INVESTIGATION
OF
OPTIMAL
CAM PROFILES
BY
DYNAMIC OPTIMIZATION

by
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ABSTRACT

In this study an investigation to find the optimal profiles for automotive cam systems has been made.The method of dynamic optimization is utilized in this objective.To formulate the problem various factors that affect the profile have been considered,namely the lift, velocity,acceleration and jerk of valve train,stresses in the system,radius of curvature of cam,lubrication between cam and tappet,inlet air quantity and other properties of valve trains.Upon determination of the constraints and the boundary conditions for the problem the solution outputs are found by numerical method.The solution of the problem by dynamic optimization technique has shown that the optimal curves are found by polynomial approximations which come out to be a better solution for the profile than any other method.After tabulation and interpretations of results recommendations are given for further investigations.

ÖZET

Bu çalışmada otomotiv kam sistemlerinde en uygun profilin bulunması için bir inceleme yapılmıştır. Bu amaçla dinamik optimizasyon metodu kullanılmıştır. Problemi formüle edebilmek için profili etkileyen çeşitli faktörler gözönüne alınmıştır; örneğin, süpapın yüksekliği hızı, ivmesi ve sıçraması, sistemdeki kuvvetler, kamın eğim yarıçapı, kam ve tapet arasındaki yağlama, emilen hava miktarı ve süpap sistemlerinin diger özellikleri gibi. Problem için sınırlamaların ve sınır şartlarının belirlenmesinden sonra çözüm nümerik metodlarla bulunmuştur. Problemin dinamik optimizasyon metodu ile çözümü bize kamlar için en uygun olan eğrilerin polinom yaklaşımları ile tarif edildiğini ve polinom metodunun diğer bütün metodlara göre daha iyi bir çözüm yolu olduğunu göstermiştir. Sonuçların listelenmesi ve irdelenmesini takiben ileri araştırmalar için tavsiyelerde bulunulmuştur.

CONTENTS

NOMENCLATURE

a - acceleration
A - matrix defining the coefficients of state vector x
A_p - piston area
A_v - valve opening area
B - matrix defining the coefficients of control vector u
b_f - coefficient of friction
c - distance of center of curvature to center of cam
D - valve head diameter
D_p - piston diameter
D_{av} - average diameter of valve opening
e - eccentricity of contact point to centerline of tappet
E - modulus of elasticity
F - matrix defining the coefficients of the differential system
F_o - force of spring at zero valve lift
F_f - frictional force
F_i - inertia force
F_s - spring force
F_t - total force on cam
h - valve lift
$\dot{h}$ - valve velocity
$\ddot{h}$ - valve acceleration
H - Hamiltonian function
$\bar{h}$ - vector defining the variation in vector x
I - identity matrix
J - functional defining the parameters to be optimized
k - valve spring constant
l - connecting rod length
m - total mass of reciprocating parts
m_1 - reciprocating mass of valve spring
m_2 - reciprocating mass of valve train except valve spring
N - system of equations defining the terminal manifold equality constraints

n - dimension of state vector x
p - Hertz pressure on cam
R - radius of curvature of flank for convex flank cam
" - matrix defining the coefficients of control vector u
R_o - base circle radius of cam
R_r - radius of roller follower
R_1 - distance between center of cam and contact point
R_2 - distance between center of cam and center of curvature
r - crank throw
q_n - scalar coefficients for elements of vector x
Q - matrix defining the coefficients of vector x
Q_v - volume flow rate of air into the cylinder
S - lubrication number
t - time
t_o - initial time
t_f - final time
u - control vector
" - distance from centro to cam center of curvature
v - velocity
$\bar{v}$ - effective translatory velocity
v_B - absolute velocity of the point of contact
v_N - absolute velocity of the sliding surface of cam
v_p - piston velocity
v_T - absolute velocity of sliding surface of tappet
V_v - velocity of air passing through valve opening
w - angular velocity of cam
$\bar{w}$ - hydrodynamic effective angular velocity
w_c - effective cam width
w_L - angular velocity of journal bearing
w_{sp} - angular velocity of contact point
w_z - angular velocity of journal shaft
W - volume of air taken into the cylinder
x - state variable vector
" - displacement of piston

α - weighting coefficient for final time t_f

Δ - system of equations defining equality constraints

ϵ - very small scalar quantity

η - variation of state variable vector x

λ - Lagrange multiplier vector

ν - Lagrange multiplier vector

Φ - matrix defining the coefficients of the solution system of equations

ϕ - integrand function of J

θ - function in t_f to be minimized

, - camshaft or crankshaft angle

φ - angle between connecting rod axis and cylinder bore axis

ρ - instanataneous radius of curvature of cam

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1.INTRODUCTION

When designing an engine there are various factors to be considered which influence the overall efficiencies of the engine.One of the most importants of these factors is the valve timing of the engine,i.e.the cam profile.

The opening and closing of valves are controlled by a cam operation.The cams on the camshaft operate through flat or roller tappets,push rods and rocker arms on the valves.With an overhead camshaft the cams operate either directly on the valve foot or on a finger in between.By the pretension on the valve spring the valves are pressed onto their seats in the closed state.And also the valve spring has the duty of controlling the valves to follow the lift curve given by the cam profile,even when the acceleration forces have the effect of deviating the valve from the appropriate path of the lift curve.It is the duty of cam calculations to find out the tappet acceleration, velocity and lift as a function of the crankshaft or camshaft angle for the given values of maximum cam lift and valve timing.

From the maximum negative tappet acceleration and the masses of the valve train one can obtain the maximum delay force considering the ratio of the rocker arm,after which the valve spring can be calculated.The maximum tappet velocity helps to determine the tappet diameter for flat tappets.The lift curve as a function of the cam angle gives the necessary data to complete the cams.

The purpose of this work is to investigate and find the optimum cam profiles and valve timing for a given valve train system.The necessary data of the system will be taken from the Otosan 1,9 lt. light diesel engine valve train.

It is known that nowadays it is the tendency to build light diesel engines of small displaced volume for passenger cars and light commercial vehicles because of economic reasons.And Otosan has intended to build a 1,9 lt. light Diesel engine,which is to be developed from the 2,0 lt. gasoline engine of Ford.Two prototypes are being built and a developement programm exists for 1982 and 1983.Hence the study is of great importance since it will also be a general study for light Diesel engine valve train system and the results will find the opportunity of being tested on prototype engines in the developement programm.

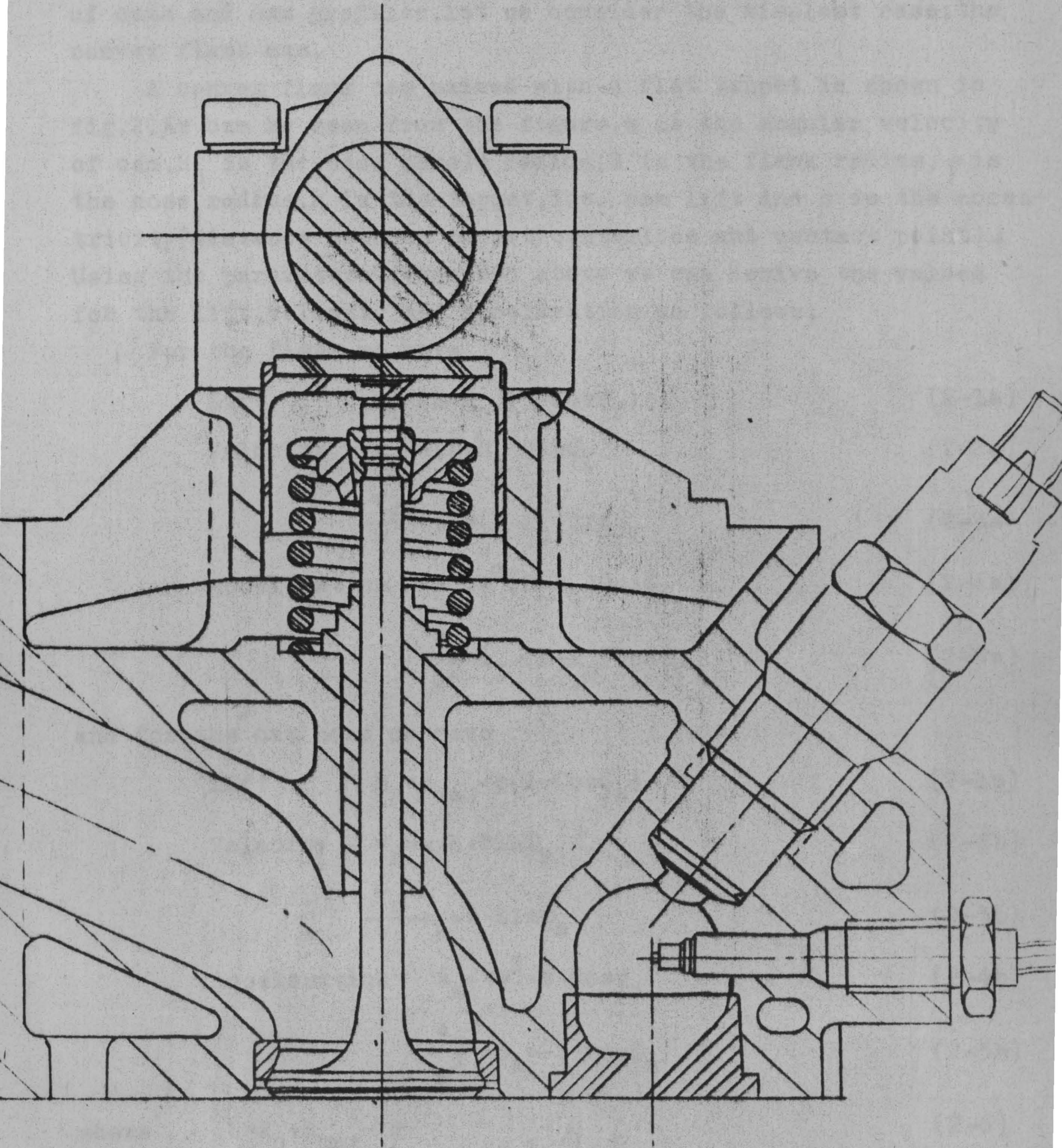


FIGURE.1.Section showing valve train

2. PROPERTIES OF CAMS AND VALVE TRAINS

2.1. THE CONVEX FLANK CAM

To investigate the factors that influence the determination of cams and cam profiles, let us consider the simplest case; the convex flank cam.

A convex flank cam paired with a flat tappet is shown in fig.2. As can be seen from the figure, w is the angular velocity of cam, R_0 is the base circle radius, R is the flank radius, ρ is the nose radius, h is the tappet, i.e. cam lift and e is the eccentricity (distance between tappet centerline and contact point). Using the parameters described above we can derive the values for the lift, velocity and acceleration as follows,

For the flank we have

Lift $h_f=(R-R_0)(1-\mathrm{Cos}\varphi_f)$ (2-1a)

Velocity $v_f=w(R-R_0)\mathrm{Sin}\varphi_f$ (2-2a)

$$\frac{v_f}{w}=e_f=(R-R_0)\mathrm{Sin}\varphi_f \qquad (2\text{-}3a)$$

Acceleration $a_f=w^2(R-R_0)\mathrm{Cos}\varphi_f$ (2-4a)

$$\frac{a_f}{w^2}=c_f=(R-R_0)\mathrm{Cos}\varphi_f \qquad (2\text{-}5a)$$

and for the cam nose we have

Lift $h_n=h_{max}-b(1-\mathrm{Cos}\varphi_n)$ (2-1b)

Velocity $v_n=w\cdot b\cdot \mathrm{Sin}\varphi_n$ (2-2b)

$$\frac{v_n}{w}=e_n=b\cdot \mathrm{Sin}\varphi_n \qquad (2\text{-}3b)$$

Acceleration $a_n=-w^2\cdot b\cdot \mathrm{Cos}\varphi_n$ (2-4b)

$$\frac{a_n}{w^2}=c_n=-b\cdot \mathrm{Cos}\varphi_n \qquad (2\text{-}5b)$$

where $b=R_0+h_{max}-\rho$ (2-6)

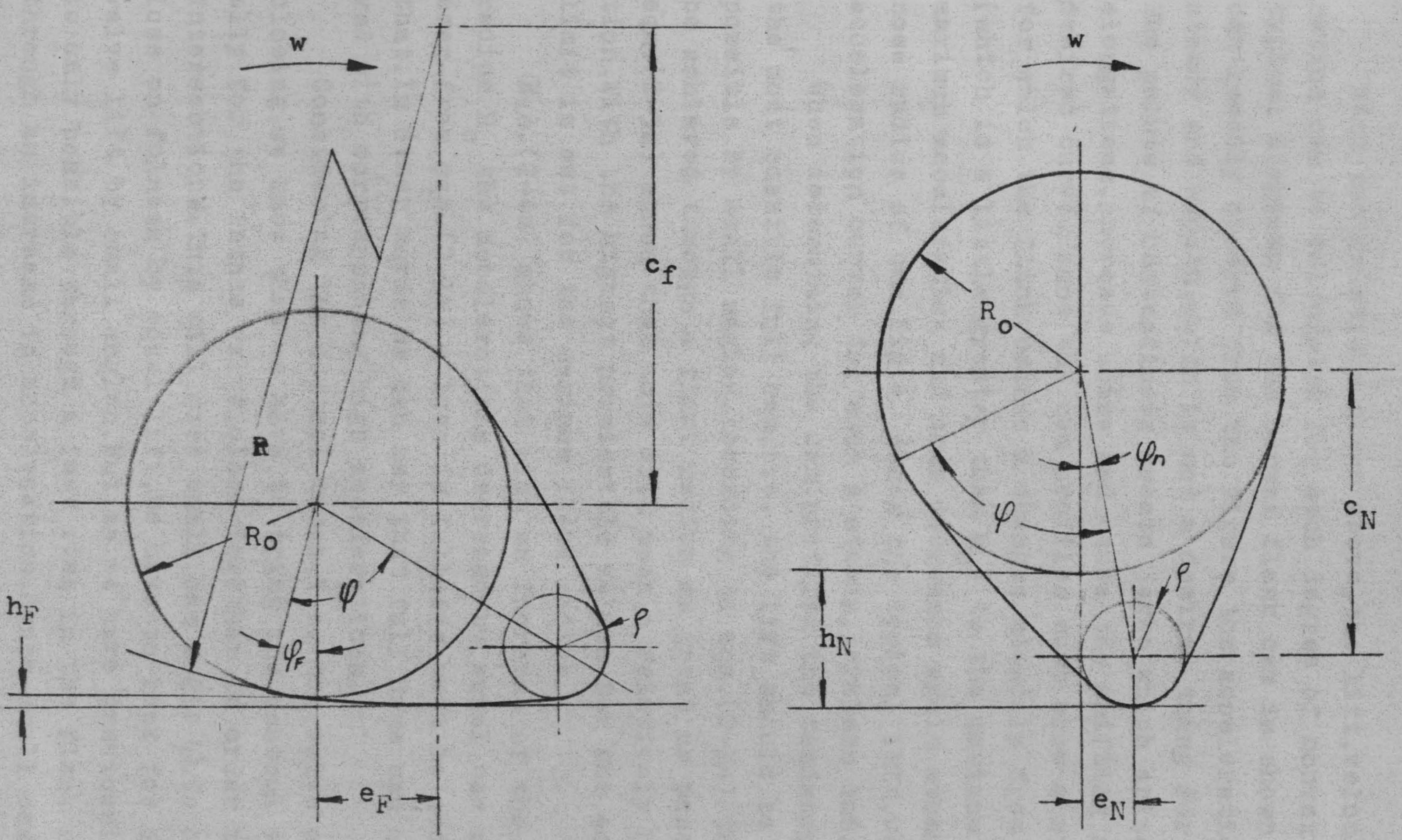


FIGURE.2.Convex flank cam with flat follower

With the relations given above, the lift, velocity and acceleration can be calculated for each degree of convex flank cam. A typical diagramm for the convex flank cam is shown in fig.3. As we can readily observe from the figure the acceleration curve is not steady and smooth, which is not a desired thing for cam profiles. The points of unsteadiness create jerks which arise great elastic elongations, increase noise and cause the shifting away from the desired curve. Hence the cam profiles must show a steady character for which the flank radius R changes steadily from a given value (which is a little greater than R_0) to the maximum value by the maximum acceleration and then decreases again steadily till the nose radius of cam. Fig.4. shows the typical lift, velocity and acceleration curves for such a steady, jerkless cam.

When determining the cam profiles the tendency is to find the most possible full cam, i.e. the lift should be as great as possible by small angles. According to eqn.(2-1a) this can only be achieved through a flank radius as great as possible. However eqn.(2-3a) shows that this will mean a relatively high acceleration. With the highest permissible values for the acceleration, a limit is set for the maximum flank radius.

Eqn.(2-4a) shows that with an increase of the base circle radius R_0 the accelerations decrease by equal cam angles , however from eqn.(2-1a) a loss in fullness must be accepted for that. In other words we can say that full cams can only be achieved with corresponding high accelerations.

Considering the control times, i.e. the valve opening and closing, we know that we have to bring the control times, especially for the intake valve, close together in order to avoid great intersections. This will give small cam angle φ. To compensate the loss on fulness by equal lift, we have to look for obtaining great valve lift by small angles. But, as we have previously stated, this is only possible through a fast rise in the flank radius R, i.e. through an increase in acceleration. Hence small cam angles will give high accelerations.

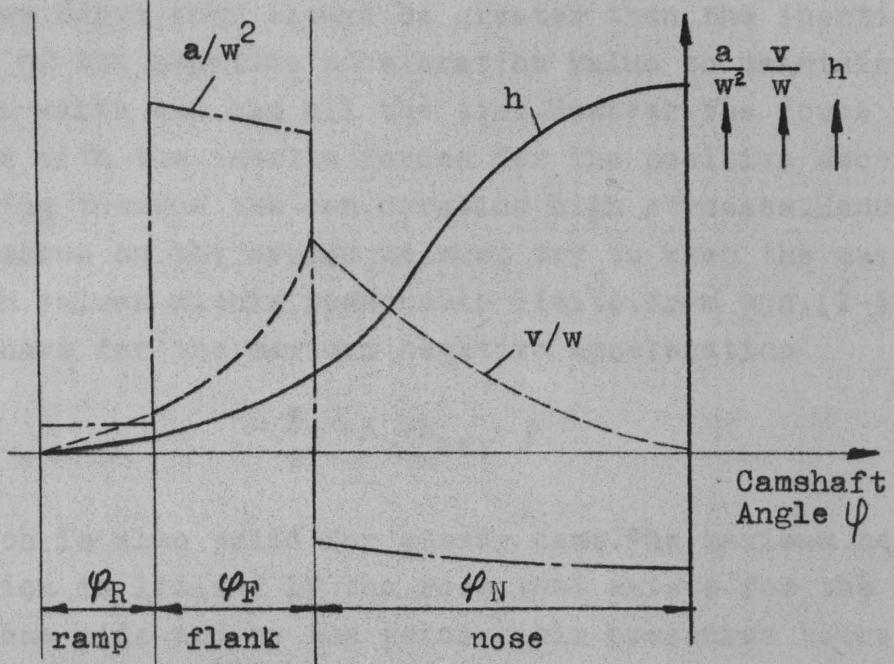


FIGURE.3.Lift,velocity and acceleration for a convex flank cam

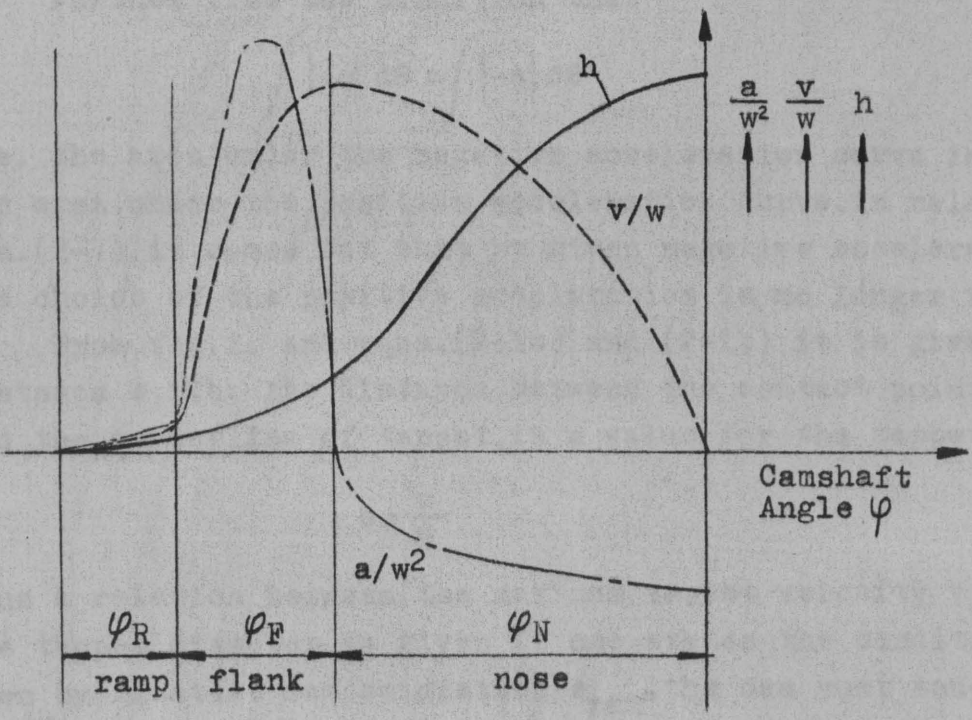


FIGURE.4.Lift,velocity and acceleration for a steady cam

We know that the negative accelerations, creating inertia forces acting away from the cam, tend to deviate the valve from its desired trajectory. this is overcome by the valve spring whose force must always be greater than the inertia force created by the negative acceleration value to maintain contact between valve and cam all the time. However the force in the spring adds with the inertia forces for the positive acceleration values acting towards the cam, creating high stresses. Hence to avoid high stresses on the system we must try to keep the maximum acceleration values within reasonable limits. From eqn.(2-5b) and (2-6) we have for the maximum negative acceleration

$$-\frac{a}{w^2} = R_o + h_{max} - \rho \qquad (2\text{-}7)$$

which is also valid for steady cams. The maximum negative acceleration is limited by the room that exists for the valve spring on one side and by the permissible pressures between cam nose and tappet on the other side. Also it can easily be seen from eqn.(2-7) that if the maximum negative acceleration and the cam nose radius are selected, choosing R_o and h_{max} are no longer free.

Further from the condition that

$$\int |+a|\, d\theta = \int |-a|\, d\theta \qquad (2\text{-}8)$$

i.e. the area under the negative acceleration curve is equal to the area under the positive acceleration curve, in relation with eqn.(2-7) it comes out that by given negative acceleration also the choice of the positive acceleration is no longer free.

From fig.2. and eqns.(2-1a) and (2-1b) it is given that the distance e, i.e. the distance between the contact point of cam and the centerline of tappet, is a value for the tappet velocity

$$e = \frac{v}{w}$$

Thus a relation between the maximum tappet velocity v_{max}/w and the tappet diameter is given if one states the condition that even by greatest cam emigration e_{max}, the cam must touch with its whole width onto the tappet. For the radius of tappet then,

$$r_{TAPPET} = \left[\left(\frac{V_{max}}{w}\right)^2 - \left(\frac{W_c}{2}\right)^2\right]^{1/2} \tag{2-9}$$

where w_c is the net width of the cam lobe. Hence this means that the tappet radius must be determined according to the maximum tappet velocity, or vice versa. Also if there exists an offset between the tappet centerline and cam centerline, this must be considered too.

One of the factors effecting the cam profile is the life of the surfaces in touch. The life of the cam and the tappet are affected by two factors mainly; The Hertz pressure between cam and tappet and the lubrication ratio between the two. Besides those the material pair play also an important role.

Leaving the investigation of Hertz pressures in some detail to later sections, let us note at this point that a low value of Hertz pressure is not sufficient between the cam nose and tappet but also the lubrication ratio between the two is important. Between the cam and flat tappet a relative oil film must be created like in a journal bearing. For the extent of this oil film the lubrication number S can be defined as follows

$$S = 2\rho - (R_0 + h) \tag{2-10}$$

If S becomes zero, the oil film between cam and tappet breaks down. Also when S is small for a long time, the lubrication is endangered and critical. To control the lubrication ratio for the cam system the lubrication number S must be derived and carried over the cam angle.

Surely it is inevitable of S becoming zero, however for a good outlayed cam system this passing through zero must be steep and should not last too long.

2.2.HERTZ PRESSURES

The valve lift curve applied on the valve by the cam must operate smoothly to obtain good results,i.e.the valve must not deviate from its given lift trajectory.This is only possible if the valve mechanism and the operating cam never lose contact during the operation of valve,from the begining of opening till the end of closing.However during the operation of valve due to the acceleration of reciprocating parts of the valve train an inertia force will act on the reciprating masses which will try to deviate the valve from its desired trajectory by causing it to lose contact.This is overcome by using a spring acting on the opposite direction of the acceleration and the force spring applies is always greater then the inertia force acting on the reciprocating parts.But for positive accelerations the inertia forces and the spring force add together to act on the cam.These forces create high stresses on the cam and tappet.

The Hertz pressure between the cam and the tappet can be calculated through the relation

$$p = 0.418 \left[\frac{F_T \cdot E}{\rho \cdot w_c} \right]^{1/2} \qquad (2\text{-}11)$$

where F_T is the force between tappet and cam,E is the modulus of elasticity,ρ is the instantaneous radius of curvature of cam at the point of interest,w_c is the length of line in contact,i.e. the effective cam width.

The forca F_T is the sum of the spring force,the frictional force and the inertia force of the reciprocating parts of the valve train,as we have stated previously.Although the force F_T during the positive acceleration,for which the spring and inertia forces are added,is the greatest;the cam nose on which F_T is the difference between spring and inertia forces is the critical point,for which the Hertz pressure must be calculated,because at this point the instantaneous radius of curvature of cam is the smallest.Since at this point $F_T = F_s - F_i$,the calculation must be

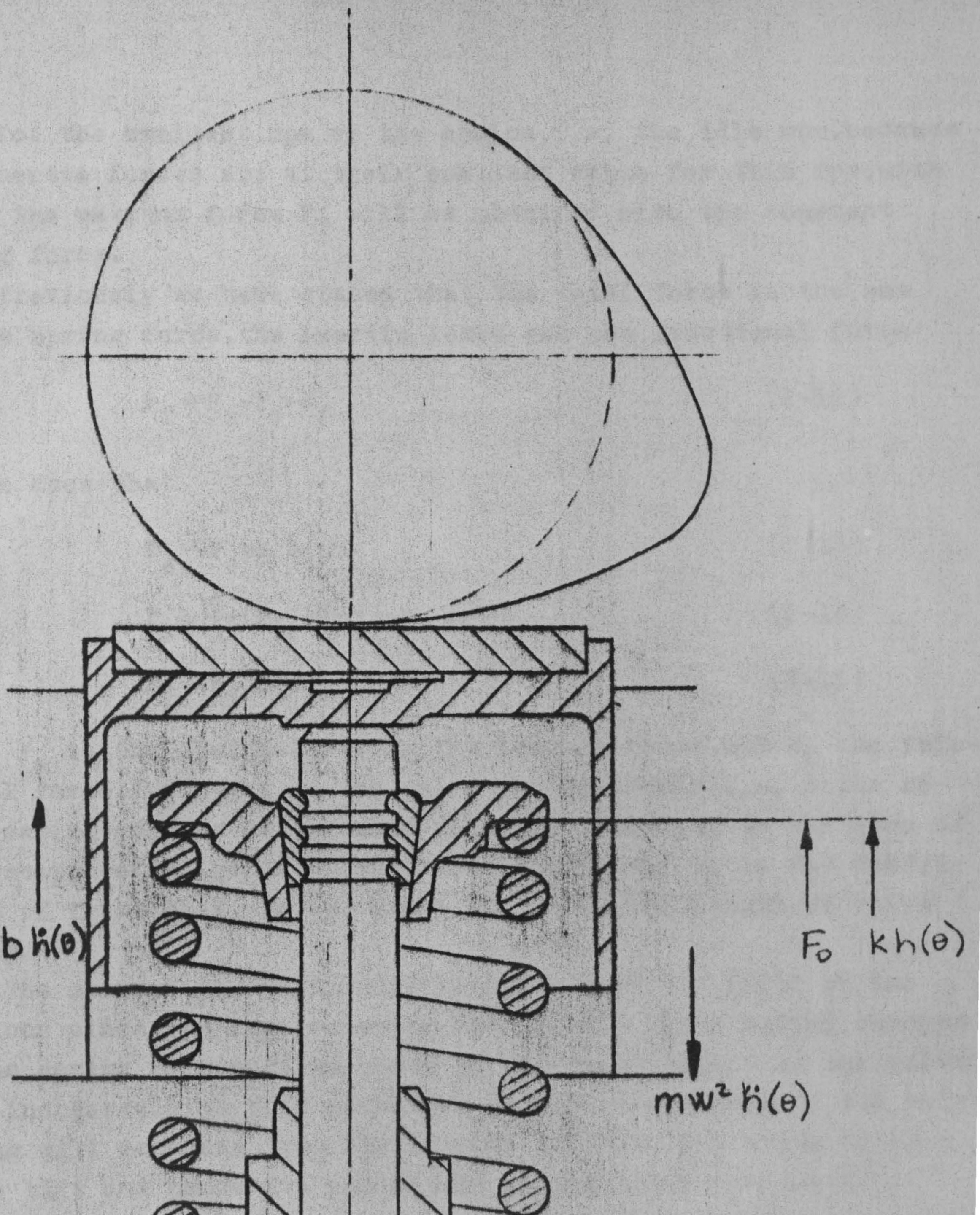


FIGURE.5.Forces acting on valvetrain

made for the smallest rpm of the engine, i.e. the idle rpm, because the inertia forces are at their smallest value for this rpm, with which the maximum force F_T will be obtained with the constant spring force.

Previously we have stated that the total force is the sum of the spring force, the inertia force and the frictional force

$$F_T = F_s - F_i - F_f \qquad (2\text{-}12)$$

and we know that

$$F_s = F_0 + k\ h(\theta) \qquad (2\text{-}13)$$

$$F_i = \frac{d}{dt}(m_1 \cdot \dot{h}(\theta)) - m_2 \cdot \ddot{h}(\theta) \qquad (2\text{-}14)$$

$$F_f = b_f \cdot \dot{h}(0) \qquad (2\text{-}15)$$

where F_s is the spring force, F_i the inertia force, and F_f the frictional force. Further k is the valve spring constant, m_1 is the reciprocating portion of the mass of valve spring, m_2 is the mass of the reciprocating parts except valve spring, and b_f is the coefficient of friction. F_0 is the force at the fitted length of valve spring.

The mass of the reciprocating parts are considered in two portions since the reciprocating mass of the valve spring changes as the spring is compressed near to its solid height as the valve lift increases. With this effective reciprocating mass of the valv spring will decrease from its initial value at the state of zero valve lift and hence its percentage on the total inertia force will also go down. However we know that, practically the usual case is that only one third of the total mass of the valve spring reciprocates with the whole system, and this rate does not change very significantly as the spring is compressed. Furthermore the percentage effect of this rate of change of mass is more negligible if the comperatively higher mass values of the valve, tappet retainers, etc. are considered. Also on the other hand, by assuming that the reciprocating part of the mass of the valve spring is always equal to the same amount of one-third of total spring mass throughout the whole opening and closing event,we shall be on the safe side with the stress calculations,which is the mostly utilized procedure.Hence we can simply assume m_1 is also constant like m_2 and define the total reciprocating mass m of valve train as

$$m=m_1+m_2 \tag{2-16}$$

and

$$F_i=m\cdot\ddot{h}(\theta) \tag{2-17}$$

Further more also assuming that we have good lubrication on the system and that the friction force is of negligible amount we obtain for total force

$$F_T=F_o+k\cdot h(\theta)-m\cdot\ddot{h}(\theta) \tag{2-18}$$

For determined k and m values the force will be a function of lift and acceleration.

$$F_T=f\left[h(\theta),\ddot{h}(\theta)\right] \tag{2-19}$$

Investigation of eqn.(2-11) shows that,what we can do to obtain low Hertz pressures,is either to decrease F_T,increase w_c or increase ρ,for given modulus of elasticity,E.

We have seen previously that the maximum available effective cam width,w_c,is limited with the maximum available tappet diameter and maximum tappet velocity.

Decreasing F_T means,from eqn.(2-18),decreasing $h(\theta)$ or $\ddot{h}(\theta)$ or both if possible.However for the desired valve lift this means minimizing the negative acceleration value.

Finally we can increase the instantaneous radius of curvature of cam nose which is,from eqn.(2-7),identical with decreasing the maximum negative acceleration value,as before.

The value of jerk,the rate of change of acceleration,plays a important role on the stress values.The rate of change of force F_T, $F_T/$ t,is called the 'impulse' and is equal to,for constant mass,the product of mass and jerk.High values of jerk mean sudden changes in acceleration values which will cause impact effect for extreme cases,creating high inertia forces.Hence,especially for the negative region of the acceleration curve,the magnitude of jerk must be retained within reasonable limits.

2.3.CAM CURVATURE

For a cam the radius of curvature of the pitch curve is in general given in polar coordinates as follows

$$\rho = \frac{\left[r^2 - \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\,\frac{d^2r}{d\theta}} \qquad (2\text{-}20)$$

Hence for a cam paired with a roller follower of radius R_r we can substitute for r

$$r = R_o + R_r + h(\theta) \qquad (2\text{-}21)$$

then eqn.(2-20) becomes

$$\rho = \frac{\left[\left[R_o + R_r + h(\theta)\right]^2 + \left[\frac{dh(\theta)}{d\theta}\right]^2\right]^{3/2}}{\left[R_o + R_r + h(\theta)\right]^2 + 2\left[\frac{dh(\theta)}{d\theta}\right]^2 - \left(R_o + R_r + h(\theta)\right)\frac{d^2h(\theta)}{d\theta^2}} \qquad (2\text{-}22)$$

or

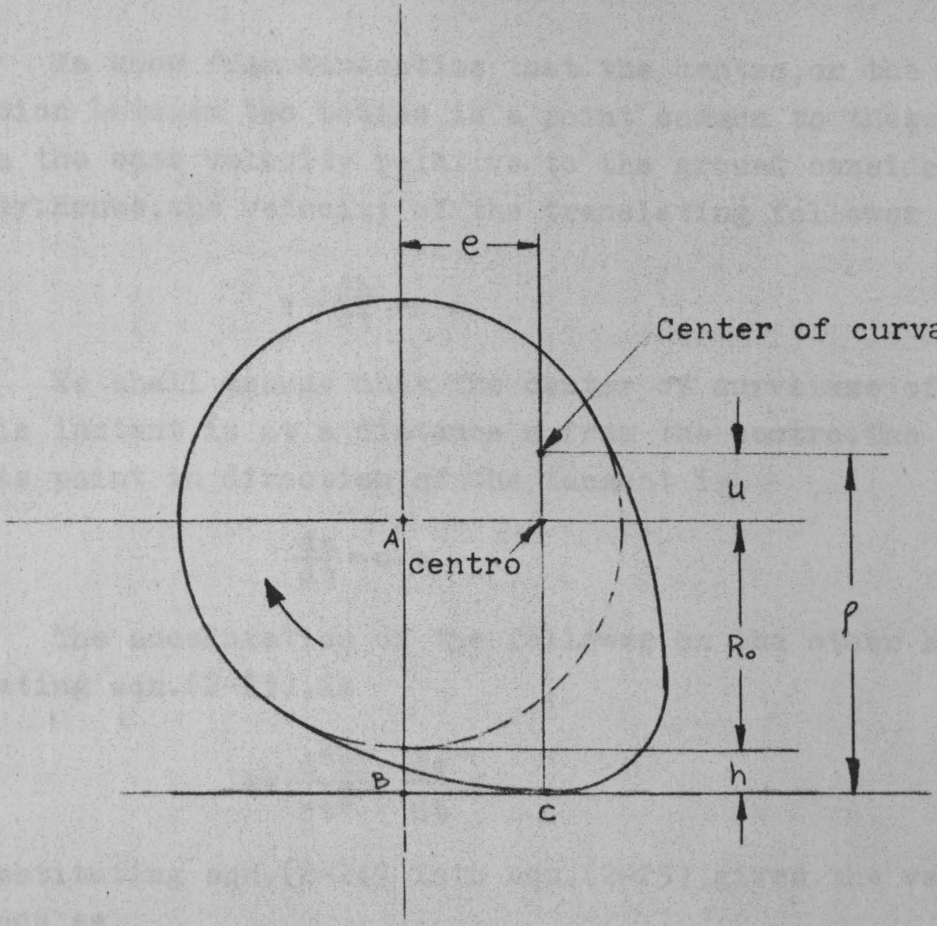
$$\rho = \frac{\left[(R_o + R_r + h)^2 + \dot{h}^2\right]^{3/2}}{(R_o + R_r + h)^2 + 2\dot{h}^2 - (R_o + R_r + h)\ddot{h}}$$

The above equation will not be valid for the cams paired with flat tappets since inserting infinity for the value of R_r gives no results.Hence a new approach must be found to establish mathematically the flat-faced follower cam curvature.In fig.6 the flat follower is shown in contact with the cam.Let,

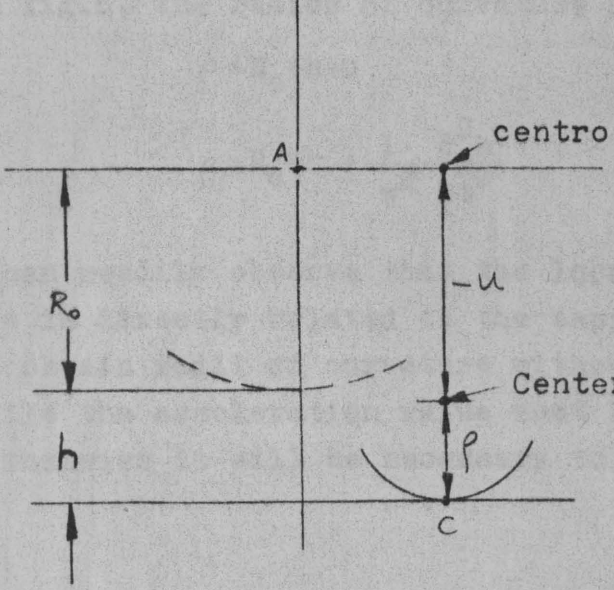
R_o :Base circle radius

e :Eccentricity of contact point from cam center

u :Distance from centro to cam center of curvature



Large radius of curvature



Small radius of curvature

FIGURE.6.Cam curvature with flat faced follower

We know from kinematics that the centro,or the center of rotation between two bodies,is a point common to these bodies and has the same velocity relative to the ground considered on either body.Hence,the velocity of the translating follower or the centro is

$$v=\frac{dh}{dt}=e\cdot w \qquad (2\text{-}23)$$

We shall assume that the center of curvature of the cam at this instant is at a distance u from the centro.The velocity of this point in direction of the tangent is

$$\frac{de}{dt}=u\cdot w \qquad (2\text{-}24)$$

The acceleration of the follower on the other hand,differentiating eqn.(2-23),is

$$a=\frac{d^2h}{dt^2}=\frac{de}{dt}\cdot w \qquad (2\text{-}25)$$

Substituting eqn.(2-24) into eqn.(2-25) gives the vertical distance as

$$u=\frac{1}{w^2}\cdot\frac{d^2h}{dt^2} \qquad (2\text{-}26)$$

and from fig.6. the radius of curvature is

$$\rho=R_0+h+u \qquad (2\text{-}27)$$

or

$$\rho=R_0+h+\frac{1}{w^2}\cdot\frac{d^2h}{dt^2} \qquad (2\text{-}28)$$

We can readily observe that the location of the center of curvature is directly related to the tappet acceleration.In other words to obtain radii of curvature without undercutting on the cam profile the acceleration value must be within reasonable limits.Otherwise it will be necessary to use a larger cam.

2.4.AMOUNT OF INLET AIR

For an engine the theoretical amount of air that must be taken into the cylinder is equal to the volume displaced by the piston for each cylinder.However this is the theoretical value and actually the amount that is taken in never reaches to that value and that actual amount defines the volumetric efficiency of the engine.There are several factors influencing the volumetric efficiency of the engine one of which is the valve lift trajectory,since it has an important effect on the air inlet.

To approach better efficiencies first of all the time duration between the opening and closing of the valve must fall in the region between the top dead center and bottom dead center of the piston movement,i.e. the intake process.It must be noted that one must avoid the valve hitting the piston when determining the opening and closing times for the valve.

Let us assume that the valve opens at θ_1 degrees and closes at θ_2 degrees of crankshaft angle.The amount of air that comes in during this time interval,is the integral of air flow per degree of crankshaft angle between θ_1 and θ_2,i.e.

$$W = \int_{\theta_1}^{\theta_2} Q_v d\theta \qquad (2\text{-}29)$$

where W is volume of air taken in,in cm^3,and Q_v is the air flow rate into the cylinder in cm^3/degrees of crankshaft angle.

The air flow rate is

$$Q_v = A_p V_p \qquad (2\text{-}30)$$

where A_p is the area of piston in cm^2,and V_p is the velocity of piston in cm/deg.

The piston area is

$$A_p = \frac{\pi}{4} D_p^2 \qquad (2\text{-}31)$$

and for the displacement of piston we have,from fig.7

$$x = r + l - r \cdot \mathrm{Cos}\theta - l \cdot \mathrm{Cos}\varphi \qquad (2\text{-}32)$$

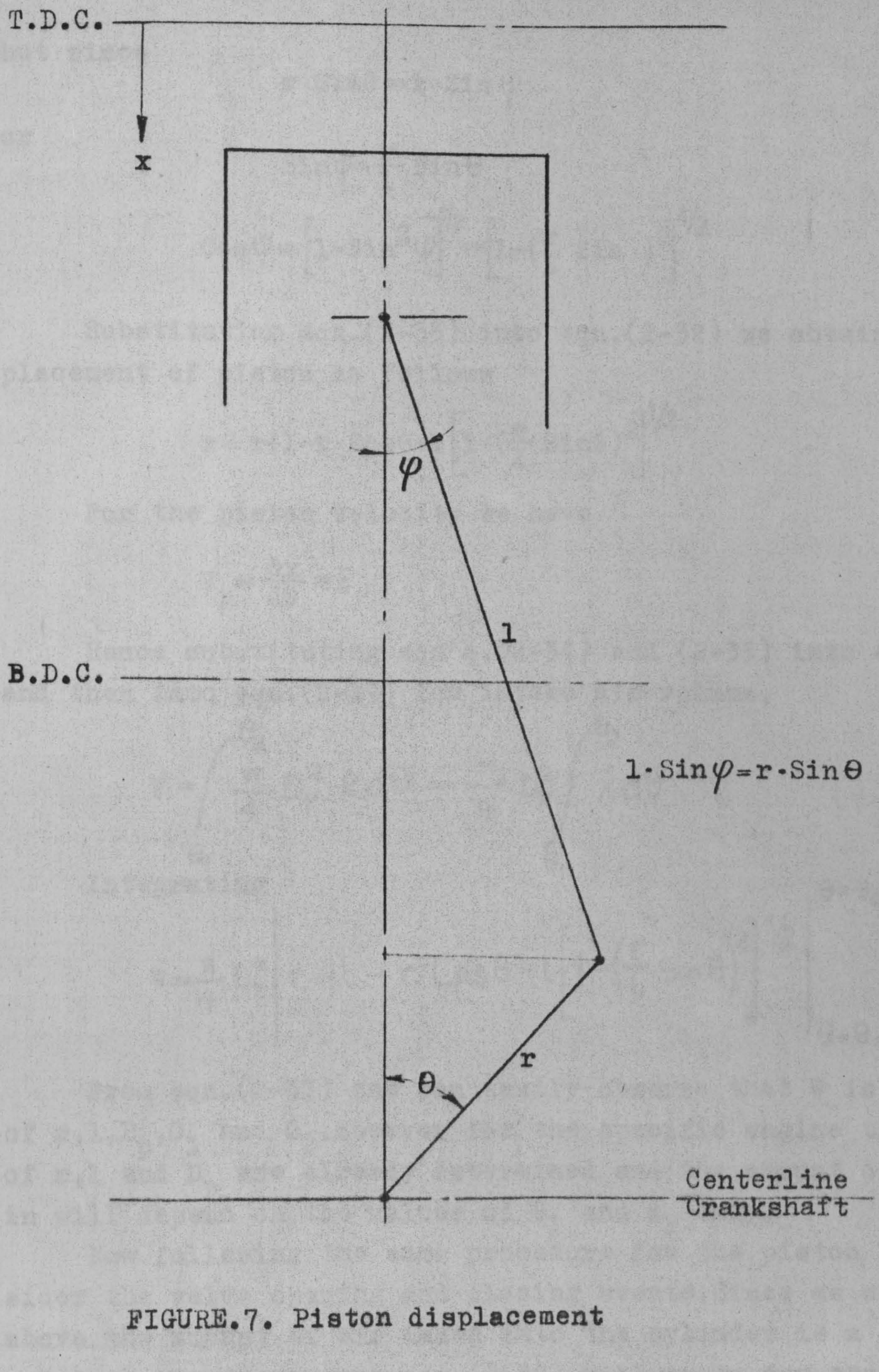


FIGURE.7. Piston displacement

but since

$$r \cdot \mathrm{Sin}\theta = l \cdot \mathrm{Sin}\,\varphi$$

or

$$\mathrm{Sin}\varphi = \frac{r}{l}\,\mathrm{Sin}\,\theta$$

$$\mathrm{Cos}\varphi = \left[1 - \mathrm{Sin}^2\varphi\right]^{1/2} = \left[1 - (\tfrac{r}{l}\ \mathrm{Sin}\ )^2\right]^{1/2} \qquad (2\text{-}33)$$

Substituting eqn.(2-33) into eqn.(2-32) we obtain the displacement of piston as follows

$$x = r + l - r \cdot \mathrm{Cos}\theta - l\left[1 - (\tfrac{r}{l}\,\mathrm{Sin}\theta)^2\right]^{1/2} \qquad (2\text{-}34)$$

For the piston velocity we have

$$V_p = \frac{dx}{d\theta} = \dot{x} \qquad (2\text{-}35)$$

Hence substituting eqn's.(2-34) and (2-35) into eqn.(2-30) and then into eqn.(2-19) for intake air volume,

$$W = \int_{\theta_1}^{\theta_2} \frac{\pi}{4} D_p^2 \cdot \dot{x}\, d\theta = \frac{\pi}{4} D_p^2 \int_{\theta_1}^{\theta_2} \dot{x}\, d\theta \qquad (2\text{-}36)$$

Integrating

$$W = \frac{\pi}{4} D_p^2 \left| r + l - r \cdot \mathrm{Cos}\theta - l\left[1 - \left(\frac{r}{l}\,\mathrm{Sin}\theta\right)^2\right]^{1/2} \right|_{\theta=\theta_1}^{\theta=\theta_2} \qquad (2\text{-}37)$$

From eqn.(2-37) one can easily observe that W is a function of r, l, D_p, O_1 and O_2. However for the specific engine the values of r, l and D_p are already determined and the amount of air taken in will depend on the values of θ_1 and θ_2 only.

Now following the same procedure for the piston let us consider the valve opening and closing events. Since as we have seen above the amount of air taken into the cylinder is a constant value the same equation, eqn.(2-29), will apply for the valve opening

$$W = \int_{\theta_1}^{\theta_2} Q_v\, d\theta = \int_{\theta_1}^{\theta_2} A_v V_v\, d\theta \qquad (2\text{-}38)$$

In eqn.(2-38) A_v is the area of the valve opening in cm^2, and V_v is the velocity of air passing through that valve opening in cm/deg. However although the same formula applies, contrary to the case for the piston, the area of valve opening is variable and it is obviously a function of valve lift, $h(\theta)$.

The area of valve opening, from fig.8. is

$$A_v = h(\theta)\cdot \mathrm{Cos}45^\circ \cdot \pi D_{av} \qquad (2\text{-}39)$$

where D_{av} is the average diameter of the valve opening and is found as follows

$$D_{av} = (\ 2D+2h(\theta).\mathrm{Cos}^2 45^\circ).0{,}5 = D + \frac{1}{2}h(\theta) \qquad (2\text{-}40)$$

Substituting Eqn.(2-40) into Eqn.(2-39) the minumum area of valve opening.

$$A_v = \pi(D-\tfrac{1}{2}h(\theta)\ ).\cos\ 45^\circ.h(\theta)$$

$$A_v = \frac{0{,}707}{2}\ h^2(\theta)-(0{,}707)\cdot\pi D.h(\theta) \qquad (2\text{-}41)$$

Since the total amount of air taken in is constant, the velocities of air flowing in will vary as the area of valve opening varies due to the lift $h(\theta)$. This leads us to the fact that the air velocity passing through intake valve is also a function of the valve lift $h(\theta)$. Hence for a constant amount of air, decreasing the area of valve opening, i.e. decreasing the valve lift, will lead us to high air velocities, which is not desired for a good efficiency.

As a result we can say that, we should maximize

$$\int_{\theta_1}^{\theta_2} A_v d\theta$$

or in other words

$$\int_{\theta_1}^{\theta_2} h^2(\theta)d\theta$$

to obtain reasonable air velocities through valves.

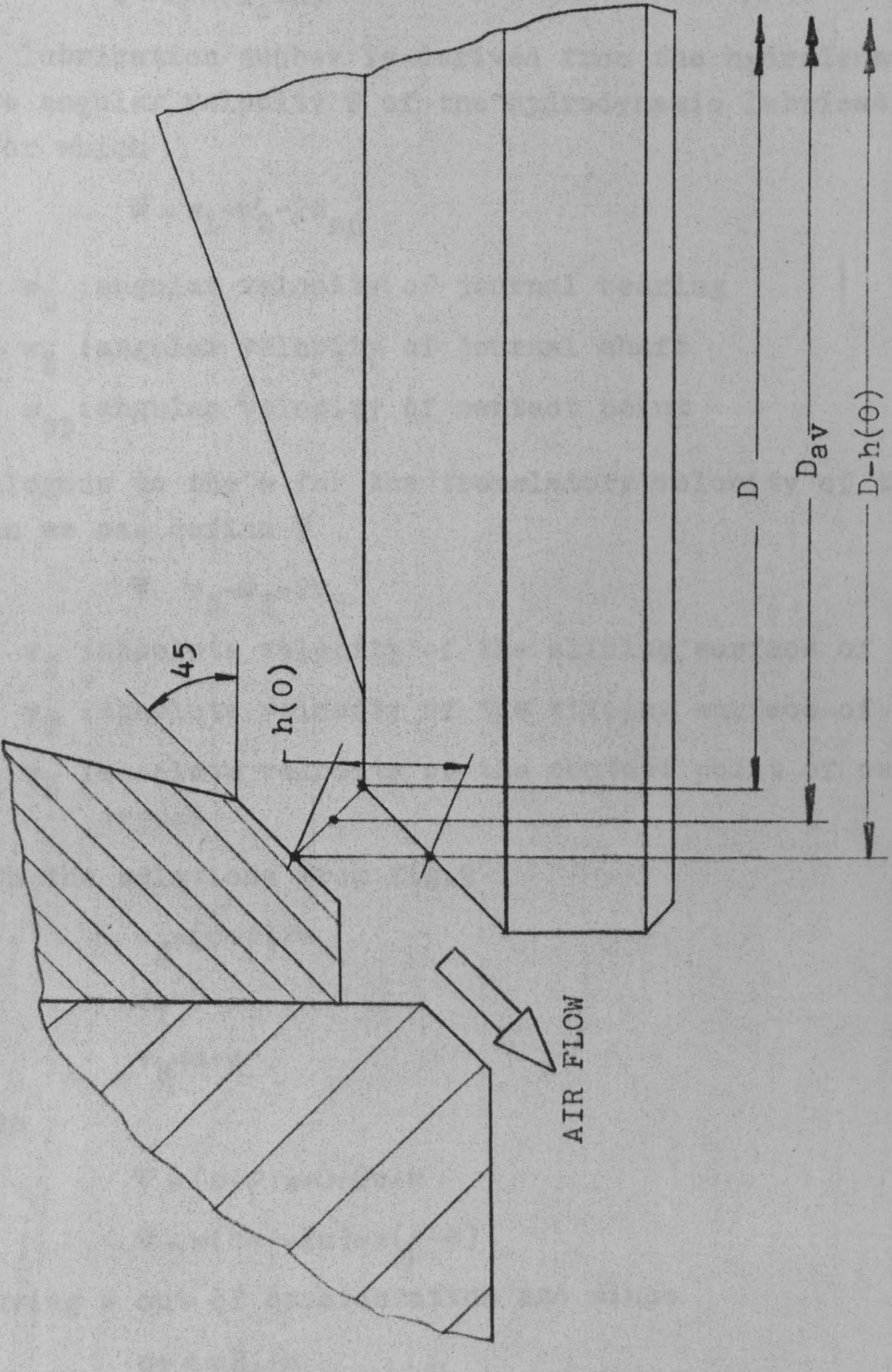


FIGURE.8.Area of valve opening

2.5.LUBRICATION NUMBER

In the previous sections we have mentioned about the lubrication number S and defined it as follows

$$S=2\rho-(R_o+h) \qquad (2\text{-}42)$$

The lubrication number is derived from the hydrodynamic effective angular velocity $\bar{w}$ of the hydrodynamic lubrication theory, for which

$$\bar{w}=w_L-w_Z-2w_{sp} \qquad (2\text{-}43)$$

where w_L ;angular velocity of journal bearing

w_Z :angular velocity of journal shaft

w_{sp}:angular velocity of contact point

Analogous to the w for the translatory velocity of the cam operation we can define $\bar{v}$

$$\bar{v}\quad v_N-v_T-2v_B \qquad (2\text{-}44)$$

where v_N :absolute velocity of the sliding surface of cam

v_T :absolute velocity of the sliding surface of tappet

v_B :absolute velocity of the contact point of cam and tappet

With the relations from fig.9

$$v_N=(c+\rho)\cdot w \qquad (2\text{-}45)$$

$$v_T=0 \qquad (2\text{-}46)$$

$$v_B=c\cdot w \qquad (2\text{-}47)$$

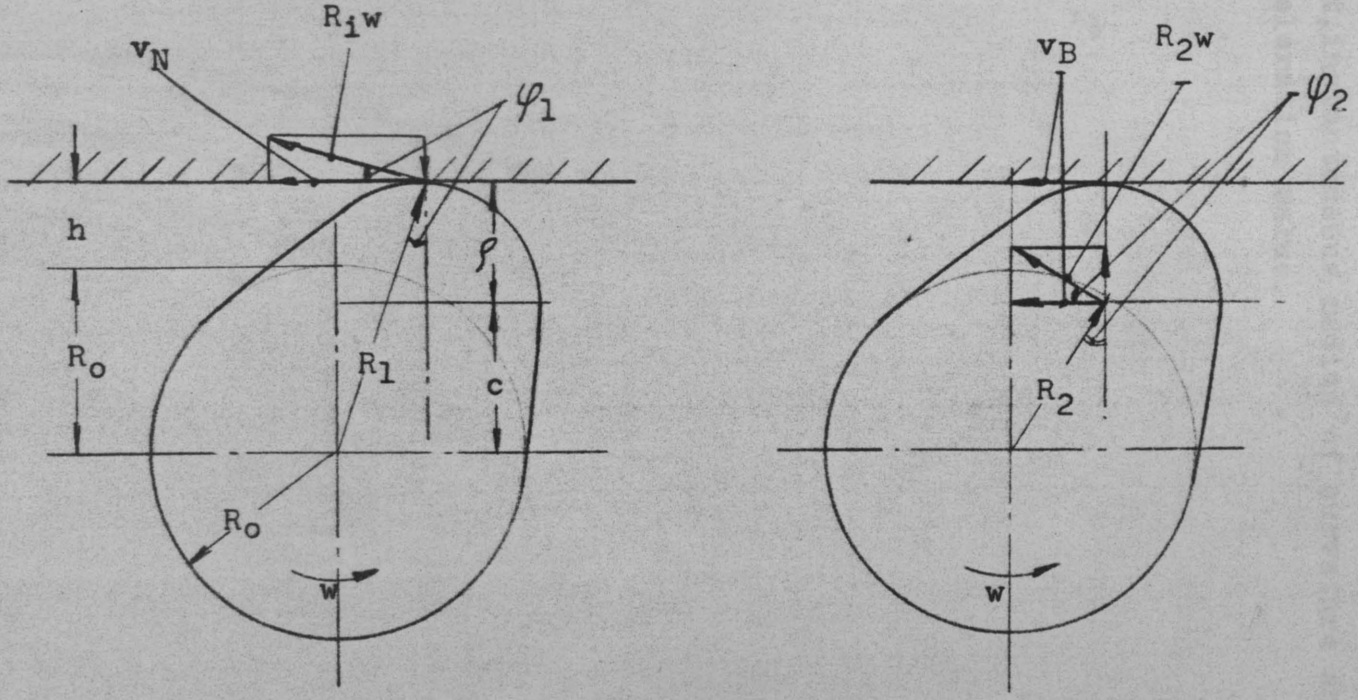
Thus

$$\bar{v}=(c+\rho)w+0-2c\cdot w \qquad (2\text{-}48)$$

$$\bar{v}=w(c+\rho-2c)=w(\rho-c) \qquad (2\text{-}49)$$

Leaving w out of consideration and since

$$c+\rho=R_o+h \qquad (2\text{-}50)$$



$$\mathrm{Cos}\,\varphi_1 = \frac{c+\rho}{R_1}$$

$$v_N = R_1 \cdot w \cdot \mathrm{Cos}\,\varphi_1$$

$$v_N = (c+\rho)\cdot w$$

$$\mathrm{Cos}\,\varphi_2 = \frac{c}{R_2}$$

$$v_B = R_2 \cdot w \cdot \mathrm{Cos}\,\varphi_2$$

$$v_B = c \cdot w$$

FIGURE.8.Derivation of lubrication number we define the lubrication number

$$S=\rho-c=2\rho-(R_0+h) \qquad (2\text{-}51)$$

of which we will try to avoid becoming zero,or staying zero for a long time.We can easily observe that this is dependent on the lift,instantaneous radius of curvature and hence in turn on the acceleration value.

2.6.BASIC CURVES

Let us now investigate some of the basic curves used for cam lift,velocity and acceleration.We shall go over some basic methods briefly and mention about their significant properties.

Constant Velocity:

Perhaps this is the most easily described motion for valve motion where the lift curve is linear .There are velocity shocks at the boundaries however between the rise and return motions.

Constant Acceleration:

This motion is sometimes called the 'gravity' curve.It produces segments of linear velocities and parabolic displacements. And it also produces acceleration shocks at the boundaries while the internal forces due to inertial loading are moderate.

Circular Arc Curves:

This type of curve is used frequently because of its simplicity to manufacture.As can be seen from fig.10-A. the velocity peak tends to be somewhat higher than for a corresponding constant acceleration curve and the intermediate acceleration shock is also worse.

Harmonic Curves:

The sine curve,best known as the 'cycloidal curve',has no acceleration and velocity shocks,but the maximum velocity is quite high as can be seen in fig.10-B.

The cosine curve,or the 'simple harmonic curve',has modest peak velocity and acceleration values.However even though small, it contains acceleration shocks at interval boundaries.

Harmonic Series Curves:

Harmonic segments may be combined to form other types of functions.A general form may be given as

$$h=\frac{C_o}{2}+\sum_{n=1}^{\infty}(C_n\cdot \mathrm{Cos}n\theta+D_n\cdot \mathrm{Sin}n\theta)$$

Harmonic curves can also be combined with other types of curves to form different functions.

Trapezoidal Specifications:

The method used for such a specification is the trapezoidal acceleration curve shown in fig.10-D. To maintain zero velocities at interval boundaries the positive and negative areas under the acceleration curve are equalised.

The Polynomial Technique:

For the desired lift curve of the valve an infinite series of the following form may be used,

$$h=\sum_{n=0}^{\infty} C_n\theta^n = C_0+C_1\theta+C_2\theta^2+C_3\theta^3+\ldots\ldots+C_n\theta^n$$

The truncated version of the above series is called a polynomial and can be used to approximate discrete data. The polynomials normally are used by cam designers to describe functions primarily dependent on interval boundary conditions.

This technique is quite widely accepted due to its advantage stemming from its high order of smoothness. Its only weakness is the lack of local control within the motion interval as a result of the present limitataion to boundary specifications. The number of boundary conditions determine the number of maximum terms in the polynomial to be used. Hence we can specify a cam system output motion over a given rotational interval by means of the polynomial

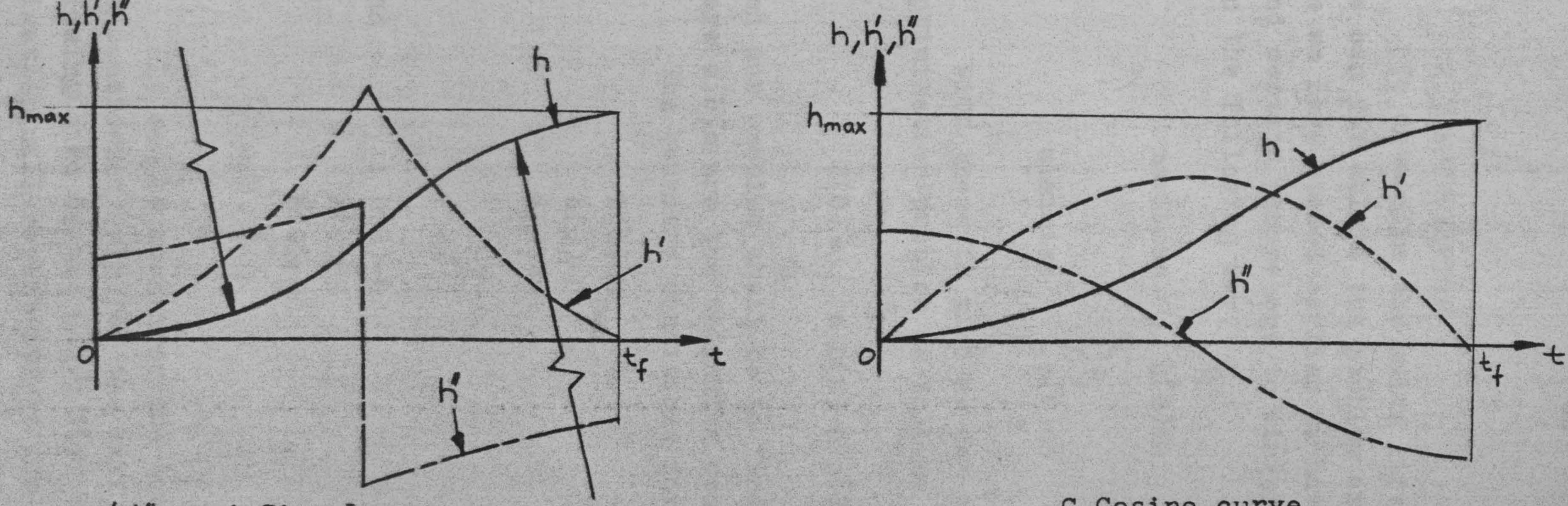
$$h = C_p\theta^p+C_q\theta^q+C_r\theta^r+\ldots\ldots+C_t\theta^t$$

where $p<q<r<\ldots<t$ are integers

The objective is to solve for the unknown polynomial coefficients $C_p, C_q, C_r, \ldots, C_t$. The derivatives of the polynomial will give the velocity, acceleration, jerk etc. values.

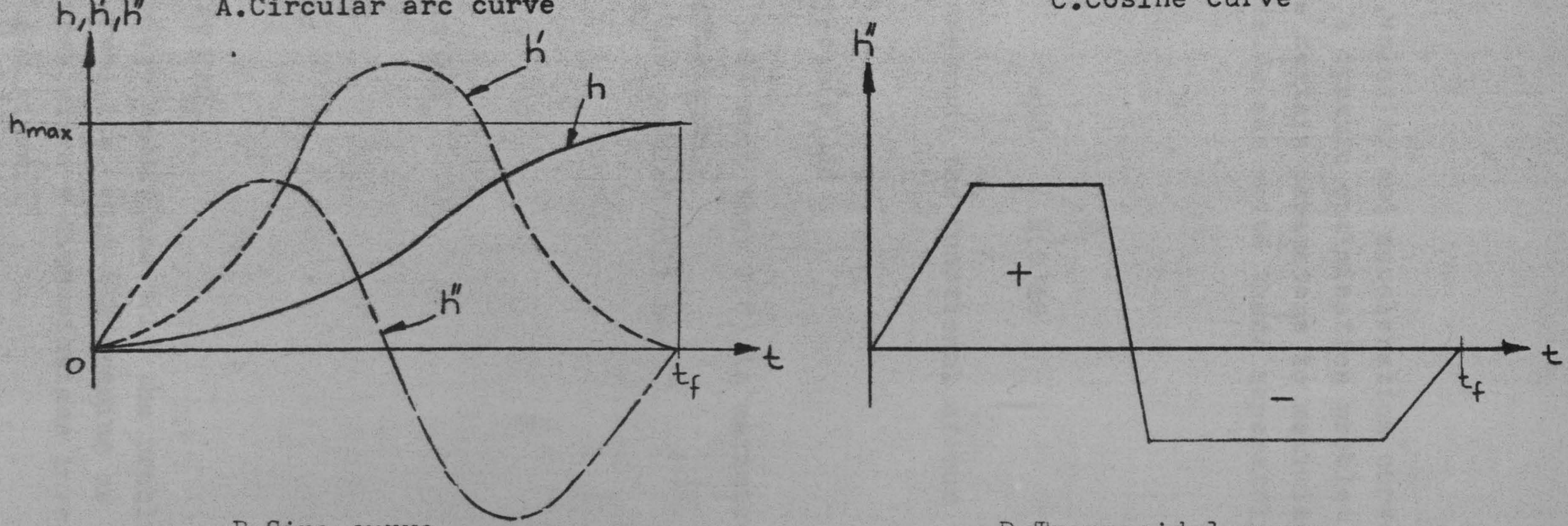
An example for the polynomial equation used in automotive cam systems is given below;

$$h = h_{max}(1-1.33\theta^2+0.15\theta^5+0.24\theta^{14}-0.049\theta^{30})$$



A.Circular arc curve

C.Cosine curve



B.Sine curve

D.Trapezoidal curve

FIGURE.10.Basic curves

2.7.FORMULATION

To determine the valve lift,velocity and acceleration curves we can for mulate the problem as a dynamic optimization problem. As we have stated earlier we have certain parameters to maximize and certain parameters to minimize.We can write these seperately in the following way,

$$\text{maximize} \quad \int_{\theta_0}^{\theta_f} h(\theta)d\theta$$

$$\text{minimize} \quad \int_{\theta_0}^{\theta_f} p(\theta)d\theta \qquad \text{and} \quad \int_{\theta_0}^{\theta_f} \ddot{h}^2(\theta)d\theta$$

With these in mind we can construct the functional of our optimization problem as follows

$$J(\theta) = \int_{\theta_0}^{\theta_f} (a\cdot h(\theta)-b\cdot p(\theta)-c\cdot \ddot{h}^2(\theta))d\theta$$

and our problem shall be to find $h(\theta)$ such that $J(\theta)$ is maximized. Here a,b,c are the weighting coefficients.

The boundary conditions for the problem will be

$$\begin{array}{ll} h(\theta)=h_{max} & \\ \dot{h}(\theta)=0 & \\ \ddot{h}(\theta)=0 & \\ \vdots & \text{at } \theta=\theta_0 \end{array}$$

and

$$\begin{array}{ll} h(\theta)=0 & \\ \dot{h}(\theta)=0 & \\ \ddot{h}(\theta)=0 & \\ \vdots & \text{at } \theta=\theta_f \end{array}$$

It can be seen from the boundary conditions that the problem is considered only for the half event,i.e. only the opening or closing side of the valve lift curve,since a dimensionless investigation will repeat itself for both sides.

The variable θ can also be taken as time t,since the two are directly related to each other.So let us denote the initial and final conditions as initial time ($t=t_0$) and final time ($t=t_f$). Hence our functional that we are going to extremize will take the form

$$J(t)=\int_{t_0}^{t_f}(a\cdot h(t)-b\cdot p(t)-c\cdot\ddot{h}^2(t))dt$$

As we have stated previously we see that all the parameters that we have to extremize are functions of lift,velocity and acceleration.Hence we can state our functional in general form as follows

$$J(t)=\int_{t_0}^{t_f}\phi\left[h(t),\dot{h}(t),\ddot{h}(t),t\right]dt$$

The method that will be used for the solution of the problem is explained in the next section.

3. DYNAMIC OPTIMIZATION

3.1.DYNAMIC OPTIMIZATION WITHOUT CONSTRAINTS

Let us examine a functional of simple form which can be written in general as follows

$$J(x)=\int_{t_0}^{t_f} \phi\left[x(t),\dot{x}(t),t\right]dt \qquad (3\text{-}1)$$

where t_0 and t_f are fixed.

What we would like to find is an x(t) so that we can extremize,i.e. maximize or minimize our given functional J(x).The solution for x(t) will be called an extremal since it will cause J(x) to have an extremum.

Let us denote this solution extremal by $\hat{x}(t)$,and define a family of curves between $t_0$and t_f including the extremal curve $\hat{x}(t)$

$$x(t)=\hat{x}(t)-\epsilon\eta(t) \qquad (3\text{-}2)$$

Here $\eta(t)$ is a variation in x(t) and ϵ is a very small number. Obviously the curves will be a minimum for $\epsilon=0$ since then

$$x(t)\Big|_{\epsilon=0}=\hat{x}(t) \qquad (3\text{-}3)$$

Hence for the extremal curves we should obtain

$$\frac{\partial J(x)}{\partial\epsilon}\Bigg|_{\epsilon=0}=0 \qquad (3\text{-}4)$$

independent of the value of variation $\eta(t)$ chosen.The extremal obtained from eqn.(3-2) will cause J(x) to have an extremum or be a stationary point.Further the condition to have a minimum or maximum is that the second derivative of J with respect to ,i.e. $\partial^2 J/\partial\epsilon^2$ be positive or negative respectively at $\epsilon=0$.However a solution for eqn.(3-4) will obviously extremize our integral J(x).

Now we can extremize our integral eqn.(3-1) by using eqn's. (3-2) and (3-4).Differentiating eqn.(3-2) with respect to t

$$\dot{x}(t)=\dot{\hat{x}}(t)+\epsilon\dot{\eta}(t) \qquad (3\text{-}5)$$

Substitute eqn's.(3-2) and (3-5) into eqn.(3-1) to obtain

$$J(x)=\int_{t_0}^{t_f} \phi\left[\hat{x}(t)+\epsilon\eta(t),\dot{\hat{x}}(t)+\epsilon\dot{\eta}(t),t\right]dt \qquad (3\text{-}6)$$

For 0 we know that $J(x)$ $J(\hat{x})$ and $x(t)$ $\hat{x}(t)$.To find the extremals we use eqn.(3-4)

$$\left.\frac{\partial J}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial}{\partial \epsilon}\int_{t_0}^{t_f} \phi\left[\hat{x}(t)+\epsilon\eta(t), \dot{\hat{x}}(t)+\epsilon\dot{\eta}(t), t\right] dt$$

$$\left.\frac{\partial J}{\partial \epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_f} \left\{ \frac{\partial \phi}{\partial(\hat{x}+\epsilon\eta)} \frac{\partial(\hat{x}+\epsilon\eta)}{\partial\epsilon} + \frac{\partial \phi}{\partial(\dot{\hat{x}}+\epsilon\dot{\eta})} \frac{\partial(\dot{\hat{x}}+\epsilon\dot{\eta})}{\partial\epsilon} + \frac{\partial \phi}{\partial t}\frac{\partial t}{\partial \epsilon} \right\} dt$$

$$\int_{t_0}^{t_f} \eta(t) \frac{\partial \phi(\hat{x},\dot{\hat{x}},t)}{\partial \hat{x}} dt + \int_{t_0}^{t_f} \dot{\eta}(t) \frac{\partial \phi(\hat{x},\dot{\hat{x}},t)}{\partial \dot{\hat{x}}} dt = 0 \tag{3-7}$$

Using integration by parts will give

$$\int_{t_0}^{t_f} \eta(t)\frac{\partial \phi}{\partial \hat{x}} dt + \left.\eta(t)\frac{\partial \phi}{\partial \dot{\hat{x}}}\right|_{t_0}^{t_f} - \int_{t_0}^{t_f} \eta(t)\frac{d}{dt}\frac{\partial \phi}{\partial \dot{\hat{x}}} dt = 0$$

Simplifying

$$\left.\eta(t)\frac{\partial \phi}{\partial \dot{\hat{x}}}\right|_{t_0}^{t_f} + \int_{t_0}^{t_f} \eta(t)\left[\frac{\partial \phi}{\partial \hat{x}} - \frac{d}{dt}\frac{\partial \phi}{\partial \dot{\hat{x}}}\right] dt = 0 \tag{3-8}$$

As we have previously mentioned eqn.(3-8) must be satisfied independent of the value chosen for the variation (t),giving

$$\frac{\partial \phi}{\partial \hat{x}} - \frac{d}{dt}\frac{\partial \phi}{\partial \dot{\hat{x}}} = 0 \tag{3-9}$$

$$\left.\frac{\partial \phi}{\partial \dot{\hat{x}}}\right|_{t=t_0} = 0 \tag{3-10a}$$

$$\left.\frac{\partial \phi}{\partial \dot{\hat{x}}}\right|_{t=t_f} = 0 \tag{3-10b}$$

These equations specify the two-point boundary value differential equation whose solution is the desired $\hat{x}$.Eqn's.(3-9) and (3-10a/b) are called the Euler-Lagrange Equations and the transversality conditions respectively.

Transversality Conditions:

For t_0 and t_f fixed let us examine eqn's.(3-10a) and (3-10b) and see the possible combinations.

a)Fixed Beginning-Terminal points:

The case for which $x(t_0)$ and $x(t_f)$ are fixed.We can easily see that $\eta(t_0)=\eta(t_f)=0$ is required for this case and every solution must pass through these points which are also specified as the correct boundary conditions.

b)Variable Beginning-Terminal Points:

The case for which $x(t_0)$ and $x(t_f)$ are variable.We see that for this case $\partial\phi/\partial\dot{x}=0$ must be satisfied for $t=t_0$ and $t=t_f$ since $\eta(t)$ can be arbitrary at these points.

c)Variable Beginning-Fixed Terminal Points:

The case for which $x(t_0)$ is variable and $x(t_f)$ is fixed. Hence from the equations we see that for $t\ t_0$ we must satisfy $\partial\phi/\partial\dot{x}=0$ and for $t=t_f$ we require $\eta(t_f)=0$ and $x(t_f)$ is specified as the correct boundary condition.

d)Fixed Beginning-Variable Terminal Points:

The case for which $x(t_0)$ is fixed but $x(t_f)$ is variable. From eqn.(3-10) we see that we must satisfy $\partial\phi/\partial\dot{x}=0$ for $t=t_f$ and we require $\eta(t_0)=0$ and $x(t_0)$is the correct boundary condition for $t=t_0$.

3.2.VECTOR FORMULATION

The results we have found for dynamic optimization can easily be expanded to the cases for which the scalar function J is a function of n-dimensional variables.In this case the functional we want to extremize is

$$J(\bar{x})=\int_{t_0}^{t_f}\phi(\bar{x},\dot{\bar{x}},t)dt \tag{3-11}$$

where $\bar{x}$ is an n-vector described as

$$\bar{x}^T=(x_1,x_2,x_3,\ldots\ldots,x_n)$$

The procedure to be followed is similar to the scalar case and the result,after setting $\partial J/\partial\epsilon=0$ for $\epsilon=0$,will be

$$\bar{\eta}(t)\frac{\partial\phi}{\partial\dot{\bar{x}}}\bigg|_{t_0}^{t_f}+\int_{t_0}^{t_f}\bar{\eta}(t)\left[\frac{\partial\phi}{\partial\bar{x}}-\frac{d}{dt}\frac{\partial\phi}{\partial\dot{\bar{x}}}\right]dt=0 \tag{3-12}$$

Again for arbitrary $\bar{\eta}(t)$,eqn.(3-12) will give the Euler-Lagrange Equation as

$$\frac{\partial\phi}{\partial\bar{x}}-\frac{d}{dt}\frac{\partial\phi}{\partial\dot{\bar{x}}}=0 \tag{3-13}$$

and the transversality conditions are

$$\bar{\eta}(t)\frac{\partial\phi}{\partial\dot{\bar{x}}}\bigg|_{t=t_0}=0 \tag{3-14a}$$

$$\bar{\eta}(t)\frac{\partial\phi}{\partial\dot{\bar{x}}}\bigg|_{t=t_f}=0 \tag{3-14b}$$

3.3.VARIATIONAL NOTATION

Using the previous results we have obtained we can easily deduct that in variational rotation the equations will appear as follows.The first variation of the functional J is

$$\delta J = \int_{t_0}^{t_f} \left[\frac{\partial \phi}{\partial \hat{x}} \delta x + \frac{\partial \phi}{\partial \dot{x}} \delta \dot{x} \right] dt \tag{3-15}$$

with the definition of the first variations of x(t) and $\dot{x}(t)$ as

$$\delta x = \epsilon \eta(t) \tag{3-16a}$$

$$\delta \dot{x} = \epsilon \dot{\eta}(t) \tag{3-16b}$$

The necessary conditions for an extremum is that the first variation of J,i.e. δJ is equal to zero as in the previous case. Thus following the same procedure we can easily derive

$$\frac{\partial \phi}{\partial \dot{x}} \delta \dot{x} \bigg|_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} \delta x \left[\frac{\partial \phi}{\partial x} - \frac{d}{dt} \frac{\partial \phi}{\partial \dot{x}} \right] dt = 0 \tag{3-17}$$

Then the Euler-Lagrange equation is

$$\frac{\partial \phi}{\partial x} - \frac{d}{dt} \frac{\partial \phi}{\partial \dot{x}} = 0 \tag{3-18}$$

and the transversality conditions are

$$\frac{\partial \phi}{\partial \dot{x}} \delta \dot{x} \bigg|_{t=t_0} = 0 \tag{3-19a}$$

$$\frac{\partial \phi}{\partial \dot{x}} \delta \dot{x} \bigg|_{t=t_f} = 0 \tag{3-19b}$$

3.4.EQUALITY CONSTRAINTS-LAGRANGE MULTİPLIERS

When the problem has some constraints to be satisfied,we can make use of the Lagrange multipliers to consider our functional and its constraints at the same time.We are seeking to extremize the functional

$$J=\int_{t_0}^{t_f}\phi(\bar{x},\dot{\bar{x}},t)dt \qquad (3\text{-}20)$$

which is subject to the equality constraints

$$\bar{\Delta}(\bar{x},\dot{\bar{x}},t)=0 \qquad (3\text{-}21)$$

where $\bar{x}$ is an n-vector and $\bar{\Delta}$ is an m-vector such that

$$\bar{x}^T=[x_1,x_2,x_3,\ldots\ldots,x_n]$$

$$\bar{\Delta}^T=[\Delta_1,\Delta_2,\Delta_3,\ldots\ldots,\Delta_n]$$

and $m\leq n$.

The solution of this type of problem is identical with the one of extremizing the functional J' of the following form

$$J'=\int_{t_0}^{t_f}\left[\phi(\bar{x},\dot{\bar{x}},t)+\bar{\lambda}^T(t)\bar{\Delta}(\bar{x},\dot{\bar{x}},t)\right]dt \qquad (3\text{-}22)$$

Here $\bar{\lambda}^T=[\lambda_1,\lambda_2,\lambda_3,\ldots\ldots,\lambda_n]$ is the Lagrange multiplier in vector form of m-dimension.Since the integrand is a function of $\bar{x},\dot{\bar{x}}$ and t,again the same procedure will be applied for the solution.

3.5.CONTINUOUS OPTIMAL CONTROL PROBLEM (BOLZA EQUATION)- FIXED BEGINNING AND UNSPECIFIED TERMINAL TIMES

Using the information we have found up to this section let us now consider the problem of continuous optimal control with equality constraints and free final time which will be applicable for our case.

We are given a nonlinear differential system operating over the interval $t\in(t_o,t_f)$.This system is defined as

$$\dot{\bar{x}}=\bar{f}\left[\bar{x}(t),\bar{u}(t),t\right] \tag{3-23}$$

Here $\bar{x}(t)$ is the n-vector state variable and is determined by the m-vector control variable $\bar{u}(t)$.

The initial condition vector is

$$\bar{x}(t_o)=\bar{x}_o \tag{3-24}$$

for $t=t_o$.For the unspecified terminal time $t=t_f$ we define the q-vector terminal manifold equation

$$\bar{N}\left[\bar{x}(t_f),t_f\right]=0 \tag{3-25}$$

which is the constraint that must be satisfied at $t=t_f$.

With these constraints and the given system we want to determine the control variable $\bar{u}(t)$ in order to minimize the functional

$$J=\theta\left[\bar{x}(t_f),t_f\right]+\int_{t_o}^{t_f}\phi\left[\bar{x}(t),\bar{u}(t),t\right]dt \tag{3-26}$$

We use the method of Lagrange multipliers to adjoin the equality constraints to the cost function J.Hence our new functional will be

$$J=\theta\left[\bar{x}(t_f),t_f\right]+\bar{\nu}^T\bar{N}^T\left[\bar{x}(t_f),t_f\right]$$
$$+\int_{t_o}^{t_f}\left\{\phi\left[\bar{x}(t),\bar{u}(t),t\right]+\bar{\lambda}^T(t)\left[\bar{f}(\bar{x}(t),\bar{u}(t),t)-\dot{\bar{x}}(t)\right]\right\}dt \tag{3-27}$$

Now for convenience in the computations let us define a scalar function H called the Hamiltonian

$$H\left[\bar{x}(t),\bar{u}(t),\bar{\lambda}(t),t\right]=\phi\left[\bar{x}(t),\bar{u}(t),t\right] + \bar{\lambda}^{T}(t)\bar{f}\left[\bar{x}(t),\bar{u}(t),t\right] \tag{3-28}$$

Hence our functional takes the form

$$J=\theta\left[\bar{x}(t_f),t_f\right]+\bar{\nu}^{T}\bar{N}\left[\bar{x}(t_f),t_f\right] + \int_{t_0}^{t_f}\left\{H\left[\bar{x}(t),\bar{u}(t),\bar{\lambda}(t),t\right]-\bar{\lambda}^{T}(t)\dot{\bar{x}}(t)\right\}dt$$

Using integration by parts we obtain for J

$$J=\theta\left[\bar{x}(t_f),t_f\right]+\bar{\nu}^{T}\bar{N}\left[\bar{x}(t_f),t_f\right]-\bar{\lambda}^{T}(t)\bar{x}(t)\Big|_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f}\left\{H\left[\bar{x}(t),\bar{u}(t),\bar{\lambda}(t),t\right]-\bar{\lambda}^{T}(t)\bar{x}(t)\right\}dt \tag{3-29}$$

At this point we let

$$\begin{aligned}\bar{x}(t)&=\hat{\bar{x}}(t)+\bar{h}(t)\\ \bar{u}(t)&=\hat{\bar{u}}(t)+\delta\bar{u}(t)\\ t_f&=\hat{t}_f+\delta t_f\end{aligned} \tag{3-30}$$

where $\bar{h}(t)$ is a variation in $\bar{x}(t)$. Now we can form the first variation of J as we have done previously,

$$\begin{aligned}\delta J=\delta t_f&\left\{H\left[\bar{x}(t_f),\bar{u}(t_f),\bar{\lambda}(t_f),t_f\right]+\frac{\partial\theta\left[\bar{x}(t_f),t_f\right]}{\partial t_f}\right.\\ &\left.+\frac{\partial\bar{N}^{T}\left[\bar{x}(t_f),t_f\right]}{\partial t_f}\bar{\nu}\right\}+\delta\bar{x}^{T}(t_f)\left\{\frac{\partial\theta\left[\bar{x}(t_f),t_f\right]}{\partial\bar{x}(t_f)}\right.\\ &\left.+\frac{\partial\bar{N}^{T}\left[\bar{x}(t_f),t_f\right]}{\partial\bar{x}(t_f)}\bar{\nu}-\bar{\lambda}(t_f)\right\}\\ &+\int^{t_f}\left\{\bar{h}^{T}(t)\left[\frac{\partial H}{\partial\bar{x}}+\dot{\bar{\lambda}}\right]+\delta\bar{u}^{T}(t)\left[\frac{\partial H}{\partial\bar{u}}\right]\right\}dt\end{aligned} \tag{3-31}$$

To obtain the necessary conditions for an extremum we must set the first variation of J equal to zero.This will give us the equations to determine the optimal control and state vector as follows

$$\frac{\partial H}{\partial \bar{\lambda}} = \dot{\bar{x}} = \bar{f}\left[\bar{x}(t), \bar{u}(t), t\right] \tag{3-32}$$

$$\frac{\partial H}{\partial \bar{x}} = -\dot{\bar{\lambda}} = \frac{\partial \bar{f}^T[\bar{x}(t), \bar{u}(t), t]}{\partial \bar{x}} \bar{\lambda}(t) + \frac{\partial \phi[\bar{x}(t), \bar{u}(t), t]}{\partial \bar{x}} \tag{3-33}$$

$$\frac{\partial H}{\partial \bar{u}} = 0 = \frac{\partial \bar{f}^T[\bar{x}(t), \bar{u}(t), t]}{\partial \bar{u}} \bar{\lambda}(t) + \frac{\partial \phi[\bar{x}(t), \bar{u}(t), t]}{\partial \bar{u}} \tag{3-34}$$

These equations together with eqn.(3-28) define the 2n differential equations for the two point boundary value problem.The boundary conditions for the fixed initial time t t_0 are

$$\bar{x}(t_0) = x_0 \tag{3-35}$$

and for the unspecified final time t_f we have

$$\bar{\lambda}(t_f) = \frac{\partial \theta[\bar{x}(t_f), t_f]}{\partial \bar{x}(t_f)} + \left[\frac{\partial \bar{N}^T[\bar{x}(t_f), t_f]}{\partial \bar{x}(t_f)}\right] \bar{\nu} \tag{3-36}$$

$$\bar{N}\left[\bar{x}(t_f), t_f\right] = 0 \tag{3-37}$$

$$H\left[\bar{x}(t_f), \bar{u}(t_f), \bar{\lambda}(t_f), t_f\right] + \frac{\partial \theta}{\partial t_f} + \left[\frac{\partial \bar{N}^T}{\partial t_f}\right] \bar{\nu} = 0 \tag{3-38}$$

4. DETERMINATION OF PROFILE

4.1. SOLUTION OF THE PROBLEM

Let us assume that our state vector is defined as

$$\dot{x} = Ax + Bu \tag{4-1}$$

and we are going to find the control vector u such that we shall minimizi the functional

$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (x^T Q x + u^T R u) dt \tag{4-2}$$

or considering an identity matrix for R

$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (x^T Q x + u^2) dt \tag{4-3}$$

The Hamiltonian will then be

$$H = \frac{1}{2} x^T Q x + \frac{1}{2} u^2 + \lambda^T [Ax + Bu] \tag{4-4}$$

To find the control vector u we use the given differential system for solution, i.e.

$$\frac{\partial H}{\partial u} = 0 = u + B^T \lambda$$

$$u = -B^T \lambda \tag{4-5}$$

$$\frac{\partial H}{\partial \lambda} = \dot{x} \qquad \frac{\partial H}{\partial x} = -\dot{\lambda}$$

obtaining as a result the following differential system

$$\dot{x} = Ax - BB^T \tag{4-6}$$

$$\dot{\lambda} = -Qx - A^T \lambda \tag{4-7}$$

or written in combined matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{4-8}$$

The solution to the above differential system is of the form

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} \Phi \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} \tag{4-9}$$

where

$$\begin{bmatrix} \Phi \end{bmatrix} = \begin{bmatrix} e^{Ft} \end{bmatrix} \tag{4-10}$$

$$\begin{bmatrix} F \end{bmatrix} = \begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} \tag{4-11}$$

Hence the equations defining $x(t)$ and $\lambda(t)$ are

$$x(t) = \Phi_{11} x(0) + \Phi_{12} \lambda(0) \tag{4-12}$$

$$\lambda(t) = \Phi_{21} x(0) + \Phi_{22} \lambda(0) \tag{4-13}$$

To find the initial values of $\lambda(t)$ we can use the final conditions on $x, x(t_f)$. Then

$$\lambda(0) = \Phi_{12}^{-1} x(t_f) - \Phi_{12}^{-1} \Phi_{11} x(0) \tag{4-14}$$

Substituting eqn.(4-14) into eqn.(4-13)

$$\lambda(t) = \Phi_{21} x(0) + \Phi_{22} \left[\Phi_{12}^{-1} x(t_f) - \Phi_{12}^{-1} \Phi_{11} x(0) \right] \tag{4-15}$$

At the final time t_f we have

$$H(t_f) + \frac{\partial \theta}{\partial t_f} + \left[\frac{\partial N^T}{\partial t_f} \right] \nu = 0 \tag{4-16}$$

For $$\theta = \frac{1}{2} \alpha t_f^2 \tag{4-17}$$

and $$N^T = \left[x(t_f) \right] = \left[0 \right] \tag{4-18}$$

$$H(t_f) + \alpha t_f = 0 \tag{4-19}$$

where $$H(t_f) = -\frac{1}{2} \lambda_n^2(t_f) \tag{4-20}$$

Hence we have for final time t_f

$$t_f = \frac{\lambda_n^2(t_f)}{2\alpha} \tag{4-21}$$

4.2.COMPUTATIONAL PROCEDURE

For the functional J given as

$$J=\frac{1}{2}\alpha t_f^2+\frac{1}{2}\int_0^{t_f}(x^TQx+u^2)dt$$

we can follow the iterative procedure described below

1-)Select values for α and Q

2-)Estimate a final time t_f (initial guess)

3-)Perform matrix F

$$F=\begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix}$$

4-)With matrix F and the estimated final time t_f compute matrix Φ by series expansion or other methods.

5-)Partition matrix Φ

$$\Phi=\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

6-)Find the inverse of Φ_{12}, Φ_{12}^{-1}

7-)With given initial values and final values of x(t) find the values of $\lambda(0)$.

8-)Compute $\lambda(t_f)$.

9-)Find the new final time according to

$$t_f=\frac{\lambda_n^2(t_f)}{2\alpha}$$

10-)Check if $\left|t_f-\frac{\lambda_n^2(t_f)}{2\alpha}\right| \leq$ tolerance

11-)a-If the check is satisfactory compute the values of the state vector x(t) for $t\in[t_0,t_f]$ with reasonable increments for t, to find the lift,velocity and acceleration curves.

b-If the check is not satisfactory apply Newton Raphson method to the equation given for final time t_f.

$$t_f-\frac{\lambda_n^2(t_f)}{2\alpha}=0=F(t_f)=2\alpha t_f-\lambda_n^2(t_f)$$

since according to the rule

$$t_f(n+1) = t_f(n) - \frac{F(t_f)}{F'(t_f)}$$

where

$$F'(t_f) = 2\alpha - 2\lambda_n(t_f)\lambda_n'(t_f)$$

With this new value of final time t_f go to item 4 and follow the same computational procedure.

After the problem is solved the expressions for x(t) will give the values of lift,velocity and acceleration as polynomial approximations since we have performed the matrix Φ=exp(Ft) in power series form.For the desired values several computer runs should be made for different values of α and Q.If t_f is given the problem will be obviously simpler.

Results of several runs have been obtained for different state vector dimensions and for different data in each case.Let us investigate the systems used for the solution for different state vector dimensions.

4.3.SYSTEM OF TWO VARIABLES

Let us first consider the simple case where the differential system is a two-vector system,i.e.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

where
$$x_1(t) = h(t)$$
$$x_2(t) = \dot{h}(t)$$
$$u(t) = \ddot{h}(t)$$

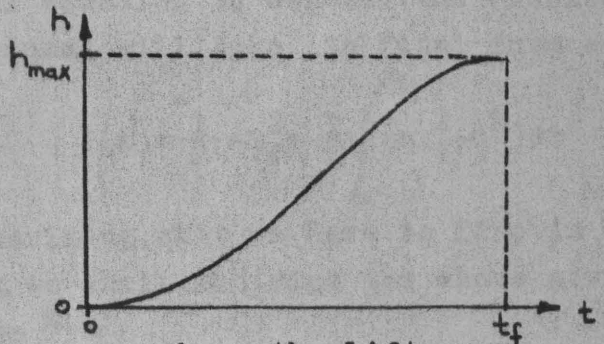
and the boundary conditions are

$$x_1(t_o) = 0 \qquad x_1(t_f) = h_{max}$$
$$x_2(t_o) = 0 \qquad x_2(t_f) = 0$$

and our functional will be of the form

$$J(t) = \int_0^{t_f} \phi\left[x_1(t), x_2(t), u(t), t\right] dt$$

A reformulation of the problem can be as follows;consider the lift curve desired for the valve motion (see figure),



For the area above the lift curve,up to h_{max} line our state variables will be

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = u$$

where
$$x_1(t) = h_{max} - h(t)$$
$$x_2(t) = \dot{h}(t)$$
$$u(t) = \ddot{h}(t)$$

with the boundary conditions

$$x_1(0)=h_{max} \qquad x_1(t_f)=0$$
$$x_2(0)=0 \qquad x_2(t_f)=0$$

and our functional is again of the following form

$$J(t)=\int_0^{t_f}\phi\left[x_1(t),x_2(t),u(t),t\right]dt$$

or for the Bolza equation of the form

$$J(t)=\theta\left[x_1(t_f),x_2(t_f),t_f\right]+\int_0^{t_f}\phi\left[x_1(t),x_2(t),u(t),t\right]dt$$

Now bearing in mind what we have to maximize consider that our two functions θ and ϕ are for the first attempt

$$\theta=t_f^2$$
$$\phi=\frac{1}{2}(x_1^2-u^2)$$

Hence our functional will be

$$J(t)= t_f^2+\frac{1}{2}\int_0^{t_f}(x_1^2-u^2)dt$$

Here we can introduce some weighting coefficients into the functional equation to control the problem by changing them. Thus our functional will take its final form as follows,

$$J(t)=\frac{1}{2}\alpha t_f^2+\frac{1}{2}\int_0^{t_f}(qx_1^2-u^2)dt$$

Summarizing, what we have to find is the control variable u(t) such that we shall minimize the above given functional J(t) for the system

$$\dot{x}_1=-x_2$$
$$\dot{x}_2=u$$

with the initial conditions

$$x_1(0)=h_{max}$$
$$x_2(0)=0$$

and the final conditions

$$x_1(t_f)=0$$
$$x_2(t_f)=0$$

with $t_o=0$ for the initial time and a free final time t_f, which will also be minimized.

Now referring to the representation of the 2n differential equations for the two point boundary value problem, the Hamiltonian is

$$H=\frac{1}{2}qx_1^2+\frac{1}{2}u^2-\lambda_1x_2+\lambda_2u$$

and our differential equations are

$$\dot{x}_1=-x_2$$
$$\dot{x}_2=u$$
$$\dot{\lambda}_1=-qx_1$$
$$\dot{\lambda}_2=\lambda_1$$

and since
$$\frac{\partial H}{\partial u}=0=u+\lambda_2$$
$$u=-\lambda_2$$

For the final time t_f we have

$$H(t_f)+\frac{\partial\theta}{\partial t_f}+\left[\frac{\partial N^T}{\partial t_f}\right]\nu=0$$

where
$$N^T=\left[x_1(t_f)\quad x_2(t_f)\right]=\left[0\quad 0\right]$$
$$\theta=\frac{1}{2}\alpha t_f^2$$

hence

$$\frac{1}{2}qx_1^2(t_f)+\frac{1}{2}u^2(t_f)-\lambda_1(t_f)x_2(t_f)+\lambda_2(t_f)u(t_f)+\alpha t_f=0$$

with
$$u=-\lambda_2$$
$$x_1(t_f)=x_2(t_f)=0$$

we obtain for the final time t_f

$$t_f = \frac{\lambda_2^2(t_f)}{2\alpha}$$

Now returning to the canonic equations in matrix notation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} F \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

with matrix F

$$F \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We can readily observe that the solution to the above differential system will be

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} e^{F \cdot t} \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

This solution will give us the desired lift, $x_1(t)$, velocity $x_2(t)$, and via $-\lambda_2(t)$ the acceleration $u(t)$ curves for this case. The computational procedure is as described previously and the exponential of the product of matrix F and time t can be computed by the exponential expansion in series as follows

$$e^{Ft} = I + Ft + \frac{(Ft)^2}{2!} + \frac{(Ft)^3}{3!} + \frac{(Ft)^4}{4!} + \ldots\ldots\ldots$$

Once the exponential term and the initial values for x and vectors are known, the system is solved. The initial values for x have been given in the initial boundary conditions. What we have to do to find the initial values of λ, i.e. $\lambda_1(0)$ and $\lambda_2(0)$; is to use the first two equations of the four equations system for the final boundary conditions on x. Writing the exponential matrix in the following form

$$e^{Ft}=\begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

Then the system of equations for the two unknowns $\lambda_1(0)$ and $\lambda_2(0)$ will be

$$x_1(t_f)=e_{11}x_1(0)+e_{12}x_2(0)+e_{13}\lambda_1(0)+e_{14}\lambda_2(0)$$

$$x_2(t_f)=e_{21}x_1(0)+e_{22}x_2(0)+e_{23}\lambda_1(0)+e_{24}\lambda_2(0)$$

Inserting the values for the boundary conditions

$$e_{13}\lambda_1(0)+e_{14}\lambda_2(0)=-e_{11}h_{Max}$$

$$e_{23}\lambda_1(0)+e_{24}\lambda_2(0)=-e_{21}h_{Max}$$

or

$$\begin{bmatrix} e_{13} & e_{14} \\ e_{23} & e_{24} \end{bmatrix}\begin{bmatrix} \lambda_1(0) \\ \lambda_2(0) \end{bmatrix}=h_{max}\begin{bmatrix} -e_{11} \\ -e_{21} \end{bmatrix}$$

which will define the values for $\lambda_1(0)$ and $\lambda_2(0)$.

4.4.SYSTEM OF THREE VARIABLES

In this case the state vector is a 3-vector and the problem is again finding the control vector u(t) which will minimize

$$J=\frac{1}{2}\alpha t_f^2+\int_0^{t_f}\frac{1}{2}(qx_1^2+u^2)dt$$

for the system

$$\dot{x}_1=-x_2$$
$$\dot{x}_2=x_3$$
$$\dot{x}_3=u$$

with the initial and final conditions as follows

$$x_1(0)=h_{max} \qquad x_1(t_f)=0$$
$$x_2(0)=0 \qquad x_2(t_f)=0$$
$$x_3(0)=0 \qquad x_3(t_f)=0$$

with $t_o=0$ for the initial time and a free final time t_f which will also be minimized.

For this case the Hamiltonian is

$$H=\frac{1}{2}qx_1^2+\frac{1}{2}u^2-\lambda_1x_2+\lambda_2x_3+\lambda_3u$$

and the canonic equations are

$$\dot{x}_1=-x_2$$
$$\dot{x}_2=x_3$$
$$\dot{x}_3=u$$
$$\dot{\lambda}_1=-qx_1$$
$$\dot{\lambda}_2=\lambda_1$$
$$\dot{\lambda}_3=-\lambda_2$$

and since

$$\frac{\partial H}{\partial u}=0=u-\lambda_3$$

$$u=-\lambda_3$$

For the final time t_f we have

$$H(t_f)+\frac{\partial \theta}{\partial t_f}=0$$

$$\frac{1}{2}qx_1^2(t_f)-\frac{1}{2}u^2(t_f)-\lambda_1(t_f)x_2(t_f)-\lambda_2(t_f)x_3(t_f)-\lambda_3(t_f)u(t_f)$$
$$+\alpha t_f=0$$

with $\quad u=-\lambda_3 \quad$ and $\quad \left[x(t_f)\right]=\left[0\right]$

$$t_f=\frac{\lambda_3^2(t_f)}{2\alpha}$$

Again using matrix notation the canonical equations can be written in the form

$$\begin{bmatrix} x \\ \lambda \end{bmatrix}=\begin{bmatrix} & F & \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}$$

with the F matrix defined as follows

$$F=\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The solution to the above differential system is again as previously

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} e^{Ft} \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

This solution will give us the lift,velocity,acceleration and the control u(t) which is the jerk for this case as a function of time.We can again write e^{Ft} in series expansion as we have done previously,i.e.

$$e^{Ft} = I + Ft + \frac{(Ft)^2}{2!} + \frac{(Ft)^3}{3!} + \frac{(Ft)^4}{4!} + \ldots\ldots\ldots$$

What we have to do next is by utilizing the final conditions for x,$x(t_f)$,to determine the initial values for λ,$\lambda(0)$.To do this we must use the first three equations we have found as a solution. Thus following the same procedure for the two-vector case and applying the boundary condition values for x vector we have

$$e_{14}\lambda_1(0) + e_{15}\lambda_2(0) + e_{16}\lambda_3(0) = -e_{11}h_{max}$$

$$e_{24}\lambda_1(0) + e_{25}\lambda_2(0) + e_{26}\lambda_3(0) = -e_{21}h_{max}$$

$$e_{34}\lambda_1(0) + e_{35}\lambda_2(0) + e_{36}\lambda_3(0) = -e_{31}h_{max}$$

or

$$\begin{bmatrix} e_{14} & e_{15} & e_{16} \\ e_{24} & e_{25} & e_{26} \\ e_{34} & e_{35} & e_{36} \end{bmatrix} \begin{bmatrix} \lambda_1(0) \\ \lambda_2(0) \\ \lambda_3(0) \end{bmatrix} = h_{max} \begin{bmatrix} -e_{11} \\ -e_{21} \\ -e_{31} \end{bmatrix}$$

and the solution of this system of equations will give the values of $\lambda_1(0)$,$\lambda_2(0)$ and $\lambda_3(0)$.

4.5. SYSTEM OF FOUR VARIABLES

Finally let us investigate the 4-vector state variable case for which we have control on higher derivatives of the lift.

We are going to find the control vector u(t) such that we will minimize the functional J, which we can write for this case in general as follows

$$J=\frac{1}{2}\alpha t_f^2+\int_0^{t_f}\frac{1}{2}(q_1x_1^2+q_2x_2^2+q_3x_3^2+q_4x_4^2+u^2)dt$$

for the system

$$\dot{x}_1=-x_2$$
$$\dot{x}_2=x_3$$
$$\dot{x}_3=x_4$$
$$\dot{x}_4=u$$

with the initial and final conditions

$$x_1(0)=h_{max} \qquad x_1(t_f)=0$$
$$x_2(0)=0 \qquad x_2(t_f)=0$$
$$x_3(0)=0 \qquad x_3(t_f)=0$$
$$x_4(0)=0 \qquad x_4(t_f)=0$$

with $t_o=0$ for the initial time and a free final time t_f which will also be minimized. The Hamiltonian for this case is

$$H=\frac{1}{2}q_1x_1^2+\frac{1}{2}q_2x_2^2+\frac{1}{2}q_3x_3^2+\frac{1}{2}q_4x_4^2$$
$$-\lambda_1x_2+\lambda_2x_3+\lambda_3x_4+\lambda_4u$$

The canonic equations are

$$\dot{x}_1=-x_2 \qquad \dot{\lambda}_1=-q_1x_1$$
$$\dot{x}_2=x_3 \qquad \dot{\lambda}_2-\lambda_1-q_2x_2$$
$$\dot{x}_3=x_4 \qquad \dot{\lambda}_3=-\lambda_2-q_3x_3$$
$$\dot{x}_4=u \qquad \dot{\lambda}_4=-\lambda_3-q_4x_4$$

From
$$\frac{\partial H}{\partial u} = 0 = u + \lambda_4$$

$$u = -\lambda_4$$

For the final time t_f

$$H(t_f) - \frac{\partial \theta}{\partial t_f} = 0$$

with $u = -\lambda_4$ and the final boundary conditions applied to the equation gives for the final time t_f

$$t_f = \frac{\lambda_4^2(t_f)}{2\alpha}$$

For the system of differential equations in the form of

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \quad F \quad \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

the matrix F will be as given below

$$F = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -q_3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -q_4 & 0 & 0 & -1 & 0 \end{bmatrix}$$

and the solution is

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} e^{Ft} \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

where e^{Ft} is found by series expansion as before and the values for $\lambda(0)$ are found by using the final boundary conditions on x

5.INTERPRETATIONS OF RESULTS

Several computer runs have been made for various cases with different data.The following conclusions may be derived from a comparison of these results.

As the matrix size,i.e. the size of the state vector x increases,the degree of the lift curve increases.Hence,the great the size of the state vector,the greater the number of continuous higher derivatives of the lift curve which can be easily interpreted with the size of state vector and its definition.

However due to these continuous high derivatives we can easily observe that we lose from the area under the lift curve and also face with higher velocity and acceleration values.But since we want to have a continuous and smooth operation of the cam on the valve as much as possible,we must admit this condition and try to minimize our functional by changing the coefficients and the final time to obtain the desired curves.

In order to be able to make an easier comparison of the results we have obtained the values are found and plotted on a dimensionless basis.That is,the lift,velocity and acceleration values are divided by the maximum valve lift,h_{max},and the time, or the angle θ,is divided by the value of the half event,i.e. the duration between the opening of valve and its maximum lift position.

One can observe from the figures that a small change in final time affects our parameters to a great extent.Decreasing the final time increases the dimensionless area under the lift curve but it also increases the maximum velocity and acceleration values significantly.On the other hand however,by increasing the final time,we gain for the maximum velocity and acceleration values but we lose from the area under the lift curve.

We also see that by changing the coefficients in our functional we can control the values of velocity and acceleration. from a comparison of the results found for different coefficien of the state vector elements,i.e. lift,velocity,acceleration an jerk,we can summarize our predictions as follows, increase in q_1 : slight increase in area under lift curve,no significant change on velocity and acceleration peaks

increase in q_3 : increase in area,increase in velocity and acceleration peaks

decrease in q_1 : no significant change especially for small decreases

decrease in q_3 : decrease in area,decrease in velocity and acceleration peaks

increase in q_1, decrease in q_3 : decrease in area,decrease in velocity and acceleration peaks

decrease in q_1 increase in q_3 : increase in area,increase in velocity and acceleration peaks

decrease in q_2 : increase in area,increase in velocity and acceleration peaks

decrease in q_4 : increase in area,increase in velocity and acceleration peaks

increase in q_2 : decrease in area,decrease in velocity and acceleration peaks

increase in q_4 : significant decrease in area,also significant decrease in velocity and acceleration peaks

decrease in q_2 decrease in q_4 : increase in area,increase in velocity and acceleration peaks

increase in q_2 increase in q_4 : significant decrease in area,also significant decrease in velocity and acceleration peaks

decrease in t_f : significant increase in area,and velocity and acceleration peaks

increase in t_f : significant decrease in area,and velocity and acceleration peaks

q_1	q_3	Area	Min.Acc.	Max.Acc.	Max.Vel.
1	1	0.50105	-7.5286	7.5286	2.1910
10	0.1	0.50014	-7.5129	7.5124	2.1862
0.1	10	0.51047	-7.6885	7.6945	2.2404
0.1	1	0.50150	-7.5286	7.5285	2.1910
1	0.1	0.50010	-7.5118	7.5116	2.1860
10	1	0.50108	-7.5004	7.5294	2.1911
1	10	0.51047	-7.6884	7.6945	2.2404
100	1	0.50136	-7.5431	7.5371	2.1927
100	0.1	0.50041	-7.5261	7.5201	2.1877

TABLE.I.Effect of coefficients of lift and acceleration on the profile(no initial ramp)

q_1	q_3	Area	Min.Acc.	Max.Acc.	Max.Vel.
1	1	0.51384	-7.2237	6.9301	2.1215
10	0.1	0.52270	-7.2089	6.9149	2.1170
0.1	10	0.52270	-7.3739	7.0861	2.1678
0.1	1	0.51384	-7.2235	6.9300	2.1215
1	0.1	0.51296	-7.2077	6.9142	2.1169
10	1	0.51387	-7.2249	6.9308	2,1216
1	10	0.52270	-7.3738	7.0862	2.1678
100	1	0.51413	-7.2373	6.9381	2.1231
100	0.1	0.51324	-7.2212	6.9221	2.1184

TABLE.II.Effect of coefficients of lift and acceleration on the profile(with initial ramp)

q_2	q_4	Area	Min.Acc.	Max.Acc.	Max.Vel.
1	0	0.51364	-7.2162	6.9257	2.1204
0.1	0	0.51382	-7.2228	6.9297	2.1214
10	0	0.51188	-7.1495	6.8859	2.1106
0	1	0.51306	-7.2076	6.9109	2,1164
0	0.1	0.51388	-7.2244	6.9303	2.1216
0	10	0.37290	-2.9239	3.9063	1.4106
1	1	0.51285	-7.1623	6.9063	2.1153
10	10	0.37262	-2.9133	3.8988	1.4088
0.1	0.1	0.51386	-7.2236	6.9298	2.1215

TABLE.III.Effect of coefficients of velocity and jerk on the profile(with initial ramp)

t_f	Area	Min.Acc.	Max.Acc.	Max.Vel.
0.8	0.6255	-11.746	11.746	2.7347
0.9	0.55632	-9.2866	9.2866	2,4323
1.1	0.45595	-6.2293	6.2293	1.9941
1.2	0.41849	-5.2425	5.2424	1.8308

TABLE.IV.Effect of final time on the profile

Acceleration check:

The results have shown that the values of maximum negative and positive accelerations increase as the degree of the lift curve increases.From the previous discussions and the results we have obtained.we can say that we can control the acceleration by changing the coefficients or the final time. However these are done for the dimensionless parameters up to now and a change in time duration,that the valve is open within that interval,will change the magnitude of the acceleration significantly.

For a numerical check consider an engine with the following data,

Half event : 70 degrees
Maximum valve lift,h_{max}: 9.0 mm.
Maximum engine rev.: 4000 rpm.

And from the results we have obtained for various cases let us assume that our maximum acceleration value is

$$a=6.5$$

We have previously stated that this is a dimensionless value and with the given data we obtain for the value of acceleration

$$a=6.5\cdot\frac{9.0}{(70)^2}=0.0119 \text{ mm/deg}^2$$

which for the maximum rpm of the engine gives

$$a=0.0119(\frac{4000}{2})^2\cdot 36\cong 1714 \text{ m/sec}^2$$

since 1 rpm=6 deg/sec

The value we have found above is an acceptable value for positive acceleration values but it is obviously a great value for the maximum negative acceleration values even for small size engines with low weight of its reciprocating masses.Hence we must try to shift the acceleration curve upwards accepting high peak values for positive acceleration to decrease the maximum negative acceleration value.

Check for radius of curvature:

In the previous sections the radius of curvature was defined with the relation

$$\rho = \frac{\left[r^2 + r'^2\right]^{3/2}}{r^2 + 2r'^2 - rr''}$$

The values corresponding to the maximum acceleration point are, in dimensionless form

$h = 0.9$
$h' = 1.0$
$h'' = -5.95$

Hence for a system with $h_{max} = 9.0$ mm. and 70 degree half event,

$h = 8.1$ mm.
$h' = 0.1286$ mm/deg.
$h'' = -0.0193$ mm/deg^2.

Inserting $w = 2\pi/360$ into the equation with the above given values we have for curvature

$$\rho = 17.01 \text{ mm.}$$

For the flat faced follower case we have for the maximum negative acceleration value

$$\rho = R_0 + h + \frac{a}{w^2}$$

with $w = \pi/180$ rad/deg. We know that in order to avoid undercutting of cam we must at least have the condition $\rho > 0$. Inserting the same values as given above we have

$$R_0 + 9.0 - \frac{0.0119}{(\pi/180)} > 0$$

$$R_0 > 30.06 \text{ mm.}$$

Obviously this is not a value within reasonable limits for the base circle radius of cam paired with a flat faced follower. Thus our previous statement applies and we have to decrease the value of maximum negative acceleration into reasonable limits in case of the flat faced follower.

6.CONCLUSION AND RECOMMENDATIONS

An investigation of the results we have obtained for various different coefficients of acceleration and jerk affect our parameters more significantly than the coefficients of lift and velocity functions.Thus the coefficients of acceleration and jerk are of greater importance.As we have previously stated,to obtain a fuller area under the lift curve we have to admit high acceleration values.However by using larger cams as much as possible this can be overcome up to an extent.Hence what we have to do is to change these coefficients further to obtain greater areas under lift curve and check for certain parameters we have discussed previously by estimating reasonable values for the base circle radius especially.

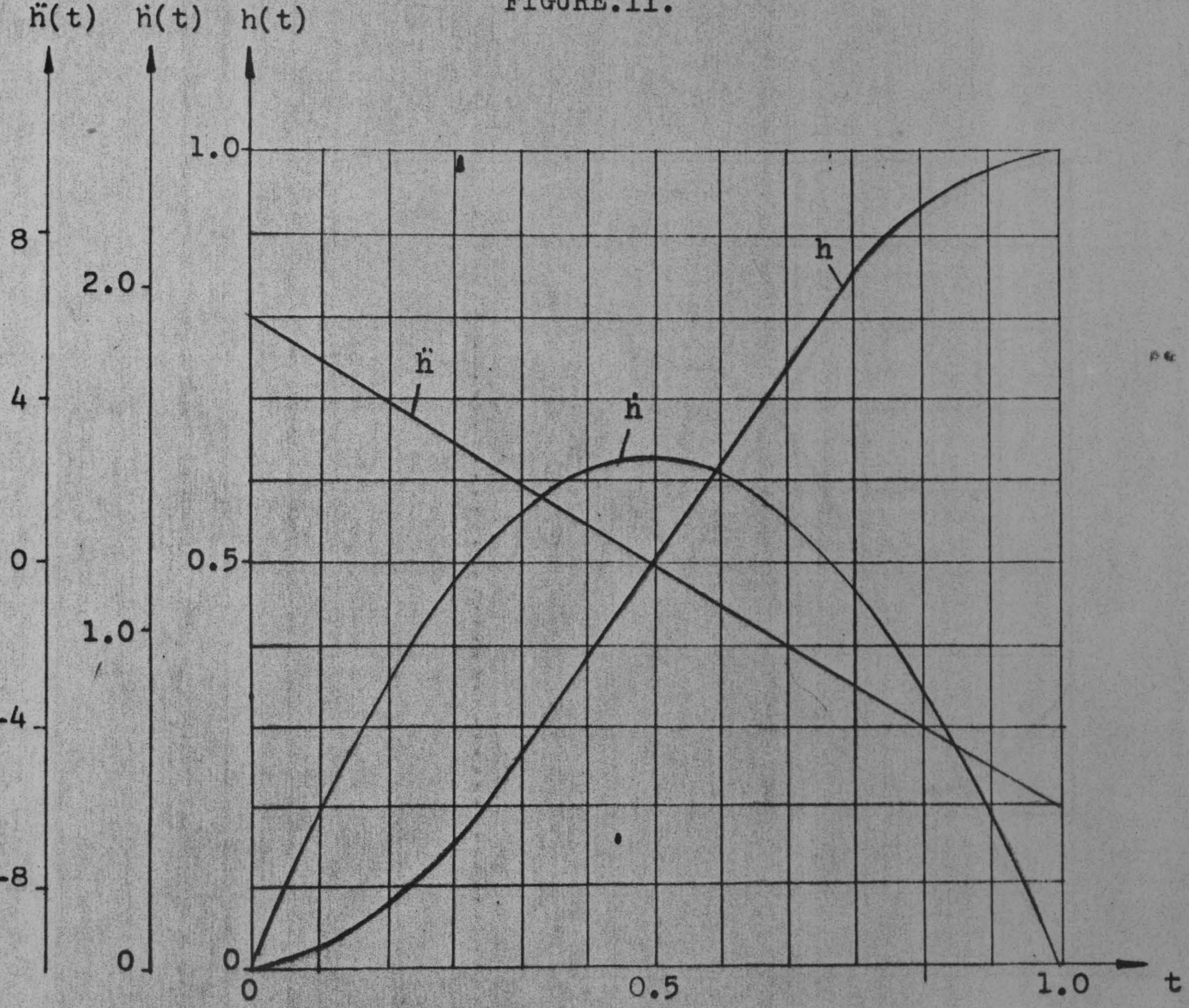
Also considering an initial ramp,i.e. giving a nonzero initial value for the velocity at $t \quad t_0$ will help us to increase the area under the lift curve and decrease our peak velocity and acceleration values.

For further investigation of optimal profiles the following recommandations may be given,

-To apply different equality constraints on the boundary conditions,especialy for acceleration at final time and velocity at initial time,for specific cases.

-To insert inequality constraints into the formulation and thus define limits to the values of the velocity,acceleration and jerk peaks additional to the specified boundary conditions.

FIGURE.11.



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (x_1^2 + u^2)\,dt$$

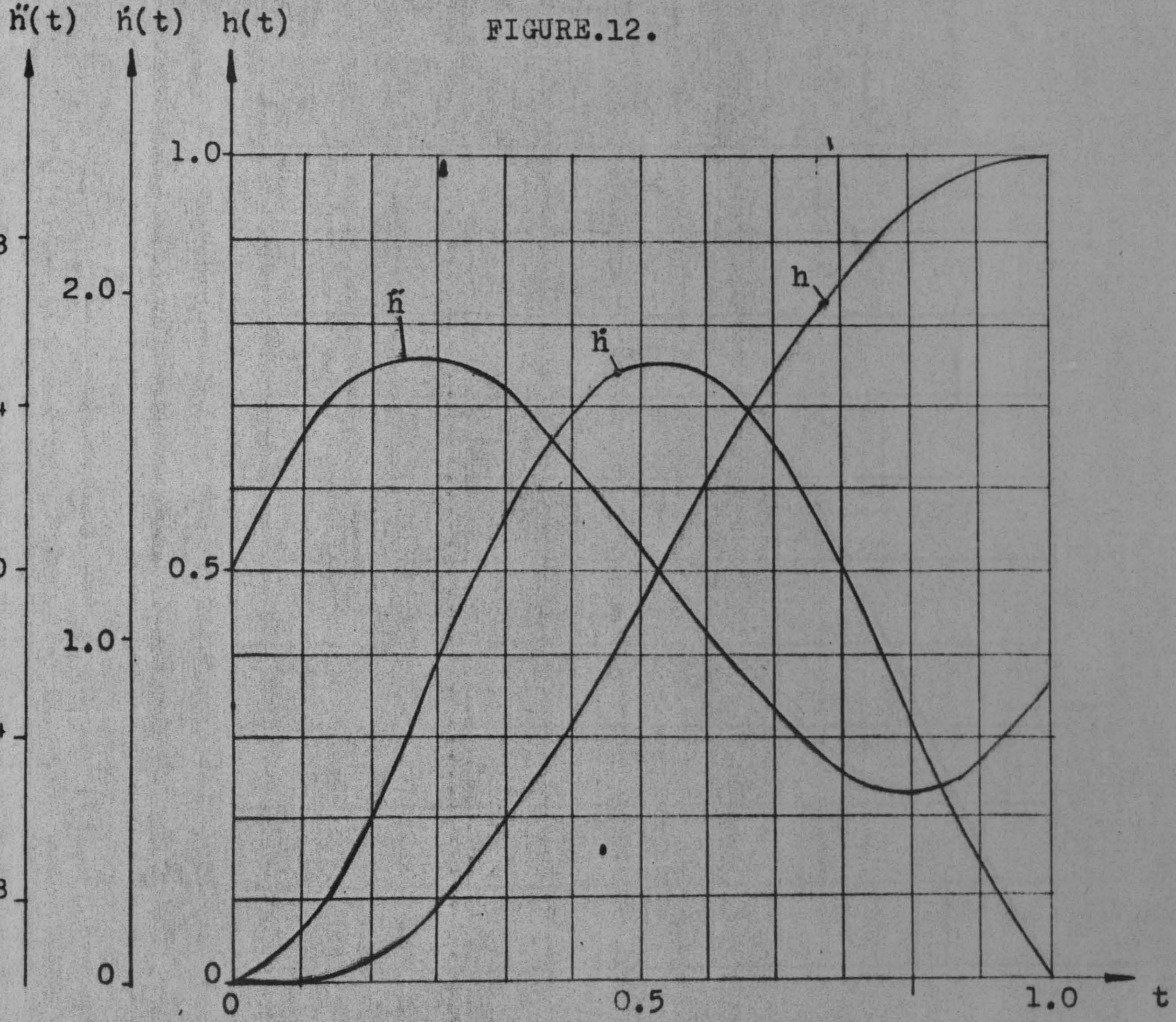
BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\qquad x_1(t_f) = 0$

$x_2(0) = 0$ $\qquad x_2(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.49236
MIN.ACCELERATION: -5.9691
MAX.ACCELERATION: 5.0523
MAX.VELOCITY: 1.4993

FIGURE.12.



$$J=\frac{1}{2}\alpha t_f^2+\frac{1}{2}\int_0^{t_f}(0.05x_1^2+u^2)dt$$

BOUNDARY CONDITIONS:

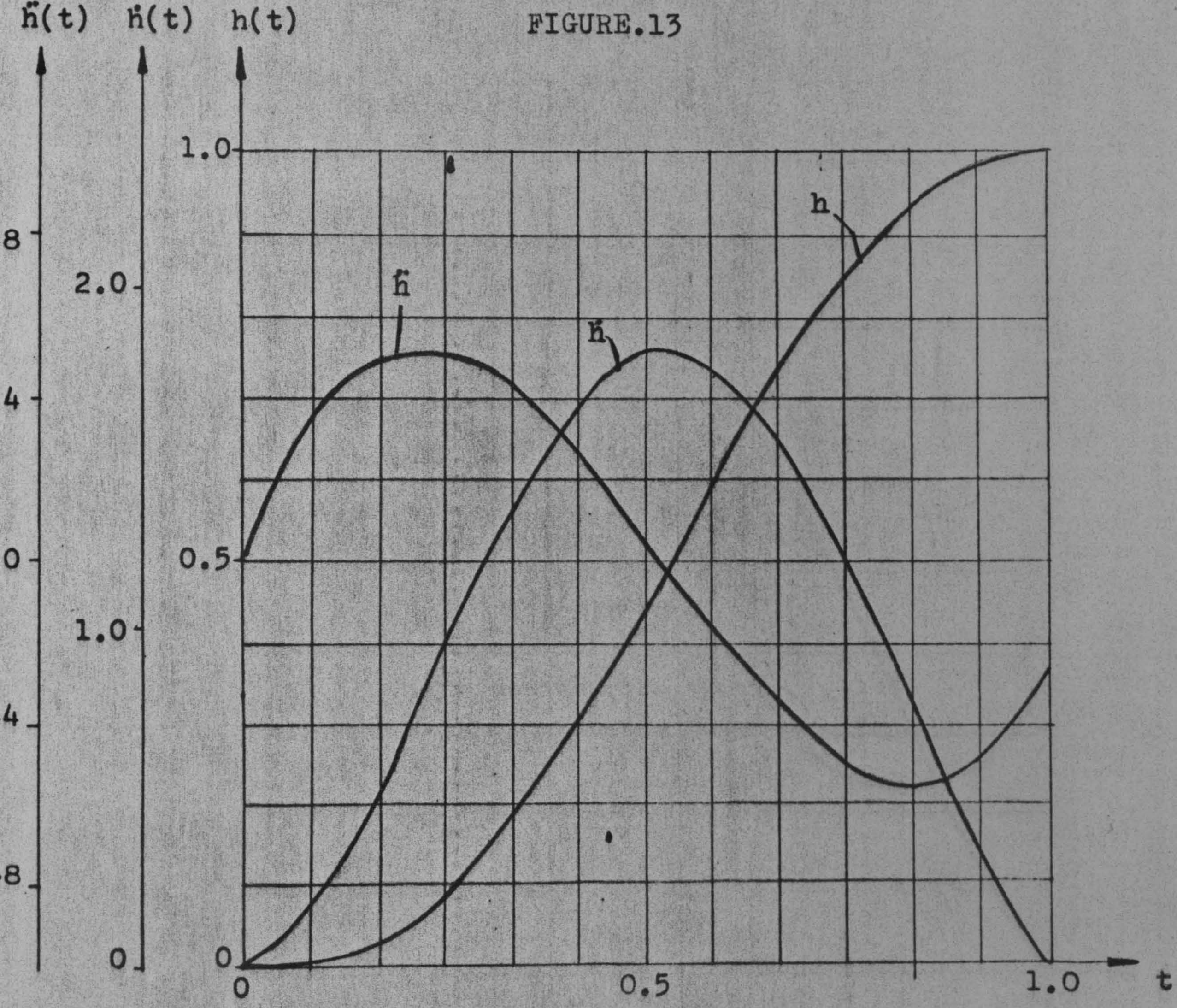
$x_1(0)=1.0$ $x_1(t_f)=0$

$x_2(0)=0$ $x_2(t_f)=0$

$x_3(0)=0$ $x_3(t_f)=-2.5$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47917
MIN.ACCELERATION: -5.4398
MAX.ACCELERATION: 5.2996
MAX.VELOCITY: 1.8036

FIGURE.13



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(0.1x_1^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

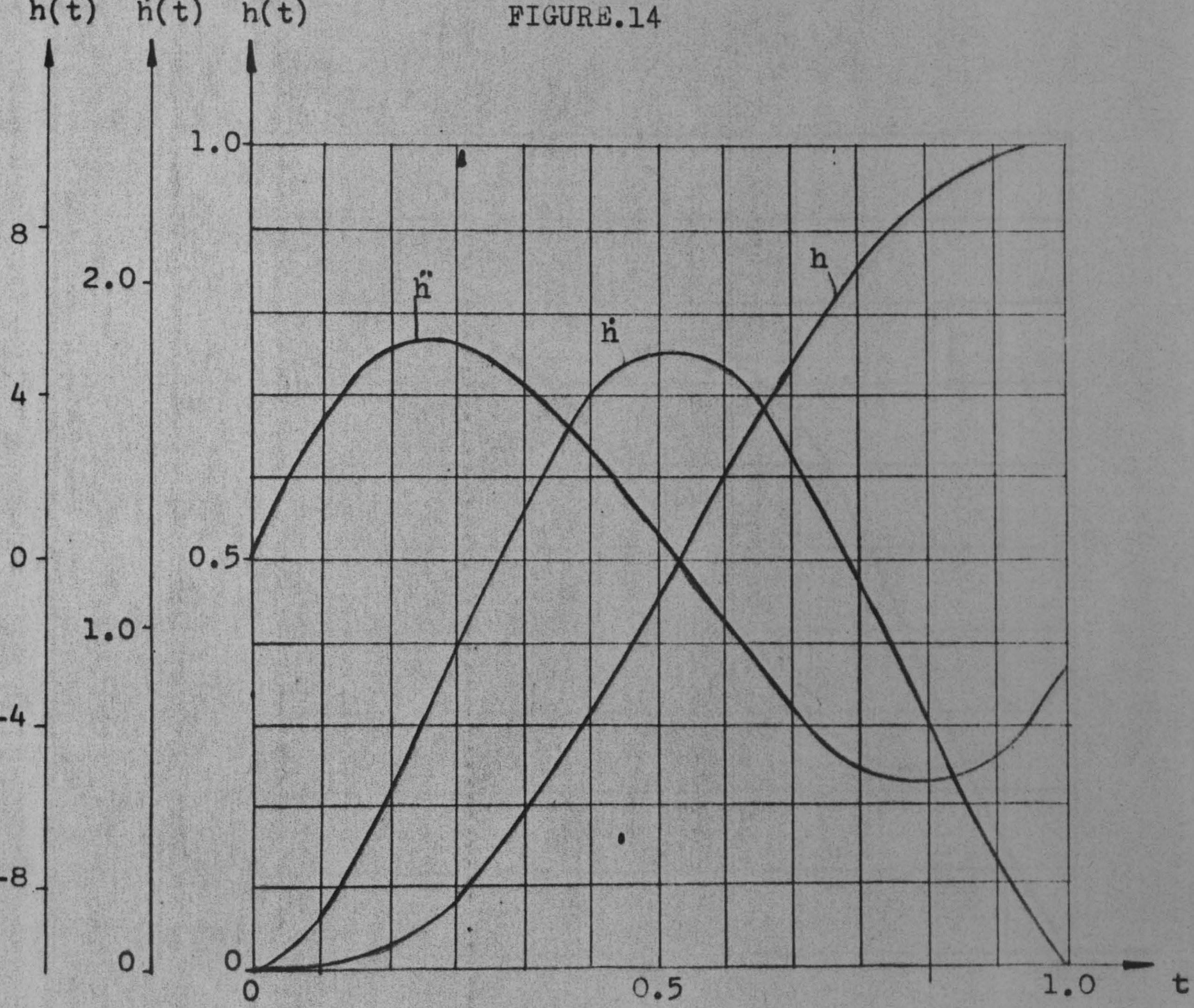
$x_1(0) = 1.0$ $\qquad$ $x_1(t_f) = 0$

$x_2(0) = 0$ $\qquad$ $x_2(t_f) = 0$

$x_3(0) = 0$ $\qquad$ $x_3(t_f) = -2.5$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47917
MIN.ACCELERATION: -5.4398
MAX.ACCELERATION: 5.2996
MAX.VELOCITY: 1.8036

FIGURE.14



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2+u^2)\,dt$$

BOUNDARY CONDITIONS:

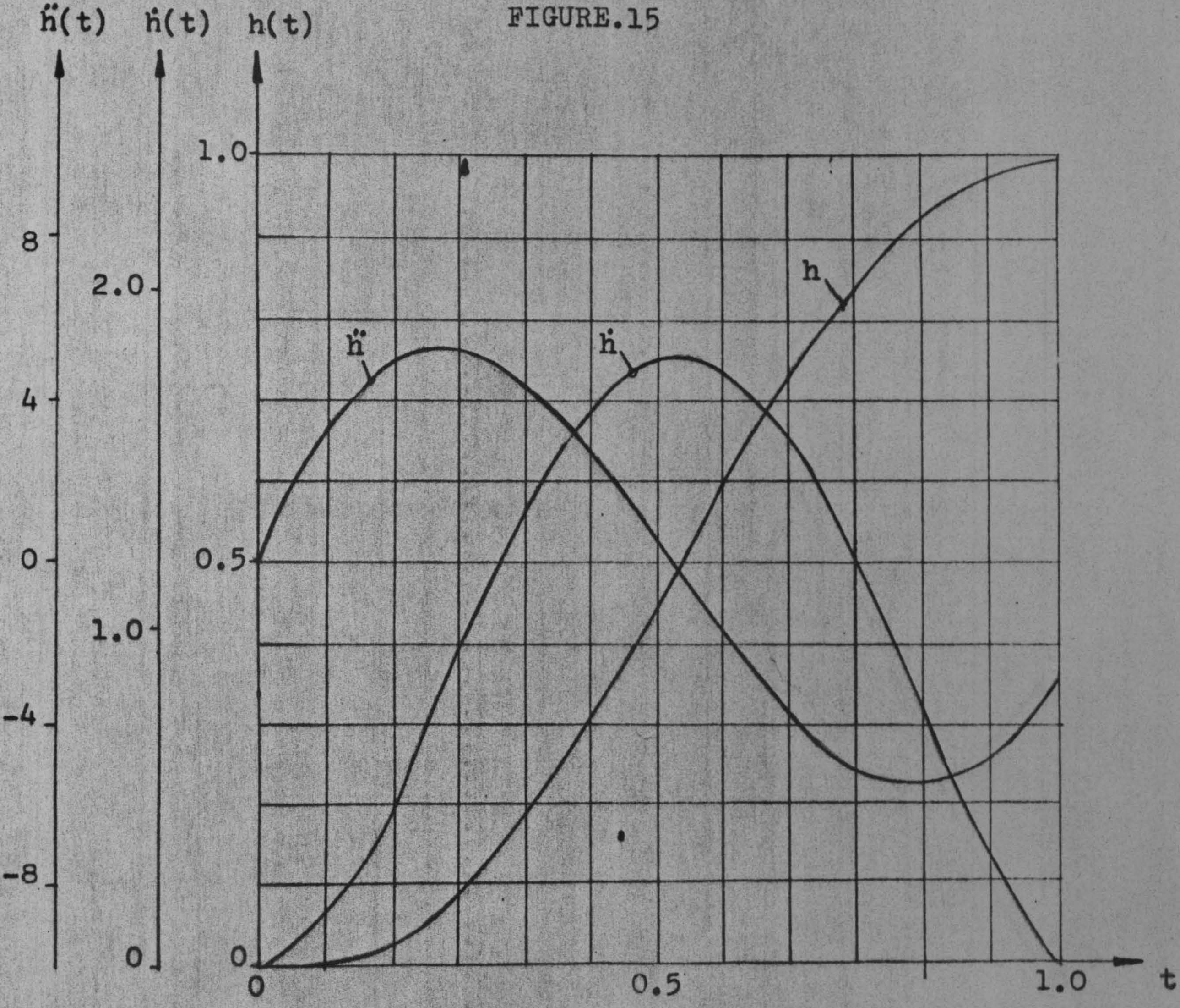
$x_1(0)=1.0$ $\qquad$ $x_1(t_f)=0$

$x_2(0)=0$ $\qquad$ $x_2(t_f)=0$

$x_3(0)=0$ $\qquad$ $x_3(t_f)=-2.5$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47918
MIN.ACCELERATION: -5.4397
MAX.ACCELERATION: 5.2998
MAX.VELOCITY: 1.8037

FIGURE.15



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(10x_1^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

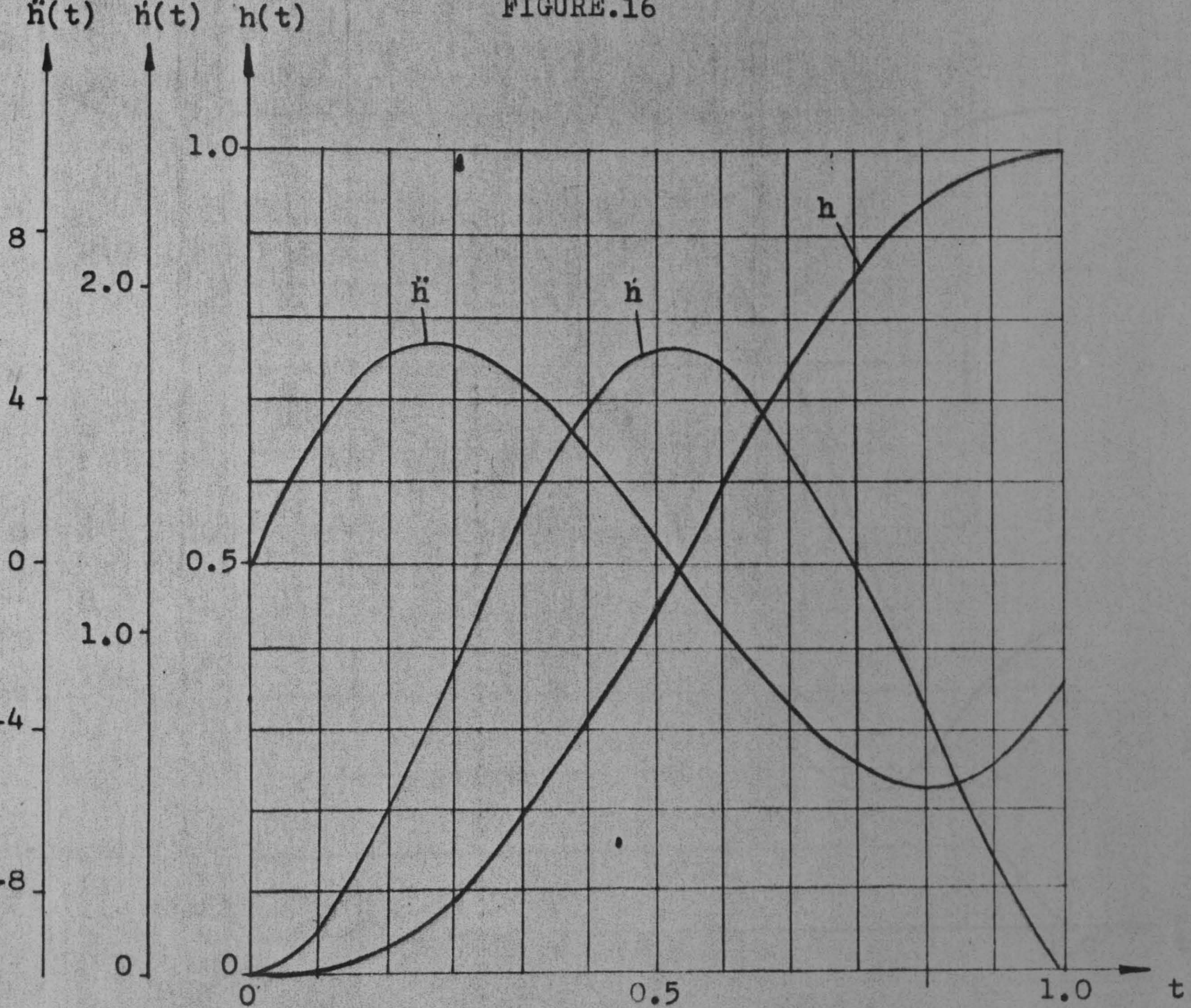
$x_1(0) = 1.0$ $\quad$ $x_1(t_f) = 0$

$x_2(0) = 0$ $\quad$ $x_2(t_f) = 0$

$x_3(0) = 0$ $\quad$ $x_3(t_f) = -2.5$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47928
MIN.ACCELERATION: -5.4388
MAX.ACCELERATION: 5.3014
MAX.VELOCITY: 1.8038

FIGURE.16



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (20x_1^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

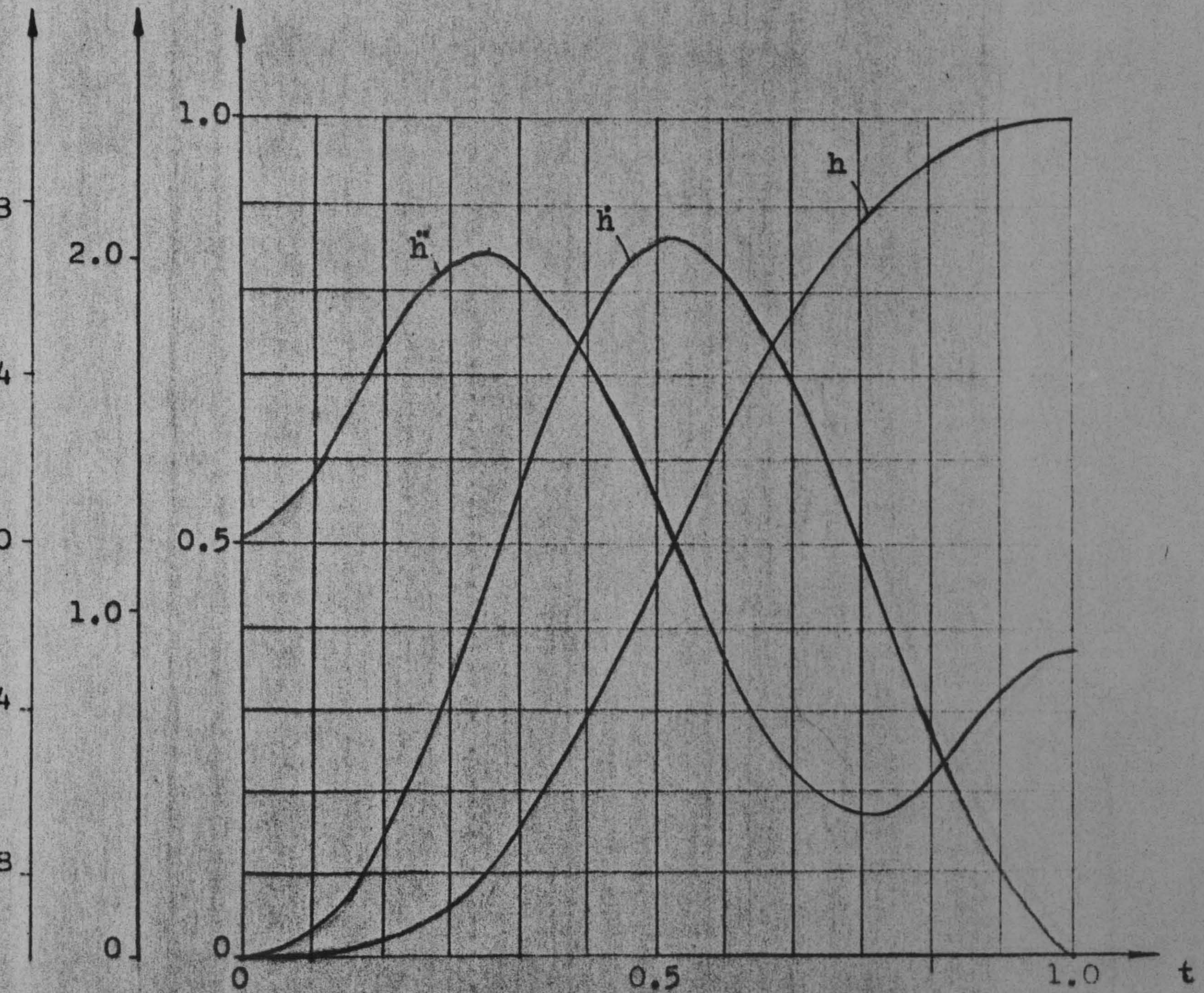
$x_1(0) = 1.0$ $\qquad$ $x_1(t_f) = 0$

$x_2(0) = 0$ $\qquad$ $x_2(t_f) = 0$

$x_3(0) = 0$ $\qquad$ $x_3(t_f) = -2.5$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47939
MIN.ACCELERATION: -5.4378
MAX.ACCELERATION: 5.3033
MAX.VELOCITY: 1.8040

FIGURE.17



$$J= \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2+u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0)=1.0$ $\quad x_1(t_f)=0$

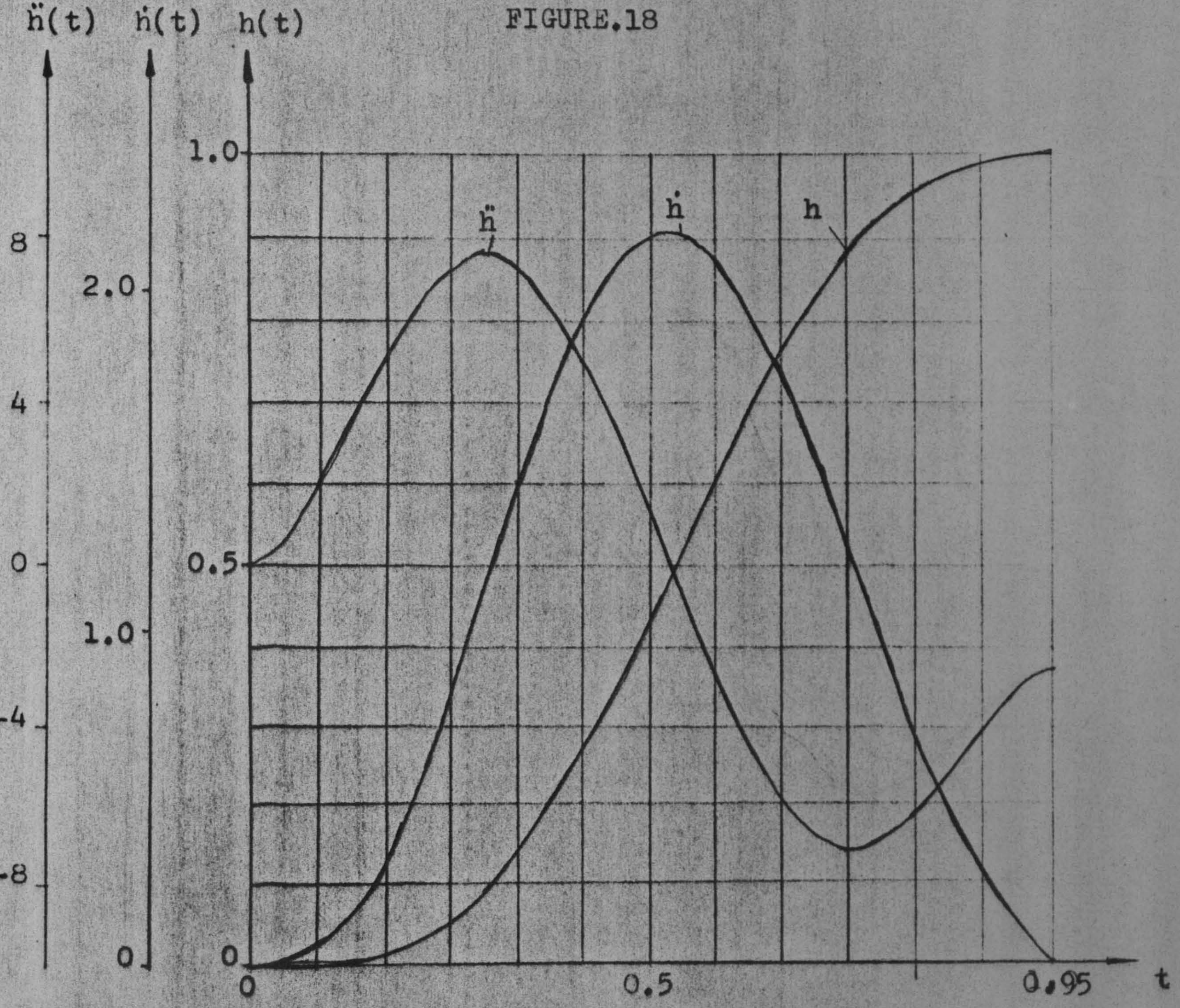
$x_2(0)=0$ $\quad x_2(t_f)=0$

$x_3(0)=0$ $\quad x_3(t_f)=-2.5$

$x_4(0)=0$ $\quad x_4(t_f)=0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47024
MIN.ACCELERATION: -6.3317
MAX.ACCELERATION: 6.686
MAX.VELOCITY: 2.0436

FIGURE.18



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (x_1^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\qquad$ $x_1(t_f) = 0$

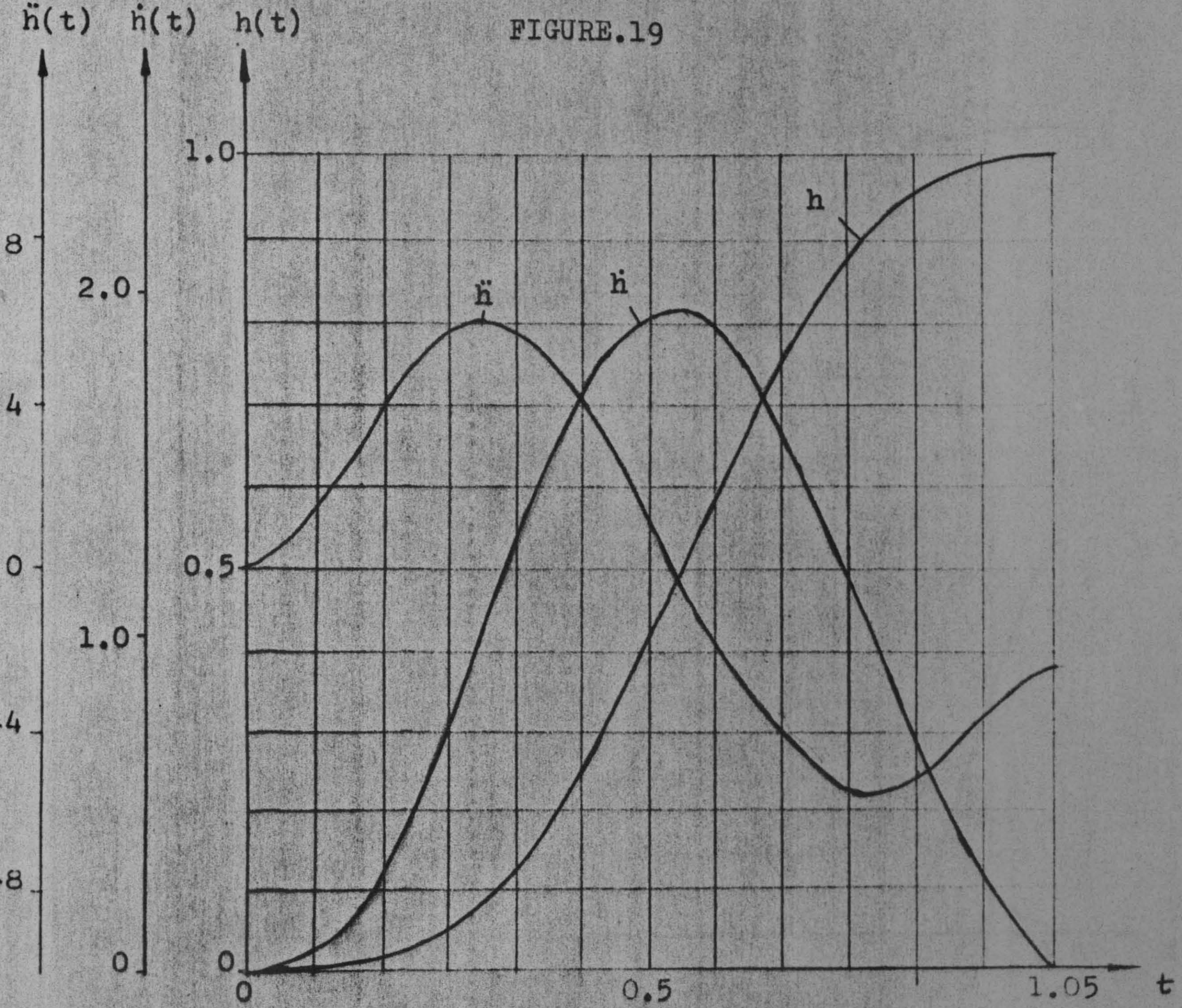
$x_2(0) = 0$ $\qquad$ $x_2(t_f) = 0$

$x_3(0) = 0$ $\qquad$ $x_3(t_f) = -2.5$

$x_4(0) = 0$ $\qquad$ $x_4(t_f) = 0$

FINAL TIME: 0.95
AREA UNDER LIFT CURVE: 0.49804
MIN.ACCELERATION: -7.1258
MAX.ACCELERATION: 7.4967
MAX.VELOCITY: 2.1651

FIGURE.19



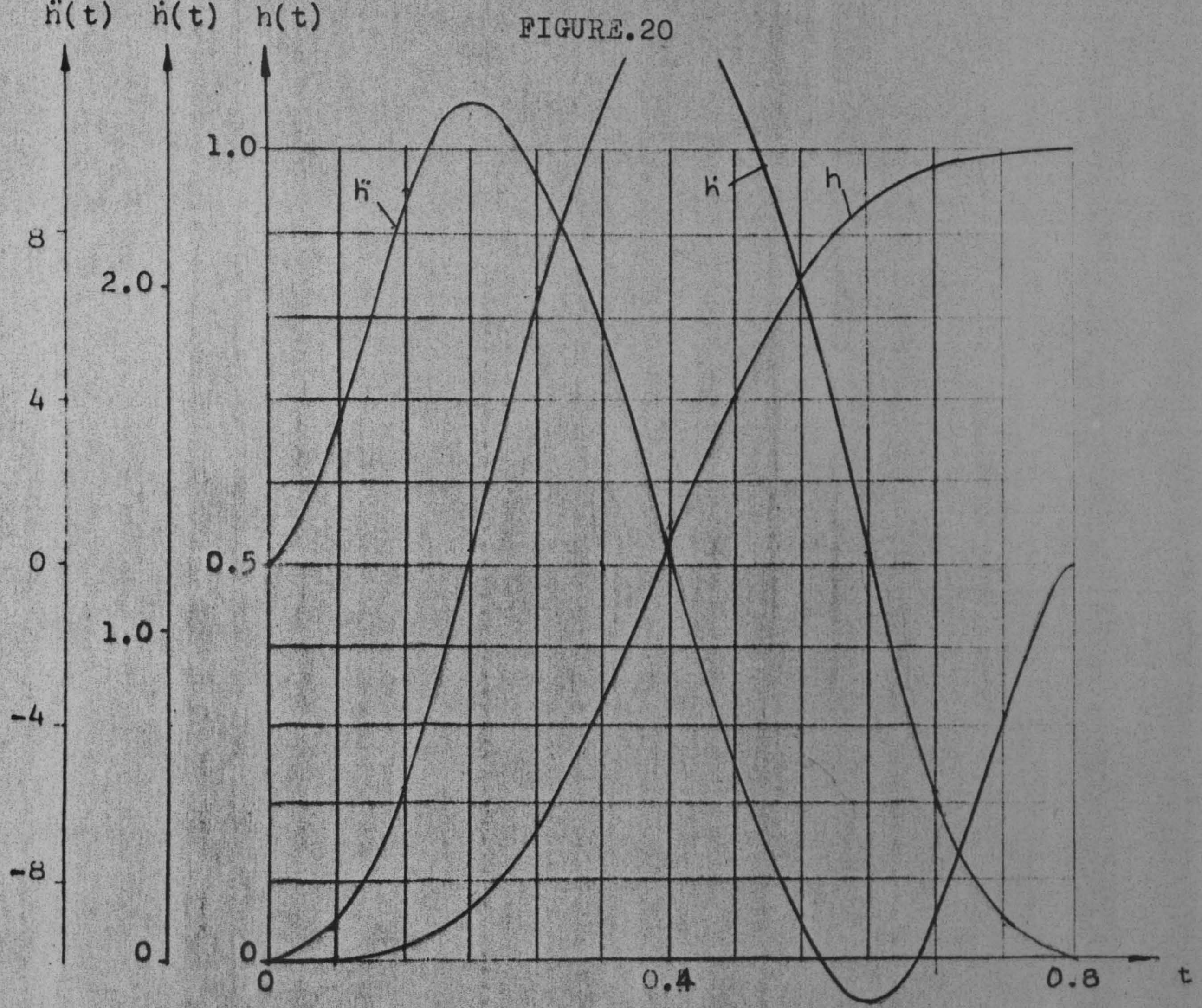
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (x_1^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$	$x_1(t_f) = 0$
$x_2(0) = 0$	$x_2(t_f) = 0$
$x_3(0) = 0$	$x_3(t_f) = -2.5$
$x_4(0) = 0$	$x_4(t_f) = 0$

FINAL TIME: 1.05
AREA UNDER LIFT CURVE: 0.44494
MIN.ACCELERATION: -5.6653
MAX.ACCELERATION: 5.9983
MAX.VELOCITY: 1.9351

FIGURE.20



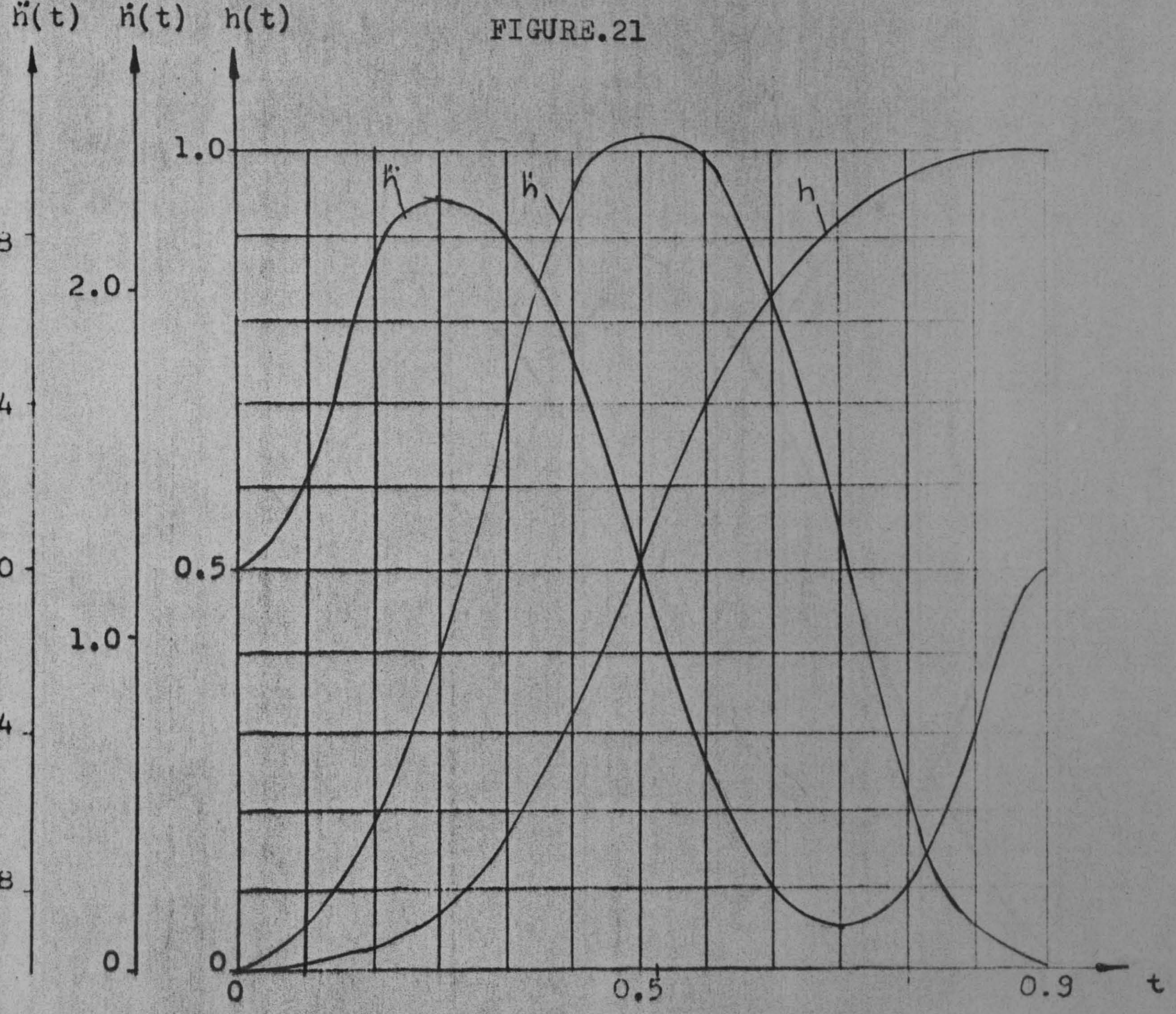
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2 + x_3^2 + u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$	$x_1(t_f) = 0$
$x_2(0) = 0$	$x_2(t_f) = 0$
$x_3(0) = 0$	$x_3(t_f) = 0$
$x_4(0) = 0$	$x_4(t_f) = 0$

FINAL TIME: 0.8
AREA UNDER LIFT CURVE: 0.6255
MIN.ACCELERATION: -11.746
MAX.ACCELERATION: 11.746
MAX.VELOCITY: 2.7347

FIGURE.21



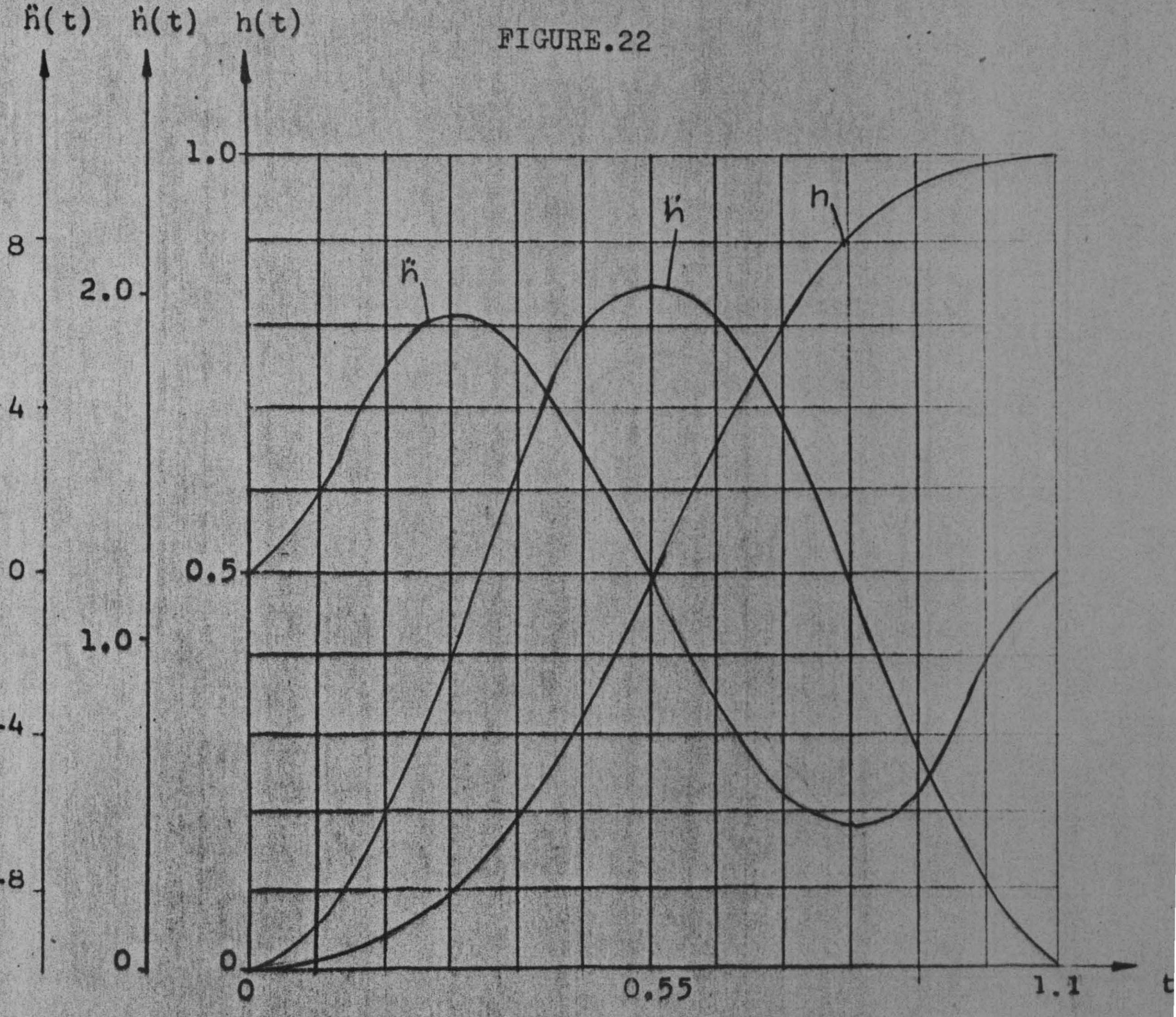
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (x_1^2 + x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\quad$ $x_1(t_f) = 0$
$x_2(0) = 0$ $\quad$ $x_2(t_f) = 0$
$x_3(0) = 0$ $\quad$ $x_3(t_f) = 0$
$x_4(0) = 0$ $\quad$ $x_4(t_f) = 0$

FINAL TIME: 0.9
AREA UNDER LIFT CURVE: 0.55632
MIN.ACCELERATION: -9.2866
MAX.ACCELERATION: 9.2866
MAX.VELOCITY: 2.4323

FIGURE.22



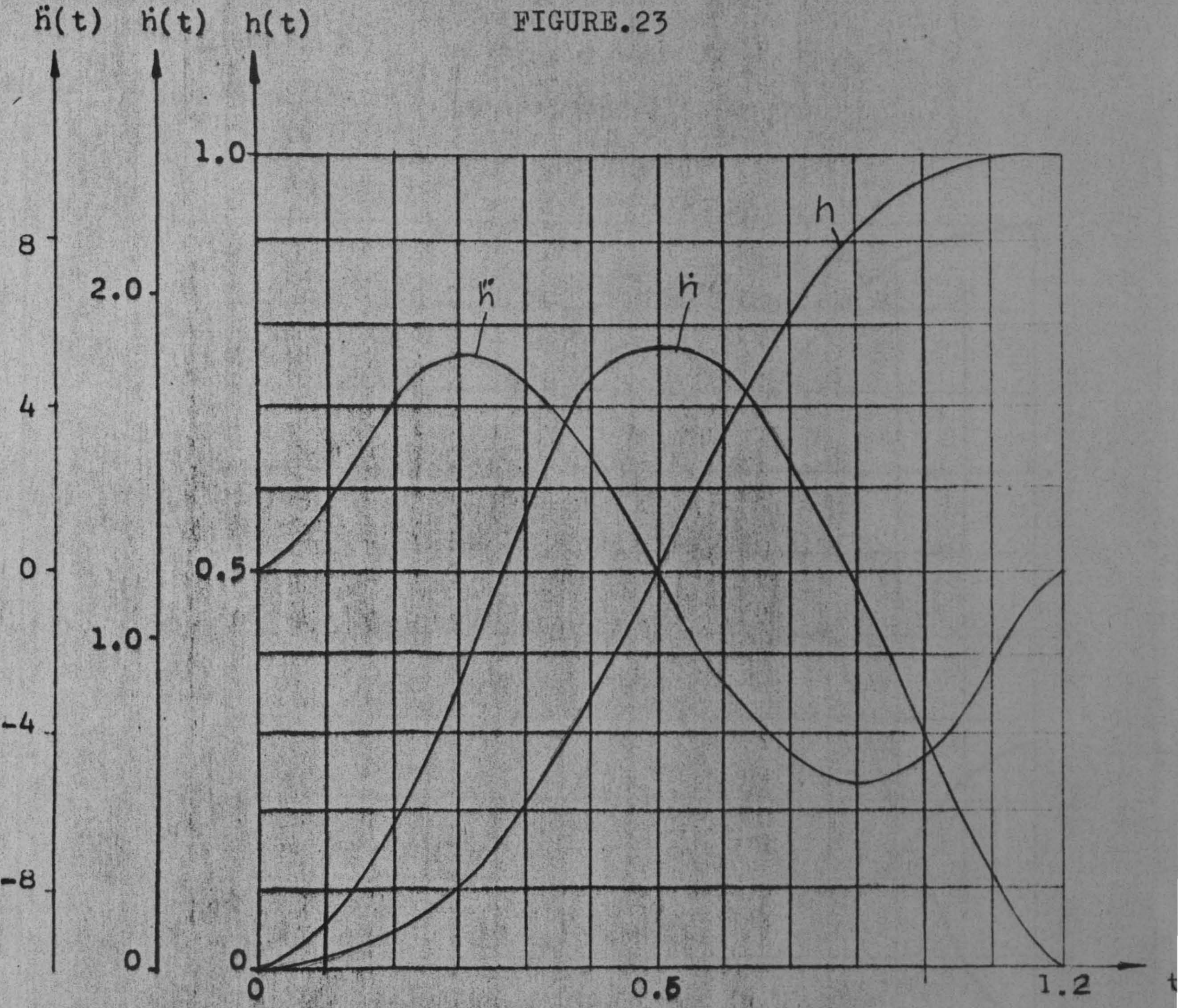
$$J= \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2+x_3^2+u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $x_1(t_f) = 0$
$x_2(0) = 0$ $x_2(t_f) = 0$
$x_3(0) = 0$ $x_3(t_f) = 0$
$x_4(0) = 0$ $x_4(t_f) = 0$

FINAL TIME: 1.1
AREA UNDER LIFT CURVE: 0.45595
MIN.ACCELERATION: -6,2293
MAX.ACCELERATION: 6.2293
MAX.VELOCITY: 1.9941

FIGURE.23



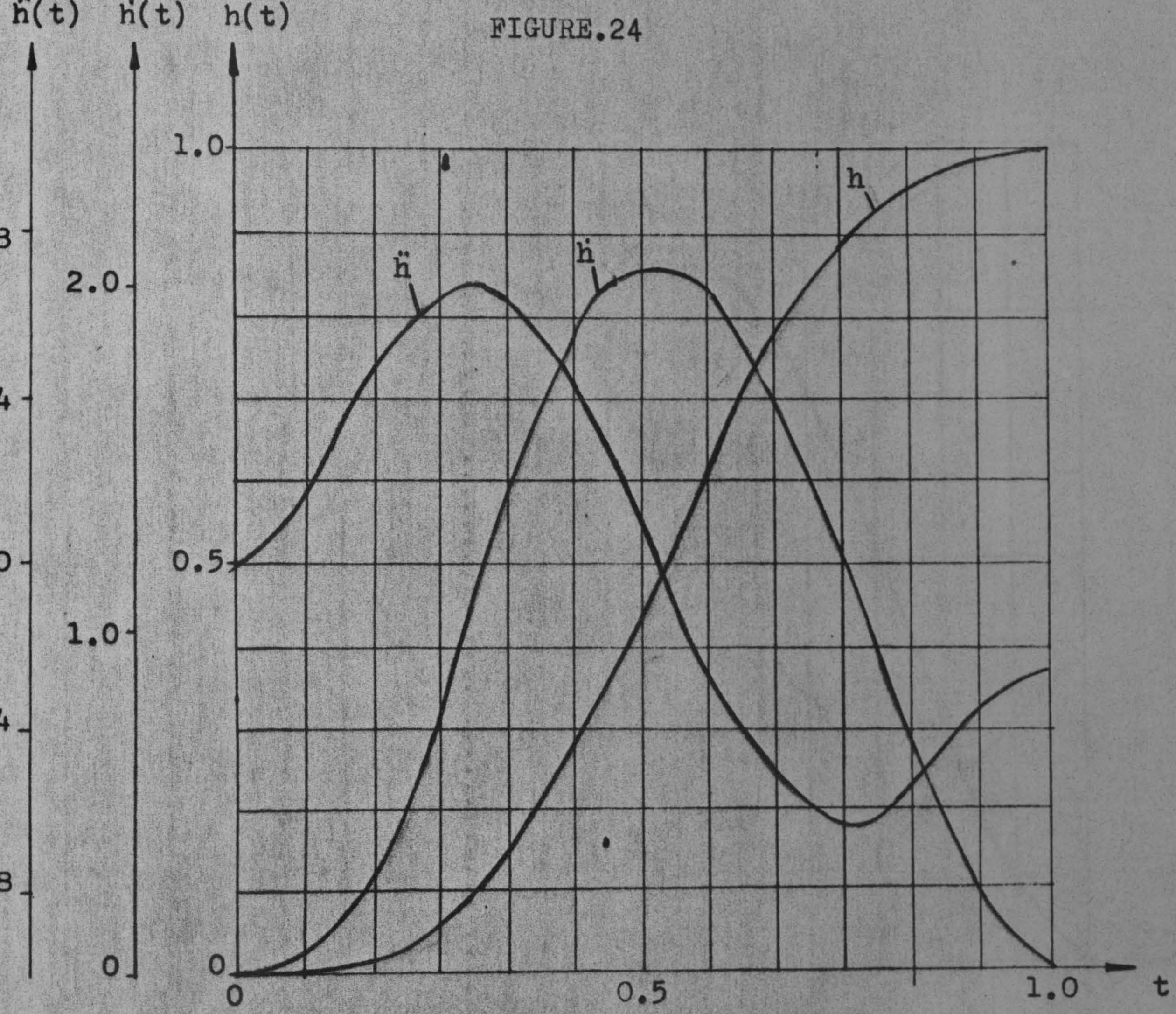
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2 + x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) - 1.0$ $x_1(t_f) = 0$
$x_2(0) = 0$ $x_2(t_f) = 0$
$x_3(0) = 0$ $x_3(t_f) = 0$
$x_4(0) = 0$ $x_4(t_f) = 0$

FINAL TIME: 1.2
AREA UNDER LIFT CURVE: 0.41849
MIN.ACCELERATION: -5.2425
MAX.ACCELERATION: 5.2424
MAX.VELOCITY: 1.8308

FIGURE.24



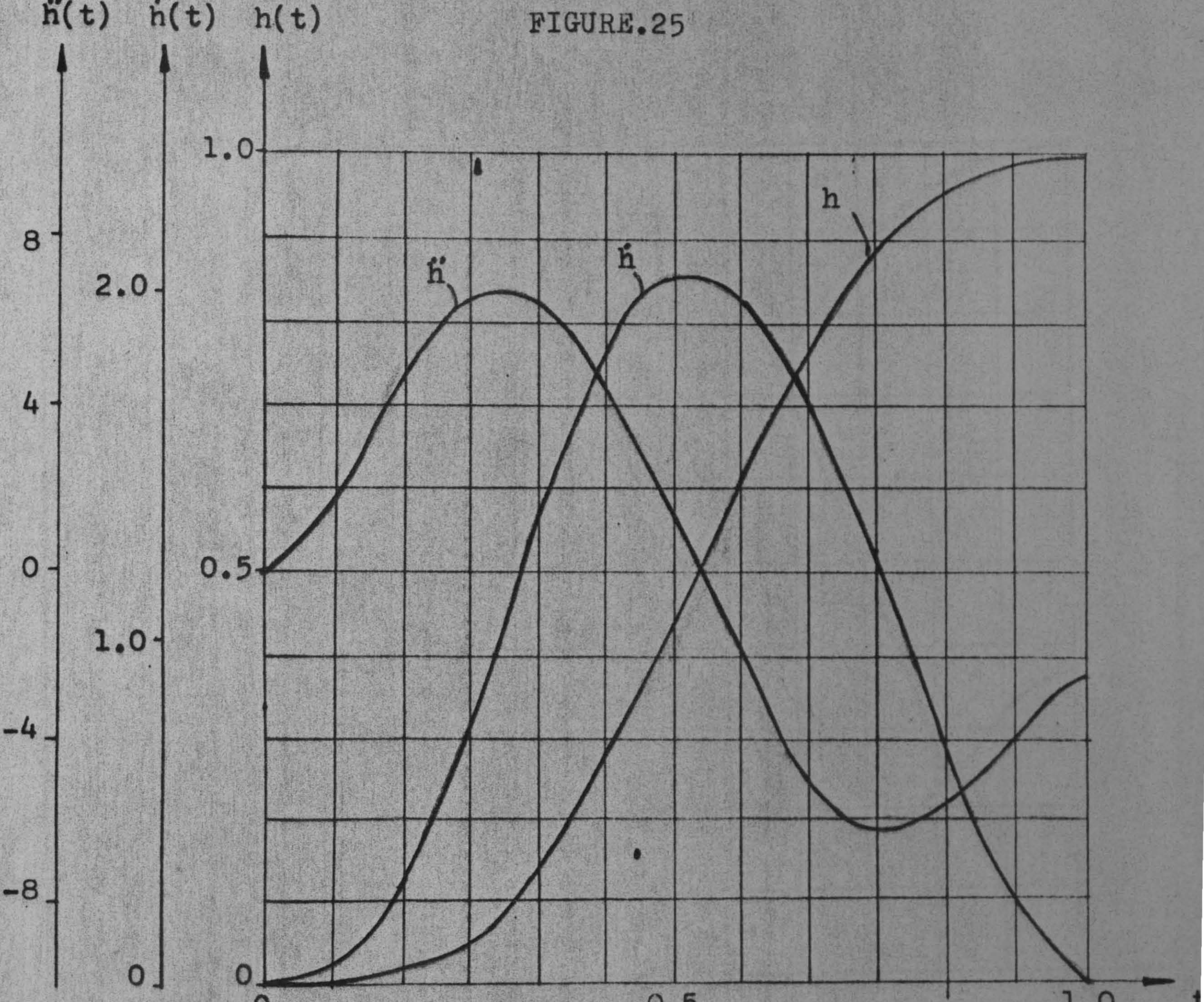
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(0.05x_1^2 + 0.05x_3^2 + u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$	$x_1(t_f) = 0$
$x_2(0) = 0$	$x_2(t_f) = 0$
$x_3(0) = 0$	$x_3(t_f) = -2.5$
$x_4(0) = 0$	$x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47028
MIN.ACCELERATION: -6.3320
MAX.ACCELERATION: 6.6867
MAX.VELOCITY: 2.0438

FIGURE.25



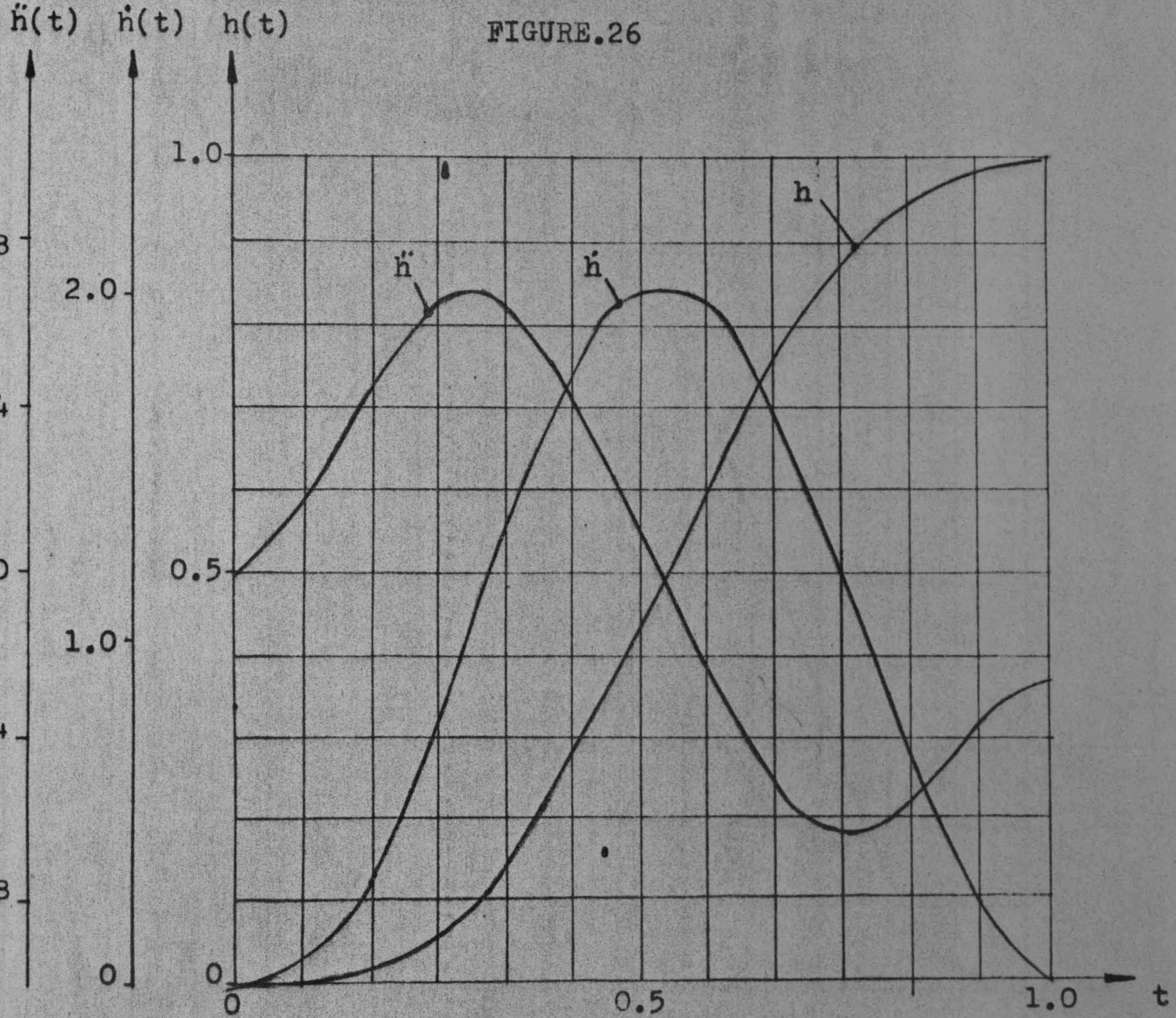
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (0.1x_1^2 + 0.1x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\qquad x_1(t_f) = 0$
$x_2(0) = 0$ $\qquad x_2(t_f) = 0$
$x_3(0) = 0$ $\qquad x_3(t_f) = -2.5$
$x_4(0) = 0$ $\qquad x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47032
MIN.ACCELERATION: -6.3326
MAX.ACCELERATION: 6.6873
MAX.VELOCITY: 2.0440

FIGURE.26



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(0.95x_1^2 + 0.95x_3^2 + u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\quad$ $x_1(t_f) = 0$

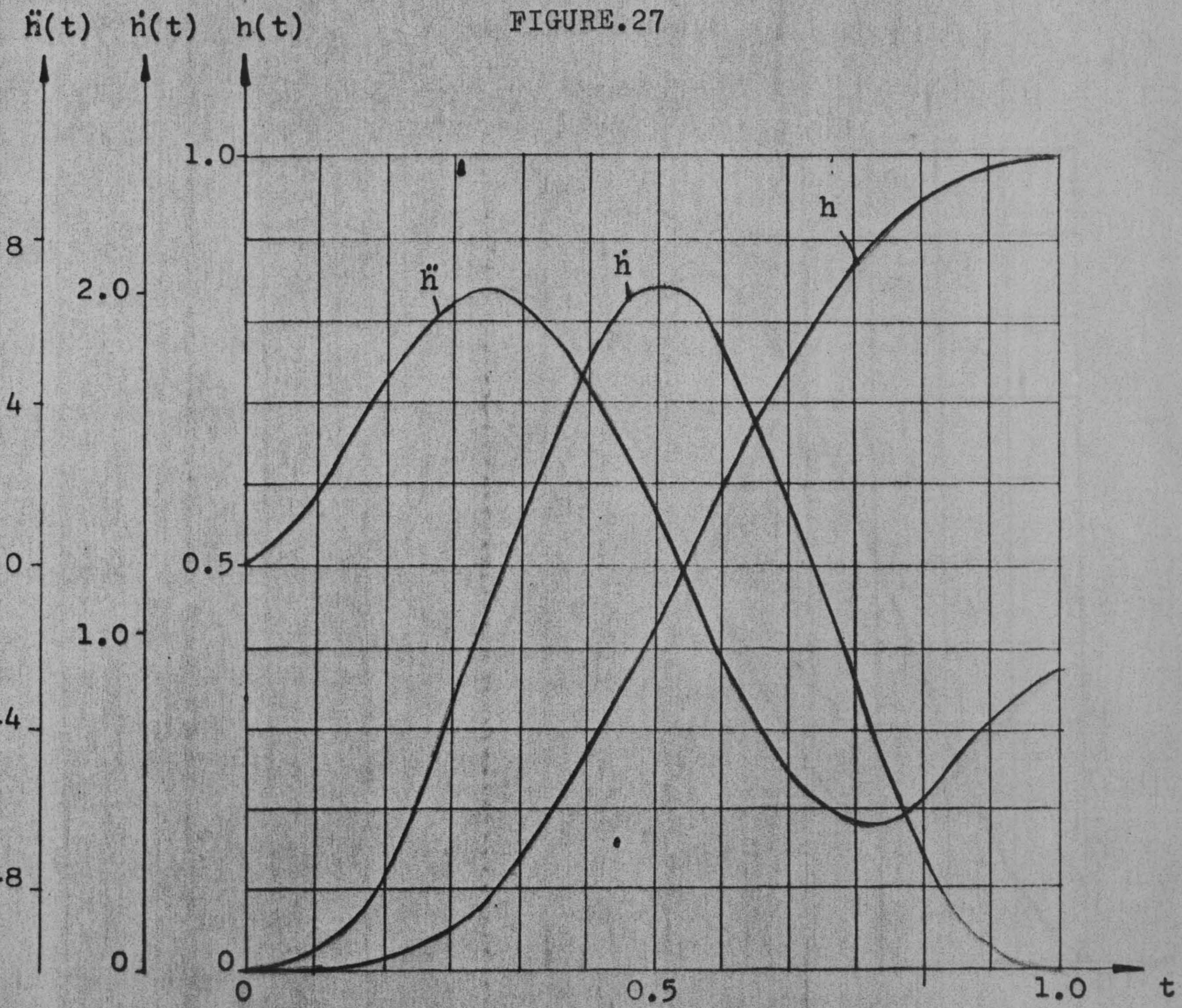
$x_2(0) = 0$ $\quad$ $x_2(t_f) = 0$

$x_3(0) = 0$ $\quad$ $x_3(t_f) = -2.5$

$x_4(0) = 0$ $\quad$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47099
MIN.ACCELERATION: -6.3442
MAX.ACCELERATION: 6.6994
MAX.VELOCITY: 2.0475

FIGURE.27



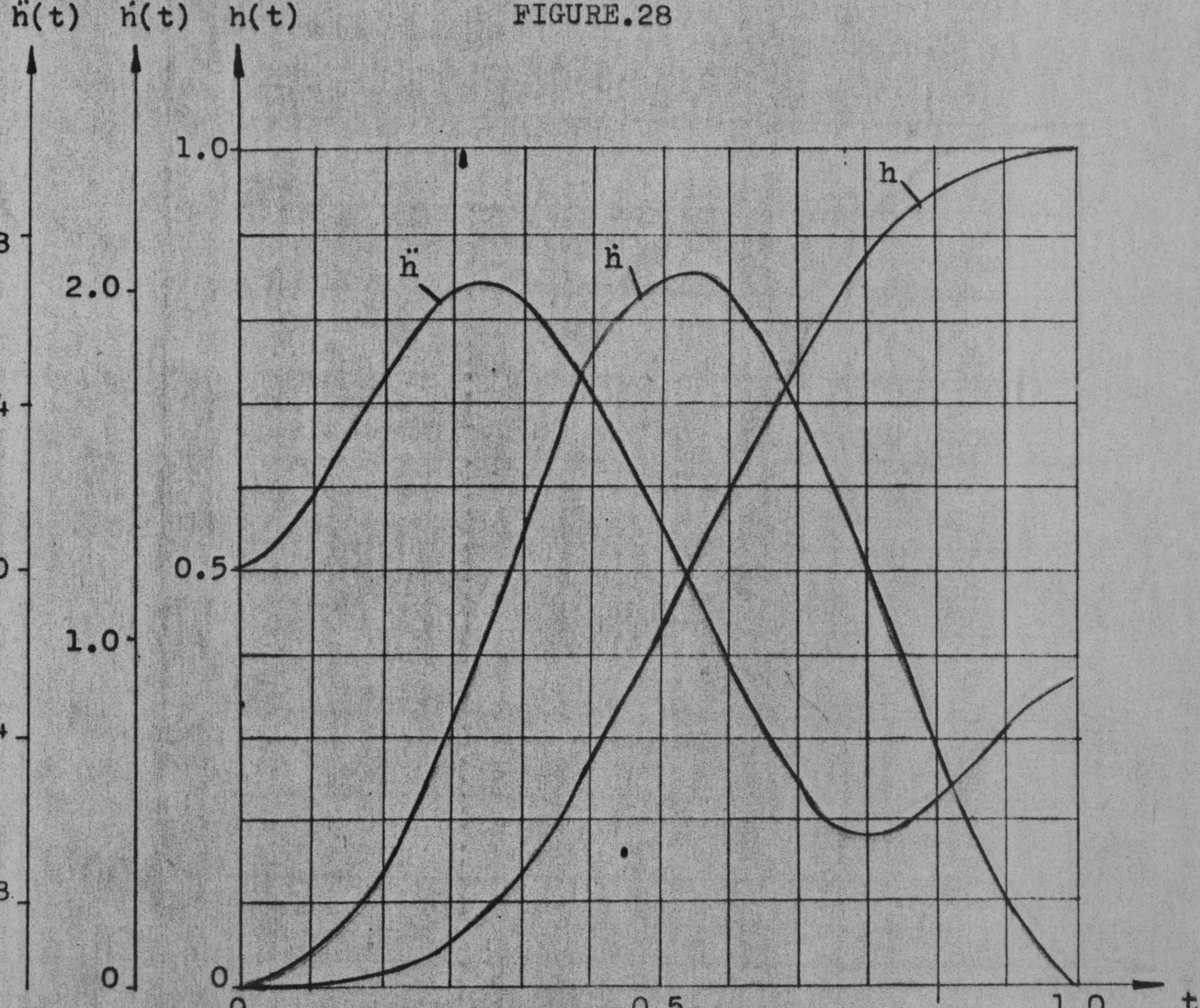
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2 + x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $x_1(t_f) = 0$
$x_2(0) = 0$ $x_2(t_f) = 0$
$x_3(0) = 0$ $x_3(t_f) = -2.5$
$x_4(0) = 0$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47103
MIN.ACCELERATION: -6.3448
MAX.ACCELERATION: 6.7001
MAX.VELOCITY: 2.0477

FIGURE.28



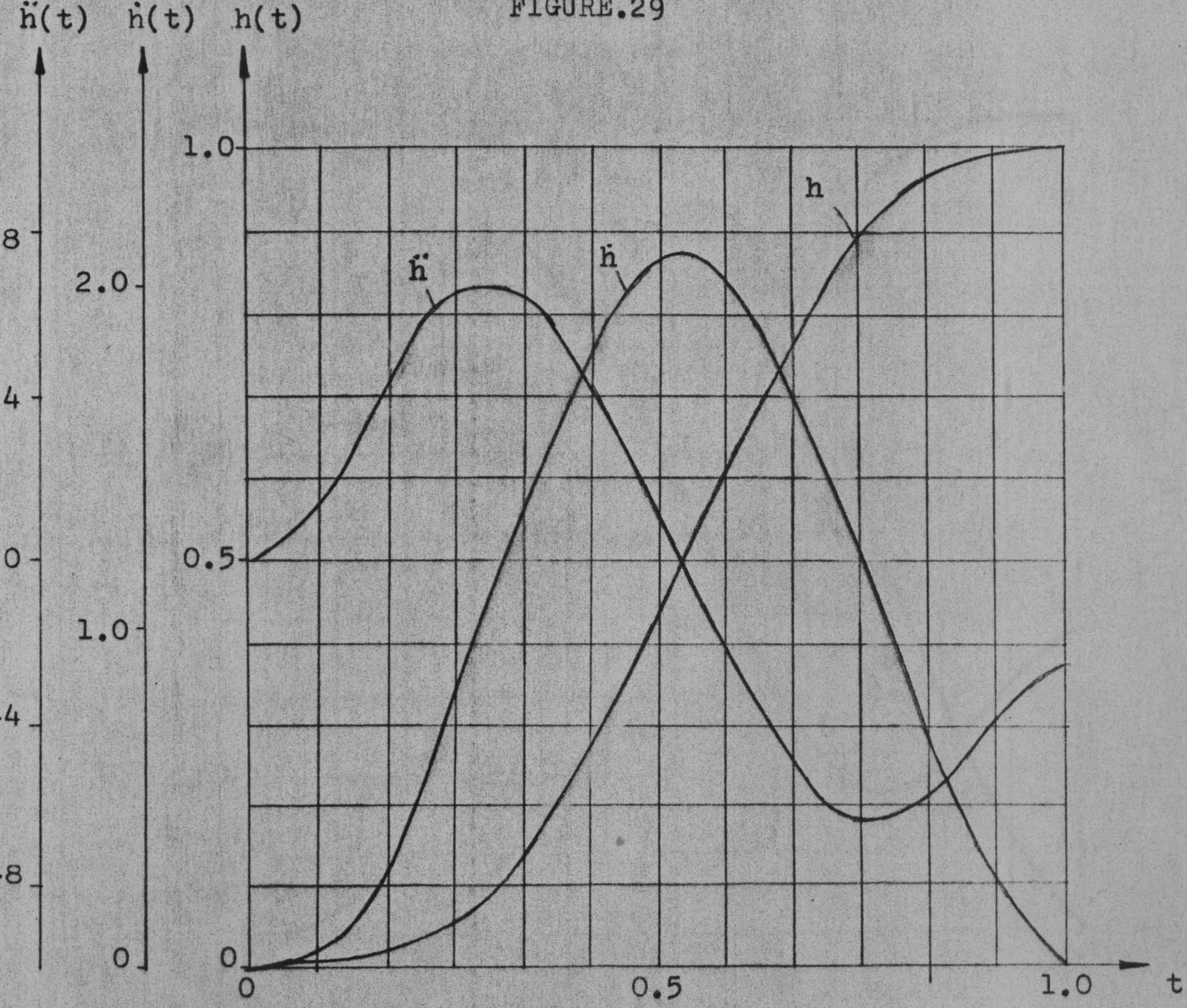
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(1.05x_1^2 + 1.05x_3^2 + u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $x_1(t_f) = 0$
$x_2(0) = 0$ $x_2(t_f) = 0$
$x_3(0) = 0$ $x_3(t_f) = -2.5$
$x_4(0) = 0$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47107
MIN.ACCELERATION: -6.3459
MAX.ACCELERATION: 6.7008
MAX.VELOCITY: 2.0479

FIGURE.29



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(10x_1^2 + 10x_3^2 + u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0 \qquad x_1(t_f) = 0$

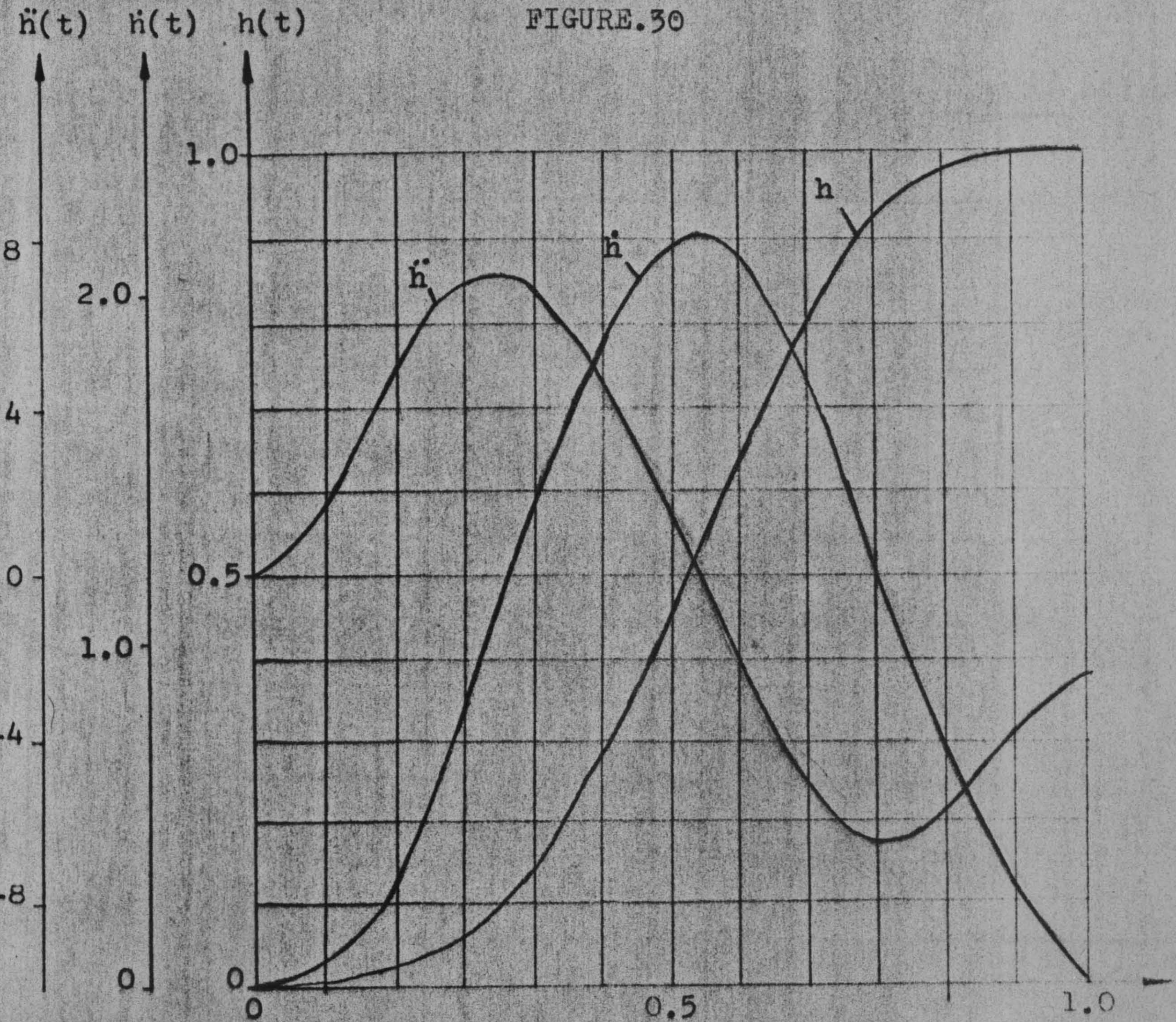
$x_2(0) = 0 \qquad x_2(t_f) = 0$

$x_3(0) = 0 \qquad x_3(t_f) = -2.5$

$x_4(0) = 0 \qquad x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.47811
MIN.ACCELERATION: -6.463
MAX.ACCELERATION: 6.825
MAX.VELOCITY: 2.0847

FIGURE.30



$$J=\frac{1}{2}\alpha t_f^2+\frac{1}{2}\int_0^{t_f}(20x_1^2+20x_3^2+u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0)=1.0$ $\qquad$ $x_1(t_f)=0$

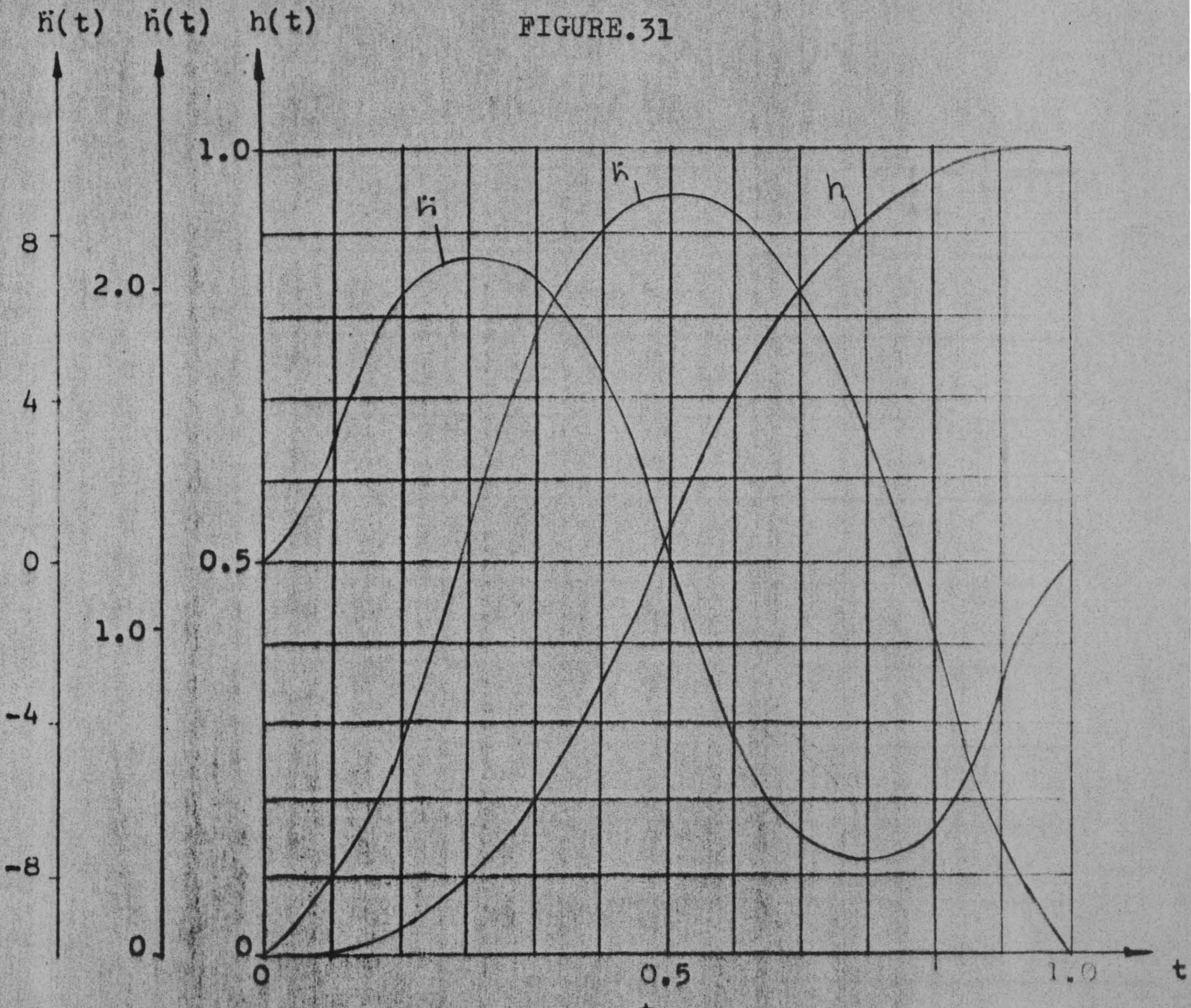
$x_2(0)=0$ $\qquad$ $x_2(t_f)=0$

$x_3(0)=0$ $\qquad$ $x_3(t_f)=-2.5$

$x_4(0)=0$ $\qquad$ $x_4(t_f)=0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.48581
MIN.ACCELERATION: -6.5839
MAX.ACCELERATION: 6.9556
MAX.VELOCITY: 2.1253

FIGURE.31



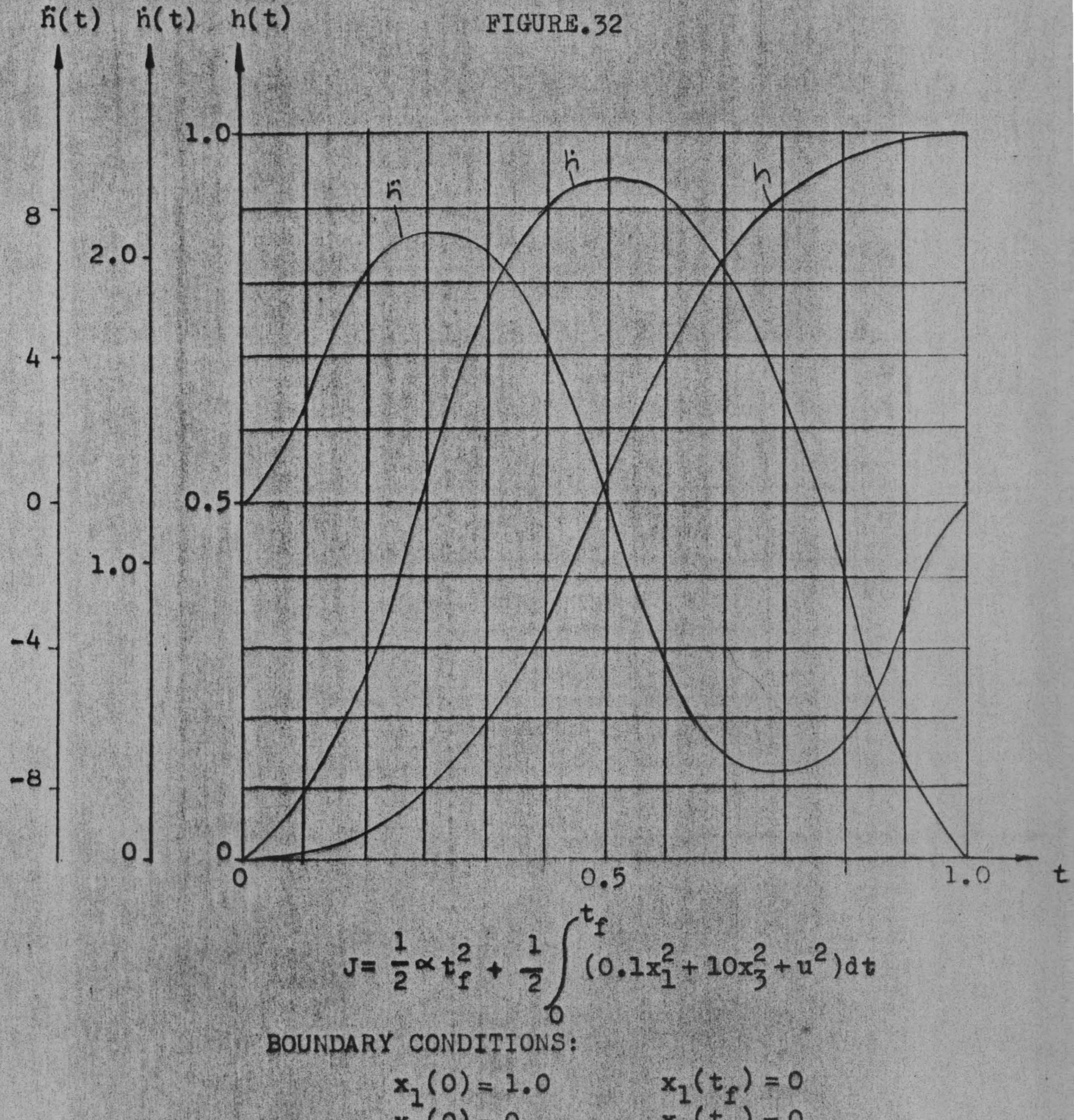
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (10x_1^2 + 0.1x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\quad$ $x_1(t_f) = 0$
$x_2(0) = 0$ $\quad$ $x_2(t_f) = 0$
$x_3(0) = 0$ $\quad$ $x_3(t_f) = 0$
$x_4(0) = 0$ $\quad$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.50014
MIN.ACCELERATION: -7.5129
MAX.ACCELERATION: 7.5124
MAX.VELOCITY: 2.1862

FIGURE.32



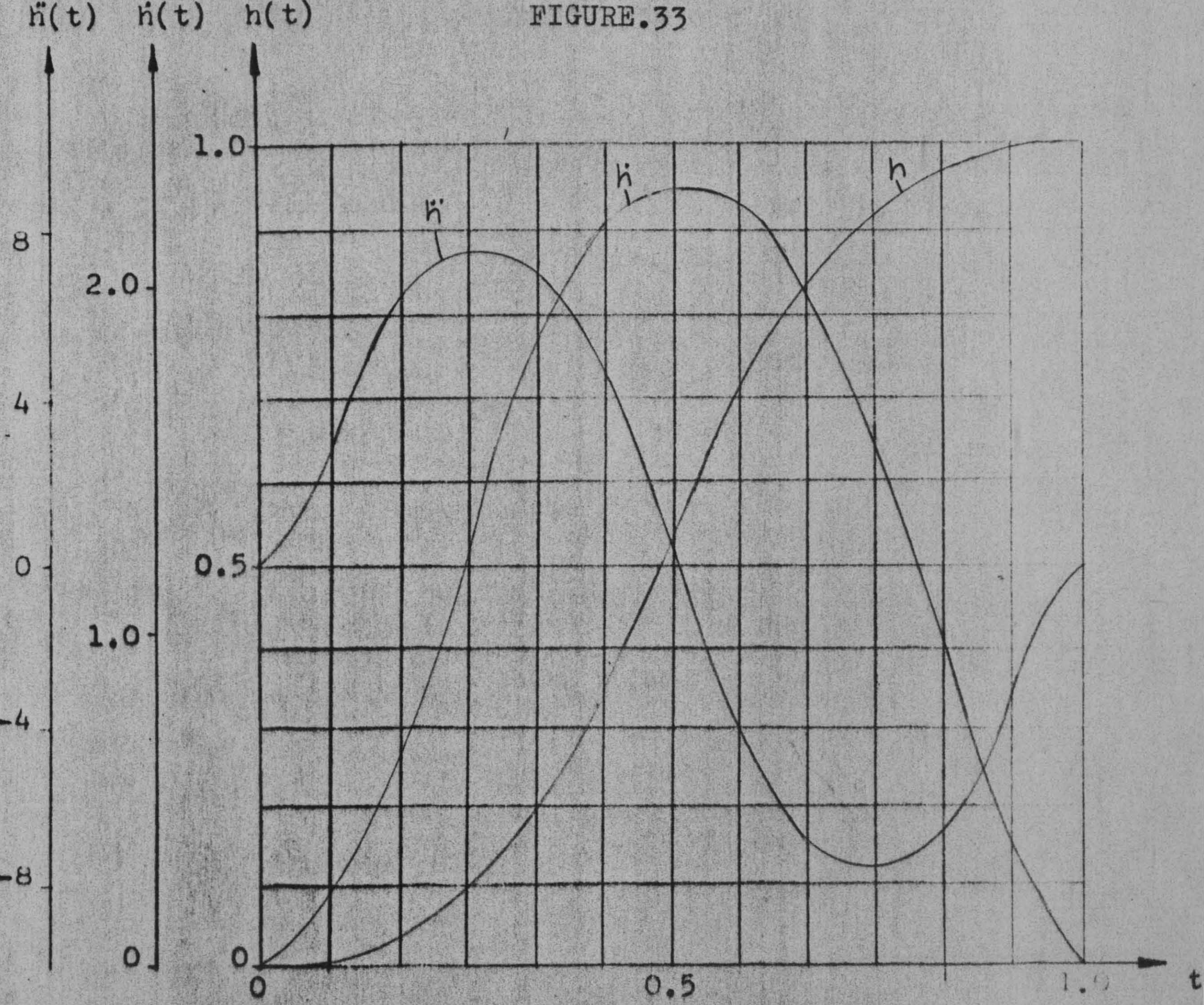
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(0.1x_1^2 + 10x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\quad$ $x_1(t_f) = 0$
$x_2(0) = 0$ $\quad$ $x_2(t_f) = 0$
$x_3(0) = 0$ $\quad$ $x_3(t_f) = 0$
$x_4(0) = 0$ $\quad$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.51047
MIN.ACCELERATION: -7.6885
MAX.ACCELERATION: 7.6945
MAX.VELOCITY: 2.2404

FIGURE.33



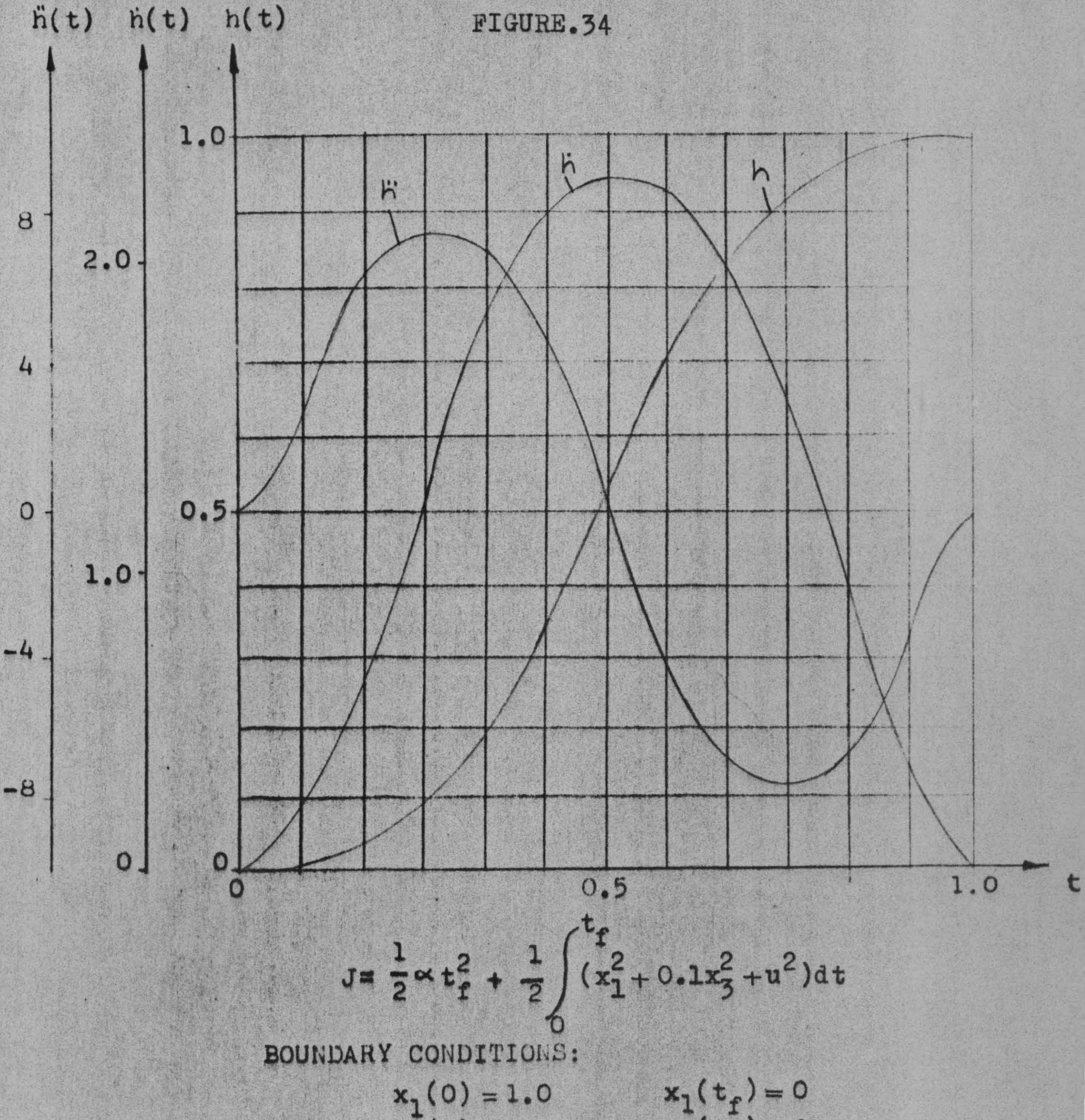
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(0.1x_1^2 + x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\quad$ $x_1(t_f) = 0$
$x_2(0) = 0$ $\quad$ $x_2(t_f) = 0$
$x_3(0) = 0$ $\quad$ $x_3(t_f) = 0$
$x_4(0) = 0$ $\quad$ $x_4(t_f) = 0$

FINAL TIME:1.0
AREA UNDER LIFT CURVE: 0.50105
MIN.ACCELERATION:-7.5286
MAX.ACCELERATION: 7.5285
MAX.VELOCITY: 2.1910

FIGURE.34



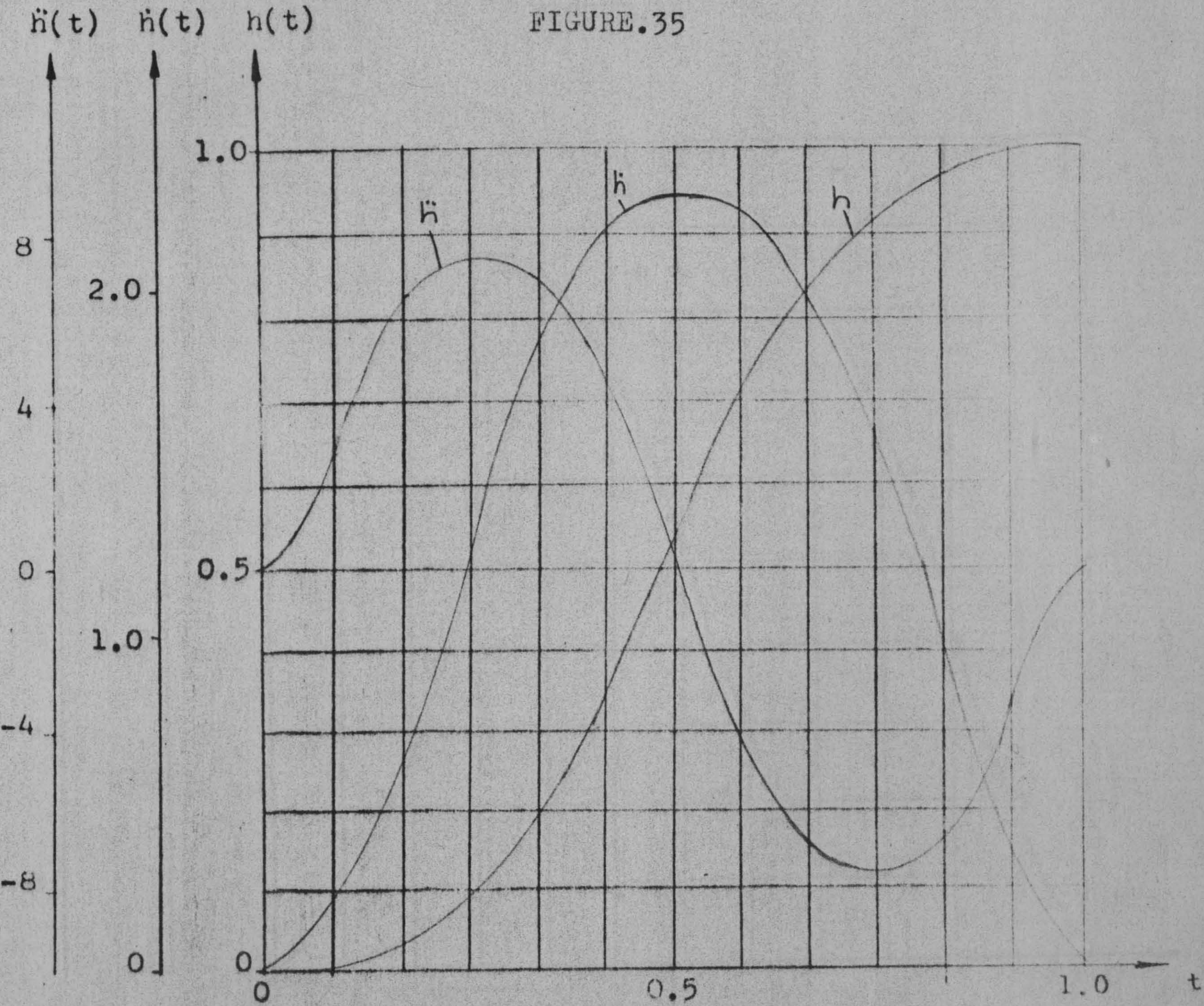
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2 + 0.1x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\quad x_1(t_f) = 0$
$x_2(0) = 0$ $\quad x_2(t_f) = 0$
$x_3(0) = 0$ $\quad x_3(t_f) = 0$
$x_4(0) = 0$ $\quad x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.50010
MIN.ACCELERATION: -7.5118
MAX.ACCELERATION: 7.5116
MAX.VELOCITY: 2.1860

FIGURE.35



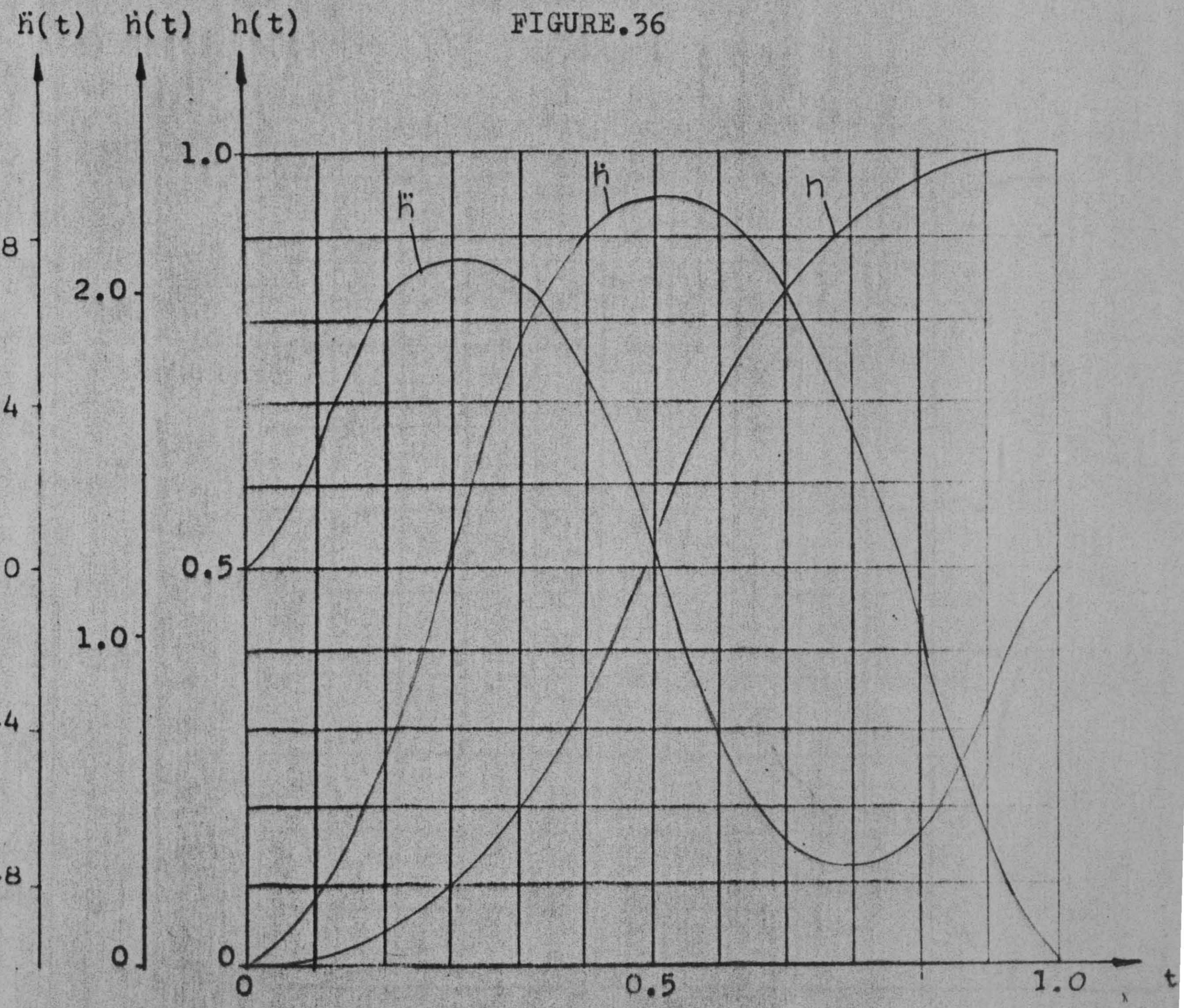
$$J= \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(10x_1^2 + x_3^2 + u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\qquad$ $x_1(t_f) = 0$
$x_2(0) = 0$ $\qquad$ $x_2(t_f) = 0$
$x_3(0) = 0$ $\qquad$ $x_3(t_f) = 0$
$x_4(0) = 0$ $\qquad$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.50108
MIN.ACCELERATION: -7.5004
MAX.ACCELERATION: 7.5294
MAX.VELOCITY: 2.1911

FIGURE.36



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2 + 10x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\quad$ $x_1(t_f) = 0$

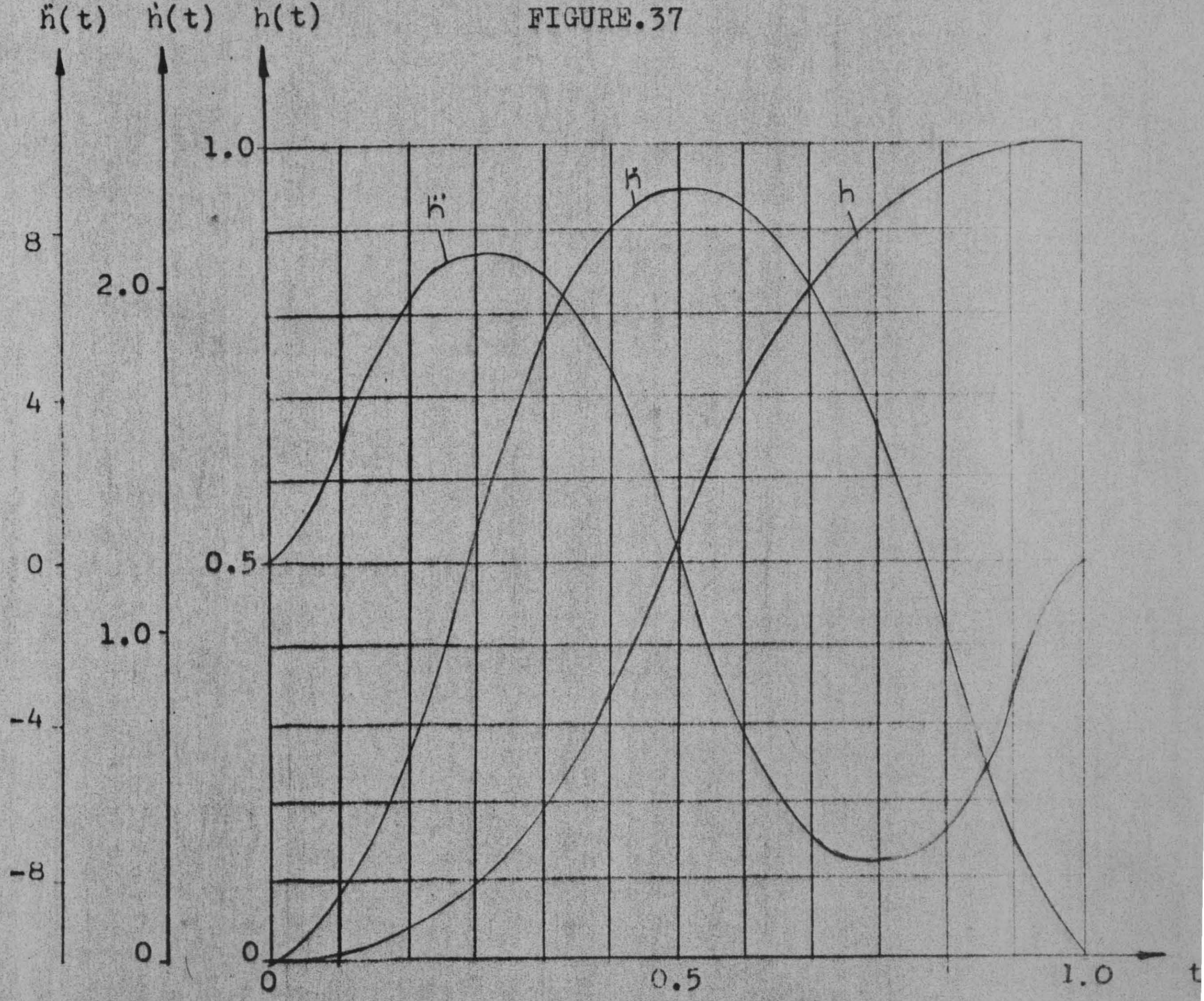
$x_2(0) = 0$ $\quad$ $x_2(t_f) = 0$

$x_3(0) = 0$ $\quad$ $x_3(t_f) = 0$

$x_4(0) = 0$ $\quad$ $x_4(t_f) = 0$

FINAL TIME:1.0
AREA UNDER LIFT CURVE: 0.51047
MIN.ACCELERATION: -7.6884
MAX.ACCELERATION: 7.6945
MAX.VELOCITY: 2.2404

FIGURE.37



$$J = \frac{1}{2}\propto t_f^2 + \frac{1}{2}\int_0^{t_f}(100x_1^2 + x_3^2 + u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\qquad$ $x_1(t_f) = 0$

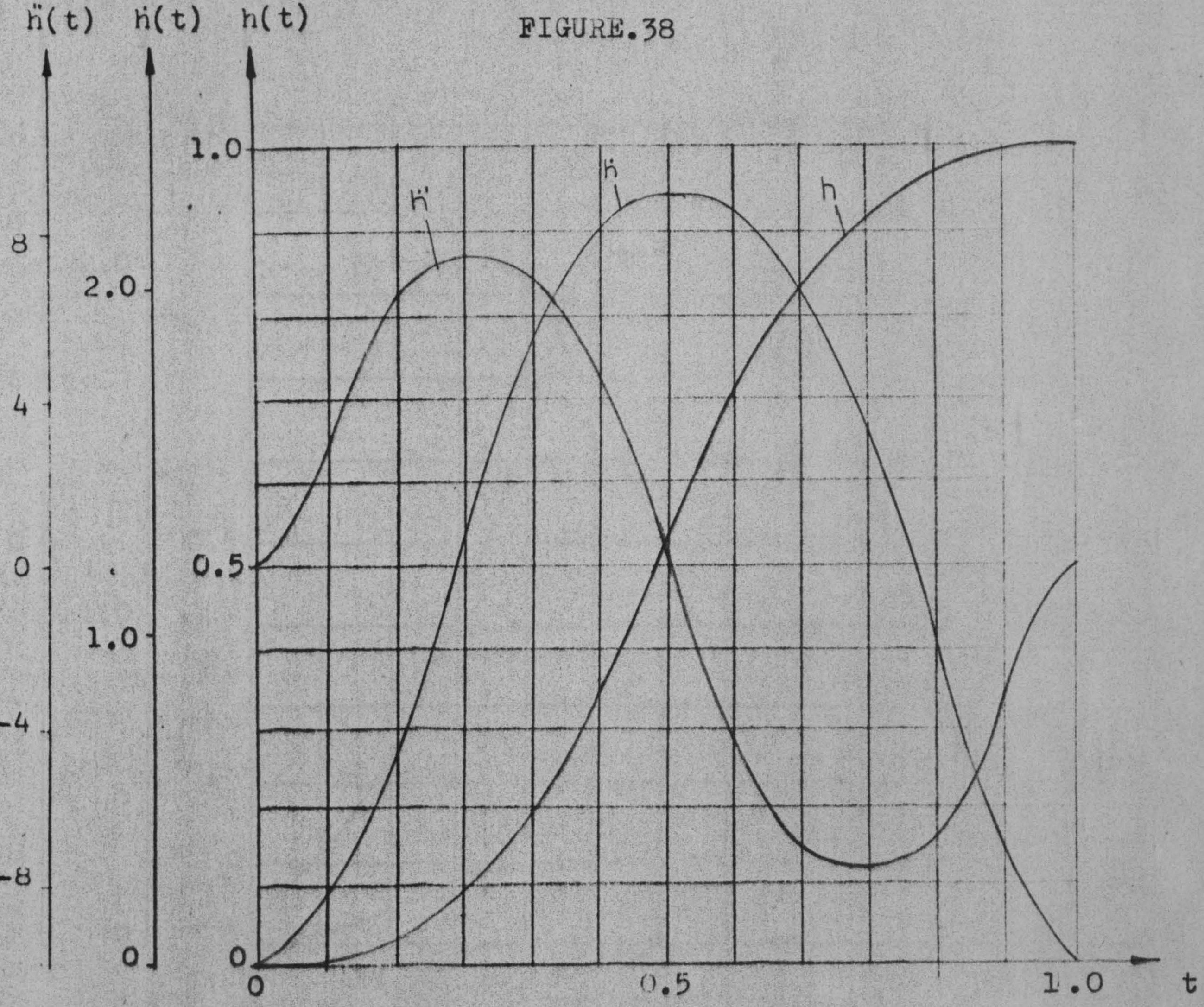
$x_2(0) = 0$ $\qquad$ $x_2(t_f) = 0$

$x_3(0) = 0$ $\qquad$ $x_3(t_f) = 0$

$x_4(0) = 0$ $\qquad$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.50136
MIN.ACCELERATION: -7.5431
MAX.ACCELERATION: 7.5371
MAX.VELOCITY: 2.1927

FIGURE.38



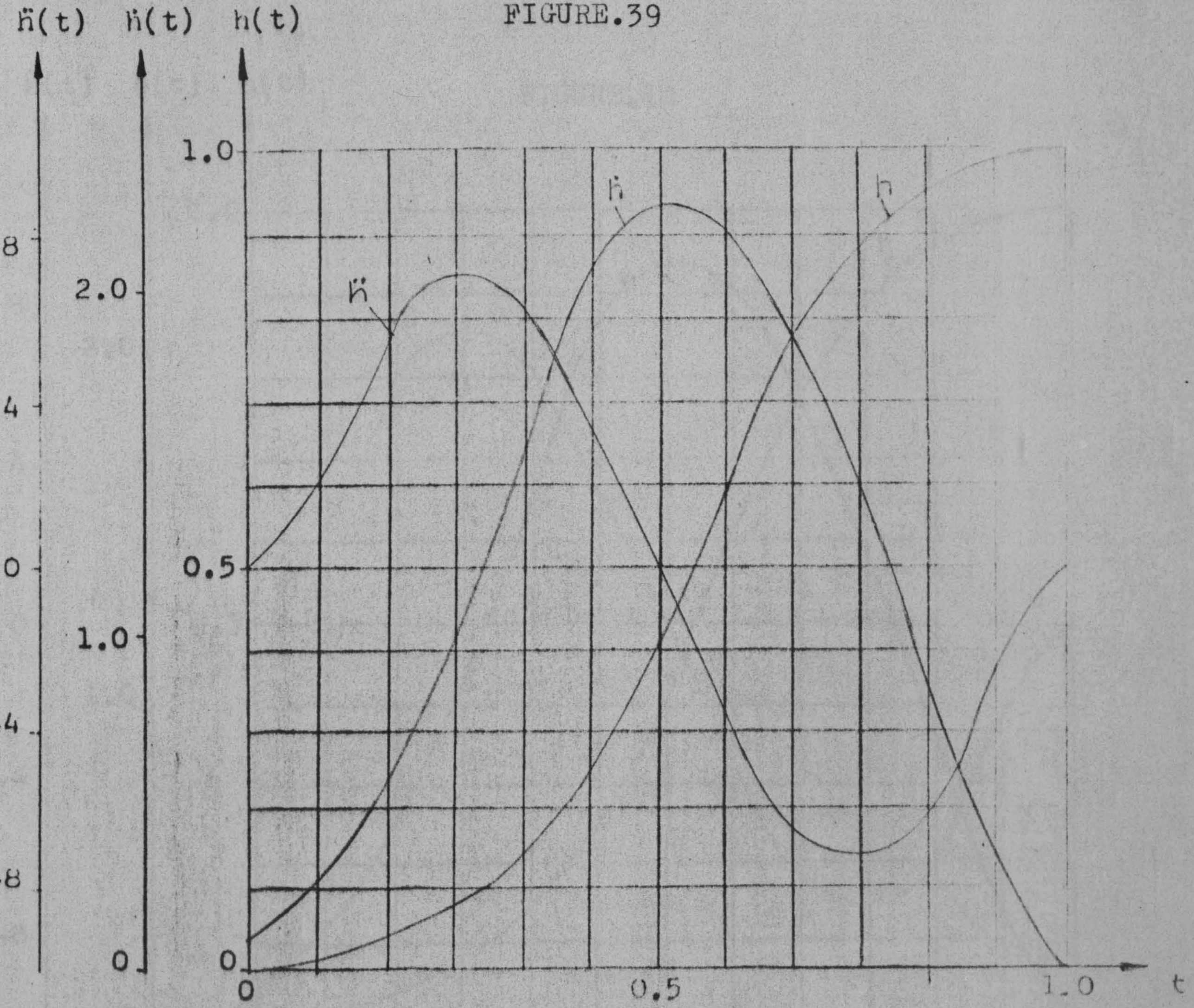
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(100x_1^2 + 0.1x_3^2 + u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\qquad$ $x_1(t_f) = 0$
$x_2(0) = 0$ $\qquad$ $x_2(t_f) = 0$
$x_3(0) = 0$ $\qquad$ $x_3(t_f) = 0$
$x_4(0) = 0$ $\qquad$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.50041
MIN.ACCELERATION: -7.5261
MAX.ACCELERATION: 7.5201
MAX.VELOCITY: 2.1877

FIGURE.39



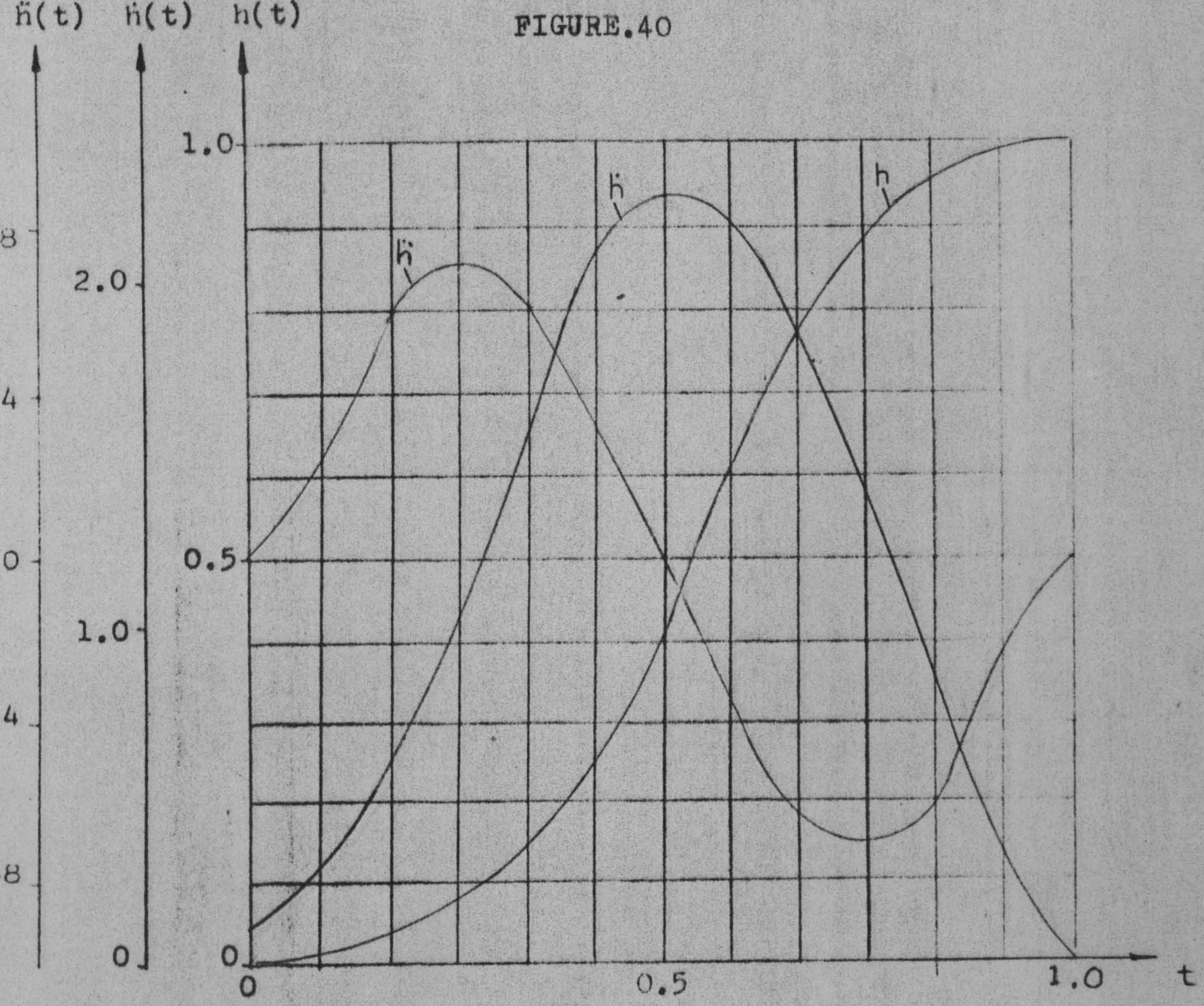
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2 + x_2^2 + x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$	$x_1(t_f) = 0$
$x_2(0) = 1.2$	$x_2(t_f) = 0$
$x_3(0) = 0$	$x_3(t_f) = 0$
$x_4(0) = 0$	$x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.51364
MIN.ACCELERATION: -7.2162
MAX.ACCELERATION: 6.9257
MAX.VELOCITY: 2.1204

FIGURE.40



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2 + 0.1x_2^2 + x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\qquad$ $x_1(t_f) = 0$

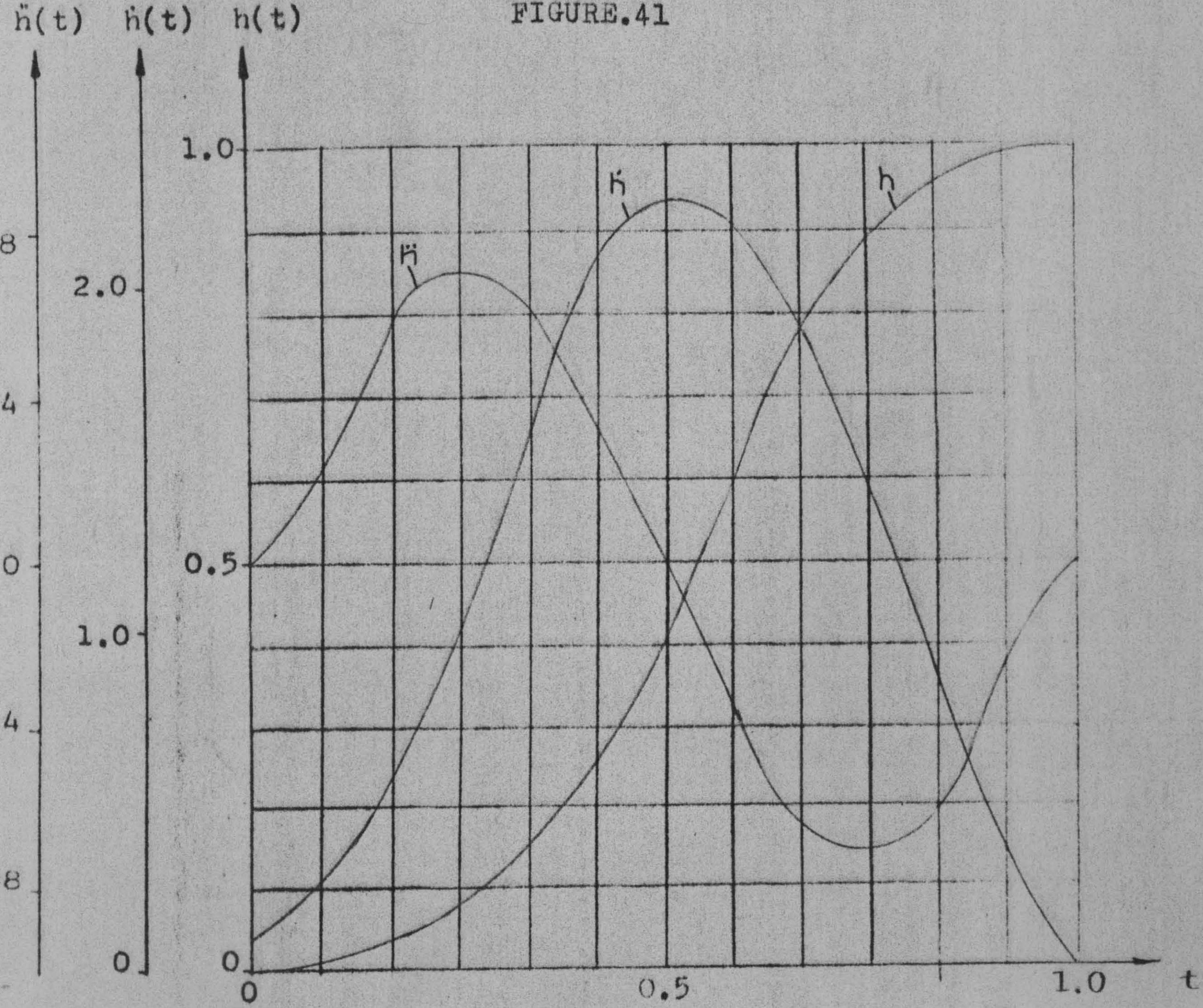
$x_2(0) = 1.2$ $\qquad$ $x_2(t_f) = 0$

$x_3(0) = 0$ $\qquad$ $x_3(t_f) = 0$

$x_4(0) = 0$ $\qquad$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.51382
MIN.ACCELERATION: -7.2228
MAX.ACCELERATION: 6.9297
MAX.VELOCITY: 2.1214

FIGURE.41



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (x_1^2 + 10x_2^2 + x_3^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $\quad$ $x_1(t_f) = 0$

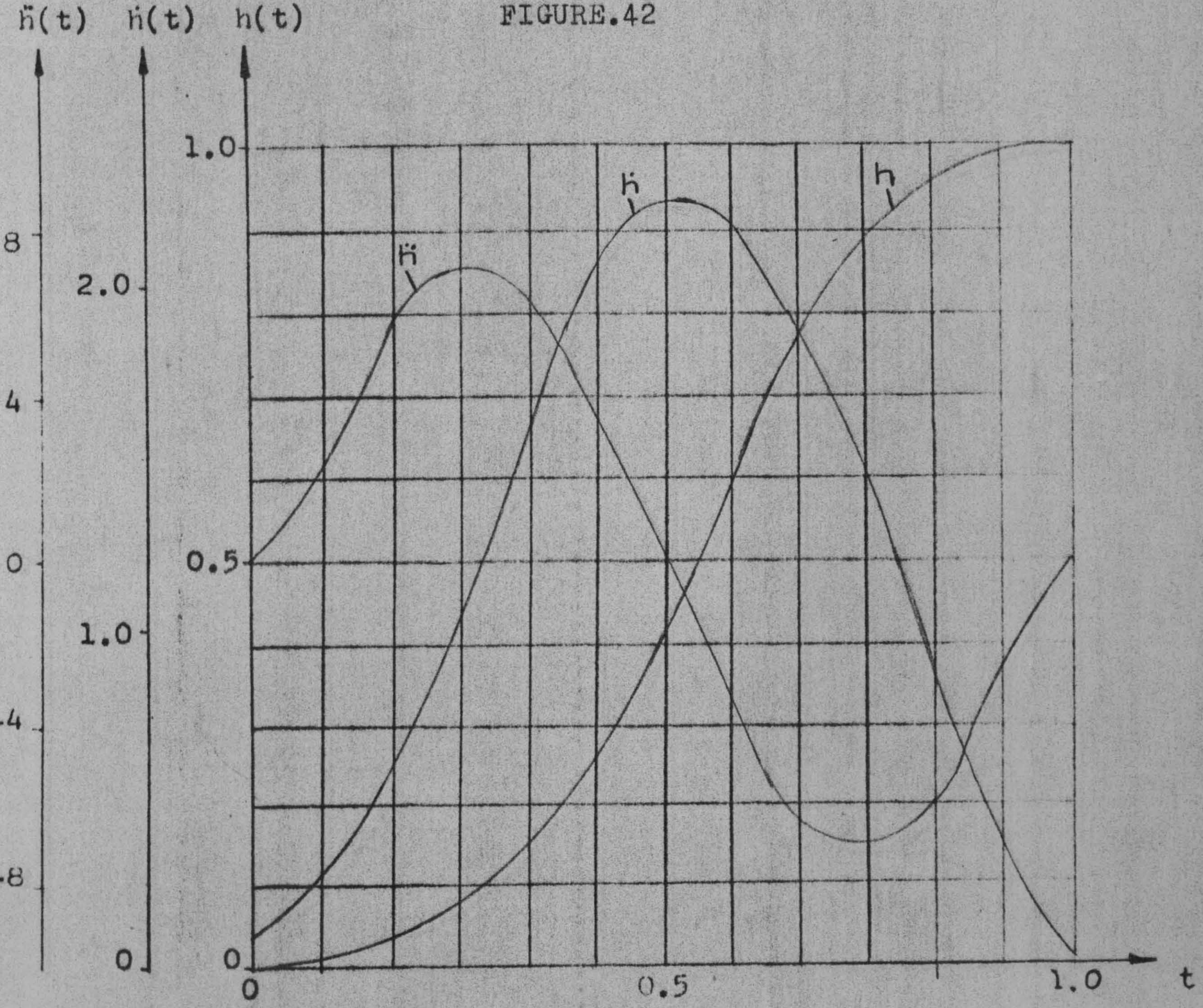
$x_2(0) = 1.2$ $\quad$ $x_2(t_f) = 0$

$x_3(0) = 0$ $\quad$ $x_3(t_f) = 0$

$x_4(0) = 0$ $\quad$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.51188
MIN.ACCELERATION: -7.1495
MAX.ACCELERATION: 6.8859
MAX.VELOCITY: 2.1106

FIGURE.42



$$J=\frac{1}{2}\alpha t_f^2+\frac{1}{2}\int_0^{t_f}(x_1^2+x_3^2+x_4^2+u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0)=1.0$ $\quad$ $x_1(t_f)=0$

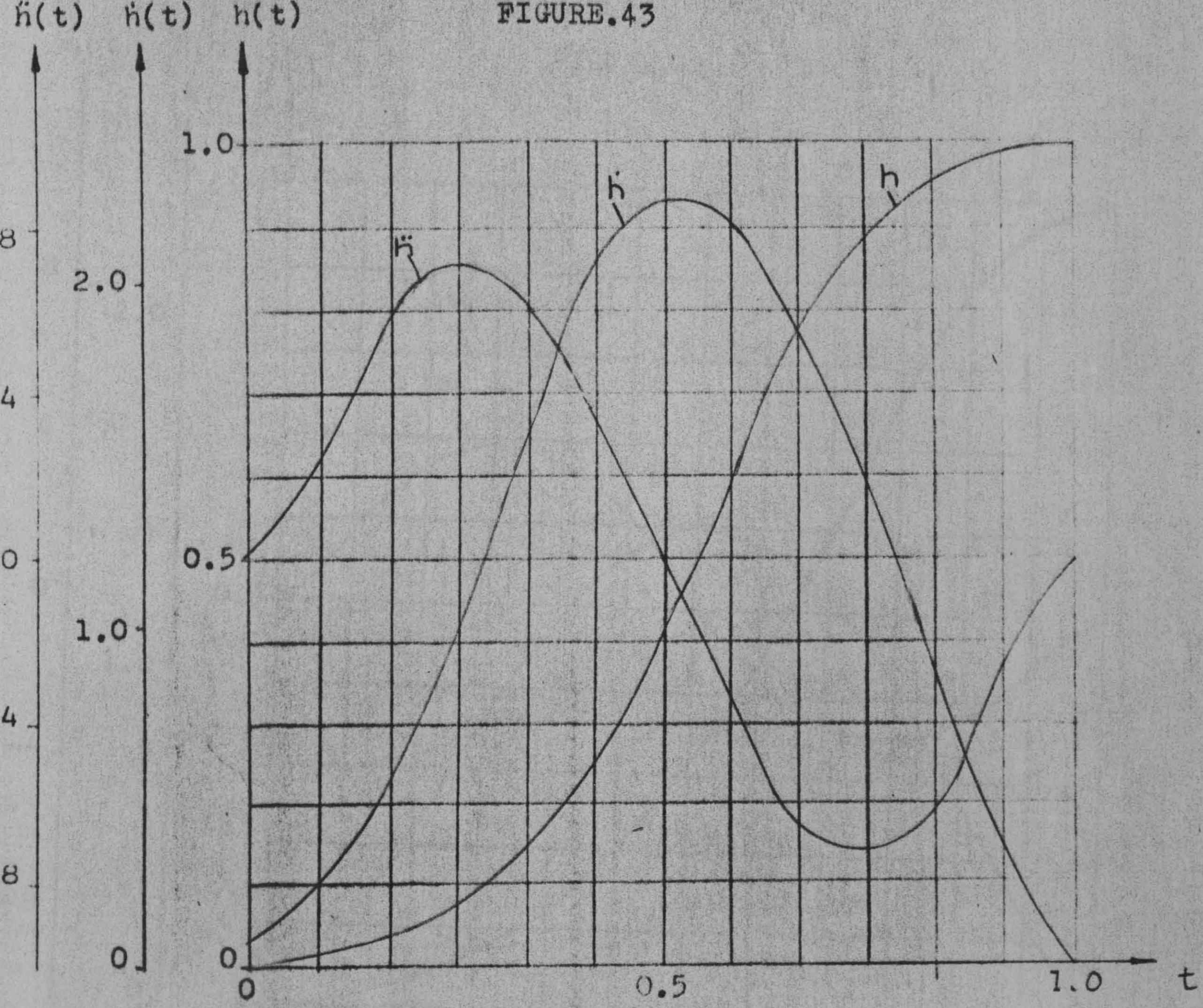
$x_2(0)=1.2$ $\quad$ $x_2(t_f)=0$

$x_3(0)=0$ $\quad$ $x_3(t_f)=0$

$x_4(0)=0$ $\quad$ $x_4(t_f)=0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.51306
MIN.ACCELERATION: -7.2076
MAX.ACCELERATION: 6.9109
MAX.VELOCITY: 2.1164

FIGURE.43



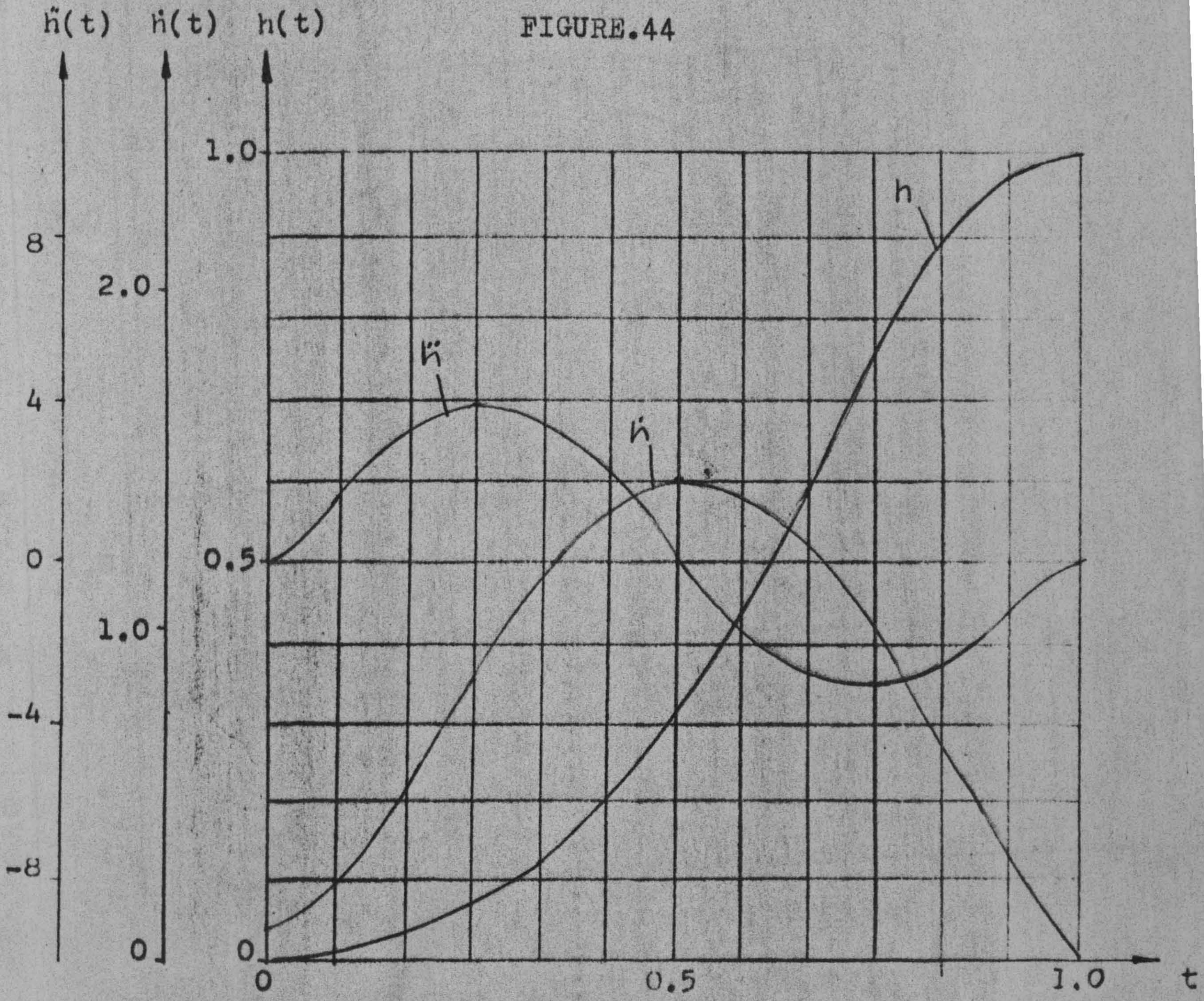
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (x_1^2 \quad x_3^2 \quad 0.1x_4^2 \quad u^2)dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$ $x_1(t_f) = 0$
$x_2(0) = 1.2$ $x_2(t_f) = 0$
$x_3(0) = 0$ $x_3(t_f) = 0$
$x_4(0) = 0$ $x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.51388
MIN.ACCELERATION: -7.2244
MAX.ACCELERATION: 6.9303
MAX.VELOCITY: 2.1216

FIGURE.44



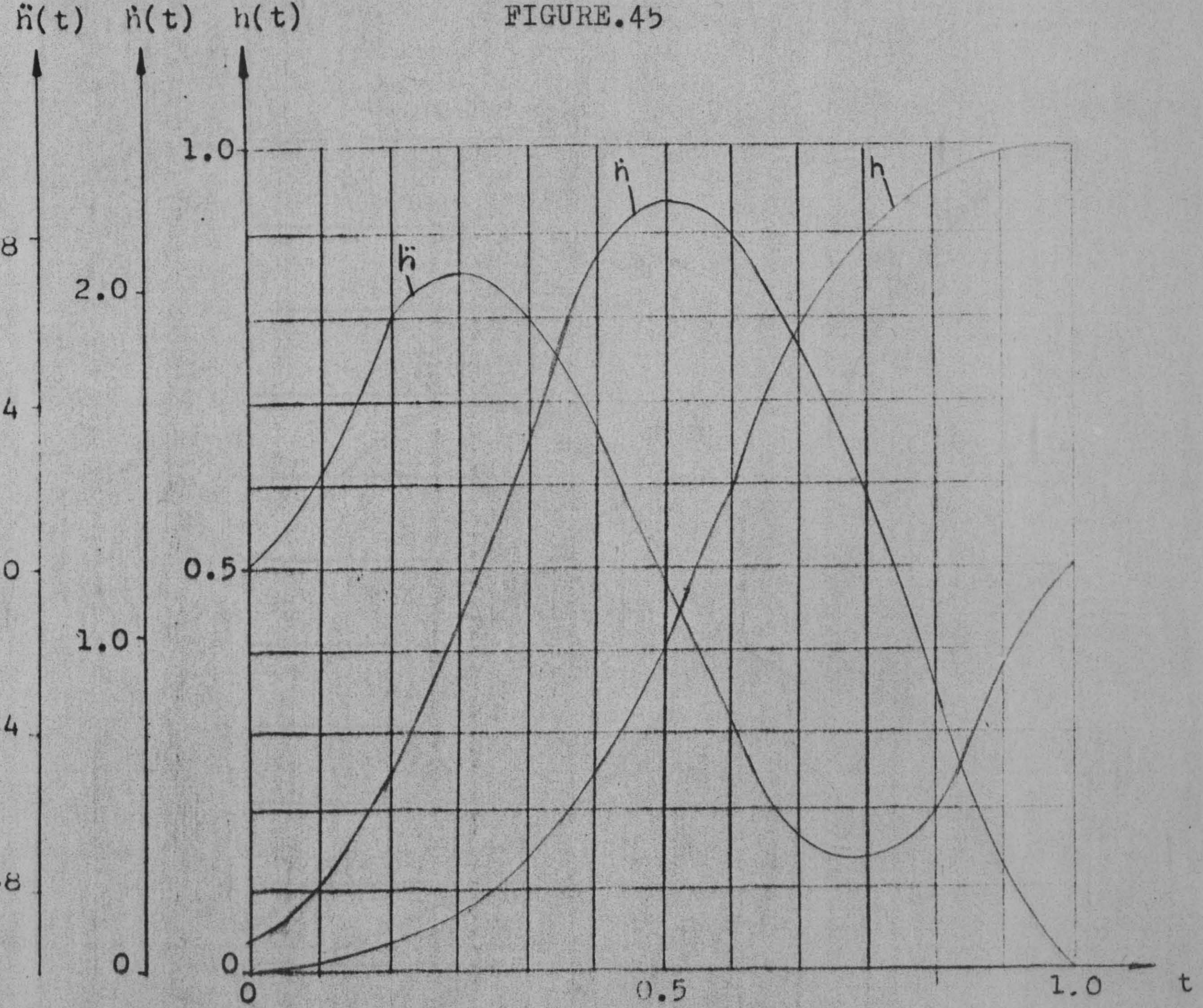
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} (x_1^2 + x_3^2 + 10x_4^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$	$x_1(t_f) = 0$
$x_2(0) = 1.2$	$x_2(t_f) = 0$
$x_3(0) = 0$	$x_3(t_f) = 0$
$x_4(0) = 0$	$x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.37290
MIN.ACCELERATION: -2.9239
MAX.ACCELERATION: 3.9063
MAX.VELOCITY: 1.4106

FIGURE.45



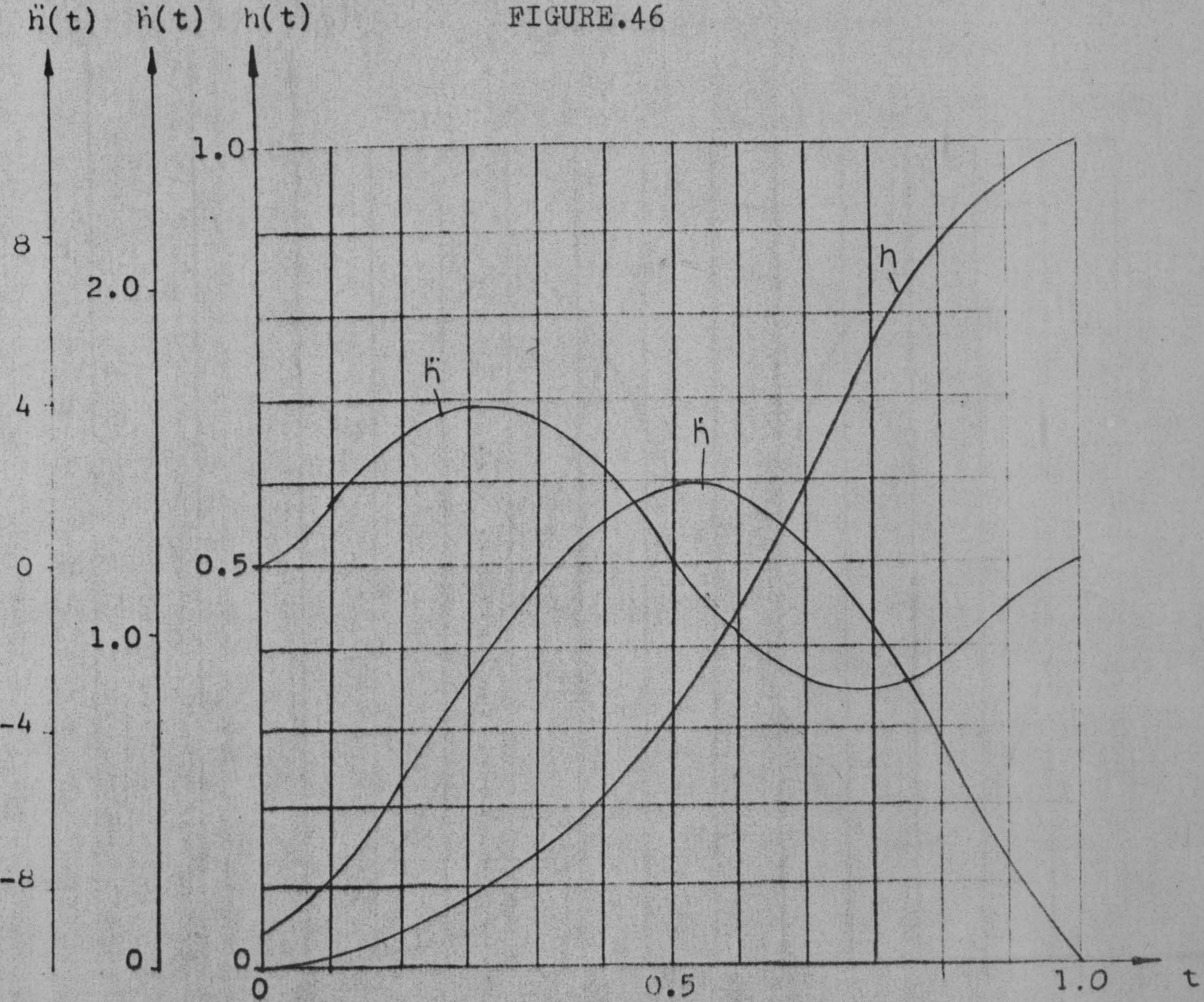
$$J=\frac{1}{2}\alpha t_f^2+\frac{1}{2}\int_0^{t_f}(x_1^2+x_2^2+x_3^2+x_4^2+u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0)=1.0$ $x_1(t_f)=0$
$x_2(0)=1.2$ $x_2(t_f)=0$
$x_3(0)=0$ $x_3(t_f)=0$
$x_4(0)=0$ $x_4(t_f)=0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.51285
MIN.ACCELERATION: -7.1624
MAX.ACCELERATION: 6.9063
MAX.VELOCITY: 2.1153

FIGURE.46



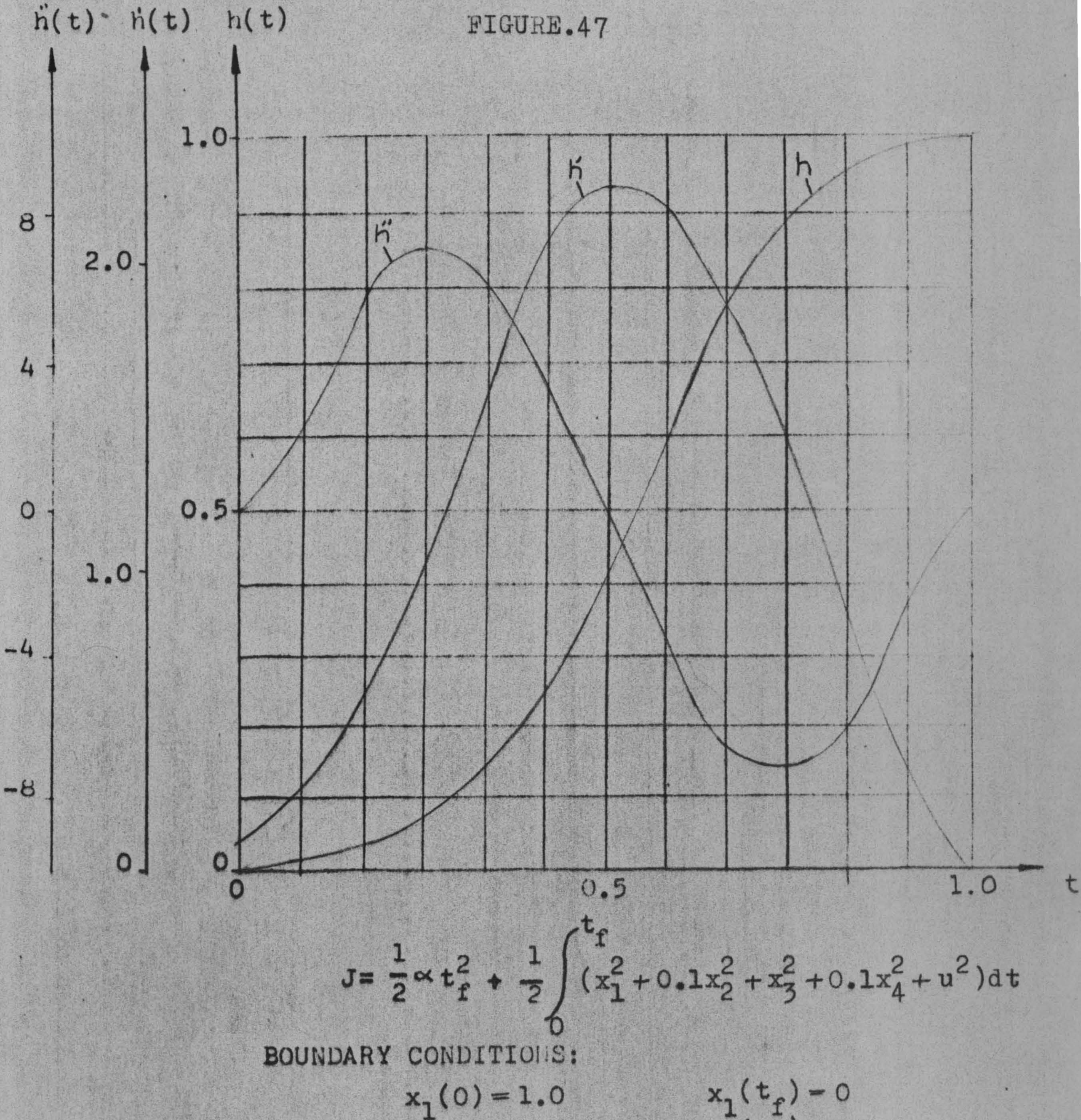
$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2 + 10x_2^2 + x_3^2 + 10x_4^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0$	$x_1(t_f) = 0$
$x_2(0) = 1.2$	$x_2(t_f) = 0$
$x_3(0) = 0$	$x_3(t_f) = 0$
$x_4(0) = 0$	$x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.37262
MIN.ACCELERATION: -2.9133
MAX.ACCELERATION: 3.8988
MAX.VELOCITY: 1.4088

FIGURE.47



$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f}(x_1^2 + 0.1x_2^2 + x_3^2 + 0.1x_4^2 + u^2)\,dt$$

BOUNDARY CONDITIONS:

$x_1(0) = 1.0 \qquad x_1(t_f) = 0$

$x_2(0) = 1.2 \qquad x_2(t_f) = 0$

$x_3(0) = 0 \qquad x_3(t_f) = 0$

$x_4(0) = 0 \qquad x_4(t_f) = 0$

FINAL TIME: 1.0
AREA UNDER LIFT CURVE: 0.51386
MIN.ACCELERATION: -7.2236
MAX.ACCELERATION: 6.9298
MAX.VELOCITY: 2.1215

APPENDIX.A.

CAM PROFILE DETERMINATION

In cam construction the informations that are generally utilized are the determination of the pressure angle,curvature, undercutting and practical factors of follower location and sizes until the ultimate cam contour is established.

The fundamental basis for all cam layouts is that the cam profile is developed by fixing the cam and moving the follower around the cam to its respective relative positions.In this manner one can obtain the same cam-follower relative motion as the cam mechanism.

Since accuracy of cam shape is essential many points are needed in constructing the actual cam.Enough points must be taken to establish the contour with confidence.By calculating the location of the cutter or grinder used to form the cam,a table of cam radii and corresponding cam angles is made.Sometimes it may be necessary to convert the coordinate system from polar to rectengular for higher accuracy of cam profile.For accurate cams the setting increments are sometimes as small as one-quarter degree.

Let us now find the exact profile dimensions for a radial cam in contact with a flat faced follower.From figure.A-1,we see that point A is the center of cam,AB is the line of follower motion and point C is the instantaneous contact point between the cam and follower.For the case of polar coordinates we are interested in finding the radius r_c,and its respective angle ψ_c, having been given the lift h as a function of the cam angle of rotation θ.Letting

R_o:Base circle radius,mm.

e :Eccentricity of point of contact from cam center,mm.

r_c:Radial distance to cam profile,mm.

We know that the point of contact C is at a distance

$$e=\frac{\dot{h}}{w} \qquad \text{(A-1)}$$

In fig.A-1 we can see that

$$r_c=\left[(R_o+h)^2+e^2\right]^{1/2} \qquad \text{(A-2)}$$

and

$$\tan \eta = \frac{e}{R_o + h} \qquad (A-3)$$

$$\psi_c = \theta - \eta \quad \text{(on the rise period of follower motion shown)} \qquad (A-4)$$

$$\psi_c = \theta - \eta \quad \text{(on the fall period of follower motion)} \qquad (A-5)$$

To determine the profile the procedure is to apply eqn's. (A-2) to (A-5) in a tabulated form.

Furthermore we can use some of the above given relationships to construct a cam for a flat faced follower if we remember that the surface BC is always perpendicular to the line of follower motion.Then the procedure is as follows,

a)For any arbitrary value of cam angle θ,the distance AB is

$$\overline{AB} = R_o + h(\theta)$$

b)For the cam angle θ,plot the value of e,obtained from eqn.(A-1) perpendicular to AB from point B.This will give point C which is on the cam profile.Other points may be found in the same manner.The more the number of points on the cam profile,the more precise will be the cam shape.

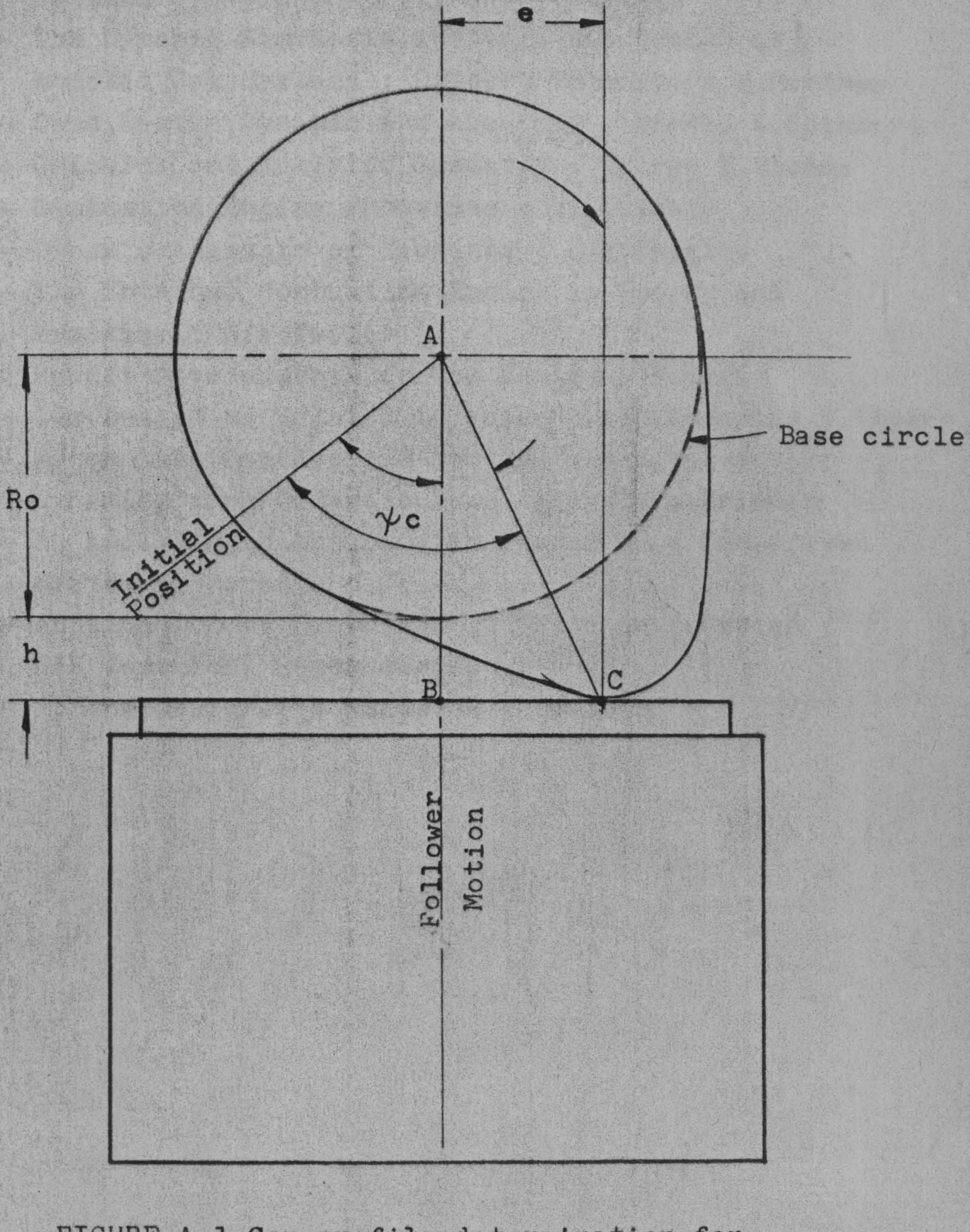


FIGURE.A-1 Cam profile determination for radialcam- translating flat follower

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- Automotive Cam Profile Synthesis and Valve Gear Dynamics from Dimensionless Analysis ; Erisman
- An Analysis of the Dynamic Forces in a Cam-Driven System ; Hrones
- Application of Computers in Valve Gear Design ; SAE Technical Paper Series
- Spring Designer's Handbook ; Carlson