DESIGN OF AN ADAPTIVE OUTPUT FEEDBACK CONTROLLER FOR LTI SYSTEMS WITH SINUSOIDAL DISTURBANCES

by

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ABSTRACT

DESIGN OF AN ADAPTIVE OUTPUT FEEDBACK CONTROLLER FOR LTI SYSTEMS WITH SINUSOIDAL DISTURBANCES

This thesis is a theoretical study which develops an adaptive output feedback controller for LTI system driven by unknown sinusoidal disturbances. Two specific systems are considered. As the first one is the unknown minimum-phase LTI system with known relative degree and system order, the second one is the known LTI system with the presence of a known input/output delay. The controller objective for both systems is to reject the disturbances and make the equilibrium of the closed loop system stable.

In the first problem, the controller design procedure is based on K-filter technique, disturbance parametrization, and adaptive backstepping. It is proven that the equilibrium at the origin is globally uniformly stable and the output signal tracks a given reference signal asymptotically. For an additive unmodelled noise, the robustness of the closed loop system is also discussed. In the second problem, the essence of the control design is composed of disturbance parametrization, state and disturbance observer design. The controller compensates the delays, rejects the disturbances and achieves the exponential stability of the equilibrium of the closed-loop system by estimating the disturbance and state perfectly.

ÖZET

SİNÜZOİDAL BOZUCU ETKİSİNDEKİ DOĞRUSAL VE ZAMANLA DEĞİŞMEYEN SİSTEMLER İÇİN UYARLAMALI VE GERİ BESLEMELİ BİR KONTROLCÜ TASARIMI

Bu tez, bilinmeyen harmonik bozucu etkisindeki doğrusal ve zamanla değişmeyen (DZD) sistemlere uyarlamalı ve geri beslemeli kontrolcü geliştiren teorik bir çalışmadır. Bu çalışmada 2 farklı problem ele alınmıştır. İlki, göreceli ve sistem derecesi bilinen fakat sistem parametreleri bilinmeyen minimum-faz DZD sistemi iken, ikincisi ise girdi/çıktı gecikmesine sahip bilinen bir DZD sistemidir. Her iki sistem için de amaç bozucuların bastırılması ve kapalı döngü sistemin kararlı hale getirilmesidir.

İlk problemde kontrolcü tasarım süreci temel olarak K-filtresi tekniği, bozucu temsili ve uyarlamalı geriadımlama içermektedir. Bu kontrolcüyle denge noktasının global düzgün kararlı olduğu ve çıktı sinyalinin verilen bir referansı asimptotik bir şekilde takip ettiği kanıtlanmıştır. Kapalı döngü sistemin, modellenmemiş ek bir gürültü karşısındaki gürbüzlüğü de tartışılmıştır. İkinci problemde geliştirilen kontrolcünün esasını ise bozucu temsili ile durum ve bozucu gözlemleyicisi tasarımları oluşturmaktadır. Bu kontrolcü, gecikmeleri telafi etmiş ve bozucuları engellemiştir. Ayrıca, durum ve bozucunun tam tahmini başararak kapalı döngü sistemin üssel kararlılığını sağlamıştır.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS . . . iii
ABSTRACT . . . iv
ÖZET . . . v
LIST OF FIGURES . . . viii
LIST OF TABLES . . . ix
LIST OF SYMBOLS . . . x
LIST OF ACRONYMS/ABBREVIATIONS . . . xii
1. INTRODUCTION . . . 1
2. OUTPUT FEEDBACK CONTROL FOR UNKNOWN LTI SYSTEMS DRIVEN BY UNKNOWN PERIODIC DISTURBANCES . . . 4
   2.1. Problem Statement . . . 4
   2.2. Disturbance Observer Design . . . 6
      2.2.1. State Representation . . . 7
      2.2.2. Parametrization of Disturbance . . . 9
   2.3. Adaptive Controller Design . . . 13
   2.4. Stability Analysis . . . 25
   2.5. Numerical Simulations . . . 30
   2.6. Conclusion . . . 32
3. REJECTION OF SINUSOIDAL DISTURBANCES FOR KNOWN LTI SYSTEMS IN THE PRESENCE OF SIMULTANEOUS INPUT-OUTPUT DELAY 34
   3.1. Problem Statement . . . 34
   3.2. Disturbance Representation . . . 36
   3.3. Parametrization of Disturbance . . . 42
   3.4. Observer Based Adaptive Controller Design . . . 45
   3.5. Stability Proof . . . 47
   3.6. Numerical Simulations . . . 51
   3.7. Conclusion . . . 52
4. CONCLUSION . . . 54
5. LIMITATIONS AND FUTURE WORK . . . 55

REFERENCES 57
APPENDIX A: SOLUTIONS OF $\lambda_0(t)$ AND $\lambda_1(x,t)$ 62

LIST OF FIGURES

Figure 2.1. The output, $y(t)$, and reference signal, $y_r(t)$. See Table 2.1 for $\nu(t)$ and $y_r(t)$. 32

Figure 2.2. The output, $y(t)$, and reference signal, $y_r(t)$. See Table 2.1 for $\nu(t)$ and $y_r(t)$. 33

Figure 3.1. Observer based adaptive controller design scheme. 47

Figure 3.2. Performance of the controller for an unstable plant with 0.6 second input and 0.7 second output delay. 52

Figure 3.3. The disturbance and its estimation 53

LIST OF TABLES

Table 2.1. Unknown disturbances $\nu(t)$ and reference signals $y_r(t)$ used in simulations 31

LIST OF SYMBOLS

0_i	$i \times 1$ column zero vector
B_i, e_i	Column vector whose ith element is 1
D_u	Input Delay
D_y	Output Delay
G	Chosen Hurwitz Matrix
g_0	Bias
g_i	Amplitude
h_ν	Output vector of exosystem
I_i	$i \times i$ identity matrix
K	Control gain
q	Number of distinct frequencies
U, u	Input
X, x	State matrix
X_i, x_i	State variable
$\hat{X}$	State estimation
$\tilde{X}$	State estimation error
$\hat{X}_d$	Filter state
$\tilde{X}_d$	Filter error
$\hat{X}_s$	State observer
$\tilde{X}_s$	State observer error
Y, y	Output
y_r	Output reference
α_i	Stabilizing function
∂_t	Time derivative of a function
∂_x	Spatial derivative of a function
θ	Unknown parameter vector
$\mathcal{L}^{-1}$	Inverse Laplace transform

$\bar{\mu}$	Mean of random signal
ν	Sinusoidal disturbance
ρ	Relative degree of plant
$\bar{\sigma}^2$	Variance of random signal
ϕ_i	Phase
χ	Random signal
ω_i	Frequency

LIST OF ACRONYMS/ABBREVIATIONS

LTI	Linear and Time Invariant
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PE	Persistently Exciting

1. INTRODUCTION

The problem of sinusoidal disturbance cancellation is observed in many real-world applications such as active suspension control [1], active noise control [2] and marine vehicles [3]– [4]. One of the techniques to approach this problem in linear systems is the internal model principle [5]– [6]. This principle suggests the disturbance to be written as the output of a known linear dynamic system (the so-called exosystem). In order to compensate for the disturbance, a reduplicated model of this exosystem is added to the feedback loop. Since the disturbance rejection with full-state feedback may not be feasible in practical applications, the studies have focused on developing control methods with output feedback.

The problem of disturbance rejection for linear systems by output feedback may be divided into four categories. The first one is a basic case in which the system parameters and the exosystem are known. This problem is studied in [7], [8]. In the second category, the system parameters are known but the disturbance is the output of an uncertain exosystem. This problem is solved for a stable plant of arbitrary relative degree with one matched sinusoidal disturbance in [9]. Multisinusoidal disturbance compensation is achieved for a system with arbitrary relative degree in [10]. The same problem is studied in [11] with an assumption that the zero dynamics of the plant is hyperbolic, which means that there is no zero on the imaginary axis of the complex plane. The systems in [10] and [11] are not restricted to be stable or minimum phase. Moreover, the problem of rejecting harmonic disturbances acting on the output signal is addressed in [12]. In the third case, which assumes that the system is uncertain whereas the exosystem is known, an adaptive algorithm is introduced for stable systems to reject the unmatched disturbance with known frequencies in [13] and [14].

The fourth, and the most complex case is the one in which the system parameters are unknown and the sinusoidal disturbance is generated by an uncertain exosystem. This is the one that we consider in Chapter 2. For a minimum phase system whose relative degree is higher than unity, an iterative algorithm is suggested in [15] to compensate for a single-frequency sinusoidal disturbance. The case of multiharmonic disturbance and arbitrary relative degree is studied in [16]. Moreover, assuming that the system is minimum phase with known relative degree, a design of an adaptive learning regulator is introduced in [17]. In [18], a biased multi-sinusoidal disturbance compensator is developed for a stable system of an unknown order and an unknown relative degree. The rejection of multisinosidal disturbances is studied in [19] for the systems which have multiple zeros at the origin. Adaptive control of unknown linear systems is dealt with state-derivative feedback in [20]– [21].

Since time delay is a common phenomenon observed in most real-world applications, the researchers have concentrated on developing control methods in which delays arise. Adaptive control design techniques for systems with unknown ODE parameters and input delay are given in [22] and [23]. The problem of adaptive stabilization is solved for the systems with unknown parameters and distributed input delay in [24]. The idea of representing time delay as dynamic of PDE is introduced in [25]. Inspired by [25], an adaptive observer for PDEs is developed in [26] with a backstepping like design technique to compensate a delay.

The cancellation of sinusoidal disturbance for known and unknown LTI systems with input delay is studied in [27], [28] and [29], respectively. The output regulation problem is addressed in [30]– [31] for output-delayed known linear systems. An observer design for output delayed systems with model parameter uncertainty is given in [32]. Moreover, for known linear systems with simultaneous state, input and output delay, disturbance cancellation algorithms are proposed in [33] and [34]. However, the studies [30]– [34] assume that exosystem is known and can be used in the controller. To the best of our knowledge, no attempt has been made to reject the disturbance, which is the output of unknown exosystem, in LTI systems with output delay.

In Chapter 2, we contribute mainly to the fourth case of output-feedback problem with no delay. In [15], the cancellation of a single-frequency disturbance is considered. The researchers in [16] drive the output to zero under matched multiharmonic disturbances, assuming a lower bound for frequencies. The adaptive learning regulator in [17] assumes that the unknown parameters are in a known bounded region. The disturbance compensator in [18] is restricted to stable linear systems. The assumption made in [19] is that the relative degree is unitary and the initial states of the exosystem are in a compact set. In Chapter 2, we consider all of these challenges simultaneously.

In Chapter 2, we consider uncertain and minimum-phase LTI systems driven by unknown and unmatched sinusoidal disturbances. We assume that the followings are known; (i) upper bound of the plant order, (ii) upper bound of the exosystem order, (iii) relative degree of the plant, (iv) the sign of the high frequency gain. We propose an algorithm to reject the disturbances and to make the output track a reference trajectory with the output feedback. We show that the equilibrium at the origin is globally uniformly stable. We also make a minor robustness analysis of the closed loop system with respect to an additive unmodelled noise.

The problem that we consider in Chapter 3 is the combination of disturbance cancellation by output feedback and delay in the measurement. Contrary to [30]–[34], the unknown disturbance in our system is generated by an uncertain exosystem. Our main contribution is to solve this type of a problem by combining two methods. We first use the technique given in [35] to express the disturbance in a parametrized form and then, employ an adaptive observer proposed in [32]. In addition to this, by employing the perfect estimation of the disturbance and the state, we design an adaptive controller that rejects the disturbance and makes the equilibrium of the closed-loop system exponentially stable.

2. OUTPUT FEEDBACK CONTROL FOR UNKNOWN LTI SYSTEMS DRIVEN BY UNKNOWN PERIODIC DISTURBANCES

This chapter considers unknown minimum-phase LTI systems with known relative degree and system order. The main aim is to reject the unknown, unmatched sinusoidal disturbances and make the output track a given trajectory with the output feedback. The essence of the control design is composed of K-filter technique [36], disturbance parametrization, and adaptive backstepping. Firstly, the K-filter approach is employed to redefine the system states. Then, the disturbance information in the output signal is parametrized and the problem is converted to an adaptive control problem. After that, an adaptive output feedback controller is designed using a backstepping approach. It is proven that the equilibrium at the origin is globally uniformly stable and the output signal tracks the reference signal asymptotically. Finally, the effectiveness of the controller is illustrated with a simulation example of a third-order unstable system. The robustness of the closed loop system with respect to an additive unmodelled noise is also discussed.

2.1. Problem Statement

We consider the following single input single output system

$$Y(s) = \frac{B(s)}{A(s)}U(s) + \frac{D(s)}{A(s)}V(s), \tag{2.1}$$

where

$$A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{2.2}$$

$$B(s) = b_m s^m + \ldots + b_1 s + b_0 \tag{2.3}$$

$$D(s) = d_n s^n + \ldots + d_1 s + d_0. \tag{2.4}$$

The representation of the system (2.1) in the observable canonical form is given by

$$\dot{x}(t) = Ax(t) - ay(t) + \begin{bmatrix} 0_{n-m-1} \\ b \end{bmatrix} u(t) + d\nu(t) \tag{2.5}$$

$$y(t) = e_1^T x(t) + d_n \nu(t), \tag{2.6}$$

where

$$A = \begin{bmatrix} 0 & & \\ \vdots & I_{n-1} & \\ 0 & \dots & 0 \end{bmatrix}, a = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}, b = \begin{bmatrix} b_m \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}, d = \begin{bmatrix} d_{n-1} \\ \vdots \\ d_1 \\ d_0 \end{bmatrix}, \tag{2.7}$$

the state $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, the input $u \in \mathbb{R}$, the output $y \in \mathbb{R}$ and the disturbance is given by

$$\nu(t) = g_0 + \sum_{i=1}^{q} g_i \cos(\omega_i t + \phi_i), \tag{2.8}$$

where $g_0,\ g_i,\ \omega_i,\ \phi_i \in \mathbb{R}$. The plant parameters a, b, d and disturbance parameters $g_0, g_i, \omega_i, \phi_i$ for $i = 1, \dots, q$ are unknown. The disturbance $\nu(t)$ is not measured. The only measured signal is $y(t)$.

Remark 2.1. *We consider a general case where all of the system states are affected by the disturbance. However, the term d_i for $i = 0, \dots, n$ can be zero. For a measurement without the disturbance, the term d_n can be selected as zero.*

The unknown disturbance $\nu(t)$ is represented as the output of an exosystem as follows

$$\dot{w}(t) = Sw(t) \tag{2.9}$$

$$\nu(t) = h_\nu^T w(t). \tag{2.10}$$

where the state $w(t) \in \mathbb{R}^{2q+1}$. The matrix S consists of the unknown frequencies of the sinusoidal disturbance, $\nu(t)$. The constant bias term g_0, amplitude g_i and phase ϕ_i determine the unknown initial condition of (2.9). Without loss of generality, one can choose output vector h_ν^T such that (h_ν^T, S) becomes an observable pair.

We have the following assumptions regarding the plant (2.5)–(2.8):

Assumption 2.2. *The plant is minimum phase, i.e., the polynomial* $B(s) = b_m s^m + \cdots + b_1 s + b_0$ *is Hurwitz.*

Assumption 2.3. *The sign of the high-frequency gain (sgn(*b_m*)) is known.*

Assumption 2.4. *The relative degree (*$\rho = n - m$*) of the plant and an upper bound for the plant order (*n*) are known.*

Assumption 2.5. *The reference signal* $y_r(t)$ *and its* ρ *derivatives are known and bounded. Moreover,* $y_r^\rho(t)$ *is piecewise continuous.*

Assumption 2.6. *The maximum number of distinct frequencies of the disturbance,* q*, is known.*

Assumptions 2.2–2.5 are necessary for a traditional model reference adaptive control design. Assumption 2.6 is used in the parametrization of the disturbance.

The aim is to design an adaptive controller which rejects the sinusoidal disturbances, keeps all the signals of the closed-loop system globally bounded, and achieves

$$\lim_{t\to\infty} y(t) - y_r(t) = 0. \tag{2.11}$$

2.2. Disturbance Observer Design

In this section, we employ the Kreisselmeier filters (K-filters), which were initially suggested in [37], to estimate the unmeasurable states of an uncertain system. By employing the K-filter, we redefine system states. Then, we extract the information of the unknown disturbance from the measured signal and parametrize it using the technique given in [35]. That enables us to approach the problem as an adaptive output feedback design problem.

2.2.1. State Representation

We rewrite the system (2.5)–(2.6) in the following form

$$\dot{x}(t) = Ax(t) + F(u(t), y(t))^T \theta_0 + d\nu(t) \tag{2.12}$$

$$y(t) = e_1^T x(t) + d_n \nu(t), \tag{2.13}$$

where

$$F(u(y), y(t))^T = \left[\begin{bmatrix} 0_{(\rho-1)\times(m+1)} \\ I_{m+1} \end{bmatrix} u(t), -y(t) I_n \right] \tag{2.14}$$

$$\theta_0 = \left[b^T, a^T \right]^T . \tag{2.15}$$

We employ the following K-filters [36]:

$$\dot{\eta}(t) = A_0 \eta(t) + e_n y(t) \tag{2.16}$$

$$\dot{\lambda}(t) = A_0 \lambda(t) + e_n u(t) \tag{2.17}$$

$$\xi(t) = -A_0^n \eta(t) \tag{2.18}$$

$$\Xi(t) = -\left[A_0^{n-1} \eta(t), \ldots, A_0 \eta(t), \eta(t) \right] \tag{2.19}$$

$$v_j(t) = A_0^j \lambda(t), \quad j = 0, \ldots, m \tag{2.20}$$

$$\Omega^T(t) = \left[v_m(t), v_{m-1}(t), \ldots, v_0(t), \Xi(t) \right], \tag{2.21}$$

where $\eta \in \mathbb{R}^n, \lambda \in \mathbb{R}^n, \xi \in \mathbb{R}^n, \Xi \in \mathbb{R}^{n\times n}, v_j \in \mathbb{R}^n, \Omega^T \in \mathbb{R}^{n\times(n+m+1)}$. Using (2.16)–(2.21), we obtain the following representation

$$\bar{x}(t) = \xi(t) + \Omega(t)^T\theta_0, \tag{2.22}$$

where $k = \begin{bmatrix} k_1, & \ldots, & k_n \end{bmatrix}^T \in \mathbb{R}^n$ is chosen such that $A_0 = A - ke_1^T$ is Hurwitz.

We take the derivative of (2.18) using (2.16) and take derivative of (2.21) using (2.16), (2.17), (2.19), (2.20). Considering the special structure of A_0 such as $A_0^j e_n = e_{n-j}$ and $A_0^n e_n = -k$, we get the following equations

$$\dot{\xi} = A_0\xi + ky \tag{2.23}$$
$$\dot{\Omega}^T = A_0\Omega^T + F(u,y)^T. \tag{2.24}$$

Defining

$$\bar{\epsilon}(t) = x(t) - \bar{x}(t), \tag{2.25}$$

and taking derivative of (2.25) in view of (2.12), (2.22)–(2.24), we get

$$\dot{\bar{\epsilon}}(t) = A_0\bar{\epsilon}(t) + d\nu(t). \tag{2.26}$$

In (2.22), K-filters (2.16)–(2.21) allow us to write the state estimation $\bar{x}(t)$ as a linear function of system parameters. In (2.26), if there was no disturbance, the state estimation error $\bar{\epsilon}(t)$ would decay exponentially. However, this error is driven by unknown sinusoidal terms. In the next step, we get the disturbance information from the output dynamics.

Noting that $y = x_1 + d_n\nu$, from (2.5), we write

$$\dot{y} = x_2 - ye_1^T a + d_n\dot{\nu} + d_{n-1}\nu. \tag{2.27}$$

Using (2.22) and (2.25), we represent x_2 as follows

$$x_2 = \xi_2 + \Omega_2^T \theta_0 + \bar{\epsilon}_2, \tag{2.28}$$

$$= b_m v_{m,2} + \xi_2 + \left[0, v_{m-1,2}, \ldots, v_{0,2}, \Xi_2\right] \theta_0 + \bar{\epsilon}_2, \tag{2.29}$$

where $\xi_2, \Omega_2, \Xi_2, \bar{\epsilon}_2$ and $v_{i,2}$ denote the second row of $\xi, \Omega, \Xi, \bar{\epsilon}$ and v_i, respectively. Substituting (2.29) into (2.27), we get the output dynamics as follows

$$\dot{y} = \xi_2 + \omega_0^T \theta_0 + \epsilon_2 \tag{2.30}$$

$$= b_m v_{m,2} + \xi_2 + \bar{\omega}_0^T \theta + \epsilon_2, \tag{2.31}$$

where

$$\epsilon_2 = \bar{\epsilon}_2 + d_n \dot{\nu} + d_{n-1} \nu \tag{2.32}$$

$$\omega_0 = \left[v_{m,2}, v_{m-1,2}, \ldots, v_{0,2}, \Xi_2 - y e_1^T\right]^T \tag{2.33}$$

$$\bar{\omega}_0 = \left[0, v_{m-1,2}, \ldots, v_{0,2}, \Xi_2 - y e_1^T\right]^T. \tag{2.34}$$

In the next section, we represent ϵ_2 as the output of the exosystem (2.9) and perform a parametrization.

2.2.2. Parametrization of Disturbance

In this step, the aim is to obtain a parametrized form of the disturbance information, ϵ_2. Recalling (2.8), the solution of (2.26) for the second element is represented as

$$\epsilon_2(t) = \epsilon_{2ss}(t) + \epsilon_{2in}(t), \tag{2.35}$$

where ϵ_{2ss} and ϵ_{2in} denote the steady state and transient responses respectively, and they are given by

$$\begin{aligned}\epsilon_{2ss}(t) =& \Big(|G_{\epsilon_2}(0)| + d_{n-1}\Big)g_0 + \sum_{i=1}^{q}\Big(|G_{\epsilon_2}(j\omega_i)|\cos(\angle G_{\epsilon_2}(j\omega_i)) + d_{n-1}\Big)g_i\cos(\omega_i t + \phi_i)\\ &- \Big(|G_{\epsilon_2}(j\omega_i)|\sin(\angle G_{\epsilon_2}(j\omega_i)) + \omega_i d_n\Big)g_i\sin(\omega_i t + \phi_i) \qquad (2.36)\\ \epsilon_{2in}(t) =& e_2^T\epsilon_{in}(t), \qquad (2.37)\end{aligned}$$

with

$$\begin{aligned}G_{\epsilon_2}(s) =& e_2^T\left(sI - A_0\right)^{-1}d \qquad (2.38)\\ \dot{\epsilon}_{in}(t) =& A_0\epsilon_{in}(t), \quad \epsilon_{in}(0) = \epsilon(0). \qquad (2.39)\end{aligned}$$

Since A_0 is a Hurwitz matrix, the effect of the initial condition decays exponentially. Moreover, $\epsilon_{2ss}(t)$ contains same frequency as $\nu(t)$. Therefore, considering (2.8), (2.9) and (2.36), there exists a vector $h_\epsilon \in \mathbb{R}^{2q+1}$ such that the pair (S, h_ϵ^T) is observable and

$$\epsilon_{2ss}(t) = h_\epsilon^T W(t). \qquad (2.40)$$

Let $G \in \mathbb{R}^{(2q+1)\times(2q+1)}$ be a Hurwitz matrix and let (G, l) be a controllable pair. Since the pair (h_ϵ^T, S) is observable and the spectra of S and G are disjoint, the matrix $M \in \mathbb{R}^{(2q+1)\times(2q+1)}$, which is the solution of the Sylvester equation given by

$$MS - GM = lh_\epsilon^T, \qquad (2.41)$$

is unique and non-singular [38]. The change of coordinates $Z(t) = MW(t)$ transforms the exosystem dynamics (2.9) and (2.40) into the form

$$\begin{aligned}\dot{Z}(t) &= GZ(t) + l\epsilon_{2ss}(t) \qquad (2.42)\\ \epsilon_{2ss}(t) &= h_\epsilon^T M^{-1}Z(t). \qquad (2.43)\end{aligned}$$

The following lemma reveals the observer designed for unmeasured signal $Z(t)$.

Lemma 2.7. *The unmeasured signal $Z(t)$ can be represented as*

$$Z(t) = \Phi(t) + \sum_{i=0}^{n-1} a_i \varphi_i(t) + \sum_{i=0}^{m} b_i \mu_i(t) + \varepsilon(t), \tag{2.44}$$

with

$$\Phi(t) = \Psi(t) + ly(t) \tag{2.45}$$

$$\dot{\Psi}(t) = G\left(\Psi(t) + ly(t)\right) - l\xi_2(t) \tag{2.46}$$

$$\dot{\varphi}_i(t) = G\varphi_i(t) - l\Xi_{2,n-i}(t) \quad i = 0, \dots, n-2 \tag{2.47}$$

$$\dot{\varphi}_{n-1}(t) = G\varphi_{n-1}(t) - l\left(\Xi_{2,1}(t) - y(t)\right) \tag{2.48}$$

$$\dot{\mu}_i(t) = G\mu_i(t) - lv_{i,2}(t) \quad i = 0, \dots, m, \tag{2.49}$$

where $\Xi_{2,n-i}(t)$ denotes the $(n-i)^{th}$ element of vector $\Xi_2(t)$. The dynamics of the estimation error is

$$\dot{\varepsilon}(t) = G\varepsilon(t) - le_2^T \epsilon_{in}(t), \tag{2.50}$$

which decays exponentially.

Proof. Taking derivative of (2.44) in view of (2.30), (2.45)–(2.49), and by recalling (2.37), we get (2.50). Let $X_\epsilon = [\varepsilon, \epsilon_{in}]^T$, derivative of X_ϵ is written as

$$\dot{X}_\epsilon(t) = \begin{bmatrix} G & -le_2^T \\ 0 & A_0 \end{bmatrix} X_\epsilon(t). \tag{2.51}$$

Since G and A_0 are Hurwitz, $X_\epsilon(t)$ vanishes exponentially. □

Using (2.43), we find the following representation

$$\epsilon_{2ss}(t) = \theta_\epsilon^T \Big(\Phi(t) + \sum_{i=0}^{n-1} a_i \varphi_i(t) + \sum_{i=0}^{m} b_i \mu_i(t) + \varepsilon(t) \Big) \tag{2.52}$$

$$= \theta_\epsilon^T \Phi(t) + \sum_{i=0}^{n-1} \theta_{a_i}^T \varphi_i(t) + \sum_{i=0}^{m} \theta_{b_i}^T \mu_i(t) + \theta_\epsilon^T \varepsilon(t), \tag{2.53}$$

where $\theta_\epsilon^T = h_\epsilon^T M^{-1}$, $\theta_{a_i}^T = \theta_\epsilon^T a_i$, $\theta_{b_i}^T = \theta_\epsilon^T b_i$.

We are able to represent the unknown signal $\epsilon_{2ss}(t)$ as the multiplication of unknown vectors with known regressors plus exponentially decaying estimation error. We use this representation in both state estimation and control law designs.

Recalling $\epsilon_2(t) = \epsilon_{2ss}(t) + \epsilon_{2in}(t)$, substituting (2.53) into (2.31) and considering (2.17), (2.20), we redefine the first ρ equations of the system so that we obtain the following system whose states are available,

$$\dot{y} = \xi_2 + \omega^T \theta + \epsilon_{2in} \tag{2.54}$$

$$= b_m v_{m,2} + \xi_2 + \bar{\omega}^T \theta + \epsilon_{2in} + \theta_\epsilon^T \varepsilon \tag{2.55}$$

$$\dot{v}_{m,2} = v_{m,3} - k_2 v_{m,1} \tag{2.56}$$

$$\vdots$$

$$\dot{v}_{m,\rho} = v_{m,\rho+1} - k_\rho v_{m,1} + u, \tag{2.57}$$

where

$$\omega^T = \left[v_{m,2}, v_{m-1,2}, \ldots, v_{0,2}, \Xi_2 - y e_1^T, \Phi^T, \varphi, \mu \right] \tag{2.58}$$

$$\bar{\omega}^T = \left[0, v_{m-1,2}, \ldots, v_{0,2}, \Xi_2 - y e_1^T, \Phi^T, \varphi, \mu \right] \tag{2.59}$$

$$\theta = \left[b^T, a^T, \theta_\epsilon^T, \theta_a, \theta_b \right]^T, \tag{2.60}$$

with

$$\varphi = \left[\varphi_0^T, \ldots, \varphi_{n-1}^T\right] \in \mathbb{R}^{1\times n(2q+1)} \tag{2.61}$$

$$\mu = \left[\mu_0^T, \ldots, \mu_m^T\right] \in \mathbb{R}^{1\times (m+1)(2q+1)} \tag{2.62}$$

$$\theta_a = \left[\theta_{a_0}, \ldots, \theta_{a_{n-1}}\right] \in \mathbb{R}^{1\times n(2q+1)} \tag{2.63}$$

$$\theta_b = [\theta_{b_0}, \ldots, \theta_{b_m}] \in \mathbb{R}^{1\times (m+1)(2q+1)}. \tag{2.64}$$

With (2.54)–(2.64), we prepare a basis for a backstepping design. Our output dynamics has a similar structure to [36] with a difference that we expand $\omega, \bar{\omega}$ and θ terms with the parametrized disturbance.

2.3. Adaptive Controller Design

In this section, we apply a backstepping technique to design an adaptive controller. We first define new error terms to get virtual control input for backstepping, i.e. stabilizing functions. Then, performing a recursive design, we propose an actual control input with the update laws, which achieves asymptotic tracking of $y_r(t)$ by $y(t)$.

We employ the following coordinate transformation

$$z_1 = y - y_r \tag{2.65}$$

$$z_i = v_{m,i} - \hat{\varrho} y_r^{(i-1)} - \alpha_{i-1} \quad i = 2, \ldots, \rho, \tag{2.66}$$

where $\hat{\varrho}$ is an estimate of $\varrho = 1/b_m$, $y_r^{(i-1)}$ denotes the $(i-1)$ derivatives of y_r. We design the stabilizing functions α_i for $i = 1, \ldots, \rho$ such that it would achieve the control objective if x_{i+1} was available as a control input. The recursive design procedure is given as follows.

Step 1 We first take the time derivative of z_1 as follows

$$\dot{z}_1 = b_m v_{m,2} + \xi_2 + \bar{\omega}^T\theta + \epsilon_{2in} + \theta_\epsilon^T\varepsilon - \dot{y}_r \tag{2.67}$$

$$= b_m\alpha_1 + \xi_2 + \bar{\omega}^T\theta + \epsilon_{2in} + \theta_\epsilon^T\varepsilon - b_m\tilde{\varrho}\dot{y}_r + b_m z_2. \tag{2.68}$$

We choose the first stabilizing function as

$$\alpha_1 = \hat{\varrho}\bar{\alpha}_1, \tag{2.69}$$

where

$$\bar{\alpha}_1 = -c_1 z_1 - h_1 z_1 - f_1 z_1 - \xi_2 - \bar{\omega}^T\hat{\theta} \tag{2.70}$$

for $c_1, h_1, f_1 > 0$. Substituting (2.69) and (2.70) into (2.68),we get

$$\dot{z}_1 = -c_1 z_1 - h_1 z_1 - f_1 z_1 + \epsilon_{2in} + \theta_\epsilon^T\varepsilon + \bar{\omega}^T\tilde{\theta} - b_m(\dot{y}_r + \bar{\alpha}_1)\tilde{\varrho} + b_m z_2. \tag{2.71}$$

By manipulating the following terms with (2.59), (2.66) and (2.69) as follows

$$\bar{\omega}^T\hat{\theta} + b_m z_2 = (\omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1)^T\hat{\theta} + \hat{b}_m z_2, \tag{2.72}$$

we rewrite (2.71) as

$$\dot{z}_1 = -c_1 z_1 - h_1 z_1 - f_1 z_1 + \epsilon_{2in} + \theta_\epsilon^T\varepsilon + (\omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1)^T\tilde{\theta} - b_m(\dot{y}_r + \bar{\alpha}_1)\tilde{\varrho} + \hat{b}_m z_2. \tag{2.73}$$

We consider the following Lyapunov function

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{|b_m|}{2\gamma_\varrho}\tilde{\varrho}^2 + \frac{1}{4h_1}\epsilon_{in}^T P_{in}\epsilon_{in} + \frac{1}{2}\epsilon_{in}^T P_{f_1}\epsilon_{in} + \frac{1}{4f_1}\varepsilon^T P_\nu\varepsilon, \tag{2.74}$$

where the positive definite matrices P_{in}, P_{f_1} and P_ν are solutions of the matrix equations

$$A_0^T P_{in} + P_{in} A_0 = -2I_n \tag{2.75}$$

$$A_0^T P_{f_1} + P_{f_1} A_0 = -\frac{1}{2f_1}\lambda_{max}\left(e_2 l^T P_\nu^T P_\nu l e_2^T\right) I_n \tag{2.76}$$

$$G^T P_\nu + P_\nu G = -\bar{q} I_{2q+1}, \tag{2.77}$$

with

$$\bar{q} > \theta_\epsilon^T \theta_\epsilon + 2. \tag{2.78}$$

The derivative of (2.74) with (2.75)–(2.77) is given by

$$\begin{aligned} \dot{V}_1 \leq & z_1\Big(- c_1 z_1 - h_1 z_1 - f_1 z_1 + \epsilon_{2in} + \theta_\epsilon^T \varepsilon + (\omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1)^T \tilde{\theta} - b_m(\dot{y}_r + \bar{\alpha}_1)\tilde{\varrho} \\ & + \hat{b}_m z_2\Big) - \tilde{\theta}^T \Gamma^{-1}\dot{\hat{\theta}} - \frac{|b_m|}{\gamma_\varrho}\tilde{\varrho}\dot{\hat{\varrho}} - \frac{2}{4h_1}\epsilon_{in}^T \epsilon_{in} - \frac{1}{4f_1}\epsilon_{in}^T\epsilon_{in}\lambda_{max}\left(e_2 l^T P_\nu^T P_\nu l e_2^T\right) \\ & - \frac{\bar{q}}{4f_1}\varepsilon^T\varepsilon + \frac{1}{4f_1}(-2\varepsilon^T P_\nu l e_2^T \epsilon_{in}). \end{aligned} \tag{2.79}$$

By applying Young's inequality for the cross term, we get

$$\begin{aligned} \dot{V}_1 \leq & - c_1 z_1^2 + \hat{b}_m z_1 z_2 - |b_m|\tilde{\varrho}\frac{1}{\gamma_\varrho}\left(\gamma_\varrho sgn(b_m)(\dot{y}_r + \bar{\alpha}_1)z_1 + \dot{\hat{\varrho}}\right) + \tilde{\theta}^T\Gamma^{-1}(\Gamma(\omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1) \\ & \times e_1)z_1 - \dot{\hat{\theta}}) - h_1 z_1^2 - f_1 z_1^2 + z_1\epsilon_{2in} - \frac{2}{4h_1}\epsilon_{in}^T\epsilon_{in} + z_1\theta_\epsilon^T\varepsilon - \frac{\bar{q}}{4f_1}\varepsilon^T\varepsilon + \frac{1}{4f_1}\varepsilon^T\varepsilon. \end{aligned} \tag{2.80}$$

We choose

$$\dot{\hat{\varrho}} = -\gamma_\varrho sgn(b_m)(\dot{y}_r + \bar{\alpha}_1)z_1, \quad \gamma_\varrho > 0 \tag{2.81}$$

and

$$\tau_1 = (\omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1)z_1. \tag{2.82}$$

Substituting (2.81) and (2.82) into (2.80), we get

$$\dot{V}_1 \leq -c_1 z_1^2 + \hat{b}_m z_1 z_2 + \tilde{\theta}^T(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}}) - h_1\left(z_1 - \frac{1}{2h_1}\epsilon_{2in}\right)^2 - \frac{1}{4h_1}\Big(\epsilon_{1in}^2 + \epsilon_{3in}^2 + \dots + \epsilon_{nin}^2\Big) - f_1\left(z_1 - \frac{1}{2f_1}\theta_\epsilon^T\varepsilon\right)^2 - \frac{1}{4h_1}\epsilon_{in}^T\epsilon_{in} + \frac{1}{4f_1}(\theta_\epsilon^T\varepsilon)^2 - \frac{\bar{q}}{4f_1}\varepsilon^T\varepsilon + \frac{1}{4f_1}\varepsilon^T\varepsilon. \tag{2.83}$$

Choosing $\bar{q}$ as (2.78), we get

$$\dot{V}_1 \leq -c_1 z_1^2 + \hat{b}_m z_1 z_2 + \tilde{\theta}^T(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}}) - \frac{1}{4h_1}\epsilon_{in}^T\epsilon_{in} - \frac{1}{4f_1}\varepsilon^T\varepsilon. \tag{2.84}$$

Through the following algebraic expression

$$v_j = A_0^j\lambda. \tag{2.85}$$

We write $v_{i,j}$ as follows

$$v_{i,j} = [*, \dots, *, 1]\,\bar{\lambda}_{i+j} \tag{2.86}$$

where $\bar{\lambda}_{i+j} = \left[\lambda_1, \dots, \lambda_{i+j}\right]^T$. For the sake of simplicity, we put $*$ to denote the elements of the matrix consisting of k terms.

Step 2 Taking the time derivative of (2.66) for $i = 2$, we get

$$\dot{z}_2 = \dot{v}_{m,2} - \hat{\varrho}\ddot{y}_r - \dot{\hat{\varrho}}\dot{y}_r - \dot{\alpha}_1(y, \eta, \hat{\theta}, \hat{\varrho}, \bar{\lambda}_{m+1}, y_r, \Phi, \varphi, \mu) \tag{2.87}$$

$$= v_{m,3} - k_2 v_{m,1} - \hat{\varrho}\ddot{y}_r - \dot{\hat{\varrho}}\dot{y}_r - \frac{\partial\alpha_1}{\partial y}(\xi_2 + \omega^T\theta + \epsilon_{2in} + \theta_\epsilon^T\varepsilon) - \frac{\partial\alpha_1}{\partial\eta}(A_0\eta + e_n y) - \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial\alpha_1}{\partial\hat{\varrho}}\dot{\hat{\varrho}} - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} - \sum_{j=1}^{m+1}\frac{\partial\alpha_1}{\partial\lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) - \frac{\partial\alpha_1}{\partial\Psi}\Big(G\Phi - l\xi_2\Big) - \sum_{i=0}^{n-1}\frac{\partial\alpha_1}{\partial\varphi_i}(G\varphi_i - l\Xi_{2,n-i}) - \sum_{i=0}^{m}\frac{\partial\alpha_1}{\partial\mu_i}(G\mu_i - l v_{i,2}) \tag{2.88}$$

$$= v_{m,3} - \beta_2 - \hat{\varrho}\ddot{y}_r - \frac{\partial\alpha_1}{\partial y}(\omega^T\tilde{\theta} + \epsilon_{2in} + \theta_\epsilon^T\varepsilon) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}, \tag{2.89}$$

where

$$\beta_2 = k_2 v_{m,1} + \frac{\partial \alpha_1}{\partial y}(\xi_2 + \omega^T \hat{\theta}) + \frac{\partial \alpha_1}{\partial \eta}(A_0 \eta + e_n y) + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \sum_{j=1}^{m+1} \frac{\partial \alpha_1}{\partial \lambda_j}(-k_j \lambda_1 + \lambda_{j+1})$$
$$+ \big(\dot{y}_r + \frac{\partial \alpha_1}{\partial \hat{\varrho}}\big)\dot{\hat{\varrho}} + \frac{\partial \alpha_1}{\partial \Psi}\Big(G\Phi - l\xi_2\Big) + \sum_{i=0}^{n-2} \frac{\partial \alpha_1}{\partial \varphi_i}(G\varphi_i - l\Xi_{2,n-i})$$
$$+ \frac{\partial \alpha_1}{\partial \varphi_{n-1}}\big(G\varphi_{n-1} - l(\Xi_{2,1} - y)\big) + \sum_{i=0}^{m} \frac{\partial \alpha_1}{\partial \mu_i}(G\mu_i - l v_{i,2}). \tag{2.90}$$

Noting from (2.66) that $v_{m,3} - \hat{\varrho}\ddot{y}_r = z_3 + \alpha_2$, we get

$$\dot{z}_2 = \alpha_2 - \beta_2 - \frac{\partial \alpha_1}{\partial y}(\omega^T \tilde{\theta} + \epsilon_{2in} + \theta_\epsilon^T \varepsilon) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} + z_3. \tag{2.91}$$

We consider the following Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{4h_2}\epsilon_{in}^T P_{in}\epsilon_{in} + \frac{1}{2}\epsilon_{in}^T P_{f_2}\epsilon_{in} + \frac{1}{4f_2}\varepsilon^T P_\nu \varepsilon, \tag{2.92}$$

where the positive definite matrix P_{f_2} is the solution of the matrix equation

$$A_0^T P_{f_2} + P_{f_2} A_0 = -\frac{1}{2f_2}\lambda_{max}\left(e_2 l^T P_\nu^T P_\nu l e_2^T\right) I_n. \tag{2.93}$$

The time derivative of (2.92) is given by

$$\dot{V}_2 \leq -c_1 z_1^2 + \hat{b}_m z_1 z_2 + \tilde{\theta}^T(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}}) + z_2\Big(\alpha_2 - \beta_2 - \frac{\partial \alpha_1}{\partial y}(\omega^T \tilde{\theta} + \epsilon_{2in} + \theta_\epsilon^T \varepsilon) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}$$
$$+ z_3\Big) - \frac{2}{4h_2}\epsilon_{in}^T \epsilon_{in} - \frac{1}{4f_2}\epsilon_{in}^T \epsilon_{in}\lambda_{max}\left(e_2 l^T P_\nu^T P_\nu l e_2^T\right) - \frac{\bar{q}}{4f_2}\varepsilon^T \varepsilon$$
$$+ \frac{1}{4f_2}(-2\varepsilon^T P_\nu l e_2^T \epsilon_{in})$$
$$\leq -c_1 z_1^2 + z_2 z_3 + \tilde{\theta}^T\Big(\tau_1 - \frac{\partial \alpha_1}{\partial y}\omega z_2 - \Gamma^{-1}\dot{\hat{\theta}}\Big) + z_2(\alpha_2 + \hat{b}_m z_1 - \beta_2 - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}})$$
$$- z_2\frac{\partial \alpha_1}{\partial y}\epsilon_{2in} - z_2\frac{\partial \alpha_1}{\partial y}\theta_\epsilon^T \varepsilon - \frac{2}{4h_2}\epsilon_{in}^T \epsilon_{in} - \frac{1}{4f_2}\epsilon_{in}^T \epsilon_{in}\lambda_{max}\left(e_2 l^T P_\nu^T P_\nu l e_2^T\right) - \frac{\bar{q}}{4f_2}\varepsilon^T \varepsilon$$
$$+ \frac{1}{4f_2}(-2\varepsilon^T P_\nu l e_2^T \epsilon_{in})$$

$$
\begin{aligned}
\le & -c_1 z_1^2 + z_2 z_3 + \tilde{\theta}^T\Big(\tau_1 - \frac{\partial \alpha_1}{\partial y}\omega z_2 - \Gamma^{-1}\dot{\hat{\theta}}\Big) + z_2(\alpha_2 + \hat{b}_m z_1 - \beta_2 - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}) \\
& + h_2(\frac{\partial \alpha_1}{\partial y})^2 z_2^2 - h_2(z_2\frac{\partial \alpha_1}{\partial y} + \frac{1}{2h_2}\epsilon_{2in})^2 - \frac{1}{4f_2}\epsilon_{in}^T\epsilon_{in}\lambda_{max}\Big(e_2 l^T P_\nu^T P_\nu l e_2^T\Big) \\
& - \frac{\bar{q}}{4f_2}\varepsilon^T\varepsilon - \frac{1}{4h_2}\epsilon_{in}^T\epsilon_{in} + \frac{1}{4f_2}(-2\varepsilon^T P_\nu l e_2^T\epsilon_{in}) + f_2(\frac{\partial \alpha_1}{\partial y})^2 z_2^2 - f_2(z_2\frac{\partial \alpha_1}{\partial y} + \frac{1}{2f_2}\theta_\epsilon^T\varepsilon)^2 \\
& + \frac{1}{4f_2}(\theta_\epsilon^T\varepsilon)^2. \qquad (2.94)
\end{aligned}
$$

We choose

$$
\begin{aligned}
\tau_2 =& \tau_1 - \frac{\partial \alpha_1}{\partial y}\omega z_2 \qquad (2.95) \\
\alpha_2 =& -c_2 z_2 - h_2(\frac{\partial \alpha_1}{\partial y})^2 z_2 - f_2(\frac{\partial \alpha_1}{\partial y})^2 z_2 - \hat{b}_m z_1 + \beta_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\tau_2. \qquad (2.96)
\end{aligned}
$$

Substituting (2.95) and (2.96) into (2.94), we get

$$
\dot{V}_2 \le -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \tilde{\theta}^T(\tau_2 - \Gamma^{-1}\dot{\hat{\theta}}) + z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma\tau_2 - \dot{\hat{\theta}}) - \frac{1}{4h_2}\epsilon_{in}^T\epsilon_{in} - \frac{1}{4f_2}\varepsilon^T\varepsilon. \qquad (2.97)
$$

Step 3 Taking the time derivative of (2.66) for $i = 3$, we get

$$
\begin{aligned}
\dot{z}_3 =& \dot{v}_{m,3} - \hat{\varrho} y_r^{(3)} - \dot{\hat{\varrho}}\ddot{y}_r - \dot{\alpha}_2(y, \eta, \hat{\theta}, \hat{\varrho}, \bar{\lambda}_{m+1}, y_r, \Phi, \varphi, \mu) \qquad (2.98) \\
=& v_{m,4} - \beta_3 - \hat{\varrho} y_r^{(3)} - \frac{\partial \alpha_2}{\partial y}(\omega^T\tilde{\theta} + \epsilon_{2in} + \theta_\epsilon^T\varepsilon) - \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}}, \qquad (2.99)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_3 =& k_3 v_{m,1} + \frac{\partial \alpha_2}{\partial y}(\xi_2 + \omega^T\hat{\theta}) + \frac{\partial \alpha_2}{\partial \eta}(A_0\eta + e_n y) + \frac{\partial \alpha_2}{\partial y_r}\dot{y}_r + \frac{\partial \alpha_2}{\partial \dot{y}_r}\ddot{y}_r + \sum_{j=1}^{m+2}\frac{\partial \alpha_2}{\partial \lambda_j}(-k_j\lambda_1 \\
& + \lambda_{j+1}) + (\ddot{y}_r + \frac{\partial \alpha_2}{\partial \hat{\varrho}})\dot{\hat{\varrho}} + \frac{\partial \alpha_2}{\partial \Psi}\Big(G\Phi - l\xi_2\Big) + \sum_{i=0}^{n-2}\frac{\partial \alpha_2}{\partial \varphi_i}(G\varphi_i - l\Xi_{2,n-i}) \\
& + \frac{\partial \alpha_2}{\partial \varphi_{n-1}}(G\varphi_{n-1} - l(\Xi_{2,1} - y)) + \sum_{i=0}^{m}\frac{\partial \alpha_2}{\partial \mu_i}(G\mu_i - l v_{i,2}). \qquad (2.100)
\end{aligned}
$$

Noting from (2.66) that $v_{m,4} - \hat{\varrho} y_r^{(3)} = z_4 + \alpha_3$, we get

$$\dot{z}_3 = \alpha_3 - \beta_3 - \frac{\partial \alpha_2}{\partial y}(\omega^T \tilde{\theta} + \epsilon_{2in} + \theta_\epsilon^T \varepsilon) - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} + z_4. \tag{2.101}$$

We consider the following Lyapunov function

$$V_3 = V_2 + \frac{1}{2} z_3^2 + \frac{1}{4h_3} \epsilon_{in}^T P_{in} \epsilon_{in} + \frac{1}{2} \epsilon_{in}^T P_{f_3} \epsilon_{in} + \frac{1}{4f_3} \varepsilon^T P_\nu \varepsilon, \tag{2.102}$$

where the positive definite matrix P_{f_3} is the solution of the matrix equation

$$A_0^T P_{f_3} + P_{f_3} A_0 = -\frac{1}{2f_3} \lambda_{max} \left(e_2 l^T P_\nu^T P_\nu l e_2^T \right) I_n. \tag{2.103}$$

The time derivative of (2.102) is given by

$$\begin{aligned} \dot{V}_3 \leq & - c_1 z_1^2 - c_2 z_2^2 + z_3 z_4 + \tilde{\theta}^T (\tau_2 - \frac{\partial \alpha_2}{\partial y} \omega z_3 - \Gamma^{-1} \dot{\hat{\theta}}) + z_3 \Big(\alpha_3 + z_2 - \beta_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} \Big) \\ & - z_3 \frac{\partial \alpha_2}{\partial y} \epsilon_{2in} - z_3 \frac{\partial \alpha_2}{\partial y} \theta_\epsilon^T \varepsilon + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) - \frac{2}{4h_3} \epsilon_{in}^T \epsilon_{in} - \frac{1}{4f_3} \epsilon_{in}^T \epsilon_{in} \\ & \times \lambda_{max} \left(e_2 l^T P_\nu^T P_\nu l e_2^T \right) - \frac{\bar{q}}{4f_3} \varepsilon^T \varepsilon + \frac{1}{4f_3} (-2 \varepsilon^T P_\nu l e_2^T \epsilon_{in}). \end{aligned} \tag{2.104}$$

By choosing

$$\tau_3 = \tau_2 - \frac{\partial \alpha_2}{\partial y} \omega z_3 \tag{2.105}$$

and noting that

$$\Gamma \tau_2 - \dot{\hat{\theta}} = \Gamma \frac{\partial \alpha_2}{\partial y} \omega z_3 + (\Gamma \tau_3 - \dot{\hat{\theta}}), \tag{2.106}$$

we can rewrite (2.104) as follows

$$\begin{aligned}\dot{V}_3 \leq & -c_1 z_1^2 - c_2 z_2^2 + z_3 z_4 + \tilde{\theta}^T(\tau_3 - \Gamma^{-1}\dot{\hat{\theta}}) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma\tau_3 - \dot{\hat{\theta}}) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_2}{\partial y}\omega z_3 \\ & + z_3\left(\alpha_3 + z_2 - \beta_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}}\right) + h_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2 z_3^2 - h_3\left(z_3\frac{\partial \alpha_2}{\partial y} + \frac{1}{2h_3}\epsilon_{2in}\right)^2 \\ & - \frac{1}{4h_3}\epsilon_{in}^T\epsilon_{in} - \frac{1}{4f_3}\epsilon_{in}^T\epsilon_{in}\lambda_{max}\left(e_2 l^T P_\nu^T P_\nu l e_2^T\right) - \frac{\bar{q}}{4f_3}\varepsilon^T\varepsilon + \frac{1}{4f_3}\left(-2\varepsilon^T P_\nu l e_2^T\epsilon_{in}\right) \\ & + f_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2 z_3^2 - f_3\left(z_3\frac{\partial \alpha_2}{\partial y} + \frac{1}{2f_3}\theta_\epsilon^T\varepsilon\right)^2 + \frac{1}{4f_3}(\theta_\epsilon^T\varepsilon)^2. \end{aligned} \tag{2.107}$$

Substitution of the following stabilizing function

$$\alpha_3 = -c_3 z_3 - h_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2 z_3 - f_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2 z_3 - z_2 + \beta_3 + \frac{\partial \alpha_2}{\partial \hat{\theta}}\Gamma\tau_3 - z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_2}{\partial y}\omega \tag{2.108}$$

into (2.107) yields

$$\begin{aligned}\dot{V}_3 \leq & -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + \tilde{\theta}^T(\tau_3 - \Gamma^{-1}\dot{\hat{\theta}}) + (z_2\frac{\partial \alpha_1}{\partial \hat{\theta}} + z_3\frac{\partial \alpha_2}{\partial \hat{\theta}})(\Gamma\tau_3 - \dot{\hat{\theta}}) \\ & - \frac{1}{4h_3}\epsilon_{in}^T\epsilon_{in} - \frac{1}{4f_3}\varepsilon^T\varepsilon. \end{aligned} \tag{2.109}$$

Step $\mathbf{i} = \mathbf{4}, \ldots, \rho$ Taking the time derivative of (2.66) for any i bigger than 3, we get

$$\begin{aligned}\dot{z}_i = & \dot{v}_{m,i} - \hat{\varrho}y_r^{(i)} - \dot{\hat{\varrho}}y_r^{(i-1)} - \dot{\alpha}_{i-1}\left(y, \eta, \hat{\theta}, \hat{\varrho}, \bar{\lambda}_{m+i-1}, \bar{y}_r^{(i-2)}, \Phi, \varphi, \mu\right) \\ = & v_{m,i+1} - \beta_i - \hat{\varrho}y_r^{(i)} - \frac{\partial \alpha_{i-1}}{\partial y}(\omega^T\tilde{\theta} + \epsilon_{2in} + \theta_\epsilon^T\varepsilon) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}, \end{aligned} \tag{2.110}$$

where

$$\begin{aligned}\beta_i = & k_i v_{m,1} + \frac{\partial \alpha_{i-1}}{\partial y}(\xi_2 + \omega^T\hat{\theta}) + \frac{\partial \alpha_{i-1}}{\partial \eta}(A_0\eta + e_n y) + \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial y_r^{(j-1)}}y_r^{(j)} \\ & + \sum_{j=1}^{m+i-1}\frac{\partial \alpha_{i-1}}{\partial \lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) + (y_r^{(i-1)} + \frac{\partial \alpha_{i-1}}{\partial \hat{\varrho}})\dot{\hat{\varrho}} + \frac{\partial \alpha_{i-1}}{\partial \Psi}\left(G\Phi - l\xi_2\right)\end{aligned}$$

$$+\sum_{i=0}^{n-2}\frac{\partial\alpha_{i-1}}{\partial\varphi_i}(G\varphi_i - l\Xi_{2,n-i}) + \frac{\partial\alpha_{i-1}}{\partial\varphi_{n-1}}\big(G\varphi_{n-1} - l(\Xi_{2,1} - y)\big) + \sum_{i=0}^{m}\frac{\partial\alpha_{i-1}}{\partial\mu_i}(G\mu_i - lv_{i,2}) \tag{2.111}$$

and $\bar{y}_r^{(i-2)} = \left[y_r, \ldots, y_r^{(i-2)}\right]$. Noting from (2.66) that $v_{m,i+1} - \hat{\varrho}y_r^{(i)} = z_{i+1} + \alpha_i$, we get

$$\dot{z}_i = \alpha_i - \beta_i - \frac{\partial\alpha_{i-1}}{\partial y}(\omega^T\tilde{\theta} + \epsilon_{2in} + \theta_\epsilon^T\varepsilon) + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} + z_{i+1}. \tag{2.112}$$

At the final step $i = \rho$, we get

$$\alpha_\rho = u + v_{m,\rho+1} - \hat{\rho}y_r^{(\rho)}, \quad z_{\rho+1} = 0. \tag{2.113}$$

We consider the following Lyapunov function

$$\begin{aligned} V_i =& V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{4h_i}\epsilon_{in}^T P_{in}\epsilon_{in} + \frac{1}{2}\epsilon_{in}^T P_{f_i}\epsilon_{in} + \frac{1}{4f_i}\varepsilon^T P_\nu\varepsilon \\ =& \frac{1}{2}\sum_{k=1}^{i}\left(z_k^2 + \frac{1}{2h_k}\epsilon_{in}^T P_{in}\epsilon_{in} + \epsilon_{in}^T P_{f_k}\epsilon_{in} + \frac{1}{2f_k}\varepsilon^T P_\nu\varepsilon\right) + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{|b_m|}{2\gamma_\varrho}\tilde{\varrho}^2. \end{aligned} \tag{2.114}$$

Similar to (2.109), we can write $\dot{V}_{i-1}$ as follows

$$\dot{V}_{i-1} \leq -\sum_{k=1}^{i-1}c_k z_k^2 + z_{i-1}z_i + \tilde{\theta}^T\left(\tau_{i-1} - \Gamma^{-1}\dot{\hat{\theta}}\right) + \left(\sum_{k=2}^{i-1}z_k\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\right)\left(\Gamma\tau_{i-1} - \dot{\hat{\theta}}\right). \tag{2.115}$$

The time derivative of (2.114) with (2.112) and (2.115) is given by

$$\begin{aligned} \dot{V}_i \leq& -\sum_{k=1}^{i-1}c_k z_k^2 + z_i z_{i+1} + \tilde{\theta}^T\left(\tau_{i-1} - \frac{\partial\alpha_{i-1}}{\partial y}\omega z_i - \Gamma^{-1}\dot{\hat{\theta}}\right) + \left(\sum_{k=2}^{i-1}z_k\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\right)\left(\Gamma\tau_{i-1} - \dot{\hat{\theta}}\right) \\ &+ z_i\left[\alpha_i + z_{i-1} - \beta_i - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right] - z_i\frac{\partial\alpha_{i-1}}{\partial y}\epsilon_{2in} - \frac{2}{4h_i}\epsilon_{in}^T\epsilon_{in} - \frac{1}{4f_i}\epsilon_{in}^T\epsilon_{in} \\ &\times\lambda_{max}\left(e_2 l^T P_\nu^T P_\nu l e_2^T\right) - \frac{\bar{q}}{4f_i}\varepsilon^T\varepsilon + \frac{1}{4f_i}(-2\varepsilon^T P_\nu l e_2^T\epsilon_{in}) - z_i\frac{\partial\alpha_{i-1}}{\partial y}\theta_\epsilon^T\varepsilon. \end{aligned} \tag{2.116}$$

Choosing

$$\tau_i = \tau_{i-1} - \frac{\partial \alpha_{i-1}}{\partial y}\omega z_i \tag{2.117}$$

and noting that

$$\Gamma\tau_{i-1} - \dot{\hat{\theta}} = \Gamma\tau_{i-1} - \Gamma\tau_i + \Gamma\tau_i - \dot{\hat{\theta}} = \Gamma\frac{\partial \alpha_{i-1}}{\alpha y}\omega z_i + \Big(\Gamma\tau_i - \dot{\hat{\theta}}\Big), \tag{2.118}$$

we can write (2.116) as follows

$$\begin{aligned}
\dot{V}_i \le & -\sum_{k=1}^{i-1} c_k z_k^2 + z_i z_{i+1} + \tilde{\theta}^T\Big(\tau_i - \Gamma^{-1}\dot{\hat{\theta}}\Big) + \Big(\sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Big)\Big(\Gamma\tau_i - \dot{\hat{\theta}}\Big) \\
& + \Big(\sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Big)\Gamma\frac{\partial \alpha_{i-1}}{\partial y}\omega z_i + z_i\Big[\alpha_i + z_{i-1} - \beta_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}\Big] - \frac{1}{4h_i}\epsilon_{in}^T\epsilon_{in} \\
& + h_i\Big(\frac{\partial \alpha_{i-1}}{\partial y}\Big)^2 z_i^2 - h_i\Big(z_i\frac{\partial \alpha_{i-1}}{\partial y} + \frac{1}{2h_i}\epsilon_{2in}\Big)^2 - \frac{1}{4f_i}\epsilon_{in}^T\epsilon_{in}\lambda_{max}\left(e_2 l^T P_\nu^T P_\nu l e_2^T\right) \\
& - \frac{\bar{q}}{4f_i}\varepsilon^T\varepsilon + \frac{1}{4f_i}(-2\varepsilon^T P_\nu l e_2^T\epsilon_{in}) + f_i\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 z_i^2 - f_i\left(z_i\frac{\partial \alpha_{i-1}}{\partial y} + \frac{1}{2f_i}\theta_\epsilon^T\varepsilon\right)^2 \\
& + \frac{1}{4f_i}(\theta_\epsilon^T\varepsilon)^2 \qquad (2.119) \\
\le & -\sum_{k=1}^{i-1} c_k z_k^2 + z_i z_{i+1} + \tilde{\theta}^T\Big(\tau_i - \Gamma^{-1}\dot{\hat{\theta}}\Big) + \Big(\sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Big)\Big(\Gamma\tau_i - \dot{\hat{\theta}}\Big) \\
& + z_i\Big[\alpha_i + z_{i-1} - \beta_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} + \Big(\sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Big)\Gamma\frac{\partial \alpha_{i-1}}{\partial y}\omega\Big] + h_i\Big(\frac{\partial \alpha_{i-1}}{\partial y}\Big)^2 z_i^2 \\
& + f_i\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 z_2^2 - \frac{1}{4h_i}\epsilon_{in}^T\epsilon_{in} - \frac{1}{4f_i}\varepsilon^T\varepsilon. \qquad (2.120)
\end{aligned}$$

Choosing the stabilizing function as follows

$$\begin{aligned}
\alpha_i = & -z_{i-1} - \left(c_i + h_i\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 + f_i\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2\right)z_i + \beta_i + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\Gamma\tau_i \\
& -\sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{i-1}}{\partial y}\omega \qquad (2.121)
\end{aligned}$$

and substituting it into (2.120), we get

$$\dot{V}_i \leq -\sum_{k=1}^{i} c_k z_k^2 + z_i z_{i+1} + \hat{\theta}^T\left(\tau_i - \Gamma^{-1}\dot{\hat{\theta}}\right) + \left(\sum_{k=2}^{i} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\right)\left(\Gamma\tau_i - \dot{\hat{\theta}}\right) - \frac{1}{4h_k}\epsilon_{in}^T\epsilon_{in} - \frac{1}{4f_k}\varepsilon^T\varepsilon. \tag{2.122}$$

The design procedure ends when $i = \rho$. Then, the update law is chosen as

$$\dot{\hat{\theta}} = \Gamma\tau_\rho. \tag{2.123}$$

By substituting (2.123) into (2.122), we get

$$\dot{V}_\rho \leq -\sum_{k=1}^{\rho} c_k z_k^2 - \frac{1}{4h_k}\epsilon_{in}^T\epsilon_{in} - \frac{1}{4f_k}\varepsilon^T\varepsilon. \tag{2.124}$$

Finally, we design a backstepping based adaptive control law as follows

$$u = \alpha_\rho - v_{m,\rho+1} + \hat{\varrho} y_r^{(\rho)}. \tag{2.125}$$

For convenience, the parameter update laws are given by

$$\dot{\hat{\varrho}} = -\gamma_\varrho \mathrm{sgn}(b_m)(\dot{y}_r + \bar{\alpha}_1)z_1, \quad \gamma_\varrho > 0 \tag{2.126}$$

$$\dot{\hat{\theta}} = \Gamma\tau_\rho, \tag{2.127}$$

where

$$\tau_1 = (\omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1)z_1 \tag{2.128}$$

$$\tau_i = \tau_{i-1} - \frac{\partial \alpha_{i-1}}{\partial y}\omega z_i, \quad i = 2, \ldots, \rho \tag{2.129}$$

for any positive definite symmetric matrix Γ.

In view of the adaptive control law (2.125), stabilizing functions (2.69), (2.96), (2.108), (2.121) and update laws (2.126)–(2.129), the resulting error system can be written in a compact form

$$\dot{z} = A_z(z,t)z + W_\epsilon(z,t)(\epsilon_{2in} + \theta_\epsilon^T \varepsilon) + W_\theta(z,t)\tilde{\theta} - b_m(\dot{y}_r + \bar{\alpha}_1)e_1\tilde{\varrho}, \tag{2.130}$$

where $A_z \in \mathbb{R}^{\rho\times\rho}, W_\epsilon \in \mathbb{R}^{\rho}, W_\theta \in \mathbb{R}^{\rho\times\rho}$ are presented as follows,

$$\begin{aligned}
W_\theta(z,t)^T =& W_\epsilon(z,t)\omega^T - \hat{\varrho}(\dot{y}_r + \alpha_1)e_1 e_1^T \in \mathbb{R}^{\rho\times\rho}\\
W_\epsilon(z,t) =& \begin{bmatrix} 1, & (-\frac{\partial\alpha_1}{\partial y})^T, & \dots, & (-\frac{\partial\alpha_{\rho-1}}{\partial y})^T \end{bmatrix}^T \in \mathbb{R}^{\rho}\\
A_z(z,t) =& \begin{bmatrix}
-c_1 - h_1 - f_1 & \hat{b}_m & 0 & \cdots & \cdots & 0\\
-\hat{b}_m & -c_2 - h_2(\frac{\partial\alpha_1}{\partial y})^2 - f_2(\frac{\partial\alpha_1}{\partial y})^2 & 1+\sigma_{23} & \sigma_{24} & \cdots & \sigma_{2,p}\\
0 & -1-\sigma_{23} & \ddots & \ddots & \ddots & \vdots\\
\vdots & -\sigma_{24} & \ddots & \ddots & \ddots & \sigma_{\rho-2,\rho}\\
\vdots & \vdots & \ddots & \ddots & \ddots & 1+\sigma_{\rho-1,\rho}\\
0 & -\sigma_{2,\rho} & \cdots & -\sigma_{\rho-2,\rho} & -1-\sigma_{\rho-1,\rho} & -c_\rho - h_\rho(\frac{\partial\alpha_1}{\partial y})^2 - f_\rho(\frac{\partial\alpha_1}{\partial y})^2
\end{bmatrix},
\end{aligned} \tag{2.131}$$

with

$$\sigma_{ij} = \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\frac{\partial\alpha_{j-1}}{\partial y}\omega. \tag{2.132}$$

Remark 2.8. *For the systems whose relative degree is one, i.e. $m = n-1$, the controller design procedure doesn't require a backstepping technique. The input term* *appears in the derivative of the output signal*

$$\dot{y} = b_m u + \xi_2 + \bar{\omega}^T \theta + \epsilon_{2in} + \theta_\epsilon^T \varepsilon. \tag{2.133}$$

For the error signal $z_1 = y - y_r$*, we choose the input as follows*

$$u = \hat{\varrho}\left(-c_1 z_1 - h_1 z_1 - f_1 z_1 - \xi_2 - \bar{\omega}^T \hat{\theta} + \dot{y}_r\right). \tag{2.134}$$

Therefore, we give the details of the case where the relative degree is greater than unity.

2.4. Stability Analysis

The main theorem is stated as follows.

Theorem 2.9. *Consider the closed-loop system consisting of the plant* (2.5), (2.6), *unknown disturbance* (2.8), *K-filters* (2.16)–(2.21), *the disturbance observer filters* (2.45)–(2.49), *the parameter update laws* (2.126)–(2.129) *and control law* (2.125). *Under Assumptions 2.2–2.6, the signals* $\hat{\varrho}, y \in \mathbb{R}, \eta, v_m, \lambda, \epsilon_{in}, \bar{x}, x \in \mathbb{R}^n, z \in \mathbb{R}^\rho, \varepsilon, Z, \Psi, \Phi \in \mathbb{R}^{2q+1}, \varphi \in \mathbb{R}^{1\times(2q+1)n}, \mu \in \mathbb{R}^{1\times(2q+1)(m+1)}, \hat{\theta} \in \mathbb{R}^{(2q+1+(n+m+1)(2q+2))}$ *are globally bounded and the asymptotic tracking is achieved, i.e.*

$$\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} (y(t) - y_r(t)) = 0. \tag{2.135}$$

Proof. Consider the Lyapunov function

$$\begin{aligned} V_\rho \;=\; & \frac{1}{2} z^T z + \frac{1}{2} \sum_{k=1}^{\rho} \left(\frac{1}{2h_k} \epsilon_{in}^T P_{in} \epsilon_{in} + \epsilon_{in}^T P_{f_k} \epsilon_{in} \right) \\ & + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \frac{|b_m|}{2\gamma_\varrho} \tilde{\varrho}^2 + \frac{1}{2} \left(\sum_{k=1}^{\rho} \frac{1}{2f_k} \varepsilon^T P_\nu \varepsilon \right), \end{aligned} \tag{2.136}$$

where $h_k, f_k > 0$ for $k = 1, \ldots, \rho$, the positive definite matrices P_{in}, P_{f_k} and P_ν are defined in (2.75)–(2.78).

Recall from (2.124) that

$$\dot{V}_\rho \leq -\left(\sum_{k=1}^{\rho} c_k z_k^2 + \frac{1}{4h_k}\epsilon_{in}^T\epsilon_{in} + \frac{1}{4f_k}\varepsilon^T\varepsilon\right) \tag{2.137}$$

for $c_k > 0$ for $k = 1, \ldots, \rho$. With LaSalle Yoshizawa theorem, we can state that $z, \epsilon_{in}, \hat{\theta}, \hat{\varrho}, \varepsilon$ are bounded and $z(t), \epsilon_{in}(t), \varepsilon(t) \to 0$ as $t \to \infty$. The boundedness of z and y_r establishes the boundedness of y from (2.65). Moreover, since $z(t) \to 0$ as $t \to \infty$, output signal, y, tracks the reference signal, y_r, perfectly.

Recalling A_0 is Hurwitz, we prove the boundedness of η from (2.16). By writing (2.17) in Laplace form and substituting (2.1), we get

$$\lambda_i(s) = \frac{s^{i-1} + k_1 s^{i-2} + \cdots + k_{i-1}}{K(s)B(s)}\Big(A(s)Y(s) - D(s)V(s)\Big). \tag{2.138}$$

where $K(s) = s^n + k_1 s^{n-1} + \cdots + k_0$. In view of the boundedness of y, ν and Assumption 2.2, (2.138) proves that $\lambda_1, \ldots, \lambda_{m+1}$ are bounded. In view of (2.20), $v_{i,j}$ can be written as

$$v_{i,j} = [*, \ldots, *, 1]\,\bar{\lambda}_{i+j}, \tag{2.139}$$

where $\bar{\lambda}_{i+j} = \big[\lambda_1, \ldots, \lambda_{i+j}\big]^T$. For the sake of simplicity, we put $*$ to denote the elements of the matrix consisting of k terms. From (2.139), $v_{i,2}$ for $i = 0, \ldots, m-1$ is bounded. Therefore, μ is bounded for $i = 0, \ldots, m-1$ from (2.49).

Recalling that ϵ_{2ss} is a bounded signal from (2.36), Z is bounded from (2.42). Recalling that y and ξ_2 are bounded, Ψ is bounded from (2.46). The boundedness of Ψ and y prove that Φ is bounded from (2.45). In view of the boundedness of η, we establish the boundedness of φ from (2.47)-(2.48). The boundedness of $Z, \Phi, \varepsilon, \varphi$ for $i = 1, \ldots, m$ and μ for $i = 1, \ldots, m-1$ establishes the boundedness of μ_m from (2.44). Recalling that $\lambda_1, \ldots, \lambda_{m+1}$ and μ_m are bounded, $v_{m,2}$ is bounded from (2.49) and λ_{m+2} is also bounded from(2.139).

The coordinate transformation is written as follows

$$v_{m,i} = z_i + \hat{\varrho} y_r^{(i-1)} + \alpha_{i-1}\Big(y, \eta, \hat{\theta}, \hat{\varrho}, \bar{\lambda}_{m+i-1}, \bar{y}_r^{(i-2)}, \Phi, \varphi, \mu\Big), \tag{2.140}$$

where $\bar{y}_r^{(i-2)} = \Big[y_r, \ldots, y_r^{(i-2)} \Big]$ for $i = 2, \ldots, \rho$. Let $i = 3$, then we prove the boundedness of $v_{m,3}$ recalling that $z_3, y, \eta, \hat{\theta}, \hat{\varrho}, \bar{\lambda}_{m+2}, y_r, \dot{y}_r$ and $\ddot{y}_r$ are bounded. From (2.139), λ_{m+3} is bounded. Recursively, we prove the boundedness of λ. With the boundedness of λ and η, we prove that $\bar{x}$ is bounded from (2.22). Considering this fact and the boundedness of ν, we prove the boundedness of x from (2.26). This proves Theorem 2.9. □

We now consider the zero dynamics and the overall error system. We define the reference signal η^r governed by

$$\dot{\eta}^r = A_0 \eta^r + e_n y_r, \tag{2.141}$$

so that the error signal $\tilde{\eta} = \eta - \eta^r$ satisfies the equation

$$\dot{\tilde{\eta}} = A_0 \tilde{\eta} + e_n z_1. \tag{2.142}$$

We define the following similarity transformation

$$\begin{bmatrix} x_1 \\ \vdots \\ x_\rho \\ \zeta \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_\rho \\ Tx \end{bmatrix} = \begin{bmatrix} I_\rho & 0_{\rho\times m} \\ & T \end{bmatrix} x, \tag{2.143}$$

where

$$T = [A_b^\rho e_1, \ldots, A_b e_1, I_m] \tag{2.144}$$

$$A_b = \begin{bmatrix} -b_{m-1}/b_m & & & \\ \vdots & I_{m-1} & & \\ -b_0/b_m & 0 & \ldots & 0 \end{bmatrix}, \tag{2.145}$$

with the following identities

$$T \begin{bmatrix} 0 \\ b \end{bmatrix} = 0, \quad TA = A_b T + TA^\rho \begin{bmatrix} 0 \\ b \end{bmatrix} e_1^T. \tag{2.146}$$

The inverse dynamics of (2.5) and (2.6) are represented as

$$\dot{\zeta} = A_b\zeta + T\left(A^\rho \begin{bmatrix} 0 \\ b \end{bmatrix} - a \right) y + Td\nu \tag{2.147}$$

$$\triangleq A_b\zeta + b_b y + b_\nu \nu. \tag{2.148}$$

We define the reference signal ζ^r governed by

$$\dot{\zeta}^r = A_b\zeta^r + b_b y_r, \tag{2.149}$$

so that the error signal $\tilde{\zeta} = \zeta - \zeta^r$ satisfies the equation

$$\dot{\tilde{\zeta}} = A_b\tilde{\zeta} + b_b z_1 + b_\nu \nu. \tag{2.150}$$

We make the following transformation

$$\zeta_b = \tilde{\zeta} - \int_0^t e^{A_b(t-\tau)} b_\nu \nu(\tau) d\tau. \tag{2.151}$$

The time derivative of ζ_b yields

$$\dot{\zeta}_b = A_b\zeta_b + b_b z_1 \tag{2.152}$$

so that we transform (2.148) to (2.152). The overall error system is summarized as follows

$$\dot{z} = A_z(z,t)z + W_\epsilon(z,t)(\epsilon_{2in} + \theta_\epsilon^T\varepsilon) + W_\theta(z,t)\tilde{\theta} - b_m(\dot{y}_r + \bar{\alpha}_1)e_1\tilde{\varrho}, \tag{2.153}$$

$$\dot{\epsilon}_{in} = A_0\epsilon_{in}, \tag{2.154}$$

$$\dot{\zeta}_b = A_b\zeta_b + b_b z_1, \tag{2.155}$$

$$\dot{\tilde{\eta}} = A_0\tilde{\eta} + e_n z_1, \tag{2.156}$$

$$\dot{\tilde{\theta}} = -\Gamma W_\theta(z,t)z, \tag{2.157}$$

$$\dot{\tilde{\varrho}} = \gamma_\varrho \mathrm{sgn}(b_m)(\dot{y}_r + \bar{\alpha}_1)e_1^T z. \tag{2.158}$$

$$\dot{\varepsilon} = G\varepsilon - le_2^T\epsilon_{in}. \tag{2.159}$$

Lemma 2.10. *The error system* (2.153)–(2.159) *is globally uniformly stable at the origin.*

Proof. We consider the following Lyapunov function

$$V = V_\rho + \frac{1}{k_\eta}\tilde{\eta}^T P_\eta \tilde{\eta} + \frac{1}{k_\zeta}\zeta_b^T P_\zeta \zeta_b, \tag{2.160}$$

where the positive definite matrices P_ζ and P_η are solutions of the matrix equations

$$P_\zeta A_b + A_b^T P_\zeta^T = -I_m \tag{2.161}$$

$$P_\eta A_0 + A_0^T P_\eta^T = -I_n \tag{2.162}$$

and k_ζ and k_η are any positive constants. The derivative of (2.160) is given by

$$\dot{V} \le -\Big(\sum_{k=1}^{\rho} c_k z_k^2 + \frac{1}{4h_k}\epsilon_{in}^T \epsilon_{in} + \frac{1}{4f_k}\varepsilon^T\varepsilon + \frac{1}{2k_\eta}\tilde{\eta}^T\tilde{\eta} + \frac{1}{2k_\zeta}\zeta_b^T\zeta_b\Big) \tag{2.163}$$

which ensures the global uniform stability of (2.153)–(2.159). □

Remark 2.11. *It should be noted that the disturbance might include a bounded noise in addition to the sinusoidal terms or that we might underestimate the maximum number of distinct frequencies q. In such a case, the parameter estimates may drift and the overall stability can be vanished. This problem may be counteracted by applying some robustification techniques that involve leakage, σ-modification, switching-σ, ϵ_1-modification, dead-zone, projection, and dynamic normalization as given in [39]. These modifications maintain the overall stability. Meanwhile, we sacrifice the convergence in the steady state. We present σ-modification technique to handle an unmodelled disturbance and give a numerical example in Section 6.*

2.5. Numerical Simulations

In this section, we present two simulations. Case one does not include any additional noise and q is underestimated. Additional noise is added to disturbance in case two to illustrate robustness. We consider the following unstable relative-degree-two plant

$$Y(s) = \frac{(s+0.8)U(s) + 0.9D(s)}{s^3 + 0.6s^2 + 0.5s + 0.4} \tag{2.164}$$

In the case one, we choose the reference signal $y_r(t)$ and the unknown disturbance $\nu(t)$ as given in Table 2.1.

The number of distinct frequencies, q, is assumed to be 2 in the design. We choose initial conditions as $X(0) = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}^T$, the controllable pair (G, l) as

Table 2.1. Unknown disturbances $\nu(t)$ and reference signals $y_r(t)$ used in simulations

	Figure 2.1		**Figure 2.2**
	$0 < t \leq 20$	$20 < t \leq 40$	$0 < t \leq 40$
$\nu(t)$	$5 + 3\sin(0.5t)$ $+4\sin(2t + \pi/6)$	$7 + 5\sin(1.5t$ $+\pi/12)$	$5 + 3\sin(0.5t)$ $+4\sin(2t + \pi/6) + \nu_a(t)$ where $\nu_a \sim \mathcal{N}(2,\, 2.25) + 2\sin(t)$
$y_r(t)$	1	2	1

$$l = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, G = \begin{bmatrix} 0_4 & I_{4\times 4} \\ 0_5^T & \end{bmatrix} + l \begin{bmatrix} -0.32 & -2.02 & -5.09 & -6.39 & -4.00 \end{bmatrix},$$

the matrix k as $\begin{bmatrix} 3.30, & 3.62, & 1.32 \end{bmatrix}^T$ and the constants $c_1,\ c_2,\ h_1,\ h_2,\ f_1,\ f_2$ as $0.3, 0.2, 0.25, 0.27, 0.1, 0.1$, respectively. Figure 2.1 shows that $y(t)$ tracks $y_r(t)$ as stated in Theorem 2.9. Note that even if we overestimate q between $20 < t \leq 40$, the output tracks the reference signal perfectly.

In the case two, we consider scenario in which we underestimate q. Furthermore, additional unmodelled noise is added as given in Table 2.1. For this case, we make a robustness analysis by considering an additional noise $\nu_a(t)$ for (2.8). We apply a σ-modification algorithm to guarantee stability as follows

$$\dot{\hat{\varrho}} = -\gamma_\varrho \mathrm{sgn}(b_m)(\dot{y}_r + \bar{\alpha}_1)z_1 - \gamma_\varrho \gamma_\sigma \hat{\varrho} \tag{2.165}$$

$$\dot{\hat{\theta}} = \Gamma W_\theta(z,t)z - \Gamma \gamma_\theta \hat{\theta}, \tag{2.166}$$

where $\gamma_\varrho, \gamma_\sigma, \gamma_\theta > 0$. Taking the time derivative of Lyapunov function (2.160), employing nonlinear damping and substituting (2.165)–(2.166) instead of (2.126)–(2.127), we get

$$\dot{V} \leq -\gamma_v V + \sum_{k=1}^{\rho} \frac{1}{4\kappa_k}\epsilon_{2,a}^2 + \gamma_\theta \frac{\theta^T\theta}{2} + |b_m|\gamma_\sigma \frac{\varrho^2}{2}, \tag{2.167}$$

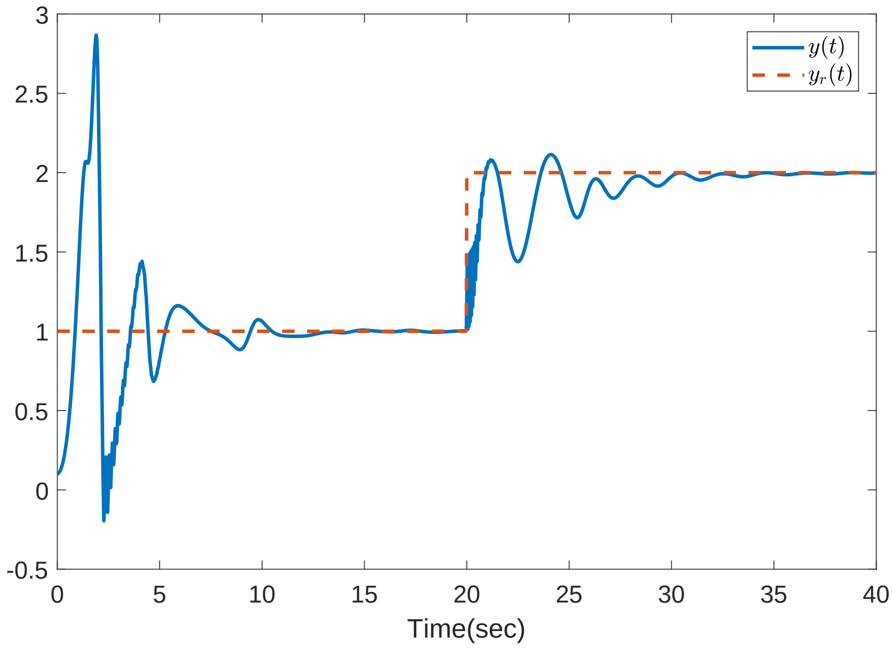

Figure 2.1. The output, $y(t)$, and reference signal, $y_r(t)$. See Table 2.1 for $\nu(t)$ and $y_r(t)$.

where $\gamma_v > 0$, $\kappa_k > 0$ for $k = 1, \ldots, \rho$ and

$$\epsilon_{2,a}(t) = \mathcal{L}^{-1}\left(e_2^T\left(sI - A_0\right)^{-1} dV_a(s)\right). \tag{2.168}$$

In the simulation, we choose the reference signal $y_r(t)$ and the unknown disturbance $\nu(t)$ as given in Table 2.1. Figure 2.2 shows that $y(t)$ does not converge, but oscillates around $y_r(t)$ due to the noise $\nu_a(t)$. The robustness is achieved at the expense of losing the asymptotic reference tracking.

Remark 2.12. *Note that Figure 2.1 and 2.2 do not have a label on y-axis although x-axis shows "Time" in seconds. This is because the states may have arbitrary units such as distance, intensity or other quantities depending on predetermined reference measurement.*

2.6. Conclusion

In this chapter, we solve the problem of an unknown sinusoidal disturbance rejection for unknown and minimum phase LTI systems with the output feedback. Firstly,

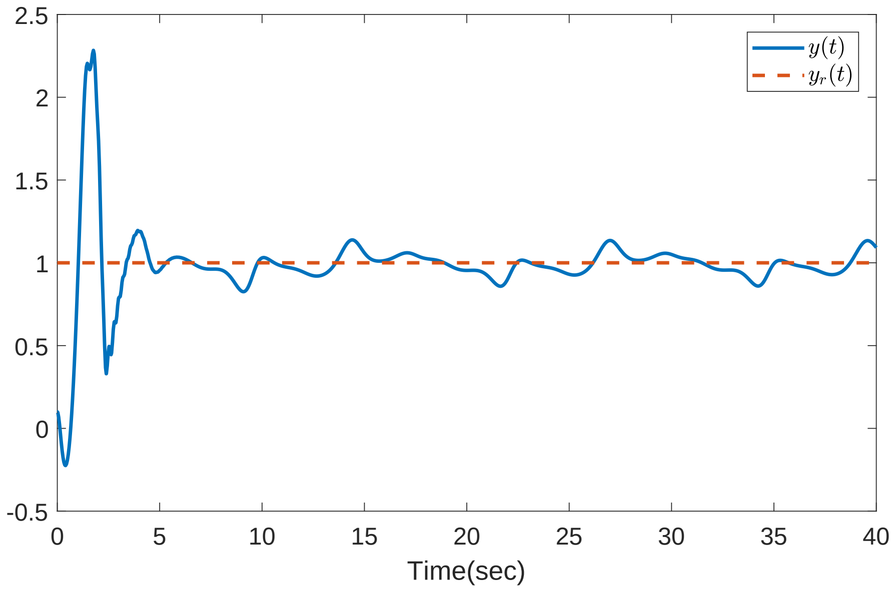

Figure 2.2. The output, $y(t)$, and reference signal, $y_r(t)$. See Table 2.1 for $\nu(t)$ and $y_r(t)$.

we employ K-filter technique to redefine the states of the plant. Then, we parametrize the disturbance information in the output signal as a multiplication of unknown constant vectors with known regressors and an exponentially decaying term so that we approach the problem as an adaptive control problem. Following the idea in [36], we propose an adaptive controller which achieves trajectory tracking despite unknown disturbances. Moreover, we prove that all the signals of the closed-loop system are globally bounded, and the asymptotic tracking is achieved. Furthermore, we show that the equilibrium at the origin is globally uniformly stable. We perform a simulation with a third-order unstable system, whose relative degree is two, to demonstrate the effects of the adaptive controller. By incorporating a σ-modification in the update laws, we also show the robustness of the system with respect to an additive unmodelled noise.

3. REJECTION OF SINUSOIDAL DISTURBANCES FOR KNOWN LTI SYSTEMS IN THE PRESENCE OF SIMULTANEOUS INPUT-OUTPUT DELAY

This chapter focuses on estimation and cancellation of unknown sinusoidal disturbances in a known LTI system with the presence of a known input/output delay. Parametrizing the disturbance and representing the delays as a transport PDE, the problem is converted to an adaptive control problem for ODE-PDE cascade. An existing state observer is used to estimate the ODE system states. The exponential stability of the equilibrium of the closed-loop and error system is proved. The perfect estimation of the disturbance and state is shown. Moreover, the convergence of the state to zero as $t \to \infty$ is achieved in the closed loop system. The effectiveness of the controller is demonstrated in a numerical simulation.

3.1. Problem Statement

We consider the single-input single-output LTI system

$$\dot{X}(t) = AX(t) + B\big(U(t - D_u) + \nu(t)\big), \tag{3.1}$$

$$Y(t) = CX(t - D_y), \tag{3.2}$$

where $D_u \in \mathbb{R}$ and $D_y \in \mathbb{R}$ are the known input and output delay, respectively. $X = [X_1, \dots, X_n]^T \in \mathbb{R}^n$ is the system state, $U(t) \in \mathbb{R}$ is the input and

$$A = \begin{bmatrix} -a_{n-1} & \\ \vdots & I_{n-1} \\ \vdots & \\ -a_0 & 0_{n-1}^T \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T \tag{3.3}$$

with $0_{n-1} = [0, \ldots, 0]^T \in \mathbb{R}^{n-1}$. The unknown sinusoidal disturbance $\nu(t) \in \mathbb{R}$ is given by

$$\nu(t) = d + \sum_{i=1}^{q} g_i \sin(w_i t + \phi_i) \tag{3.4}$$

where $d, g_i, w_i, \phi_i \in \mathbb{R}$ are unknown. The output delay can be modelled by the following first-order hyperbolic PDE

$$\partial_t y(x,t) = \partial_x y(x,t), \quad x \in \left[0, D_y\right] \tag{3.5}$$

$$y(D,t) = CX(t). \tag{3.6}$$

The solution of these equations is $y(x,t) = CX(t + x - D)$.

The sinusoidal disturbance $\nu(t)$ can be represented as the output of a linear exosystem,

$$\dot{W}(t) = SW(t), \tag{3.7}$$

$$\nu(t) = h_\nu^T W(t), \tag{3.8}$$

where the state $W(t) \in \mathbb{R}^{2q+1}$. The matrix S comprises the unknown frequency of the sinusoidal disturbance $\nu(t)$. Constant bias term d, amplitude g_i and phase ϕ_i are determined by initial condition of (3.7), are thus unknown. Without loss of generality, one can choose output vector h_ν^T such that (h_ν^T, S) becomes observable pair.

The disturbance $\nu(t)$ is not measured. The output $Y(t)$ is the only available measurement. Regarding the plant (3.1)–(3.2) and the exosystem (3.7)–(3.8), we make the following assumptions:

Assumption 3.1. *The frequencies of the disturbance are distinct, $\omega_i \neq \omega_j$ for $i \neq j$, and the number of the distinct frequencies q is known.*

Assumption 3.2. *The bias $d \neq 0$ and amplitude $g_i \neq 0$ for all $i \in \{1, \ldots, q\}$.*

Our ultimate goal is to design an observer achieving accurate online estimation of state $X(t)$ as well as the disturbance $\nu(t)$. Using the observer states, we design a controller stabilizing the equilibrium of the closed loop system. Moreover, we aim the state $X(t)$ to converge to zero as $t \to \infty$ in the presence of simultaneous input-output delay and unmeasured sinusoidal disturbance.

3.2. Disturbance Representation

Our main interest here is a preparation for disturbance observer design which is presented in the next section. Firstly, we employ a filter introduced in [26] for systems under no disturbance effect. However, because of the unknown disturbance in our system, we show that the error between the system states and the filter states is driven by unknown sinusoidal terms. Main motivation of obtaining this error is to use it in disturbance representation and then, disturbance parametrization.

Inspiring [26], we propose the following filter

$$\dot{\hat{X}}_d(t) = A\hat{X}_d(t) + BU(t - D_u) + e^{AD_y}L(Y(t) - \hat{y}_d(0,t)), \tag{3.9}$$

$$\partial_t \hat{y}_d(x,t) = \partial_x \hat{y}_d(x,t) + Ce^{Ax}L\big(Y(t) - \hat{y}_d(0,t)\big), \tag{3.10}$$

$$\hat{y}_d(D_y,t) = C\hat{X}_d(t), \tag{3.11}$$

where L is chosen such that $A - LC$ is Hurwitz. Since the pair (A, C) is observable, there exists an L such that this condition is satisfied. The error is given as follows,

$$\tilde{X}_d(t) = \hat{X}_d(t) - X(t), \tag{3.12}$$

$$\dot{\tilde{X}}_d(t) = A\tilde{X}_d - e^{AD_y}L\tilde{y}_d(0,t) - B\nu(t), \tag{3.13}$$

$$\partial_t \tilde{y}_d(x,t) = \partial_x \tilde{y}_d(x,t) - Ce^{Ax}L\tilde{y}_d(0,t), \tag{3.14}$$

$$\partial_t \tilde{y}_d(D_y,t) = C\tilde{X}_d(t). \tag{3.15}$$

The following transformation

$$\tilde{w}(x,t) = \tilde{y}_d(x,t) - Ce^{A(x-D_y)}\tilde{X}_d(t) \tag{3.16}$$

transforms (3.12), (3.13) into the form of

$$\dot{\tilde{X}}_d(t) = A_e\tilde{X}_d(t) - e^{AD_y}L\tilde{w}(0,t) - B\nu(t), \tag{3.17}$$

$$\partial_t\tilde{w}(x,t) = \partial_x\tilde{w}(x,t) + Ce^{A(x-D_y)}B\nu(t), \tag{3.18}$$

$$\tilde{w}(D_y,t) = 0, \tag{3.19}$$

where $A_e = A - e^{AD_y}LCe^{-AD_y}$. By using similarity transformation e^{AD_y} and noting that $A - LC$ is Hurwitz, it can be proved that A_e is Hurwitz.

If there is no disturbance in the system as it is shown in [26], the error $\tilde{X}_d(t)$ converges to 0 as $t \to \infty$. However, $\tilde{X}_d(t)$ is driven by $\nu(t)$ and $\tilde{w}(0,t)$ as seen in (3.17). In Lemma 3.3, we show that $\tilde{\omega}(0,t)$ can be expressed as a sum of sinusoidal signals whose frequencies are same as $\nu(t)$.

Lemma 3.3. *The signal $\tilde{w}(0,t)$ can be expressed in the following form*

$$\tilde{w}(0,t) = \bar{d} + \sum_{i=1}^{q} \bar{g}_i \sin(w_i t + \bar{\phi}_i) \tag{3.20}$$

where

$$\bar{g}_i = \sqrt{(\bar{g}_i^s)^2 + (\bar{g}_i^c)^2} \tag{3.21}$$

$$\bar{\phi}_i = \phi_i + \arctan(\frac{\bar{g}_i^c}{\bar{g}_i^s}), \tag{3.22}$$

$$\bar{d} = \Big(C\int_0^{D_y} e^{A(\xi-D_y)}Bd\xi\Big)d \tag{3.23}$$

$$\bar{g}_i^s = \Big(C\int_0^{D_y} e^{A(\xi-D_y)}B\cos(w_i\xi)d\xi\Big)g_i, \tag{3.24}$$

$$\bar{g}_i^c = -\Big(C\int_0^{D_y} e^{A(\xi-D_y)}B\sin(w_i\xi)d\xi\Big)g_i. \tag{3.25}$$

Proof. Solution of $\tilde{w}(x,t)$ with Laplace Transformation method gives us

$$\tilde{w}(x,t) = \int_x^{D_y} Ce^{A(\xi - D_y)} B\nu(t + x - \xi)d\xi. \tag{3.26}$$

Using necessary trigonometric formulas, $\nu(t-\xi)$ can be written as

$$\nu(t-\xi) = d + \sum_{i=1}^{q} g_i \sin(w_i t + \phi_i)\cos(w_i \xi) - g_i \cos(w_i t + \phi_i)\sin(w_i \xi). \tag{3.27}$$

Substituting (3.27) into (3.26), writing for $x = 0$ and using trigonometric identities, we get (3.20). □

From Lemma 3.3, we show that the signal $\tilde{w}(0,t)$ excites at the same frequencies with the disturbance $\nu(t)$. From (3.26), we prove the boundedness of $\tilde{\omega}(0,t)$ with the boundedness of $\nu(t)$. Considering the boundedness of $\tilde{w}(0,t)$ and $\nu(t)$ and noting that A_e is Hurwitz, from (3.17), we can conclude that $\tilde{X}_d(t)$ is bounded and driven by the unknown sinusoidal terms.

Second step of this section deals with unknown sinusoidal terms in the output dynamics. For this aim, we write the derivative of the output using (3.3) as the following

$$\dot{Y}(t) = -a_{n-1}X_1(t - D_y) + X_2(t - D_y) + b_1\nu(t - D_y) + b_1 U(t - D_u - D_y). \tag{3.28}$$

Using (3.12), we get

$$X_2(t) = \hat{X}_{d2}(t) - \tilde{X}_{d2}(t) \tag{3.29}$$

where $\hat{X}_{d2}(t) = B_2^T \hat{X}_d(t) \in \mathbb{R}$ and $\tilde{X}_{d2}(t) = B_2^T \tilde{X}_d(t) \in \mathbb{R}$ with the column vector B_i whose i^{th} element is 1 and the rest is 0. Substituting (3.29) into (3.28), we get

$$\begin{aligned} \dot{Y}(t) =& - a_{n-1}X_1(t-D_y) + \hat{X}_{d2}(t-D_y) - \tilde{X}_{d2}(t-D_y) \\ &+ b_1U(t-D_u-D_y) + b_1\nu(t-D_y). \end{aligned} \tag{3.30}$$

In (3.30), the output dynamics consists of output signal, known filter state, input and two unknown terms $\tilde{X}_{d2}(t-D_y)$ and $\nu(t-D_y)$. Therefore, we need to obtain the response of $\tilde{X}_{d2}(t)$ from (3.17), so we represent signal $\tilde{X}_{d2}(t)$ as summation of the steady state $\tilde{X}_{d2}^{ss}(t)$ and transient responses $\tilde{X}_{d2}^{in}(t)$,

$$\tilde{X}_{d2}(t) = \tilde{X}_{d2}^{ss}(t) + \tilde{X}_{d2}^{in}(t). \tag{3.31}$$

The states $\tilde{X}_{d2}^{ss}(t), \tilde{X}_{d2}^{in}(t)$ are given in the proof of next lemma. Substituting (3.31) with the delay D_y into (3.30), output dynamics is rewritten as

$$\begin{aligned} \dot{Y}(t) =& - a_{n-1}X_1(t-D_y) + \hat{X}_{d2}(t-D_y) + \epsilon_x(t-D_y) \\ &- \tilde{X}_{d2}^{in}(t-D_y) + b_1U(t-D_u-D_y) \end{aligned} \tag{3.32}$$

where

$$\epsilon_x(t) = b_1\nu(t) - \tilde{X}_{d2}^{ss}(t). \tag{3.33}$$

The representation of $\epsilon_x(t)$ is given in the following lemma.

Lemma 3.4. *The signal $\epsilon_x(t)$ can be represented in the form*

$$\epsilon_x(t) = d_\epsilon + \sum_{i=1}^{q} g_{\epsilon i}^s \sin\left(w_i t + \phi_i\right) + g_{\epsilon i}^c \cos\left(w_i t + \phi_i\right) \tag{3.34}$$

where $d_\epsilon = b_1 d - d_{\tilde{x}}, g^s_{\epsilon_i} = b_1 g_i - g^s_{\tilde{x}_i}, g^c_{\epsilon_i} = -g^c_{\tilde{x}_i}$ *with*

$$\begin{aligned} d_{\tilde{x}} =& |G_{d_1}(0)|d + |G_{d_2}(0)|\bar{d}, && (3.35)\\ g^s_{\tilde{x}_i} =& |G_{d_2}(jw_i)|\Big(\bar{g}^s_i \cos(\underline{/G}_{d_2}(jw_i)) - \bar{g}^c_i \sin(\underline{/G}_{d_2}(jw_i))\Big)\\ &+ |G_{d_1}(jw_i)|g_i \cos(\underline{/G}_{d_1}(jw_i)), && (3.36)\\ g^c_{\tilde{x}_i} =& |G_{d_2}(jw_i)|\Big(\bar{g}^s_i \sin(\underline{/G}_{d_2}(jw_i)) + \bar{g}^c_i \cos(\underline{/G}_{d_2}(jw_i))\Big)\\ &+ |G_{d_1}(jw_i)|g_i \sin(\underline{/G}_{d_1}(jw_i)), && (3.37) \end{aligned}$$

and

$$\begin{aligned} G_{d_1}(s) =& - B_2^T(sI - A_e)^{-1}B, && (3.38)\\ G_{d_2}(s) =& - B_2^T(sI - A_e)^{-1}(e^{AD_y}L). && (3.39) \end{aligned}$$

Proof. The states $\tilde{X}^{ss}_{d2}(t)$ and $\tilde{X}^{in}_{d2}(t)$ in (3.31) are given by

$$\begin{aligned} \tilde{X}^{ss}_{d2}(t) =& |G_{d_1}(0)|d + \sum_{i=1}^{q} |G_{d_1}(jw_i)|g_i\Big(\cos(\underline{/G}_{d_1}(jw_i))\\ & \times \sin(w_i t + \phi_i) + \sin(\underline{/G}_{d_1}(jw_i)) \cos(w_i t + \phi_i)\Big)\\ & + |G_{d_2}(0)|\bar{d} + \sum_{i=1}^{q} |G_{d_2}(jw_i)|\bar{g}^s_i\Big(\cos(\underline{/G}_{d_2}(jw_i))\\ & \times \sin(w_i t + \phi_i) + \sin(\underline{/G}_{d_2}(jw_i)) \cos(w_i t + \phi_i)\Big)\\ & + \sum_{i=1}^{q} |G_{d_2}(jw_i)|\bar{g}^c_i\Big(\cos(\underline{/G}_{d_2}(jw_i))\times\\ & \cos(w_i t + \phi_i) - \sin(\underline{/G}_{d_2}(jw_i)) \sin(w_i t + \phi_i)\Big), && (3.40)\\ \tilde{X}^{in}_{d2}(t) =& B_2^T \tilde{X}^{in}_d(t), && (3.41) \end{aligned}$$

with (3.38), (3.39) and

$$\dot{\tilde{X}}^{in}_d(t) = A_e \tilde{X}^{in}_d(t), \quad \tilde{X}^{in}_d(0) = \tilde{X}_d(0). \tag{3.42}$$

We can rewrite the expression (3.40) in a more compact form

$$\tilde{X}_{d2}^{ss}(t) = d_{\tilde{x}} + \sum_{i=1}^{q} g_{\tilde{x}_i}^{s} \sin(w_i t + \phi_i) + \sum_{i=1}^{q} g_{\tilde{x}_i}^{c} \cos(w_i t + \phi_i) \tag{3.43}$$

where $d_{\tilde{x}}, g_{\tilde{x}_i}^{s}$ and $g_{\tilde{x}_i}^{c}$ are given in (3.35)–(3.37). Substituting (3.43) and (3.4) into (3.33), we get (3.34). □

From (3.34), it is shown that $\epsilon_x(t)$ is a sinusoidal signal with a bias term, which excites at the same frequency as the disturbance $\nu(t)$.

In the final step of this section, we write $\epsilon_x(t)$ signal as the input of a linear stable system whose system state is unknown and estimated in the next section. Considering the exosystem (3.7)–(3.8), the signal $\epsilon_x(t)$ can be represented as $\epsilon_x(t) = h_\epsilon^T W(t)$ where (h_ϵ, S) is an observable pair. Let $G \in \mathbb{R}^{(2q+1)\times(2q+1)}$ be a Hurwitz matrix and let (G, l) be a controllable pair. Since the spectra of S and G are disjoint, this guarantees that the unique solution $M \in \mathbb{R}^{(2q+1)\times(2q+1)}$ of the Sylvester equation

$$MS - GM = lh_\epsilon^T \tag{3.44}$$

is invertible [38]. The change of coordinates

$$Z(t) = MW(t) \tag{3.45}$$

transforms the exosystem (3.7), (3.8) into the form

$$\dot{Z}(t) = GZ(t) + l\epsilon_x(t) \tag{3.46}$$

where $\epsilon_x(t) = h_\epsilon^T M^{-1} Z(t)$. Using (3.45), we rewrite (3.8) as

$$\nu(t) = h_\nu^T M^{-1} Z(t). \tag{3.47}$$

Using (3.47), we write the disturbance in terms of the unknown constant vectors and the unknown $Z(t)$ signal. In the next section, we design a disturbance observer so that we cancel the unknown sinusoidal effect with an observer based adaptive controller.

3.3. Parametrization of Disturbance

In this section, since we cannot estimate $Z(t)$ due to the delay in the output, we design filters to estimate $Z(t-D_y)$ signal. Then, we represent the unknown disturbance by using the filter states. The following lemma presents the estimation of $Z(t-D_y)$ signal and establishes the properties of the filters.

Lemma 3.5. *The unmeasured signal $Z(t-D_y)$ can be represented as*

$$Z(t-D_y) = \Xi(t) + \epsilon_\nu(t) \tag{3.48}$$

with

$$\Xi(t) = \eta(t) + lY(t), \tag{3.49}$$

$$\dot{\eta}(t) = G\Xi(t) - l\Big(\hat{X}_{d2}(t-D_y) - a_{n-1}X_1(t-D_y) + b_1U(t-D_u-D_y)\Big). \tag{3.50}$$

The estimation error defined by $\epsilon_\nu = Z(t-D_y) - \Xi(t)$ obeys the equation

$$\dot{\epsilon}_\nu(t) = G\epsilon_\nu(t) + lB_2^T e^{-A_e D_y}\tilde{X}_d^{in}(t). \tag{3.51}$$

Proof. Since (3.46) is linear and G is a Hurwitz matrix, we can write

$$\dot{Z}(t-D_y) = GZ(t-D_y) + l\epsilon_x(t-D_y). \tag{3.52}$$

Taking derivative of (3.48) in view of (3.32), (3.49), (3.50), (3.52) and recalling (3.41), we get

$$\dot{\epsilon}_\nu(t) = G\epsilon_\nu(t) + lB_2^T\tilde{X}_d^{in}(t - D_y). \tag{3.53}$$

Solution of (3.42) gives $\tilde{X}_d^{in}(t) = e^{A_e t}\tilde{X}_d^{in}(0)$, the delayed signal is written by

$$\begin{aligned}\tilde{X}_d^{in}(t - D_y) =& e^{A_e(t-D_y)}\tilde{X}_d^{in}(0) = e^{-A_e D_y}e^{A_e t}\tilde{X}_d^{in}(0)\\ =& e^{-A_e D_y}\tilde{X}_d^{in}(t).\end{aligned} \tag{3.54}$$

Substitution of (3.54) into (3.53) yields (3.51). □

In the following lemma, we write the disturbance signal $\nu(t)$ by using (3.48).

Lemma 3.6. The unknown disturbance $\nu(t)$ can be represented in the form

$$\nu(t) = \theta_1^T\Xi(t - D_u) + \theta_2^T\epsilon_\nu(t) + \theta_3^T\tilde{X}_d^{in}(t) \tag{3.55}$$

where

$$\theta_1^T = h_\nu^T e^{S(D_u+D_y)}M^{-1} \tag{3.56}$$

$$\theta_2^T = \theta_1^T e^{-GD_u} \tag{3.57}$$

$$\theta_3^T = -\theta_2^T\int_0^{D_u} e^{G\tau}lB_2^T e^{-A_e(\tau+D_u)}d\tau \tag{3.58}$$

are unknown.

Proof. Solving (3.7), we get $W(t) = e^{St}W(0)$. The delayed signal is given by

$$\begin{aligned}W(t - D_u - D_y) &= e^{S(t-D_u-D_y)}W(0) = e^{-S(D_u+D_y)}e^{St}W(0)\\ &= e^{-S(D_u+D_y)}W(t).\end{aligned} \tag{3.59}$$

Using (3.45) and (3.59), $Z(t-D)$ is expressed as follows

$$Z(t-D_u-D_y) = MW(t-D_u-D_y) = Me^{-S(D_u+D_y)}W(t). \tag{3.60}$$

Substitution of $W(t) = M^{-1}Z(t)$ into (3.60) and writing for $Z(t)$ gives

$$Z(t) = Me^{S(D_u+D_y)}M^{-1}Z(t-D_u-D_y). \tag{3.61}$$

Substituting (3.61) into (3.47), we obtain

$$\nu(t) = h_\nu^T e^{S(D_u+D_y)}M^{-1}Z(t-D_u-D_y). \tag{3.62}$$

Representing $Z(t-D_y-D_u)$ by using (3.48) and substituting it into (3.62), we get (3.55). □

Lemma 3.6 gives us the representation of the unknown disturbance as the multiplication of unknown constant vector with a known regressor and two exponentially vanishing terms. This method converts the disturbance cancellation problem to an adaptive control problem. In the next section, we propose an adaptive controller together with a disturbance and state observer.

Remark 3.7. *Our state observer design given in the next section is based on the idea introduced in [32]. Contrary to [32] where a parametric uncertainty is considered, here we consider an unknown sinusoidal disturbance as given in* (3.4). *Using Lemmas 3.3–3.6, we formulate the problem that is similar to one given in [32].*

3.4. Observer Based Adaptive Controller Design

Substituting the representation of the disturbance given in (3.55) into (3.1), we get

$$\dot{X}(t) = AX(t) + B\Big(u(0,t) + \theta_1^T \xi(0,t) + \theta_2^T \epsilon_\nu(t) + \theta_3^T \tilde{X}_d^{in}(t)\Big) \tag{3.63}$$

$$\partial_t u(x,t) = \partial_x u(x,t), \quad x \in [0, D_u] \tag{3.64}$$

$$u(D_u,t) = U(t) \tag{3.65}$$

$$\partial_t \xi(x,t) = \partial_x \xi(x,t), \quad x \in [0, D_u] \tag{3.66}$$

$$\xi(D_u,t) = \Xi(t) \tag{3.67}$$

The solutions of the transport PDEs are $u(x,t) = U(x+t-D_u), \xi(x,t) = \Xi(x+t-D_u)$. Considering (3.63)–(3.67), (3.5)–(3.6) and following the idea given in [32], we design the following observer based adaptive controller

$$U(t) = Ke^{AD_u}\hat{X}_s(t) - \hat{\theta}_1^T(t)\Xi(t) + K\int_0^{D_u} e^{A(D_u-y)}B\left(u(y,t) + \hat{\theta}_1^T(t)\xi(y,t)\right)dy \tag{3.68}$$

where the state observer is

$$\begin{aligned}\dot{\hat{X}}_s(t) =& A\hat{X}_s(t) + BU(t-D_u) + B\Xi^T(t-D_u)\hat{\theta}_1(t) + e^{AD_y}L(Y(t) - \hat{y}_s(0,t)) \\ &+ \lambda_0(t)\dot{\hat{\theta}}_1(t),\end{aligned} \tag{3.69}$$

$$\begin{aligned}\partial_t \hat{y}_s(x,t) =& \partial_x \hat{y}_s(x,t) + Ce^{Ax}L(Y(t) - \hat{y}_s(0,t)) \\ &+ \Big(\lambda_1(x,t) + Ce^{(x-D_y)A}\lambda_0(t)\Big)\dot{\hat{\theta}}_1(t),\end{aligned} \tag{3.70}$$

$$\hat{y}_s(D_y,t) = C\hat{X}_s(t), \tag{3.71}$$

with the auxiliary states

$$\dot{\lambda}_0(t) = A_e\lambda_0(t) + B\Xi^T(t-D_u) - e^{AD_y}L\lambda_1(0,t), \tag{3.72}$$

$$\partial_t\lambda_1(x,t) = \partial_x\lambda_1(x,t) - Ce^{(x-D_y)A}B\Xi^T(t-D_u) \tag{3.73}$$

$$\lambda_1(D_y,t) = 0 \in \mathbb{R}^{1\times(2q+1)},\ \lambda_1(x,0) = 0. \tag{3.74}$$

The control gain $K \in \mathbb{R}^{1\times n}$ is chosen such that $(A + BK)$ becomes Hurwitz. The least-squares parameter update law is given by

$$\dot{\hat{\theta}}_1(t) = -\rho R(t)\Lambda(t)\tilde{y}_s(0,t), \tag{3.75}$$

$$\dot{R}(t) = R(t) - R(t)\Lambda(t)\Lambda(t)^T R(t), \tag{3.76}$$

$$\Lambda(t) = (Ce^{-AD_y}\lambda_0(t) + \lambda_1(0,t))^T, \tag{3.77}$$

where $\rho > 0$. The closed loop system is schematically given in Figure 3.1. Unknown sinusoidal disturbance information is obtained with Filter 1. This information is fed to Filter 2 which is used in the state observer and the parameter update law through the auxiliary states. The signals coming from the disturbance observer, the update law and the state observer compose our adaptive controller.

We now state the main theorem and then prove it in the next section.

Theorem 3.8. *Consider the closed-loop system consisting of the plant* (3.1), (3.2), *the unknown disturbance* (3.4), *the filters* (3.9)–(3.11), (3.49), (3.50), *the control law* (3.68)–(3.74) *and the update law* (3.75)–(3.77). *Under Assumptions 3.1–3.2, the following holds*

(a) For any $\alpha > 0$, $\Upsilon(t) \le e^{-\alpha t}\Upsilon(0) \quad \forall t \ge 0$ *where*

$$\begin{aligned}\Upsilon(t) =& \left|\tilde{X}_s(t) - \lambda_0(t)\tilde{\theta}_1(t)\right|^2 + \int_0^{D_y} (\tilde{y}_s(x,t) - Ce^{(x-D_y)A}\tilde{X}_s(t) - \lambda_1(x,t)\tilde{\theta}_1(t))^2 dx \\ &+ \left|\tilde{y}_s(0,t) - Ce^{-AD_y}\tilde{X}_s(t) - \lambda_1(0,t)\tilde{\theta}_1(t)\right|^2 + \left|\Lambda^T(t)\tilde{\theta}_1(t)\right|^2 + \left|\epsilon_\nu(t)\right|^2 \\ &+ \left|\tilde{X}_d^{in}(t)\right|^2,\end{aligned}$$

(b) $\left|\hat{X}_s(t) - X(t)\right|^2$, $\left|X(t)\right|^2$ *and* $\left|\hat{y}_s(x,t) - y(x,t)\right|^2 \to 0$ *as* $t \to \infty$. *Moreover, we achieve perfect estimation of the unknown disturbance, i.e.* $\hat{\theta}_1^T(t - D_u)\Xi(t - D_u) - \nu(t) \to 0$ *as* $t \to \infty$.

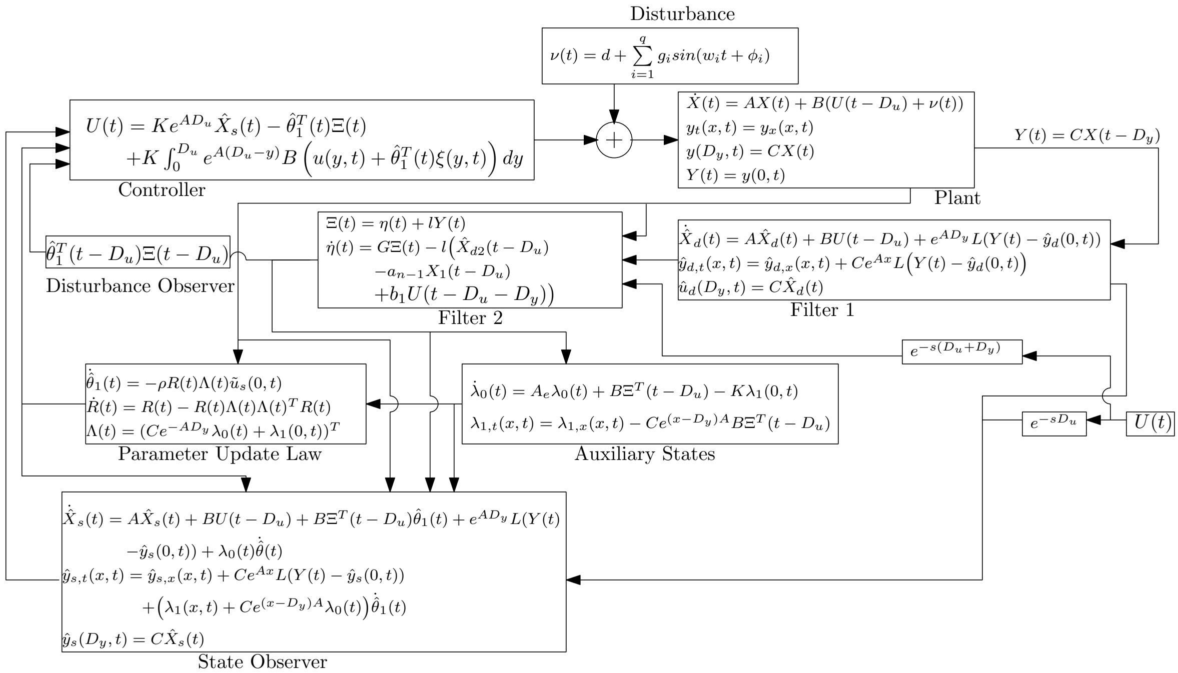

Figure 3.1. Observer based adaptive controller design scheme.

3.5. Stability Proof

The state observer error system is given by

$$\begin{aligned}\dot{\tilde{X}}_s(t) =& A\tilde{X}_s(t) + B\Xi^T(t - D_u)\tilde{\theta}_1(t) + e^{AD_y}L(Y(t) - \hat{y}_s(0,t)) \\ &+ \lambda_0(t)\dot{\hat{\theta}}_1(t) - \epsilon_\theta(t), \qquad (3.78)\\ \partial_t\tilde{y}_s(x,t) =& \partial_x\tilde{y}_s(x,t) + Ce^{Ax}L(Y(t) - \hat{y}_s(0,t)) \\ &+ \left(\lambda_1(x,t) + Ce^{(x-D_y)A}\lambda_0(t)\right)\dot{\hat{\theta}}_1(t), \qquad (3.79)\\ \tilde{y}_s(D_y,t) =& C\tilde{X}_s(t). \qquad (3.80)\end{aligned}$$

where $\epsilon_\theta(t) = B\theta_2^T\epsilon_\nu(t) + B\theta_3^T\tilde{X}_d^{in}(t)$. Consider the following backstepping transformation proposed in [32]

$$\Phi(t) = \tilde{X}_s(t) - \lambda_0(t)\tilde{\theta}_1(t), \qquad (3.81)$$

$$\varepsilon(x,t) = \tilde{y}_s(x,t) - Ce^{(x-D_y)A}\tilde{X}_s(t) - \lambda_1(x,t)\tilde{\theta}_1(t) \qquad (3.82)$$

This transformation yields a closed-loop system in the following form

$$\dot{\Phi}(t) = A_e\Phi(t) - e^{AD_y}L\varepsilon(0,t) - \epsilon_\theta(t), \tag{3.83}$$

$$\partial_t\varepsilon(x,t) = \partial_x\varepsilon(x,t) + Ce^{(x-D_y)A}\epsilon_\theta(t), \tag{3.84}$$

$$\varepsilon(D,t) = 0. \tag{3.85}$$

Using the backstepping transformations (3.81) and (3.82), recalling $\tilde{\theta} = \hat{\theta} - \theta$ and noting that $\dot{\hat{\theta}} = \dot{\tilde{\theta}}$, the update law (3.75) is rewritten in terms of the signals $\varepsilon(0,t)$, $\Phi(t)$ and $\tilde{\theta}(t)$,

$$\dot{\tilde{\theta}}(t) = -\rho R(t)\Lambda(t)\Lambda^T(t)\tilde{\theta}(t) - \rho R(t)\Lambda(t)\varepsilon(0,t) - \rho R(t)\Lambda(t)Ce^{-AD}\Phi(t). \tag{3.86}$$

In order to ensure the exponential convergence of the state observer, we need to establish persistent excitation of signal $\Lambda(t)$. The following lemma states this property.

Lemma 3.9. *The signal vector $\Lambda(t)$ is persistently exciting (PE) and there exist $\delta_0, \delta_1, T_0 > 0$ such that the following holds*

$$\delta_1 I \geq \int_t^{t+T_0} \Lambda(\tau)\Lambda^T(\tau)d\tau \geq \delta_0 I \quad \forall t \geq 0. \tag{3.87}$$

Proof. Considering Assumptions 3.1–3.2, (3.17), (3.20) and noting that A_e is Hurwitz, we can state that the signal $\tilde{X}_d(t)$ is sufficiently rich in the order of $2q+1$. From (3.33), it follows that $\epsilon_x(t)$ is also sufficiently rich in the order of $2q+1$. Considering this fact and controllable pair (G,l) and noting that $Z(t) \in \mathbb{R}^{2q+1}$, we can show that $Z(t)$ is PE signal from (3.46). This implies that $\Xi(t)$ satisfies the condition of persistent excitation from (3.48) in view of (3.53). In Appendix A, we give the solution of $\lambda_1(x,t)$ and $\lambda_0(t)$. Since $\lambda_1(0,t)$ and $\lambda_0(t)$ are dependent on $\Xi(t)$ signal as seen in (A.1)–(A.2), we ensure that $\lambda_1(0,t)$ and $\lambda_0(t)$ are bounded and sufficiently rich in the order of $2q+1$. As seen in (3.77), the signal $\Lambda(t)$ consists of two bounded PE signals $C^{-AD_y}\lambda_0(t), \lambda_1(0,t)$. Therefore, the signal $\Lambda(t)$ only depends on the signal $\Xi(t)$ and satisfies (3.87). □

As stated in [39], if persistent excitation condition is satisfied, then covariance matrix inverse $R^{-1}(t)$ is a bounded positive definite matrix. Its dynamics is given by

$$\frac{dR^{-1}}{dt} = -R^{-1} + \Lambda(t)\Lambda(t)^T. \tag{3.88}$$

This guarantees that $\rho_l \leq R^{-1}(t) \leq \rho_u$, $\forall t \geq 0$ for lower and upper bounds $\rho_l, \rho_u > 0$. The proof of Theorem 3.8 is given as follows.

Proof. Consider the following Lyapunov function

$$\begin{aligned} V =& \mu_1 \Phi^T P \Phi + \mu_2 \int_0^{D_y} (1+x)\varepsilon(x,t)^2 dx \\ &+ \mu_3 \tilde{\theta}_1^T R^{-1} \tilde{\theta}_1 + q_1 \epsilon_v^T P_G \epsilon_v + q_2 (\tilde{X}_d^{in})^T P \tilde{X}_d^{in}, \end{aligned} \tag{3.89}$$

where

$$\mu_3 = \frac{\beta_3}{\rho(2-\zeta_1)-1} \tag{3.90}$$

$$\mu_1 = (\beta_1 + \frac{\mu_3 \rho}{2\zeta_1}(Ce^{-AD_y}e^{-(AD_y)^T}C^T)(\mu_P - 3)^{-1} \tag{3.91}$$

$$\mu_2 = \mu_1 \lambda_{max}(Pe^{AD_y}LL^T(e^{AD_y})^T P) + \frac{\mu_3 \rho}{2\zeta_1} + \beta_2 \tag{3.92}$$

$$q_1 = (\beta_4 + \mu_1 \lambda_{max}(PB\theta_2^T\theta_2 B^T P) + \frac{\mu_2}{2\zeta_2}\lambda_{max}(\theta_2\theta_2^T))(\mu_G - 1)^{-1} \tag{3.93}$$

$$\begin{aligned} q_2 =& (q_1 \lambda_{max}\big(P_G l B_2^T e^{-A_e D_y} e^{-(A_e D_y)^T} B_2 l^T P_G) \\ &+ \mu_1 \lambda_{max}(PB\theta_3^T\theta_3 B^T P) + \frac{\mu_2}{2\zeta_2}\lambda_{max}(\theta_3\theta_3^T) + \beta_5\big)(\mu_P)^{-1}, \end{aligned} \tag{3.94}$$

with

$$PA_e + A_e^T P = -\mu_P I, \tag{3.95}$$

$$G^T P_G + P_G G = -\mu_G I \tag{3.96}$$

$$\zeta_2 = (1-\beta_6)/(\int_0^{D_y} (1+x)Ce^{(x-D_y)A}Bdx)^2 \tag{3.97}$$

for parameters β_1, β_2, β_3, β_4, β_5 and $\zeta_1 > 0$. Taking the time derivative of (3.89) by virtue of (3.42), (3.53), (3.83)–(3.86), (3.88), (3.90)–(3.94) and using the Young's inequality for the cross terms, we obtain

$$\begin{aligned}\dot{V}(t) \leq & -\beta_1\Phi^T\Phi - \mu_3\tilde{\theta}_1^T R^{-1}\tilde{\theta}_1 - \beta_2\varepsilon^2(0,t) - \beta_3\left|\Lambda^T\tilde{\theta}_1\right|^2 - \beta_4\epsilon_\nu^T\epsilon_\nu - \beta_5(\tilde{X}_d^{in})^T\tilde{X}_d^{in} \\ & - \frac{\beta_6}{1+D_y}\int_0^{D_y}(1+x)\varepsilon^2(x,t)dx. \end{aligned} \tag{3.98}$$

Considering (3.89) and noting that $|\Lambda|$ is a bounded and PE signal from Lemma 3.9, (3.98) can be written as

$$\dot{V}(t) \leq -\alpha V(t) \tag{3.99}$$

for $\alpha > 0$.

Recalling (3.81), (3.82), from (3.99), we prove part (a) of Theorem 3.8. Recalling the boundedness of $\nu(t)$ and $\epsilon_\nu(t)$, from (3.55), $\Xi(t)$ is bounded. Recalling that $\lambda_0(t)$ is bounded and $\tilde{\theta}_1(t), \Phi(t) \to 0$ as $t \to \infty$, from (3.81), we can show that $\tilde{X}_s(t)$ is bounded and goes to zero as $t \to \infty$. In view of boundedness of $\lambda_1(0,t)$, this fact implies that, from (3.82), $\tilde{y}_s(x,t)$ is bounded and goes to zero as $t \to \infty$. We obtain the boundedness of $U(t)$ and $\dot{\hat{\theta}}_1(t)$ from (3.68) and (3.75)–(3.77). Note that input (3.68) transforms system (3.63) to

$$\dot{X}(t) = (A+BK)X(t) + B\big(K\tilde{X}_s(t) - \tilde{\theta}_1^T(t)\xi(0,t) + w_u(0,t) + \theta_2^T\epsilon_\nu(t) + \theta_3^T\tilde{X}_d^{in}(t)\big), \tag{3.100}$$

where

$$\begin{aligned} w_u(x,t) =& u(x,t) + \hat{\theta}_1(t)\xi(x,t) - Ke^{Ax}\hat{X}_s(t) \\ & - K\int_0^x e^{A(x-y)}B\left(u(y,t) + \hat{\theta}_1^T(t)\xi(y,t)\right)dy \end{aligned} \tag{3.101}$$

which satisfies

$$\partial_t w_u(x,t) = \partial_x w_u(x,t) + \dot{\hat{\theta}}_1^T(t)\Big(\xi(x,t) - \int_0^x \xi(y,t)Ke^{A(x-y)}Bdy\Big) + Ke^{A(x-D_y)}L\tilde{y}_s(0,t)$$
$$- Ke^{Ax}\lambda_0(t)\dot{\hat{\theta}}_1(t), \tag{3.102}$$
$$w_u(D_u,t) = 0. \tag{3.103}$$

The solution of (3.102)–(3.103) is given by

$$w_u(x,t) = \int_x^{D_u}\Big[\dot{\hat{\theta}}_1(t+x-v)\Big(\Xi(t+x-D_u) - \int_0^v \Xi(y+t-D_u)K^{A(v-y)}Bdy\Big)$$
$$+ Ke^{A(v-D_u)}L\tilde{y}_s(0,t+x-v) - Ke^{Av}\lambda_0(t+x-v)\dot{\hat{\theta}}_1(t+x-v)\Big]dv. \tag{3.104}$$

From (3.104), we prove that $w_u(0,t)$ is bounded and converges to zero as $t \to \infty$ since $\dot{\hat{\theta}}_1^T(t), \Xi(t), \tilde{y}_s(0,t), \lambda_0(t)$ are bounded and $\dot{\hat{\theta}}_1^T(t), \tilde{y}_s(0,t) \to 0$ as $t \to \infty$. Considering this fact and noting that $(A+BK)$ is Hurwitz, from (3.100), we obtain that $X(t)$ is bounded and goes to zero as $t \to \infty$. Moreover, noting that $\tilde{\theta}_1(t) \to 0$ as $t \to \infty$, from Lemma 3.6, we obtain that $\hat{\theta}_1^T(t-D_u)\Xi(t-D_u) - \nu(t) \to 0$ exponentially, i.e. we achieve perfect estimation of the unknown disturbance. This proves part (b) of Theorem 3.8. □

3.6. Numerical Simulations

We test the performance of the controller with an unstable second-order system with $A = \begin{bmatrix} 0.3 & 1 \\ 0.2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, the unknown disturbance $\nu(t) = 4 + 8\sin(0.5t + \pi/6) + 18\sin(2.5t + \pi/5)$, the known input delay $D_u = 0.6$, the known output delay $D_y = 0.7$ and the initial conditions $X(0) = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$. We choose ρ as 1.5 and eigenvalues of $A - LC$ as -2 and -2.5. We choose the controllable pair (G, l) as $l = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$, $G = \begin{bmatrix} 0_4 & I_4 \\ 0_5^T & \end{bmatrix} + l\begin{bmatrix} -19.96 & -54.85 & -60.28 & -33.12 & -9.10 \end{bmatrix}$.

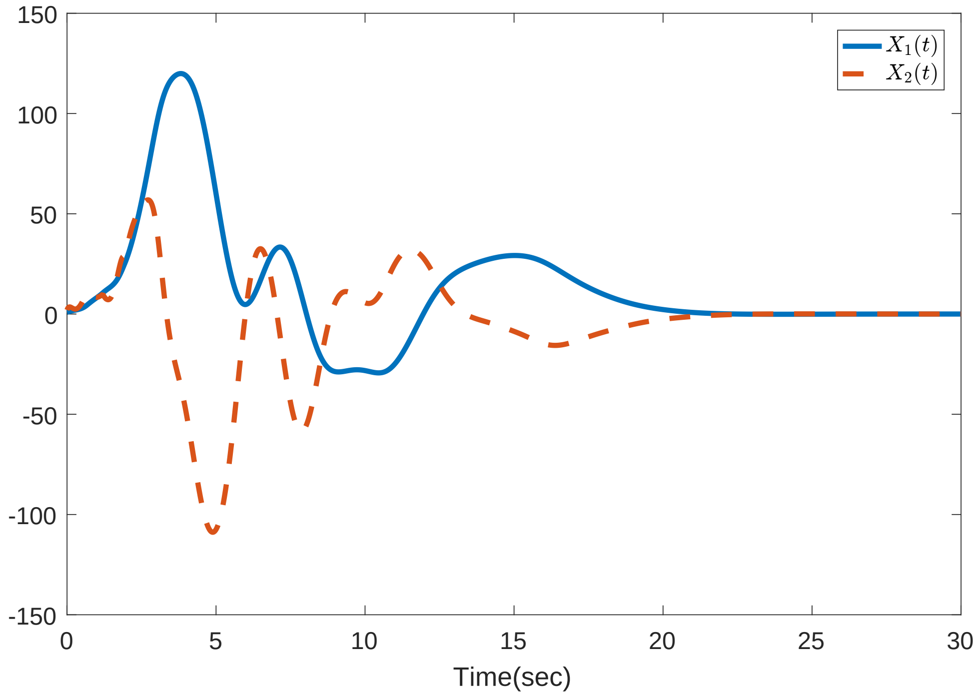

Figure 3.2. Performance of the controller for an unstable plant with 0.6 second input and 0.7 second output delay.

The control gain K is chosen such that the eigenvalues of $A + BK$ are -1.5 and -2. Figure 3.2 and 3.3 show that $X(t)$ and $\hat{\theta}_1(t - D_u)\Xi(t - D_u) - \nu(t)$ converge to zero as stated in Theorem 3.8.

3.7. Conclusion

In this chapter, we solve the problem of unknown sinusoidal disturbance rejection for LTI systems in the presence of simultaneous input-output delay. We parametrize the disturbance as a multiplication of unknown constant vector with a known regressor and exponentially vanishing terms so that we approach the problem as an adaptive control problem. Following the idea in [26], we represent the delays as a transport PDE. We utilize the adaptive observer given in [32], which is a backstepping like design technique for an ODE-PDE cascade. Combining the parametrization and observers, we propose an adaptive controller. We prove that the equilibrium of the adaptive closed-loop system is exponentially stable. We perform a simulation to demonstrate effects of the adaptive observer and controller designs.

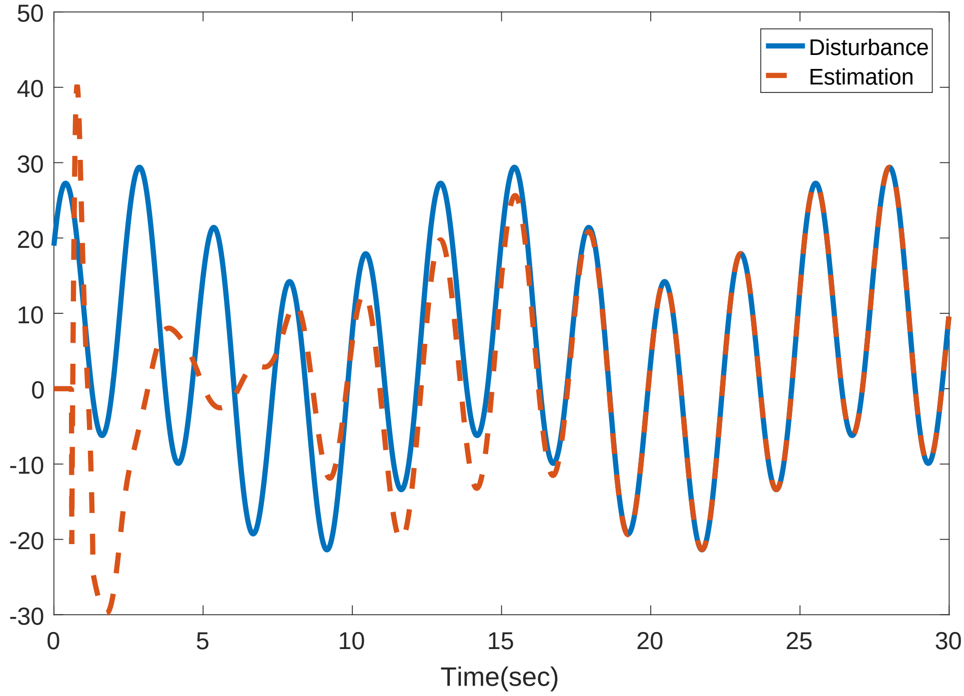

Figure 3.3. The disturbance and its estimation

4. CONCLUSION

This dissertation develops an adaptive output feedback algorithm for two different problems. The first problem is unknown sinusoidal disturbance rejection for unknown and minimum phase LTI systems with the output feedback. For this problem, we firstly employ K-filter technique to redefine the states of the plant. Then, we parametrize the disturbance information in the output signal as a multiplication of unknown constant vectors with known regressors and an exponentially decaying term so that we approach the problem as an adaptive control problem. Following the idea in [36], we propose an adaptive controller which achieves trajectory tracking despite unknown disturbances. Moreover, we prove that all the signals of the closed-loop system are globally bounded, and the asymptotic tracking is achieved. Furthermore, we show that the equilibrium at the origin is globally uniformly stable. We perform a simulation with a third-order unstable system, whose relative degree is two, to demonstrate the effects of the adaptive controller. By incorporating a σ-modification in the update laws, we also show the robustness of the system with respect to an additive unmodelled noise.

The second problem is unknown sinusoidal disturbance rejection for LTI systems in the presence of input/output delay. For this problem, we again parametrize the disturbance. Following the idea in [26], we represent the delays as a transport PDE. We utilize the adaptive observer given in [32], which is a backstepping like design technique for an ODE-PDE cascade. Combining the parametrization and observers, we propose an adaptive controller. We prove that the equilibrium of the adaptive closed-loop system is exponentially stable. We perform a simulation to demonstrate effects of the adaptive observer and controller designs.

5. LIMITATIONS AND FUTURE WORK

In this study, it is theoretically proven that the adaptive controllers achieve the closed loop stability despite unknown system parameters, time delays and sinusoidal disturbances. However, it is also necessary to implement these controllers on real dynamical systems, which operate under a variety of inherent constraints and limitations. One of the physical limitations is input saturation. The dynamical systems have hard limit constraints on the amplitude and changing rate of control inputs. To make the designed controller more realistic, an upper and lower saturation values may be used for the boundedness of the controller effort.

Second limitation is the computational time needed to compute the control law. In the second problem, the controller law is infinite-dimensional, because it includes the distributed delay term involving past controls. When the time delay is increased, the computation of this term becomes more difficult task and requires a significant amount of computational effort. Therefore, an upper bound for time delays may be determined to keep the computational effort within desired limits.

Third limitation is the convergence rate of the closed loop system. In this study, it is proven that system states or output signal converge to zero or track a given trajectory as time goes to infinity. However, there is strict limits for the convergence rate in the practical engineering applications. The some parameters such as initial states of the systems, the chosen update gains and the chosen filter matrices greatly effect the convergence time of the system. Therefore, a desired convergent rate can be achieved with a new controller algorithm which chooses optimum parameters and works large initial conditions.

This study can be extended in several directions. For the each problem discussed in the study, the possible improvements are presented as follows.

- In the first problem, the minimum phase assumption for the plant can be removed to obtain more general solution.
- In the second problem, it is assumed that the time-delays are known. However, in the most of dynamical systems, these delays vary over time. For that, the robustness of the closed-loop stability to a small mismatch in the delays can be discussed. A bound can be determined for the stability of the closed-loop system.
- The methods used for first and second problems can be combined to design an output-feedback adaptive controller for unknown LTI systems, which rejects the sinusoidal disturbances, compensates the simultaneous input-output delay and achieves the asymptotic tracking of the reference signal by the output signal.

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APPENDIX A: SOLUTIONS OF $\lambda_0(t)$ AND $\lambda_1(x,t)$

To implement the designed observer, the solutions of $\lambda_0(t)$ and $\lambda_1(x,t)$ need to be given. In this regard, we solve (3.73) using Laplace Transform with the initial conditions (3.74) as follows

$$\lambda_1(x,t) = -\int_x^D Ce^{(\sigma-D)A}B\Xi^T(t+x-\sigma)d\sigma. \tag{A.1}$$

Using state transition matrix, the solution of (3.72) with the initial condition $\lambda_0(0) = 0$ is given by

$$\lambda_0(t) = \int_0^t e^{A_{aug}(t-\tau)}\Big(B\Xi^T(\tau) - e^{AD}L\lambda_1(0,\tau)\Big)d\tau. \tag{A.2}$$

Recalling that $\Xi(t)$ is bounded, we guarantee that, from (A.1), $\lambda_1(0,t)$ is bounded as well. This implies the boundedness of $\lambda_0(t)$ from (A.2).