PSEUDO-ANOSOV DIFFEOMORPHISMS AND THE MONODROMY OF AN OPEN BOOK

by

Sinan Eden

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Estragon: We always find something, eh Didi, to give us the impression we exist? Endgame, Samuel Beckett.

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ABSTRACT

PSEUDO-ANOSOV DIFFEOMORPHISMS AND THE MONODROMY OF AN OPEN BOOK

In this thesis, we study the proof of the fact that any mapping class on a compact oriented surface with non-empty boundary can be made pseudo-Anosov after a sequence of positive stabilizations. In the language of contact topology, it means that an abstract open book can be stabilized in order to make its monodromy isotopic to a pseudo-Anosov homeomorphism. To attain this goal we use the curve complex of the surface and the classification of surface diffeomorphisms, the latter of which is the secondary goal of this thesis. In order to classify surface diffeomorphisms, we study Thurston's compactification of the Teichmüller space, which uses essential curves and measured foliations.

ÖZET

ANOSOVUMSU DİFEOMORFİZMLER VE AÇIK KİTAPLARIN MONODROMİLERİ

Bu tezde, bir tıkız yönlendirilebilir kenarlı yüzeyin herhangi bir gönderim sınıfının, pozitif dengelemeler aracılığıyla Anosovumsu yapılabileceğinin ispatı incelenmiştir. Temas topolojisi dilinde bu, bir soyut açık kitabın, monodromisi bir Anosovumsu homeomorfizme izotopik yapılacak şekilde dengelenebileceği anlamına gelir. Bu amaca ulaşmak için, yüzeylerin eğri kompleksi ve yüzey difeomorfizmlerinin sınıflandırılması kullanılmıştır. Yüzey difeomorfizmlerinin sınıflandırılması, bu tezin ikincil amacını oluşturmaktadır ve bunun için, esas eğriler ve ölçülü folyolama kullanılarak gerçekleştirilen, Teichmüller uzayının tıkızlaştırılması incelenmiştir.

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LIST OF SYMBOLS

$A(P^2)$	The set of isotopy classes of arcs $I \subset P^2$ with $\partial I \subset \partial P^2$ each end free to move on the respective connected component of P^2 and representing the nontrivial elements of $\pi_1(P^2, \partial P^2)$
$A'(P^2)$	The same as $A(P^2)$ but with several pairwise disjoint arcs
$\mathbb{D}^2$	The Poincaré disk model for the 2-dimensional hyperbolic space
$\mathrm{Diff}(D^2, rel(K, \partial))$	The set of diffeomorphisms $\varphi : D^2 \to D^2$ such that $\varphi\|_{K \cup \partial D^2} = id$
$\mathrm{Diff}(D^2, K, rel\partial)$	The set of diffeomorphisms $\psi : D^2 \to D^2$ such that $\psi(K) = K$ and $\psi\|_{\partial D^2} = id$
$\mathrm{Diff}(X)$	The set of diffeomorphisms of X
$\mathrm{Diff}^+(X)$	The set of orientation-preserving diffeomorphisms of X
$\mathrm{Diff}_0(X)$	The set of diffeomorphisms of X that are isotopic to identity
$\mathcal{E}$	The space of hyperbolic metric on a hexagon which make the edges geodesics and all inner angles right angles
$\mathcal{H}$	The set of hyperbolic metrics on the given surface
$\mathbb{H}^2$	The upper half-plane model for the 2-dimensional hyperbolic space
$i(\alpha, \beta)$	The geometric intersection number between α and β
$\mathcal{MF}$	The space of measured foliations
$\mathcal{NF}$	The space of normal forms
P^2	A pair of pants
$\mathcal{P}$	The set hyperbolic metrics on P^2 that make the boundary components geodesics
$\mathbb{R}_+$	The set of non-negative real numbers
$\mathbb{R}^*$	The set of non-zero real numbers
$\mathbb{R}_+^{\mathcal{S}}$	The set of functions from $\mathcal{S}$ into $\mathbb{R}_+$
$\mathcal{S}$	The space of essential curves
$\mathcal{S}'$	The space of multicurves
$sing\mathcal{F}$	The set of singularities of the foliation $\mathcal{F}$

$\mathcal{T}$	Teichmüller space
$\overline{\mathcal{T}}$	The compactified Teichmüller space
$\overline{\pi}_1(S)$	The group of free-homotopy classes of loops in S
$\pi_n(X)$	The n^{th} homotopy group of X
$\partial_1, \partial_2, \partial_3$	The boundary components of P^2
$\approx$	Used interchangably to denote a homeomorphism between topological spaces or to mean "approximately equal to"
$\star$	A point
$\rightsquigarrow$	is deformed by a free-homotopy to
$(\leq \triangledown)$	The region of triangle inequality

1. INTRODUCTION

In [1] E. Giroux demonstrated a 1-1 correspondence between isomorphism classes of contact structures on closed 3-manifolds and equivalence classes of open books, up to positive stabilization and conjugation; obtaining a link between 3-dimensional topology and 3-dimensional differential geometry.

An *(abstract) open book* is defined to be a pair (S, h) where S is a compact oriented surface with non-empty, possibly disconnected, boundary ∂S, and $h : S \to S$ is a diffeomorphism for which $h|_{\partial S} = id$. As described in [2], the pair (S, h) can be interpreted as an *open book decomposition* (B, π) of a 3-manifold M, where B is an oriented link in M and $\pi : M \setminus B \to S^1$ is a fibration such that $\pi^{-1}(\theta)$ is the interior of a compact surface $\Sigma \subset M$ and $\partial\Sigma = B$ for all $\theta \in S^1$. By a theorem of J. W. Alexander [3], every closed oriented 3-manifold has an open book decomposition. Hence, in order to understand 3-dimensional manifolds, it is natural to study compact oriented surfaces and their diffeomorphisms.

Our primary goal in this thesis is to comprehend a theorem of V. Colin and K. Honda [4] stating that any mapping class on a compact oriented surface with non-empty boundary can be made pseudo-Anosov after a sequence of positive stabilizations. We give the appropriate setup and results in Chapter 6.

Pursuing this direction led us to a secondary goal of understanding the classification of surface diffeomorphisms given by W. P. Thurston [5]. In the proof of the classification of surface diffeomorphisms, we will utilize Thurston's compactification of the Teichmüller space, following the detailed exposition given by L. Fathi, F. Laudenbach and V. Poenaru [6]. To this end, we will first introduce the *Teichmüller space* $\mathcal{T}$ and propose a way of parametrization in Chapter 2. In Chapter 3, we will present the space $\mathcal{S}$ of *essential curves* and the space $\mathcal{MF}$ of *measured foliations*, and discuss the link between the two. Then, in Chapter 4, we give the compactification of the Teichmüller space. Finally in Chapter 5, we outline the classification of surface diffeomorphisms.

The idea of the classification of surface diffeomorphisms is to see the projectivization $P\mathcal{MF}$ of the space of measured foliations as the boundary of $\mathcal{T}$. We will see that for a closed orientable surface S of genus $g > 1$, the Teichmüller space is homeomorphic to $\mathbb{R}^{6g-6}$ (Corollary 2.20) and the projectivization of the space of measured foliations is homeomorphic to S^{6g-7} (Theorem 3.57). We will then glue $P\mathcal{MF}$ and $\mathcal{T}$ to get a closed disk D^{6g-6} (Theorem 4.11). The ambient space for this gluing process will be the projective space $P(\mathbb{R}_+^{\mathcal{S}})$ of functions from $\mathcal{S}$ into $\mathbb{R}$. Then the classification follows from analyzing the fixed points of the natural action of $\pi_0(\mathrm{Diff}(S))$ on the compactified Teichmüller space. More precisely, if a diffeomorphism $\varphi : S \to S$ has a fixed point in $\mathcal{T}$, then we show that φ is isotopic to a periodic diffeomorphism. Otherwise if the fixed point is in $P\mathcal{MF}$, this means that φ fixes a foliation and stretches the transverse invariant measure defined on the foliation by a scalar λ. Investigating the effect of φ on the measured foliation and the scalar λ; we will see that φ is isotopic to either a periodic diffeomorphism, or a diffeomorphism that fixes a system of simple curves which are mutually disjoint and not homotopic to a point (i.e. it is *reducible*), or a *pseudo-Anosov diffeomorphism*. Pseudo-Anosov diffeomorphisms are defined in Section 5.4, and some of their important properties are given in Section 5.6.

In Section 1.1, we will treat the particularly simple case of the torus T^2 for the classification of surface diffeomorphisms, which, we suppose, will provide insight for the general case. In the rest of the thesis, we will concentrate on compact orientable surfaces of genus $g > 1$. For this, we will fix our basic setup and notation in Section 1.2.

1.1. Classification of the Diffeomorphisms of the Torus

This section relies on [7]. We let T^2 denote the 2-dimensional torus, and $\mathrm{Diff}(T^2)$ denote the topological group of diffeomorphisms of T^2. It is known that $\mathrm{Diff}(T^2)$ can be identified with $GL(2, \mathbb{Z})$. As the path component of the identity is a normal subgroup in any topological group, we set $\mathcal{T}(T^2) \doteq \mathrm{Diff}(T^2)\ /\ \mathrm{Diff}_0(T^2)$. Then $\mathcal{T}(T^2)$ is identified with $SL(2,\mathbb{Z})$.

Under this identification a diffeomorphism $\varphi \in \mathcal{T}(T^2)$ corresponds to a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and $\det(A) = 1$. We now study the eigenvalues $\lambda, \lambda^{-1} \in \mathbb{C}$.

- Case1: $\lambda, \lambda^{-1} \notin \mathbb{R}$. Since $\mathrm{tr}(A) = \lambda + \lambda^{-1} = a + d \in \mathbb{R}$, it follows that λ and λ^{-1} are on the unit circle so that $|\mathrm{tr}(A)| < 2$. As $a, d \in \mathbb{Z}$, we get that either $\mathrm{tr}(A) = 0$ and thus $A^4 = I$, or $\mathrm{tr}(A) = \pm 1$ and thus $A^6 = I$. We say that φ has *finite order.*

- Case2: $\lambda = \lambda^{-1} = 1$. In this case, we either have $\varphi = id$ or A is conjugate to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some $n \in \mathbb{Z}$. We observe that A fixes $(1, 0)$, which can be seen as one of the standard basis vectors of $\pi_1(T^2)$. Such a φ is called *reducible.*

We remark that if $\lambda = \lambda^{-1} = -1$, then we get either $\varphi = -id$ or A is conjugate to a matrix taking $(1, 0)$ to its inverse. Such a φ is also called reducible.

Here, the term reducible refers to the observation that φ fixes a loop in T^2 and if we cut the surface along this loop and *reduce* it to a simpler surface, we will also get another surface on which the diffeomorphism φ is *reduced* to satisfy one of the other cases.

- Case3: $\lambda, \lambda^{-1} \in \mathbb{R} \setminus \{1\}$. Recalling $\det(A) = \lambda\lambda^{-1} = 1$, we suppose without loss of generality that $\lambda > 1 > \lambda^{-1}$. Then A has two distinct eigenvectors $e^{\pm}$. We let $\mathcal{F}^+$ and $\mathcal{F}^-$ be the linear foliations of T^2 by lines parallel to e^+ and e^-, respectively. Now, we note that A takes leaves of $\mathcal{F}^{\pm}$ to themselves, stretching the leaves of $\mathcal{F}^+$ by a factor of λ and the leaves of $\mathcal{F}^-$ by a factor of λ^{-1}. For some choice of Euclidean structure on T^2, these foliations can be taken perpendicular. We will call such a diffeomorphism *Anosov.*

We thus proved the following theorem (compare with Theorem 5.16).

Theorem 1.1. *For any orientation-preserving diffeomorphism φ of T^2, one of the following conditions holds:*

(i) *φ is periodic; i.e. some finite power of φ is isotopic to identity.*
(ii) *φ is reducible; i.e. there is a simple closed curve which is taken to itself by φ, up to isotopy.*
(iii) *φ is Anosov.*

1.2. Basic Setup and Notations

We assume that the reader is familiar with the basics of algebraic topology (see, for example, [8]) and differentiable manifolds (see, for example, [9], [10]).

Until the end of Chapter 5, S will denote a closed orientable surface of genus $g > 1$. In Chapter 6, we will include the case when S has non-empty boundary. We recall that the 2-dimensional hyperbolic space can be considered as the universal covering of S. We denote the universal covering of S by $p : \tilde{S} \to S$. As models for the hyperbolic space, we will use the upper half-plane model $\mathbb{H}^2$ and the Poincaré disk model $\mathbb{D}^2$ interchangably, depending on which serves best for visualizing.

We reserve the notation $\pi : X \to P(X)$ for the projectivization of a space X.

We will repeatedly make use of the Uniformization Theorem, which states that any simply connected Riemann surface admits a Riemannian metric of constant curvature (see, for example, [11], Corollary 2.10). As we concentrate on *hyperbolic* geometry due to our choice of S, when talking about geodesics on S, we will have a Riemannian metric of constant curvature -1 in the background.

We recall that two loops α_0' and α_1' in S are *free-homotopic if* there exists a continuous function $F : [0,1] \times [0,1] \to S$ such that for all $t, s \in [0,1]$, $F(t,0) = \alpha_0'(t)$, $F(t,1) = \alpha_1'(t)$, $F(0,s) = F(1,s)$. We also recall that α_0' and α_1' are called *isotopic if* there exists a diffeomorphism $\Phi : S \times [0,1] \to S \times [0,1]$ such that $\Phi(x,t) = (\phi_t(x), t)$

for all x in S and t in $[0,1]$, where ϕ is a family of diffeomorphisms satisfying $\phi_0 = id$ and $\phi_1(\alpha'_0(s)) = \alpha'_1(s)$ for all $s \in [0,1]$.

The following observations are essential in the sequel (see, for example, [12]).

Theorem 1.2. *Two simple closed curves on a connected surface which are free-homotopic are isotopic.*

Theorem 1.3. *On a surface N with boundary, if γ' and γ'' are two embedded arcs with $\partial\gamma' = \partial\gamma'' = \gamma' \cap \partial N = \gamma'' \cap \partial N$, homotopic with endpoints fixed, then γ' and γ'' are isotopic relative to boundary.*

Theorem 1.4. *A two-sided embedding of the circle in a surface is never homotopic to a k-fold cover of a two-sided simple curve, for $k > 1$.*

Remark 1.5. We will use the following notations for curves in S. By α, β, γ etc., we denote the free-homotopy classes of curves. For representatives of the curve α, we will use α', α'' etc. A lift of the curve α' is denoted by $\tilde{\alpha}'$.

When a curve is denoted by a Latin letter such as the curves $\{K_j\}_{j=1,2,\ldots,3g-3}$ in Section 3.3, we will denote its isotopy class by $[K_j]$.

2. TEICHMÜLLER SPACE

In this chapter, we will construct the Teichmüller space of the surface S and formalize its parametrization. For simplicity, we will additionally assume S to be connected.

2.1. Construction of the Teichmüller Space

There are three equivalent ways to construct the Teichmüller space of S. We will only use one of them in the following chapters, while the others make some of the arguments in the next sections more natural.

(i) Let $\mathcal{H}$ denote the set of all hyperbolic metrics on S. On $\mathcal{H}$, we define an equivalence relation $\sim$ as follows:

$h_1 \sim h_2$ *if and only if* there exists an isometric diffeomorphism
$\phi : (S, h_1) \to (S, h_2)$ isotopic to identity.

We put $\mathcal{T}(S) \doteq \mathcal{H}/\sim$.

(ii) Let $\mathcal{H}'$ be defined by

$\mathcal{H}' \doteq \{ (S',h,f) : S'$ connected, compact, genus g surface; h hyperbolic metric on S'; f orientation-preserving diffeomorphism of S' onto S $\}$

On $\mathcal{H}'$, we define an equivalence relation $\sim'$:

$(S_1,h_1,f_1) \sim' (S_2,h_2,f_2)$ *if and only if* there exists an isometric diffeomorphism
$\phi : (S_1, h_1) \to (S_2, h_2)$ such that $f_2 \circ \phi \circ f_1$ is isotopic to identity.

We then put $\mathcal{T}(S) \doteq \mathcal{H}'/\sim'$.

(iii) On $\mathrm{Diff}^+(S)$, we put the topology of C^∞ convergence. We let $\mathrm{Diff}_0(S)$ denote the connected component of id_S with respect to this topology.

Then $\mathrm{Diff}_0(S)$ acts on $\mathcal{H}$ as follows: For $f \in \mathrm{Diff}_0(S)$ and $h \in \mathcal{H}$; $f_*(h)_x(v,w) = h_{f^{-1}(x)}(d(f_x^{-1}(v), d(f_x^{-1}(w))$.

We define $\mathcal{T}(S) \doteq \mathcal{H}/\mathrm{Diff}_0(S)$.

Remark 2.1. The third construction allows us to endow $\mathcal{T}$ with a topological structure: We consider $\mathcal{H}$ as a subspace of the fiber bundle $T(S)^* \otimes T(S)^*$ and put the C^∞ topology. The group $\mathrm{Diff}^+(S)$ acts on $\mathcal{H}$ continuously as a group of homeomorphisms. So, $\mathcal{T}$ can be equipped with the quotient topology.

2.2. Some Reminders on Hyperbolic Geometry

In the next section, we will give the Fenchel-Nielsen coordinates for the Teichmüller space. The basic idea is to cut the surface into smaller pieces called *pairs of pants*, study the hyperbolic structures on each of these pieces and then glue them back. We define a *pair of pants* to be a thrice-punctured sphere. We first state a topological fact:

Proposition 2.2. *There exist pairwise disjoint loops $\alpha'_1, ...\alpha'_h$ on S such that $S \backslash \bigcup_{i=1}^h \alpha'_i$ consists of k connected components each having closure homeomorphic to a pair of pants.*
Here, $h = 3(g-1)$ and $k = 2(g-1) = \chi(S)$.

Proof. This is easily observable. See Figure 2.1. □

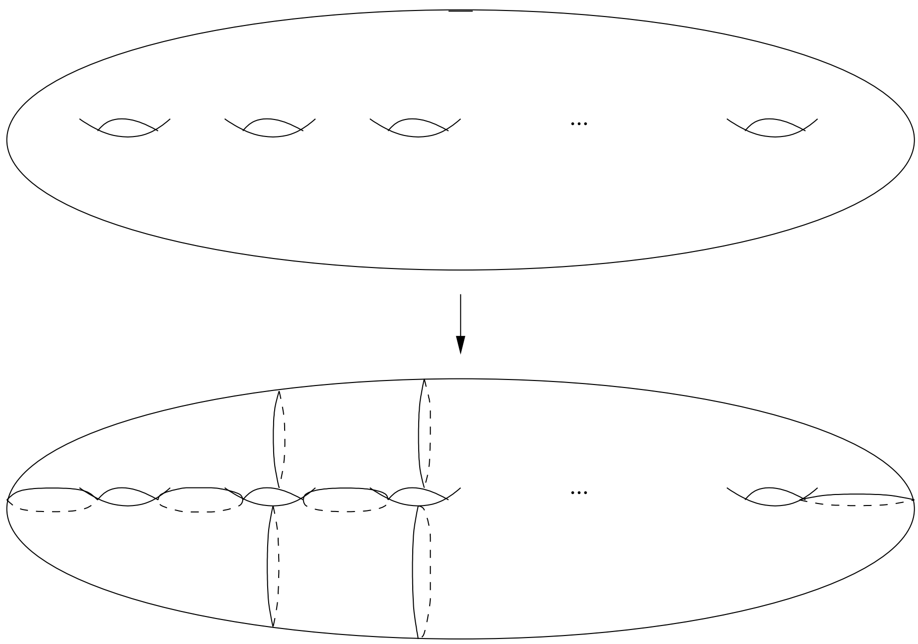


Figure 2.1. Pair of pants decomposition of a genus g surface.

The above construction is called a *pair of pants decomposition* of the surface. The particular decomposition demonstrated in Figure 2.1 is clearly not unique, yet is sufficient for our purposes. When we talk about a pair of pants decomposition of a surface, we will always mean the decomposition given in Figure 2.1.

We are going to show that a point in the Teichmüller space can be parametrized by the lengths of the $3g-3$ curves, plus another $3g-3$ coordinates which record the twisting of each gluing curve. But first, we shall see that the loops in Figure 2.1 can be realized by geodesics.

By $\overline{\pi}_1(S)$, we denote the group of free-homotopy classes of loops in S, while $\pi_1(S)$ denotes the fundamental group of S. We also recall that for a group G, $Int(G)$ denotes the group of homomorphisms of the form $g \mapsto h^{-1}gh$ for *suitable* h in G. (For $\pi_1(S)$, suitable will correspond to elements that can be concatenated with g so that the resulting loop is again in $\pi_1(S)$.)

Proposition 2.3. *Consider the universal covering $p : \tilde{S} \to S$ of S. Then p induces a bijection $\Phi : \pi_1(S)/Int(\pi_1(S)) \longrightarrow \overline{\pi}_1(S)$.*

Proof. We start by observing that for a loop α', and a path β' in S satisfying $\beta'(1) = \alpha'(0)$, the loop $\beta'^{-1}\alpha'\beta'$ is free-homotopic to α', where by β'^{-1} we mean the inverse of β' in $\pi_1(S)$. To see this, it suffices to consider the scheme in Figure 2.2.

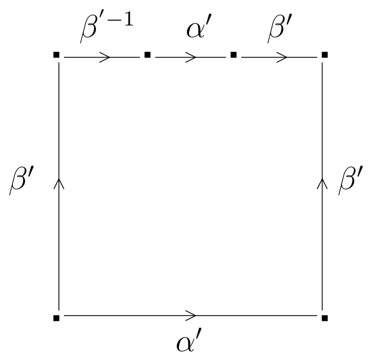


Figure 2.2. The loop $\beta'^{-1}\alpha'\beta'$ is free-homotopic to α'.

We note that the map p induces an action of $\pi_1(S)$ as a group of diffeomorphisms of $\tilde{S}$. Given $T \in \pi_1(S)$, we fix an arbitrary point $\tilde{x} \in \tilde{S}$ and consider the path $\tilde{\alpha}'$

joining $\tilde{x}$ to $T(\tilde{x})$. We then set $\alpha' = p \circ \tilde{\alpha'}$. We denote the free-homotopy class of α' by α, and the coset corresponding to T in $\pi_1(S)/Int(\pi_1(S))$ by $< T >$. Lastly, we define $\Phi(< T >) \doteq \alpha$.

We first show that Φ is well-defined: The definition is independent of the lift $\tilde{\alpha'}$ since $\tilde{S}$ is simply-connected. It is also independent of the choice of $\tilde{x}$ by simply-connectedness of $\tilde{S}$ and the very first observation we made. Now if we take $U^{-1} \circ T \circ U$ as a representative of $< T >$, we can take the path $U^{-1}(\tilde{\alpha'})$ starting at $U^{-1}(\tilde{x})$ and get the same loop in S.

We now construct the inverse of Φ. Given a loop α' in S, we fix a point $\tilde{x}$ in $p^{-1}(\alpha'(0))$ and lift α' to the curve $\tilde{\alpha'}$ starting at $\tilde{x}$. We then select the unique element T in $\pi_1(S)$ satisfying $\tilde{\alpha'}(1) = T(\tilde{x})$.

We claim that the map $\Psi(\alpha) \doteq < T >$ is well defined: If we start with another point $U(\tilde{x})$ in $p^{-1}(\alpha'(0))$, we obtain $U^{-1} \circ T \circ U$; giving the independence from the choice of representative of $< T >$. If we take another loop α'' in α, we can lift the whole free-homotopy from α' to α'' to get the same $< T >$.

Finally, it is clear that the maps Φ and Ψ are inverses of each other. □

Lemma 2.4. *The nontrivial elements of $\pi_1(S)$ act on the hyperbolic space as isometries of hyperbolic type, that is, they have no fixed points inside the hyperbolic space and two fixed points at infinity.*

Proof. We use the upper half-plane model $\mathbb{H}^2$ for the hyperbolic space. It is known that its isometries are classified as elliptic, parabolic and hyperbolic. (See, for example, [13].) We first notice that a nontrivial element of $\pi_1(S)$ cannot have an action of elliptic type since $\pi_1(S)$ acts freely on $\mathbb{H}^2$. We suppose that the nontrivial element T has an action of parabolic type and obtain a contradiction. Under this assumption, T has no fixed points in $\mathbb{H}^2$ and one fixed point at infinity. This yields a sequence x_n such that $\mathrm{d}(x_n, T(x_n)) \longrightarrow 0$. (For example the sequence $\{x_n\}$ given by $x_n = (0, n)$ works.) Thus we can find arbitrarily short nontrivial loops in S. This contradicts to the fact that S is compact; since compactness of S yields that it can be covered by a finite number of open sets that are isometric to balls in $\mathbb{H}^2$, so a loop that is short enough will be contained in one of these sets and hence must be trivial. □

Lemma 2.5. *Each nontrivial free-homotopy class in S contains a unique geodesic loop.*

Proof. Let α' be a nontrivial loop in S and let $T \in \Phi^{-1}(\alpha)$, Φ being as in Proposition 2.3. That is, $T((\tilde{\alpha'}(0))) = \tilde{\alpha}'(1)$. By Lemma 2.4, T is hyperbolic. So there is a unique T-invariant geodesic line $\tilde{\gamma}'$. The projection of $\tilde{\gamma}'$ is a geodesic loop in the class of α'. Now, let γ_1' be a geodesic loop in S representing α. We lift γ_1' it to the universal covering to get $\tilde{\gamma}_1'$, and take T' in $\Phi^{-1}(\alpha)$ so that $T'(\tilde{\gamma}_1'(0)) = \tilde{\gamma}_1'(1)$. Then, since $\tilde{\gamma}_1'$ is a geodesic line, it is T'-invariant: $T'(\tilde{\gamma}_1') = \tilde{\gamma}_1'$. By Proposition 2.3, we notice that $T' = S^{-1} \circ T \circ S$ for some S, which yields $S^{-1}TS(\tilde{\gamma}_1') = \tilde{\gamma}_1'$ so that $TS(\tilde{\gamma}_1') = S(\tilde{\gamma}_1')$. Now, by uniqueness of $\tilde{\gamma}'$, $S(\tilde{\gamma}_1') = \tilde{\gamma}'$. But since the action of $\pi_1(S)$ is free, we obtain $\gamma_1' = \gamma'$. □

We now prove the main result of this section, which asserts that the loops used in the pair of pants decomposition can be realized by geodesics. We state the theorem in full for future reference.

Proposition 2.6. *Let $\alpha_1',\ldots,\alpha_\nu'$ be pairwise non-intersecting and non-isotopic nontrivial simple loops in S. Then we can find pairwise non-intersecting and non-isotopic simple geodesic loops $\gamma_1',\ldots,\gamma_\nu'$ such that each γ_i' is isotopic to α_i'. Moreover, γ_i''s are unique.*

Proof. Uniqueness follows from Lemma 2.5 and Theorem 1.2. We just check that the geodesic loop in the free-homotopy class of α_i' satisfies the following conditions:

(i) If $i \neq j$ then γ_i' and γ_j' are not isotopic.

(ii) If $i \neq j$ then $\gamma_i' \cap \gamma_j' = \emptyset$.

(iii) γ_i''s are simple.

The first item is clear. We start by making an observation.

We recall the construction of $\tilde{\gamma}'$ in the proof of Lemma 2.5. (We lift α' to $\tilde{\alpha}'$ and consider $T_{\alpha'}$ in $\pi_1(S)$ such that $T_{\alpha'}(\tilde{\alpha}'(0)) = \tilde{\alpha}'(1)$. Then we define $\tilde{\gamma}'$ as the unique $T_{\alpha'}$-invariant geodesic line.) Now, we notice that if we extend $\tilde{\alpha}'$ on $\mathbb{R}$, then $\tilde{\alpha}'$ and $\tilde{\gamma}'$ have the same endpoints at infinity: To see this, let $\tilde{\alpha}'_0$ and $\tilde{\gamma}'_0$ be the restrictions of $\tilde{\alpha}'$ and $\tilde{\gamma}'$ to $[0,1]$. We observe the following: $\sup\{d(x,\tilde{\gamma}'_0) : x \in \tilde{\alpha}'_0\} < \infty$ and $\sup\{d(x,\tilde{\alpha}'_0) : x \in \tilde{\gamma}'_0\} < \infty$. Also, $\tilde{\alpha}' = \bigcup_{n\in\mathbb{N}} T^n_{\alpha'}(\tilde{\alpha}'_0)$ and $\tilde{\gamma}' = \bigcup_{n\in\mathbb{N}} T^n_{\alpha'}(\tilde{\gamma}'_0)$. So, $\sup\{d(x,\tilde{\gamma}) : x \in \tilde{\alpha}\} < \infty$ and $\sup\{d(x,\tilde{\alpha}') : x \in \tilde{\gamma}'\} < \infty$. Moreover, we remark that for any lift $\tilde{\gamma}'$ of γ', there is a lift $\tilde{\alpha}'$ of α' having the same endpoints at infinity as $\tilde{\gamma}'$.

Now we show (ii): Suppose $\gamma'_i \cap \gamma'_j \neq \emptyset$. We lift them with $\tilde{\gamma}'_i \cap \tilde{\gamma}'_j \neq \emptyset$. Then we lift α'_i and α'_j as above so that $\tilde{\alpha}'_i \cap \tilde{\alpha}'_j \neq \emptyset$. So, $\alpha'_i \cap \alpha'_j \neq \emptyset$.

Finally if γ' is not simple, we lift it to $\tilde{\gamma}'^1$ and $\tilde{\gamma}'^2$, both passing from x_0, say. We then lift α' as in the above observation to get $\tilde{\alpha}'^1$ and $\tilde{\alpha}'^2$. Then $\tilde{\alpha}'^1 \cap \tilde{\alpha}'^2 \neq \emptyset$, and hence α' is not simple. This yields (iii) and the proof is complete. □

Remark 2.7. In fact, after some work, one can prove a more general result. Let α_1,...,α_k be isotopy classes of simple, closed, connected curves in S which are not homotopic to a point or a boundary component of S. We suppose that for each i and j, one can find representatives satisfying $card(\alpha'_i \cap \alpha'_j) \leq 1$. We consider the complex $\Gamma(\alpha_1, ..., \alpha_k)$ having as vertices the α_1,...,α_k; the vertices α_l and α_q are joined by an edge if they do <u>not</u> have any representatives with no intersection points. We suppose in addition that $\Gamma(\alpha_1, ..., \alpha_k)$ is a tree. For each j, we let $\alpha'_j, \alpha''_j \in \alpha_j$ in such a way that they satisfy the minimal intersection number in their isotopy classes; i.e. for each j, $card(\alpha'_l \cap \alpha'_q) = card(\alpha''_l \cap \alpha''_q)$ is the least number attained in the isotopy classes of α_l and α_q. Under these assumptions, we conclude that there exists a diffeomorphism of S, isotopic to identity, which transforms $\bigcup \alpha'_j$ into $\bigcup \alpha''_j$. (See [6], Lemma 3.19 for a proof.) In particular, we deduce that the collection of curves indicated in Figure 2.3 is realizable by geodesics.

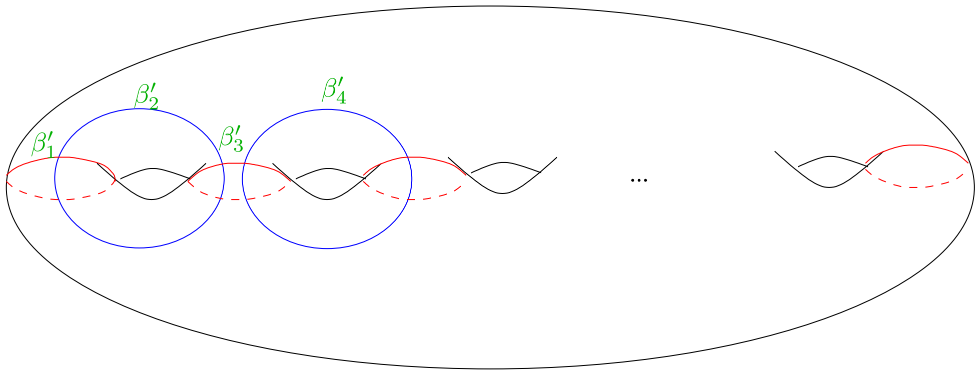


Figure 2.3. Generators of the homology group as geodesics.

2.3. Fenchel-Nielsen Coordinates

This section mainly relies on [BP92]. We fix a pair of pants decomposition of S as in Section 2.2 . We denote the loops by α'_1,...,α'_{3g-3}. We define

$$L : \mathcal{H} \to \mathbb{R}_+^{3(g-1)},\ L(h) = (L^{(h)}(\gamma_1^{'(h)}), ..., L^{(h)}(\gamma_{3(g-1)}^{'(h)}))$$

where $L^{(h)}(\gamma_i^{'(h)})$ is the length of the h-geodesic loop $\gamma_i^{'(h)}$ with respect to the metric h.

Proposition 2.8. *If h_1 and h_2 are two hyperbolic metrics on S such that there exists an isometry $\phi : (S, h_1) \to (S, h_2)$ isotopic to identity, then $L(h_1) = L(h_2)$.*

Proof. As ϕ is an isometry, $\phi(\gamma_1^{'(h_1)})$,...,$\phi(\gamma_{3(g-1)}^{'(h_1)})$ are h_2-geodesics; moreover they are pairwise disjoint and non-isopotic. Also, for each i, the loop $\phi(\gamma_i^{'(h_1)})$ is isotopic to α'_i whenever $\gamma_1^{'(h_2)}$ is isotopic to α'_i. Hence $\gamma_i^{'(h_2)} = \phi(\gamma_i^{'(h_1)})$. □

Therefore, L induces a well-defined map on $\mathcal{T}$, again denoted by L. We now fix a pair of pants P^2 with boundary components ∂_1, ∂_2, ∂_3; and consider the set $\mathcal{P}$ of all hyperbolic structures on P^2 making the boundary components geodesics. We define an equivalence relation S on $\mathcal{P}$ as follows: $k_1 \mathrm{S} k_2$ *if and only if* there exists an isometry $\phi : (P^2, k_1) \to (P^2, k_2)$ isotopic to identity via an isotopy Φ satisfying $\Phi(\partial_i \times \{t\}) = \partial_i \times \{t\}$.

Remark 2.9. We will now study the Teichmüller space of a pair of pants instead of the whole surface. We keep in mind that to glue two pairs of pants, the only necessary condition is that the edges to be glued have the same length.

Lemma 2.10. *For k in $\mathcal{P}$ and $1 \leq i < j \leq 3$, there exists a unique k-geodesic arc $C_{i,j}^{(k)}$ joining ∂_i and ∂_j which is orthogonal to both.*
Moreover if $\{i,j\} \neq \{i',j'\}$, then $C_{i,j}^{(k)} \cap C_{i',j'}^{(k)} = \emptyset$. In particular, the endpoints of $C_{i,j}^{(k)}$'s divide each of the ∂_l's into two nontrivial arcs.

Proof. We take an arc α joining ∂_1 and ∂_2, and consider the genus 2 surface Q obtained by doubling the pair of pants (see Figure 2.4). In this way, we obtain a closed curve α_1. We then take the geodesic γ_1 isotopic to α_1, and denote its restriction to P^2 by γ. The curve γ is the desired arc in P^2. Orthogonality follows from the symmetry with respect to the identified edges in Q. □

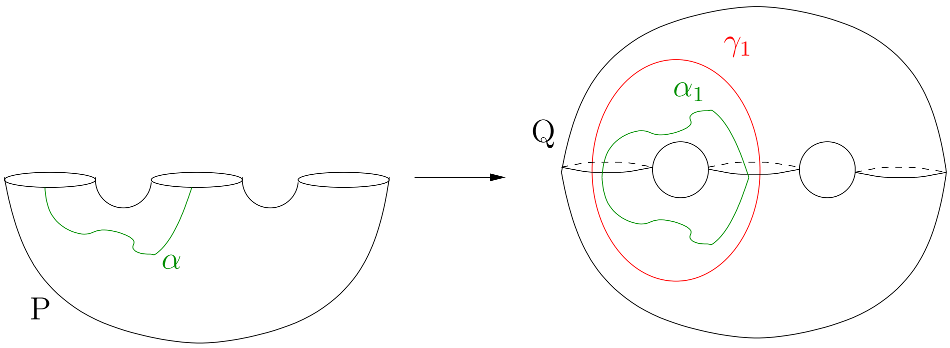


Figure 2.4. Doubling the pair of pants to stretch an arc tight.

Using Lemma 2.10, we cut P^2 into two right hexagons as in Figure 2.5.

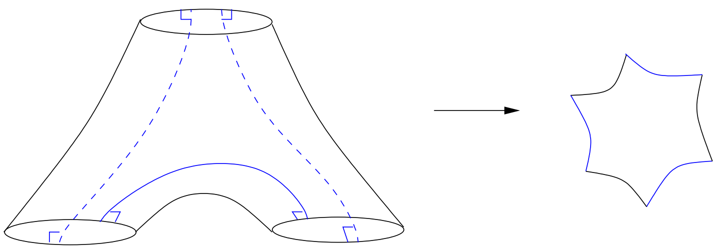


Figure 2.5. Cutting the pair of pants into hexagons.

Our study now reduces to the study of hyperbolic hexagons. We fix a hyperbolic hexagon oriented in the natural way E and label its edges as a_1,b_1,a_2,b_2,a_3,b_3 clockwise.

We denote by $\mathcal{E}$ the space of hyperbolic structures on E which make the edges geodesics and all inner angles right angles. As above, we define an equivalence relation R on $\mathcal{E}$ by identification via an isometry isotopic to identity where the isotopy keeps the vertices fixed and maps each edge onto itself.

Lemma 2.11. *For e in $\mathcal{E}$, (E, e) can be embedded isometrically into $\mathbb{H}^2$.*

Proof. We glue two copies of E to obtain the pair of pants P^2, then glue two copies of P^2 to get a genus 2 surface. The hexagon E is isometrically embedded in this surface. Noting that E is simply-connected, we may lift E globally to $\mathbb{H}^2$. $\square$

We consider the mapping $A : \mathcal{E} \to \mathbb{R}^3_+$, $A(e) = (L^{(e)}(a_1), L^{(e)}(a_2), L^{(e)}(a_3))$.

Proposition 2.12. *If $e_1 R e_2$, then $A(e_1) = A(e_2)$. So A induces a map $\hat{A} : \mathcal{E}/R \to \mathbb{R}^3_+$. Moreover, $\hat{A}$ is bijective.*

Proof. The map $\hat{A}$ is well-defined by definition of R. We first show $\hat{A}$ is surjective: We fix l_1,l_2,l_3 in $\mathbb{R}^3_+$, and take two geodesic lines γ'_1, γ'_2 in $\mathbb{D}^2$ such that $\gamma'_1 \perp \gamma'_2$. Say $\gamma'_1 \cap \gamma'_2 = \{x_0\}$. Then we take x_2 on γ'_2 such that $d(x_0, x_2) = l_1$ and let γ'_3 be orthogonal to γ'_2 at x_2 (see Figure 2.6).

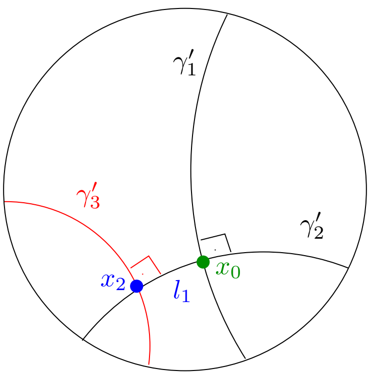


Figure 2.6. Construction of the geodesic γ'_3.

We now make the following construction: For x on γ'_1, we consider the geodesic line β'_x orthogonal to γ'_1 at x (see Figure 2.7). We then find x_1 on γ'_1 such that β'_{x_1} is asymptotically parallel to γ'_3. We put $w(l_1) = d(x_0, x_1)$ and denote this line by δ'_1

(see Figure 2.8). Note that $w(l_1)$ does not depend on the choice of x_0,γ_1',γ_2' and x_2.[1] Similarly, we define $w(l_3)$. Let $\lambda > 0$. In $\mathbb{D}^2$, we consider an arbitrary geodesic line and fix on it a segment of length $w(l_1) + \lambda + w(l_3)$. We thus obtain Figure 2.9.

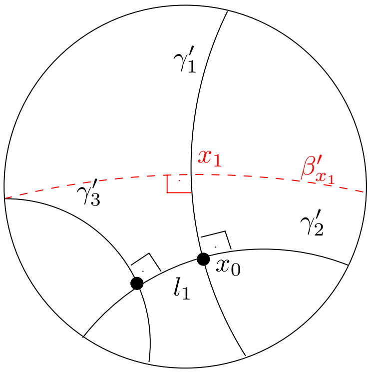


Figure 2.7. Construction of the geodesic β_x'.

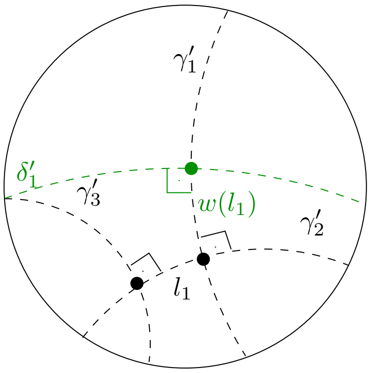


Figure 2.8. Construction of the geodesic δ_1'.

Let use define the function μ as $\mu(\lambda) \doteq d(\delta_1', \delta_2')$. We observe that μ is a continuous function of λ satisfying $\lim_{\lambda\to 0} \mu(\lambda) = 0$ and $\lim_{\lambda\to\infty} \mu(\lambda) = \infty$. So, there exists λ_0 such that $d(\delta_1', \delta_2') = l_2$. We now obtained the desired right hexagon.

Finally, we show $\hat{A}$ is injective: We observe that μ is strictly increasing so that λ_0 above is in fact unique, and so is b_3. In a similar way, we conclude that all edges are determined uniquely. Then Lemma 2.11 identifies all such hexagons to prove injectivity.

□

Proposition 2.13. *The map* $B : \mathcal{P} \to \mathbb{R}^3_+$, $B(k) = (L^{(k)}(\partial_1), L^{(k)}(\partial_2), L^{(k)}(\partial_3))$ *induces a bijection* $\hat{B} : \mathcal{P}/S \to \mathbb{R}^3_+$.

[1] This statement is easier to observe in the half space model.

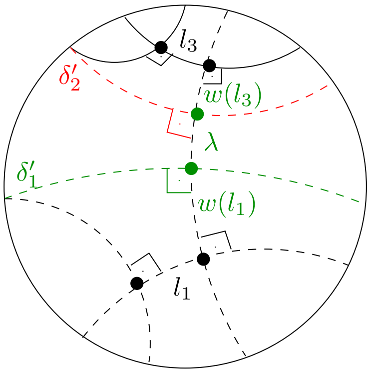


Figure 2.9. Construction of the variable λ.

Proof. The map $\hat{B}$ is clearly well-defined. Given k in $\mathcal{P}$, we associate to k two elements of $\mathcal{E}$ with opposite orientations and get

$$L(a_1) = L(a_1') = L(C_{1,3}^{(k)})$$
$$L(a_2) = L(a_2') = L(C_{1,2}^{(k)})$$
$$L(a_3) = L(a_3') = L(C_{2,3}^{(k)})$$

where $C_{i,j}^{(k)}$'s are as in Lemma 2.10. So, the two hexagons are in fact the same. In particular $L(b_i) = L(b_i') = L(\partial_i)/2$. Now, Proposition 2.12 yields injectivity. Furthermore, $\hat{B}$ is surjective since we just glue hexagons with $l(b_i) = l(b_i') = l_i/2$. □

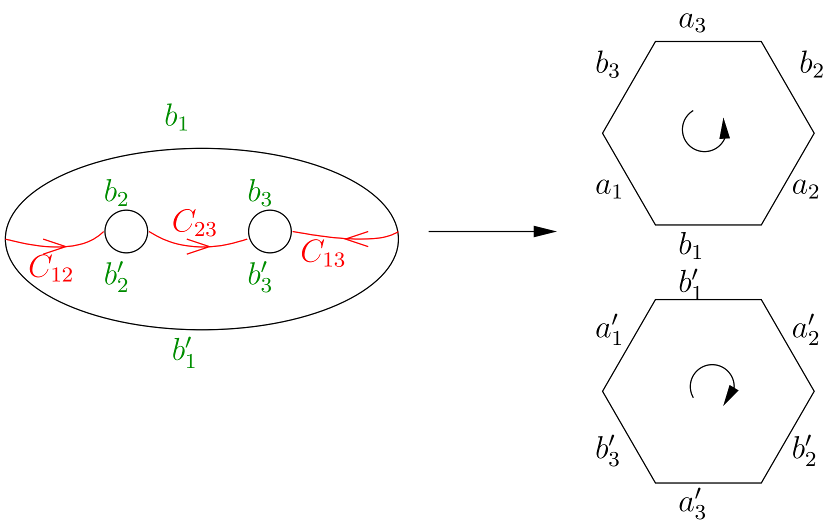


Figure 2.10. Parametrizing $\mathcal{P}$.

Proposition 2.14. *The map $L : \mathcal{H} \to \mathbb{R}_+^{3(g-1)}$ as defined in the beginning of the section is surjective.*

Proof. Given l_i, $i = 1, ...3(g-1)$, we decompose the surface into pairs of pants, use Proposition 2.13 and glue them back. As noted in Remark 2.9, the metric structures on each pair of pants will be preserved in the gluing since the edges have the same length. □

Remark 2.15. The map L is onto; so we only need to study its fibers; since the only part that was not unique in the construction was the gluing of $2(g-1)$ hyperbolic pairs of pants, we focus on this part. Informally, we glue two edges as follows: We choose a point and proceed as the orientation suggests. Since we can twist each edge, we get a factor of $\mathbb{R}^{3(g-1)}$. As all such structures are pairwise non-equivalent, $\mathcal{T}$ can be parametrised by $\mathbb{R}_+^{3(g-1)} \times \mathbb{R}^{3(g-1)}$.

We take a section $\sigma : \mathbb{R}_+^{3(g-1)} \to \mathcal{T}$ of L, i.e. we have $L \circ \sigma = id$.

Theorem 2.16. *There exists an action* Θ *of* $\mathbb{R}^{3(g-1)}$ *on* $\mathcal{T}$ *such that the map*

$$\Psi : \mathbb{R}_+^{3(g-1)} \times \mathbb{R}^{3(g-1)} \to \mathcal{T}, \;\; \Psi(l, v) = \Theta_v(\sigma(l))$$

is bijective.

Proof. We will define Θ in such a way that for each fixed l, the map $\mathbb{R}^{3(g-1)} \to L^{-1}(l)$, $v \mapsto \Theta_v(\sigma(l))$ is bijective.

For $v \in \mathbb{R}^{3(g-1)}$, we define Θ_v first on $\mathcal{H}$: Let $h \in \mathcal{H}$ and $\{\gamma_i'\}$ be the h-geodesic loops of the pair of pants decomposition. The metric $\Theta_v(h)$ will differ from h only in a small neighbourhood of γ_i''s.

Fix i. For $x \in \gamma_i'$, we let δ_x' to be the orthogonal geodesic to γ_i' at x such that the pair $(\dot{\gamma}_i'(x), \dot{\delta}_x'(x))$ is positive. For simplicity, we suppose γ_i' and δ_x' are parametrized by arc length. Since γ_i' is compact, we can find $\varepsilon > 0$ such that the map $\gamma_i' \times [0, 3\varepsilon] \to C_i$, $(x, t) \longmapsto \delta_x'(t)$ is a diffeomorphism, where by C_i we denote a collar based on γ_i'.

We now define a diffeomorphism ϕ_i of C_i by instead defining it on $\gamma_i' \times [0, 3\varepsilon]$ as fol-

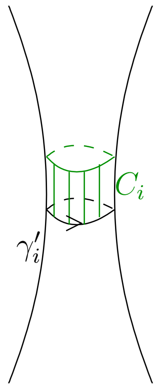


Figure 2.11. Collar neighborhood of the loop γ_i'.

lows: Consider the universal covering $\mathbb{R}$ of γ_i' with the identification $x \mapsto x+\text{length}(\gamma_i')$. We see $\mathbb{R} \times [0, 3\varepsilon]$ as the universal covering of $\gamma_i' \times [0, 3\varepsilon]$.

Consider $\tilde{\phi}_i : \mathbb{R} \times [0, 3\varepsilon] \to \mathbb{R} \times [0, 3\varepsilon]$ such that

(i) $\tilde{\phi}_i(t, s) = (t + v_i)$ for $s \leq \varepsilon$,
(ii) $\tilde{\phi}_i(t, s) = (t, s)$ for $s \geq 2\varepsilon$,
(iii) $\tilde{\phi}_i(t_1 + t_2, s) = \tilde{\phi}_i(t_1, s) + (t_2, 0)$.

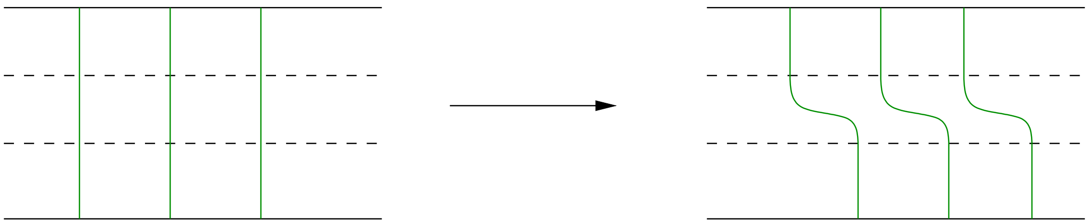


Figure 2.12. Twisting the collars while gluing the pairs of pants.

The effect of $\tilde{\phi}_i$ can be seen in Figure 2.12. By (iii), we get a diffeomorphism ϕ_i of $\gamma_i' \times [0, 3\varepsilon]$. We now observe that ϕ_i is the identity in $\gamma_i' \times [2\varepsilon, 3\varepsilon]$ and an h-isometry in $\gamma_i' \times [0, \varepsilon]$. So, we get a well-defined hyperbolic metric

$$h_i' = \begin{cases} h & \text{outside } C_i \\ \phi_i^*(h) & \text{on } C_i \end{cases}$$

We define these for each i to get the metric $\Theta_v(h)$.

We remark that $\Theta_v(h)$ is well-defined only in $\mathcal{T}$ (and not $\mathcal{H}$) since different choices of ε's and ϕ_i's produce different but equivalent metrics. Also, it can be checked easily that $\Theta_v(h_1) \sim \Theta_v(h_2)$ whenever $h_1 \sim h_2$. The definition of Θ is now complete.

We observe that $L(\Theta_v(h)) = L(h)$ for all v so that Θ operates on each fiber of L; since by construction γ_i''s have the same length as before.

Surjectivity of Ψ is straightforward from the construction of Θ. To finish the proof we now show injectivity of Ψ. Since Θ keeps fibers of L invariant, we will only show that for $h \in \mathcal{T}$ and $v \neq v'$, $\Theta_v(h)$ cannot be equivalent to $\Theta_{v'}(h)$.

For β' a non-trivial loop in S, we put $\tilde{\Lambda}^h(\beta')$ = length of the h-geodesic in free-homotopy class of β'. We will use the following technical lemma, the proof of which can be found in [[14], Lemma B.4.18].

Lemma 2.17. *For $i = 1, ..., 3(g-1)$, the loop β_i' shown in Figure 2.13 satisfies the following condition: For any $h \in \mathcal{T}$, the map $\mathbb{R} \to \mathbb{R}$, $v_i \mapsto \tilde{\Lambda}^{\Theta_{0,\ldots,v_i,\ldots 0}(h)}(\beta_i')$ is strictly convex and has a minimum.*

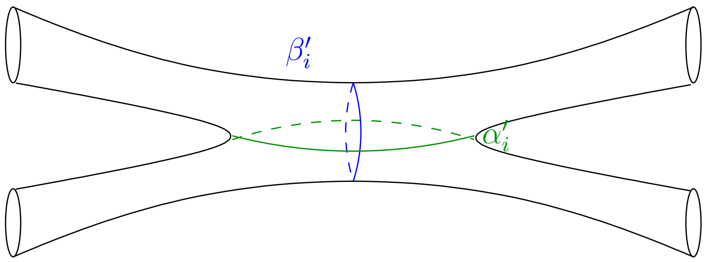


Figure 2.13. Gluing pairs of pants.

Now, by way of contradiction, we suppose that for $h \in \mathcal{T}$ and $v \neq v'$ (say $v_i \neq v_i'$), we have $\Theta_v(h) \nsim \Theta_{v'}(h)$. Let $\hat{v}$ and $\hat{v'}$ denote v and v' with v_i and v_i' replaced by 0's, respectively. We then set $h_1 = \Theta_{\hat{v}}(h)$ and $h_2 = \Theta_{\hat{v'}}(h)$ so that $\Theta_{0,\ldots,v_i,\ldots 0}(h_1) = \Theta_{0,\ldots,v_i',\ldots 0}(h_2)$. Hence for each n in $\mathbb{Z}$, we have

$$\tilde{\Lambda}^{\Theta_{0,\ldots,n+v_i,\ldots 0}(h_1)}(\beta_i') = \tilde{\Lambda}^{\Theta_{0,\ldots,n+v_i',\ldots 0}(h_2)}(\beta_i')$$

since in Figure 2.13, a twist by an integer leaves β_i' unaffected. As h_1 and h_2 coincide with h in the neighborhood of α_i', we may substitute h for h_1 and h_2 in the above equality to get

$$\tilde{\Lambda}^{\Theta_{0,\ldots,n+v_i,\ldots 0}(h)}(\beta_i) = \tilde{\Lambda}^{\Theta_{0,\ldots,n+v_i',\ldots 0}(h)}(\beta_i')$$

for all n in $\mathbb{Z}$; but by Proposition 2.13, this cannot hold for $n > \max\{|v_i|, |v_i'|\}$. □

Remark 2.18. In fact, one can prove more on the function L. Recall that in Remark 2.1, we expressed that one can endow $\mathcal{T}$ with a topology. One can easily observe that L is continuous with respect to this topology. Furthermore, with some more work, one can show that it has continuous local sections $\sigma : \mathbb{R}_+^{3(g-1)} \to \mathcal{T}$. Referring to [6] for a detailed discussion, we will sketch the idea of this last statement.

We make an intrinsic and continuous choice for the points of gluing the pairs of pants. We fix the isotopy classes of (oriented) β_i''s in the beginning (β_i' as in Figure 2.13). Given a hyperbolic structure on a pair of pants with edges ∂_i, we double it and glue via the identity map.

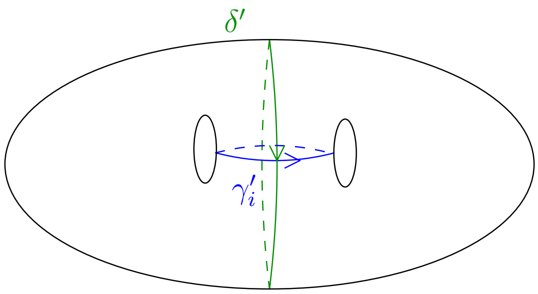


Figure 2.14. Parametrizing the Teichmüller space continuously.

We thus get the geodesic loop δ' as in Figure 2.14. Notice that $card(\delta' \cap \gamma_i') = 2$. Moreover, $(\dot{\delta}', \dot{\gamma}_i') > 0$ for exactly one of the intersection points. We choose one without ambiguity. Now with this choice, σ becomes continuous.

Finally, we state the parametrization of the Teichmüller space.

Proposition 2.19. *The map $L : \mathcal{T} \to (\mathbb{R}_+^*)^{3g-3}$ is a principle fibration of the group $\mathbb{R}^{3g-3}$ acting by Θ.*

Corollary 2.20. *The Teichmüller space of a closed surface of genus g is homeomorphic to* $\mathbb{R}^{6g-6}$.

2.4. Classification of Surface Diffeomorphisms: First Step

We are now ready to exhaust the first case of Thurston's classification of surface diffeomorphisms. As briefly mentioned in the Introduction, the classification of surface diffeomorphisms relies on analyzing the fixed points of the natural action of $\pi_0(\mathrm{Diff}(S))$ on the compactified Teichmüller space $\overline{\mathcal{T}} \doteq \mathcal{T} \sqcup P\mathcal{MF}$. We will now deal with the case when a diffeomorphism φ of S has a fixed point in $\mathcal{T}$.

Theorem 2.21. *Let ρ be a metric of curvature -1 on S. Let $I(S,\rho)$ denote the group of isometries of ρ. Then $I(S,\rho)$ is finite.*

Proof. Consider $S^S = \{f : S \to S\}$ with the topology of pointwise convergence. Since S is compact, S^S is compact. On $I(S,\rho)$, the topology of pointwise convergence coincides with the topology of uniform convergence. We get that $I(S,\rho)$ is closed in S^S, and hence compact.

We now claim that an isometry ϕ isotopic to identity must equal identity: ϕ acts on the isotopy classes of simple, closed curves trivially. In each class α, there is a unique geodesic g_α. So, $\phi(g_\alpha) = g_\alpha$ for each isotopy class α. Therefore ϕ is the identity on each geodesic. Hence it must be identity on the cell that is complementary to the loops in Figure 2.3.

It follows that $I(S,\rho)$ is discrete since the identity is isolated. But a discrete compact set must be finite. □

Corollary 2.22. *Let φ be a diffeomorphism of S. Let $\mathcal{T}(\varphi)$ be the natural action of φ on $\mathcal{T}$. If $\mathcal{T}(\varphi)$ has a fixed point, then there is a periodic diffeomorphism of S isotopic to φ.*

The context of this discussion will be clear in the following chapters. We find it sufficient to say that the first case of our analysis will be when the fixed point is inside the Teichmüller space, while the other cases when the fixed point is in $P\mathcal{MF}$ requires more work. Corollary 2.22 states that if the action of the diffeomorphism φ on $\mathcal{T}$ has a fixed point, then φ is isotopic to a periodic diffeomorphism.

3. ESSENTIAL CURVES AND MEASURED FOLIATIONS

In this chapter, we study the space $\mathcal{S}$ of essential curves, the space $\mathcal{MF}$ of measured foliations, and the link between the two. This chapter heavily relies on [6].

3.1. The Space $\mathcal{S}$ of Essential Curves

Definition 3.1. We define *the space $\mathcal{S}$ of essential curves* in S as the set of isotopy classes of simple, closed, connected curves in S which are not homotopic to a point.

Remark 3.2. Using Theorem 1.2, we can replace "isotopy classes" in the definition of $\mathcal{S}$ with "free-homotopy classes".

Definition 3.3. For $\alpha, \beta \in \mathcal{S}$, we define their *geometric intersection number*, denoted by $i(\alpha, \beta)$, as the minimal number of intersection points of a representative of α with a representative of β.

Definition 3.4. The geometric intersection induces a map $i_* : \mathcal{S} \to \mathbb{R}_+^{\mathcal{S}}$ defined by $i_*(\alpha)(\beta) = i(\alpha, \beta)$.

Proposition 3.5. *Let α'_0,α'_1 be transverse simple curves in S not homotopic to a point. Suppose their isotopy classes α_0 and α_1 are distinct. Then the following statements are equivalent:*

(i) $card(\alpha'_0 \cap \alpha'_1) = i(\alpha_0, \alpha_1)$.

(ii) All simple closed curves formed from an arc of α'_0 and an arc of α'_1 are not homotopic to a point in S.

(iii) On letting $\tilde{\alpha_0}$,$\tilde{\alpha_1}$ be connected components of $p^{-1}(\alpha'_0)$ and $p^{-1}(\alpha'_1)$ in the universal covering $p : \tilde{S} \to S$, we have $card(\tilde{\alpha_0} \cap \tilde{\alpha_1}) \leq 1$.

(iv) There exists a Riemannian metric on S with constant curvature -1 such that α'_0 and α'_1 are geodesics.

Proof. (i) $\Rightarrow$ (ii): We prove this statement by contraposition. Suppose there is a curve

γ' homotopic to a point. Then we can isotope α_0' and α_1' as in Figure 3.1 to obtain $card(\alpha_0' \cap \alpha_1') \neq i(\alpha_0, \alpha_1)$.

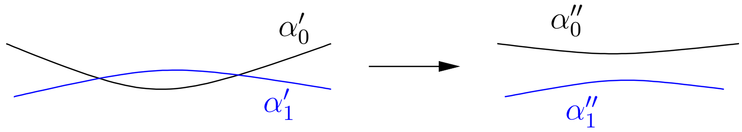


Figure 3.1. Eliminating intersections.

(iii) $\Rightarrow$ (ii): We proceed again by contraposition. If there is γ' homotopic to a point, then via the homotopy $\star \rightsquigarrow \gamma'$ we lift the simple closed curve to get two distinct points in $\tilde{\alpha}_0 \cap \tilde{\alpha}_1$.

(iv) $\Rightarrow$ (ii): We prove a lemma which directly implies the desired implication:

Lemma 3.6. *For α',β' distinct geodesic arcs in S with same endpoints, the closed curve $\alpha' \cup \beta'$ cannot be homotopic to a point.*

Proof. Suppose $\alpha' \cup \beta'$ is homotopic to a point. We consider its lift in $\tilde{S}$. So, while $\star \rightsquigarrow \alpha' \cup \beta'$ in S, we have $\tilde{\star} \rightsquigarrow$ some closed curve in $\tilde{S}$. Hence $\tilde{\alpha'} \cup \tilde{\beta'}$ is a closed curve in $\tilde{S}$. But $\tilde{\alpha'}$ and $\tilde{\beta'}$ are geodesics meeting at two distinct points in $\mathbb{H}^2$. By uniqueness of geodesics, we must have $\tilde{\alpha'} = \tilde{\beta'}$, and thus α' and β' cannot be distinct. $\square$

(iv) $\Rightarrow$ (iii): If α_0' and α_1' are geodesics, then by uniqueness of geodesics in $\tilde{S}$ we have $card(\tilde{\alpha}_0 \cap \tilde{\alpha}_1) \leq 1$.

(ii) $\Rightarrow$ (i): We prove the contrapositive statement. Suppose $card(\alpha_0' \cap \alpha_1') > i(\alpha_0, \alpha_1)$. So, there is a homotopy $h_t : S^1 \to S$ such that $h_0 = \alpha_0'$ and $card(h_1(S^1) \cap \alpha_1') < card(\alpha_0' \cap \alpha_1')$. We may suppose $h : S^1 \times [0, 1] \to S$ is transverse to α_1'. Then $h^{-1}(\alpha_1')$ is a submanifold of dimension 1 whose connected components look as in Figure 3.2 below.

By assumption, (I) exists. We put $\Gamma_1 =$ (I) and let Γ_0 be the arc joining q_0 and q_1 to get the result.

(ii) $\Rightarrow$ (iii): We prove the contrapositive statement. Suppose $card(\tilde{\alpha}_0 \cap \tilde{\alpha}_1) > 1$. We can find an embedded disk $\triangle$ in $\tilde{S}$ formed by arcs of $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$. On $\triangle$, we see

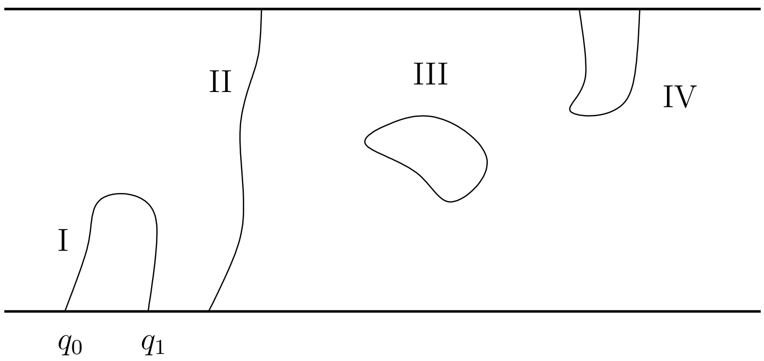


Figure 3.2. Configurations of $h^{-1}(\alpha_1')$.

$p^{-1}(\alpha_0' \cup \alpha_1')$ as in Figure 3.3. We choose a minimal such disk δ so that $int(\delta) \cap p^{-1}(\alpha_0' \cup$

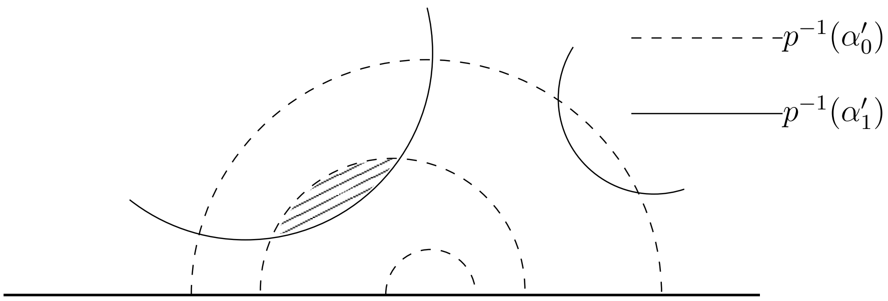


Figure 3.3. Configuration of $p^{-1}(\alpha_0' \cup \alpha_1')$.

$\alpha_1') = \emptyset$. We observe that the covering map $p : \tilde{S} \to S$ is an immersion of codimension 0 that embeds $\partial\delta$. Hence p embeds δ, since the number of points for the fiber is locally constant. This yields a curve formed from arcs of α_0' and α_1' that is homotopic to a point in S.

(i) $\Rightarrow$ (iv): The proof of this implication requires some work.

Proposition 3.7. *Let $\alpha_0', \alpha_0'' \in \alpha_0$ and $\alpha_1' \in \alpha_1$ be simple curves not homotopic to a point. Suppose $\alpha_0 \neq \alpha_1$ and $card(\alpha_0' \cap \alpha_1') = card(\alpha_0'' \cap \alpha_1') = i(\alpha_0, \alpha_1)$. Then there exists an ambient isotopy of (S, α_1') which pushes α_0' onto α_0''.*

Proof. Let $h : S^1 \times [0,1] \to S$ be a map transverse to α_1' such that $h|_{S^1 \times \{0\}} = \alpha_0'$ and $h|_{S^1 \times \{1\}} = \alpha_0''$.

Lemma 3.8. *The closed components of $h^{-1}(\alpha_1')$ are homotopic to a point in $S^1 \times [0,1]$.*

Proof. We prove this statement by contradiction. Suppose γ' is a closed component of $h^{-1}(\alpha_1')$ not homotopic to a point in $S^1 \times [0,1]$. Then γ' is isotopic to the boundary. We let d be the degree of $h : \gamma' \to \alpha_1'$. $d = 0$ implies that α_0' is homotopic to a point. $|d| = 1$ implies α_0' and α_1' are homotopic. $|d| > 1$ implies that a nontrivial multiple of α_1' is free-homotopic to an embedded curve, i.e. α_0', but this is impossible by Theorem 1.4. We obtained a contradiction in each case. □

Now, we again get Figure 3.2. By the second hypothesis, we are left with (II) and (III). We kill type (III) arcs as $\pi_2(S, \alpha_1') = 0$. If $h^{-1}(\alpha_1') = \emptyset$, the result is straightforward. Otherwise, a slight improvement of Theorem 1.3 yields the desired isotopy (see [6], Lemma 3.15). □

Theorem 3.9. *If S has a metric of constant curvature -1, then all simple curves not homotopic to a point are isotopic to a simple geodesic. Moreover, two simple geodesics meet in the minimal number of points of intersection in their isotopy classes.*

Proof. We let $f : S^1 \to S$ be an embedding not homotopic to a point. By Lemma 2.5, it is homotopic to a geodesic immersion g. We lift f to $\tilde{f}_0, \tilde{f}_1 : \mathbb{R} \to \tilde{S}$, proper embeddings with distinct images. They are homotopic to geodesics $\tilde{g}_0, \tilde{g}_1$, which by Lemma 3.6 meet at most at one point.

In fact we have $\tilde{g}_0 \cap \tilde{g}_1 = \emptyset$: We visualize $\tilde{S}$ by the Poincaré disk model $\mathbb{D}^2$. The geodesics $\tilde{g}_i$ have two limit points. Since the homotopy $\tilde{f}_i \rightsquigarrow \tilde{g}_i$ is via lifting a homotopy in S where S is compact, the hyperbolic distance from $\tilde{g}_i(x)$ to $\tilde{f}_i(x)$ is uniformly bounded for $x \in \mathbb{R}$. Thus $\tilde{f}_i$ must have the same limit point as $\tilde{g}_i$. Now if $\tilde{g}_0$ and $\tilde{g}_1$ have a common point, then $\tilde{f}_0$ and $\tilde{f}_1$ must meet again (Jordan theorem), which contradicts to f being an embedding. So, the image of the map g in S is a simple curve covered by g a certain number of times. But then, Theorem 1.4 implies that g is an embedding. □

Finally, we are ready to show (i) $\Rightarrow$ (iv): Suppose $card(\alpha_0' \cap \alpha_1') = i(\alpha_0, \alpha_1)$.

Using Theorem 3.9, we isotope α_0' and α_1' to geodesics g_0 and g_1. Then $card(g_0 \cap g_1) = card(\alpha_0' \cap \alpha_1') = i(\alpha_0, \alpha_1)$. So, in fact α_i and g_i are ambient isotopic by Proposition 3.7. We then isotope the metric via this ambient isotopy to obtain another metric, in which case α_0' and α_1' become geodesics. □

Corollary 3.10. *Let α_0',α_1' be simple curves in S that intersect transversely. Suppose they have components $\tilde{\alpha}_i$ in $p^{-1}(\alpha_i')$ such that $card(\tilde{\alpha}_0 \cap \tilde{\alpha}_1) = \infty$. Then $\alpha_0 = \alpha_1$.*

Proof. By transversality we have $card(\alpha_0' \cap \alpha_1') < \infty$. So, there exist $\star \in \alpha_0' \cap \alpha_1'$ and $x, y \in \tilde{\alpha}_0 \cap \tilde{\alpha}_1$ such that $x \neq y$ and $p(x) = p(y) = \star$. We orient every arc $\tilde{\alpha}_i$ from x to y and then orient α_i' as $\tilde{\alpha}_i$. Considering α_0 and α_1 inside $\pi_1(S, \star)$, we see that the segment from x to y on $\tilde{\alpha}_0$ (resp.$\tilde{\alpha}_1$) covers α_0' (resp.α_1') k-times (resp.l-times); so that $\alpha_0^k = \alpha_1^l$ in $\pi_1(S, \star)$. Now, we let g_i be the geodesic in $\tilde{S}$ invariant under T_{α_i}, where T_{α_i} is as in Lemma 2.4. Then $T_{\alpha_0^k} = T_{\alpha_1^l}$ leaves g_0 and g_1 invariant. Hence $g_0 = g_1$. Thus α_0' and α_1' are homotopic to the same geodesic $p(g_0) = p(g_1)$ in S. □

3.2. $\mathcal{S}$ vs. $\mathbb{R}_+^{\mathcal{S}}$

Proposition 3.11. *(i) $i_*(\mathcal{S}) \subset \mathbb{R}_+^{\mathcal{S}} - \{0\}$*
(ii) $\pi i_* : \mathcal{S} \to P(\mathbb{R}_+^{\mathcal{S}})$ is injective.

Proof. We first prove the following stronger claim: For $\alpha_1, \alpha_2 \in \mathcal{S}$ with $\alpha_1 \neq \alpha_2$, there exists $\beta \in \mathcal{S}$ such that $i(\alpha_1, \beta) = 0 \neq i(\alpha_2, \beta)$. To see this, we first note that if $i(\alpha_1, \alpha_2) \neq 0$, then $\beta = \alpha_1$ works. We suppose $i(\alpha_1, \alpha_2) = 0$; i.e. that there exists α_1', α_2' with $\alpha_1' \cap \alpha_2' = \emptyset$. We cut S along α_1' to obtain a surface N containing α_2' in its interior. As $\alpha_1 \neq \alpha_2$, there is a loop β' in N such that α_2' and β' cannot be separated in N. (Otherwise α_2' would be homotopic to the boundary of N, which is α_1'.) If α_2' does not separate N, we take β' such that $card(\beta' \cap \alpha_2') = 1$. If α_2' separates N into N_1 and N_2, we take $\beta' = I_1 \cup I_2$ where I_i is an arc representing a nontrivial element of $\pi_1(N_i, \alpha_2')$. (This is possible as N_i is not annulus or disk.) Now we take β as the isotopy class of β'. Then $i(\alpha_1, \beta) = 0$ is obvious and $i(\alpha_2, \beta) \neq 0$ follows from Proposition 3.5.

The claim proves (ii). It also implies that for any α in $\mathcal{S}$ there is β in $\mathcal{S}$ such that $i(\alpha, \beta) \neq 0$, which gives (i). $\square$

Definition 3.12. We extend the function $i_* : \mathcal{S} \to \mathbb{R}_+^{\mathcal{S}}$ to

$$i_* : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+^{\mathcal{S}} \text{ by } i_*(\lambda, \alpha)(\beta) = \lambda i(\alpha, \beta).$$

Remark 3.13. We note that $\pi(\overline{i_*(\mathbb{R}_+ \times \mathcal{S})} - \{0\}) = \overline{\pi i_*(\mathcal{S})}$, where $\overline{i_*(\mathbb{R}_+ \times \mathcal{S})}$ is the closure of $i_*(\mathbb{R}_+ \times \mathcal{S})$ in $\mathbb{R}_+^{\mathcal{S}}$.

Proposition 3.14. *The set $\pi i_*(\mathcal{S})$ is precompact in $P(\mathbb{R}_+^{\mathcal{S}})$.*

Proof. We fix a metric ρ on S, and let $l(\alpha)$ denote the ρ-length of the unique geodesic in the class of $\alpha \in \mathcal{S}$.

Lemma 3.15. *There exists a constant $C = C_{S,\rho}$ such that for all $\alpha, \beta \in \mathcal{S}$ $i(\alpha, \beta) \leq Cl(\alpha)l(\beta)$.*

Proof. For $\alpha = \beta$, there is nothing to prove. We assume $\alpha \neq \beta$. We let ε be smaller than the radius of normal neighborhoods. The geodesic g_α in the class α may be covered by fewer than $\frac{l(\alpha)}{\varepsilon} + 1$ small arcs, each of which is inside a geodesic disk. Same is true for g_β. A small arc of g_α intersects a small arc of g_β in at most one point. So, in a small arc of g_α, there are at most $\frac{l(\beta)}{\varepsilon} + 1$ intersection points with g_β. Hence, $i(\alpha, \beta) = card(g_\alpha \cap g_\beta) \leq (\frac{l(\alpha)}{\varepsilon} + 1)(\frac{l(\beta)}{\varepsilon} + 1)$ where $l(\alpha) > \varepsilon$. $\square$

Now we set $\beta_1, ..., \beta_{2g+1} \in \mathcal{S}$ via geodesics as Figure 3.4 below. (cf. Figure 2.3 after Remark 2.7.)

Lemma 3.16. *There is a constant c such that for any $\alpha \in \mathcal{S}$ we have $\sum_j i(\alpha, \beta_j) \geq c\, l(\alpha)$.*

Proof. The geodesics $\{g_{\beta_j}\}_j$ decompose S into simply-connected regions. In each of these regions, a geodesic arc is bounded, say by L. Then, we take $c = \frac{1}{L}$. $\square$

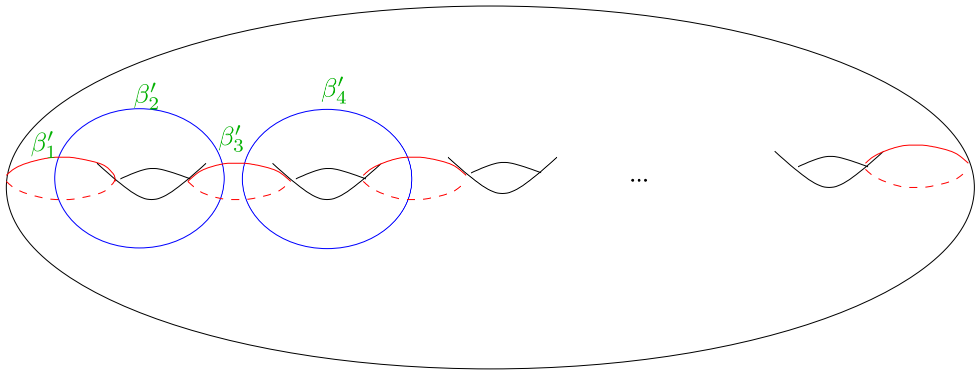


Figure 3.4. Generators of the homology group as geodesics.

Finally, we let C as in Lemma 3.15, and put $S(C)$ to be set of all functionals f in $\mathbb{R}_+^{\mathcal{S}}$ such that $f(\beta) \leq C\ l(\beta)$ for all β in $\mathcal{S}$. The set $S(C)$ is compact by Tychonov's theorem. Consider the set $S_0 \subset S(C)$ defined as the closure (in $\mathbb{R}_+^{\mathcal{S}}$) of the set of functionals of type $\dfrac{i_*(\alpha)}{l(\alpha)}$. By Lemma 3.16, $S_0 \subset \mathbb{R}_+ - \{0\}$. Also, S_0 is compact. Then so is $\pi(S_0)$. By Lemma 3.15, $\pi i_*(\mathcal{S}) \subset \pi(S_0)$. Therefore $\overline{\pi i_*(\mathcal{S})}$ is compact. $\square$

We now introduce the space $\mathcal{S}'$ of multicurves, which are easier to study than $\mathcal{S}$.

Definition 3.17. We define $\mathcal{S}'$ to be the space of isotopy classes of closed submanifolds of dimension 1 where no component is homotopic to a point. An element of $\mathcal{S}'$ is called a *multicurve*.

In a similar fashion as in $\mathcal{S}$, we define i on $\mathcal{S}' \times \mathcal{S}$ to get $i_* : \mathcal{S}' \to \mathbb{R}_+^{\mathcal{S}}$ and $\pi i_* : \mathcal{S}' \to P(\mathbb{R}_+^{\mathcal{S}})$.

Theorem 3.18. *In $P(\mathbb{R}_+^{\mathcal{S}})$, we have $\overline{\pi i_*(\mathcal{S})} = \overline{\pi i_*(\mathcal{S}')}$.*

Proof. It suffices to show that $\pi i_*(\mathcal{S})$ is dense in $\pi i_*(\mathcal{S}')$. We let $\alpha \in \mathcal{S}'$ be composed of distinct curves α'_1,...,α'_k. We choose a simple curve γ' such that for all j, $card(\gamma' \cap \alpha'_j) = i(\gamma, \alpha_j)$ and $card(\gamma' \cap \alpha'_j) \neq 0$.
For $n_1, ..., n_k \in \mathbb{N}$, we construct $\Gamma(n_1, ..., n_k) \in \mathcal{S}$ by replacing small arcs of γ' as in Figure 3.5 below.

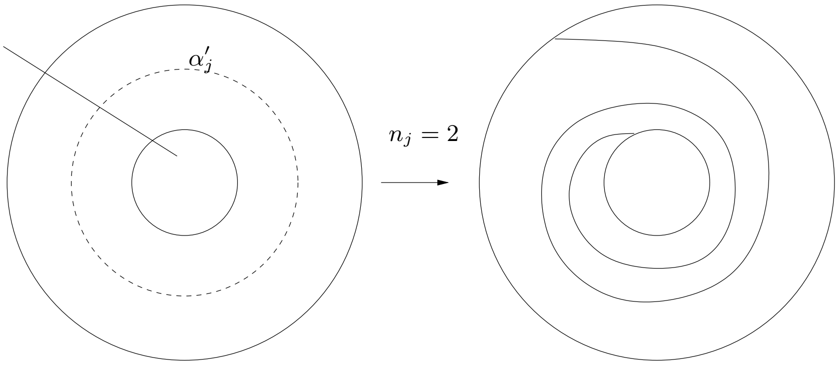


Figure 3.5. Spiralling essential curves around multicurves.

Also, one can show (See [6], Appendix A) that for $\beta \in \mathcal{S}$;

$$| \ i(\Gamma(n_1, ...n_k), \beta) - \sum_j n_j \ i(\gamma, \alpha_j) \ i(\alpha_j, \beta) \ | \ \leq \ i(\gamma, \beta)$$

Now for $n \in \mathbb{N}$, we take $n_j = n \ \prod_{l \neq j} i(\gamma, \alpha_l)$ and obtain a curve $\Gamma(n)$ satisfying

$$| \ i(\Gamma(n), \beta) - n \ \prod_j i(\gamma, \alpha_j) \ [\sum_j i(\alpha_j, \beta)] \ | \ \leq \ i(\gamma, \beta)$$

This means, when we projectivize and let $n \longrightarrow \infty$, the effect of γ to the intersection becomes negligible. Thus, $\pi i_*(\Gamma(n)) \longrightarrow \pi i_*(\alpha)$ as desired. □

Corollary 3.19. *In* $\mathbb{R}_+^{\mathcal{S}}$, *we have* $i_*(\mathcal{S}') \subset \overline{i_*(\mathbb{R}_+ \times \mathcal{S})}$.

Proof. This is immediate from Remark 3.13 and Theorem 3.18. □

3.3. A Parametrization of the Space $\mathcal{S}'$ of Multicurves

As in Section 2.3, we decompose S into pairs of pants, study the pairs of pants separately and glue back. To this end, we first recall some definitions and basic results on the space of diffeomorphisms of the disk D^2.

For K a subset of $int(D^2)$ with k elements, we set

$$\text{Diff}(D^2, rel(K,\partial)) = \{\varphi : D^2 \to D^2 \text{ diffeomorphism such that } \varphi|_{K\cup\partial D^2} = id\}$$
$$\text{Diff}(D^2, K, rel\partial) = \{\psi : D^2 \to D^2 \text{ diffeomorphism such that } \psi(K) = K \text{ and } \psi|_{\partial D^2} = id\}$$

We denote the boundary components of P^2 by $\partial_1, \partial_2, \partial_3$. The proofs of the following two propositions can be found in [6].

Proposition 3.20. *We have* $\pi_0(\textit{Diff}(P^2, rel\partial)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Proposition 3.21. *The space* $\textit{Diff}_+(P^2, \partial_1, \partial_2, \partial_3)$ *is contractible.*

Now we define $A(P^2)$ to be the set of isotopy classes of arcs $I \subset P^2$ with $\partial I \subset \partial P^2$, each end free to move on the respective connected component of ∂P^2 and representing the nontrivial elements of $\pi_1(P^2, \partial P^2)$; and we set $A'(P^2)$ to be the same as above but with several pairwise disjoint arcs.

Corollary 3.22. $A(P^2)$ *consists of 6 elements, classified by the connected components of* ∂P^2 *in which the endpoints of the respective arcs fall (see Figure 3.6).*

Proof. We start with two representatives τ, τ' of elements of $A(P^2)$ with endpoints in the same connected component of ∂P^2. We take an orientation-preserving diffeomorphism taking τ to τ'. By Proposition 3.21, this diffeomorphism is isotopic to the identity. □

On writing $A' = \{(a_1, a_2, a_3) \mid a_i \in \mathbb{Z}_+ \ \ \sum a_i \equiv 0 \ (mod\ 2)\}$, we define a function $i : A'(P^2) \to A'$ by $i(\tau) = (i(\tau, \partial_1), i(\tau, \partial_2), i(\tau, \partial_3))$ where $i(\tau, \partial_i)$ denotes the number of points τ has common with ∂_i.

Theorem 3.23. *The map* $i : A'(P^2) \to A'$ *is bijective.*

Proof. To prove surjectivity, we will construct a map $\tau : A' \to A'(P^2)$ such that $i(\tau(a_1, a_2, a_3)) = (a_1, a_2, a_3)$. For (a_1, a_2, a_3), the point $(\frac{a_1}{\sum a_i}, \frac{a_2}{\sum a_i}, \frac{a_3}{\sum a_i})$ falls in one

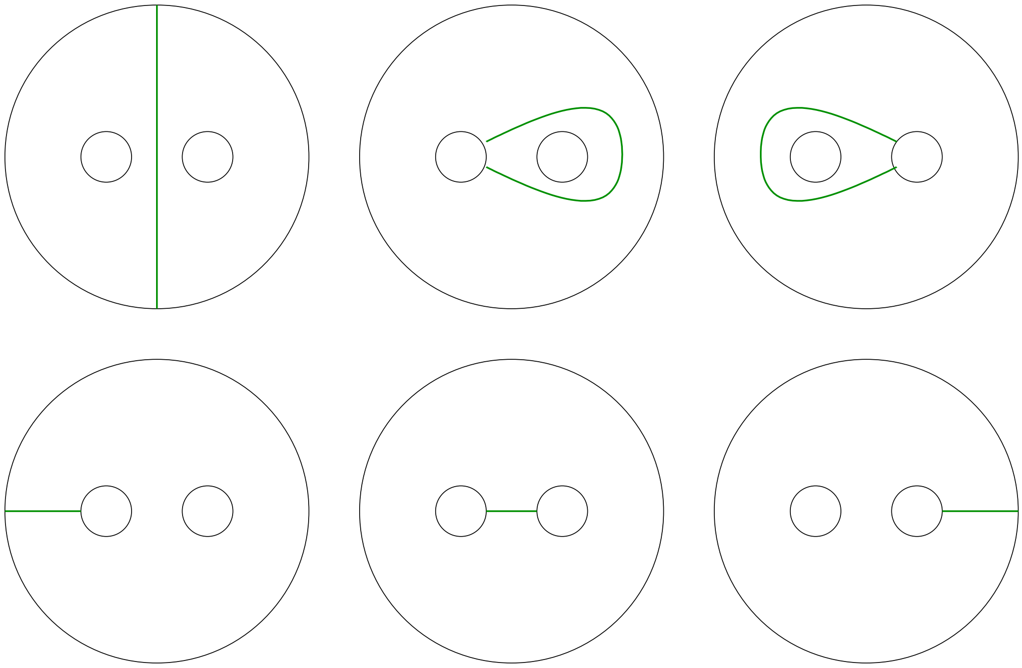


Figure 3.6. Elements of $A(P^2)$.

of the regions in Figure 3.7 below. Here and in the following sections, we will abbreviate the region of triangle inequality by $(\leq \bigtriangledown)$.

We treat each region separately: In $(\leq \bigtriangledown)$; we set $x_{12} = x_{21} = \frac{1}{2}(a_1 + a_2 - a_3)$, $x_{23} = x_{32} = \frac{1}{2}(a_2 + a_3 - a_1)$, $x_{13} = x_{31} = \frac{1}{2}(a_3 + a_1 - a_2)$; and define $\tau(a_1, a_2, a_3)$ as the element of $A'(P^2)$ which consists of x_{ij} segments of the type τ_{ij}, for $i \neq j$. For $a_1 \geq a_2 + a_3$, we set $x_{11} = \frac{1}{2}(a_1 - a_2 - a_3)$, $x_{12} = a_2$, $x_{13} = a_3$ and define $\tau(a_1, a_2, a_3)$ as in Figure 3.8. The other two regions are treated similarly. One can easily observe that the definitions of τ on $\partial(\leq \bigtriangledown)$ agree and that $i \circ \tau = id$.

It remains to show injectivity. We start by observing that the four regions in Figure 3.7 correspond to the triangles in Figure 3.9 below. We let $x_{\alpha\beta}$ denote the number of segments of type $\tau_{\alpha\beta}$ in $\tau \in A'(P^2)$. We observe that $x_{\alpha\alpha} = 0$ for all α implies $i(\tau) \in (\leq \bigtriangledown)$, and $x_{11} \neq 0$ implies $a_1 > a_2 + a_3$. Now, we take $\tau_1, \tau_2 \in A'(P^2)$ with $i(\tau_1) = i(\tau_2) \in A'$. By the above correspondence, $x_{\alpha\beta}$'s of τ_1 and τ_2 coincide. It is sufficient to show that if $\tau_1, \tau_2 \in A'(P^2)$ have all $x_{\alpha\beta}$ equal, then they must be equal. This is straightforward when $\sum_{\alpha \leq \beta} x_{\alpha\beta} = 1$. The general case follows by induction. $\square$

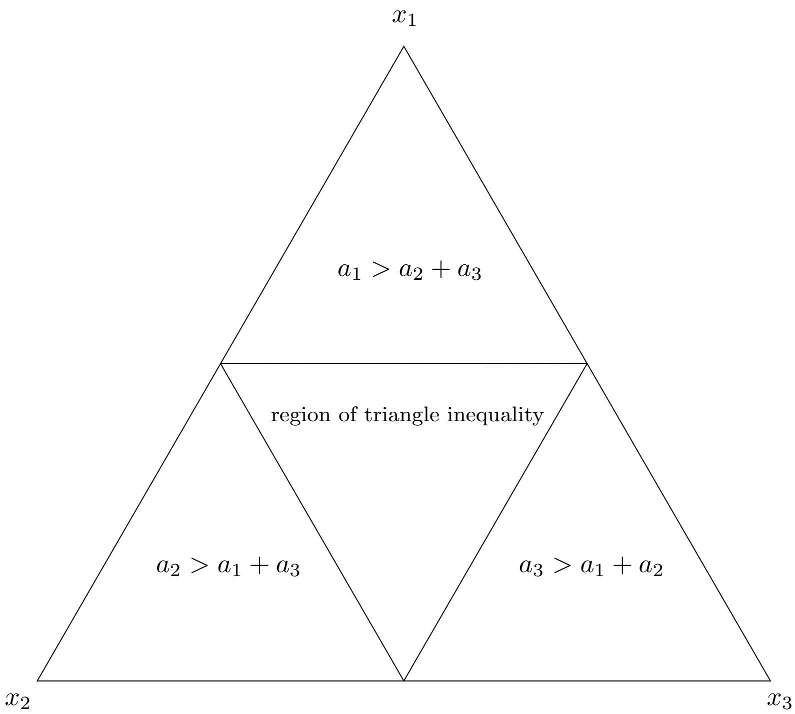


Figure 3.7. Parametrizing $A'(P^2)$.

We are now ready to study $\mathcal{S}'$. We will need some technical ingredients for this part.

In each class of $A'(P^2)$, we choose a canonical representative as in Figure 3.10 below. For each $\tau \in A'(P^2)$ and each ∂_j, we choose an arc x_j (a connected component of $\partial_j - \tau$) again as in the figure. This choice is well-defined since there are no nontrivial orientation-preserving automorphisms of (P^2, τ). Also, for each model τ, we choose an "arc jaune" $J_1 = J_1(\tau)$ with the following properties:

- J_1 is a simple arc joining ∂_1 to itself and which cuts P^2 into two regions, one of which contains ∂_2 and the other contains ∂_3.
- J_1 has one endpoint in the arc $x_1(J_1)$.
- J_1 has minimal intersection with τ.

J_2 and J_3 are defined similarly.

We parametrize $\mathcal{S}'$ as follows:

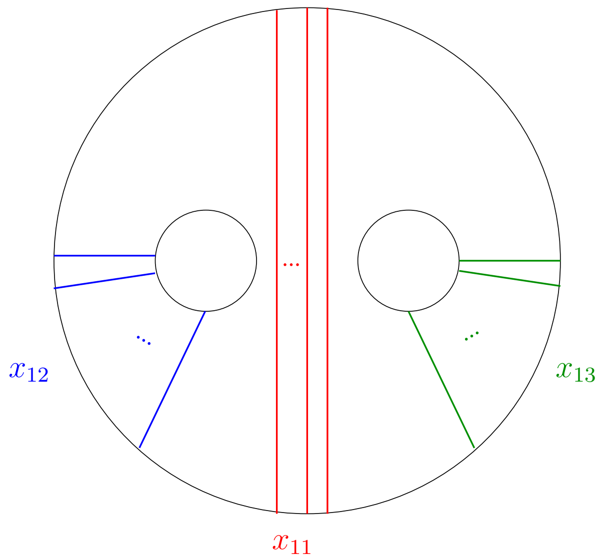


Figure 3.8. Parametrizing $A'(P^2)$: $a_1 \geq a_2 + a_3$.

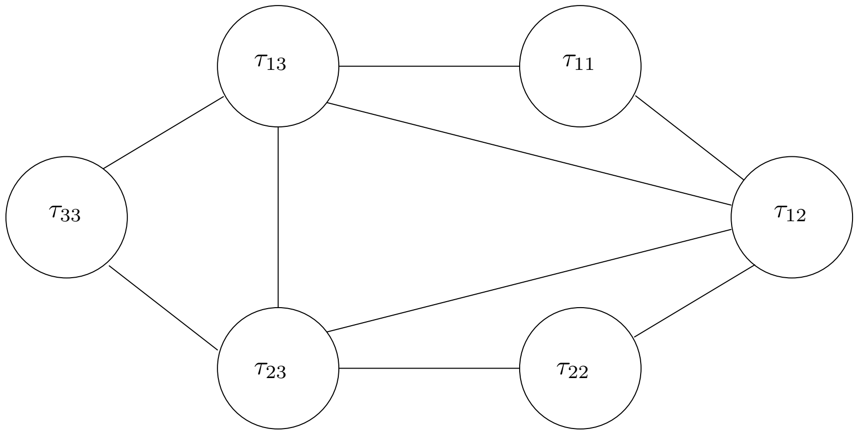


Figure 3.9. Parametrizing $A'(P^2)$: injectivity of the map τ.

(i) We choose simple curves $K_1, ..., K_{3g-3}$ for a pair of pants decomposition of M into $2g-2$ pairs of pants R_j as in Proposition 2.2.

(ii) For each K_j we choose two simple curves K_j' and K_j'' as in Figure 3.11 below. (K_j' and K_j'' differ by a positive Dehn twist along K_j.)

(iii) We take disjoint tubular neighbourhoods $K_j \times [-1,1]$ of K_j's, and set $R_j' = R_j \setminus \bigcup_i (K_i \times [-1,1])$.

(iv) Each R_j' is parametrized by P^2 via a coordinate map ϕ_j. We consider

$$B = \{(m_i, s_i, t_i) \mid i = 1, ..., 3g{-}3,\ m_i, s_i, t_i \geq 0,\ (m_i, s_i, t_i) \in \partial(\leq \bigtriangledown)\} \subset \mathbb{R}_+^{9g-9} \tag{3.3.1}$$

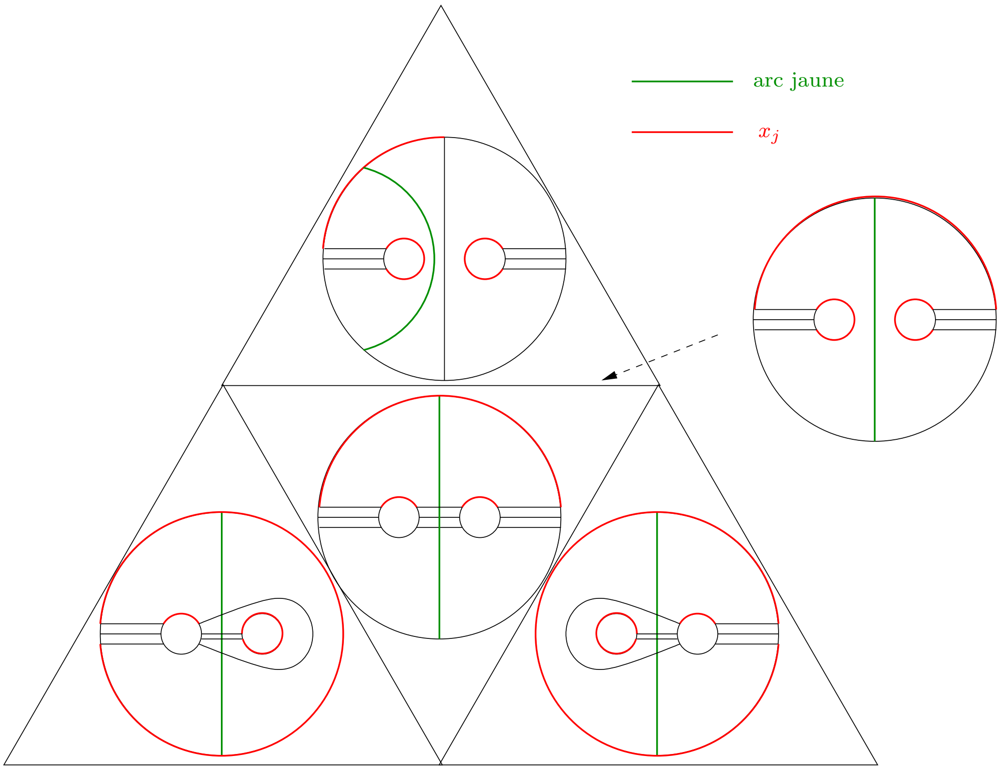


Figure 3.10. Arcs jaune.

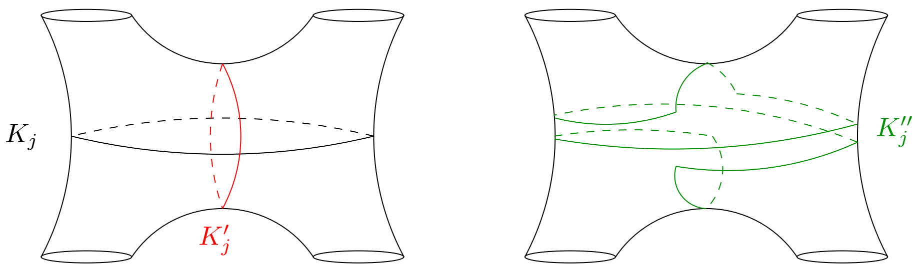


Figure 3.11. Dehn twists along decomposing curves.

We notice that B is a cone in $\mathbb{R}_+^{9g-9}$ which is homeomorphic to $\mathbb{R}^{6g-6}$.

Finally, we construct the classification map $\Phi : \mathcal{S}' \to B$:

We fix $\beta \in \mathcal{S}'$ and put

$$m_j(\beta) \doteq i(\beta, K_j)$$

We choose the representative β_0 of β to be equal to the model (P^2, ϕ_k) in all pairs of pants R'_k. We say β_0 is in "normal form." An easy extension of Proposition 3.7 yields that the normal form β_0 of β is unique; i.e. if β_1 is another representative of β in normal form, then $\beta_0 \cap K_j \times [-1,1]$ and $\beta_1 \cap K_j \times [-1,1]$ are isotopic relative to the boundary, for each j. Now $\beta_0 \cap R_l$'s are equipped with their arcs jaune. We consider the curve K_j and the adjacent pairs of pants R_1, R_2. In R'_1 and R'_2, we have two arcs jaune J_1 and J_2 emanating from the boundaries parallel to K_j.

In $K_j \times [-1,1]$, there exist simple arcs S_j, S'_j, T_j, T'_j such that

- $J_1 \cup S_j \cup J_2 \cup S'_j$ is isotopic to K'_j
- $J_1 \cup T_j \cup J_2 \cup T'_j$ is isotopic to K''_j

We impose the additional conditions $\partial S_j = \partial T_j$, $\partial S'_j = \partial T'_j$, $S_j \cap S'_j = \emptyset$ and $T_j \cap T'_j = \emptyset$; so that $S_j \cup S'_j$ and $T_j \cup T'_j$ are unique up to isotopy relative to the boundary. Moreover, $T_j \cup T'_j$ can be obtained from $S_j \cup S'_j$ by a positive Dehn twist in the annulus. As the endpoints of these arcs are not in β_0, we can assume they are in general position with respect to β_0.

Then we put

$$s_j(\beta) \doteq card(\beta_0 \cap S_j)$$
$$t_j(\beta) \doteq card(\beta_0 \cap T_j)$$

Lemma 3.24. *For each $j = 1, ..., 3g-3$, we have $(m_j(\beta), s_j(\beta), t_j(\beta)) \in \partial(\leq \bigtriangledown)$.*

Proof. We cut the annulus $K_j \times [-1,1]$ along S_j to get Figure 3.12 below. □

We consider the subset B_0 of B consisting of the nonzero points with integer coordinates satisfying the condition that if K_{j_1},K_{j_2},K_{j_3} are on the boundary of the same pair of pants, then $m_{j_1} + m_{j_2} + m_{j_3}$ is even.

Theorem 3.25. *The map $\Phi : \mathcal{S}' \to B$ is a bijection of $\mathcal{S}'$ onto B_0.*

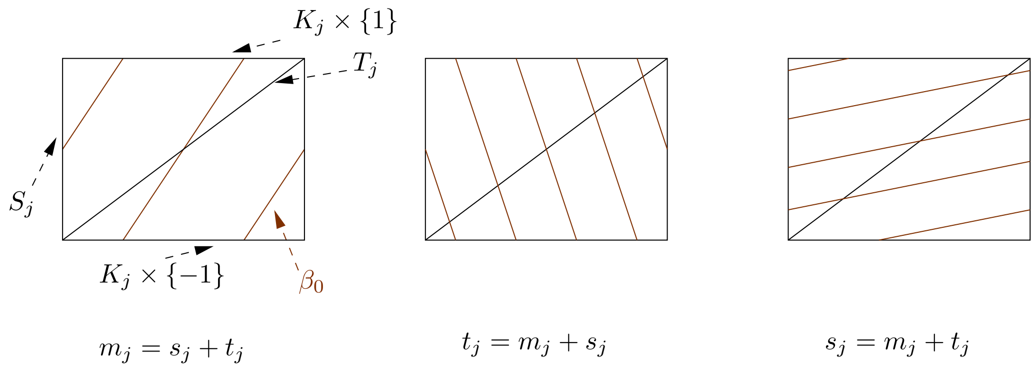


Figure 3.12. The parameters m_j,s_j,t_j satisfy the triangle inequality.

Proof. Let $\beta \in \mathcal{S}'$. We first notice that by Theorem 3.23, m_j's completely determine the behavior of β away from K_j's. It remains to investigate its behavior inside $K_j \times [-1, 1]$. The parameter s_j determines how many times β revolves around K_j. But it cannot distinguish between β_1 and β_2 in Figure 3.13 below:

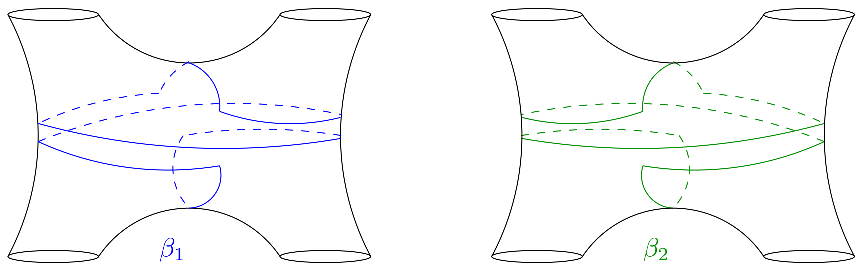


Figure 3.13. The parameter t_j for $\mathcal{S}'$.

To this end, we introduce the parameter t_j. This completes the parametrization of $\mathcal{S}'$. □

As Φ is homogeneous of degree 1, we may extend Φ to

$$\Phi : \mathbb{R}_+^* \times \mathcal{S}' \to B \tag{3.3.2}$$

Corollary 3.26. *The map* $\Phi : \mathbb{R}_+^* \times \mathcal{S}' \to B$ *is injective.*

Proof. We start with α_0, α_1 in $\mathcal{S}$ and $\lambda > 0$ such that $\Phi(\alpha_0) = \lambda\Phi(\alpha_1)$. In fact, one can easily observe that λ is in $\mathbb{Q}$. So, there are integers n_0, n_1 such that $\Phi(n_0, \alpha_0) =$

$\Phi(n_1, \alpha_1)$. By Theorem 3.25, we get $n_0\alpha_0 = n_1\alpha_1$ and the results follows. □

Remark 3.27. In Section 3.6, we will prove that there exists a cone C in $\mathbb{R}_+^{\mathcal{S}}$ and a continuous function $\theta_C : C \to B$ positively homogeneous of degree 1 such that the following diagram commutes:

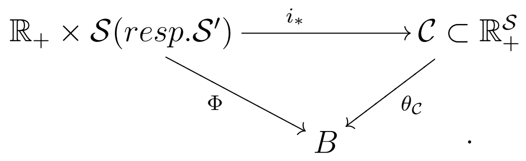


Figure 3.14. Parametrizing $\mathbb{R}_+ \times \mathcal{S}$.

In fact, θ_C induces a homeomorphism of $\overline{i_*(\mathbb{R}_+ \times \mathcal{S}')}$ onto B.

Remark 3.28. By the above remark, we will then deduce the following facts:

- The set $\Phi(\mathbb{R}_+ \times \mathcal{S})$ is dense in B.
- $\overline{\pi i_*(\mathcal{S})} \approx S^{6g-7}$.

3.4. Measured Foliations: Definitions and Some Basic Results

In this section, we will introduce the notion of measured foliations. Measured foliations build up an appropriate framework to study multicurves. While they have important applications in the dynamics of pseudo-Anosov diffeomorphisms, we are mainly concentrated on how Thurston made use of them to compactify the Teichmüller space. This is why we will state some of the theorems without proof and refer the reader to [6].

Definition 3.29. For a foliation $\mathcal{F}$ of S with isolated singularities; by a *transverse invariant measure*, we mean a measure μ defined on each arc transverse to the foliation and satisfying the following property:

If α, β are two arcs transverse to $\mathcal{F}$ which are isotopic through transverse arcs whose endpoints remain in the same leaf, then $\mu(\alpha) = \mu(\beta)$.

Remark 3.30. We will restrict our attention to the case where the measure is regular with respect to the Lebesgue measure. That is, every regular point admits a smooth chart (x, y) such that the foliation is given by dy and the measure is induced by dy.

In this work, we will permit only the following singularities: For a singularity in the interior, we suppose that the foliation is defined by the 1-form $Im(z^{\frac{k}{2}})dz$. A singularity on the boundary is assumed to admit a chart having this foliation in the upper half-plane if k is even and in the half-plane of negative reals if k is odd. Hence, all the charts for regular points and permissible singularities are given (up to isomorphism of foliations) in Figure 3.15 below. (In the sequel, we will cut surfaces along certain curves and obtain surfaces with non-empty boundary. This is why we define permissible singularities on the boundary.)

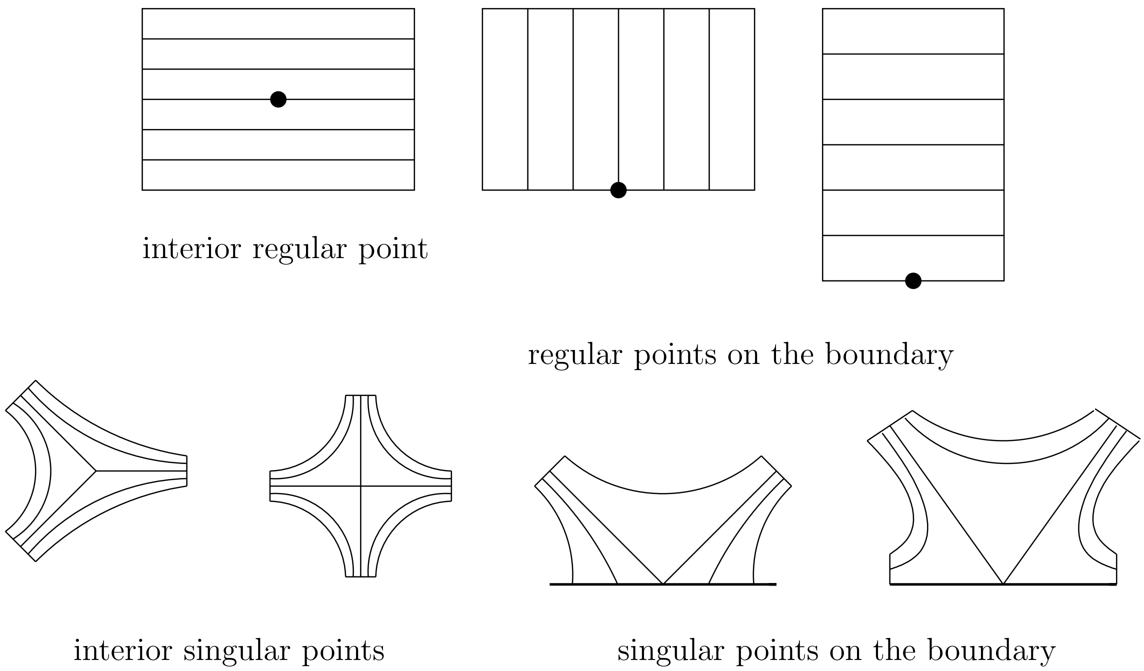


Figure 3.15. Regular points and permissible singularities.

Theorem 3.31. *(Poincaré recurrence) Let S be equipped with a measured foliation $(\mathcal{F}, \mu)$. Let α' be an arc of ∂M, transverse to $\mathcal{F}$ at all points of $int(\alpha')$, and let x be one of its endpoints. Then the leaf leaving from x either goes to a singularity point or to the boundary.*

Remark 3.32. The Poincaré recurrence implies that each boundary component of a measured foliation $(\mathcal{F}, \mu)$ on a compact surface is either transverse to $\mathcal{F}$ or a *cycle of leaves*, i.e. a finite union of leaves and singular points.

Definition 3.33. A curve γ' is said to be *quasitransverse to* $\mathcal{F}$ if each connected component of $\gamma' - sing\mathcal{F}$ is either a leaf or is transverse to $\mathcal{F}$.

Proposition 3.34. *There does not exist a disk D with corners, with $\partial D = \alpha' \cup \beta'$, where α' is an arc contained in a leaf and β' is a quasitransverse arc.*

By $\mathcal{MF}$, we denote the set of measured foliations on a given compact surface quotiented by isotopies and Whitehead operations described in Figure 3.16. (In the former Whitehead operation, we collapse two singularities on the boundary only when the connecting leaf is on the boundary.) We remind that we will cut the surface S along certain curves and get surfaces with boundary so that the definition of Whitehead operations for singularities on the boundary is essential here.

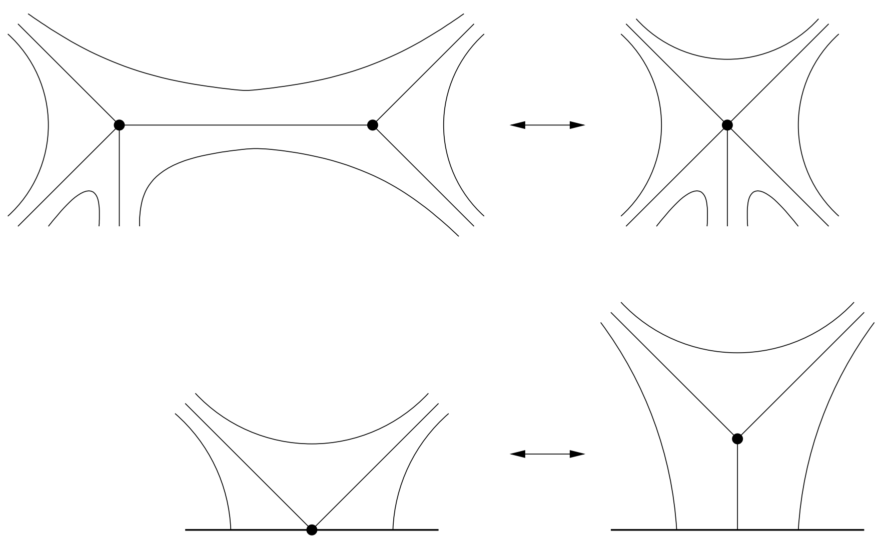


Figure 3.16. Whitehead operations.

Given a closed curve γ'; by $\mu(\gamma')$ (or $\int_{\gamma'} \mathcal{F}$) we mean the total variation of the y coordinate along γ' in an atlas. Then, for $\gamma \in \mathcal{S}$, we set

$$I(\mathcal{F}, \mu; \gamma) \doteq \inf_{\gamma' \in \gamma} \mu(\gamma')$$

This definition is clearly invariant under isotopies and Whitehead operations so that we get a map $I_* : \mathcal{MF} \to \mathbb{R}_+^{\mathcal{S}}$ defined by

$$I_*(\mathcal{F}, \mu)(\gamma) \doteq I(\mathcal{F}, \mu; \gamma)$$

The map I_* constitutes one of the main ingredients in the compactification of the Teichmüller space. It will allow us to see $\mathcal{MF}$ inside $\mathbb{R}_+^{\mathcal{S}}$ and realize that $\overline{\mathcal{MF}}$ is homeomorphic to $\mathbb{R}^{6g-6}$ via a homeomorphism which is positively homogeneous of degree 1 (compare with Remark 3.27). This implies $P\mathcal{MF} \approx S^{6g-6}$. Now, on recalling $\mathcal{T} \approx \mathbb{R}^{6g-6}$ (Corollary 2.20), we will treat $P\mathcal{MF}$ as the "Thurston boundary" of $\mathcal{T}$. The following two propositions investigate how I_* operates, by specifying its connection to quasitransverse curves.

Proposition 3.35. *If γ' is a quasitransverse curve to $\mathcal{F}$, then $\mu(\gamma') = I(\mathcal{F}, \mu; \gamma)$.*

Proof. We take another representative γ''. We start with the case when γ' and γ'' are disjoint. Then they bound an annulus A. By Theorem 3.31, almost every leaf entering A at a point of γ' must meet the boundary of A again. By Proposition 3.34, it cannot meet γ' again. Hence, $\mu(\gamma') \leq \mu(\gamma'')$. Now we suppose γ' and γ'' have a common point. We may assume that they are in general position with respect to each other. (This would change $\mu(\gamma'')$ by an arbitrarily small amount.) As γ' and γ'' are homotopic, we can find arcs α' and α'' such that $int(\alpha') \cap int(\alpha'') = \emptyset$ and $\alpha' \cup \alpha''$ bounds a disk D^2. Again by Theorem 3.31, almost every leaf entering D^2 at a point of α' must meet the boundary of D^2 again so that $\mu(\alpha') < \mu(\alpha'')$. Finally, as γ' and γ'' are in general position, $\gamma'' - \gamma'$ is a finite union of open intervals; and the result follows by induction. □

We recall that a *spine* of a manifold is a 1-complex onto which the manifold deformation retracts.

Proposition 3.36. *Let γ' be a simple closed curve in S and $(\mathcal{F}, \mu)$ be a measured foliation of S. If γ' separates S, we write $S = S_1 \cup_{\gamma'} S_2$ and denote a spine of S_i by Σ_i.*

- *If $I(\mathcal{F},\mu;\gamma) \neq 0$, then there exists $(\mathcal{F}',\mu')$ equivalent to $(\mathcal{F},\mu)$ such that γ' is transverse to $\mathcal{F}'$ and avoids the singularities.*
- *If $I(\mathcal{F},\mu;\gamma) = 0$, then there exists $(\mathcal{F}',\mu')$ equivalent to $(\mathcal{F},\mu)$ such that one of the following (non-exclusive) conditions holds:*
  (i) γ' is a cycle of leaves of $\mathcal{F}'$.
  (ii) γ' separates, and Σ_i is an invariant set of $\mathcal{F}'$ for $i = 1$ or $i = 2$.

Our next step is to introduce "the enlargement" procedure. We will make use of this construction in the next section when we classify measured foliations. We start with a 2-dimensional submanifold S_0 of S such that $S - S_0$ does not have any contractible components. We let Σ be a spine of $\overline{S - S_0}$. We may assume that each singularity of Σ has three branches leaving from it. Then we construct a surjective map $j : S_0 \to S$ satisfying

- j is a (piecewise differentiable) immersion.
- $j|_{int(S_0)}$ is a diffeomorphism onto $S - \Sigma$.
- $j(\partial S_0 - \partial S) = \Sigma$.
- j is the identity outside a small collar of $\partial S_0 - \partial S$.

A typical example is given in Figure 3.17 below.

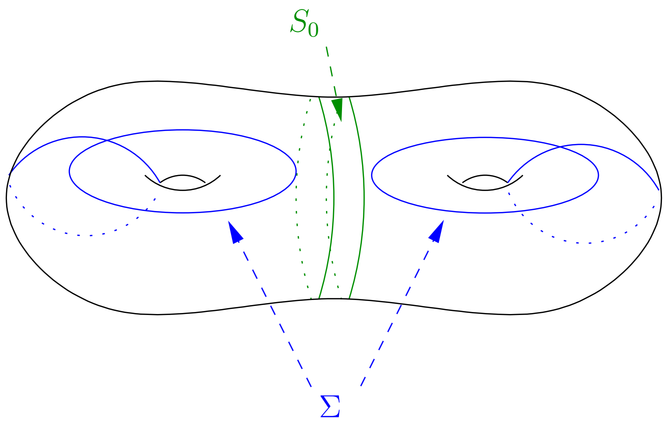


Figure 3.17. Building blocks of the enlargement procedure.

We take a measured foliation $\mathcal{F}_0$ of S_0 satisfying the condition that each component of $\partial S_0 - \partial S$ is an invariant set. We "enlarge" it to $\mathcal{F} \doteq j_*\mathcal{F}_0$ in such a way that

- Σ is an invariant set of $\mathcal{F}$.
- $j|_{int(S_0)}$ conjugates $\mathcal{F}_0|_{int(S_0)}$ and $\mathcal{F}|_{S-\Sigma}$ as measured foliations.

On noting that this procedure is independent of the choice of the spine Σ, we obtain a map

$$\mathcal{MF}(S_0, \partial S_0 - \partial S) \longrightarrow \mathcal{MF}(S) \tag{3.4.1}$$

Lemma 3.37. *Suppose that a measured foliation $(\mathcal{F}, \mu)$ of S is obtained from $(\mathcal{F}_0, \mu_0)$ by enlarging S_0. Let γ' be a simple curve in S. Then $I(\mathcal{F}, \mu; \gamma) = \inf \mu_0(\gamma'' \cap S_0)$ where γ'' is isotopic to γ.*

Proof. It suffices to observe that for each curve C in S_0, there is a curve C' in S isotopic to C such that $C' \cap S_0 = j^{-1}(C)$. □

We conclude this section by defining an inclusion $\mathbb{R}_+^* \times \mathcal{S} \hookrightarrow \mathcal{MF}$. We fix $C \in \mathcal{S}$ and $\lambda \in \mathbb{R}_+^*$, and take a tubular neighborhood S_0 of C. We foliate S_0 by circles parallel to C. As the invariant transverse measure on S_0, we take μ_0 so that the width of S_0 is λ. This procedure defines a measured foliation of S_0, unique up to isotopy. We then enlarge it to get a measured foliation of S, denoted by $(\mathcal{F}_{\lambda,C}, \mu)$.

Proposition 3.38. *Let γ' be a simple curve in S. Then we have $I(\mathcal{F}_{\lambda,C}, \mu; \gamma) = \lambda\ i(C, \gamma)$.*

Proof. We take a component α' of $\gamma' \cap S_0$. If α' goes from one boundary component of S_0 to another, then $\mu_0(\alpha') \geq \lambda$ must hold. We then isotope α' inside S_0 to make it transverse to the foliation; $\mu_0(\alpha'') = \lambda$. If α' does not touch a component of the boundary, then we can push it outside of S_0. So, we apply Lemma 3.37 to get $I(\mathcal{F}_{\lambda,C}, \mu; \gamma) \geq \lambda\ i(C, \gamma)$. Finally, for the equality, we isotope γ' to get minimal intersection with C. □

Remark 3.39. By the above proposition, we get that the diagram in Figure 3.18 commutes.

$$\begin{array}{ccc} \mathbb{R}_+^* \times \mathcal{S} & \longrightarrow & \mathcal{MF} \\ & \searrow^{i_*} & \downarrow I_* \\ & & \mathbb{R}_+^{\mathcal{S}}. \end{array}$$

Figure 3.18. Embedding $\mathbb{R}_+^* \times \mathcal{S}$ into $\mathcal{MF}$.

We have already seen that i_* is injective. We now get that the map $\mathbb{R}_+^* \times \mathcal{S} \hookrightarrow \mathcal{MF}$ defined above is injective.

3.5. Classification of Measured Foliations

In this section, we will classify all the measured foliations on a S up to isotopy and Whitehead operations. In the previous sections, we realized that once we fix a pair of pants decomposition R_j via the curves K_i, we encounter two different type of topological spaces: The collar neighborhoods of K_i's, i.e. annuli; and the pairs of pants R_j''s. So, there is a technical difference between the studies of $\mathcal{T}$ and $\mathcal{MF}$. In $\mathcal{T}$, studying P^2 was sufficient, the gluing only added an $\mathbb{R}$ factor for each K_i, $i = 1, ..., 3g-3$. On the other hand, we encounter three tasks to accomplish the classification of measured foliations: We classify measured foliations of the annulus and of a pair of pants, and then combining them will be rather complicated and requires some work.

We start by recalling a general fact on foliations:

Let M be a compact surface. Let $\mathcal{F}$ be a foliation of M with permissible singularities only. Each component of ∂M is (i) either transverse to $\mathcal{F}$ (ii) or a cycle of leaves (that is, a finite union of leaves and singularity points. See Remark 3.32). To each $s \in sing\mathcal{F}$, we associate

$$P_s = \begin{cases} number \;\; of \;\; separatrices & \text{if } s \in int(M) \text{ or if } s \in \partial M (\text{case (ii)}), \\ number \;\; of \;\; separatrices + 1 & \text{if } s \in \partial M (\text{case (i)}). \end{cases}$$

Then we have the Euler-Poincaré formula, a proof of which can be found in [6]:

$$2\chi(M) = \sum_{sing\mathcal{F}} (2 - P_s) \tag{3.5.1}$$

It directly follows from (3.5.1) that a measured foliation of the annulus $S^1 \times [0,1]$ cannot have any singularities. If $S^1 \times \{0\}$ is a leaf, then all the leaves are closed curves. $S^1 \times \{0\}$ is transverse to the foliation, then all the leaves go from one boundary component to the other. So if we denote coordinates of $S^1 \times [0,1]$ by (θ, x), then all measured foliations are isotopic to $\lambda d\theta$ or λdx, $\lambda \in \mathbb{R}^*$.

This was of course elementary. We omitted the fact that we will parametrize $\mathcal{MF}$. So, we cannot allow every isotopy. We therefore classify the measured foliations of A modulo the action of $\text{Diff}_0(A \;, rel\partial A)$. To this end, we fix γ' and $\overline{\gamma'}$ as in the Figure 3.19 below.

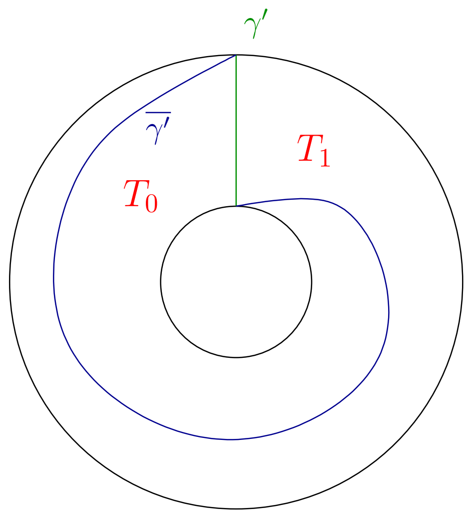


Figure 3.19. Classifying measured foliations on the annulus.

Then, for a measured foliation $(\mathcal{F}, \mu)$, we set

$$m = \mu(S^1 \times \{0\}) = \mu(S^1 \times \{1\}) = I(\mathcal{F}, \mu; [S^1 \times \{0\}])$$
$$s = \inf\{\mu(\gamma'') \mid \gamma'' \ is \ isotopic \ to \ \gamma' \ with \ endpoints \ fixed\}$$
$$t = \inf\{\mu(\gamma'') \mid \gamma'' \ is \ isotopic \ to \ \overline{\gamma'} \ with \ endpoints \ fixed\}$$

Lemma 3.40. *$(m, s, t) \in \partial(\leq \bigtriangledown)$ if and only if (m, s, t) is associated to a measured foliation of A.*

Proof. Straightforward from Figure 3.19 on noting the triangles T_0 and T_1. □

Proposition 3.41. *Let $(\mathcal{F}, \mu)$, $(\mathcal{F}', \mu')$ be two measured foliations of A transverse to ∂A and coinciding on ∂A. Then, $\mathcal{F}$ and $\mathcal{F}'$ are isotopic via an isotopy which is constant on the boundary if and only if $(s, t)(\mathcal{F}) = (s, t)(\mathcal{F}')$.*

Proof. We deform $\mathcal{F}$ and $\mathcal{F}'$ until $s = \mu(\gamma') = \mu'(\gamma')$ and $t = \mu(\overline{\gamma'}) = \mu'(\overline{\gamma'})$. This makes the arcs γ' and $\overline{\gamma'}$ transverse to the foliations. We isotope further to make the measures on γ' and $\overline{\gamma'}$ coincide. Referring to Figure 3.19, we see that the foliations on the boundary of the two triangles T_1 and T_0 match up, which finishes the proof. The reverse implication is clear. □

Next, we pass to the measured foliations on P^2. By Euler-Poincaré formula (3.5.1), we notice that a foliation of P^2 either has one singularity (with 4 separatrices) or two singularities (each with 3 separatrices). Again, we are not interested in all the measured foliations of P^2. Instead, we restrict our attention to those foliations where each boundary component of P^2 is not a smooth leaf. We call such foliations *good foliations*. We denote the set of equivalence classes of good measured foliations of P^2 by $\mathcal{MF}_0(P^2)$.

Lemma 3.42. *Let $\mathcal{F}$ be a measured foliation of P^2. Then*

(i) every leaf is closed in $P^2 - sing\mathcal{F}$.

(ii) If, further, $\mathcal{F}$ is a good foliation, then there does not exist a cycle of leaves interior to P^2.

Proof. (i): We suppose that there is a non-closed leaf L. Then we can find an arc β' on L whose endpoints are joinable by a transversal α'. This yields two possibilities:

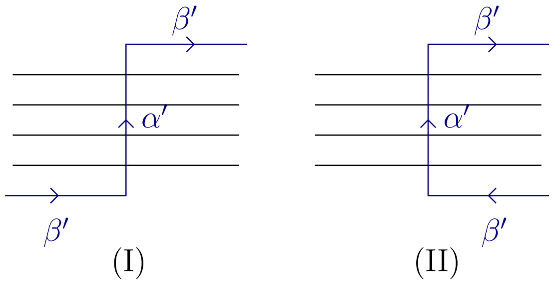


Figure 3.20. Constructing a loop from a non-closed leaf.

Lemma 3.43. *Suppose a compact surface M is foliated by $\mathcal{F}$. If a leaf L of $\mathcal{F}$ is not closed in $M - sing\mathcal{F}$ and if α' is an arc transverse to $\mathcal{F}$ cutting L, then $\alpha \cap L$ is infinite.*

Proof. We will show that $\alpha' \cap L$ cannot be an endpoint of α'. To this end, we cut M along $int(\alpha')$ to get M'. If C is the curve of $\partial M'$ starting from α', the foliation $\mathcal{F}'$ along C induced by $\mathcal{F}$ has the configuration of Figure 3.21. By the Poincaré recurrence theorem (Theorem 3.31), L_g and L_d reach a singularity of $\mathcal{F}'$ or the boundary of M'. This implies that L is closed. □

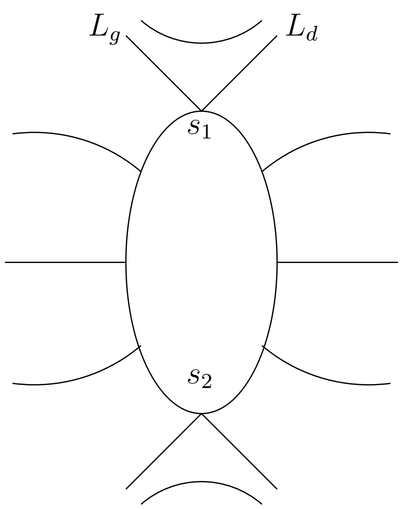


Figure 3.21. Cutting the surface M along $int(\alpha)$.

Now, by Lemma 3.43, $L \cap \alpha'$ is infinite. Since $\alpha' \cup \beta'$ disconnects P^2, (II) is reduced to (I). We approximate $\alpha' \cup \beta'$ by a closed curve γ' transverse to $\mathcal{F}$. The curve γ' intersects L infinitely many times. By Proposition 3.34, γ' cannot bound a disk. Hence, it bounds an annulus with one of the boundary components ∂_i. But each leaf intersecting γ' intersects the boundary component. So, γ' cannot intersect ∂_i infinitely many times; a contradiction.

(ii): Now if γ' is an interior cycle of leaves, $\gamma' \cup \partial_i$ bounds an annulus A. In the neighborhood of γ' in A, the leaves are smooth and closed. So, ∂_i is a smooth closed leaf, which means that $\mathcal{F}$ cannot be a good foliation. □

Corollary 3.44. *Every leaf of a good foliation goes*

(i) either from the boundary to the boundary,

(ii) or from the boundary to a singularity,

(iii) or from a singularity to a singularity.

Definition 3.45. By a *reduced good foliation* of P^2, we mean a good foliation satisfying the following properties:

- if ∂_i is transverse then it avoids singularities,
- the singularities of the boundary are simple (i.e. they have 3 separatrices),
- there are no connections between two singularities where at least one is in the interior.

Lemma 3.46. *In each class of $\mathcal{MF}_0(P^2)$, there is a unique reduced good foliation up to isotopy.*

Proof. This is easily observable. □

Theorem 3.47. *The function*

$$\mathcal{MF}_0(P^2) \longrightarrow \mathbb{R}^3_+ - \{0\}$$
$$(\mathcal{F}, \mu) \mapsto (\mu(\partial_1), \mu(\partial_2), \mu(\partial_3))$$

is a bijection.

Proof. We give the inverse function in Figure 3.22. □

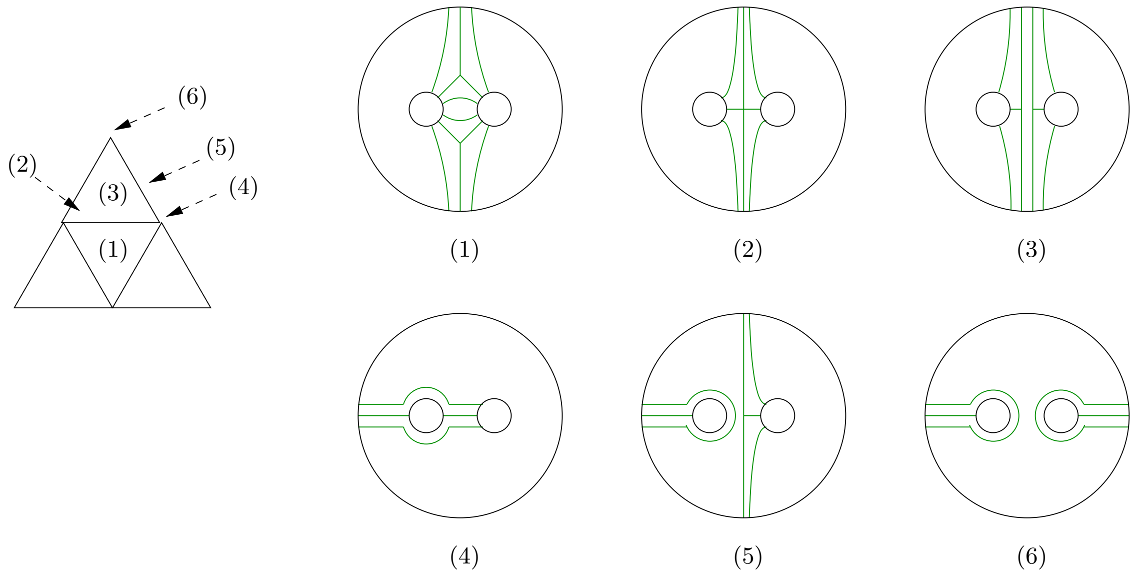


Figure 3.22. Parametrizing good foliations on P^2.

Proposition 3.48. *Any measured foliation $\mathcal{F}$ of P^2 can be obtained by enlarging a foliation $\mathcal{F}_0$ of a submanifold P_0 having the following property: Each connected component C of P_0 is*

- *either a pair of pants and $\mathcal{F}_0|_C$ is a good foliation,*
- *or a collar neighborhood of a curve of ∂P^2 and $\mathcal{F}_0|_C$ is a foliation by circles.*

Proof. If each component of the boundary is not smooth, then there is nothing to prove. We simply take $P_0 = P^2$. Otherwise, we consider a smooth leaf γ_1 in ∂P^2. We then consider the maximal "annulus" A associated to γ_1. If $A = P^2$, then we let P_0 be a collar neighborhood of γ_1. If $A \neq P^2$, then there is a leaf L of $\mathcal{F}$ going from a singularity s_0 to a singularity s_1. If $s_0 = s_1$, $\overline{L}$ is a cycle of leaves that bounds an annulus A' foliated by circles. Then $\overline{P^2 - A'}$ is a pair of pants where the foliation is a good foliation. If $s_0 \neq s_1$, we have the configuration in Figure 3.23. There is a leaf L' leaving from s_1. If L' returns to s_1, $\overline{L'}$ is an embedded cycle; if not, it goes to s_0 and $\overline{L \cup L'}$ bounds an annulus. In each case, we may continue as above. □

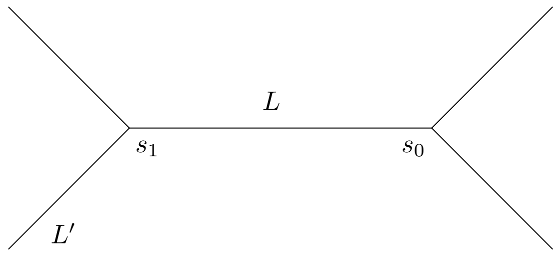
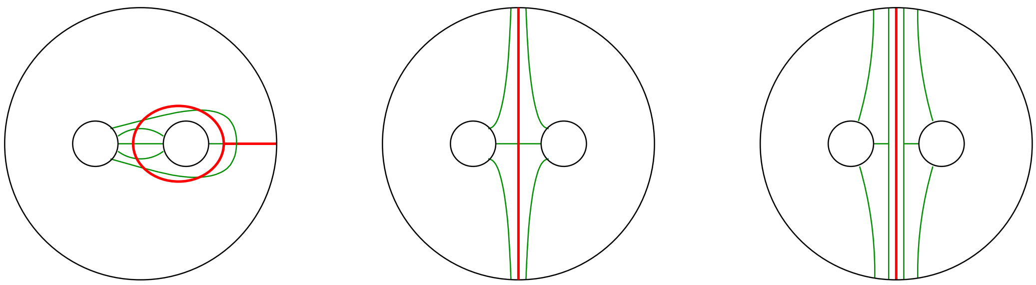


Figure 3.23. Obtaining measured foliations of P^2 using enlargement.

This finishes the study of measured foliations of P^2. In Theorem 3.47, we classified all the good foliations of P^2; and in Proposition 3.48, we saw that good foliations are indeed the building blocks of measured foliations of P^2. It now remains to conjoin the annuli and pairs of pants to get the closed surface we started with. For this, we will need two important concepts; the arcs jaune and the normal forms.

Definition 3.49. For each boundary component ∂_i of P^2 and for each type of good foliation of P^2, we choose a quasitransverse arc with endpoints in ∂_i and not homotopic to an arc of the boundary. We call it the *arc jaune.*

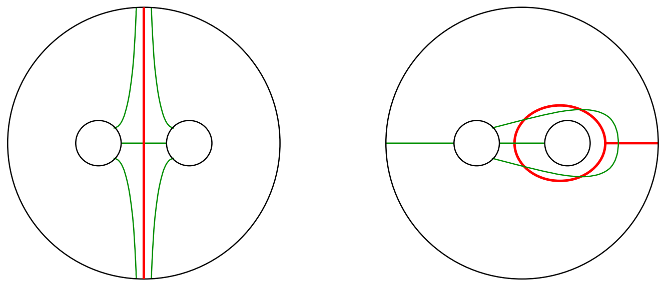
For ∂_1, we choose arcs jaune as indicated in Figures 3.24-27 below.



$m_2 > m_1 + m_3$ (or $m_3 > m_1 + m_2$) | (m_1, m_2, m_3) satisfies the triangle inequality | $m_1 > m_2 + m_3$

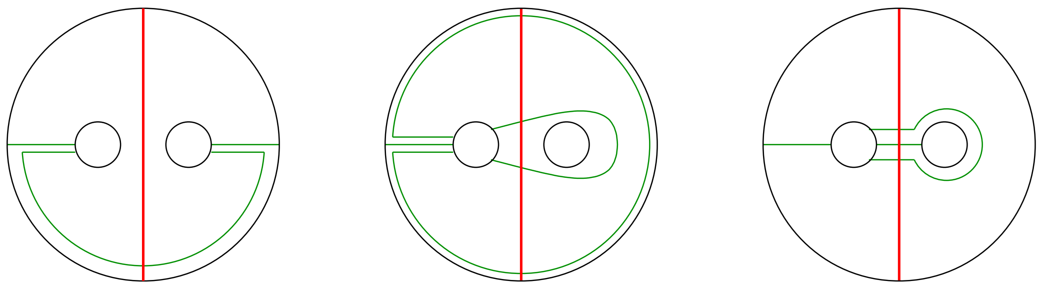
Figure 3.24. Arcs jaune: Generic case.

As before, we start by decomposing S into $2g-2$ pairs of pants $\{R_j\}$ using $3g-3$ curves $\{K_i\}$. We further assume that each R_j is the image of an embedded P^2, i.e. each K_i does not lie in the same pair of pants on its two sides. We will call such a decomposition "permissible."



$m_1 = m_2 + m_3$ $\qquad$ $m_2 = m_1 + m_3$ (or $m_3 = m_1 + m_2$)

Figure 3.25. Arcs jaune: $(m_1, m_2, m_3) \in \partial(\leq \bigtriangledown)$ and $m_1 \neq 0$,$m_2 \neq 0$, $m_3 \neq 0$.



$m_2 = m_3$ $\qquad$ $m_2 > m_3 > 0$ (or $m_3 > m_2 > 0$) $\qquad$ $m_3 = 0$ (or $m_2 = 0$)

Figure 3.26. Arcs jaune: $m_1 = 0$.

Again we take take disjoint tubular neighbourhoods $K_i \times [-1, 1]$ of K_i's, and denote the connected components of $S - \bigcup(K_i \times [-1, 1])$ by R'_j.

Definition 3.50. A compact submanifold S_0 of S endowed with a measured foliation $\mathcal{F}_0$ is in *normal form* (with respect to a permissible decomposition) if

(i) each component of ∂S_0 is a cycle of leaves,

(ii) $S_0 \cap R'_j = \emptyset$, or $S_0 \cap R'_j = R'_j$ with $\mathcal{F}|_{R'_j}$ good foliation,

(iii) $S_0 \cap K_i \times (-1, 1)$ is

- either empty,
- or $K_i \times (-1, 1)$ and $\mathcal{F}_0$ is transverse to $K_i \times \{t\}$, $t \in [-1, 1]$ (Here, S_0 contains the pairs of pants adjacent to the annulus.),
- or $K_i \times [\frac{1}{2}, \frac{1}{2}]$ and $\mathcal{F}_0$ has $K_i \times \{t\}$ as leaves, $t \in [\frac{1}{2}, \frac{1}{2}]$.

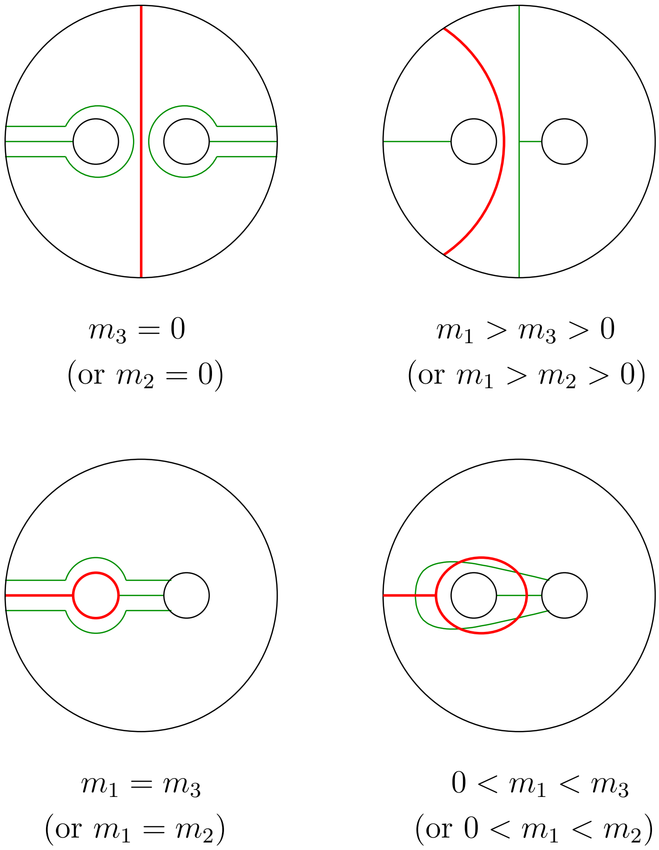


Figure 3.27. Arcs jaune: $m_2 = 0$ (or $m_3 = 0$).

We will say that $(S_0, \mathcal{F}_0)$ is a *normal form of* $\mathcal{F}$ if $(S_0, \mathcal{F}_0)$ is a normal form and $\mathcal{F}$ is obtained by enlargement of $(S_0, \mathcal{F}_0)$.

Proposition 3.51. *Every measured foliation on S has a normal form.*

We now introduce an equivalence relation compatible with the notion of normal forms. $(S_0, \mathcal{F}_0)$ and $(S_0', \mathcal{F}_0')$ are equivalent if $S_0 = S_0'$ and if $\mathcal{F}_0'$ can be obtained from $\mathcal{F}_0$ by a finite sequence of the operations below:

- Whitehead operation with support in one of the R_j',
- isotopy with support in $K_i \times [-1, 1] \cap S_0$,
- isotopy with support in R_j.

We denote the equivalence classes by $\mathcal{NF}$.

Remark 3.52. The enlargement function induces a map $\mathcal{NF} \to \mathcal{MF}$. It is surjective by Proposition 3.51. In fact, we will see that it is bijective.

To parametrize $\mathcal{MF}$, we will first parametrize $\mathcal{NF}$. We recall the curves K'_j and K''_j from the previous section. (Figure 3.28.)

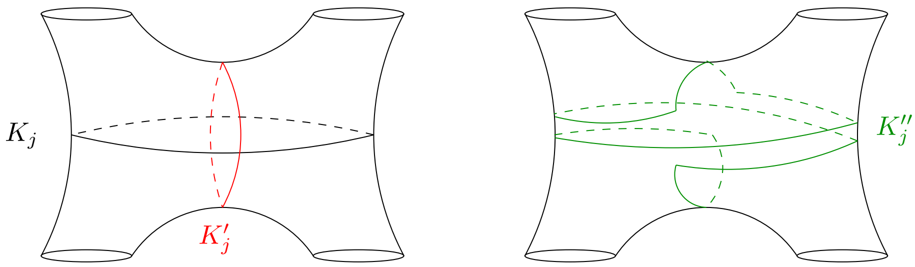


Figure 3.28. Parametrizing $\mathcal{NF}$ using the curves K_j, K'_j, K''_j.

We also recall

$$B \;=\; \{(m_i, s_i, t_i) \mid i = 1, ..., 3g-3 \;\; m_i, s_i, t_i \geq 0 \;\; (m_i, s_i, t_i) \in \partial(\leq \bigtriangledown)\} \qquad (3.5.2)$$

which is a cone in $\mathbb{R}_+^{9g-9}$ homeomorphic to $\mathbb{R}^{6g-6}$.

We construct a function $\mathcal{NF} \to B - \{0\}$: We let $(S_0, \mathcal{F}_0)$ be the representative of an element of $\mathcal{NF}$. We may suppose that $\mathcal{F}_0|_{R'_j}$ is a canonical form for all j. We set m_i be the measure of K_i (and zero if $K_i \cap S_0$). To define s_i and t_i, we consider the annulus $K_i \times [-1, 1]$. If $K_i \times [-1, 1] \cap S_0 \subset K_i \times \{-1\} \cup \{1\}$, we set $s_i = t_i = 0$. If $K_i \times [-\frac{1}{2}, \frac{1}{2}] = K_i \times (-1, 1) \cap S_0$, then the annulus is foliated by circles and $s_i = t_i$ is the width of the annulus. We observe that in both of the cases above, we have $m_i = 0$ so that $(m_i, s_i, t_i) \in \partial(\leq \bigtriangledown)$ holds. We are now left with the case when $S_0 \cap K_i \times [-1, 1] = K_i \times [-1, 1]$. Then, the foliation is transverse to $K_i \times \{x\}$ for all $x \in [-1, 1]$ and S_0 contains adjacent pairs of pants R'_k and R'_l. The pairs of pants R'_k and R'_l come with their arcs jaune J_k and J_l. As in the parametrization of the space $\mathcal{S}'$ of multicurves (Section 3.3), we take arcs S_i and S'_i in $K_i \times [-1, 1]$ such that $J_k \cup S_i \cup J_l \cup S'_i$ is a closed curve homotopic to K'_i. Similarly, we take arcs T_i and T'_i such that $J_k \cup T_i \cup J_l \cup T'_i$ is homotopic to K''_i. Here, we take S_i and S'_i to be arcs of minimal length. Finally, we set $s_i = \mu_0(S_i)$ and $t_i = \mu_0(T_i)$. As we already have the classification of measured foliations on the annulus, by Lemma 3.40, we conclude that $(m_i, s_i, t_i) \in \partial(\leq \bigtriangledown)$.

Lemma 3.53. *The function constructed above $\mathcal{NF} \to B - \{0\}$ is a bijection.*

Proof. It is easy to see that the invariants m_i, s_i, t_i are independent of the choice of the representative $(S_0, \mathcal{F}_0)$. We must demonstrate that the image of the function does not contain 0: If for some i, $S_0 \cap K_i \times (-1, 1) = K_i \times [-\frac{1}{2}, \frac{1}{2}]$, then we have $(s_i, t_i) \neq 0$. Otherwise if $R'_j \subset S_0$ for some j, then the induces foliation is a good foliation so that one of the boundary curves has nonzero measure. Finally, that it is bijective is straightforward from the classifications of measured foliations of the annulus and P^2. □

Proposition 3.54. *There exists a continuous function $\theta : I_*(\mathcal{MF}) \to B$, positively homogeneous of degree 1 (i.e. $\theta(\lambda x) = \lambda\, \theta(x)$ for $\lambda > 0$) which makes the diagram in Figure 3.29 commutative.*

$$\begin{array}{ccc} \mathcal{NF} \longrightarrow \mathcal{MF} & \xrightarrow{\quad I_* \quad} & I_*(\mathcal{MF}) \subset \mathbb{R}_+^{\mathcal{S}} \\ & \searrow & \downarrow \theta \\ & & B. \end{array}$$

Figure 3.29. Parametrizing $\mathcal{NF}$.

Proof. The proof relies on determining the invariants s_i,t_i via homogeneous continuous formulas. We omit this rather technical computation and refer the reader to [6]. On the other hand, we clearly have $m_i = I(\mathcal{F}_0, \mu_0; [K_i])$ for any foliation $(\mathcal{F}_0, \mu_0)$. □

As $\mathcal{NF} \to B$ is an injection, we obtain two immediate consequences.

Theorem 3.55. *Two measured foliations $(\mathcal{F}, \mu)$ and $(\mathcal{F}', \mu')$ are Whitehead equivalent if and only if for all simple curves γ' on S we have $I(\mathcal{F}, \mu, \gamma) = I(\mathcal{F}', \mu', \gamma)$.*

Proposition 3.56. *The enlargement function $\mathcal{NF} \to \mathcal{MF}$ is a bijection.*

Finally, we are ready to state the classification of measured foliations. We identify $\mathcal{MF}$ with $I_*(\mathcal{MF})$ to provide it with the topology induced by $\mathbb{R}_+^{\mathcal{S}}$ and to complete it to $\overline{\mathcal{MF}} = \mathcal{MF} \cup \{0\}$.

Theorem 3.57. *The function θ is a homeomorphism of $\overline{\mathcal{MF}}$ onto $B \approx \mathbb{R}^{6g-6}$, positively homogeneous of degree 1. Consequently, $P\mathcal{MF} \approx S^{6g-7}$.*

Proof. We will be satisfied with giving a sketch of the proof. We already know that θ is a continuous bijection. Around each point $(\mathcal{F}, \mu)$ in $\mathcal{MF}$, we will construct a coordinate chart. This will imply that $I_*(\mathcal{MF})$ is a topological manifold. Then, by the invariance of domain theorem (see for example [15]), we conclude that θ is in fact a homeomorphism. To this end, we first observe that if we are given a measured foliation with some $m_i = 0$, we can change the pair of pants decomposition to make $m_i \neq 0$. The exact procedure is given in Figure 3.30 below. We consider the adjacent pairs of

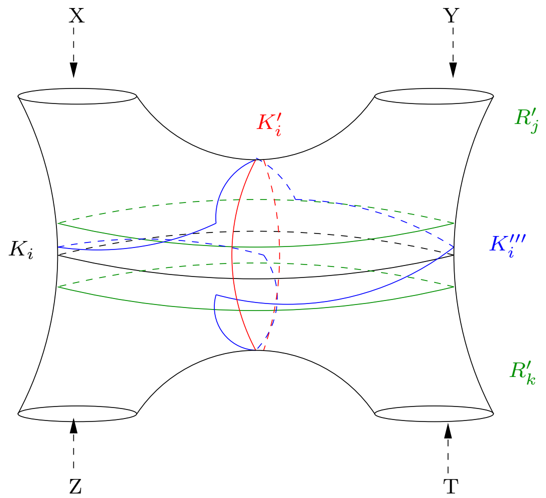


Figure 3.30. Classifying measured foliations.

pants, and replace K_i with either K_i''' or K_i', depending on whether the curves X and Z bound an annulus or not. Then one can show that the inverse of θ is continuous at points with coordinates $(m_i, s_i, t_i)_{i=1,\ldots,3g-3}$ where $m_i \neq 0$, and the result follows. □

3.6. $\mathcal{MF}$ vs. $\mathbb{R}_+^{\mathcal{S}}$

We have two commutative diagrams given in Figure 3.31 and Figure 3.32.

$$\begin{array}{ccccc} & & \mathbb{R}_+^* \times \mathcal{S} & & \\ & & {\scriptstyle enlargement}\downarrow & \searrow^{i_*} & \\ \mathcal{NF} & \longrightarrow & \mathcal{MF} & \longrightarrow & \mathbb{R}_+^{\mathcal{S}}. \end{array}$$

Figure 3.31. $\mathcal{MF}$ versus $\mathbb{R}_+^{\mathcal{S}}$.

$$\begin{array}{ccccc} \mathcal{NF} & \longrightarrow \mathcal{MF} & \xrightarrow{I_*} & I_*(\mathcal{MF}) & \subset \mathbb{R}_+^{\mathcal{S}} \\ & \searrow & & \downarrow{\scriptstyle \theta} & \\ & & & B. & \end{array}$$

Figure 3.32. Parametrizing $\mathcal{NF}$.

Combining them, we get $\overline{\Phi} : \mathbb{R}_+^* \times \mathcal{S}$ (resp. $\mathcal{S}'$) $\to B$, which to $\beta \in \mathcal{S}$ associates $\{(\overline{m_j}(\beta), \overline{s_j}(\beta), \overline{t_j}(\beta)) \mid j = 1, ..., 3g-3\ \}$. (Here we use the fact that θ is positively homogeneous of degree 1. We define $\overline{\Phi}$ on $\mathcal{S}$ and extend it by homogeneity.)

We remark here that $\overline{\Phi}$ does not coincide with the function $\Phi : \mathcal{S}' \to B$ defined in Section 3.3. The reason for this is that for multicurves and for foliations, we chose the arcs jaune differently. Yet, one can implement a modification in the formulas to obtain a homeomorphism

$$\theta_{\mathcal{C}} : \overline{I_*(\mathcal{MF})} \to B \tag{3.6.1}$$

which makes the diagram in Figure 3.33 commutative.

$$\begin{array}{ccc} \mathcal{S}' & \xrightarrow{i_*} & I_*(\mathcal{MF}) \\ & {\scriptstyle \Phi}\searrow & \downarrow{\scriptstyle \theta_{\mathcal{C}}} \\ & & B. \end{array}$$

Figure 3.33. Parametrizing $\mathcal{MF}$.

Combining this with the fact $i_*(\mathcal{S}') \subset \overline{i_*(\mathbb{R}_+ \times \mathcal{S})}$ (Corollary 3.19), we collect the following important results:

Theorem 3.58. *There exists a closed cone $\mathcal{C}$ in $\mathbb{R}_+^{\mathcal{S}}$ and a continuous map $\theta_{\mathcal{C}} : \mathcal{C} \to B$ positively homogeneous of degree 1, which makes the diagram in Figure 3.34 commutative. Furthermore, $\theta_{\mathcal{C}}$ induces a homeomorphism of $\overline{i_*(\mathbb{R}_+ \times \mathcal{S}')}$ onto B.*

$$\begin{array}{ccc} \mathbb{R}_+ \times \mathcal{S}(resp.\mathcal{S}') & \xrightarrow{\ i_*\ } & \mathcal{C} \subset \mathbb{R}_+^{\mathcal{S}} \\ & \searrow^{\Phi} \quad \swarrow_{\theta_{\mathcal{C}}} & \\ & B & . \end{array}$$

Figure 3.34. Parametrizing $\mathbb{R}_+ \times \mathcal{S}$.

Corollary 3.59. *The set $\Phi(\mathbb{R}_+ \times \mathcal{S})$ is dense in B.*

Corollary 3.60. $\overline{\pi i_*(\mathcal{S})} \approx S^{6g-7}$.

Proposition 3.61. *In $P(\mathbb{R}_+^{\mathcal{S}})$, the set $\pi i_*(\mathcal{S}) = \mathcal{S}$ is dense in $\pi I_*(\mathcal{MF})$. Consequently, $I_*(\mathcal{MF}) \cup \{0\} = \overline{i_*(\mathbb{R}_+ \times \mathcal{S})}$.*

4. THURSTON'S COMPACTIFICATION OF THE TEICHMÜLLER SPACE

In Chapter 2, we showed that $\mathcal{T}$ is homeomorphic to $\mathbb{R}^{6g-6}$. Then in Chapter 3, we saw $\mathcal{MF}$ inside $\mathbb{R}_+^{\mathcal{S}}$ and proved that $P\mathcal{MF}$ is homeomorphic to S^{6g-7}. The idea in Thurston's compactification of the Teichmüller space is to see $\mathcal{T}$ and $P\mathcal{MF}$ glued together in such a way that they form a closed disk D^{6g-6}, $P\mathcal{MF}$ constituting the boundary. The ambient space for this gluing process will be $P(\mathbb{R}_+^{\mathcal{S}})$. In Section 4.1, we will embed $\mathcal{T}$ into $\mathbb{R}_+^{\mathcal{S}}$. In Section 4.2, we will observe that $\mathcal{T}$ and $\mathcal{MF}$ lie *nicely* inside $\mathbb{R}_+^{\mathcal{S}}$. And finally in Section 4.3, we will introduce the topology on $\overline{\mathcal{T}} \doteq \mathcal{T} \sqcup P\mathcal{MF}$ which makes it a closed disk.

4.1. $\mathcal{T}$ vs. $\mathbb{R}_+^{\mathcal{S}}$

For a hyperbolic metric h and $\alpha \in \mathcal{S}$, we define $l(h, \alpha)$ to be the length of the unique geodesic in the isotopy class α (see Lemma 2.5 and Remark 3.2). We thus get a map $l_* : \mathcal{T} \to \mathbb{R}_+^{\mathcal{S}}$ defined by $l_*([h])(\alpha) = \inf\{l(h, \alpha) \ : \ h \in [h]\}$. We studied l_* in the proof of Theorem 2.16 when we parametrized the Teichmüller space. In this section, we will prove that l_* defines a proper map which is a homeomorphism onto its image. We do this in two steps. We fix a pair of pants decomposition $\{K_i\}$ and choose the curves K_i' and K_i'' as before.

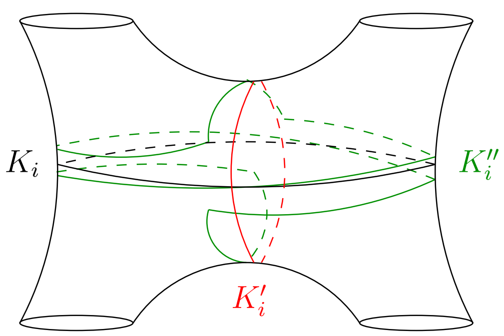


Figure 4.1. Linking $\mathcal{T}$ to $\mathbb{R}_+^{\mathcal{S}}$.

Proposition 4.1. *The map*

$$\Lambda : \mathcal{T} \to \mathbb{R}_+^{9g-9} \ , \ m \mapsto (l(m,[K_i]), l(m,[K_i']), l(m,[K_i'']))$$

is injective and proper (hence a homeomorphism onto its image).

Proof. We already saw in Section 2.3 that the function l is strictly increasing and proper. □

Recalling the correspondence between $\mathbb{R}_+^{\mathcal{S}}$ and $\mathbb{R}_+^{9g-9}$ (Section 3.6), Proposition 4.1 yields that l_* is proper. We already know that it is continuous and has continuous sections. It is left to show that l_* is injective.

Proposition 4.2. $\pi \circ l_* : \mathcal{T} \to P(\mathbb{R}_+^{\mathcal{S}})$ *is injective.*

Proof. We consider the upper half-space model $\mathbb{H}^2$ with $ds = \dfrac{dx^2 + dy^y}{y^2}$. The group of isometries is $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm Id\}$. Here the action is given by $z \mapsto \dfrac{az+b}{cz+d}$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For a hyperbolic element A, we set $l(A) = \inf_{z\in\mathbb{H}^2} d(z, A\cdot z)$. The minimum is attained on the invariant geodesic of A. We have $\mathrm{Tr}(A) = 2\cosh(\dfrac{l(A)}{2})$ and $\mathrm{Tr}(A)\cdot\mathrm{Tr}(B) = \mathrm{Tr}(AB) + \mathrm{Tr}(A^{-1}B)$, which can be proven by direct calculations. We take two simple curves γ_1' and γ_2' which intersect transversely at the base point. We write $\gamma_3' = \gamma_1' * \gamma_2'$ and $\gamma_4' = \gamma_1'^{-1} * \gamma_2'$. We may assume that γ_3' and γ_4' are simple curves. Now, we fix a metric h of curvature -1. Then γ_i''s correspond to hyperbolic isometries of $\mathbb{H}^2$ with $l_i = l(h,\gamma_i)$. By the above formulas, we get

$$\cosh(\frac{l_1+l_2}{2}) + \cosh(\frac{l_1-l_2}{2}) = \cosh(\frac{l_3}{2}) + \cosh(\frac{l_4}{2}) \qquad (4.1.1)$$

We claim that there cannot another metric for which the lengths of all closed geodesics are multiplied by $k \neq 1$. Otherwise, inserting (4.1.1), we would get

$$\cosh(k\frac{l_1+l_2}{2}) + \cosh(k\frac{l_1-l_2}{2}) = \cosh(k\frac{l_3}{2}) + \cosh(k\frac{l_4}{2}) \tag{4.1.2}$$

which, together with (4.1.1), implies $\{l_1+l_2, l_1-l_2\} = \{l_3, l_4\}$. This is impossible as the angle between γ_1' and γ_2' is nonzero. □

We thus proved the following theorem:

Theorem 4.3. *The map $l_* : \mathcal{T} \to \mathbb{R}_+^{\mathcal{S}}$ is a proper map which is a homeomorphism onto its image.*

4.2. $\mathcal{T}$ vs. $\mathcal{MF}$

Proposition 4.4. *In $\mathbb{R}_+^{\mathcal{S}}$, the spaces $\mathcal{T}$ and $\mathcal{MF}$ are disjoint.*

Proof. We consider the elements of $\mathcal{T}$ and $\mathcal{MF}$ inside $\mathbb{R}_+^{\mathcal{S}}$. If $f \in \mathcal{T}$, then the set $\{i(f,\alpha) \mid \alpha \in \mathcal{S}\}$ is bounded below by a positive constant $c > 0$ since the surface is compact. To prove $\mathcal{T}$ and $\mathcal{MF}$ are disjoint; we will show that for any $f \in \mathcal{MF}$, $0 \in \overline{\{i(f,\alpha) \mid \alpha \in \mathcal{S}\}}$.

We take $f \in \mathcal{MF}$, let $(\mathcal{F},\mu)$ represent f and fix $\varepsilon > 0$. Then we consider a small arc δ transverse to $\mathcal{F}$ with $\mu(\delta) \leq \varepsilon$. By Poincaré recurrence theorem (Theorem 3.31), almost every leaf departing from a point of δ returns to δ. So, we obtain a simple closed curve γ' formed by an arc of δ and an arc of a leaf of $\mathcal{F}$. We thus get $i(f,\gamma) \leq \mu(\gamma') \leq \mu(\delta) \leq \varepsilon$, as desired. □

We now introduce a projection $q : \mathcal{T} \to \mathcal{MF}$ that will be utilized to define the charts for the manifold $\overline{\mathcal{T}}$. We fix a pair of pants decomposition $\{R_j\}_{j=1,\dots,2g-2}$ of the surface via mutually disjoint simple curves $\mathcal{K} = \{K_1, ..., K_{3g-3}\}$. We take an arbitrary element $[h]$ of $\mathcal{T}$ and represent it by the metric $\bar{h}$ for which K_j's are geodesics. We define $q([h]) \in \mathcal{MF}$ as follows:

We first impose $i(q([h]), K_j) = i([h], K_j)$ for all j. Then, as always, we focus on a single pair of pants R. On writing $\partial R = K_1 \cup K_2 \cup K_3$, we set $2m_j = i([h], K_j)$. By $g_{jj'}$, we denote the simple $\bar{h}$-geodesic of R orthogonal to K_j and to $K_{j'}$.

• Case 1: $(m_1, m_2, m_3) \in (\leq \bigtriangledown)$. We let T_{12} be the closed geodesic tube of points in R at a distance from g_{12} at most $\dfrac{m_1 + m_2 + m_3}{2}$. We foliate it in the natural way, and the distance induces the transverse measure. We apply the same procedure to T_{23} and T_{13}. The tubes have only two common points which are on the boundary and the configuration is as in Figure 4.2 below.

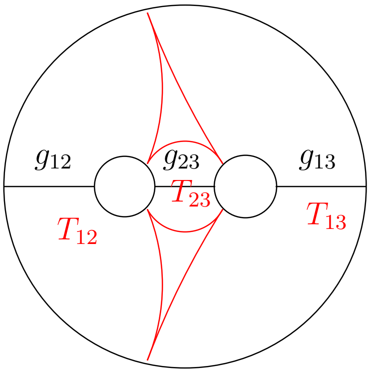


Figure 4.2. $\mathcal{T}$ versus $\mathcal{MF}$, Case 1.

• Case 2: $m_1 > m_2 + m_3$. Here, T_{12} is the tube of radius m_2 and T_{13} is the tube of radius m_3. We then let T_{11} be the union of lines of equal distance to g_{11}. Then $T_{11} \cap K_1 = K_1 \setminus (T_{13} \cup T_{12})$ and the configuration is as in Figure 4.3.

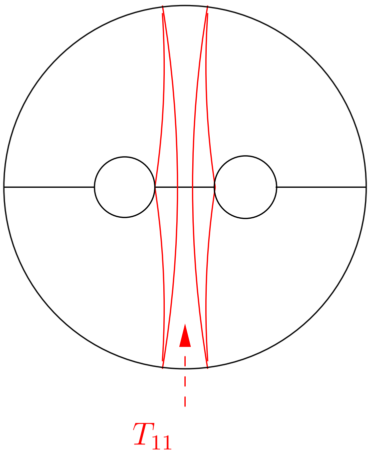


Figure 4.3. $\mathcal{T}$ versus $\mathcal{MF}$, Case 2.

We treat the other cases as in Case 2. Now in both cases, we get a *partial* measured foliation $\mathcal{F}_{\bar{h}}$ (see Definition 5.2). We define $q([h])$ to be its equivalence class.

Proposition 4.5. *The map q is a homeomorphism of $\mathcal{T}$ onto the open set $\mathcal{U}(\mathcal{K})$ of $\mathcal{MF}$ consisting of the functionals taking nonzero values on each component of $\mathcal{K}$.*

Proof. We think that it is clear from the construction that the assignment q is a continuous surjection. By the classification of hyperbolic metrics of pairs of pants; we see that if two hyperbolic metrics are in the preimage of a measured foliation, then they are conjugate by a diffeomorphism isotopic to identity, by an isotopy which is constant on the boundary. (Here we work with a fixed measured foliation and fixed metrics instead of the corresponding equivalence classes.) So, regluing the pairs of pants, we conclude that $q : \mathcal{T} \to \mathcal{MF}$ (as a map between equivalence classes) is injective. As both $\mathcal{T}$ and $\mathcal{U}(\mathcal{K})$ are topological manifolds, by the invariance of domain theorem (see for example [15]), q is a homeomorphism. □

4.3. The Manifold $\overline{\mathcal{T}}$

Before discussing the topological structure of $\overline{\mathcal{T}}$, we state two facts the proofs of which can be found in [6].

For a pair of pants decomposition $\mathcal{K}$ and $\varepsilon > 0$, we let $V(\mathcal{K}, \varepsilon)$ denote the open set of $\mathcal{T}$ defined by the metrics for which each component of $\mathcal{K}$ is a geodesic of length $> \varepsilon$.

Lemma 4.6. *For all $\varepsilon > 0$ and for each $\alpha \in \mathcal{S}$, there exists a constant C such that for all $[h] \in V(\mathcal{K}, \varepsilon)$ we have*

$$i(q([h]), \alpha) \leq i([h], \alpha) \leq i(q([h]), \alpha) + C.$$

Corollary 4.7. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $V(\mathcal{K}, \epsilon)$ tending to infinity in $\mathcal{T}$. Then, $\pi(x_n)$ converges if and only if $\pi \circ q(x_n)$, with the same limit in the case of convergence.*

We now consider the disjoint union $\mathcal{T} \sqcup P\mathcal{MF}$. We endow it with a topology as follows: We take the open sets in $\mathcal{T}$ and the sets of the form $(\mathcal{T} \cap \pi^{-1}(U)) \cup (P\mathcal{MF} \cap U)$ where U is an open subset of $P(\mathbb{R}_+^{\mathcal{S}})$. They form a basis for a topology on $\mathcal{T} \sqcup P\mathcal{MF}$. We denote this topological space by $\overline{\mathcal{T}}$.

We recall that $\pi \circ l_*$ injects $\mathcal{T}$ into $P(\mathbb{R}_+^{\mathcal{S}})$ (Theorem 4.3) and $\pi \circ l_*(\mathcal{T})$ avoids $P\mathcal{MF}$ (Proposition 4.4).

Remark 4.8. The topology on $\overline{\mathcal{T}}$ clearly satisfies the second axiom of countability.

Next we investigate how $\overline{\mathcal{T}}$ looks like. We start with $f \in P\mathcal{MF}$. As in the proof of Theorem 3.57, we choose a pair of pants decomposition along a system $\mathcal{K}$ of curves $\{K_j\}$ so that $i(\overline{f}, [K_j]) \neq 0$ for each j, where $\overline{f}$ is a lift of f to $\mathbb{R}_+^{\mathcal{S}}$. (Here, we note that $i(f, [K_j]) \neq 0$ implies $i(\overline{f}, [K_j]) \neq 0$; this is a general fact in projective geometry.) We let $\{K_j', K_j''\}$ be the system of curves utilized to parametrize $\mathcal{T}$.
We fix the following notation:

- $\mathcal{U}(\mathcal{K})$ is the open set consisting of the functionals in $\mathcal{MF}$ taking nonzero values on K_i's.
- $V(\mathcal{K}, \varepsilon)$ is the open set in $\mathcal{T}$ defined by the metrics for which each K_i is a geodesic of length $> \varepsilon$.
- W is the open set in $P\mathcal{MF}$ of the projective functionals which are nonzero on K_i's.

Then, we observe tht $\pi^{-1}(W) = \mathcal{U}(\mathcal{K})$ and $\pi \circ q(V(\mathcal{K}, \varepsilon)) = W$.

Lemma 4.9. *The set $W \cup V(\mathcal{K}, \varepsilon)$ is open in $\overline{\mathcal{T}}$.*

Proof. For an element of $\mathcal{T}$, there is nothing to show since, by definition of $\overline{\mathcal{T}}$, the open neighborhoods of $\mathcal{T}$ and open neighborhoods in $\overline{\mathcal{T}}$ coincide inside $\mathcal{T}$. We will only show that for $x \in W$, $W \cup V(\mathcal{K}, \epsilon)$ defines a neighborhood of x in $\overline{\mathcal{T}}$. This we prove by contradiction. We suppose that for $x \in W$ we can construct a sequence $\{x_n\}$

in $\mathcal{T}$ such that $x_n \notin V(\mathcal{K}, \varepsilon)$ yet $\pi(x_n) \to x$. The hypothesis $x_n \notin V(\mathcal{K}, \varepsilon)$ means that, up to renumeration and finding a subsequence, we have $i(x_n, K_1) < \varepsilon$. The sequence $\{x_n\}$ is in $\mathcal{T}$, and $\mathcal{T}$ and $\mathcal{MF}$ are disjoint by Proposition 4.4. Hence, x_n cannot a subsequence converging to a point of $\mathcal{T}$. This means that x_n tends to infinity in $\mathcal{T}$. But we assumed $\pi(x_n) \to x$. So, we must have a sequence λ_n in $\mathbb{R}_+$ such that $\lambda_n x_n \to f$ where f is a measured foliation in the fiber of x. Since x_n tends to infinity, we get $\lambda_n \to 0$. Therefore $i(f, K_i) = 0$, because $i(\lambda_n x_n, K_1) = \lambda_n i(x_n, K_1) \to 0$. This contradicts to our assumption that $x \in W$. $\square$

We now define

$$\phi : W \cup V(\mathcal{K}, \varepsilon) \to W \times [0, 1]$$

by $\phi(x) = (x, 0)$ for $x \in W$,
and $\phi(x) = (\pi \circ q(x), e^{-\sum\{i(q(x),K_j)+i(q(x),K'_j)+i(q(x),K''_j)\}})$ for $x \in V(\mathcal{K}, \varepsilon)$.

(Here, as we stated previously, we identify $\mathcal{MF}$ with $I_*(\mathcal{MF}) \subset \mathbb{R}_+^{\mathcal{S}}$ so that an element of $\mathcal{MF}$ is in fact a functional. This is how we interpret $i(q(x), K_j)$ in the definition of ϕ.)

Lemma 4.10. *The map ϕ defined above is a homeomorphism onto an open subset of $W \times [0, 1]$.*

Proof. Continuity on W is clear. We study the continuity of ϕ on $V(\mathcal{K}, \varepsilon)$.

The continuity of the first component is easy: In the proof of Lemma 4.9, we showed that if a sequence $\{x_n\}$ in $\mathcal{T}$ converges to $x \in W$ in $\overline{\mathcal{T}}$, then $\{x_n\}$ tends to infinity in $\mathcal{T}$. As $\{x_n\} \subset V(\mathcal{K}, \epsilon)$, by Corollary 4.7, $\pi(x_n) \to x$ implies $\pi \circ q(x_n) \to x$.

Similarly, the second component of ϕ is continuous: If x_n is a sequence in $V(\mathcal{K}, \epsilon)$ converging to x in $\overline{\mathcal{T}}$, then $\sum\{i(q(x), K_j) + i(q(x), K'_j) + i(q(x), K''_j)\} \to \infty$ since $\{x_n\}$ tends to infinity in $\mathcal{T}$.

We thus conclude that ϕ is continuous.

The injectivity of ϕ is trivial for elements $x, y \in W$ and for elements $x \in W$, $y \in \mathcal{T}$. We suppose $x, y \in \mathcal{T}$. The equality $\pi \circ q(x) = \pi \circ q(y)$ yields $q(x) = \lambda q(y)$; and $e^{-\sum\{i(q(x),K_j)+i(q(x),K'_j)+i(q(x),K''_j)\}} = e^{-\sum\{i(q(y),K_j)+i(q(y),K'_j)+i(q(y),K''_j)\}}$ implies $\lambda = 1$. So, $q(x) = q(y)$. Since q is injective, $x = y$ as desired.

We observe that for $\phi(x) = (z, t)$, we $\phi(q^{-1}(\lambda q(x))) = (z, t^{\lambda})$.

Now, to see that ϕ is a homeomorphism onto an open subset of $W \times [0, 1]$, we first take a continuous section $\sigma : W \to \mathcal{MF}$ of π (i.e. for $x \in W$, $\pi \circ \sigma(x) = x$). In the sequel, we will only deal with elements inside $q(V(\mathcal{K}, \varepsilon))$ by scaling appropriately. $\phi \circ\ q^{-1} \circ \sigma(W)$ is the graph in $W \times [0, 1]$ of a strictly positively function on W by the following computation:

$$
\begin{aligned}
&x \in W,\\
&\sigma(x) \in q(V(\mathcal{K}, \epsilon)) \subset \mathcal{U}(\mathcal{K}) \subset \mathcal{MF},\\
&q^{-1} \circ \sigma(x) \in V(\mathcal{K}, \epsilon),\\
&\phi \circ q^{-1} \circ \sigma(x) \in W \times [0, 1] \subset P\mathcal{MF} \times [0, 1],\\
&\phi \circ q^{-1} \circ \sigma(x) = (\pi \circ q \circ q^{-1} \circ \sigma(x),\\
&\qquad e^{-\sum\{i(q\circ q^{-1}\circ\sigma(x),K_j)+i(q\circ q^{-1}\circ\sigma(x),K'_j)+i(q\circ q^{-1}\circ\sigma(x),K''_j)\}})\\
&\qquad = (x, e^{-\sum\{i(\sigma(x),K_j)+i(\sigma(x),K'_j)+i(\sigma(x),K''_j)\}}).
\end{aligned}
$$

We consider a neighborhood of $W \times \{0\}$, by taking the set bounded by this graph. By the above observation, this neighborhood is indeed contained in the image of ϕ.

Finally, the inverse of ϕ is continuous on this neighborhood: We take a sequence $(z_n, t_n) \to (z, 0)$. Since $t_n \to 0$, we get that $q \circ \phi^{-1}(z_n, t_n)$ tends to infinity. By Lemma 4.6, $\phi^{-1}(z_n, t_n)$ tends to infinity in $\mathcal{T}$. Since $z_n = \pi \circ q \circ \phi^{-1}(z_n, t_n) \to z$, by Corollary 4.7, $\pi \circ \phi^{-1}(z_n, t_n) \to z$, as desired. □

Lemma 4.9 and Lemma 4.10 yield that $\overline{\mathcal{T}}$ is a topological manifold with boundary and $\partial\overline{\mathcal{T}} = P\mathcal{MF}$. In particular, $\overline{\mathcal{T}}$ is locally compact and by Remark 4.8 it is paracompact. By [16]; the boundary admits a collar neighborhood and a generalization of the Schönflies theorem gives that it bounds a sphere. We conclude that $\overline{\mathcal{T}}$ is homeomorphic to a ball; in particular it is compact. We finally get Thurston's compactification of the Teichmüller space:

Theorem 4.11. *The space $\overline{\mathcal{T}} = P\mathcal{MF} \sqcup \mathcal{T}$, given the topology induced by $P(\mathbb{R}_+^{\mathcal{S}})$, is a compact manifold with boundary, homeomorphic to a ball and bounded by $P\mathcal{MF}$.*
If the surface is closed and of genus $g > 1$, then $\overline{\mathcal{T}} \approx D^{6g-6}$.
The group of isotopy classes of diffeomorphisms of the surface acts continuously on $\overline{\mathcal{T}}$, by the transposed action of the direct image action on $\mathcal{S}$.

Proof. We already showed everything except that the topology on $\overline{\mathcal{T}}$ is induced by the topology on $P(\mathbb{R}_+^{\mathcal{S}})$, which is a direct consequence of Proposition 4.2 and Proposition 4.4. $\square$

5. THE CLASSIFICATION OF SURFACE DIFFEOMORPHISMS

5.1. Preliminaries

We combine the natural actions of $\pi_0(\text{Diff}(S))$ on S and on $P\mathcal{MF}$ to obtain a continuous action on $\overline{\mathcal{T}}$.

We take $\varphi \in$Diff(S). Since $\overline{\mathcal{T}}$ is homeomorphic to D^{6g-6} (Theorem 4.11), by the Brouwer Fixed Point Theorem, $[\varphi]$ has a fixed point. That is to say, there exists $x \in \overline{\mathcal{T}}$ such that $[\varphi] \cdot x = x$. In Section 2.4, we have dealt with the case when $x \in \mathcal{T}$ and showed that in that case φ is isotopic to a diffeomorphism of finite order (Corollary 2.22). In this chapter, we will study the case when $x \in P\mathcal{MF}$. Then $[\varphi] \cdot x = x$ means that there exists a measured foliation $(\mathcal{F}, \mu)$ and a scalar $\lambda \in \mathbb{R}$ such that

$$\varphi(\mathcal{F}, \mu) \sim (\mathcal{F}, \lambda\mu) \sim \lambda(\mathcal{F}, \mu). \tag{5.1.1}$$

Remark 5.1. We recall that in the above expression the equivalence relation means that the two measured foliations represent the same functional in $\mathbb{R}_+^{\mathcal{S}}$, or equivalently that they are Whitehead equivalent.

Definition 5.2. A *partial measured foliation* of S is a measured foliation $(\mathcal{F}', \mu')$ supported on a compact submanifold N of dimension 2, satisfying

i) every connected component of ∂N is a cycle of leaves,
ii) if Γ is a component of ∂N which bounds a disk in $int(S) \setminus int(N)$, then the number of separatrices which leave $sing(\mathcal{F}' \cap \Gamma)$ and enter N is at least 2.

We start with the measured foliation $(\mathcal{F}, \mu)$ given in (5.1.1). We "unglue" $\mathcal{F}$ along the leaves joining the singularities, and "blow-up" the singularities which are not touched by these connections. The obtained partial measured foliation is called the *unglue* of $(\mathcal{F}, \mu)$, denoted by $U(\mathcal{F}, \mu)$. We observe that all the singularities of $U(\mathcal{F}, \mu)$ are on the boundary of the resulting submanifold.

Remark 5.3. We note the following properties of $U(\mathcal{F}, \mu)$:

(i) $I_*(\mathcal{F}, \mu) = I_*(U(\mathcal{F}, \mu)) \in \mathbb{R}_+^{\mathcal{S}}$.
(ii) if $I_*(\mathcal{F}_1, \mu_1) = I_*(\mathcal{F}_2, \mu_2)$ then $U(\mathcal{F}_1, \mu_1)$ is isotopic to $U(\mathcal{F}_2, \mu_2)$.

By $\beta U(\mathcal{F}, \mu)$, we denote the union of the boundary components of the support of $U(\mathcal{F}, \mu)$ which do not bound a disk. We will see $\beta U(\mathcal{F}, \mu)$ as an element of $\mathcal{S}'$. By Remark 5.3, $\beta U(\mathcal{F}, \mu)$ only depends on the class of $(\mathcal{F}, \mu)$. In what follows, we will distinguish the three cases below and study them separately:

- $\beta U(\mathcal{F}, \mu) \neq \emptyset$.
- $\beta U(\mathcal{F}, \mu) = \emptyset$ and $\lambda = 1$.
- $\beta U(\mathcal{F}, \mu) = \emptyset$ and $\lambda \neq 1$.

We will call a measured foliation $(\mathcal{F}, \mu)$ *arational* if $\beta U(\mathcal{F}, \mu) = \emptyset$. Before analyzing the three cases above, we collect several facts on the behavior of arational measured foliations.

Lemma 5.4. *Suppose $(\mathcal{F}, \mu)$ is arational. Then we have*

(i) the compact invariant set X consisting of all singularities and the leaves joining two singularities only has connected components that are contractible.
(ii) $\mathcal{F}$ does not have any closed smooth leaves.

Proof. Suppose $\beta U(\mathcal{F}, \mu) = \emptyset$. Then $\overline{S - \text{supp} U(\mathcal{F}, \mu)}$ collapses onto X, by definition of $U(\mathcal{F}, \mu)$. This is (i).

To show (ii), we take a smooth leaf Γ. We thicken Γ by stability and find a maximal cylinder $\Phi : \Gamma \times [0, 1] \to M$ such that $\Phi(\Gamma \times \{0\}) = \Gamma$, and $\Phi(\Gamma \times [0, 1))$ is an embedding on one side of Γ. We have $\Phi(\Gamma \times \{1\}) \subset X$ since $g > 1$. Then by (i), $\Phi(\Gamma \times [0,1]) \approx D^2$ with spine $\Phi(\Gamma \times \{1\})$. But there is no measured foliation of D^2 such that ∂D^2 is a leaf. This means that Γ cannot be closed. □

Corollary 5.5. *Suppose $(\mathcal{F}, \mu)$ is arational. Then there exists a measured foliation $(\mathcal{F}', \mu')$ (unique up to isotopy) equivalent to $(\mathcal{F}, \mu)$, which has no connection between singularities.*

Proof. Given $(\mathcal{F}, \mu)$, we collapse the connected components of the set X in Lemma 5.4 to get $(\mathcal{F}', \mu')$. Uniqueness follows from the fact that Whitehead smoothings of X leads to an isotopy of $(\mathcal{F}', \mu')$. □

The measured foliation $(\mathcal{F}', \mu')$ in Corollary 5.5 is called the *canonical model* for the arational measured foliation $(\mathcal{F}, \mu)$. In the sequel, we will always represent arational measured foliations by their canonical models.

Lemma 5.6. *Suppose $(\mathcal{F}, \mu)$ is the canonical model of an arational measured foliation and φ is a diffeomorphism satisfying $\varphi(\mathcal{F}, \mu) \sim \lambda(\mathcal{F}, \mu)$ for $\lambda \in \mathbb{R}^*_+$. Then there exists a diffeomorphism φ', isotopic to φ, such that $\varphi'(\mathcal{F}, \mu) = (\mathcal{F}, \lambda\mu)$.*

Proof. The measured foliations $\varphi(\mathcal{F}, \mu)$ and $(\mathcal{F}, \lambda\mu)$ are canonical models of the same type; hence they are isotopic. □

Definition 5.7. For a measured foliation $(\mathcal{F}, \mu)$, an $\mathcal{F}$*-rectangle* is the image of an immersion $\varphi : [0,1] \times [0,1] \to M$ with the following properties:

i) $\varphi|_{(0,1)\times(0,1)}$ is a smooth embedding.
ii) $\varphi(\{t\} \times [0,1])$ is contained in a finite union of leaves and singularities; if $t \in (0,1)$, then the image is contained in a single leaf.
iii) $\varphi([0,1] \times \{0\})$ and $\varphi([0,1] \times \{1\})$ are transverse to $\mathcal{F}$.

For an $\mathcal{F}$-rectangle R, we use the following notations:

- $\partial_{\mathcal{F}} R = \varphi(\{0,1\} \times [0,1])$
- $\partial^0_{\mathcal{F}} R = \varphi(\{0\} \times [0,1])$
- $\partial^1_{\mathcal{F}} R = \varphi(\{1\} \times [0,1])$
- $\partial_\tau R = \varphi([0,1] \times \{0,1\})$
- $\partial^0_\tau R = \varphi([0,1] \times \{0\})$
- $\partial^1_\tau R = \varphi([0,1] \times \{1\})$
- $intR = \varphi((0,1) \times (0,1))$

Definition 5.8. A *good system of transversals* for $\mathcal{F}$ is a finite system $\tau = \{\tau'_i \ : \ i \in I\}$ of simple arcs with the following properties:

i) Each τ'_i is transverse to $\mathcal{F}$ and may only meet a singularity at one of its endpoints.

ii) Two arcs cannot meet except at a single endpoint; if this is a singularity, then the two arcs fall into two distinct sectors.

Lemma 5.9. *Given a measured foliation $\mathcal{F}$ and a good system of transversals τ, there exists a system of $\mathcal{F}$-rectangles $R_1, ..., R_N$ such that*

(i) $intR_i \cap intR_j = \emptyset$ for $i \neq j$,

(ii) $\partial^\varepsilon_\tau R_i$ is contained in a single arc of τ (for $\varepsilon \in \{0,1\}$),

(iii) every $\partial^\varepsilon_{\mathcal{F}} R_i$ contains a points of $sing\mathcal{F} \cup \partial\tau$, i.e. R_i's are maximal with respect to condition (ii),

(iv) the two sides of each τ'_k are covered by the R_i's.

The system $(R_1, ...R_N)$ is unique.
If, in addition, $\mathcal{F}$ is arational, then we have

(v) $R_1 \cup ... \cup R_N = S$.

Proof. We cut the surface as in Figure 5.1 below to get a surface $\hat{S}$ with boundary, equipped with a foliation $\mathcal{F}'$. The boundary $\hat{\tau}$ of $\hat{S}$ is the double of τ. We consider the finite subset Z of $\hat{\tau}$, defined by any one of the following conditions:

(i) $x \in sing\mathcal{F}'$,

(ii) x is one of the points giving an endpoint of τ,

(iii) The leaves departing from x run into a singularity of $\mathcal{F}$ or a point which gives an endpoint of τ.

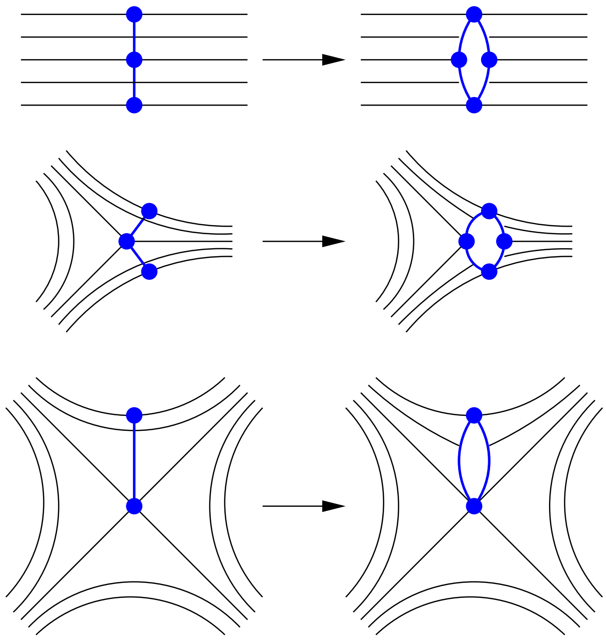


Figure 5.1. Cutting the manifold around singularities.

By Poincaré recurrence theorem (Theorem 3.31), almost all leaves departing from a point of $\hat{\tau} - Z$ return to $\hat{\tau} - Z$. We thicken each component α_i' of $\hat{\tau} - Z$ by stability to find a rectangle R_i such that $\partial_\tau^0 R_i = \overline{\alpha_i'}$ and $\partial_\tau^1 R_i$ gets attached to another component of $\hat{\tau} - Z$. This procedure yields the desired $\mathcal{F}$-rectangles. Uniqueness is straightforward from the construction.

We now suppose $\mathcal{F}$ is arational. We first notice that $\bigcup_{i=1}^N R_i$ is a closed $\mathcal{F}$-invariant set. If the boundary is nonempty, there would exist a closed $\mathcal{F}$-invariant set consisting of cycles of leaves. Since $\mathcal{F}$ is arational, it follows that the boundary is empty. Therefore $\bigcup_{i=1}^N R_i = S$, as desired. □

5.2. Classification of Surface Diffeomorphisms: Second Step

In this section, we deal with the case $\beta U(\mathcal{F}, \mu) \neq \emptyset$.

Proposition 5.10. *Suppose φ is a diffeomorphism and $(\mathcal{F}, \mu)$ is a measured foliation such that $\varphi(\mathcal{F}, \mu) \sim \varphi(\mathcal{F}, \lambda\mu) = \lambda(\mathcal{F}, \mu)$. If $\beta U(\mathcal{F}, \mu) \neq \emptyset$, then φ is isotopic to a diffeomorphism that fixes a system of simple curves which are mutually disjoint and not homotopic to a point.*
We call such a diffeomorphism reducible.

Proof. By assumption we know that $U(\varphi(\mathcal{F}, \mu))$ and $U(\mathcal{F}, \lambda\mu)$ are isotopic so that $\beta U(\varphi(\mathcal{F}, \mu)) = \beta U(\mathcal{F}, \lambda\mu)$. Also, we observe that $\varphi(\beta U(\mathcal{F}, \mu) = \beta U(\varphi(\mathcal{F}, \mu))$ and $\beta U(\mathcal{F}, \mu) = \beta U(\mathcal{F}, \lambda\mu)$. So, $\beta U(\mathcal{F}, \mu) \in \mathcal{S}'$ is invariant under $[\varphi]$.

We sketch the construction of φ': Cutting S along $\beta U(\mathcal{F}, \mu)$, we obtain a (possibly disconnected) submanifold with boundary. Since in this work we skipped the generalization of Thurston's theory of surfaces to surfaces with boundary, we refer the reader to [6]. We find it satisfactory to say that the connected components of the resulting submanifold have either smaller genus than S or smaller Euler characteristic in absolute value (hence the word *reducible*), and that therefore we can construct φ' in a finite number of steps. □

5.3. Classification of Surface Diffeomorphisms: Third Step

In this section, we study the case when $(\mathcal{F}, \mu)$ is arational and $\lambda = 1$.

Proposition 5.11. *If φ is a diffeomorphism and $(\mathcal{F}, \mu)$ is an arational measured foliation such that $\varphi(\mathcal{F}, \mu) = (\mathcal{F}, \mu)$, then φ is isotopic to a diffeomorphism of finite order which preserves $(\mathcal{F}, \mu)$.*

Proof. Around each singularity, we take a transverse arc in each sector, all having the same length, as indicated in Figure 5.2.

Since $\lambda = 1$, we may choose τ so that $\varphi(\tau) = \tau$ (perhaps after an isotopy through diffeomorphisms which preserve $\mathcal{F}$). The collection τ is a good system of transversals.

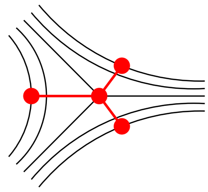


Figure 5.2. Transverse arcs emanating from singularities.

Using Lemma 5.9, we find $\mathcal{F}$-rectangles $R_1, ..., R_N$. Since $\varphi(\tau) = \tau$ and $\varphi(\mathcal{F}) = \mathcal{F}$, every $\varphi(R_i)$ is again an $\mathcal{F}$-rectangle. Moreover, φ permutes the R_i's. We first isotope φ to get a diffeomorphism which is periodic on $\bigcup \partial R_i$, and then make a second isotopy to get the desired periodic diffeomorphism on S. □

5.4. Classification of Surface Diffeomorphisms: Fourth Step

In this section, we study the case when $(\mathcal{F}, \mu)$ is arational and $\lambda \neq 1$. This is the most complicated case and we will see that φ is isotopic to a pseudo-Anosov "diffeomorphism." We take $(\mathcal{F}, \mu)$ as the canonical model (Corollary 5.5) and by Lemma 5.6 we suppose $\varphi(\mathcal{F}, \mu) = (\mathcal{F}, \lambda\mu)$. By changing φ to φ^{-1} if necessary, we further assume that $\lambda > 1$.

Definition 5.12. A diffeomorphism $\varphi : S \to S$ is *pseudo-Anosov* if there exist transverse measured foliations $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$, and a scalar $\lambda > 1$ such that

- $\varphi(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \frac{1}{\lambda}\mu^s)$
- $\varphi(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u)$

We call $\mathcal{F}^s$ and $\mathcal{F}^u$ the *stable* and the *unstable* foliations, respectively.

Remark 5.13. A pseudo-Anosov diffeomorphism is a true diffeomorphism on $S - sing\mathcal{F}^s$ but is never C^1 at the singularities. This is why we will write diffeomorphism inside quotation marks in the sequel.

We will be satisfied with giving the idea of the proof of the following proposition. Details of each step in the proof can be found in [6].

Proposition 5.14. *If for a diffeomorphism φ there exist an arational measured foliation $(\mathcal{F}, \mu)$ and a scalar $\lambda \neq 1$ such that $\varphi(\mathcal{F}, \mu) \sim (\mathcal{F}, \lambda\mu)$, then φ is isotopic to a pseudo-Anosov "diffeomorphism."*

Proof. By the above assumptions, we already have $\mathcal{F}^u$. We will make an isotopy of φ preserving $\mathcal{F}^u$ and construct $\mathcal{F}^s$. The tools of Section 5.1 will not suffice for this task and we will improve them using the assumption $\lambda > 1$.

• Step 1: We construct a good system of transversals τ satisfying the following properties:

(i) In every sector of a singularity, there is an arc of τ.
(ii) $\varphi(\tau) \subset \tau$.
(iii) If $x \in \partial - sing\mathcal{F}^u$, then x belongs to the separatrices of a singularity.

We denote by F_x the arc of the leaf joining x to $sing\mathcal{F}^u$.

(iv) Every separatrix contains an F_x.
(v) $\bigcup F_x \subset \varphi(\bigcup F_x)$.

The construction starts as in the proof of Proposition 5.11. We then modify τ to get the desired additional properties.

• Step 2: Free to make an isotopy of φ preserving $\mathcal{F}^u$, we construct a collection $R_1, ..., R_m$ of rectangles such that

(i) $intR_i \cap intR_j = \emptyset$ for $i \neq j$,
(ii) $\bigcup R_i = M$,
(iii) $\varphi(\bigcup \partial_\tau R_i) \subset \bigcup \partial_\tau R_i$,
(iv) $\varphi^{-1}(\bigcup \partial_\mathcal{F} R_i) \subset \bigcup \partial_\mathcal{F} R_i$,
(v) for each $i = 1, ...m$ and $\varepsilon = 0, 1$, $\varphi(\partial_\tau^\varepsilon R_i)$ (resp. $\varphi^{-1}(\partial_\mathcal{F}^\varepsilon R_i)$) is covered on the side of $\varphi(R_i)$ (resp. $\varphi^{-1}(R_i)$) by a single rectangle.

We call such a collection a "Markov pre-partition" for $(\mathcal{F}^u, \varphi)$. In proving Step 2, we first notice that a collection satisfying (i)-(iv) automatically satisfies (v). For $x \in \partial\tau - sing\mathcal{F}^u$ and L the leaf containing F_x; leaving from the singularity s of L, we consider the first point y in $\tau - F_x$. We denote by F'_x the segment from s to y on L and set $F' = \bigcup F'_x$. Now, using the good system of transversals τ above, we get a system $R'_1, ..., R'_N$ of rectangles by Lemma 5.9. We then construct the desired system of rectangles by taking closures of the connected components of $\bigcup intR'_i - F'$.

• Step 3: We fix a Markov pre-partition $R_1, ..., R_m$ for $(\mathcal{F}^u, \varphi)$. We denote by x_i the μ^u-length of R_i, and by a_{ij} the number of times that $\varphi(intR_i)$ crosses $intR_j$ (i.e. the number of components of $\varphi(intR_i) \cap intR_j$).
We then realize that $\lambda x_j = \sum_i x_i a_{ij}$, i.e. the vector $[x_j]$ is an eigenvector, with eigenvalue λ, for the transpose matrix of $A = [a_{ij}]$.

• Step 4: There exist numbers $\xi > 0$ and $y_1, ..., y_m > 0$ such that $y_i = \xi \sum_j a_{ij} y_j$; i.e. the matrix A admits an eigenvalue $\xi^{-1} > 0$ with an eigenvector whose coordinates are all positive.

• Step 5: We foliate each rectangle R_i by the natural transverse foliation $\mathcal{F}^s$ and endow it with an invariant measure μ^s such that the $\mathcal{F}^s$-width of R_i is the y_i above. We then provide each R_i with a system of coordinates X^i, Y^i with $R_i = \{0 \leq X^i \leq x_i \ , \ 0 \leq Y^i \leq y_i\}$ and thus see that the rectangles $R_1, ..., R_m$ give an atlas for the foliation $(\mathcal{F}^s, \mu^s)$.

•Step 6: $\xi = \dfrac{1}{\lambda}$. To see this, we endow each rectangle with the product measure $\mu = \mu^u \otimes \mu^s$. Then $\varphi_* \mu = \lambda \xi \mu$. But as S is compact, μ is a finite measure. Hence $\lambda \xi = 1$.

• Step 7: We make an isotopy of φ to get the pseudo-Anosov "diffeomorphism" φ'.
We define φ' by the following conditions:

- $\varphi'(R_i) = \varphi(R_i)$.
- $\varphi'(X^i = c) = \varphi(X^i = c)$.
- For V a component of $R_i \cap \varphi^{-1}(R_j)$, we have $\varphi'(V \cap (Y^i = \mathit{constant})) \subset (Y^j = \mathit{constant})$.
- For $p, q \in V$, we have $\xi\ |Y^j(\varphi'(p)) - Y^j(\varphi'(q))| = |Y^i(p) - Y^i(q)|$.

This isotopy clearly preserves $\mathcal{F}^u$. □

Remark 5.15. There exists a measured foliation $(\mathcal{F}, \mu)$ having a cycle of leaves and a diffeomorphism φ not isotopic ot a pseudo-Anosov "diffeomorphism" such that $\varphi(\mathcal{F}, \mu) \sim (\mathcal{F}, \lambda\mu)$ with $\lambda \neq 1$. Therefore, the hypothesis that $\mathcal{F}$ is arational is essential here.

5.5. Classification of Surface Diffeomorphisms

Theorem 5.16. *Let φ be a diffeomorphism of a surface of genus $g > 1$. Up to isotopy, φ is in one of the following situations:*

(i) Isometry for a hyperbolic structure.

(ii) "Reducible" (i.e. φ fixes a system of simple curves which are mutually disjoint and not homotopic to a point.)

(iii) Pseudo-Anosov.

The situations (i)-(iii) and (ii)-(iii) are mutually exclusive.

Proof. Sections 2.5, 5.2, 5.3 and 5.4 combine to give the classification. We only need to prove the exclusions. $\varphi(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda\mu^s)$ with $\lambda \neq 1$ prohibits the isotopy class of φ from being periodic; from which it follows that (i) and (iii) are incompatible. We lastly prove that (ii) and (iii) are mutually exclusive. We proceed by contradiction. We suppose φ is pseudo-Anosov and fixes an element of $\mathcal{S}'$. Replacing φ by one of its powers if necessary, we suppose φ fixes the isotopy class of the curve γ'. We get $I(\mathcal{F}^s, \mu^s; \gamma) = \lambda I(\mathcal{F}^s, \mu^s; \varphi(\gamma)) = \lambda I(\mathcal{F}^s, \mu^s; \gamma)$ so that $I(\mathcal{F}^s, \mu^s; \gamma) = 0$. By Proposition 3.36, $\mathcal{F}^s$ is equivalent to a foliation that has a cycle of leaves. By Remark 5.3, this contradicts to $\mathcal{F}^s$ being arational. □

Remark 5.17. As we stated in Section 1.2, Thurston's theory of surfaces can be generalized to surfaces with boundary by appropriate modifications in the definitions of $\mathcal{T}$ and $\mathcal{S}$. Namely; $\mathcal{T}$ is defined using hyperbolic metrics for which the boundary components become geodesics (See Section 2.1.), and $\mathcal{S}$ is defined as the set of isotopy classes of simple, closed, connected curves in S which are not homotopic to a point or a boundary component of M (compare with Definition 3.1). We omit this generalization in this study and refer the reader to [6]. Yet, in the next chapter, we will utilize Theorem 5.16 for a surface with non-empty boundary. This is the reason why we stated Theorem 5.16 in its most generality.

5.6. Properties of Pseudo-Anosov Diffeomorphisms

We finish this chapter by giving some of the interesting properties of pseudo-Anosov "diffeomorphisms." This section is intended to motivate the reader about why pseudo-Anosov "diffeomorphisms" are important. We will briefly survey the results in [7] and [6], and omit the proofs.

Lemma 5.18. *The stable and unstable foliations of pseudo-Anosov "diffeomorphisms" do not have connections between singularities; they are thus canonical models for the classes of arational foliations.*

Proposition 5.19. *If U is a nonempty open set that is invariant under a pseudo-Anosov "diffeomorphism," then U is dense.*

Corollary 5.20. *A pseudo-Anosov "diffeomorphism" is topologically transitive; i.e. there exists a dense orbit.*

Proposition 5.21. *The periodic points of a pseudo-Anosov "diffeomorphism" are dense.*

Proposition 5.22. *Let g be a Riemannian metric on S, α be an essential curve on S. By $l_g(\alpha)$, we denote the length of the g-geodesic in the class α. Suppose φ is a* *pseudo-Anosov "diffeomorphism" and let λ be defined as before. Then we have*

$$\lim_{n\to\infty} l_g(\varphi^n(\alpha))^{1/n} = \lambda$$

The last two propositions give an insight about the dynamics of pseudo-Anosov "diffeomorphisms." We now relate the measured foliations we discussed previously with geodesic laminations.

By a *measured geodesic lamination*, we mean a compact geodesic lamination Λ, which is endowed with a transverse measure whose support is all of Λ.

Remark 5.23. A pseudo-Anosov "diffeomorphism" φ is equipped with the stable and unstable measured geodesic laminations (Λ^s, μ^s) and (Λ^u, μ^u) such that $\varphi(\Lambda^s, \mu^s) = (\Lambda^s, \lambda\mu^s)$ and $\varphi(\Lambda^u, \mu^u) = (\Lambda^u, \lambda^{-1}\mu^u)$. Let Λ denote Λ^s or Λ^u. We have the following properties:

- Λ is *minimal*; i.e. it does not contain any proper sublaminations.
- Λ does not have closed or isolated leaves.
- Λ is disjoint from the boundary ∂S.
- $S - \Lambda$ is either an open disk or a semi-open annulus containing a component of ∂S. We say Λ is *filling*.
- In particular, every leaf of Λ is dense in Λ.
- For any simple geodesic γ, we have the following dynamical behaviour of a pseudo-Anosov "diffeomorphism" φ:

$$\varphi_*^i(\gamma) \to \Lambda^s \text{ as } i \to \infty,$$
$$\varphi_*^i(\gamma) \to \Lambda^u \text{ as } \ i \to -\infty.$$

6. OPEN BOOKS AND PSEUDO-ANOSOV DIFFEOMORPHISMS

In this chapter, we pursue the second aim of our study and relate pseudo-Anosov "diffeomorphisms" with open books. This chapter mainly relies on [4].

6.1. Basic Definitions

Definition 6.1. An *(abstract) open book* is a pair (S, h), where S is an oriented compact surface with nonempty boundary and $h : S \to S$ is a diffeomorphism which is the identity near ∂S. The diffeomorphism h is called the *monodromy* of the open book.

A *positive stabilization* of (S, h) is defined as follows: Let S' be the surface obtained from S by attaching a 1-handle B along ∂S. Let γ be a simple closed curve in S' which intersects the co-core of B at exactly one point and let $id_B \cup h$ be the extension of h by the identity map to S'. Then, we define $h' = \tau_\gamma \circ (id_B \cup h)$, where τ_γ is a positive Dehn twist about γ. The pair (S', h') is called an *elementary positive stabilization* of (S, h). More generally, we say that (S', h') is a *positive stabilization* of (S, h) if it is obtained from (S, h) by a sequence of elementary positive stabilizations.

6.2. Pseudo-Anosov Diffeomorphisms and Open Books

In this section we will state and prove [Theorem1.1., [4]], which was one of our main objectives in this study. We fix a compact oriented surface S with nonempty boundary such that $g(S) > 1$.

Definition 6.2. The *curve complex* $\mathcal{C}(S)$ of S is defined as follows: k-simplices of $\mathcal{C}(S)$ are $(k+1)$-tuples $\{\gamma_0, \gamma_1, ..., \gamma_k\}$ of distinct essential curves which can be realized disjointly.

We let C_k denote the k-skeleton of $\mathcal{C}(S)$. We turn C_1 into a metric space by letting each edge have length 1, and defining the distance function d by taking shortest paths. In fact, C_1 is a geodesic metric space, i.e. it is a path-connected metric space in which any two points x,y are connected by an isometric image of an interval in $\mathbb{R}$, called a geodesic and denoted $[xy]$.

Remark 6.3. By [17], we know that $\mathcal{C}(S)$ is δ-hyperbolic, i.e. there is $\delta \geq 0$ such that for any geodesic triangle $[xy] \cup [yz] \cup [xz]$ in $\mathcal{C}(S)$, each edge is contained in a δ-neighborhood of the other two.

Remark 6.4. We note that here we slightly changed our notation. The space $\mathcal{S}$ of essential curves is now interpreted as the 0-skeleton C_0 of the curve complex.

Remark 6.5. While we did not study the classification of surface diffeomorphisms for a surface with nonempty boundary, we stated in the beginning of Chapter 2 that all the results can be adapted appropriately. We should now make this statement more explicit. We extend the definition of the geometric intersection number to isotopy classes of arcs that begin and end at the boundary. Here we are free to move the end-points in the isotopy. Moreover, while we stabilize an open book, we change the surface by attaching a 1-handle to it. To distinguish geometric intersection numbers in these surfaces, in the sequel we will use the notation $i_\Sigma(\cdot,\cdot)$ for the geometric intersection number on a surface Σ.

Lemma 6.6. *Let S be a surface of genus g with p punctures such that the cases $g = 0$, $p \leq 4$ and $g = 1$, $p \leq 1$ are excluded.*
(i) If α, $\beta \in C_0(S)$, then $d(\alpha, \beta) \leq 2i(\alpha, \beta) + 1$.
(ii) If $d(\alpha, \beta) \geq 3$, then α and β fill S, i.e. any essential curve must intersect α or β.

Proof. (i): To prove the first claim, we start by assuming that α and β are realized in minimal intersection number.

• Case 1: $i(\alpha, \beta) = 1$. In this case, a regular neighborhood of $\alpha \cup \beta$ is a punctured torus. Since the torus and punctured torus cases are excluded, there is a curve γ not intersecting any of α and β. Hence we have $d(\alpha, \beta) = 2$.

• Case 2: $i(\alpha, \beta) \geq 2$. We take two intersection points of α and β that are adjacent in α. We replace a segment of β with the segment of α between the intersection points (notice that there are two ways to do it). Thus we obtain two closed curves β_1 and β_2 with $i(\alpha, \beta_j) \leq i(\alpha, \beta) - 1$. If the two intersections have the same orientation, then $i(\beta, \beta_j) = 1$; which implies $d(\alpha, \beta) \leq 2 + d(\alpha, \beta_j)$ and we are done by induction. Otherwise if the intersections have opposite signs, we get $i(\alpha, \beta_j) \leq i(\alpha, \beta) - 2$ and $i(\beta, \beta_j) = 0$. Now if at least one β_j is essential, we can apply induction. Finally we suppose that both β_i's are free-homotopic to boundary components. This means that β bounds a twice punctured disk on the side containing the α segment between the intersection points. We continue by considering the segment of α between the intersections; if it is also free-homotopic to a boundary component, then β bounds a 4-times punctured disk, a case we excluded.

(ii): We prove this statement by contraposition. Let γ be a curve that does not intersect α and β. Then α and β can be connected in $C_1(S)$ via a path through γ, which yields $d(\alpha, \beta) \leq 2$. □

Lemma 6.7. *For any $h \in Map(S, \partial S)$ not freely isotopic to a periodic diffeomorphism, there exists $\alpha \in C_0(S)$ such that $d(\alpha, h(\alpha))$ is arbitrarily large.*

Proof. We fix a reference hyperbolic metric on S making the boundary geodesic in order to talk about geodesics. By $\mathcal{EL}(S)$, we denote the space of minimal filling laminations in S, viewed as a subset of the space of measured geodesic laminations.

- Claim1: There exists $\mu \in \mathcal{EL}(S)$ such that $h(\mu) \neq \mu$.

If h is pseudo-Anosov, we simply pick μ minimal which is not the stable lamination or the unstable lamination of h, and we are done. Hence we suppose h is reducible. Taking h^n (n large enough), we get a collection of disjoint, homotopically nontrivial annuli $A_1, ..., A_m$ such that h^n on each component S_j of $S - \bigcup_i A_i$ is either identity or pseudo-Anosov, and $h^n|_{A_i} = \tau_{\gamma_i}^{ni}$, $n_i \in \mathbb{Z}$, with γ_i the core curve of A_i. In this case, any minimal filling lamination μ with ideal 3-gon and once punctured ideal monogon complementary regions would work. We treat two cases separately:

Case(i): There exists a pseudo-Anosov piece S_j. In this case, μ restricted to S_j is a finite union of disjoint non-parallel arc types which cut up S_j into 3-gons and once-punctured monogons. We temporarily denote by μ_{S_j} the union of arcs, one from each type. Then we see that $i(h(\mu)|_{S_j}, \mu|_{S_j}) \neq 0$ and hence $h(\mu) \neq \mu$. In fact, if $i(h(\mu)|_{S_j}, \mu|_{S_j}) = 0$; then $\mu|_{S_j} = h(\mu)|_{S_j}$, since $S_j - \mu|_{S_j}$ consists of 3-gons and once-punctured monogons. This implies that h is periodic on S_j, contradicting to the assumption that h is pseudo-Anosov on S_j.

Case(ii): There are no pseudo-Anosov components. This means that h^n is given as a product of Dehn twists. We let $\overline{S_j}$ denote the union of S_j and all its adjacent annuli A_i. For simplicity, we assume that no A_i bounds same S_j along both boundary components. Let δ be a properly embedded arc of $\overline{S_j}$ from A_i to another $A_{i'}$, and let δ' be a parallel push-off of δ. Now, if n_i and $n_{i'}$ have the same sign, then $h(\delta')$ and δ have minimal intersection; and if n_i and $n_{i'}$ have opposite signs, then $h(\delta')$ and δ can be realized in minimal intersection number after a push-off to one side, and they have two additional intersections for a push-off to the other side. So, we conclude that δ and $h(\delta')$ intersect nontrivially, provided $|n_i| + |n_{i'}| >> 0$. Same argumentation holds for μ. This finishes the proof of Claim 1.

- Claim2: For every $\mu \in \mathcal{EL}(S)$, there exists a sequence of simple closed curves converging to μ.

We pick a point p in μ and a short transversal δ such that $p \in int(\delta)$ and take a neighborhood $N(p)$ of p with the following properties:

- $N(p) \cong [-1, 1] \times [-1, 1]$.
- leaves of $\mu \cap N(p)$ are given by $[-1, 1] \times \{point\}$.
- $\delta = \{0\} \times [-1, 1]$.
- $p = (0, 0)$.

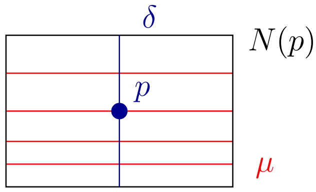


Figure 6.1. Rectangular neighborhood of a point.

Let β' be an arc constructed by starting at p and following along μ until the first return to δ. If μ is orientable, then we can freely homotope β' to obtain a closed curve arbitrarily close to μ and we are done. If μ is not orientable, β' will return to δ at a point p_1 from the "same side". We put $\delta_0 = \delta$ and consider $\delta_1 \subset \delta_0$ containing p and has p_1 as an endpoint. Continuing β' past p_1, we reach p_2 on δ_1. If it comes from the same side again, we take a shorter $\delta_2 \subset \delta_1$. But if all δ_i's in this process are on the same side, this means we can actually orient μ, a contradiction. This finishes the proof of Claim 2.

By Remark 6.3, we can consider the boundary at infinity $\partial_\infty \mathcal{C}(S)$ of the curve complex. By [18]; we have that $\mathcal{EL}(S) \approx \partial_\infty \mathcal{C}(S)$, and $(\beta_i)_{i\in\mathbb{N}} \subset C_0(S)$ converges to $\beta \in \partial_\infty \mathcal{C}(S)$ if and only if it converges to $\beta \in \mathcal{EL}(S)$ in the topology of measured geodesic laminations. Finally, using Claim 1 we take $\mu \in \mathcal{EL}(S)$ such that $h(\mu) \neq \mu$. Using Claim 2, we take a sequence (α_n) of simple closed curves converging to μ in the space of measured geodesic laminations. Then, $h(\alpha_n)$ converges to $h(\mu)$. Since $\mu \neq h(\mu)$ in $\partial_\infty \mathcal{C}(S)$, we must have $d(\alpha_n, h(\alpha_n)) \to \infty$. □

Theorem 6.8. *Let (S, h) be an open book. Then there exists a positive stabilization (S', h') of (S, h) such that $\partial S'$ is connected and h' is freely homotopic to a pseudo-Anosov diffeomorphism.*

Proof. Case 1: ∂S is connected.

We suppose that h is reducible, see Remark 6.9. By Lemma 6.7, we take an essential curve $\gamma_0 \in C_0(S)$ such that $d(h(\gamma_0), \gamma_0) = N \gg 0$. Let γ be a properly embedded arc in S which becomes γ_0 after conjoining with an arc of ∂S which connects the endpoints of γ.

Stabilizing along γ, we get $(S', h' = \tau_{\gamma'} \circ h)$ where γ' is the extension of γ to the 1-handle. We will show that h' is pseudo-Anosov. (We remark that in this construction $\partial S'$ has two components. We will return to this issue in Case 2.)

- Claim1: h' is not reducible; i.e. for any multicurve δ in S', we have $h'(\delta) \neq \delta$.

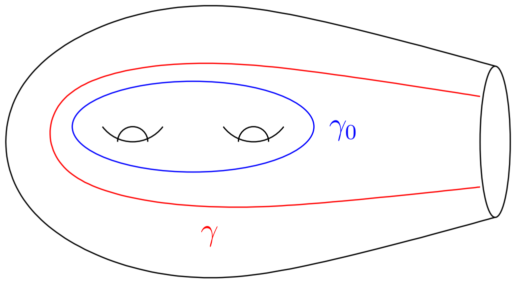


Figure 6.2. Constructing the arc of stabilization.

Proof of Claim 1: Let $\delta \in \mathcal{S}'$. The idea is to distinguish $h'(\delta)$ and δ by intersecting with the co-core a of the 1-handle and with the curve γ'.

Case(i): $\delta \subset S$; i.e. $i_{S'}(\delta, a) = 0$. We put $i_{S'}(h(\delta), \gamma') = m$. We claim that $i_{S'}(\tau_{\gamma'} \circ h(\delta), a) = m$. Indeed; it is easy to see that there is a curve g in the class of $\tau_{\gamma'} \circ h(\delta)$ which intersects a at m points. To see that this intersection is minimal, we suppose there is another representative strictly less than m intersections. This means that there is a bigon consisting of a subarc of a and an arc of g. But then, there is a bigon consisting of an arc of γ and an arc of $h(\delta)$, a contradiction.

Now if $m > 0$; $i_{S'}(\delta, a) = 0$ and $i_{S'}(h'(\delta), a) = m$ yield $h'(\delta) \neq \delta$. We suppose that $m = 0$. Then, for each component α of δ, we have $i_S(h(\alpha), \gamma_0) = 0$ and $d(h(\alpha), \gamma_0) = 1$. Since $d(h(\gamma_0), \gamma_0) = N$, we have $d(h(\alpha), h(\gamma_0)) \approx N$. (Here and in the sequel, we follow the notation of [4] and use $\approx$ to mean "approximately equal to.") Composing with h^{-1}, we get $d(\alpha, \gamma_0) \approx N$. By Lemma 6.6, we obtain $i_S(\alpha, \gamma_0) \gtrapprox \frac{N-1}{2}$. Hence $i_S(\alpha, \gamma) = i_{S'}(\alpha, \gamma') \gtrapprox \frac{N-1}{2}$ for each component α of δ. But $m = 0$ yields $i_{S'}(h'(\delta), \gamma') = 0$. Therefore $h'(\delta) \neq \delta$, as desired. This finishes the study of Case(i).

Case(ii): $\delta \subsetneq S$. We let $k = i_{S'}(\delta, a)$. We will write the 1-handle $B = S' - int(S)$ as $[-1, 1] \times [-1, 1]$ so that $\{\pm 1\} \times [-1, 1] \subseteq \partial S'$, $a = [-1, 1] \times \{0\}$, and $\gamma' \cap B = \{0\} \times [-1, 1]$.

We now describe how to "normalize" δ: We first realize γ', a and δ in minimal intersection numbers; and then subdivide δ into a nonordered collection of arc $\delta_1, ..., \delta_k, \delta'_1, ..., \delta'_k$ where $\delta_i \subset S$ and $\delta'_i \subset B$. Here each δ'_i is a linear arc with endpoints on $[-1,1]\times\{\pm 1\}$. If $\delta'_i \cap (\gamma' \cap B) = \emptyset$, we assume $\delta'_i = \{point\} \times [-1,1]$. We further assume that there is no triangle in S with boundary consisting of a subarc of δ_i, a subarc of γ and a subarc of $[-1,1]\times\{\pm 1\}$. (This we do by pushing such triangles into B.) Similarly, we normalize $h(\delta)$ to get $(h(\delta))_1, ..., (h(\delta))_k, (h(\delta))'_1, ..., (h(\delta))'_k$.

Claim1a: If any δ_i is an arc in S that is free-homotopic to a boundary component of S, then δ must have a component free-homotopic to $\partial S'$.

Proof of Claim1a: Suppose δ_i is an arc in S that is free-homotopic to a boundary component of S.

Case(a): δ_i has endpoints on both components of $[-1,1]\times\{\pm 1\}$. It suffices to observe that if the endpoints of δ_i are not connected by a single δ'_j, then δ will be spiralling towards one component of $\partial S'$, a contradiction.

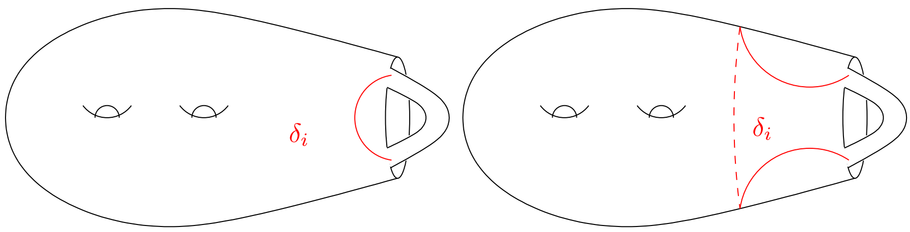


Figure 6.3. δ_i has endpoints on both components of $[-1,1]\times\{\pm 1\}$.

Case(b): δ_i begins and ends on a single component. (Here we should pay attention to the fact that δ_i may not form a bigon together with some arc of $[-1,1]\times\{\pm 1\}$.) In this case, we have that one of δ'_{j_1} and δ'_{j_2} beginning at endpoints of δ_i should continue to spiral around one component of $\partial S'$, a contradiction. This proves Claim1a.

Now, since δ is a multicurve in S', we conclude that no δ_i is free-homotopic to a boundary component of S. We split our discussion into two cases.

Case A: Some $(h(\delta))'_i$ has negative slope. We let $m > 0$ be the number of such $(h(\delta))'_i$'s. The rest will have slope $+\infty$. We put $n = \Sigma_{j=1}^{k} i_S((h(\delta))_j, \gamma)$. We will show that $i_{S'}(h'(\delta), a) = k + m + n$. Since $i_{S'}(\delta, a) = k$, this would show $h'(\delta) \neq \delta$. The effect of $\tau_{\gamma'}$ on $h(\delta)$ is described in Figure 6.4 below.

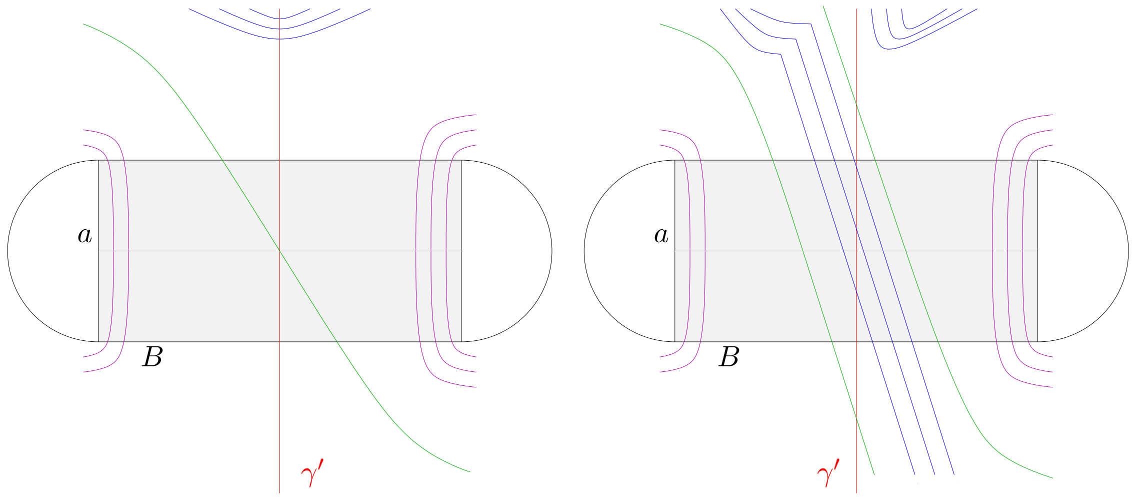


Figure 6.4. The effect of $\tau_{\gamma'}$ on $h(\delta)$ in Case A for $k = 6$, $m = 1$, and $n = 3$.

In Figure 6.4, we already have $k + m + n$ intersection points. It suffices to show the intersection is minimal; i.e. no subarc of $(h'(\delta))_i$ bounds a bigon together with a subarc of a: Consider the surface S'' defined as the union of B with a neighborhood of γ' containing support of $\tau_{\gamma'}$. Then $h'(\delta) \cap S''$ consists of arcs that are not free-homotopic to a boundary component, except possibly the vertical arcs in B. We observe that $h'(\delta)|_{S'-S''} = h(\delta)|_{S'-S''}$. So, it suffices to see that there is component of $(S' - S'') \cap h(\delta)$ which is free-homotopic to a boundary component and cobounds a bigon together with a subarc of $\partial S'' - \partial S'$. Such an arc is impossible by normalization.
Case B: No $(h(\delta))'_i$ has negative slope. We let m to be the number of $(h(\delta))'_i$'s with positive slope, and $n = \Sigma_j i_S((h(\delta))_j, \gamma)$. As above, we can show that $i_{S'}(h'(\delta), a) = k - m + n$ (see Figure 6.5).

If $m \neq n$, we are done. So, we assume $m = n$. Then $i_{S'}(h'(\delta), \gamma') = m+n = 2m \leq 2k$. We will show that $i_{S'}(\delta, \gamma') \neq i_{S'}(h'(\delta), \gamma')$: By way of contradiction, we suppose $i_{S'}(\delta, \gamma') = i_{S'}(h'(\delta), \gamma') \leq 2k$. So, there is some index i such that $i_S(\delta_i, \gamma) \leq \frac{2k}{k} = 2$ since $i = 1, ..., k$. We close up δ_i by conjoining with an arc of ∂S to get $\overline{\delta_i}$. Then $i_S(\overline{\delta_i}, \gamma_0) \leq 3$. By Lemma 6.6, $d(\overline{\delta_i}, \gamma_0) \leq 7$. For simplicity, we will write $d(\overline{\delta_i}, \gamma_0) \approx 0$. As δ_i and δ_j are disjoint, we have $d(\overline{\delta_i}, \overline{\delta_j}) \approx 0$ for all i, j so that $d(\overline{\delta_j}, \gamma_0) \approx 0$ for all j. Since $d(h(\gamma_0), \gamma_0) = N \gg 0$, we conclude that $d(h(\overline{\delta_j}), \gamma_0) \approx N$ for all j. Again by Lemma 6.6, $i_S(h(\delta_j), \gamma) \gtrapprox \frac{N}{2}$ so that $i_{S'}(h'(\delta), \gamma') = i_{S'}(h(\delta), \gamma') \gtrapprox \frac{N}{2}k \gg 2k$, a contradiction. This finishes the proof of Claim 1. h' cannot be reducible.

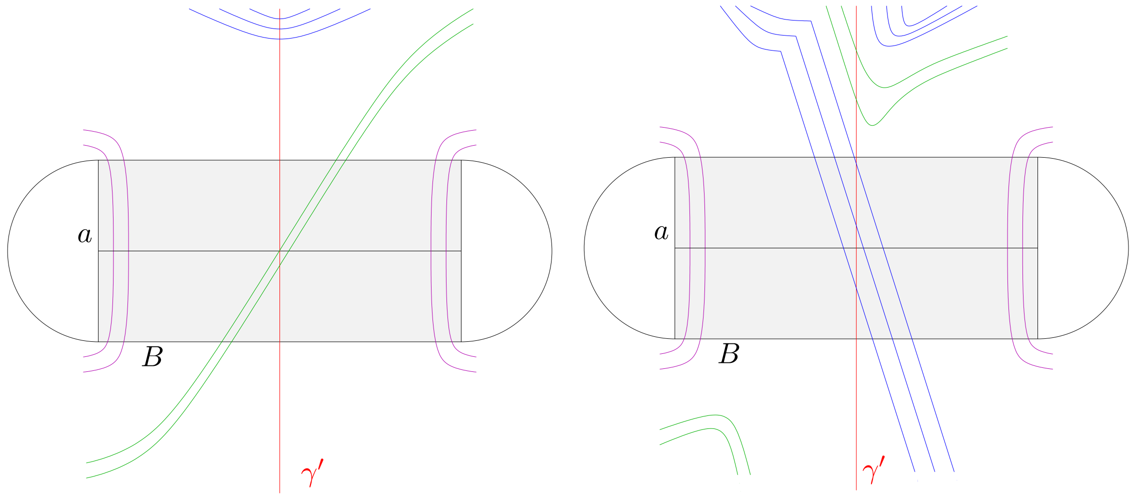


Figure 6.5. The effect of $\tau_{\gamma'}$ on $h(\delta)$ in Case B for $k = 6$, $m = 2$, and $n = 3$.

- Claim2: h' is not periodic; i.e. there is a curve δ such that for all j, we have $(h')^j(\delta) \neq \delta$.

Proof of Claim 2: We take $\delta = \partial S$. We normalize δ as before. The curve $h'(\delta) = \tau_{\gamma'}(\delta)$ has $k_1 = 2$ intersections with a, and $n_1 = 0$ intersections with γ' away from B. Clearly, $h'(\delta) \neq \delta$. We then proceed inductively: Let k_i and n_i be as above for $(h')^i(\delta)$; and suppose $\frac{n_i}{k_i} \ll 1$ and $k_i \to \infty$. Then for each component $((h')^i(\delta))_j$ of $(h')^i \cap S$, we have $d(\overline{((h')^i(\delta))_j}, \gamma_0) \approx 0$ and $d(h(\overline{((h')^i(\delta))_j}), \gamma_0) \approx N$. Hence $(h \circ (h')^i)(\delta)$ has k_i intersections with a, and $\gtrapprox \frac{N}{2}k_i$ intersections with γ' away from B. Similar as above, we see that $(h')^{i+1}(\delta)$ has $\gtrapprox \frac{N}{2}k_i$ intersections with a and at most k_i intersections with γ' away from B. Hence $\frac{n_{i+1}}{k_{i+1}} \ll 1$, $k_{i+1} > k_i$, and $(h')^i(\delta) \neq \delta$ for each i. This completes the proof of Claim 2.

We are finished with Case 1: ∂S is connected.

Case 2: ∂S has two components.

For an arc γ connecting the two components, we construct a closed curve α_γ as follows: We take a pair of pants neighborhood P of $\gamma \cup \partial S$ and let $\alpha = \alpha_\gamma$ be the component of ∂P which is not a subset of ∂S. We note that given a suitable α, we can recover γ satisfying $\alpha = \alpha_\gamma$.

We now describe how to pick γ: Let γ_0 be any arc connecting the two components. Let $\alpha_0 = \alpha_{\gamma_0}$. We set μ to be the stable lamination of a pseudo-Anosov diffeomorphism g, where g is not free-homotopic to h. Then we know that $g^i(\alpha_0) \to \mu$. By Lemma 6.7, there is $\alpha = g^n(\alpha_0)$ ($n \gg 0$) such that $d(\alpha, h(\alpha)) = N \gg 0$. Now, we put γ so that $\alpha_\gamma = \alpha$. We remark that $i_S(\gamma, h(\delta))$ and $i_S(\alpha, h(\alpha))$ are proportional by a factor of 4.

Once γ is picked with $d(\alpha_\gamma, h(\alpha_\gamma)) = N$, the rest is identical to Case 1. The only difference is the definition of $\overline{\delta_i}$. We will have an additional case where endpoints of δ_i are on distinct components of ∂S. In that case we set $\overline{\delta_i} = \alpha_{\delta_i}$. □

Remark 6.9. In fact, in [4], the authors combine their results with [Proposition 6.2, [19]] to conclude that the monodromy h' additionally satisfies the property of being *right-veering*. As we skipped this part of the discussion in this work, we refer the reader to [19] for a comprehensive study of right-veering diffeomorphisms.

The construction in [19] makes the monodromy map reducible so that in the proof of Theorem 6.8 we may assume that the h is reducible.

Remark 6.10. When studying open books, we are mainly motivated by 3-manifolds, where we interpret (S, h) as an open book decomposition of the 3-manifold. In [1], Giroux demonstrated a one-to-one correspondence between isomorphism classes of contact structures on closed 3-manifolds and equivalence classes of pairs (S, h) up to positive stabilization and conjugation. Combining Theorem 6.8 with Remark 6.9; we state the corresponding result in contact geometry without entering into the details.

Corollary 6.11. *On a closed oriented 3-manifold M, every oriented, positive contact structure is carried by an open book whose binding is connected, and whose monodromy is right-veering and free-homotopic to a pseudo-Anosov diffeomorphism.*

7. CONCLUSIONS

We have thus completed the study of the result that an abstract open book can be stabilized in order to make its monodromy isotopic to a pseudo-Anosov homeomorphism. As briefly mentioned in Section 5.6, pseudo-Anosov homeomorphisms are the most interesting ones among those given in Theorem 5.16. In view of Remark 6.10 and Corollary 6.11, we have a connection between pseudo-Anosov homeomorphisms and the contact structures. An inspirational study after this point could be to investigate the effect of the monodromy being pseudo-Anosov on the contact structure.

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