and a unique control pair satisfying the sufficiency condition of Corollary (1.2), and that the limits of $\{v_n\}$ and $\{(r_n,p_n)\}$ exist and are optimal in $\mathcal{M}_\infty$. Our approach is similar to ÖZEKİCİ [35], and the mathematics involved follows PLISKA [27].

LEMMA (2.1). For any $n \in \mathbb{N}_+$ $\bar{x} \in \mathbb{R}_+$, $\bar{v} \in \mathbb{R}$ there exists a unique bounded function v on $[\bar{x}, \bar{x}+\bar{t}]$ where $\bar{t} = 1/2n_\alpha$ which satisfies:

i. v is differentiable with a bounded Lipschitz continuous derivative v' on $[\bar{x},\bar{x}+\bar{t}]$;

ii. $$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P - R)v'(x)\} + L_1(x) - \alpha v(x) = 0 , \quad x \varepsilon (\bar{x},\bar{x}+\bar{t}]$$

$$\alpha v(\bar{x}) = \bar{v} .$$

Proof. For any $x \varepsilon (\bar{x},\bar{x}+\bar{t}]$ it is clear that

$$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P - R)v'(x)\} + L_1(x) - \alpha v(x) = 0 \tag{2.2}$$

if and only if

$$L_2(P) + L_3(R) + (P - R)v'(x) + L_1(x) - \alpha v(x) \leq 0 \tag{2.3}$$

For all $P \varepsilon [0,\bar{p}]$ and $R \varepsilon [P+(1/n),\bar{r}]$ with equality holding for some $P \varepsilon [0,\bar{p}]$ and $R \varepsilon [P+(1/n),\bar{r}]$, if and only if

$$v'(x) \geq \frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v(x)] \tag{2.4}$$

for all $P \varepsilon [0,\bar{p}]$ and $R \varepsilon [P+(1/n),\bar{r}]$ with equality holding for some $P \varepsilon [0,\bar{p}]$ and $R \varepsilon [P+(1/n),\bar{r}]$. It follows from (2.4) that

$$v'(x) = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v(x)]\} \tag{2.5}$$

for all $x \varepsilon (\bar{x},\bar{x}+\bar{t}]$. Hence v satisfies (ii) if and only if it satisfies (2.5).

We let B be the Banach space of all bounded continuous functions on $[\overline{x},\overline{x}+\overline{t}]$ with the usual supremum norm $||\cdot||$ and define two mappings Γ_1 and Γ_2 on B such that for any $f \varepsilon B$

$$\Gamma_1(f)(x) = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{\frac{1}{R - P}[L_2(P) + L_3(R) + L_1(x) - \alpha f(x)]\} \tag{2.6}$$

$$\Gamma_2(f)(x) = \overline{v} + \int_{\overline{x}}^{x} \Gamma_1(f)(s)ds \ . \tag{2.7}$$

Now we need to show that Γ_2 is a contraction mapping, so that it possesses a unique fixed point. It is obvious that for all $f,g \ \varepsilon \ B$, $P \ \varepsilon \ [0,\overline{p}]$, $R \ \varepsilon \ [P+(1/n),\overline{r}]$ and x_1, x_2, $x \ \varepsilon \ [\overline{x},\overline{x}+\overline{t}]$

$$|\Gamma_1(f)(x_1) - \Gamma_1(f)(x_2)| \leq \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{|\frac{1}{R - P}[L_1(x_1) - L_1(x_2)| - \alpha|f(x_1)-f(x_2)|\} \tag{2.8}$$

$$\leq n|L_1(x_1) - L_1(x_2)| + n\alpha|f(x_1) - f(x_2)|.$$

Similarly,

$$|\Gamma_1(f-g)(x)| \leq \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{|\frac{1}{R - P}[\alpha g(x) - \alpha f(x)]|\} \tag{2.9}$$

$$\leq \ n\alpha|g(x) - f(x)| \leq n|| \ g - f||.$$

From the definition of Γ_2 it follows that Γ_2 is a mapping of B into B and

$$|\Gamma_2(f - g)| \leq \int_{\overline{x}}^{x} |\Gamma_1(f - g)(s)|ds \ . \tag{2.10}$$

By (2.10), we obtain

$$||\Gamma_2(f - g)|| \leq n\alpha\int_{\overline{x}}^{x} ||f - g||ds \; . \qquad (2.11)$$

Since $x - \overline{x} \leq \overline{t}$ for $x \in [\overline{x},\overline{x}+\overline{t}]$,

$$||\Gamma_2(f - g)|| \leq n\alpha\overline{t}||f - g||. \qquad (2.12)$$

Letting $\overline{t} = 1/2n\alpha$,

$$||\Gamma_2(f - g)|| \leq \frac{1}{2} ||f - g||.$$

So Γ_2 is a contraction mapping, which implies that there is a unique $v \in B$ such that

$$v'(x) = \Gamma_1(v)(x) \; , \qquad v(x) = \Gamma_2(v)(x)$$

for all $x \in [\overline{x},\overline{x}+\overline{t}]$. Since v satisfies (2.5)-(2.7), v satisfies Condition (ii) of the Lemma. Since L_1 is Lipschitz continuous on R_+ and v' is bounded by

$$||v'|| = ||\Gamma_1(v)|| \leq n[L_2(0) + L_3(\overline{r}) + ||L_1|| + \alpha||v||] \; ,$$

v is Lipschitz continuous on $[\overline{x},\overline{x}+\overline{t}]$. Furthermore v' is also Lipschitz continuous by (2.6). □

Although it would be preferrable to directly obtain a global existence and uniqueness result over M, the necessity to use M_n is apparent, since the possibility that $r(\overline{x}) = p(\overline{x})$ for some $\overline{x} \in \mathbb{R}_+$ is allowed, in which case Γ_2 would not be a contraction mapping as it can be seen in the proof of Lemma (2.1). This presents a difficulty, namely, the solution v might be unbounded in a neighborhood of x, or even if not the limit of v(x) as $x \to \overline{x}$ might not exist.

REMARK (2.1). An important point to note here is that the properties of L_1 are inherent in v' although v always turns out to be absolutely continuous whatever L_1 is. The continuity of L_1 implies the continuity of v', and the Lipschitz property of L_1 implies the Lipschitz property of v'. Furthermore if L_1 is left-continuous, $D^- v$ is also left-continuous. $\Box$

The following lemma now extends the result of Lemma (2.1) to [0,s] for any s > 0. Note that we need a boundary condition at zero as it can be seen below.

LEMMA (2.2). For any $n \in \mathbb{N}_+$, $s \in \mathbb{R}_0$ there exists a unique function v on [0,s] which satisfies:

i. v is differentiable with a bounded Lipschitz continuous derivative v' on [0,s];

ii. $$\sup_{\substack{P\in[0,\overline{p}]\\ R\in[P+(1/n),\overline{r}]}} \{L_2(P) + L_3(R) + (P - R)v'(x)\} + L_1(x) - \alpha v(x) = 0 \ , \quad x \in [0,s]$$

$$\sup_{\substack{P\in[0,\overline{p}]\\ R=P}} \{L_2(P) + L_3(R) + (P - R)v'(0)\} + L_1(0) - \alpha v(0) = 0.$$

Proof. Note that the requirement $r_n(0) = P_n(0)$ implies

$$\sup_{\substack{P\in[0,\overline{p}]\\ R=P}} \{L_2(P) + L_3(R) + (P - R)\, v_n'(0)\} + L_1(0) - \alpha v_n(0) = 0$$

if and only if

$$\sup_{\substack{P\in[0,\overline{p}]\\ R=P}} \{L_2(P) + L_3(R)\} + L_1(0) - \alpha v_n(0) = 0.$$

So it is obvious that $p_n(0) = p(0)$ for all $n \in \mathbb{N}_+$ independent of n, which implies that the boundary conditions in M_n for all $n \in \mathbb{N}_+$ coincide. Thus the optimal input rate at zero content level is determined by

$$L_2(p(0)) + L_3(p(0)) = \sup_{P\varepsilon[0,\overline{p}]} \{L_2(P) + L_3(P)\}. \tag{2.13}$$

In other words if p(0) maximizes the right-hand-side of (2.13), then

$$\begin{aligned} p(0) &= 0 && \text{if} \quad -L_2'(0) > L_3'(0) ; \\ p(0) &= \overline{p} && \text{if} \quad -L_2'(\overline{p}) < L_3'(\overline{p}) ; \\ L_3'(p(0)) &= -L_2'(p(0)) && \text{otherwise.} \end{aligned} \tag{2.14}$$

The strict concavity of both L_2 and L_3 ensures the uniqueness of p(0) given by (2.14). Then r(0) which is the optimal output rate at zero content level is determined by setting r(0) = p(0) independent of n. With this argument note that

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R=P}} \{L_2(P) + L_3(R) + (P - R)v'(0)\} + L_1(0) - \alpha v(0) = 0$$

if and only if

$$L_2(p(0)) + L_3(p(0)) + L_1(0) - \alpha v(0) = 0 \tag{2.15}$$

where p(0) is defined by (2.14). Since the boundary condition required by the Lemma is now provided by (2.15), we are in a position to prove the Lemma. Using Lemma (2.1) we will iteratively construct a unique function v on [0,s].

On $[0,(1/2n\alpha)]$ let v be the unique function of Lemma (2.1) with $\overline{x} = 0$ and $\overline{v} = (L_2(p(0)) + L_3(p(0)) + L_1(0))/\alpha$ which follows from (2.15). Repeating this argument and using Lemma (2.1) recursively by taking $\overline{x} = k/2n\alpha$ and $\overline{v} = v(k/2n\alpha)$ for $k = 1,2,\ldots, [s/(2n\alpha)] + 1$, we define v on [0,s] in a finite number of steps.

Note that v is bounded on [0,s] since it is bounded on intervals $[k/2n\alpha, (k+1)/2n\alpha]$ and v is continuous because of our construction. The continuity of v' at points $\overline{x} = k/2n\alpha$ follows from the continuity of v and L_1. The Lipschitz continuity of v' follows from the facts that v' is bounded, L_1 is Lipschitz continuous and for any $(x_1,x_2) \varepsilon [0,s]$

$$|v'(x_1) - v'(x_2)| \leq n[|L_1(x_1) - L_1(x_2)| + \alpha|v(x_1) - v(x_2)|].$$

Uniqueness follows trivially from Lemma (2.2).

Letting v_s be the unique function on [0,s] of Lemma (2.2), we realize that for all $s \varepsilon \mathbb{R}_0$

$$||v_s|| = \sup_{x\varepsilon[0,s]} |v_s(x)| < \infty \tag{2.16}$$

$$||v_s'|| = \sup_{x\varepsilon[0,s]} |v_s'(x)| \leq n[L_2(0) + L_3(\overline{r}) + ||L_1|| + \alpha||v_s||].$$

It is obvious that if $0 < s < t$, then $v_s(x) = v_t(x)$ for $x \leq s$, i.e. v_s and v_t coincide on [0,s].

Now we state some bounds on v_s and study its limiting behaviour.

<u>LEMMA (2.3)</u>. Let $\overline{L}_1 = \sup_{x\varepsilon\mathbb{R}_+} L_1(x)$ and $\underline{L}_1 = \inf_{x\varepsilon\mathbb{R}_+} L_1(x)$. Then for every $s \varepsilon \mathbb{R}_0$;

i. $L_2(\overline{p}) + L_3(0) + \underline{L}_1 \leq \alpha v_s(s) \leq L_2(0) + L_3(\overline{r}) + \overline{L}_1$;

ii. $\lim_{s\to\infty} \alpha v_s(s) = L_2(0) + L_3(\overline{r}) + L_1(\infty)$.

Proof. i) We will first prove that $\alpha v_s(s) \leq L_2(0) + L_3(\bar{r}) + \bar{L}_1$ by contradiction, so assume that

$$\alpha v_s(s) > L_2(0) + L_3(\bar{r}) + \bar{L}_1 \tag{2.17}$$

which implies that

$$\alpha v_s(s) \geq L_2(P) + L_3(R) + L_1(s) \tag{2.18}$$

for all $P \in [0,\bar{p}]$ and $R \in [P+(1/n),\bar{r}]$ because of Assumption (V.1.1) and the definition of $\bar{L}_1$.

Recall that

$$v_s'(x) = \sup_{\substack{P\in[0,\bar{p}] \\ R\in[P+(1/n),\bar{r}]}} \left\{ \frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v_s(x)]\right\} , \; x \in [0,s]. \tag{2.19}$$

If (2.18) holds true, it follows from (2.19) that $v_s'(s) < 0$. Now it suffices to show that v_s is decreasing on $[0,s]$. If it is not, then

$$\bar{x} = \sup\{x \in [0,s]: v_s'(x) = 0\} > 0$$

and

$$\alpha v_s(\bar{x}) > \alpha v_s(s) > L_2(0) + L_3(\bar{r}) + \bar{L}_1 \geq L_2(P) + L_3(R) + L_1(\bar{x})$$

for all $P \in [0,\bar{p}]$, $R \in [P+(1/n),\bar{r}]$. But then (2.19) yields $v_s'(\bar{x}) < 0$ which contradicts the definition of $\bar{x}$. Hence v_s is decreasing on $[0,s]$. Then for all $x \leq s$

$$\alpha v_s(x) > v_s(s) > L_2(0) + L_3(\bar{r}) + \bar{L}_1 \geq L_2(0) + L_3(\bar{r}) + L_1(0) \tag{2.20}$$

Taking $x = 0$, it follows from (2.20) that

$$\alpha v_s(0) > L_2(0) + L_3(\overline{r}) + L_1(0) \geq L_2(P) + L_3(R) + L_1(0)$$

for all $P \varepsilon [0,\overline{p}]$, $R \varepsilon [0,\overline{r}]$, which then contradicts the boundary condition given by (2.15).

In a similar manner we now assume that

$$\alpha v_s(s) < L_2(\overline{p}) + L_3(0) + \underline{L}_1 \tag{2.21}$$

which implies that for all $P \varepsilon [0,\overline{p}]$, $R \varepsilon [P + (1/n),\overline{r}]$

$$\alpha v_s(s) \leq L_2(P) + L_3(R) + L_1(s) .$$

So it turns out that $v_s'(s) > 0$ in (2.19). If v_s is not increasing on $[0,s]$, then

$$\overline{x} = \sup\{x \varepsilon [0,s]: v_s'(x) = 0\} > 0$$

and

$$\alpha v_s(\overline{x}) < \alpha v_s(s) < L_2(\overline{p}) + L_3(0) + \underline{L}_1 \leq L_2(P) + L_3(R) + L_1(\overline{x})$$

for all $P \varepsilon [0,\overline{p}]$, $R \varepsilon [P+(1/n),\overline{r}]$. This implies that $v_s'(\overline{x}) > 0$ which contradicts the definition of $\overline{x}$. Hence we conclude that v_s is increasing on $[0,s]$. Then for all $x \leq s$

$$\alpha v_s(x) \leq \alpha v_s(s) < L_2(\overline{p}) + L_3(0) + \underline{L}_1 \leq L_2(\overline{p}) + L_2(0) + L_1(0).$$

Setting $x = 0$, we obtain

$$\alpha v_s(0) < L_2(\overline{p}) + L_3(0) + L_1(0) \leq L_2(P) + L_3(R) + L_1(0) \tag{2.22}$$

for all $P \varepsilon [0,\overline{p}]$, $R \varepsilon [0,\overline{r}]$. On the other hand, (2.22) contradicts the boundary condition given by (2.14).

ii. Let $\varepsilon > 0$ be arbitrary and assume without loss of generality $\overline{L}_1 - L_1(\infty) > \varepsilon$ and $[L_2(0) - L_2(\overline{p})] + [L_3(\overline{r}) - L_3(0)] + [L_1(\infty) - \underline{L}_1] > \varepsilon$. Let x_ε be defined such that for all $x \geq x_\varepsilon$

$$|L_1(x) - L_1(\infty)| < \varepsilon \quad .$$

Now we define y_1, y_2 on $[x_\varepsilon,\infty)$ by

$$y_1(x) = C_1 e^{-n\alpha x} + \frac{1}{\alpha}[L_2(0) + L_3(\overline{r}) + L_1(\infty) + \varepsilon]; \qquad (2.23)$$

$$y_1(x_\varepsilon) = \frac{1}{\alpha}[L_2(0) + L_3(\overline{r}) + \overline{L}_1];$$

$$y_2(x) = C_2 e^{-n\alpha x} + \frac{1}{\alpha}[L_2(0) + L_3(\overline{r}) + L_1(\infty) - \varepsilon];$$

$$y_2(x_\varepsilon) = \frac{1}{\alpha}[L_2(\overline{p}) + L_3(0) + \underline{L}_1];$$

where C_1 and C_2 are suitable constants determined by the boundary conditions. It can be easily shown that under the assumptions made above $C_1 \geq 0$ and $C_2 \leq 0$, and y_1 turns out to be convex decreasing and y_2 turns out to be concave increasing on $[x_\varepsilon,\infty)$. It is obvious that

$$\lim_{x\to\infty} y_1(x) = \frac{1}{\alpha}[L_2(0) + L_3(\overline{r}) + L_1(\infty) + \varepsilon]$$

$$\lim_{x\to\infty} y_2(x) = \frac{1}{\alpha}[L_2(0) + L_3(\overline{r}) + L_1(\infty) - \varepsilon].$$

First of all it is necessary to show $y_2(s) \leq v_s(s) \leq y_1(s)$ for all $s \geq x_\varepsilon$. We will present the argument only for $v_s(s) \leq y_1(s)$, leaving the other part to the reader.

If $s = x_\varepsilon$, $y_1(s) = (1/\alpha)[L_2(0) + L_3(\overline{r}) + \overline{L}_1]$ and the desired result follows immediately from Condition (i) of the Lemma. Now assume $v_s(s) > y_1(s)$ for some $s > x_\varepsilon$, then

$$v_s'(s) = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(s) - \alpha v_s(s)]\}$$

$$\leq n[L_2(0) + L_3(\overline{r}) + L_1(\infty) + \varepsilon - \alpha v_s(s)]$$

$$< n[L_2(0) + L_3(\overline{r}) + L_1(\infty) + \varepsilon - \alpha y_1(s)] = y_1'(s)$$

where the last statement follows from (2.23). This implies that $v_{x_\varepsilon}(x_\varepsilon) > y_1(x_\varepsilon)$ by repeating the same argument for

$$\overline{x} = \sup\{x\varepsilon[x_\varepsilon,s]: v_x'(x) = y_1'(x)\} \vee x_\varepsilon$$

and showing that $\overline{x} = x_\varepsilon$. Thus we reach a contradiction to part (i) of the Lemma, i.e. $\alpha v_s(s) \leq L_2(0) + L_3(\overline{r}) + \overline{L}_1$. Hence we have shown that for all $s \geq x_\varepsilon$ $v_s(s) \leq y_1(s)$.

Now we let $\delta > 0$ be arbitrary, take $\varepsilon = \delta/4$ and let x_ε, y_1, y_2 be defined accordingly as before. Let

$$x_\delta = \inf\{x \ \varepsilon \ [x_\varepsilon,\infty):y_1(x) - y_1(\infty) < 2\varepsilon,\ y_2(\infty) - y_2(x) < 2\varepsilon\},$$

then

$$|\alpha v_s(s) - [L_2(0) + L_3(\overline{r}) + L_1(\infty)]| < \delta$$

for all $s \geq x_\delta$. This completes the proof. Δ

REMARK (2.2). Note that if $\overline{L}_1 = L_1(\infty)$, then we let $C_1 = 0$ and define y_1 on $[x_\varepsilon,\infty)$ independent of ε. The same results would still hold true. Δ

Now it is necessary to extend v_s to v_n on R_+, so define v_n on R_+ such that

$$v_n(x) = v_s(x) \quad \text{for any} \quad x \ \varepsilon \ \mathbb{R}_+ \text{ and } \ s > x \tag{2.24}$$

where v_s is the unique function of Lemma (2.2). We are now in a position to construct the control pair (r_n, p_n) in $\mathcal{M}_n$ satisfying Condition (ii) of Corollary (1.2). For $x \in \mathbb{R}_0$ define $(r_n(x), p_n(x))$ such that

$$L_2(p_n(x)) + L_3(r_n(x)) + [p_n(x) - r_n(x)]v_n'(x) = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)v_n'(x)\} \qquad (2.25)$$

with $0 \leq p_n(x) \leq \overline{p}$, $p_n(x)+(1/n) \leq r_n(x) \leq \overline{r}$. By Kuhn-Tucker conditions, if $(r_n(x), p_n(x))$ satisfies (2.25), then there exist Lagrange multipliers $\{\lambda_i, i=1,2,3\}$ such that $(r_n(x), p_n(x), \lambda_1, \lambda_2, \lambda_3)$ satisfies:

$$L_2'(p_n(x)) + v_n'(x) - \lambda_1 - \lambda_3 \leq 0 \qquad (2.26)$$

$$L_3'(r_n(x)) - v_n'(x) + \lambda_1 - \lambda_2 = 0 \qquad (2.27)$$

$$p_n(x)[L_2'(p_n(x)) + v_n'(x) - \lambda_1 - \lambda_3] = 0 \qquad (2.28)$$

$$r_n(x) - p_n(x) - \frac{1}{n} \geq 0 \qquad (2.29)$$

$$\overline{r} - r_n(x) \geq 0 \qquad (2.30)$$

$$\overline{p} - p_n(x) \geq 0 \qquad (2.31)$$

$$\lambda_1[r_n(x) - p_n(x) - \frac{1}{n}] = 0 \qquad (2.32)$$

$$\lambda_2[\overline{r} - r_n(x)] = 0 \qquad (2.33)$$

$$\lambda_3[\overline{p} - p_n(x)] = 0 \qquad (2.34)$$

where $r_n(x)$, $p_n(x)$, λ_1, λ_2, $\lambda_3 \in \mathbb{R}_+$.

Moreover the sufficient conditions for $(r_n(x), p_n(x))$ to be the unique optimal solution satisfying (2.25) require the Hessian matrix

$$H = \begin{vmatrix} L_2''(p_n(x)) & 0 \\ & \\ 0 & L_3''(r_n(x)) \end{vmatrix} \tag{2.35}$$

to be negative definite on the tangent subspace of active constraints at $(r_n(x), p_n(x))$. This assertion implies that

$$y_1^2 L_2''(p_n(x) + y_2^2 L_3''(r_n(x)) < 0 \tag{2.36}$$

for all possible values of y_1, y_2. However the concavity assumptions imposed upon L_2 and L_3 ensure us that (2.36) holds true for all y_1 and y_2. So it sufficies to set $r(0) = p(0)$ where $p(0)$ is as defined by (2.14) and to determine $(r_n(x),p_n(x))$ for $x \in \mathbb{R}_0$ by solving (2.26)-(2.34).

<u>REMARK (2.3).</u> Note that for $x \in \mathbb{R}_0$ and $v'(x)$ fixed, the control pair $(r_n(x),p_n(x))$ satisfying (2.25) is characterized by:

i) $r_n(x) = \frac{1}{n}$, $p_n(x) = 0$ if $v_n'(x) \leq -L_2'(0)$
$v_n'(x) \geq L_3'(1/n)$

ii) $r_n(x) = \overline{r}$, $p_n(x) = 0$ if $v_n'(x) \leq -L_2'(0)$
$v_n'(x) \leq L_3'(\overline{r})$

iii) $L_3'(r_n(x)) = v_n'(x)$, $p_n(x) = 0$ if $v_n'(x) \leq -L_2'(0)$
$L_3'(\overline{r}) < v_n'(x) < L_3'(1/n)$

iv) $L_3'(r_n(x)) = v_n'(x) - \lambda_1 + \lambda_2$ if $-L_2'(0) < v_n'(x) < -L_2'(\overline{p})$
$-L_2'(p_n(x)) = v_n'(x) - \lambda_1$ $v_n'(x) \geq L_3'(1/n)$

$$\lambda_1[r_n(x) - p_n(x) - \frac{1}{n}] = 0$$
$$\lambda_2[\overline{r} - r_n(x)] = 0$$

v) $r_n(x) = \overline{r}$, $-L_2'(p_n(x)) = v_n'(x)$ if $-L_2'(0) < v_n'(x) < -L_2'(\overline{p})$

$v_n'(x) \leq L_3'(\overline{r})$

vi) $L_3'(r_n(x)) = v_n'(x) - \lambda_1$ if $-L_2'(0) < v_n'(x) < -L_2'(\overline{p})$

$-L_2'(p_n(x)) = v_n'(x) - \lambda_1$ $L_3'(\overline{r}) < v_n'(x) < L_3'(1/n)$

$\lambda_1[r_n(x) - p_n(x) - (1/n)] = 0$

vii) $L_3'(r_n(x)) = v_n'(x) - \lambda_1 + \lambda_2$ if $v_n'(x) \geq -L_2'(\overline{p})$

$-L_2'(p_n(x)) = v_n'(x) - \lambda_1 - \lambda_3$ $v_n'(x) \geq L_3'(1/n)$

$\lambda_1[r_n(x) - p_n(x) - (1/n)] = 0$

$\lambda_2[\overline{r} - r_n(x)] = 0$

$\lambda_3[\overline{p} - p_n(x)] = 0$

viii) $r_n(x) = \overline{r}$, $p_n(x) = \overline{p}$ if $v_n'(x) \geq -L_2'(\overline{p})$

$v_n'(x) \leq L_3'(\overline{r})$

ix) $L_3'(r_n(x)) = v_n'(x) - \lambda_1$ if $v_n'(x) \geq -L_2'(\overline{p})$

$-L_2'(p_n(x)) = v_n'(x) - \lambda_1 - \lambda_3$ $L_3'(\overline{r}) < v_n'(x) < L_3'(1/n)$

$\lambda_1[r_n(x) - p_n(x) - (1/n)] = 0$

$\lambda_3[\overline{p} - p_n(x)] = 0$.

It follows from this characterization that $0 \leq p_n(x) \leq \overline{p}$, $0 \leq r_n(x) \leq \overline{r}$, $r_n(x) \geq p_n(x)+(1/n)$ and $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}_+$ for all $x \in \mathbb{R}_0$. $\square$

The following result provides an insight as to what properties $(r_n(x),p_n(x))$ as defined by Remark (2.3) possesses. So for $x \in \mathbb{R}_+$ define $(\hat{r}_n(x),\hat{p}_n(x))$ by

$$L_3'(\hat{r}_n(x)) = v_n'(x) \quad \text{and} \quad -L_2'(\hat{p}_n(x)) = v_n'(x) . \tag{2.37}$$

It is clear from (2.37) that $(\hat{r}_n,\hat{p}_n)$ is the solution of the unconstrained optimization problem.

<u>LEMMA (2.4)</u>. For any x_1, $x_2 \in \mathbb{R}_+$, $n \in \mathbb{N}_+$;

i. if $v_n'(x_1) \geq v_n'(x_2)$, then $p_n(x_1) \geq p_n(x_2)$ and $r_n(x_1) \leq r_n(x_2)$;

ii. if $v_n'(x_1) \leq v_n'(x_2)$, then $p_n(x_1) \leq p_n(x_2)$ and $r_n(x_1) \geq r_n(x_2)$.

<u>Proof</u>. We prove only part (i) of the Lemma for the case described by (vi) of Remark (2.3) where the optimal controls do not occur at the boundaries. The other cases can be shown in a similar manner, but will be omitted here to avoid repetition.

Let $(\hat{r}_n(x_1),\hat{p}_n(x_1))$ and $(\hat{r}_n(x_2),\hat{p}_n(x_2))$ be the tangent points as defined by (2.37). Furthermore let $(r(x_1),p(x_1))$ and $(r(x_2),p(x_2))$ be the solutions determined by Remark (2.3.(iv)) while λ_{11} and λ_{12} are the corresponding Lagrange multipliers. So

$$L_3'(r_n(x_1)) = v_n'(x_1) - \lambda_{11} , \quad -L_2'(p_n(x_1)) = v_n'(x_1) - \lambda_{11} ,$$
$$\lambda_{11}[r_n(x_1) - p_n(x_1) - (1/n)] = 0 \tag{2.38}$$

and

$$L_3'(r_n(x_2)) = v_n'(x_2) - \lambda_{12} , \quad -L_2'(p_n(x_2)) = v_n'(x_2) - \lambda_{12} ,$$

$$\lambda_{12}[r_n(x_2) - p_n(x_2) - (1/n)] = 0 \quad (2.39)$$

Three possible cases should be considered:

a) $\hat{r}_n(x_1) - \hat{p}_n(x_1) > 1/n$ implies that $\hat{r}_n(x_2) - \hat{p}_n(x_2) > 1/n$ since $v_n'(x_1) \geq v_n'(x_2)$ and L_2 and L_3 are strictly concave. Hence $\lambda_{11} = \lambda_{12} = 0$ and $(r_n(x_1),p_n(x_1)) = (\hat{r}_n(x_1),\hat{p}_n(x_1))$, $(r_n(x_2),p_n(x_2)) = (\hat{r}_n(x_2),\hat{p}_n(x_2))$. It follows from (2.38) and (2.39) that $-L_2'(p_n(x_1)) \geq -L_2'(p_n(x_2))$ and $L_3'(r_n(x_1)) \geq L_3'(r_n(x_2))$. By the concavity of L_2 and L_3 this implies that $p_n(x_1) \geq p_n(x_2)$ and $r_n(x_1) \leq r_n(x_2)$;

b) $\hat{r}_n(x_1) - \hat{p}_n(x_1) < 1/n$ and $\hat{r}_n(x_2) - \hat{p}_n(x_2) > 1/n$ imply that $\lambda_{11} > 0$ and $\lambda_{12} = 0$. It obviously follows that $(r_n(x_2),p_n(x_2)) = (\hat{r}_n(x_2),\hat{p}_n(x_2))$ and $r_n(x_1) = p_n(x_1) + 1/n$ satisfying (2.38). It is clear that $v_n'(x_1) - \lambda_{11} \geq v_n'(x_2)$ since $v_n'(x_1) - \lambda_{11} < v_n'(x_2)$ contradicts the fact that $\hat{r}_n(x_2) - \hat{p}_n(x_2) > 1/n$. So $-L_2'(p_n(x_1)) \geq -L_2'(p_n(x_2))$ and $L_3'(r_n(x_1)) \geq L_3'(r_n(x_2))$ and the desired result follows immediately;

c) $\hat{r}_n(x_1) - \hat{p}_n(x_1) < 1/n$ and $\hat{r}_n(x_2) - \hat{p}_n(x_2) < 1/n$ imply that $\lambda_{11} > 0$ and $\lambda_{12} > 0$. Then $r_n(x_1) = p_n(x_1) + 1/n$ and $r_n(x_2) = p_n(x_2) + 1/n$ satisfying (2.38) and (2.39), respectively. Now we show by contradiction that $v_n'(x_1) - \lambda_{11} = v_n'(x_2) - \lambda_{12}$. First assume that $v_n'(x_1) - \lambda_{11} < v_n'(x_2) - \lambda_{12}$ which implies that $L_3'(r_n(x_1)) < L_3'(r_n(x_2))$ and $-L_2'(p_n(x_1)) < -L_2'(p_n(x_2))$; hence $r_n(x_1) > r_n(x_2)$ and $p_n(x_1) < p_n(x_2)$

which contradicts the fact that $r_n(x_1) - p_n(x_1) = 1/n$ and $r_n(x_2) - p_n(x_2) = 1/n$. Now assume that $v_n'(x_1) - \lambda_{11} > v_n'(x_2) - \lambda_{12}$ which implies that $L_3'(r_n(x_1)) > L_3'(r_n(x_2))$ and $-L_2'(p_n(x_1)) > -L_2'(p_n(x_2))$; so it follows that $r_n(x_1) < r_n(x_2)$ and $p_n(x_1) > p_n(x_2)$ which again contradicts $r_n(x_1) - p_n(x_1) = 1/n$ and $r_n(x_2) - p_n(x_2) = 1/n$. So $v_n'(x_1) - \lambda_{11} = v_n'(x_2) - \lambda_{12}$ and by the strict concavity of L_2 and L_3, $r_n(x_1) = r_n(x_2)$ and $p_n(x_1) = p_n(x_2)$. Δ

REMARK (2.4). Note that $\lambda_{11} \geq \lambda_{12}$ if $v_n'(x_1) \geq v_n'(x_2)$ and $\lambda_{11} \leq \lambda_{12}$ if $v_n'(x_1) \leq v_n'(x_2)$. This is trivially apparent in Cases (a) and (b) of Lemma (2.4) and to see this in Case (c), note that $\lambda_{11} - \lambda_{12} = v_n'(x_1) - v_n'(x_2) \geq 0$. Δ

The proof of the following result follows from Lemmas (2.2), (2.3) and (2.4). Note that v_n is the return function as defined by (2.24).

THEOREM (2.1). For every $n \in \mathbb{N}_+$ there exists a unique bounded function v_n on R_+ which satisfies:

i. v_n is differentiable with a bounded Lipschitz continuous derivative v_n' on $\mathbb{R}_+$;

ii. $$\sup_{\substack{P\in[0,\bar{p}] \\ R\in[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P - R)v_n'(x)\} + L_1(x) - \alpha v_n(x) = 0 , \quad x > 0$$

$$\sup_{\substack{P\in[0,\bar{p}] \\ R=P}} \{L_2(P) + L_3(R) + (P - R)v_n'(0)\} + L_1(0) - \alpha v_n(0) = 0 ;$$

iii. Furthermore there exists a unique optimal control pair (r_n,p_n) in M_n such that $v_n = v_{r_n p_n}$, so (r_n,p_n) are optimal in M_n.

Proof. i) Boundedness of v_n follows from Lemma (2.3). The differentiability of v_n follows from Lemma (2.2) since v_s is differentiable on [0,s] for every $s \in \mathbf{R}_0$. The boundedness and Lipschitz property of v_n' follows from the boundedness of v_n and L_1 and Lipschitz continuity of L_1. It is obvious that v_n is unique since v_s is unique on [0,s];

ii) v_n satisfies Condition (ii) of the Theorem since v_s satisfies Condition (ii) of Lemma (2.2);

iii) For every $x \in \mathbb{R}_0$, $(r_n(x),p_n(x))$ is as defined by Remark (2.3) and $r(0) = p(0)$ where $p(0)$ is given by (2.14). Hence it suffices to show that $(r_n(x),p_n(x))$ obtained accordingly are Lipschitz continuous on R_0.

The assumption that L_2 and L_3 are strictly concave implies that L_2' and L_3' are strictly decreasing. This together with the fact that v_n' is continuous on R_0 amounts to saying that $(r_n(x),p_n(x))$ is continuous on R_0. Let m be the Lipschitz constant of v_n' and ε be such that $|L_2''| \geq \varepsilon$ and $|L_3''| \geq \varepsilon$ as given by Assumption (V.1.1).

We show the Lipschitz property of p_n only for the case (vi) of Remark (2.3), leaving the other cases to the reader. Letting $x_1, x_2 \in \mathbb{R}_0$ be arbitrary, assume without loss of generality that $v_n'(x_2) \leq v_n'(x_1)$. So,

$$-L_2'(0) < v_n'(x_2) \leq v_n'(x_1) < -L_2'(\overline{p})$$
$$L_3'(\overline{r}) < v_n'(x_2) \leq v_n'(x_1) < L_3'(1/n) .$$

It follows from Lemma (2.4) that $p_n(x_1) \geq p_n(x_2)$ since $v_n'(x_1) \geq v_n'(x_2)$. Then,

$$0 \leq L_2'(p_n(x_2)) - L_2'(p_n(x_1)) = \int_{p_n(x_1)}^{p_n(x_2)} L_2''(s)ds \geq \int_{p_n(x_2)}^{p_n(x_1)} \varepsilon . ds$$

$$= \varepsilon[p_n(x_1) - p_n(x_2)].$$

Thus,

$$0 \leq p_n(x_1) - p_n(x_2) \leq \frac{1}{\varepsilon} [L_2'(p_n(x_2)) - L_2'(p_n(x_1))]. \tag{2.40}$$

Letting $\lambda_{11}, \lambda_{12} \in \mathbb{R}_+$ be the multipliers in (vi) of Remark (2.3) corresponding to $x = x_1$ and $x = x_2$, respectively, (2.40) implies that

$$0 \leq p_n(x_1) - p_n(x_2) \leq \frac{1}{\varepsilon} [v_n'(x_1) - v_n'(x_2) + \lambda_{12} - \lambda_{11}].$$

By Remark (2.4), $\lambda_{12} - \lambda_{11} \leq 0$ and thus

$$0 \leq p_n(x_1) - p_n(x_2) \leq \frac{1}{\varepsilon} [v_n'(x_1) - v_n'(x_2)] \leq \frac{m}{\varepsilon} |x_2 - x_1| .$$

The Lipschitz property of r_n on R_0 can be established using a similar procedure. Note that (r_n, p_n) is Lipschitz continuous only on $(0,\infty)$ and not continuous at $x = 0$, but they are still admissible. Ω

REMARK (2.5). Theorem (2.1) clearly demonstrates the reasons why we have assumed the reward function to satisfy Assumption (V.1.1). Those conditions on L_1, L_2 and L_3 enable us to find admissible controls in M_n. If L_1 is taken to be piecewise Lipschitz, then both v_n' and (r_n, p_n) turn out to be piecewise Lipschitz, preserving the admissibility. However if L_1 is continuous only, we cannot be assured about the admissibility of (r_n, p_n). If on the other hand L_1 is both decreasing and continuous, then (r_n, p_n) is certainly admissible as it will be considered in Section 3. Ω

COROLLARY (2.1). For every $n \varepsilon \mathbb{N}_+$, v_n satisfies:

i. $L_2(\bar{p}) + L_3(0) + \underline{L}_1 \leq \alpha v_n(x) \leq L_2(0) + L_3(\bar{r}) + \bar{L}_1$, $x \varepsilon \mathbb{R}_+$;

ii. $\lim_{x\to\infty} \alpha v_n(x) = L_2(0) + L_3(\bar{r}) + L_1(\infty)$;

iii. $||v_n'|| \leq n[L_2(0) + L_3(\bar{r}) + ||L_1|| + \alpha||v_n||] \leq nC$

for some $C > 0$ independent of n;

iv. $\lim_{x\to\infty} v_n'(x) = 0$.

Proof. Here (i) and (ii) follow directly from Lemma (2.3) while (iii) follows from (2.16). To see (iv), note that

$$v_n'(x) = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v_n(x)]\}.$$

So,

$$\lim_{x\to\infty} v_n'(x) = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(\infty) - \alpha v_n(\infty)]\}$$

$$= \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) - L_2(0) - L_3(\bar{r})]\} = 0. \square$$

So far we have created a sequence of locally optimal return functions $\{v_n\}$ and control pairs $\{(r_n,p_n)\}$. Now it is necessary to show that the sequence $\{v_n\}$ converges to some function in $\mathcal{M}_\infty = \bigcup_{n>1} \mathcal{M}_n$ as defined by (2.1). Since $\mathcal{M}_n \subset \mathcal{M}_{n+1}$, it is obvious that $v_{n+1} \geq v_n$ for every $n \varepsilon \mathbb{N}_+$. Therefore

$$v(x) = \lim_{n\to\infty} v_n(x) \tag{2.41}$$

exists. Now the following proposition states that the limiting function is optimal in $\mathcal{M}_\infty$, but does not necessarily satisfy the sufficient optimality condition.

PROPOSITION (2.1). $v \geq v_{rp}$ for every $(r,p) \in \mathcal{M}_\infty$. In particular, $v(x) \geq L_1(x) + L_2(p(x)) + L_3(p(x))$ for every $x \in \mathbb{R}_+$.

Proof. Let $(r,p) \in \mathcal{M}_\infty$ and define

$$r_n(x) = \begin{cases} p(x) & \text{if } x = 0, \\ p(x) + \frac{1}{n} & \text{if } x > 0,\ r(x) \leq p(x)+(1/n), \\ r(x) & \text{if } x > 0,\ r(x) > p(x)+(1/n), \end{cases}$$

and

$$p_n(x) = p(x) \quad , \quad x \geq 0$$

for every $n \in \mathbb{N}_+$, which implies that $(r_n,p_n) \in \mathcal{M}_n$ since $(r,p) \in \mathcal{M}_\infty$. It is obvious that $|r_n(x) - r(x)| \leq 1/n$ and $r_n(x) \geq r_{n+1}(x)$ for every $x \in \mathbb{R}_+$. So $\{r_n\}$ decreases to r.

For fixed $x \in \mathbb{R}_+$ let $f_n(t)$ and $f(t)$ be the content level of the store at time t with initial content x when the controls being used are (r_n,p_n) and (r,p) respectively. In other words, f_n and f are the unique solutions of

$$f_n(t) = x + \int_0^t (p_n - r_n)(f_n(s))ds \quad , \ t \geq 0$$

$$f(t) = x + \int_0^t (p - r)(f(s))ds \quad , \ t \geq 0 .$$

Since $r_n \geq r_{n+1} \geq p_{n+1} = p_n$ for every $n \in \mathbb{N}_+$, $0 \leq f_n \leq f_{n+1} \leq x$ and $\{f_n\}$ increases to h which is defined by

$$h(t) = \lim_{n\to\infty} f_n(t) .$$

Note that for all $t \geq 0$

$$\begin{aligned} |(r_n-p_n)(f_n(t)) - (r-p)(h(t))| &= |(r_n-p_n)(f_n(t)) - (r-p)(f_n(t)) \\ &\quad + (r-p)(f_n(t)) - (r-p)(h(t))| \\ &\leq |(r_n-p_n)(f_n(t)) - (r-p)(f_n(t))| \\ &\quad +|(r-p)(f_n(t)) - (r-p)(h)t))| \\ &\leq (1/n) + |(r-p)(f_n(t)) - (r-p)(h(t))|, \end{aligned}$$

and similarly

$$\begin{aligned} |r_n(f_n(t)) - r(h(t))| &\leq |r_n(f_n(t)) - r(f_n(t))| + |r(f_n(t)) - r(h(t))| \\ &\leq (1/n) + |r(f_n(t)) - r(h(t))| \end{aligned}$$

since $|r_n(x) - r(x)| \leq 1/n$ and $p_n(x) = p(x)$. Then the left-continuity of both r(.) and p(.) on $\{r(.) \geq p(.)\}$ by Admissibility Condition 3 and the fact that $f_n \uparrow h$ imply that

$$\lim_{n\to\infty}|(r-p)(f_n(t)) - (r-p)(h(t))| = 0 \quad , \quad t \geq 0 ,$$

and

$$\lim_{n\to\infty}|r(f_n(t)) - r(h(t))| = 0 \quad , \quad t \geq 0 . \tag{2.42}$$

Therefore by the bounded convergence theorem,

$$h(t) = x + \int_0^t (p-r)(h(s))ds \quad , \quad t \geq 0$$

which implies that $h = f$ by uniqueness and $f_n \uparrow f$. Then by the left-continuity of L_1, L_2 and L_3, we have

$$\lim_{n\to\infty} L_1(f_n(t)) = L_1(f(t))$$

and

$$\lim_{n\to\infty} L_2(p_n(f_n(t))) = L_2(p(f(t)))$$

since $p_n(x) = p(x)$ for all $x \in \mathbb{R}_+$. Moreover it follows from (2.42) that

$$\lim_{n\to\infty} L_3(r_n(f_n(t))) = L_3(r(f(t))) .$$

By the bounded convergence theorem

$$\lim_{n\to\infty} v_{r_n p_n}(x) = v_{rp}(x)$$

since by definition

$$v_{r_n p_n}(x) = \int_0^\infty e^{-\alpha t}[L_1(f_n(t)) + L_2(p_n(f_n(t))) + L_3(r_n)(f_n(t)))]dt$$

and

$$v_{rp}(x) = \int_0^\infty e^{-\alpha t}[L_1(f(t)) + L_2(p(f(t))) + L_3(r(f(t)))]dt .$$

By Theorem (2.1), $v_n(x) \geq v_{r_n p_n}(x)$ which implies

$$v(x) = \lim_{n\to\infty} v_n(x) \geq \lim_{n\to\infty} v_{r_n p_n}(x) = v_{rp}(x). \tag{2.43}$$

Now take $r(x) = p(x)$ for all $x \in \mathbf{R}_+$ which implies that $f(t) = x$ for all $t \geq 0$. So

$$v_{rp}(x) = \int_0^\infty e^{-\alpha t}[L_1(x) + L_2(p(x)) + L_3(p(x))]dt$$

$$= \frac{1}{\alpha}[L_1(x) + L_2(p(x)) + L_3(p(x))]$$

for all $x \in \mathbb{R}_+$. Then it follows from (2.43) that

$$\alpha v(x) \geq L_1(x) + L_2(p(x)) + L_3(p(x)) \,. \Delta$$

As Proposition (2.1) reveals, we can be assured of the optimality of the limiting function and the corresponding controls in $\mathcal{M}_\infty$ only. In general it is difficult to find controls $(r,p) \in \mathcal{M}_\infty$ such that $v = v_{rp}$. This problem can be solved by imposing some monotonicity requirements upon L_1, which has been taken to be quite general so far, and will be discussed in detail in Section 3.

6.3 CONSTRUCTION OF GLOBALLY OPTIMAL CONTROLS

In this section we show that there exists a unique return function satisfying the sufficient condition of Corollary (1.1) under some additional restrictions imposed upon L_1. Proposition (2.1) shows that the limiting return function is not guaranteed to satisfy the sufficient condition of optimality in $\mathcal{M}$ if L_1 is taken to be simply Lipschitz continuous. To overcome this problem, we assume that L_1 is a decreasing function and then verify the existence and uniqueness of the optimal return function and the optimal control pair in $\mathcal{M}$. We first analyze what further properties are satisfied by v_n and (r_n,p_n) under the assumption that L_1 is decreasing.

<u>COROLLARY (3.1).</u> For any $n \in \mathbb{N}_+$, let v_n be the optimal return function as given by Theorem (2.1) and $(r_n,p_n) \in \mathcal{M}_n$ be the corresponding optimal controls. If L_1 is decreasing,

i. $\alpha v_n(x) \geq L_1(x) + L_2(p_n(x)) + L_3(p_n(x))$ for $x \in \mathbb{R}_+$;

ii. $(1/\overline{r})[\underline{L}_1 - \overline{L}_1] \leq v_n'(x) \leq L_3'(p_n(x)) \leq L_3'(0)$ for $x \in \mathbb{R}_+$;

iii. r_n is increasing and p_n is decreasing.

Proof. (i) Let (r_n, p_n) be the optimal controls determined by Remark (2.3). It is clear that the assertion is true for $x = 0$ since $r_n(0) = p_n(0) = p(0)$ where $p(0)$ is as defined by (2.14) and $f_n(t) = x$ for all $t \geq 0$ where f_n is the unique solution of

$$f_n(t) = x + \int_0^t (p_n - r_n)(f_n(s))ds \quad , \quad t \geq 0 .$$

For fixed $x_0 \in \mathbb{R}_0$, define

$$\hat{r}_n(x) = \begin{cases} p_n(x_0) + (1/n) & \text{if} \quad x > 0 \\ p_n(x_0) & \text{if} \quad x = 0 \end{cases} \tag{3.1}$$

and

$$\hat{p}_n(x) = p_n(x) \quad , \quad x \geq 0.$$

It is obvious that $(\hat{r}_n, \hat{p}_n) \in M_n$ and by Theorem (2.1)

$$v_n(x_0) \geq \int_0^{nx_0} e^{-\alpha t}[L_1(f_n(t)) + L_2(\hat{p}_n(f_n(t))) + L_3(\hat{r}_n(f_n(t)))]dt$$

$$+ \int_{nx_0}^{\infty} e^{-\alpha t}[L_1(f_n(t)) + L_2(\hat{p}_n(f_n(t))) + L_3(\hat{r}_n(f_n(t)))]dt$$

for every $n \in \mathbb{N}_+$ and $x_0 \in \mathbb{R}_0$. It follows from the definition of $(\hat{r}_n, \hat{p}_n)$ that

$$f_n(t) = \begin{cases} x_o-(1/n)t & \text{if} \quad t \le nx_o \\ 0 & \text{if} \quad t > nx_o \end{cases}$$

Therefore,

$$v_n(x_o) \ge \int_0^{nx_o} e^{-\alpha t}[L_1(x_o - (1/n)t) + L_2(p_n(x_o)) + L_3(p_n(x_o) + (1/n))]dt$$

$$+ \int_{nx_o}^{\infty} e^{-\alpha t}[L_1(0) + L_2(p_n(x_o)) + L_3(p_n(x_o))]dt$$

$$\ge \frac{L_1(x_o)}{\alpha} + \frac{L_2(p_n(x_o))}{\alpha} + \frac{L_3(p_n(x_o))}{\alpha}$$

since L_1 is decreasing and L_3 is increasing. This completes the proof of (i) since x_o is arbitrary.

ii) Note that

$$v_n'(x) = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v_n(x)]\}$$

$$\le \sup_{R\varepsilon[p_n(x)+(1/n),\overline{r}]} \{\frac{1}{R - p_n(x)} [L_3(R) - L_3(p_n(x))]\}$$

$$= L_3'(p_n(x)) \le L_3'(0)$$

by part (i) of the Corollary and the fact that L_3' is decreasing. Also by Corollary (2.1),

$$v_n'(x) \ge \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \overline{L}_1 - L_2(0) - L_3(\overline{r})]\}.$$

Letting P = 0 and R = $\overline{r}$, we obtain

$$v_n'(x) \geq \frac{1}{r}\,[\underline{L}_1 - \overline{L}_1]$$

for every $x \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$. So for every $n \in \mathbb{N}_+$, $||v_n'|| \leq C$ for some constant C defined by

$$C = \max\{|L_3'(0)|, \quad 1/\overline{r}|\underline{L}_1 - \overline{L}_1|\} > 0.$$

(ii) If v_n is increasing on some interval $[0,\overline{x}_n]$ where $\overline{x}_n$ is defined by

$$\overline{x}_n = \inf\{x \in \mathbb{R}_+ : v_n'(u) \leq 0 \quad \text{for all } u \geq x\}, \tag{3.2}$$

then v_n must be concave on $[0,\overline{x}_n]$. To verify this it suffices to show for $x \in \mathbb{R}_+$

$$v_n'(x) = \sup_{u \in [x,\overline{x}_n]} v_n'(u) .$$

Now assume that there exists some $x < \hat{x} < \overline{x}_n$ such that $v_n'(\hat{x}) \geq v_n'(u)$ for all $u \in [x,\overline{x}_n]$ which implies that

$$\sup_{\substack{P\in[0,\overline{p}]\\ R\in[P+(1/n),\overline{r}]}}\{L_2(P) + L_3(R) + (P - R)v_n'(\hat{x})\} \leq \sup_{\substack{P\in[0,\overline{p}]\\ R\in[P+(1/n),\overline{r}]}}\{L_2(P) + L_3(R) + (P-R)v_n'(u)\}.$$

Since $L_1(\hat{x}) \leq L_1(u)$ for $u \in [x,\hat{x}]$,

$$\sup_{\substack{P\in[0,\overline{p}]\\ R\in[P+(1/n),\overline{r}]}}\{L_2(P) + L_3(R) + (P-R)v_n'(\hat{x})\} + L_1(\hat{x})$$

$$\leq \sup_{\substack{P\in[0,\overline{p}]\\ R\in[P+(1/n),\overline{r}]}}\{L_2(P) + L_3(R) + (P-R)v_n'(u)\} + L_1(u).$$

So $\alpha v_n(\hat{x}) \leq \alpha v_n(u)$ which contradicts the fact that v_n is increasing on $[x,\overline{x}_n]$; hence v_n must be concave increasing on $[0,\overline{x}_n]$ for some $\overline{x}_n \geq 0$ possibly infinite and decreasing on $[\overline{x}_n,\infty)$. Furthermore by the concave increasing property of v_n on $[0,\overline{x}_n]$ v_n' is decreasing and positive on $[0,\overline{x}_n]$. Thus the characterization of (r_n,p_n) as given by Remark (2.3) and Lemma (2.4) implies that as the content level of the store increases r_n increases and approaches its upper limit $\overline{r}$ while p_n decreases and approaches its lower limit zero.

Moreover (r_n,p_n) is continuous on $\mathbb{R}_0^2$ by the strict concavity of both L_2 and L_3 and the continuity of L_1. It follows from (2.19) that

$$\alpha v_n(\overline{x}) = L_2(0) + L_3(\overline{r}) + L_1(\overline{x}_n)$$

since $v_n'(\overline{x}_n) = 0$. It is further obvious that for all $x \geq x_n$

$$\alpha v_n(x) \geq L_2(0) + L_3(\overline{r}) + L_1(x).$$

This argument revals that $(r_n,p_n) \in M_n$, and r_n is increasing and p_n is decreasing as functions of the content level . $\Box$

REMARK (3.1). Note that the same characterization could be made even if L_1 were assumed to be simply continuous and decreasing, but not Lipschitz. Although (r_n,p_n) would not turn out to be Lipschitz, by Proposition (IV.2.1) and Remark (IV.2.1) their admissibility would be preserved since r_n would still be increasing and p_n would still be decreasing. $\Box$

We now aim at analyzing the limiting function v by acquiring a detailed insight into how the sequence $\{v_n\}$ is proceeding. In order to achieve this, the behaviour of (r_n,p_n) should be studied in more detail. So define

$$\hat{x}_n = \sup\{x \in \mathbb{R}_+;\ r_n(x) = p_n(x) + (1/n)\} \tag{3.3}$$

for all $n \in \mathbb{N}_+$. Corollary (3.1) and the fact that $v_n'(x) \geq v_n'(\hat{x}_n)$ for $x \leq \hat{x}_n$ imply that as the content level decreases, r_n decreases, p_n increases until $\hat{x}_n$ is reached, and then both stay at respective levels satisfying $r_n(\hat{x}_n) = p_n(\hat{x}_n) + (1/n)$ for all $x \in (0,\hat{x}_n]$. An important result on the behaviour of (r_n,p_n) is stated by Lemma (3.1).

LEMMA (3.1). For fixed $n \in \mathbb{N}_+$ and $x \in (0,\hat{x}_n]$, (r_n,p_n) satisfies:

i. $p_n(x) \leq p_n(0) \leq p_n(x)+(1/n)$;

ii. $p_n(0) = p(0) \leq r_n(x)$.

Proof. As it is given by (2.14), p(0) is found by solving

$$L_2'(p(0) + L_3'(0)) = 0 , \tag{3.4}$$

and $p_n(0) = p(0)$ for all $n \in \mathbb{N}_+$. Note that for all $x \in (0,\hat{x}_n]$

$$r_n(x) = p_n(x) + (1/n) \tag{3.5}$$

and

$$\sup_{P\in[0,\bar{p}]} \{L_2(P) + L_3(P+(1/n)) + (1/n)v_n'(x)\} + L_1(x) - \alpha v_n(x) = 0 \tag{3.6}$$

which implies that

$$-L_2'(p_n(x) = L_3'(p_n(x) + (1/n)) , \quad x \in (0,\hat{x}_n] \tag{3.7}$$

where $p_n(x) = p_n(0+)$ on $(0,\hat{x}_n]$.

i. By the concavity of L_3, it follows from (3.4)

$$L_2'(p(0)) + L_3'(p(0) + (1/n)) \leq 0$$

which together with (3.7) implies that $p(0) \geq p_n(x)$. On the other hand the concavity of L_2 implies that

$$L_2'(p(0)) - (1/n)) + L_3'(p(0)) \geq 0 ,$$

so $p(0) \leq p_n(x)+(1/n)$ by (3.7);

ii. This follows directly from (i) by noting that $r_n(x) = p_n(x)+(1/n)$.

By Lemma (3.1) the definition of $\hat{x}_n$ can be restated as

$$\hat{x}_n = \sup\{x \in \mathbb{R}_+ : v_n'(x) = L_3'(p_n(x)+(1/n)) = -L_2'(p_n(x))\}.$$

We are now in a position to provide a pictorial description of the relationship between r_n and p_n in M_n. Note that both r_n and p_n are discontinuous only at x = 0 in Figure (3.1) which depicts the observations made by Corollary (3.1) and Lemma (3.1).

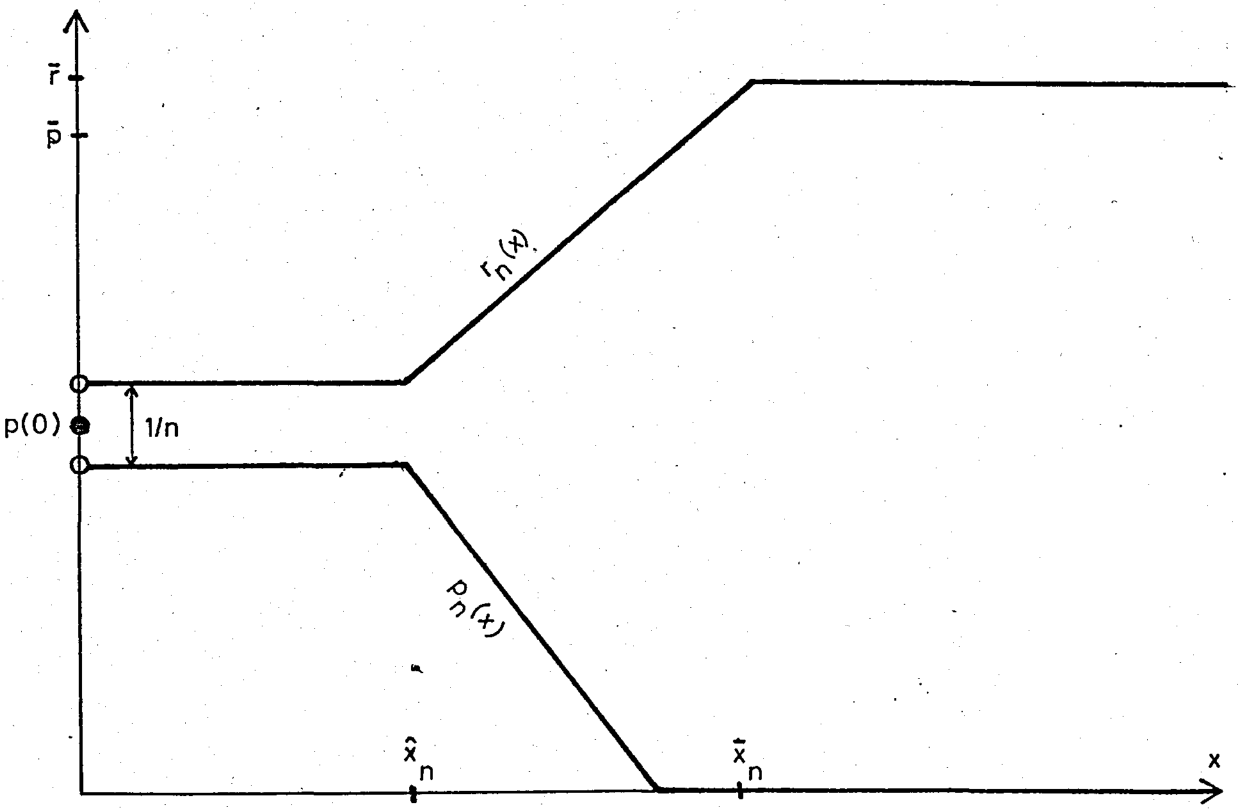

FIGURE 3.1 - Optimal Control pair in M_n $(r_n(.),p_n(.))$.

It remains to study the limiting behaviour of $(r_n(x),p_n(x))$ for $x \in \mathbb{R}_+$, so for $m \geq n$ define

$$x_{mn} = \inf\{x \in \mathbb{R}_+ : r_m(x) = p_m(x)+(1/n)\} \tag{3.8}$$

which is the content level at which the difference between r_m and p_m becomes exactly $1/n$ for the first time.

The following result clarifies the relationship between v_m, v_n and (r_m,p_m), (r_n,p_n).

<u>LEMMA (3.2)</u>. For $m,n \in \mathbb{N}_+$ and $m \geq n$,

i. $v_m'(x) \leq v_n'(x)$ for $x \in [x_{mn},\infty)$;

ii. $r_m(x) \geq r_n(x)$ and $p_m(x) \leq p_n(x)$ for $x \in [x_{mn},\infty)$;

iii. $r_m(x) \leq r_n(x)$ and $p_m(x) \geq p_n(x)$ for $x \in (0,x_{mn}]$.

<u>Proof.</u> (i) Recall that for $x \in \mathbb{R}_+$

$$v_m'(x) = \sup_{\substack{P\in[0,\bar{p}] \\ R\in[P+(1/m),\bar{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v_m(x)\}.$$

On $[x_{mn},\infty)$ $r_m(x) \geq p_m(x)+(1/n)$ by the definition of x_{mn}. So it follows that for $m,n \in \mathbb{N}_+$ and $m \geq n$

$$\begin{aligned} v_m'(x) &= \sup_{\substack{P\in[0,\bar{p}] \\ R\in[P+(1/n),\bar{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v_m(x)]\} \\ &\leq \sup_{\substack{P\in[0,\bar{p}] \\ R\in[P+(1/n),\bar{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v_n(x)]\} = v_n'(x) \end{aligned}$$

since $v_m(x) \geq v_n(x)$ for $m \geq n$ by Proposition (2.1);

ii) follows immediately from Condition (i) of the Lemma and Lemma (2.4);

iii) Since $r_m(x) \geq r_n(x)$ on $[x_{mn}, \ )$ by (ii), $r_m(x_{mn}) \geq r_n(x)$ for $x \leq x_{mn}$; furthermore since $p_m(x) \leq p_n(x)$ on $[x_{mn},\infty)$ by (ii) and $r_m(x_{mn}) = p_m(x_{mn}) + (1/n)$ by (3.8), $p_m(x_{mn}) \leq p_n(x)$ for $x \leq x_{mn}$. Then it follows from $r_n(x) \geq p_n(x) + (1/n)$ that

$$1/n = r_m(x_{mn}) - p_m(x_{mn}) \geq r_n(x) - p_n(x) \geq 1/n$$

which implies that

$$r_m(x_{mn}) = r_n(x_{mn}) \text{ and } p_m(x_{mn}) = p_n(x_{mn}).$$

So

$$r_m(x_{mn}) = p_m(x_{mn}) + (1/n) \quad \text{and} \quad r_n(x_{mn}) = p_n(x_{mn}) + (1/n) .$$

Then $r_n(x) = r_n(x_{mn})$ and $p_n(x) = p_n(x_{mn})$ for all $x \in (0,x_{mn}]$. Also $r_m(x_{mn}) \geq r_m(x)$ and $p_m(x_{mn}) \leq p_m(x)$ for $x \in (0,x_{mn}]$ which follows from (iii) of Corollary (3.1). Then the desired result follows directly. Δ

Figure (3.2) given below depicts the argument of Lemmas (3.1) and (3.2) for (r_n,p_n) and (r_m,p_m), illustrating the relationship between them.

By Figure (3.2) it is apparent that the sequence $\{\hat{x}_n\}$ is decreasing, and its limit exists and is defined by

$$\hat{x} = \lim_{n\to\infty} \hat{x}_n \ . \qquad (3.9)$$

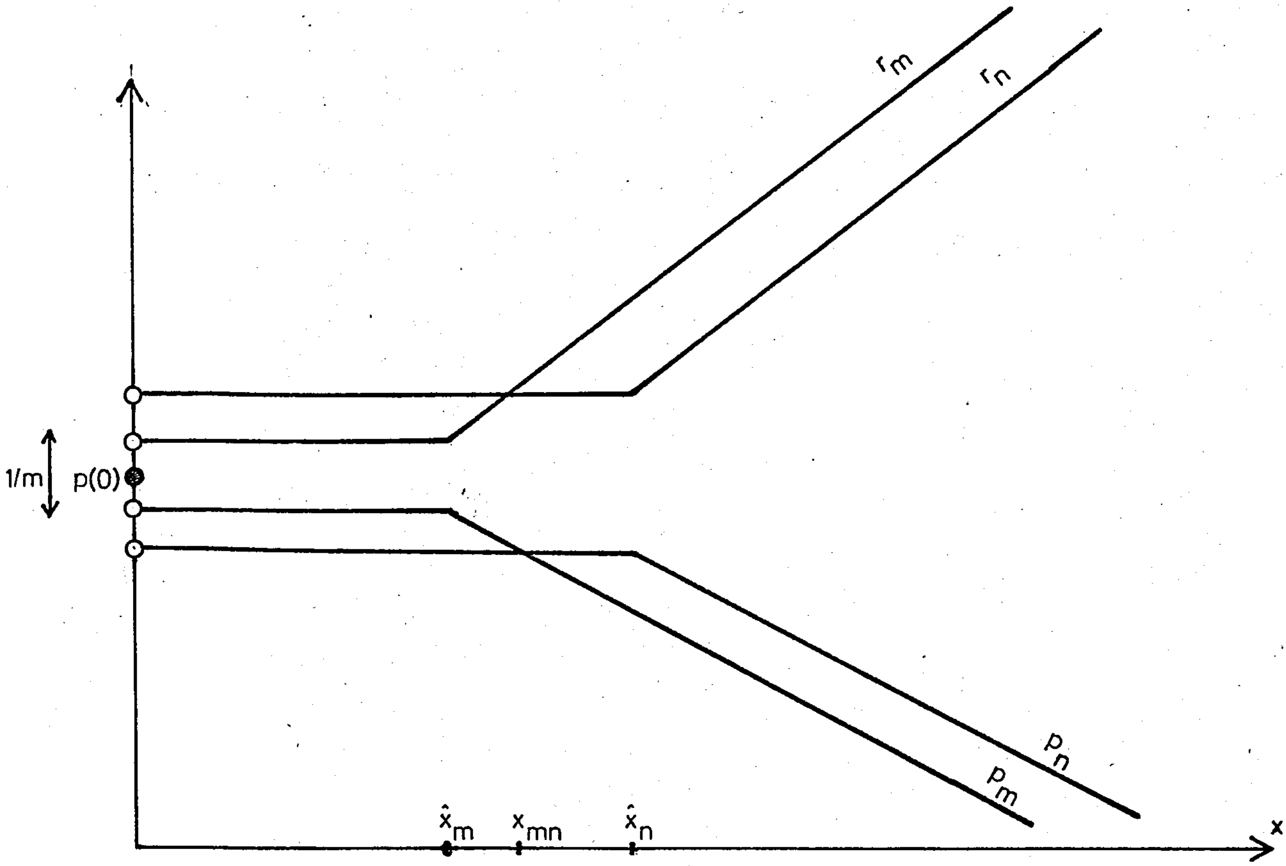

FIGURE 3.2 - The relationship between (r_n, p_n) and (r_m, p_m) for $m \geq n$.

We are now in a position to show that the limiting function v is the one we are seeking for.

THEOREM (3.1). If L_1 is decreasing, then

i) v is the unique function satisfying the sufficiency condition of Corollary (1.1);

ii) There exist unique optimal controls (r^*, p^*) such that $v = v_{r^*p^*}$. Furthermore (r^*, p^*) $\mathcal{M}_\infty$, and r^* is increasing while p^* is decreasing

Proof. (i) First it is obvious that v is bounded since v_n is bounded by (i) of Corollary (2.1) for every $n \in \mathbb{N}_+$. Next it is necessary to show that v is continuous on R_+.

Note that $v_m(0) = v_n(0)$ and $v_m(\infty) = v_n(\infty)$ for every $m \geq n$ implies that there exists some $0 < \bar{x} < \infty$ such that

$$||v_m - v_n|| = v_m(\bar{x}) - v_n(\bar{x}) \text{ with } v_m'(\bar{x}) = v_n'(\bar{x}).$$

Then it follows that

$$0 \leq \alpha[v_m(\bar{x}) - v_n(\bar{x})] = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/m),\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v_n'(\bar{x})\}$$

$$- \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v_n'(\bar{x})\}.$$

Recall the definition of(r,p) given by (2.37) and consider the three possible cases:

1) $\hat{r}-\hat{p} > 1/n$ implies that $\hat{r}-\hat{p} > 1/m$. So $(r_m,p_m) = (r_n,p_n) = (\hat{r},\hat{p})$. Therefore, $\alpha[v_m(\bar{x}) - v_n(\bar{x})] = 0$;

2) $\hat{r}-\hat{p} > 1/m$ and $\hat{r}-\hat{p} < 1/n$ implies that $(r_m,p_m) = (\hat{r},\hat{p})$ and $r_n = p_n+(1/n)$. So $\alpha[v_m(\bar{x})-v_n(\bar{x})] = L_2(p_m) + L_3(r_m) + (p_m-r_m)v_n'(\bar{x})$

$$- L_2(p_n) - L_3(r_n) - (p_n-r_n)v_n'(\bar{x})$$

$$= L_2(p_m) - L_2(p_n) + L_3(r_m) - L_3(r_n) + (p_m-r_m)v_n'(\bar{x}) + (1/n)v_n'(\bar{x})$$

$$\leq L_2(p_m) - L_2(p_n) + L_3(r_m) - L_3(r_n) + [\frac{1}{n} - \frac{1}{m}]v_n'(\bar{x})$$

$$\leq [\frac{1}{n} - \frac{1}{m}]v_n'(\bar{x})$$

where the last statement follows from (2.38) and the fact that L_2 is decreasing and L_3 is increasing;

3) $\hat{r}-\hat{p} < 1/m$ implies that $\hat{r}-\hat{p} < 1/n$, $r_m = p_m + (1/m)$ and $r_n = p_n + (1/n)$. It is obvious that $r_n > r_m$ and $p_n < p_m$. Similarly we obtain

$$\alpha[v_m(\bar{x}) - v_n(\bar{x})] = L_2(p_m) - L_2(p_n) + L_3(r_m) - L_3(r_n) + [\frac{1}{n} - \frac{1}{m}]v_n'(\bar{x})$$
$$\leq [\frac{1}{n} - \frac{1}{m}]v_n'(x).$$

Then it follows from Corollary (3.1) that

$$\alpha||v_m - v_n|| \leq [\frac{1}{n} + \frac{1}{m}]C$$

which implies that $\{v_n\}$ converges uniformly to v. This in turn implies v is continuous.

Now it remains to show the continuous differentiability of v on R_+. Define for $x \varepsilon \mathbb{R}_+$

$$f(x) = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P,\bar{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v(x)]\}. \qquad (3.10)$$

Let us assume for now that $\hat{x} = 0$ and prove the results under this assumption. We shall later show that this is in fact true to complete the proof. For any $x \varepsilon \mathbb{R}_o$ we can find an N_x given by

$$N_x = \inf\{n: \hat{x}_n < x\}$$

so that $r_n(x) > p_n(x)+(1/n)$ for $n \geq N_x$. Therefore

$$v_n'(x) = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{ \frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v_n(x)]\}$$

$$= \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P,\bar{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha v_n(x)]\}$$

for $n \varepsilon N_x$. Since $v_n \uparrow v$ by Proposition (2.1), $v_n' \to f$ pointwise as $n \to \infty$. By Theorem (2.1) for every $n \varepsilon \mathbb{N}_+$

$$v_n(x) = v_n(x_1) + \int_{x_1}^{x} v_n'(s)ds .$$

Hence by the bounded convergence theorem

$$v(x) = v(x_1) + \int_{x_1}^{x} f(s)ds$$

which implies that v is differentiable with a bounded derivative $v' = f$ on R_0. The boundedness of v' follows from the fact $v_n' \downarrow v'$ and $||v_n'|| \leq C$ for all $n \varepsilon \mathbb{N}_+$ and $x \varepsilon \mathbb{R}_+$.

Now define for $x \varepsilon \mathbb{R}_0$

$$p(x) = \lim_{n\to\infty} p_n(x) \tag{3.11}$$

and

$$r(x) = \lim_{n\to\infty} r_n(x) . \tag{3.12}$$

Note that these limits exist since $\{p_n\}$ is a decreasing sequence and $\{r_n\}$ is an increasing sequence. Furthermore $r(x) > p(x)$ since for all $n \geq N_x$ $r_n(x) > p_n(x) + (1/n)$.

It suffices to show that the limits given by (3.11) and (3.12) are the solution to (3.10). If (r_n, p_n) is the optimal control pair, then

$$\alpha v_n(x) = L_1(x) + L_2(p_n(x)) + L_3(r_n(x)) + [p_n(x) - r_n(x)]v_n'(x) \tag{3.13}$$

which in the limit converges to

$$\alpha v(x) = L_1(x) + L_2(p(x)) + L_3(r(x)) + [p(x) - r(x)]v'(x) \tag{3.14}$$

since v_n converges uniformly to v and v_n' converges pointwise to v'.

On the other hand, for $n \geq N_x$

$$\alpha v_n(x) = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v_n'(x)\} + L_1(x) \tag{3.15}$$

which implies that as $n \to \infty$

$$\alpha v(x) = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\} + L_1(x) . \tag{3.16}$$

Since the assertions made by (3.14) and (3.16) are equivalent, the limiting control pair (r,p) as given by (3.11) and (3.12) respectively solves (3.10).

Furthermore it follows from Corollary (3.1) that

$$\alpha v(x) \geq L_1(x) + L_2(p(x)) + L_3(p(x)) \tag{3.17}$$

where p(x) is the solution to (3.10). So

$$v'(x) \leq L_3'(p(x)) . \tag{3.18}$$

Note that for any $x \varepsilon \mathbb{R}_0$ there is some $r(x) > p(x)$ which maximizes the right hand side of (3.10). Thus for any u_1, $u_2 \varepsilon \mathbb{R}_0$

$$|v'(u_1)-v'(u_2)| \leq \sup_{\substack{P\ [0,\bar{p}] \\ R\ [P,\bar{r}]}} \left|\frac{1}{R - P} [L_1(u_1) - L_1(u_2) - \alpha v(u_1) + \alpha v(u_2)]\right|$$

$$\leq \frac{1}{r(u_1)\Lambda r(u_2)-p(u_1)Vp(u_2)}[|L_1(u_1)-L_1(u_2)|+\alpha|v(u_1)-v(u_2)|]$$

which shows the continuity of v' on R_0. The behaviour of v' near zero level is important in our analysis. We first show that p is continuous at zero. This follows by noting that for $x \in (0,\hat{x}_n]$ and any $n \in \mathbb{N}_+$

$$p(0) - (1/n) \leq p_n(x) \leq p(0)$$

by Lemma (3.1). So

$$\lim_{x\downarrow 0} \lim_{n\to\infty}[p(0) - (1/n)] \leq \lim_{x\downarrow 0} \lim_{n\to\infty} p_n(x) \leq \lim_{x\downarrow 0} \lim_{n\to\infty} p(0)$$

which implies that

$$\lim_{x\downarrow 0} p(x) = p(0) .$$

We will now show that $v'(0) \equiv \lim_{u\downarrow 0} v'(u) = L_3'(p(0))$. Let $\{u_n\} \subset \mathbb{R}_+$ with $u_n \downarrow 0$ and recall that $\alpha v(0) = \alpha v_n(0) = L_1(0) + L_2(p(0)) + L_3(p(0))$. For any $u_n > 0$ and $m \in \mathbb{N}_+$,

$$\begin{aligned} v'(u_n) &= \sup_{\substack{P\in[0,\bar{p}] \\ R\in[P,\bar{r}]}} \left\{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(u_n) - \alpha v(u_n)]\right\} \\ &\geq \sup_{\substack{P\in[0,\bar{p}] \\ R\in[P+(1/m),\bar{r}]}} \left\{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(u_n) - \alpha v(u_n)]\right\} \\ &\geq m[L_2(p(u_n)) + L_3(p(u_n) + (1/m)) + L_1(u_n) - \alpha v(u_n)] . \end{aligned}$$

Thus,

$$\begin{aligned} &\liminf_{n\to\infty} m[L_2(p(u_n)) + L_3(p(u_n)+(1/m)) + L_1(u_n) - \alpha v(u_n)] \\ &\qquad \leq \liminf_{n\to\infty} v'(u_n) \\ &\qquad \leq \limsup_{n\to\infty} v'(u_n) \leq L_3'(p(0)) \end{aligned}$$

since p is continuous at zero. Then the continuity of L_1 and v implies that

$$m[L_3(p(0)+(1/m)) - L_3(p(0))] \leq \operatorname*{Lim\,inf}_{n\to\infty} v'(u_n) \leq \operatorname*{Lim\,sup}_{n\to\infty} v'(u_n) \leq L_3'(p(0))$$

for every $m \in \mathbb{N}_+$. Taking the limit as $m \uparrow \infty$, we obtain

$$\operatorname*{Lim}_{m\to\infty} m[L_3(p(0)+(1/m)) - L_3(p(0))] = L_3'(p(0)) .$$

So we have

$$\operatorname*{Lim}_{n\to\infty} v'(u_n) = L_3'(p(0)) .$$

Since v has a bounded continuous derivative v' given by

$$v'(x) = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P,\overline{r}]}} \{\frac{1}{R-P}[L_2(P) + L_3(R) + L_1(x) - \alpha v(x)]\} ,$$

then it satisfies

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P,\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\} + L_1(x) - \alpha v(x) = 0 , \quad x > 0.$$

Furthermore it follows from $v'(x) \leq L_3'(p(x))$ for all $x > 0$ that the optimal $r(x)$ occurs in the interval $[p(x),\overline{r}]$ since L_3 is a concave function. Thus the optimization over $[0,\overline{r}]$ yields the same solution as the optimization over $[p(x),\overline{r}]$, and consequently we have

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P,\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\} = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\}.$$

Also $v'(0) = L_3'(p(0))$ implies that

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R=P}} \{L_2(P) + L_3(R) + (P-R)v'(0)\} = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,p]}} \{L_2(P) + L_3(R) + (P-R)v'(0)\}.$$

So v(0) satisfies

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,p]}} \{L_2(P) + L_3(R) + (P-R)v'(0)\} + L_1(0) - \alpha v(0) = 0. \qquad (3.20)$$

It immediately follows from (3.19) and (3.20) that v satisfies the sufficiency condition of Corollary (1.1).

To complete the proof of (i) we need to show that $\hat{x}$ is in fact equal to zero as assumed before.

If $\hat{x} > 0$, then for any $n \varepsilon \mathbb{N}_+$ and all $x \leq \hat{x}_n$

$$\alpha v_n(x) = L_1(x) + L_2(p_n(x)) + L_3(p_n(x)+(1/n)) + (1/n)v_n'(x)$$

which implies

$$\alpha v(x) = L_1(x) + L_2(p(0)) + L_3(p(0)) \quad , \quad x \varepsilon [0,\hat{x}_n]$$

by taking the limit since $||v_n'||$ is bounded independent of n. But this contradicts the fact that $v \geq v_n$ for any n since by taking $(\hat{r}_n,\hat{p}_n) \varepsilon \mathcal{M}_n$ as

$$\hat{r}_n(x) = \begin{cases} p(0)+(1/n) & , \quad x > 0 \\ p(0) & , \quad x = 0 \end{cases}$$

and

$$\hat{p}_n(x) = p(0) \qquad , \quad x \geq 0 ,$$

one can easily see that

$$v_n(x) \geq v_{\hat{r}_n\hat{p}_n}(x) = \int_0^{nt} e^{-\alpha t}[L_1(x-(1/n)t) + L_2(p(0)) + L_3(p(0)+(1/n))]dt$$

$$+ \int_{nt}^{\infty} e^{-\alpha t}[L_1(0) + L_2(p(0)) + L_3(p(0))]dt$$

$$> \frac{1}{\alpha}[L_1(x) + L_2(p(0)) + L_3(p(0))] = v(x).$$

Now there only remains to show the uniqueness of v. Assume there exists another function u satisfying the sufficiency condition of Corollary (1.1), so that $u \geq v$. Then for every $x \in \mathbb{R}_+$

$$\sup_{s\varepsilon[0,x]} \{u(s) - v(s)\} = u(x) - v(x) \geq 0 \tag{3.21}$$

which amounts to saying that $u(x) - v(x)$ is increasing in x and $(u-v)' \geq 0$. To see (3.21), assume that this maximum is attained at some $0 < s < x$. Then $u'(s) = v'(s)$ which implies $u(s) - v(s) = 0 \leq u(x) - v(x)$.

Now assume that $u(\bar{x}) > v(\bar{x})$ for some $\bar{x} > 0$. Then for $x \geq \bar{x}$ $u'(x) < L_3'(p(x))$ implies

$$\begin{aligned}\alpha[u(x)-v(x)] &= \sup_{\substack{P\varepsilon[0,\bar{p}]\\ R\varepsilon[0,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)u'(x)\} \\ &\qquad - \sup_{\substack{P\varepsilon[0,\bar{p}]\\ R\varepsilon[0,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\} \\ &= \sup_{\substack{P\varepsilon[0,\bar{p}]\\ R\varepsilon[P,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)u'(x)\} \\ &\qquad - \sup_{\substack{P\varepsilon[0,\bar{p}]\\ R\varepsilon[P,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\} \\ &\leq 0\end{aligned}$$

Since $u'(x) \geq v'(x)$ by the above argument and by (3.21). So $u'(x) \geq L_3'(p(x))$ for all $x \geq \overline{x}$ which in turn contradicts the boundedness of u, so in fact $u = v$ and v is unique.

ii) We have already defined (r^*,p^*) as the limiting functions by (3.11) and (3.12). This definition and (3.19) imply that (r^*,p^*) satisfies

$$L_2(p^*(x)) + L_3(r^*(x)) + [p^*(x) - r^*(x)]v'(x) = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\}$$

for $x \in \mathbb{R}_+$. Furthermore $v'(x) \leq L_3'(p^*(x))$ implies $r^*(x) \geq p^*(x)$ for $x > 0$ and $v'(0) = L_3'(p(0))$ implies $r^*(0) = p^*(0)$. The strict concavity of L_2 and L_3 ensures us about the uniqueness of (r^*,p^*) for any given v'.

Note that $v'(0) = L_3'(p(0)) > 0$, and the procedure of Corollary (3.1) can be repeated here to show that if v is increasing on $[0,\overline{x}]$ for some $\overline{x} > 0$ given by

$$\overline{x} = \inf\{u \in \mathbb{R}_+ : v'(u) = 0\},$$

then v' is decreasing, so that v is concave on $[0,\overline{x}]$. Furthermore, by our definition of (r^*,p^*), r^* is increasing, p^* is decreasing and $(r^*(x),p^*(x)) = (\overline{r},0)$ for all $x \geq \overline{x}$. So $(r^*,p^*) \in \mathcal{M}_\infty \cap \mathcal{M}_i$. Also $v = v_{r^*p^*}$ which follows by noting that (r^*,p^*) is continuous on R_+ by the strict concavity of L_2 and L_3.

Then it follows from (3.10) that

$$\alpha v(\overline{x}) = L_2(0) + L_3(\overline{r}) + L_1(\overline{x})$$

and

$$\alpha v(x) \geq L_2(0) + L_3(\overline{r}) + L_1(x), \qquad x \geq \overline{x}. \; \Delta$$

REMARK (3.2). Since v is either concave increasing on $\mathbb{R}_+$ or decreasing on $[\overline{x},\infty)$ for $\overline{x} > 0$, $v(\infty) = \lim_{x\to\infty} \alpha v(x)$ exists. By uniform convergence of $\{v_n\}$ to v and (ii) of Corollary (2.1)

$$\alpha v(\infty) = L_2(0) + L_3(\overline{r}) + L_1(\infty). \square$$

So far we accomplished to construct locally and globally optimal controls in the deterministic problem. If L_1 is arbitrary, then for every $n \in \mathbb{N}_+$ there exists a unique optimal control pair (r_n,p_n) in $\mathcal{M}_n \cap \mathcal{M}_\ell$, and the return function v_n is shown to be the unique solution of the sufficiency condition of Corollary (1.2). We later showed that v_n converges to the optimal return function in $\mathcal{M}_\infty$ and established the optimality of the corresponding controls in $\mathcal{M}_\infty$. However it is perceived that the optimality of the limiting function in $\mathcal{M}$ requires further conditions upon L_1. So L_1 is assumed to be decreasing, and v is shown to satisfy the sufficient condition of Corollary (1.1) under this assumption. We then characterized the optimal return function and the optimal control pair explicitly. Note that $r^* \geq p^*$ implies that one has no incentive to increase the content level because of the decreasing property of L_1.

VII. THE STOCHASTIC OPTIMAL CONTROL PROBLEM

This chapter is devoted to an extensive analysis of the original stochastic storage model. The results obtained in the deterministic problem are fully used to verify whether there exists a unique optimal return function $v \varepsilon b(R_+)$ satisfying the sufficient condition of Corollary (V.2.1) and to characterize the optimal controls which yield this return if there exists any. In Section 1 the emphasis is on the construction of suboptimal return function and suboptimal control pair. It is demonstrated that there exists a unique return function and a unique control pair optimal in M_n satisfying Corollary (V.2.2) if L_1 is assumed to be arbitrary. Again the limit of the sequence of suboptimal return functions does not turn out to be optimal in M. So in Section 2 the decreasing requirement is imposed upon L_1; however it is not technically possible to employ our argument of the deterministic problem in the presence of random output jumps. So the stochastic output process B is excluded in our procedure of proving the desired result, and the existence and the uniqueness of globally optimal return function and control pair is proven for a storage model whose randomness arises only from the input process A.

7.1 CONSTRUCTION OF SUBOPTIMAL CONTROLS

Our main concern in this section is to verify that even in the presence of two stochastic processes there exists a unique solution to the functional differential equation characterization of the maximum expected discounted return defined over M_n while the basic assumptions on the cost structure are as given by (V.1.1). In fact, this unique solution is proven to be identical with the maximum return of the unique suboptimal control pair. Such a study is first carried out by MORAIS [28] and later by DESHMUKH and PLISKA [31] who consider the optimal control of nonrenewable resources. Our procedure follows ÖZEKİCİ [38] who studies a similar situation with Markov additive inputs.

Recall that in the stochastic problem for $(r,p) \varepsilon M$ and $x \varepsilon \mathbb{R}_+$ v_{rp} satisfies

$$\hat{\alpha} v_{rp}(x) = L_1(x) + Kv_{rp}(x) + L_2(p(x)) + L_3(r(x)) + [p(x)-r(x)]v'_{rp}(x)$$

where

$$\hat{\alpha} = \alpha + \lambda_a + \lambda_b$$

$$Kv_{rp}(x) = \lambda_a \int_0^\infty v(x+y)G_a(dy) + \lambda_b \int_0^x v(x-y)G_b(dy) + \lambda_b v(0)[1 - G_b(x)].$$

Now it is convenient to define for $x \varepsilon \mathbb{R}_+$

$$\hat{L}_1(v)(x) = L_1(x) + \lambda_a \int_0^\infty v(x+y)G_a(dy) + \lambda_b \int_0^x v(x-y)G_b(dy)$$
$$+ \lambda_b v(0)[1 - G_b(x)] \qquad (1.1)$$

and treat $\hat{L}_1(v)(x)$ as $L_1(x)$ in the deterministic problem. Note that if v is Lipschitz continuous, then $L_1(v)(x)$ is Lipschitz continuous since L_1 is Lipschitz by Assumption (V.1.1) and Kv is Lipschitz continuous by Lemma (V.2.1).

The following result is equivalent to Theorem (VI.2.1).

COROLLARY (1.1). For arbitrary L_1 and every $n \in \mathbb{N}_+$;

i) there exists a unique function $v_n \in b(R_+)$ satisfying the sufficiency condition of Corollary (V.2.2);

ii) furthermore v_n satisfies

$$\lim_{x\to\infty} \alpha v_n(x) = L_2(0) + L_3(\bar{r}) + L_1(\infty)$$

and

$$L_2(\bar{p}) + L_3(0) + \underline{L}_1 \leq \alpha v_n(x) \leq L_2(0) + L_3(\bar{r}) + \bar{L}_1 \quad \text{for} \quad x \in \mathbb{R}_+;$$

iii) there exists a unique optimal control pair $(r_n, p_n) \in M_n$ such that $v_n = v_{r_n p_n}$.

Proof. (i) For fixed $n \in \mathbf{N}_+$ let B be a Banach space with the usual supremum norm and define the mapping Γ on B so that for every $f \in B$, $u(.) = \Gamma(f)(.)$ is the unique solution of

$$\sup_{\substack{P\in[0,\bar{p}] \\ R\in[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)u'(x)\} + \hat{L}_1(f)(x) - \hat{\alpha}u(x) = 0 \qquad (1.2)$$

where $L_1(f)(.)$ is as defined by (1.1). It suffices to show that Γ is a contraction mapping. Let $f_1, f_2 \in B$ and $u_1 = \Gamma(f_1)$ and $u_2 = \Gamma(f_2)$, then

$$\hat{\alpha}[u_1(x)-u_2(x)] = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)u_1'(x)\} \tag{1.3}$$

$$- \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)u_2'(x)\} + \hat{L}_1(f_1)(x) - \hat{L}_1(f_2)(x) \ , \quad x > 0$$

and

$$\hat{\alpha}[u_1(0)-u_2(0)] = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R=P}} \{L_2(P) + L_3(R) + (P-R)u_1'(0)\}$$

$$- \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R=P}} \{L_2(P) + L_3(R) + (P-R)u_2'(0)\} + \hat{L}_1(f_1)(0) - \hat{L}_1(f_2)(0) \ . \tag{1.4}$$

Note that $|u_1(.)-u_2(.)|$ is maximized either at some $\overline{x} > 0$ in which case $u_1'(\overline{x}) = u_2'(\overline{x})$ or at $\overline{x} = 0$ in which case $u_1'(0) \leq u_2'(0)$ assuming without loss of generality that $u_1(0) \geq u_2(0)$. The fact that $\overline{x}$ should be finite follows from the boundedness of u_1 and u_2 so that $u_1(\infty) = u_2(\infty)$. Therefore it follows from (1.3) and (1.4)

$$\hat{\alpha}|u_1(x)-u_2(x)| \leq |\hat{L}_1(f_1)(x) - \hat{L}_1(f_2)(x)|. \tag{1.5}$$

On the other hand, we know that

$$|\hat{L}_1(f_1)(x)-\hat{L}_1(f_2)(x)| = \lambda_a \int_0^\infty [f_1(x+y)-f_2(x+y)]G_a(dy)$$

$$+ \lambda_b \int_0^x [f_1(x-y)-f_2(x-y)]G_b(dy)$$

$$+ \lambda_b[1 - G_b(x)][f_1(0)-f_2(0)]$$

$$\leq \lambda_a||f_1-f_2||+\lambda_b||f_1-f_2||G_b(x)+\lambda_b[1-G_b(x)]||f_1-f_2||$$

$$= (\lambda_a + \lambda_b)||f_1-f_2||$$

which implies that

$$|u_1(x)-u_2(x)| \leq \frac{\lambda_a + \lambda_b}{\hat{\alpha}} ||f_1-f_2||$$

or

$$||u_1-u_2|| \leq \frac{\lambda_a + \lambda_b}{\alpha + \lambda_a + \lambda_b} ||f_1-f_2||.$$

Since $(\lambda_a+\lambda_b)/(\alpha+\lambda_a+\lambda_b) < 1$, Γ is a contraction so that there is a $v^* \varepsilon B$ such that $v^* = \Gamma(v^*)$. This proves the uniqueness of v^* in B. Assume now that $u \notin B$ is another function satisfying the sufficient condition of Corollary (V.2.2) such that $u \geq v^*$.

If $u(.)-v^*(.)$ is maximized at some $\overline{x} \geq 0$, then it follows from (1.2) and $u'(\overline{x}) = v^{*\prime}(\overline{x})$ that

$$\hat{\alpha}[u(\overline{x}) - v^*(\overline{x})] \leq (\lambda_a + \lambda_b)||u-v^*||. \tag{1.6}$$

So

$$||u-v^*|| \leq \frac{\lambda_a + \lambda_b}{\alpha + \lambda_a + \lambda_b} ||u-v^*||$$

which implies that if $||u-v^*|| > 0$, then $||u-v^*|| = \infty$. Then u is unbounded since v^* is bounded; on the other hand the unboundedness of u is a contradiction and consequently $||u-v^*|| = 0$.

Now to see (1.6) note that if $u(.)-v^*(.)$ is maximized at the infinity, then $\sup_{x\varepsilon R_+} \{u(x)-v^*(x)\} = \lim_{n\to\infty}\{u(x_n)-v^*(x_n)\}$ for some sequence $\{x_n\} \subset \mathbb{R}_+$ with $x_n \uparrow \infty$. This limit may be infinite, i.e. $\lim_{n\to\infty}\{u(x_n)-v^*(x_n)\} = \infty$ in which case $||u-v^*|| = \infty$ and the above argument holds true immediately. However if $\lim_{n\to\infty}\{u(x_n)-v^*(x_n)\} < \infty$, we should consider two possible cases:

1. $u'(x_n) = v^{*\prime}(x_n)$ for all n implies from (1.2)

$$\lim_{n\to\infty} \hat{\alpha}\{u(x_n)-v^*(x_n)\} = \hat{L}_1(u)(x_n) - \hat{L}_1(v^*)(x_n)$$

$$\leq (\lambda_a + \lambda_b)||u-v^*||.$$

2. $\lim\{u'(x)-v^*(x)\} = 0$ implies that $v^{*\prime}(x) \leq u'(x) \leq L_3'(p(x))$ for all $x \geq \bar{x}$ for some $\bar{x} \geq 0$. By (1.2) and the fact that

$$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)u'(x)\} - \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)u^{*\prime}(x)\} \leq 0$$

for $x \geq \bar{x}$,

$$\lim_{n\to\infty} \hat{\alpha}\{u(x_n)-v^*(x_n)\} \leq \hat{L}_1(u)(x_n) - \hat{L}_1(v^*)(x_n) \leq (\lambda_a+\lambda_b)||u-v^*||.$$

This proves the uniqueness of v^* in M_n.

ii) Letting v_n be v^* of part (i), it is easy to show that $\hat{L}_1(v_n)(.)$ satisfies

$$\hat{L}_1(v_n)(.) \leq \bar{L}_1 + \frac{\lambda_a + \lambda_b}{\alpha} [L_2(0) + L_3(\bar{r}) + \bar{L}_1]; \qquad (1.7)$$

$$\hat{L}_1(v_n)(.) \geq \underline{L}_1 + \frac{\lambda_a + \lambda_b}{\alpha} [L_2(\bar{p}) + L_3(0) + \underline{L}_1];$$

$$\lim_{x\to\infty} \hat{L}_1(v_n)(x) = L_1(\infty) + \frac{\lambda_a + \lambda_b}{\alpha} [L_2(0) + L_3(\bar{r}) + L_1(\infty)]$$

where the last statement requires $G_b(\infty) = 1$. It follows from (1.7) that $\hat{L}_1$ is bounded and its limit exists. Note that by (i) of the Corollary

$$||\Gamma(v_n)'|| \leq n[||L_2|| + ||L_3|| + ||\hat{L}_1(v_n)|| + \hat{\alpha}||v_n||]$$

$$\leq n[||L_2|| + ||L_3|| + ||L_1|| + (\lambda_a+\lambda_b)||v_n|| + \hat{\alpha}||v_n||]$$

$$\leq \frac{2n}{\alpha}(\alpha+\lambda_a+\lambda_b)[||L_1|| + ||L_2|| + ||L_3||] \equiv M$$

since $\alpha||v_n|| \leq ||L_1|| + ||L_2|| + ||L_3||$. This implies that

$$|v_n(x_1) - v_n(x_2)| \leq M|x_1 - x_2|.$$

Then by Lemma (V.2.1) $\hat{L}_1$ is Lipschitz continuous and by Corollary (VI.2.1) v_n satisfies

$$\hat{\alpha}v_n(\infty) = L_2(0) + L_3(\overline{r}) + \hat{L}_1(v_n)(\infty); \qquad (1.8)$$

$$\hat{\alpha}v_n(\cdot) \geq \underline{\hat{L}}_1(v_n) + L_2(\overline{p}) + L_3(0) ;$$

$$\hat{\alpha}v_n(\cdot) \leq \overline{\hat{L}}_1(v_n) + L_2(0) + L_3(\overline{r}).$$

By some straightforward calculations it is clear that (1.8) implies

$$\alpha v_n(\infty) = L_2(0) + L_3(\overline{r}) + L_1(\infty) ; \qquad (1.9)$$

$$\alpha v_n(\cdot) \geq \underline{L}_1 + L_2(\overline{p}) + L_3(0) ;$$

$$\alpha v_n(\cdot) \leq \overline{L}_1 + L_2(0) + L_3(\overline{r}).$$

iii) Define (r_n,p_n) by Remark (VI.2v3) as we have done in the deterministic problem. By the Lipschitz property of $\hat{L}_1$, (r_n,p_n) are Lipschitz continuous as before so that $(r_n,p_n) \in \mathcal{M}_L$. Thus we have $v_n = v_{r_n p_n}$. The uniqueness of (r_n,p_n) follows from the uniqueness of v_n and the strict concavity of L_2 and L_3. ◻

Corollary (1.1) enables us to create a sequence $\{v_n\}$ of return functions locally optimal in M_n and to characterize the corresponding controls (r_n,p_n). Meanwhile we observe that the input and output processes of the stochastic problem do not affect the basic features of our construction of the suboptimal controls in M_n.

Letting v_n be the optimal return function in M_n and (r_n,p_n) be the corresponding optimal control pair in M_n, it is obvious that $v_{n+1} \geq v_n$ since $M_{n+1} \subset M_n$ for every $n \in \mathbb{N}_+$.

Let $v = \lim_{n\to\infty} v_n$. It is clear that

$$L_2(\overline{p}) + L_3(0) + \underline{L}_1 \leq \alpha v(x) \leq L_2(0) + L_3(\overline{r}) + \overline{L}_1 .$$

<u>PROPOSITION (1.1).</u> $v \geq v_{rp}$ for every $(r,p) \in M_\infty$.

<u>Proof.</u> Let $(r,p) \in M_\infty$ and define

$$r_n(x) = \begin{cases} p(x) & \text{if} \quad x = 0 \\ p(x)+(1/n) & \text{if} \quad x > 0, \ r(x) \leq p(x)+(1/n) \\ r(x) & \text{if} \quad x > 0, \ r(x) > p(x)+(1/n) \end{cases}$$

and

$$p_n(x) = p(x) \quad , \quad x \geq 0$$

for every $n \in \mathbb{N}_+$. Obviously $(r_n,p_n) \in M_n$. We can prove the desired result by using the same argument as in Proposition (VI.2.1) and showing that for fixed $x \in \mathbb{R}_+$ and almost every $\omega \in \Omega$ $f_n \uparrow f$ where f_n and f are the unique solutions of

$$f_n(t) = x + A_t(\omega) - B_t(\omega) + \int_0^t (p_n-r_n)(f_n(s))ds \quad , \quad t \geq 0$$

$$f(t) = x + A_t(\omega) - B_t(\omega) + \int_0^t (p-r)(f(s))ds \quad , \qquad t \geq 0$$

respectively. Δ

Proposition (1.1) states that the sequence $\{v_n\}$ converges to the optimal return function in M_∞. However, in order to guarantee that the limiting function is optimal in M, we need to place some monotonicity restrictions upon L_1. Thus, as we have done in the deterministic case, L_1 is taken to be decreasing to see whether the limiting return function satisfies the sufficient optimality condition of Corollary (V.2.1).

7.2 CONSTRUCTION OF GLOBALLY OPTIMAL CONTROLS

In our procedure of showing that there exists a unique return function optimal in M satisfying the sufficient condition of Corollary (V.2.1) under the additional assumption that L_1 is decreasing, we realize that the generalization of the results obtained in the deterministic control problem cannot be done immediately and a different analysis is required in the presence of jump outputs which decrease the content level of the store randomly. So in order not to digress from the proposed solution approach, we restrict our attention to a storage process where there are only random input jumps and there does not exist any form of random output in proving the following result.

<u>THEOREM (2.1).</u> If L_1 is decreasing and there does not exist a random output process, then

i) v is the unique function satisfying the sufficiency condition of Corollary (V.2.1);

ii) there exists a unique optimal control pair (r^*,p^*) such that $v = v_{r^*p^*}$. Furthermore $(r^*,p^*) \varepsilon M_\infty$ and r^* is increasing while p^* is decreasing.

Proof. Note that this corresponds to the case where $\lambda_b = 0$ and there are no random outputs. Therefore, the function $\hat{L}_1(v_n)$ is now equal to

$$\hat{L}_1(v_n)(x) = L_1(x) + \lambda_a \int_0^\infty v_n(x+y)G_a(dy). \qquad (2.1)$$

i) We will first show that $v_n(.)$ is concave increasing on $[0,\overline{x}_n]$ and decreasing on $[\overline{x}_n,\infty)$ where $\overline{x}_n$ is given by

$$\overline{x}_n = \inf\{x \varepsilon \mathbb{R}_+;\ v_n'(u) \leq 0 \text{ for all } u \geq x\}.$$

To do this, it is necessary to show that for any $0 \leq x \leq \overline{x}_n$

$$v_n'(x) = \sup_{u\varepsilon[x,\overline{x}_n]} v_n'(u) > 0 .$$

By the definition of $\overline{x}_n$ this supremum is greater than zero. Now assume there exists some $x < \hat{x} < \overline{x}_n$ such that $v_n'(\hat{x}) \geq v_n'(u)$ for all $u \varepsilon [x,\overline{x}_n]$ which implies that

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)v_n'(\hat{x})\} \leq \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)v_n'(u)\}$$

$$u \varepsilon [x,x] .$$

Then

$$\hat{\alpha}[v_n(x)-v_n(u)] \leq \hat{L}_1(v_n)(\hat{x}) - \hat{L}_1(v_n)(u)$$

$$\leq \lambda_a \int_0^\infty v_n(\hat{x}+y)G_a(dy) - \lambda_a \int_0^\infty v_n(u+y)G_a(dy)$$

for $u \in [x,\hat{x}]$ since L_1 is decreasing. Then dividing by $\hat{x}$-u and taking the limit, we have

$$\hat{\alpha}v_n'(\hat{x}) \leq \lambda_a \int_0^\infty v_n'(\hat{x}+y)G_a(dy) \leq \lambda_a v_n'(\hat{x})$$

by the bounded convergence theorem. But this is a contradiction since $v_n'(\hat{x}) > 0$. So $v_n(.)$ is concave increasing on $[0,\overline{x}_n]$ for some $\overline{x}_n > 0$ and decreasing on $[\overline{x}_n,\infty)$.

Let (r_n,p_n) be the optimal control pair of (ii) of Corollary (1.1). Then this result implies that as functions of the content level r_n is increasing and p_n is decreasing. Now note that for $n \in \mathbb{N}_+$ v_n satisfies the boundary condition

$$\hat{\alpha}v_n(0) = L_2(p(0)) + L_3(p(0)) + \hat{L}_1(v_n)(0)$$

where p(0) is the optimal input control at zero level and defined by (VI.2.4). Then for $x \in \mathbb{R}_+$

$$v_n'(x) = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P+(1/n),\overline{r}]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + \hat{L}_1(v_n)(x) - \hat{\alpha}v_n(x)]\}$$

implies that

$$v_n'(x) \leq v_n'(0) = n[L_3(p(0) + (1/n)) - L_3(p(0)] \leq L_3'(p(0)) \leq L_3'(p_n(x)) \tag{2.2}$$

since L_3 is concave and p_n is decreasing. So $v_n'(.)$ is bounded below uniformly in n and $|v_n'(.)| \leq C$ for some $C > 0$ for all $n \in \mathbb{N}_+$.

Now the procedure of Theorem (VI.3.1) can be repeated here to show that v is continuous. Let $\overline{x}$ be so that for $m \geq n$

$$\sup_{x\in\mathbb{R}_+} \{v_m(x) - v_n(x)\} = v_m(\bar{x}) - v_n(\bar{x}).$$

If this supremum occurs at $\bar{x} = 0$ for some $m \geq n$, then

$$\hat{\alpha}[v_m(0) - v_n(0)] = \hat{L}_1(v_m)(0) - \hat{L}_1(v_n)(0)$$

$$= \lambda_a \int_0^\infty [v_m(y) - v_n(y)]G_a(dy)$$

$$\leq \lambda_a[v_m(0) - v_n(0)]$$

which implies $v_m(0) = v_n(0)$ by the definition of $\hat{\alpha}$.

If $\bar{x} > 0$, then $v_m'(\bar{x}) = v_n'(\bar{x})$ and

$$0 \leq \hat{\alpha}[v_m(\bar{x}) - v_n(\bar{x})] \leq (\frac{1}{n} + \frac{1}{m})C + Kv_m(\bar{x}) - Kv_n(\bar{x})$$

which follows by recalling that in the deterministic problem

$$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v_m'(\bar{x})\} - \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P+(1/n),\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v_n'(\bar{x})\}$$

$$\leq (\frac{1}{n} + \frac{1}{m})C.$$

On the other hand,

$$Kv_m(\bar{x}) - Kv_n(\bar{x}) = \lambda_a \int_0^\infty [v_m(\bar{x}+y) - v_n(\bar{x}+y)]G_a(dy)$$

$$\leq \lambda_a[v_m(\bar{x}) - v_n(\bar{x})].$$

Therefore

$$0 \leq \hat{\alpha}[v_m(\bar{x}) - v_n(\bar{x})] \leq (\frac{1}{n} + \frac{1}{m})C + \lambda_a[v_m(\bar{x}) - v_n(\bar{x})]$$

or

$$0 \leq \alpha[v_m(\bar{x}) - v_n(\bar{x})] \leq (\frac{1}{n} + \frac{1}{m})C$$

since $\hat{\alpha} = \alpha + \lambda_a$. So the sequence $\{v_n\}$ converges uniformly to v and thus v(.) is continuous.

Now define for $x \in \mathbb{R}_o$

$$p(x) = \lim_{n\to\infty} p_n(x) \tag{2.3}$$

and

$$r(x) = \lim_{n\to\infty} r_n(x) \ . \tag{2.4}$$

We can further define $\{\hat{x}_n\}$ in the same manner as we have done in the deterministic problem and restate the definition of $\hat{x}$ as

$$\hat{x} = \sup\{x \in [0,\infty) : \hat{\alpha}v(x) = \hat{L}_1(v)(x) + L_2(p(x)) + L_3(p(x))\}.$$

To proceed further, we assume that $\hat{x} = 0$ and show the results under this assumption, deferring its proof until later. The proof of Theorem (VI.3.1) can be repeated here to show that v is continuously differentiable with a derivative given by

$$v'(x) = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P,\overline{r}]}} \left\{\frac{1}{R - P}\left[L_2(P) + L_3(R) + \hat{L}_1(v)(x) - \hat{\alpha}v(x)\right]\right\} \tag{2.5}$$

for $x \in [0,\infty)$. Note that $v_n'(x) \leq L_3'(p_n(x))$ by (2.2) implies

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P,\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)v_n'(x)\} \geq L_2(p_n(x)) + L_3(p_n(x))$$

for $x \in \mathbb{R}_+$, every $n \in \mathbb{N}_+$. Then it is obvious that

$$\hat{\alpha}v_n(x) \geq \hat{L}_1(v_n)(x) + L_2(p_n(x)) + L_3(p_n(x))$$

and by Proposition (1.1) we obtain

$$\hat{\alpha}v(x) \geq \hat{L}_1(v)(x) + L_2(p(x)) + L_3(p(x))$$

where p is as defined by (2.3). This together with (2.5) implies that $v'(x) \leq L_3'(p(x))$ for $x \in \mathbb{R}_0$. So $v'(.)$ is in fact bounded. The continuity of v' and (r,p) on $\mathbb{R}_+$ can be shown by using the argument of Theorem (VI.3.1); thus we omit it to avoid repetition.

Now there remains to show that $\hat{x} = 0$ to complete our proof of (i). Since

$$\hat{\alpha}v(\infty) = \hat{L}_1(v)(\infty) + L_2(0) + L_3(\overline{r}) > \hat{L}_1(v)(\infty) + L_2(p(x)) + L_3(p(x))$$

for any $x \in \mathbb{R}_+$, it is obvious that $\hat{x}$ is finite.

Note that $v'(\hat{x}) = L_3'(p(\hat{x}))$ which can be shown by using a sequence $\{u_n\} \subseteq R_+$ with $u_n \downarrow \hat{x}$ as we have done in Theorem (VI.3.1). Furthermore by the same argument given at the beginning of the proof it can be shown that v is concave increasing on $[\hat{x},\overline{x}]$ for $\overline{x} > \hat{x}$ and decreasing on $[\overline{x},\infty)$. Now define $(r^*(x),p^*(x))$ on $\mathbb{R}_+$ such that

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R=P}} \{L_2(P) + L_3(R) + (P-R)v'(x)\} = L_2(p^*(x)) + L_3(r^*(x)) \quad \text{for } 0 \leq x \leq \hat{x} \tag{2.6}$$

and

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P,\overline{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\} = L_2(p^*(x)) + L_3(r^*(x)) + [p^*(x) - r^*(x)]v'(x \quad \text{for } x > \hat{x} . \tag{2.7}$$

Now it is obvious that $(r^*,p^*) \in M_\infty$ since $r^*(.)$ is increasing and $p^*(.)$ is decreasing and $v(x) = v_{r^*p^*}(x)$ for every $x \geq \hat{x}$.

Now assume that $\hat{x} > 0$ and for arbitrary $\Delta < \hat{x}$ define $(r_\Delta(x), p_\Delta(x))$ on $\mathbb{R}_+$ such that

$$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R=P}}\{L_2(P) + L_3(R) + (P-R)u'(x)\} = L_2(p_\Delta(x)) + L_3(r_\Delta(x)) \quad \text{for} \quad 0 \leq x \leq \hat{x}-\Delta$$

and

$$(r_\Delta(x), p_\Delta(x)) = (r^*(x+\Delta), p^*(x+\Delta)) \quad \text{for} \quad x > \bar{x}-\Delta \; .$$

For fixed $\omega \in \Omega$, let $f_1(t,\omega)$ and $f_2(t,\omega)$ be the unique solutions of

$$f_1(t,\omega) = \hat{x} + A_t + \int_0^t (p^*-r^*)(f_1(s,\omega))ds \; ,$$

$$f_2(t,\omega) = \hat{x} - \Delta + A_t + \int_0^t (p_\Delta - r_\Delta)(f_2(s,\omega))ds$$

respectively. Then $f_2(t,\omega) \leq f_1(t,\omega) \leq f_2(t,\omega)+\Delta$, and consequently $r^*(f_1(t,\omega)) \leq r_\Delta(f_2(t,\omega))$ and $p^*(f_1(t,\omega)) \geq p_\Delta(f_2(t,\omega))$ for all t. This implies that

$$\int_0^\infty e^{-\alpha t}[L_1(f_2(t,\omega)) + L_2(p_\Delta(f_2(t,\omega)) + L_3(r_\Delta(f_2(t,\omega))]dt$$

$$\geq \int_0^\infty e^{-\alpha t}[L_1(f_1(t,\omega)) + L_2(p^*(f_1(t,\omega)) + L_3(r^*(f_1(t,\omega)))]dt$$

since L_1 is decreasing, L_2 is decreasing and L_3 is increasing. Thus

$$v_{r_\Delta p_\Delta}(\hat{x}-\Delta) \geq v_{r^*p^*}(\hat{x}) \; .$$

From our definition of (r^*,p^*), $v_{r^*p^*}(\hat{x}-\Delta) \geq v_{r_\Delta p_\Delta}(\hat{x}-\Delta)$ which implies that

$$v_{r^*p^*}(\hat{x} - \Delta) \geq v_{r^*p^*}(\hat{x}). \tag{2.8}$$

Assertion (2.8) is a contradiction since v is concave increasing on $[0,\bar{x}]$ where $\bar{x} > \hat{x}$. So our assumption $\hat{x} > 0$ fails. Then by (2.5) v satisfies on $\mathbb{R}_o$

$$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\} + \hat{L}_1(v)(x) - \hat{\alpha}v(x) = 0 ,$$

and furthermore by the fact that $v'(.) \leq L_3'(p(.))$ we have

$$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[P,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\} = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[0,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)v'(x)\}$$

The fact $\hat{x} = 0$ implies $v'(0) = L_3'(p(0))$ which in turn satisfies

$$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ R=P}} \{L_2(P) + L_3(R) + (P-R)v'(0)\} = \sup_{\substack{P\varepsilon[0,\bar{p}] \\ R\varepsilon[0;p]}} \{L_2(P) + L_3(R) + (P-R)v'(0)\}$$

and

$$\sup_{\substack{P\varepsilon[0,\bar{p}] \\ P\varepsilon[0,p]}} \{L_2(P) + L_3(R) + (P-R)v'(0)\} + \hat{L}_1(v)(0) - \hat{\alpha}v(0)$$

$$= L_2(p^*(0)) + L_3(p^*(0)) + \hat{L}_1(v)(0) - \hat{\alpha}v(0).$$

It then immediately follows that v satisfies the sufficiency condition of Corollary (V.2.1).

The proof of (i) is completed by noting that the uniqueness of v can be shown as in the deterministic problem by using the procedure of Theorem (VI.3.1).

ii) The optimal control pair (r^*,p^*) are as defined by (2.6) and (2.7). It follows from $v'(x) \leq L_3'(p(x))$ and $v'(0) = L_3'(p(0))$ that $r^*(x) \geq p^*(x)$ for $x \in \mathbb{R}_0$ and $r^*(0) = p^*(0)$. The strict concavity of L_2 and L_3 ensures us about the continuity of (r^*,p^*) on $\mathbb{R}_+$ and the uniqueness of (r^*,p^*) for any given v'. Since v is shown to be concave increasing on $[0,\bar{x}]$ and decreasing on $[\bar{x},\infty)$ for some $\bar{x} > 0$, v' is decreasing which implies that r^* is increasing, p^* is decreasing on $[0,\bar{x}]$ and $(r^*(x),p^*(x)) = (\bar{r},0)$ for all $x \geq \bar{x}$. So $(r^*,p^*) \in M_\infty \subset M_1$ and $v = v_{r^*p^*}$. Δ

<u>REMARK (2.1).</u> Note that $v'(x)$ is the marginal contribution of an additional unit of the content level to the total optimal return and is interpreted as the shadow price of the content at the content level x. Since $v(.)$ is concave on $[0,\bar{x}]$, this price is decreasing in the content level on $[0,\bar{x}]$ and becomes negative on $[\bar{x},\infty)$. Thus the lower the shadow price of the content level, the higher is the optimal output rate and the lower is the optimal input rate. The characterization of the optimal control pair (r^*,p^*) as $r^*(x) \geq p^*(x)$ for $x \in \mathbb{R}_+$ implies that the greater the content level, the faster it should be diminished. Also the fact that $v'(x) \leq L_3'(p^*(x))$ yields that at the optimum the marginal value of the content level is less than or equal to the marginal utility of consuming the content. Δ

Theorem (2.1) resolves the optimal control problem of the generalized storage model by disregarding the stochastic output process. The existence and uniqueness of an optimal return function v satisfying the sufficiency condition of Corollary (V.2.1) is verified under the assumption that there is no uncontrolled output from the store and L_1 is decreasing. Unfortunately it seems difficult to obtain similar results in the presence of a random output process by extending the procedure of the deterministic problem. However, the same line of reasoning can be utilized to study various storage processes with different model attributes, a few of which will be discussed in the following chapters.

VIII. GENERALIZATIONS

The main point in this chapter is to reveal to what extent our procedure of constructing the optimal return function and the associated optimal controls can be employed in various applications of storage models. In Section 1, stores with finite physical capacity will be considered, and the applicability of our construction will be discussed briefly. It will be pointed out that a different analysis should be developed to handle with the finite capacity restriction. The concept of backlogging will be introduced in Section 2. The no-backlogging condition which does not permit the content level to fall below zero will be relaxed, and our procedure will prove to be efficient in treating the problem of finite backlogging; however in case of infinite backlogging the boundary condition invokes a complication which our procedure cannot solve adequately. To overcome that difficulty one can incorporate a content-dependent output process which in turn changes the basic features of our argument. Finally in Section 3 restrictions on cost and reward structures are modified, and the construction of the optimal return function and the optimal control pair with these modifications is outlined by simply providing rough sketches of the proofs. The optimal control problem will be overviewed first by assuming L_1 to be a concave increasing function and then by assuming L_2 and L_3 to be nondifferentiable at a finite number of points.

8.1 FINITE CAPACITY STORES

In our analysis we have constructed the unique optimal return function and the associated optimal control pair under the assumption that the store has infinite physical capacity. However it is common experience to encounter stores with finite storage capacity. Then our construction methodology is naturally expected to change considerably, and the admissible controls and consequently the optimal control problem should be redefined to analyze storage models with a finite capacity level K. We will not present a formal argument, but simply point out some basic considerations that should be taken into account in such an analysis.

One formulation possibility is to fail the system and to incur a lumpsum cost as soon as the content of the store exceeds its capacity. If we define

$$T = \inf\{t \geq 0 : X_t \geq K\}$$

as the hitting time of level K, then the storage process X is made to remain above K forever after T. In other words, $X_t = \Delta$ for $t > T$ where $\Delta = [K,\infty) \subset \mathbb{R}_+$.

Another possibility on the other hand is to imagine the existence of another infinite store which simply records the behaviour of X until routine operations are resumed. So it is assumed that any excess input over the capacity K will simply overflow into the imaginary store and will not enter the controlled model. Although the problem of choosing optimal controls is somewhat artificial with this convention, its results are useful since controls are needed only when $X_t \in [0,K]$. In the process of treating this situation, it suffices to express the output control as the sum of two deterministic functions, one of which is specified by our choice of an output rate r while the other is specified by a given deterministic function f. Thus if we define $(\hat{r},\hat{p})$ by

$$\hat{r}(x) = \begin{cases} r(x) & , x \leq K \\ r(x)+f(x-K) & , x > K \end{cases}$$

and

$$\hat{p}(x) = p(x) \qquad , x \in \mathbb{R}_+ ,$$

we can replace (r,p) by $(\hat{r},\hat{p})$ in our analysis. If the content level of the store is less than or equal to the physical capacity K, then the output rate is given by r(x) and the input rate is given by p(x). Otherwise if the content level exceeds the capacity, there is an additional output stream at a rate f(x-K) into the imaginary store while the input rate remains the same. It is preferable to take f(.) to be continuous increasing function with f(0) = 0. In the dam models f possesses a well-known interpretation and corresponds to the flooding rate.

This discussion reveals that our formulation of the storage model and the structure of the optimal control problem changes considerably, and a different line of reasoning is required to characterize the optimal return function and the optimal controls in case of stores with finite capacity.

8.2 STORES WITH BACKLOGGING

In our analysis of the generalized storage model we have assumed that there cannot be any output from the store when it is empty. This no-backlogging requirement has prevented the content level of the store from falling below zero. This is accomplished by assuming that $r(0) \leq p(0)$. Although we had to exclude the random output process B in the proof of Theorem (VII.2.1), the output process B is inherent in the generalized storage model, so it should be further considered in our analysis. As far as the stochastic output process B is concerned, the random outputs whose jump magnitudes exceed the current content level of the store are avoided at the point of emptiness in our analysis. Hence the jump outputs are made dependent upon the content level, so that the distribution of the output jump magnitudes in fact is made to satisfy

$$G_b^x(dy) = \begin{cases} G_b(dy) & , \; y < x \\ \int_x^\infty G_b(dy) & , \; y = x \end{cases} \tag{2.1}$$

if the content level of the store is x. This convention is depicted in Figure (2.1) where a typical distribution function $G_b^x(.)$ is given. Note that $G_b^{x_1}(u) = G_b^{x_2}(u)$ for $u \leq x_1 \leq x_2$.

An obvious generalization of this convention would be to allow for finite backlogging. If a backorder level of I units is permitted, the content level may fall below zero down to -I units. So the state space of the content process X becomes $[-I,\infty)$ for some $I \geq 0$. In such a case our construction can be directly repeated so as to include finite backlogging of I units by simply extending the domains of input and output controls and the cost and reward functions. Then it suffices to change the boundary condition by assuming that $r(-I) \leq p(-I)$. In treating the finite backlogging situation, similar results can be obtained by employing our procedure of the nobacklogging case; however we should point out that this argument holds true only when the convention introduced by (2.1) is used. Otherwise a totally different analysis would result when a different kind of distribution were assumed for the output jump magnitudes. For example a more complicated output process would be obtained by considering content-dependent outputs, and a different solution procedure would be required.

In stores with content-dependent outputs, the times between successive outputs are independent and identically distributed, but the magnitudes of successive outputs depend upon the content level of the store at the particular instant. So $\{Z_n\}$ are conditionally independent given $\mathcal{F}$ and satisfy

$$\mathbb{P}\{Z_n \in D | X_{\tau_n} = x\} = \int_D G_b^x(dy) \quad , \quad D \in \mathbb{R}_+ \, , \qquad n \geq 1 \tag{2.2}$$

for some family $\{G_b^x(dy)\}$ of distribution functions on $\mathbb{R}_+$ with $G_b^x(x) = 1$. A content-dependent output process affects the essence of the generalized storage model, so a different argument should be employed to accomplish the construction of the storage process and the characterization of the optimal return function.

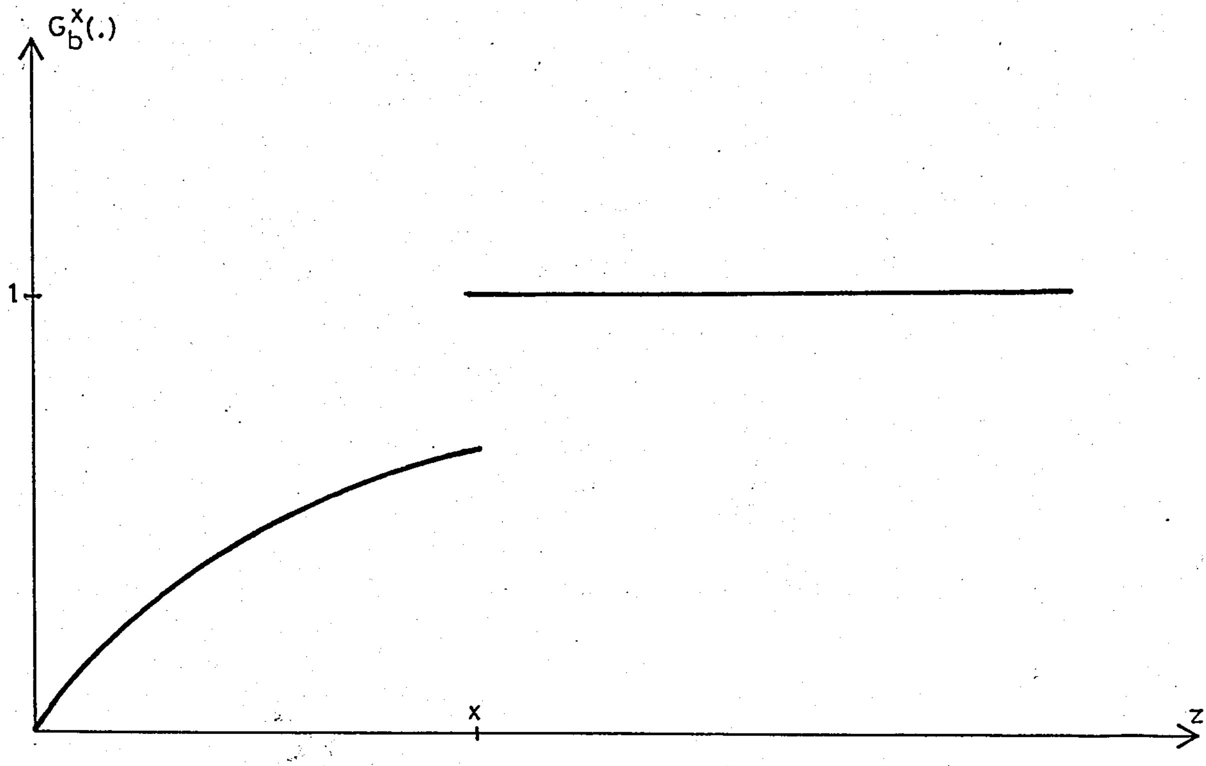

FIGURE 2.1 - For a given distribution $G_b(.)$ and a fixed $x \in \mathbb{R}_+$, the distribution of output jump magnitude $G_b^x(.)$.

8.3 GENERALIZATIONS ON THE COST AND REWARD STRUCTURE

The procedure developed so far has achieved the construction of the optimal return function and the optimal control pair satisfying the sufficient optimality condition of Corollary (V.2.1) under the restrictions specified by Assumption (V.1.1) and the additional assumption that L_1 is decreasing. Naturally these conditions eliminate some interesting cases encountered frequently in storage models. Thus in this section the validity of our procedure will be established for some problems with different cost and reward structures. This will be done by discussing these new applications and pointing out the differences in the construction rather than providing a detailed proof of the results.

We showed that the limit of return functions locally optimal in M_n might not turn out to be optimal in M in case L_1 is an arbitrary function and pointed out that monotonicity properties on L_1 are required to guarantee the global optimality in M. Hence L_1 was assumed to be decreasing, and only then the existence and uniqueness of the global return function and the control pair in M were established. Naturally it is possible to consider the same problem with a different monotonicity restriction imposed on L_1 and try to achieve the construction by a similar argument.

COROLLARY (3.1). If L_1 is concave increasing, then in the deterministic case

i) v is concave increasing;

ii) v is the unique function satisfying the sufficient condition of Corollary (VI.1.1);

iii) there exist unique optimal control pair (r^*,p^*) such that $v = v_{r^*p^*}$. Furthermore $(r^*,p^*) \in M_i$.

Proof. (i) Consider the deterministic problem discussed in Chapter VI. For every $n \in \mathbb{N}_+$ the construction of the unique optimal return function v_n and the unique optimal control pair $(r_n,p_n) \in M_n$ can be exactly as in Theorem (VI.2.1). Also $v = \lim_{n\to\infty} v_n$. Now for fixed $n \in \mathbb{N}_+$, $0 \le x_1 < x_2$, $0 < \lambda < 1$, define

$$f_1(t) = x_1 + \int_0^t (p_n - r_n)(f_1(s))ds \quad , \quad t \ge 0 , \tag{3.1}$$

$$f_2(t) = x_2 + \int_0^t (p_n - r_n)(f_2(s))ds \quad , \quad t \ge 0 ,$$

$$f(t) = \lambda f_1(t) + (1 - \lambda)f_2(t) \ , \quad t \geq 0$$

$$t_1 = \inf\{t \geq 0 : f_1(t) = 0\} \ ,$$

$$t_2 = \inf\{t \geq 0 : f_2(t) = 0\} \ .$$

Then it is obvious that f_2 and f are strictly decreasing on $[0,t_2]$ while f_1 is decreasing on $[0,t_1]$, and $f_1(t) = f_2(t) = f(t) = 0$ for $t \geq t_2$ and $0 \leq t_1 < t_2 < \infty$ since $(r_n,p_n) \in \mathcal{M}_n$. Now define $f^{-1}(x)$ to be the functional inverse of f, so that for $0 \leq x \leq \lambda x_1 + (1-\lambda)x_2$

$$f^{-1}(x) = \inf\{0 \leq t \leq t_2 : f(t) \leq x\}.$$

Since f is strictly decreasing and continuous on $[0,t_2]$, f^{-1} is strictly decreasing and continuous on $[0,\lambda x_1 + (1-\lambda)x_2]$. Define $(\hat{r},\hat{p})$ by

$$\hat{r}(x) = \begin{cases} \lambda r_n(x_1)+(1-\lambda)r_n(x_2) & , \ x \in (\lambda x_1+(1-\lambda)x_2,\infty) \\ \lambda r_n(f_1(f^{-1}(x)))+(1-\lambda)r_n(f_2(f^{-1}(x))), & x \in [0,\lambda x_1+(1-\lambda)x_2] \end{cases}$$

and

$$\hat{p}(x) = \begin{cases} \lambda p_n(x_1)+(1-\lambda)p_n(x_2) & , \ x \in (\lambda x_1+(1-\lambda)x_2,\infty) \\ \lambda p_n(f_1(f^{-1}(x)))+(1-\lambda)r_n(f_2(f^{-1}(x))), & x \in [0,\lambda x_1 + (1-\lambda)x_2]. \end{cases}$$

Note that $\hat{r}(0) = \hat{p}(0)$ since $f^{-1}(0) = t_2$, $f_1(t_2) = f_2(t_2) = 0$ and $r_n(0) = p_n(0)$. Also the fact that $f^{-1}(\lambda x_1+(1-\lambda)x_2) = 0$ implies that $(\hat{r},\hat{p})$ are continuous at $\lambda x_1+(1-\lambda)x_2$. Within this set-up, the procedure of ÖZEKİCİ [35] can be utilized to show that $(\hat{r},\hat{p})$ are both Lipschitz continuous on $(0,\lambda x_1+(1-\lambda)x_2)$ which implies that $(\hat{r},\hat{p}) \in \mathcal{M}_\infty$. We omit the proof to avoid repetition, but rely upon the Lipschitz continuity result.

By definition f_1 and f_2 are the content levels of the store with initial levels x_1 and x_2, respectively, where (r_n, p_n) are the controls being used. Now note that

$$\begin{aligned} f(t) &= \lambda f_1(t) + (1-\lambda)f_2(t) \\ &= \lambda x_1 + (1-\lambda)x_2 + \int_0^t [\lambda p_n(f_1(s)) + (1-\lambda)p_n(f_2(s)) \\ &\qquad - \lambda r_n(f_1(s)) - (1-\lambda)r_n(f_2(s))]ds \\ &= \lambda x_1 + (1-\lambda)x_2 + \int_0^t (\hat{p}-\hat{r})(f(s))ds \end{aligned} \tag{3.2}$$

where the third equality follows from

$$\hat{p}(f(s)) = \lambda p_n(f_1(s)) + (1-\lambda)p_n(f_2(s))$$

and

$$\hat{r}(f(s)) = \lambda r_n(f_1(s)) + (1-\lambda)r_n(f_2(s)).$$

So f is the content level of the store with initial level $\lambda x_1 + (1-\lambda)x_2$ where $(\hat{r}, \hat{p})$ are the controls being used.

The concavity of v can be shown by noting that

$$\begin{aligned} v_{\hat{r}\hat{p}}(\lambda x_1 + (1-\lambda)x_2) &= \int_0^\infty e^{-\alpha t}[L_1(f(t)) + L_2(\hat{p}(f(t))) + L_3(\hat{r}(f(t)))]dt \\ &= \int_0^\infty e^{-\alpha t}[L_1(\lambda f_1(t) + (1-\lambda)f_2(t)) + L_2(\lambda p_n(f_1(t)) \\ &\qquad + (1-\lambda)p_n(f_2(t))) + L_3(\lambda r_n(f_1(t)) \\ &\qquad + (1-\lambda)r_n(f_2(t)))]dt. \end{aligned}$$

By the concavity of L_1, L_2 and L_3, we obtain

$$v_{\hat{r}\hat{p}}(\lambda x_1+(1-\lambda)x_2) \geq \lambda\int_0^\infty e^{-\alpha t}[L_1(f_1(t)) + L_2(p_n(f_1(t))) + L_3(r_n(f_1(t)))]dt$$

$$+ (1-\lambda)\int_0^\infty e^{-\alpha t}[L_1(f_2(t)) + L_2(p_n(f_2(t)))$$

$$+ L_3(r_n(f_2(t)))]dt$$

$$= \lambda v_{r_n p_n}(x_1) + (1-\lambda)v_{r_n p_n}(x_2) = \lambda v_n(x_1) + (1-\lambda)v_n(x_2).$$

Then it follows from Proposition (VI.2.1) that

$$v(\lambda x_1+(1-\lambda)x_2) \geq v_{\hat{r}\hat{p}}(\lambda x_1+(1-\lambda)x_2) \geq \lambda v_n(x_1) + (1-\lambda)v_n(x_2) \qquad (3.3)$$

since $v \geq v_{\hat{r}\hat{p}}$ for all $(\hat{r},\hat{p}) \in \mathcal{M}_\infty$. The pointwise convergence of v_n to v together with (3.3) implies that

$$v(\lambda x_1 + (1-\lambda)x_2) \geq \lambda v(x_1) + (1-\lambda)v(x_2) . \qquad (3.4)$$

Now note that $\alpha v_n(x) \leq L_2(0) + L_3(\overline{r}) + L_1(x)$ for every $x \in \mathbb{R}_+$ since $0 \leq p_n \leq r_n \leq \overline{r}$, L_1 and L_3 are increasing and L_2 is decreasing. Then for $x \in \mathbb{R}_+$

$$v_n'(x) = \sup_{\substack{P\in[0,\overline{p}] \\ R\in[P+(1/n),\overline{r}]}} \left\{\frac{1}{R - P}[L_2(P) + L_3(R) + L_1(x) - \alpha v_n(x)]\right\}$$

$$\geq \frac{1}{\overline{r}}[L_2(0) + L_3(\overline{r}) + L_1(x) - \alpha v_n(x)] \geq 0.$$

Thus v is concave increasing.

ii) The concave increasing property of L_1 tempts one to increase the content level of the store when it is below a certain level, so one is justified to claim that for $n \in \mathbb{N}_+$ there exists some $\overline{x}_n \in \mathbb{R}_+$ such that

$$\alpha v_n(x) > L_2(p_n(x)) + L_3(p_n(x)) + L_1(x) \quad \text{for every} \quad x > \overline{x}_n \qquad (3.5)$$

$$\alpha v_n(x) \leq L_2(p_n(x)) + L_3(p_n(x)) + L_1(x) \quad \text{for every} \quad x \leq \overline{x}_n \qquad (3.6)$$

So define $\overline{x}_n$ for arbitrary $n \in \mathbb{N}_+$ by

$$\overline{x}_n = \inf\{x \in [0,\infty): \alpha v_n(u) > L_2(p_n(u)) + L_3(p_n(u)) + L_1(u) \quad \text{for all} \quad u > x\}, \qquad (3.7)$$

and note that $0 \leq \overline{x}_n < \infty$ since $\alpha v_n(\infty) = L_2(0) + L_3(\overline{r}) + L_1(\infty) > L_2(p_n(x)) + L_3(p_n(x)) + L_1(\infty)$ for any $x \in \mathbb{R}_+$ and $\alpha v_n(0) = L_2(p(0)) + L_3(p(0)) + L_1(0)$.

Then the argument of Theorem (VI.3.1) together with the bounded convergence theorem can be used here to show that v is continuously differentiable with a bounded derivative v' given by

$$v'(x) = \sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[P,\overline{r}]}} \{\frac{1}{R - P}[L_2(P) + L_3(R) + L_1(x) - \alpha v(x)]\}$$

for all $x \in [\overline{x},\infty)$ where $\overline{x} = \lim_{n\to\infty} \overline{x}_n$. The optimal control pair $(r^*(x),p^*(x))$ are still as defined by (VII.2.7) such that $v(x) = v_{r^*p^*}(x)$ on $[\overline{x},\infty)$.

However the construction of the optimal return function and the optimal control pair on $[0,\overline{x}]$ necessarily introduces for $n \in \mathbb{N}_+$

$$\hat{M}_n = \{(r,p) \in M: r(x) \leq p(x)-(1/n) \quad \text{for} \quad x < \overline{x}, \ (r(x),p(x)) = (r^*(x),p^*(x) \ \text{for} \ x \geq \overline{x}\} \qquad (3.8)$$

where (r^*,p^*) is as defined by (VII.2.7). The procedure of Chapter VI is employed to yield symmetrical results. So within the framework of the approach outlined previously, we create a sequence of return functions $\{u_n\}$ locally optimal in $\hat{M}_n$ such that it satisfies

a) u_n is differentiable with a bounded Lipschitz continuous derivative u_n' on $[0,\overline{x}]$;

b)
$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,P-(1/n)]}} \{L_2(P) + L_3(R) + (P-R)u_n'(x)\} + L_1(x) - \alpha u_n(x) = 0 \ , \quad 0 \leq x \leq \overline{x}$$

$$u_n(x) = v_{r^*p^*}(x) = v(x) \qquad , \quad x \geq \overline{x};$$

c) $L_2(\overline{p}) + L_3(0) + \underline{L}_1 \leq \alpha u_n(x) \leq L_2(0) + L_3(\overline{r}) + \overline{L}_1$ for all $x \in [0,\overline{x}]$;

d) there exist unique optimal control pair $(\hat{r}_n,\hat{p}_n)$ in $\hat{M}_n$ such that $u_n = v_{\hat{r}\hat{p}}$, so $(\hat{r},\hat{p})$ are optimal in $\hat{M}_n$.

So (a), (b), and (c) can be proven repeating the steps of Lemmas (VI.2.1), (VI.2.2), (VI.2.3) and (VI.2.4),Theorem (VI.2.1) and Corollary (VI.2.1) by noting that u_n satisfies (b) if and only if

$$u_n'(x) = \inf_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,P-(1/n)]}} \{\frac{1}{R - P} [L_2(P) + L_3(R) + L_1(x) - \alpha u_n(x)]\} \ , \quad x \in [0,\overline{x}].$$

To show (d), define $(\hat{r}_n(x),\hat{p}_n(x))$ such that

$$\sup_{\substack{P\varepsilon[0,\overline{p}] \\ R\varepsilon[0,P-(1/n)]}} \{L_2(P) + L_3(R) + (P-R)u_n'(x)\} = L_2(\hat{p}_n(x)) + L_3(\hat{r}_n(x)) + [\hat{p}_n(x) - \hat{r}_n(x)]u_n'(x), \quad (3.9)$$

$$x < \overline{x}$$

$$(\hat{r}_n(x),\hat{p}_n(x)) = (r^*(x),p^*(x)) \quad , \qquad x \geq \bar{x} .$$

An explicit characterization of $(\hat{r}_n,\hat{p}_n)$ on $[0,\bar{x}]$ similar to Remark (VI.2.3) can be made directly by using the Kuhn-Tucker approach, but is omitted here to avoid repetition.

Next the argument of Proposition (VI.2.1) can be repeated here to show that the sequence $\{u_n\}$ converges to some function u optimal in $\hat{M}_\infty = \bigcup_{n\geq 1} \hat{M}_n$ given by

$$\hat{M}_\infty = \{(r,p) \in M: r(x) \leq p(x) \text{ for } x \in [0,\bar{x}];\ (r(x),p(x)) = (r^*(x),p^*(x)) \text{ for } x \in [\bar{x},\infty)\} . \qquad (3.10)$$

Finally u can be shown to satisfy the sufficiency condition of Corollary (VI.1.1) by employing an argument similar to Corollary (VI.3.1), Lemmas (VI.3.1) and (VI.3.2) and Theorem (VI.3.1). So u turns out to be continuously differentiable with a bounded derivative $u'(x) = f(x)$ given by

$$f(x) = \inf_{\substack{P \in [0,\bar{p}] \\ R \in [0,p]}} \{\frac{1}{R - P}[L_2(P) + L_3(R) + L_1(x) - \alpha u(x)]\} , \ x \in [0,\bar{x}] \qquad (3.11)$$

where (r,p) is the limit of $(\hat{r}_n,\hat{p}_n)$ defined by (3.9). The fact that $u_n'(x)$ converges pointwise to (3.11) follows from

$$\alpha u(x) > L_2(p(x)) + L_3(p(x)) + L_1(x) \quad , \quad x \in [0,\bar{x}]$$

which together with (3.11) and by the bounded convergence theorem implies

$$u'(x) = f(x) \geq L_3'(p(x)) .$$

Note that the concavity of u on $[0,\overline{x}]$ can be shown by the procedure of (i) of the Corollary. For $n \in \mathbb{N}_+$ and $0 \leq x \leq \overline{x}$ $u_n(x) \geq L_2(p_n(x)) + L_3(p_n(x)-(1/n)) + L_1(x)$ which implies that

$$u_n'(x) \geq \inf_{R\varepsilon[0,p_n(x)-(1/n)]} \{\frac{1}{R - p_n(x)} [L_3(R) - L_3(p_n(x) - (1/n))]\} \geq 0.$$

So u must be concave increasing on $\mathbb{R}_+$ since $u(x) = v(x)$ for all $x \geq \overline{x}$.

iii) The optimal control pair $(r^*(x),p^*(x))$ on $[\overline{x},\infty)$ is defined by (VII.2.7), and $r^*(x) \geq p^*(x)$ on $[\overline{x},\infty)$ since $u'(x) = v'(x) \leq L_3'(p(x))$ on $[\overline{x},\infty)$. However for $x \in [0,\overline{x}](r^*(x),p^*(x))$ is characterized as the limiting function of $(r_n(x),p_n(x))$ defined by (3.9), so that it becomes optimal to choose some $r^*(x) \leq p^*(x)$. Furthermore the concavity of u on $[0,\infty)$ implies that r^* is increasing and p^* is decreasing in the content level. The uniqueness of (r^*,p^*) is ensured by the strict concavity of L_2 and L_3. So $(r^*,p^*) \in M_i$. □

Now there remains to generalize the results of Corollary (3.1) to include the stochastic processes; however, the output process B creates a difficulty in constructing the optimal return function and the optimal control pair which satisfy the sufficiency condition of Corollary (V.2.1). So it becomes necessary to exclude the consideration of the random output process B in this generalization as stated below.

COROLLARY (3.2). If L_1 is concave increasing and there does not exist any stochastic output from the store, then

i) there exists a unique function $v \in b(R_+)$ that satisfies the sufficiency condition of Corollary (V.2.1);

ii) furthermore, v(.) is concave increasing and there exist a unique optimal control pair (r^*,p^*) such that $v = v_{r^*p^*}$.

Proof. Let B be the set of all $f \in b(R_+)$ which satisfy:

1. f(.) is concave increasing;

2. $\lim_{x\to\infty} \alpha f(x) = L_2(0) + L_3(\bar{r}) + L_1(\infty)$;

3. for every $x \in \mathbb{R}_+$

$$L_2(\bar{p}) + L_3(0) + \underline{L}_1 \le \alpha f(x) \le L_2(0) + L_3(\bar{r}) + \bar{L}_1 \;;$$

4. furthermore f(.) is Lipschitz continuous.

Define a mapping Γ on B so that for $f \in B$, $\Gamma(f)(.) = u(.)$ is the unique solution of

$$\sup_{\substack{P\in[0,\bar{p}] \\ R\in[0,\bar{r}]}} \{L_2(P) + L_3(R) + (P-R)u'(x)\} + L_1(f)(x) - \alpha u(x) = 0\,, \quad x > 0\,, \tag{3.12}$$

$$\sup_{\substack{P\in[0,\bar{p}] \\ R\in[0,P]}} \{L_2(P) + L_3(R) + (P-R)u'(0)\} + \hat{L}_1(f)(0) - \hat{\alpha}u(0) = 0\,, \tag{3.13}$$

where

$$\hat{L}_1(f)(x) = L_1(x) + \lambda_a \int_0^\infty f(x+y)G_a(dy)$$

and

$$\hat{\alpha} = \alpha + \lambda_a .$$

Then there is a unique solution u(.) of (3.12) by Corollary (3.1) that is concave increasing by noting that $\hat{L}_1(f)(.)$ is concave increasing since f ε B and

$$\hat{L}_1(f)(.) \leq \overline{L}_1 + (\lambda_a/\alpha)[L_2(0) + L_3(\overline{r}) + \overline{L}_1] ,$$

$$\hat{L}_1(f)(.) \geq \underline{L}_1 + (\lambda_a/\alpha)[L_2(\overline{p}) + L_3(0) + \underline{L}_1] ,$$

$$\lim_{x\to\infty} \hat{L}_1(f)(x) = L_1(\infty) + (\lambda_a/\alpha)[L_2(0) + L_3(\overline{r}) + L_1(\infty)].$$

The rest follows similarly as in Corollary (VII.1.1) and Theorem (VII.2.1). Δ

Another generalization on the cost and reward structure would be to consider the situation where L_2 and L_3 are not necessarily continuously differentiable on $\mathbb{R}_+$. Suppose $L_2(.)$ and $L_3(.)$ are differentiable except at a finite number of points. The construction of the optimal return function v and the optimal control pair (r*,p*) follow the procedure of the previous chapters. The optimal return function which satisfies the sufficiency condition of Corollary (VI.1.1) for the deterministic problem under the assumption that L_1 is decreasing can be constructed by Corollary (VI.3.1), Lemmas (VI.3.1) and (VI.3.2), and Theorem (VI.3.1). Then the argument of Theorem (VII.2.1) can be employed for the stochastic problem. As before, v can be shown to be concave increasing on $[0,\overline{x}]$ and decreasing on $[\overline{x},\infty)$ for some $\overline{x} \geq 0$, and (r*,p*) can be shown to be in M_I. Under these new conditions on

$L_2(.)$ and $L_3(.)$, the characterization of (r^*,p^*) can be still done by (VII.2.6) and (VII.2.7) where $L_2'(.)$ and $L_3'(.)$ are replaced by $D^+L_2(.)$ and $D^+L_3(.)$, respectively. The fact that $v'(.) \leq D^+L_3(p^*(.))$ still holds true, so $r^*(.) \geq p^*(.)$. The difficulty arising in this situation is that the optimal controls might not be unique in M.

<u>REMARK (3.1).</u> Note that piecewise linear functions are not differentiable at a finite number of points and constitute a special class to which the above argument applies directly. If $L_2(.)$ and $L_3(.)$ are assumed to be piecewise linear functions, then $L_2(.)$ is linear with slope γ_i on nonoverlapping intervals $I_i \subset [0,\overline{p}]$, and $L_3(.)$ is linear with slope β_i on nonoverlapping intervals $K_i \subset [0,\overline{r}]$. Then the above argument can be used to show that there exists a unique optimal return function v and an optimal control pair (r^*,p^*) whose uniqueness on the other hand might fail. However, (r^*,p^*) are observed to possess a finite number of jumps; in fact they are bang-bang controls. Δ

The bang-bang controls consitute a significant class of admissible controls and deserve more emphasis. So in the next chapter we will dwell upon the construction of the bang-bang controls in the presence of piecewise linear cost and reward structures.

IX. BANG-BANG CONTROLS

In this chapter the primary emphasis will be on presenting and analyzing a special class of controls, namely bang-bang controls. In Section 1, the characterization of optimal bang-bang controls will be shown to be an immediate consequence of the results obtained so far, and an algorithmic procedure will be provided. In Section 2, the methodology developed will be employed to obtain the explicit expressions for the optimal return function and the associated optimal control pair in some numerical problems.

9.1 A THEORETICAL FRAMEWORK

In this section we study the conditions imposed on the cost and reward structure under which bang-bang controls arise and try to construct them by utilizing the argument presented in Section VIII.3. Here $L_2(.)$ is taken to be piecewise linear and concave decreasing, $L_3(.)$ is taken to be piecewise linear and concave increasing, and $L_1(.)$ is taken to be decreasing. The piecewise linearity of $L_2(.)$ and $L_3(.)$ implies that they are differentiable except at a finite number of points and satisfy

$$L_2'(P) = \begin{cases} \gamma_1 & , \; P \subset (P_o, P_1) \\ \gamma_2 & , \; P \subset (P_1, P_2) \\ \vdots & \\ \gamma_n & , \; P \subset (P_{n-1}, P_n) \end{cases} \qquad (1.1)$$

and

$$L_3'(R) = \begin{cases} \beta_1 & , \; R \subset (R_o, R_1) \\ \beta_2 & , \; R \subset (R_1, R_2) \\ \vdots & \\ \beta_m & , \; R \subset (R_{m-1}, R_m) \end{cases}$$

for some $0 = P_o < P_1 < \dots < P_n = \overline{p}$, $0 = R_o < R_1 < \dots < R_m = \overline{r}$ $0 \geq \gamma_1 > \gamma_2 > \dots > \gamma_n$, and $\beta_1 > \beta_2 > \dots > \beta_m \geq 0$, respectively.

By Remark (VIII.3.1), the optimal return function v is concave increasing on $[0,\overline{x}]$ and decreasing on $[\overline{x},\infty)$ for some $\overline{x} \geq 0$, and it is continuously differentiable on $\mathbb{R}_+$. The optimal controls (r^*,p^*) are as defined by (VII.2.6) and(VII.2.7) where $L_2'(.)$ and $L_3'(.)$ are replaced by $D^+L_2(.)$ and $D^+L_3(.)$, respectively. Then they increase or decrease by jumps only, so they both have bang-bang structures. The fact $v'(.) \leq D^+L_3(p^*(.))$ still holds true and implies that $r^*(x) \geq p^*(x)$ on $\mathbb{R}_+$. Although they are not unique any longer, $(r^*,p^*) \in M_i$.

For $j = 1,\dots,n$ and $L = 1,\dots,m$, define

$$z_j = \sup\{x \in [0,\infty): D^+v(x) \geq -\gamma_{n+1-j}\} \vee 0 \;, \qquad (1.2)$$

and

$$y_L = \sup\{x \in [0,\infty): D^+v(x) \geq \beta_L\} \vee 0 , \qquad (1.3)$$

respectively. Then it follows that $0 \leq z_1 \leq z_2 \leq \cdots \leq z_n$ and $0 = y_1 \leq y_2 \leq \cdots \leq y_m$. Now assume without loss of generality that p(0) defined by (VI.2.14) satisfies

$$P_{k-1} < P(0) \leq P_k \quad \text{and} \quad R_{i-1} \leq P(0) < R_i$$

for some $1 \leq k \leq n$ and $1 \leq i \leq m$, respectively. Then it follows from the definition of (r*,p*) that

$$P^*(x) = \begin{cases} P(0) & , \ x = 0 \\ P_{k-1} & , \ x \in (0, z_{n-k+2}] \\ P_{k-2} & , \ x \in (z_{n-k+2}, z_{n-k+3}] \\ \cdot & \\ \cdot & \\ P_1 & , \ x \in (z_{n-1}, z_n] \\ P_o & , \ x \in (z_n, \infty) \end{cases} \qquad (1.4)$$

and

$$R^*(x) = \begin{cases} P(0) & , \ x = 0 \\ R_i & , \ x \in (0, y_{i+1}] \\ R_{i+2} & , \ x \in (y_{i+1}, y_{i+2}] \\ \cdot & \\ \cdot & \\ R_{m-1} & , \ x \in (y_{m-1}, y_m] \\ R_m & , \ x \in (y_m, \infty) \ . \end{cases} \qquad (1.5)$$

This characterization reveals that (r*,p*) are bang-bang controls.

REMARK (1.1). Note that (r^*,p^*) are unique if v is strictly concave on $[0,\bar{x}]$. For any i, L, the sets $\{D^+v = \beta_i\}$ and $\{D^+v = -\gamma_{n+1-L}\}$ are isolated points of $\mathbb{R}_+$ since then D^+v is strictly decreasing on $[0,\bar{x}]$. If for some $1 \leq j \leq n$ and $x \in [0,\bar{x}]$, $-\gamma_{n+1-j} < D^+v(x) < -\gamma_{n+2-j}$, then $p^*(x) = P_{n+1-j}$ which implies that $p^*(.)$ is uniquely determined on the set $\bigcup_{j=1}^{n} \{D^+v(.) \neq \gamma_{n+1-j}\}$. Then the set $\bigcup_{j=1}^{n} \{D^+v(.) = -\gamma_{n+1-j}\}$ comprises a finite number of points on $\mathbb{R}_+$. Similarly if $\beta_L < D^+v(x) < \beta_{L-1}$, then $r^*(x) = R_{L-1}$ is the unique point on $[0,\bar{r}]$. So $r^*(x)$ is uniquely determined on the set $\bigcup_{L=1}^{m} \{D^+v(.) \neq \beta_L\}$, and similarly the set $\bigcup_{L=1}^{m} \{D^+v(.) = \beta_L\}$ includes a finite number of isolated points on $\mathbb{R}_+$.

Then Admissibility Condition 3 allows us to define (r^*,p^*) uniquely on these sets as the left-hand limits since both r^* and p^* are left-continuous on $\{r(.) \geq p(.)\}$. Δ

Now the problem reduces to determining $p(0)$ by (VI.2.14) and finding a function v and two sequences of points $\{z_j\}$ and $\{y_L\}$ such that:

i) $v(.)$ is a bounded continuously differentiable function on $\mathbb{R}_+$ which is concave increasing on $[0,\bar{x}]$ and decreasing on $[\bar{x},\infty)$ for some $x \geq 0$ given by

$$\bar{x} = \max(z_n, y_m) \; ; \tag{1.6}$$

ii) $L_2(P_j) + L_3(R_L) + (P_j - R_L)v'(x) + Kv(x) = 0$ whenever $x \in (z_{n-j}, z_{n-j+1}]$ and $x \in (y_L, y_{L+1}]$ for $k \leq j \leq n-1$ and $i \leq L \leq m-1$, and

$$v'(z_{n-j}) = -\gamma_{j+1} \quad , \quad v'(y_L) = \beta_L$$
$$v'(z_{n-j+1}) = -\gamma_j \quad , \quad v'(y_{L+1}) = \beta_{L+1} \; ;$$

iii) $L_2(0) + L_3(\bar{r}) - \bar{r}\cdot v'(x) + Kv(x) = 0$ whenever $x \in (\bar{x},\infty)$, and furthermore $v'(z_n) = -\gamma_1$ and $v'(y_m) = \beta_m$.

This characterization of v and (r*,p*) constitutes a methodology which enables us to obtain explicit solutions to some problems.

9.2 EXPLICIT SOLUTIONS TO SOME DETERMINISTIC PROBLEMS

The results of the previous section reveal that in case L_2 and L_3 possess piecewise linear structures, there exist unique optimal controls which are of the bang-bang type. Our aim in this section is to construct the optimal return function and the associated optimal controls explicitly by employing the characterization given by (1.6). This can be accomplished rather easily in deterministic storage models where there are no jump inputs and outputs to the store, and the procedure will be described below to identify the optimal return function v and the optimal control pair (r*,p*) in two deterministic problems. However in the stochastic optimal control problem in which either a random input or a random output process prevails, it is computationally difficult to retain our methodology.

<u>EXAMPLE (2.1).</u> Let $L_1 = 0$, $\bar{r} = 4$, $\bar{p} = 2$ and

$$L_2(P) = \begin{cases} \gamma_1 P & , \quad 0 \leq P \leq 1 \\ \gamma_1 + \gamma_2(P-1) & , \quad 0 \leq P \leq 2 \end{cases}$$

for some $0 \geq \gamma_1 \geq \gamma_2$. Furthermore let

$$L_3(R) = \begin{cases} \beta_1 R & , \quad 0 \leq R \leq 2 \\ 2\beta_1+\beta_2(R-2) & , \quad 2 \leq R \leq 3 \\ 2\beta_1+\beta_2+\beta_3(R-3) & , \quad 3 \leq R \leq 4 \end{cases}$$

for some $\beta_1 \geq \beta_2 \geq \beta_3 \geq 0$. We will illustrate the solution procedure for $\beta_2 \geq -\gamma_2 \geq -\gamma_1$, leaving the other cases for the reader. As a first step it is necessary to find $\bar{p}(0)$ which maximizes $L_2(P) + L_3(P)$ on $[0,2]$. By straightforward calculations it can be shown that $L_2(P) + L_3(P)$ attains its maximum at $P = 2$. Hence $p(0) = r(0) = 2$.

Then the characterization given by (1.4) and (1.5) implies that (r^*,p^*) are of the form

$$r^*(x) = \begin{cases} 2 & , \quad x = 0 \\ 3 & , \quad x \in (0,y_3] \\ 4 & , \quad x \in (y_3,\infty) \end{cases}$$

and

$$p^*(x) = \begin{cases} 2 & , \quad x = 0 \\ 1 & , \quad x \in (0,y_3] \\ 0 & , \quad x \in (y_3,\infty) \end{cases}$$

for some $z_1 > 0$ and $y_3 > 0$. It is obvious that $\alpha v(0) = \gamma_1+\gamma_2+2\beta_1$. Now it follows from (1.6) that the optimal return function v satisfies

$$\gamma_1 + 2\beta_1 + \beta_2 - 2v'(x) - \alpha v(x) = 0 \quad , \quad x \in (0,z_1\Lambda y_3] \tag{2.1}$$

which implies that

$$v(x) = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + C_1 e^{-(\alpha/2)x} \tag{2.2}$$

and

$$v' = -(\alpha/2)C_1 e^{-(\alpha/2)x}$$

for all $x \in (0, z_1 \Lambda y_3]$. The constant of integration C_1 must be chosen so that v is continuous at zero. i.e.,

$$v(0+) = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + C_1 = \frac{\gamma_1 + \gamma_2 + 2\beta_1}{\alpha} \quad . \tag{2.3}$$

So C_1 turns out to be $(\gamma_2 - \beta_2)/\alpha$, and

$$v(x) = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + \frac{\gamma_2 - \beta_2}{\alpha} e^{-(\alpha/2)x}$$

and

$$v'(x) = \frac{\beta_2 - \gamma_2}{2} e^{-(\alpha/2)x}$$

for all $x \in (0, z_1 \Lambda y_3]$. It is obvious that v is concave increasing on $(0, z_1 \Lambda y_3]$. Now the fact that $v'(y_3) = \beta_3$ and $v'(z_1) = -\gamma_1$ implies that

$$y_3 = \frac{2}{\alpha} \ln\left[\frac{\beta_2 - \gamma_2}{2\beta_3}\right] \tag{2.4}$$

and

$$z_1 = \frac{2}{\alpha} \ln\left[\frac{\beta_2 - \gamma_2}{-2\beta_1}\right] \tag{2.5}$$

Now to proceed further we should consider the following cases:

<u>Case 1.</u> $-\gamma_1 > \beta_3$. Then it follows from (2.4) and (2.5) that $y_3 > z_1$ and consequently $(r^*(x), p^*(x)) = (3,0)$ for $x \in (z_1, y_3]$. Then the optimal return function satisfies

$$2\beta_1 + \beta_2 - 3v'(x) - \alpha v(x) = 0 \quad , \qquad x \in (z_1,y_3] \; . \tag{2.6}$$

Consequently,

$$v(x) = \frac{2\beta_1 + \beta_2}{\alpha} + C_2e^{-(\alpha/3)x}$$

and

$$v'(x) = -\frac{\alpha}{3} C_2e^{-(\alpha/3)x}$$

for all $x \in [z_1,y_3]$. Then it follows from the continuity of v at 2 that

$$\frac{2\beta_1 + \beta_2}{\alpha} + C_2e^{-(\alpha/3)z_1} = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + \frac{\gamma_2 - \beta_2}{\alpha} e^{-(\alpha/2)z_1} \tag{2.7}$$

where z_1 is as given by (2.5). Solving (2.7), we obtain for

$$C_2 = -\frac{3}{2\alpha}(-2\gamma_1)^{1/3}(\beta_2 - \gamma_2)^{2/3} < 0.$$

Then

$$v(x) = \frac{2\beta_1 + \beta_2}{\alpha} - \frac{3}{2\alpha} (-2\gamma_1)^{1/3}(\beta_2 - \gamma_2)^{2/3} e^{-(\alpha/3)x}$$

and

$$v'(x) = \frac{1}{2} (-2\gamma_1)^{1/3}(\beta_2 - \gamma_2)^{2/3}e^{-(\alpha/3)x}$$

for all $x \in (z_1,y_3]$. The boundary condition $v'(y_3) = \beta_3$ implies that

$$v'(y_3) = \beta_3 = \frac{1}{2} (-2\gamma_1)^{1/3}(\beta_2 - \gamma_2)^{2/3} e^{-(\alpha/3)y_3}$$

or

$$y_3 = \frac{1}{\alpha} \ln[\frac{-\gamma_1(\beta_2 - \gamma_2)^2}{4\beta_3^3}]. \tag{2.8}$$

It can be easily verified that $y_3 > z_1$ since $(-\gamma_1(\beta_2-\gamma_2)^2/4\beta_3^3) > (\beta_2-\gamma_2)^2/4\gamma_1^2$. Now note that $(r^*(x),p^*(x)) = (4,0)$ on (y_3,∞), and the optimal return function v satisfies

$$2\beta_1 + \beta_2 + \beta_3 - 4v'(x) - \alpha v(x) = 0 \qquad (2.9)$$

for $x \in (y_3,\infty)$. Solving (2.9) we obtain

$$v(x) = \frac{2\beta_1 + \beta_2 + \beta_3}{\alpha} + C_3 e^{-(\alpha/4)x} \quad , \quad x \in [y_3,\infty)$$

and

$$v'(x) = -\frac{\alpha}{4} C_3 e^{-(\alpha/4)x} \quad , \quad x \in [y_3,\infty) \ .$$

Then the boundary condition $v'(y_3) = \beta_3$ implies that

$$\beta_3 = -\frac{\alpha}{4} C_3 e^{-(\alpha/4)y_3}$$

where y_3 is as given by (2.8). So

$$C_3 = \frac{-2^{5/4}}{\alpha} (-2\gamma_1\beta_3)^{1/4}(\beta_2 - \gamma_2)^{1/2} < 0 \ ,$$

and then we have

$$v(x) = \frac{2\beta_1 + \beta_2 + \beta_3}{\alpha} - \frac{2^{5/4}}{\alpha} (-2\gamma_1\beta_3)^{1/4}(\beta_2 - \gamma_2)^{1/2} e^{-(\alpha/4)x}$$

and

$$v'(x) = \frac{1}{2^{3/4}} (-2\gamma_1\beta_3)^{1/4}(\beta_2 - \gamma_2)^{1/2} e^{-(\alpha/4)x}$$

for all $x \in (y_3,\infty)$. These results can be summarized as follows:

$$v(x) = \begin{cases} (\gamma_1+\gamma_2+2\beta_1)/\alpha & ; x = 0 \\ ((\gamma_1+2\beta_1+\beta_2)/\alpha) - ((\beta_2-\gamma_2)/\alpha)e^{-(\alpha/2)x} & , x \in (0, \frac{2}{\alpha} \ln[\frac{\beta_2-\gamma_2}{-2\gamma_1}]] \\ \frac{2\beta_1 + \beta_2}{\alpha} - \frac{3}{2^{2/3}}(-\gamma_1)^{1/3}(\beta_2-\gamma_2)^{2/3}e^{-(\alpha/3)x} & , x \in (\frac{2}{\alpha} \ln[\frac{\beta_2-\gamma_2}{-2\gamma_1}] , \frac{1}{\alpha} \ln[\frac{-\gamma_1(\beta_2-\gamma_2)^2}{4\beta_3^3}]] \\ \frac{2\beta_1+\beta_2+\beta_3}{\alpha} - \frac{4^{3/4}}{\alpha}(-\gamma_1\beta_3)^{1/4}(\beta_2-\gamma_2)^{1/2}e^{-(\alpha/4)x} & , x \in (\frac{1}{\alpha} \ln[\frac{-\gamma_1(\beta_2-\gamma_2)^2}{4\beta_3^3}],\infty) \end{cases}$$

$$v'(x) = \begin{cases} \frac{\beta_2-\gamma_2}{2} e^{-(\alpha/2)x} & , x \in (0, \frac{2}{\alpha} \ln[\frac{\beta_2-\gamma_2}{-2\gamma_1}]] \\ \frac{1}{2^{2/3}}(-\gamma_1)^{1/3}(\beta_2-\gamma_2)^{2/3}e^{-(\alpha/3)x} & , x \in (\frac{2}{\alpha} \ln[\frac{\beta_2-\gamma_2}{-2\gamma_1}], \frac{1}{\alpha} \ln[\frac{-\gamma_1(\beta_2-\gamma_2)^2}{4\beta_3^3}]] \\ \frac{1}{4^{1/4}}(-\gamma_1\beta_3)^{1/4}(\beta_2-\gamma_2)^{1/2}e^{-(\alpha/4)x} & , x \in (\frac{1}{\alpha} \ln[\frac{-\gamma_1(\beta_2-\gamma_2)^2}{4\beta_3^3}],\infty) \end{cases}$$

$$P^*(x) = \begin{cases} 2 & , \quad x = 0 \\ 1 & , \quad x \in (0, (2/\alpha)\ln[\frac{\beta_2-\gamma_2}{-2\gamma_1}]] \\ 0 & , \quad x \in ((2/\alpha)\ln[\frac{\beta_2-\gamma_2}{-2\gamma_1}],\infty) \end{cases}$$

$$R^*(x) = \begin{cases} 2 & , \quad x = 0 \\ 3 & , \quad x \quad (0,(1/\alpha)\ln[\frac{-\gamma_1(\beta_2-\gamma_2)^2}{4\beta_3^3}]] \\ 4 & , \quad x \in ((1/\alpha)\ln[\frac{-\gamma_1 (\beta_2-\gamma_2)^2}{4\beta_3^3}],\infty) \end{cases} .$$

Case 2. $-\gamma_1 < \beta_3$. Then it follows from (2.4) and (2.5) that $y_3 < z_1$, and $(r^*(x),p^*(x)) = (4,1)$ for $x \in (y_3,z_1]$. Then v satisfies

$$\gamma_1 + 2\beta_1 + \beta_2 + \beta_3 - 3v'(x) - \alpha v(x) = 0\ , \quad x \in (y_3,z_1] \qquad (2.10)$$

which yields

$$v(x) = \frac{\gamma_1 + 2\beta_1 + \beta_2 + \beta_3}{\alpha} + C_4 e^{-(\alpha/3)x} \qquad (2.11)$$

and

$$v'(x) = -\frac{\alpha}{3} C_4 e^{-(\alpha/3)x}$$

for all $x \in (y_3,z_1]$. Then it follows from the continuity of v that

$$\frac{\gamma_1 + 2\beta_1 + \beta_2 + \beta_3}{\alpha} + C_4 e^{-(\alpha/3)y_3} = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + \frac{\gamma_2 - \beta_2}{\alpha} e^{-(\alpha/2)y_3} \qquad (2.12)$$

where y_3 is as given by (2.4). Solving (2.12) for C_4, we obtain

$$C_4 = -\frac{3}{2^{2/3}\alpha} (\beta_3)^{1/3} (\beta_2 - \gamma_2)^{2/3} < 0\ .$$

Then

$$v(x) = \frac{\gamma_1 + 2\beta_1 + \beta_2 + \beta_3}{\alpha} - \frac{3}{2^{2/3}\alpha} (\beta_3)^{1/3} (\beta_2 - \gamma_2)^{2/3} e^{-(\alpha/3)x}$$

and

$$v'(x) = \frac{1}{2^{2/3}} (\beta_3)^{1/3} (\beta_2 - \gamma_2)^{2/3} e^{-(\alpha/3)x}$$

for all $x \in (y_3,z_1]$. Now the boundary condition $v'(z_1) = -\gamma_1$ implies

$$v'(z_1) = -\gamma_1 = \frac{1}{2^{2/3}} (\beta_3)^{1/3} (\beta_2 - \gamma_2)^{2/3} e^{-(\alpha/3)z_1}$$

or

$$z_1 = \frac{1}{\alpha} \ln[\frac{\beta_3(\beta_2 - \gamma_2)^2}{-4\gamma_1^3}] . \tag{2.13}$$

It immediately follows that $(r^*(x),p^*(x)) = (4,0)$ on (z_1,∞), and v then satisfies

$$2\beta_1 + \beta_2 + \beta_3 - 4v'(x) - \alpha v(x) = 0 \quad , \quad x \in (z_1,\infty) \tag{2.14}$$

which can be shown to equal

$$v(x) = \frac{2\beta_1 + \beta_2 + \beta_3}{\alpha} + C_5 e^{-(\alpha/4)x} \quad , \quad x \in (z_1,\infty)$$

and

$$v'(x) = -\frac{\alpha}{4} C_5 e^{-(\alpha/4)x} \quad , \quad x \in (z_1,\infty) .$$

By the boundary condition $v'(z_1) = -\gamma_1$, we obtain

$$C_5 = -\frac{4^{3/4}}{\alpha}(-\gamma_1\beta_3)^{1/4}(\beta_2 - \gamma_2)^{1/2} < 0 .$$

Then

$$v(x) = \frac{2\beta_1 + \beta_2 + \beta_3}{\alpha} - \frac{4^{3/4}}{\alpha}(-\gamma_1\beta_3)^{1/4}(\beta_2 - \gamma_2)^{1/2}e^{-(\alpha/4)x}$$

and

$$v'(x) = \frac{1}{4^{1/4}}(-\gamma_1\beta_3)^{1/4}(\beta_2 - \gamma_2)^{1/2}e^{-(\alpha/4)x}$$

for all $x \in (z_1,\infty)$.

Similarly we can summarize the results as follows:

$$v(x) = \begin{cases} (\gamma_1+\gamma_2+2\beta_1)/\alpha & , \ x = 0 \\ \dfrac{\gamma_1+2\beta_1+\beta_2}{\alpha} - \dfrac{\beta_2-\gamma_2}{\alpha} e^{-(\alpha/2)x} & , \ x \in (0, \dfrac{2}{\alpha} \ln[\dfrac{\beta_2-\gamma_2}{2\beta_3}]] \\ \dfrac{\gamma_1+2\beta_1+\beta_2+\beta_3}{\alpha} - \dfrac{3}{2^{2/3}\alpha}(\beta_3)^{1/3}(\beta_2-\gamma_2)^{2/3}e^{-(\alpha/3)x} & , \ x \in (\dfrac{2}{\alpha}\ln[\dfrac{\beta_2-\gamma_2}{2\beta_3}], \\ & \quad \dfrac{1}{\alpha}\ln[\dfrac{\beta_3(\beta_2-\gamma_2)^2}{-4\gamma_1^3}]] \\ \dfrac{2\beta_1+\beta_2+\beta_3}{\alpha} - \dfrac{4^{3/4}}{\alpha}(-\gamma_1\beta_3)^{1/4}(\beta_2-\gamma_2)^{1/2}e^{-(\alpha/4)x} & , \ x \in (\dfrac{1}{\alpha}\ln[\dfrac{\beta_3(\beta_2-\gamma_2)^2}{-4\gamma_1^3}],\infty). \end{cases}$$

$$v'(x) = \begin{cases} \dfrac{\beta_2 - \gamma_2}{\alpha} e^{(-\alpha/2)x} & , \ x \in (0, \dfrac{2}{\alpha}\ln[\dfrac{\beta_2-\gamma_2}{2\beta_3}]] \\ \dfrac{1}{2^{2/3}}(\beta_3)^{1/3}(\beta_2-\gamma_2)^{2/3}e^{-(\alpha/3)x} & , \ x \in (\dfrac{2}{\alpha}\ln[\dfrac{\beta_2-\gamma_2}{2\beta_3}], \dfrac{1}{\alpha}\ln[\dfrac{\beta_3(\beta_2-\gamma_2)^2}{-4\gamma_1^3}]] \\ \dfrac{1}{4^{1/4}}(-\gamma_1\beta_3)^{1/4}(\beta_2-\gamma_2)^{1/2}e^{-(\alpha/4)x} & , \ x \in (\dfrac{1}{\alpha}\ln[\dfrac{\beta_3(\beta_2-\gamma_2)^2}{-4\gamma_1^3}],\infty). \end{cases}$$

$$P^*(x) = \begin{cases} 2 & , \ x = 0 \\ 1 & , \ x \in (0, \dfrac{1}{\alpha} \ln[\dfrac{\beta_3(\beta_2-\gamma_2)^2}{-4\gamma_1^3}]] \\ 0 & , \ x \in (\dfrac{1}{\alpha}\ln[\dfrac{\beta_3(\beta_2-\gamma_2)^2}{-4\gamma_1^3}],\infty) . \end{cases}$$

$$R^*(x) = \begin{cases} 2 & , \quad x = 0 \\ 3 & , \quad x \in (0, \frac{2}{\alpha} \ln[\frac{\beta_2-\gamma_2}{2\beta_3}]] \\ 4 & , \quad x \in (\frac{2}{\alpha} \ln[\frac{\beta_2-\gamma_2}{2\beta_3}], \infty) . \end{cases}$$

Note that v is concave increasing and differentiable with a bounded continuous derivative. It is also obvious from the characterizations that $\alpha v(\infty) = 2\beta_1+\beta_2+\beta_3$, and $v'(\infty) = 0.$ Δ

REMARK (2.1). In Example (2.1) C_1 is chosen so as to guarantee the continuity of v at zero; however this convention results in

$$v'(0) = \frac{\beta_2 - \gamma_2}{2} < \beta_2 = D^+L_3(p(0)). \tag{2.15}$$

So in the Bang-Bang case $v'(0) = D^+L_3(p(0))$ condition may fail because for any $n \in \mathbb{N}_+$ $\lim_{x\downarrow 0} p_n(x) = 1 \neq p(0)$ and

$\lim_{x\downarrow 0} r_n(x) = 3 \neq p(0)$. However $v'(0) \leq D^+L_3(p(0)) = \beta_2$ must still hold true. Δ

EXAMPLE (2.2). Let $\overline{r} = 4$, $\overline{p} = 2$, L_2 and L_3 be as given in Example (2.1) and define

$$L_1(x) = \begin{cases} -kx & , \quad x \in [0,x_0] \\ -kx_0 & , \quad x \in [x_0,\infty) \end{cases} \tag{2.16}$$

for some $k \geq 0$ and $x_0 \in \mathbb{R}_+$. Let the optimal control pair (r^*,p^*) be defined as in Example (2.1). Then by (1.6) v satisfies

$$\gamma_1 + 2\beta_1 + \beta_2 - 2v'(x) - kx - \alpha v(x) = 0 , \quad x \in (0,z_1 \wedge y_3 \wedge x_0]$$

which implies that

$$v(x) = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + \frac{2k}{\alpha^2} - \frac{k}{\alpha} x + C_1 e^{-(\alpha/2)x}$$

and

$$v'(x) = -\frac{k}{\alpha} - \frac{\alpha}{2} C_1 e^{-(\alpha/2)x}$$

for all $x \in (0, z_1 \wedge y_3 \wedge x_0]$. Recall that $\alpha v(0) = \gamma_1 + \gamma_2 + 2\beta_1$ which implies by the continuity of v at zero that

$$\frac{\gamma_1 + \gamma_2 + 2\beta_1}{\alpha} = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + \frac{2k}{\alpha^2} + C_1 .$$

Then the constant of integration C_1 turns out to be equal to $[-2k - \alpha(\beta_2 - \gamma_2)]/\alpha^2$, so

$$v(x) = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + \frac{2k}{\alpha^2} - \frac{k}{\alpha} x - \frac{2k}{\alpha^2} e^{-(\alpha/2)x} - \frac{\beta_2 - \gamma_2}{\alpha} e^{-(\alpha/2)x}$$

and

$$v'(x) = -\frac{k}{\alpha} + \frac{k}{\alpha} e^{-(\alpha/2)x} + \frac{\beta_2 - \gamma_2}{2} e^{-(\alpha/2)x}$$

for all $x \in (0, z_1 \wedge y_3 \wedge x_0]$. By the boundary conditions $v'(y_3) = \beta_3$ and $v'(z_1) = -\gamma_1$, we obtain

$$y_3 = \frac{2}{\alpha} \ln\left[\frac{2k + \alpha(\beta_2 - \gamma_2)}{2(\alpha\beta_3 + k)}\right] \qquad (2.17)$$

and

$$z_1 = \frac{2}{\alpha} \ln\left[\frac{2k + \alpha(\beta_2 - \gamma_2)}{2(-\alpha\gamma_1 + k)}\right]. \qquad (2.18)$$

Again possible cases should be taken into consideration.

Case 1. $-\gamma_1 > \beta_3$. Then it follows from (2.17) and (2.18) that $y_3 > z_1$. If $x_o > z_1$, then z_1 as given by (2.18) is the optimal solution. Consequently $(r^*(x),p^*(x)) = (3,0)$ for $x \in (z_1, y_3 \wedge x_o]$. Then the optimal return function satisfies

$$2\beta_1 + \beta_2 - 3v'(x) - kx - \alpha v(x) = 0 \ , \quad x \in (z_1, y_3 \wedge x_o] \qquad (2.19)$$

which is equal to

$$v(x) = \frac{2\beta_1 + \beta_2}{\alpha} + \frac{3k}{\alpha^2} - \frac{k}{\alpha} x + C_2 e^{-(\alpha/3)x} \quad , \quad x \in (z_1, y_3 \wedge x_o]$$

and

$$v'(x) = -\frac{k}{\alpha} - \frac{\alpha}{3} C_2 e^{-(\alpha/3)x} \quad , \quad x \in (z_1, y_3 \wedge x_o] \ .$$

Then it follows from the continuity of v that

$$\frac{2\beta_1 + \beta_2}{\alpha} + \frac{3k}{\alpha^2} - \frac{k}{\alpha} z_1 + C_2 e^{-(\alpha/3)z_1} = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + \frac{2k}{\alpha^2} - \frac{k}{\alpha} z_1 - \frac{2k}{\alpha^2} e^{-(\alpha/2)z_1} - \frac{\beta_2 - \gamma_2}{\alpha} e^{-(\alpha/2)z_1} \qquad (2.20)$$

where z_1 is as given by (2.18). Solving (2.20), we obtain

$$C_2 = -\frac{3}{2^{2/3}\alpha^2} (-\alpha\gamma_1 + k)^{1/3} (2k + \alpha(\beta_2 - \gamma_2))^{2/3} < 0.$$

Then it becomes that

$$v(x) = \frac{2\beta_1 + \beta_2}{\alpha} + \frac{3k}{\alpha^2} - \frac{k}{\alpha} x - \frac{3}{2^{2/3}\alpha^2} (-\alpha\gamma_1 + k)^{1/3} (2k + \alpha(\beta_2 - \gamma_2))^{2/3} e^{-(\alpha/3}$$

and

$$v'(x) = -\frac{k}{\alpha} + \frac{1}{2^{2/3}\alpha}(-\alpha\gamma_1 + k)^{1/3}(2k + \alpha(\beta_2 - \gamma_2))^{2/3}e^{-(\alpha/3)x}$$

for $x \in (z_1, y_3 \wedge x_0]$. It then follows from the boundary condition $v'(y_3) = \beta_3$ that

$$v'(y_3) = \beta_3 = -\frac{k}{\alpha} + \frac{1}{2^{2/3}\alpha}(-\alpha\gamma_1 + k)^{1/3}(2k + \alpha(\beta_2 - \gamma_2))^{2/3}e^{-(\alpha/3)y_3}$$

or

$$y_3 = \frac{1}{\alpha}\ln\left[\frac{(-\alpha\gamma_1 + k)(2k + \alpha(\beta_2 - \gamma_2))^2}{2^2(\alpha\beta_3 + k)^3}\right]. \qquad (2.21)$$

If $x_0 > y_3$ as given by (2.21), then y_3 is the optimal solution. If $x_0 \leq y_3$, then v satisfies

$$v(x) = \frac{2\beta_1 + \beta_2 - kx_0}{\alpha} + C_3e^{-(\alpha/3)x} \quad , \quad x \in [x_0, y_3] .$$

Then by the continuity of v at x_0, we obtain

$$v(x) = \frac{2\beta_1 + \beta_2 - kx_0}{\alpha} + \frac{3k}{\alpha^2}e^{-(\alpha/3)(x-x_0)}$$

$$- \frac{3}{2^{2/3}\alpha^2}(-\alpha\gamma_1 + k)^{1/3}(2k + \alpha(\beta_2 - \gamma_2))^{2/3}e^{-(\alpha/3)x}$$

for $x \in [x_0, y_3]$. Now $v'(y_3) = \beta_3$ implies

$$v'(y_3) = \beta_3 = -\frac{k}{\alpha}e^{-(\alpha/3)(y_3-x_0)} + \frac{1}{2^{2/3}\alpha}(-\alpha\gamma_1+k)^{1/3}(2k+\alpha(\beta_2-\gamma_2))^{2/3}e^{-(\alpha/3)y_3}$$

or

$$y_3 = \frac{1}{\alpha}\ln\left[\frac{(-\alpha\gamma_1+k)^{1/3}(2k + \alpha(\beta_2-\gamma_2))^{2/3} - 2^{2/3}ke^{(\alpha/3)x_0}}{2^{2/3}\alpha\beta_3}\right]^3. \qquad (2.22)$$

On the other hand if $x_o \leq z_1$ which is given by (2.18), then v satisfies

$$\gamma_1 + 2\beta_1 + \beta_2 - 2v'(x) - kx_o - \alpha v(x) = 0 , \quad x \in [x_o,z_1] \qquad (2.23)$$

or

$$v(x) = \frac{\gamma_1 + 2\beta_1 + \beta_2 - kx_o}{\alpha} + C_4 e^{-(\alpha/2)x} , \quad x \in [x_o,z_1] .$$

The continuity of v at x_o implies that

$$\frac{\gamma_1 + 2\beta_1 + \beta_2 - kx_o}{\alpha} + C_4 e^{-(\alpha/2)x_o} = \frac{\gamma_1 + 2\beta_1 + \beta_2}{\alpha} + \frac{2k}{\alpha^2} - \frac{k}{\alpha} x_o$$
$$- \frac{2k}{\alpha^2} e^{-(\alpha/2)x_o} - \frac{\beta_2 - \gamma_2}{\alpha} e^{-(\alpha/2)x_o} \qquad (2.24)$$

Solving (2.24) for C_4, we obtain

$$C_4 = \frac{2k}{\alpha^2} e^{(\alpha/2)x_o} - \frac{2k}{\alpha^2} - \frac{\beta_2 - \gamma_2}{\alpha}$$

and

$$v(x) = \frac{\gamma_1 + 2\beta_1 + \beta_2 - kx_o}{\alpha} + \frac{2k}{\alpha^2} e^{-(\alpha/2)(x-x_o)} - \frac{2k}{\alpha^2} e^{-(\alpha/2)x} - \frac{\beta_2-\gamma_2}{\alpha} e^{-(\alpha}$$

for $x \in [x_o,z_1]$. Then $v'(z_1) = -\gamma_1$ implies that

$$v'(z_1) = - \gamma_1 = - \frac{k}{\alpha} e^{-(\alpha/2)(z_1-x_o)} + \frac{k}{\alpha} e^{-(\alpha/2)z_1} + \frac{\beta_2 - \gamma_2}{2} e^{-(\alpha/2)z_1}$$

or

$$z_1 = \frac{1}{\alpha} \ln\left[\frac{\alpha(\beta_2 - \gamma_2) + 2k - 2ke^{(\alpha/2)x_o}}{-2\alpha\gamma_1}\right]^2 \qquad (2.25)$$

which turns out to be the optimal z_1 in this case. Then $(r^*(x),p^*(x)) = (3,0)$ for $x \in (z_1,y_3]$, and the optimal return function v satisfies

$$2\beta_1 + \beta_2 - 3v'(x) - kx_o - \alpha v(x) = 0 \quad , \quad x \in (z_1, y_3] \tag{2.26}$$

which in turn equals

$$v(x) = \frac{2\beta_1 + \beta_2 - kx_o}{\alpha} + C_5 e^{-(\alpha/3)x} \quad , \quad x \in (z_1, y_3]$$

and

$$v'(x) = -\frac{\alpha}{3} C_5 e^{-(\alpha/3)x} \quad , \quad x \in (z_1, y_3]$$

Then by the continuity of v at z_1 we have

$$\frac{2\beta_1 + \beta_2 - kx_o}{\alpha} + C_5 e^{-(\alpha/3)z_1} = \frac{\gamma_1 + 2\beta_1 + \beta_2 - kx_o}{\alpha} + \frac{2k}{\alpha^2} e^{-(\alpha/2)(z_1-x}$$
$$- \frac{2k}{\alpha^2} e^{-(\alpha/2)z_1} - \frac{\beta_2 - \gamma_2}{\alpha} e^{-(\alpha/2)z_1} \tag{2.27}$$

where z_1 is as given by (2.25). Solving (2.27) for C_5, we get

$$C_5 = -\frac{3}{2^{2/3}\alpha^2}(-\alpha\gamma_1)^{1/3}(\alpha(\beta_2 - \gamma_2) + 2k - 2ke^{(\alpha/2)x_o})^{2/3} < 0$$

and

$$v'(x) = \frac{1}{2^{2/3}\alpha}(-\alpha\gamma_1)^{1/3}(\alpha(\beta_2 - \gamma_2) + 2k - 2ke^{(\alpha/2)x_o})^{2/3} e^{-(\alpha/3)x}$$

for $x \in (z_1, y_3]$. Then the boundary condition $v'(y_3) = \beta_3$ implies

$$v'(y_3) = \beta_3 = \frac{1}{2^{2/3}\alpha}(-\alpha\gamma_1)^{1/3}(\alpha(\beta_2 - \gamma_2) + 2k - 2ke^{(\alpha/2)x_o})^{2/3} e^{-(\alpha/3)y_3}$$

or

$$y_3 = \frac{1}{\alpha} \ln\left[\frac{(-\alpha\gamma_1)(\alpha(\beta_2 - \gamma_2) + 2k - 2ke^{(\alpha/2)x_o})^2}{2^2\alpha^3\beta_3^3}\right] .$$

In summary,

(1) If $x_o > \frac{2}{\alpha}\ln[\frac{2k + \alpha(\beta_2 - \gamma_2)}{2(-\alpha\gamma_1 + k)}]$, then $z_1 = \frac{2}{\alpha}\ln[\frac{2k + \alpha(\beta_2 - \gamma_2)}{2(-\alpha\gamma_1 + k)}]$;

(i) Then if $x_o > \frac{1}{\alpha}\ln[\frac{(-\alpha\gamma_1 + k)(2k + \alpha(\beta_2 - \gamma_2))^2}{2^2(\alpha\beta_3 + k)^3}]$,

$$y_3 = \frac{1}{\alpha}\ln[\frac{(-\alpha\gamma_1 + k)(2k + \alpha(\beta_2 - \gamma_2))^2}{2^2(\alpha\beta_3 + k)^3}];$$

(ii) If $x_o \leq \frac{1}{\alpha}\ln[\frac{(-\alpha\gamma_1 + k)(2k + \alpha(\beta_2 - \gamma_2))^2}{2^2(\alpha\beta_3 + k)^3}]$,

$$y_3 = \frac{1}{\alpha}\ln[\frac{(-\alpha\gamma_1 + k)^{1/3}(2k + \alpha(\beta_2 - \gamma_2))^{2/3} - 2^{2/3}ke^{(\alpha/3)x_o}}{2^{2/3}\alpha\beta_3}]^3;$$

(2) If $x_o \leq \frac{2}{\alpha}\ln[\frac{2k + \alpha(\beta_2 - \gamma_2)}{2(-\alpha\gamma_1 + k)}]$, then

$$z_1 = \frac{1}{\alpha}\ln[\frac{\alpha(\beta_2 - \gamma_2) + 2k - 2ke^{(\alpha/2)x_o}}{-2\alpha\gamma_1}]^2 \text{ and}$$

$$y_3 = \frac{1}{\alpha}\ln[\frac{(-\alpha\gamma_1)(\alpha(\beta_2 - \gamma_2) + 2k - 2ke^{(\alpha/2)x_o}}{2^2\alpha^3\beta_3^3}].$$

<u>Case 2.</u> $-\gamma_1 \leq \beta_3$. Then it follows from (2.17) and (2.18) that $y_3 \leq z_1$. The same argument presented above can be repeated here to obtain symmetrical results.

<u>REMARK (2.2)</u>. It follows from the examples given above that when there are no jump inputs to the store and no jump outputs from the store the optimal return function v is obtained by recursively solving an ordinary first order differential equation. However in the presence of stochastic output and input processes it becomes necessary to solve a functional differential equation which complicates the procedure greatly, so that the computations involved become cumbersome. Δ

X. SUMMARY OF RESULTS

In this dissertation the optimal control problem of the generalized storage model subject to both random jump inputs and outputs is considered where the content level of the store can be controlled through proper choices of input and output rates. Under mild conditions the existence and uniqueness of optimal input and output control functions which maximize the expected infinite time discounted earnings and the associated optimal return function is proven for the deterministic problem. The solution procedure is then extended to include the stochastic processes, but proven to be inefficient in handling with the stochastic output process, although it can be easily applied to the stochastic models where there exists only a random input process. The results obtained both in the deterministic case and the stochastic case with a random input process are symmetrical such that as functions of the content level of the store the optimal output rate is shown to be increasing while the optimal input rate is decreasing. The optimal return function is shown to be concave increasing until a certain level is reached and decreasing from then on.

Once the generalized storage process is introduced by (I.1), a brief review of studies carried out so far on storage theory is provided in Chapter II, and possible applications of the generalized storage model are discussed in Chapter III. Our emphasis in Chapter IV is on analyzing the uncontrolled storage model. We first specify the properties of the input and output processes as two independent compound Poisson processes and then characterize the set of admissible controls so as to meet the model requirements and to guarantee the existence of a unique solution to the generalized storage model. Next the storage process is constructed and shown to be a Hunt process for any given admissible control pair. Its generator together with its domain and range is specified to enable us to employ Markov decision theory in the optimal control problem.

The basic features of the optimal control problem are introduced in Chapter V where the cost and reward structure is specified by the assumptions imposed on them. Then a Markov decision theoretic approach is employed to express the sufficient condition of global optimality in terms of a functional differential equation. In a similar manner the sufficient condition of local optimality, optimality with respect to a subset of admissible controls, is derived. The first step of our procedure to analyze the optimal control problem is to study the corresponding deterministic problem in Chapter VI. By showing the existence and uniqueness of a return function which satisfies the sufficient condition of local optimality, we thus create a sequence of locally optimal return functions. Later L_1 is assumed to be decreasing in order to guarantee that the limit of this locally optimal return functions is the global optimal return function. Moreover the local and global optimal control pairs are characterized as functions of the marginal utility.

The results and the procedure of the deterministic problem are extended in Chapter VII so as to include the stochastic input process involved in the original generalized storage model. Again the existence and uniqueness of local and global optimal control pairs are proven under the condition that L_1 is decreasing. In Chapter VIII some natural generalizations of the model characteristics are presented, and some solution procedures are briefly discussed. Finally in Chapter IX, the theory is extended for the case when both L_2 and L_3 have piecewise linear structures, and the optimal control pair is shown to be of the Bang-Bang form whose uniqueness may in general fail. The procedure outlined is illustrated with some example problems of the deterministic case.

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