AN INVESTIGATION OF FRESHMEN AND SENIOR MATHEMATICS AND TEACHING MATHEMATICS STUDENTS' CONCEPTIONS AND PRACTICES REGARDING PROOF

by

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ABSTRACT

AN INVESTIGATION OF FRESHMEN AND SENIOR MATHEMATICS AND TEACHING MATHEMATICS STUDENTS' CONCEPTIONS AND PRACTICES REGARDING PROOF

This study was conducted with 93 freshmen and 82 senior students from Mathematics, Primary Education and Secondary Education Teaching Mathematics Programs in Boğaziçi University in order to investigate their attitudes and beliefs regarding proof, the types of reasoning and proof methods they use while constructing proof, as well as their proof evaluation practices. Three instruments were developed for this purpose: *Attitudes and Beliefs Scale* (ABS), *Proof Exam* (PE) and *Proof Evaluation Exam* (PEE). Data were collected in 2009-2010 fall and spring semesters during lecture sessions with the presence of the researcher. Factor analysis on ABS yielded four components, labeled *background, attitude, self efficacy* and *beliefs*. Students' responses to PE were categorized with respect to the types of proof they attempted and also scored according to their reasoning styles, ranging from experimental-inductive using numerical examples to formal deductive using symbolic mathematical language. PEE responses were categorized by examining whether the students think the provided argument proves the statement is true for all cases, true for some cases or does not prove the statement, and scored according to their accuracy of evaluating the given arguments. Results of related descriptive statistics regarding data collected by these three instruments are also reported. Comparisons of the results with respect to grade (freshmen and seniors) and department (Mathematics, Secondary Education Teaching Mathematics, and Primary Education Teaching Mathematics Programs) are done using appropriate quantitative analysis methods.

Results indicate statistically significant differences among senior and freshmen students' *background, attitude, self efficacy* and *beliefs* subscales, where freshmen have higher *background* scores than seniors but for the other three subscales seniors' scores are higher. Seniors also have statistically significant higher PE scores than freshmen. While there are no significant differences among freshmen with respect to department in neither

ABS nor PE scores; senior Mathematics, Secondary Education Teaching Mathematics and Primary Education Teaching Mathematics students' mean scores differ significantly both in PE and PEE, as well as some subscales of ABS.

It has been observed that freshmen students mostly rely on inductive reasoning while attempting to prove given mathematical statements. Seniors are generally aware of the necessity of generalizing their results and attempting to use procedures involving deductive reasoning. Nonetheless seniors still have difficulties in constructing and evaluating proofs. Significant differences observed between senior Mathematics and Secondary and Primary Education Teaching Mathematics students' proof construction and evaluation practices are in favor of Mathematics majors. Implications for teaching and further studies are discussed.

ÖZET

BİRİNCİ VE SON SINIF MATEMATİK VE MATEMATİK ÖĞRETMENLİĞİ ÖĞRENCİLERİNİN İSPATA İLGİLİ KAVRAMSALLAŞTIRMA VE BECERİLERİNİN İNCELENMESİ

Boğaziçi Üniversitesi İlköğretim Matematik Öğretmenliği, Ortaöğretim Matematik Öğretmenliği ve Matematik programlarında okuyan 93 birinci ve 82 son sınıf öğrencisinin katılımları ile gerçekleştirilmiş olan bu çalışma, öğrencilerin matematiksel ispat konusundaki tutum ve inançlarını, ispat yaparken kullandıkları yöntem ve akıl yürütme şekillerini, ayrıca başkaları tarafından yapılan ispatları nasıl değerlendirdiklerini araştırmayı hedeflemiştir. Araştırmada kullanılmak üzere üç ölçek geliştirilmiştir: *Tutum ve İnanç Ölçeği* (TİÖ), *İspat Sınavı* (İS) ve *İspat Değerlendirme Sınavı* (İDS). Veriler 2009-2010 güz ve ilkbahar dönemlerinde, ders esnasında, araştırmacının gözetimi altında toplanmıştır. Faktör analizi sonucunda TİÖ' nin dört tane alt boyutu olduğu ortaya çıkmıştır. Bu alt boyutlar *altyapı*, *tutum*, *öz yeterlik* ve *inanç* olarak adlandırılmıştır. İS' na verilen yanıtlar kullanılan ispat tiplerine göre sınıflandırılmış ve ayrıca akıl yürütme (tümevarımsal-deneysel sayısal örneğe dayalı ile matematiğin sembolik dilini kullanan tümdengelimsel arasında değişen) stillerine göre puanlandırılmıştır. Öğrencilerin İDS yanıtları ise, verilen argümanları "önermenin tüm durumlar için doğru olduğunu gösterir", "önermenin bazı durumlar için doğru olduğunu gösterir" veya "yanlıştır (önermeyi ispatlamaz)" seçeneklerinden hangisine uygun düştüğüne dair düşüncelerine göre sınıflandırılmış, yapılan değerlendirmelerin tutarlılığına göre puanlandırılmıştır. Üç ölçekten elde edilen verilerle ilgili ayrıca betimsel istatistikler de yürütülmüştür. Bulgular arasında sınıflara (birinci ve sonuncu sınıf) ve programlara (matematik, ilköğretim matematik öğretmenliği ve ortaöğretim matematik öğretmenliği) göre gerekli nicel analiz yöntemleri kullanılarak karşılaştırmalar yapılmıştır.

Araştırma sonuçları, birinci sınıf ve son sınıf öğrencilerinin *altyapı*, *tutum*, *öz yeterlik* ve *inanç* puanları arasında istatistiksel olarak anlamlı farklar olduğunu ortaya çıkarmıştır. Birinci sınıfların *altyapı* puanları son sınıflara göre daha yüksekken, diğer alt boyutlarda son sınıflar daha yüksek puanlar elde etmişlerdir. Son sınıf öğrencilerinin İS puanları da birinci sınıflara göre anlamlı derecede yüksek çıkmıştır. Farklı programların birinci sınıfları arasında TİÖ ve İS'na göre anlamlı farklılıklar gözlemlenmemiştir. Son sınıflarda ise İS ve İDS puanlarında ve TİÖ' nün bazı alt boyutlarında anlamlı farklılıklar ortaya çıkmıştır.

Birinci sınıfların ispat yaparken daha çok tümevarımsal akıl yürütme kullanmaya eğilimli oldukları gözlemlenmiştir. Son sınıflar ise çoğunlukla genelleme yapma gereği duyup tümdengelimsel yöntemler kullanmaya çalışmışlardır. Yine de son sınıfların ispat yapma ve değerlendirmede hala bazı zorluklar yaşadıkları ortaya çıkmıştır. Matematik programı öğrencileri ile öğretmen adayı öğrenciler arasında ispat yapma ve değerlendirme konusunda bulunan anlamlı farklar matematik programı öğrencileri lehinedir. Araştırmanın bulguları ve gelecekteki çalışmalar için öneriler tartışılmıştır.

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ABS	Attitudes and Beliefs Scale
MATH	Mathematics
PE	Proof Exam
PEE	Proof Evaluation Exam
PRED	Primary Education Teaching Mathematics Program
SCED	Secondary Education Teaching Mathematics Program

1. INTRODUCTION

Undoubtedly, proof is an essential part of mathematics, a process which is certainly complex. Mathematical processes start with looking for patterns, discovering relationships, apprehending by intuition, making a conjecture, and ends with more formal processes such as proving and defining (Dreyfus 1991; Schoenfeld 1994).

The term “proof” can take various meanings in different contexts (Recio, 2001; Healy and Hoyles, 2000). Views differ even within mathematics community. In the broadest sense, proof can be thought as establishing the truth of a certain claim. The debate arises from how that “establishment” is achieved. Alibert and Thomas (1991) see proof as “a means of convincing oneself whilst trying to convince others”. They point out that two fundamental aspects of a mathematician’s work are formulation of conjectures and development of proofs. These aspects show a dual character; the personal side aiming to clarify the researcher’s own understanding through statement of explicit hypothesis, and the collective side, where researcher propose a conjecture for the reflection of other mathematicians to share ideas. Section 2 elaborates more on meanings and functions of proof.

Proof also plays an important role in mathematics education, and many studies have been conducted in recent years investigating teachers’ and students’ understanding of proof across all grades (e.g. Knuth 2002; Miyazaki 2000; Morris 2000, 2002; Healy and Hoyles, 2000; Weber, 2001; Hoyles and Küchemann, 2002; Stylianides and Stylianides, 2009). Most of these studies report that students have a poor understanding of proof and have difficulties in constructing their own proofs.

Although there is a large scope of research on role of proof in mathematics education in the United States and other countries, only a limited amount of studies have been conducted in Turkey related to this topic (Özer and Arıkan, 2002; Sarı, et al., 2007; Baştürk, 2010). Nevertheless, it is seen that current Turkish mathematics curriculum highlights the importance of active, meaningful learning. The students are expected to engage in mathematics actively, and learn how to solve problems, share, explain and defend their solutions and thoughts; find relations within mathematics and also between mathematical and other subjects (MEB, 2005). Among the general aims of this curriculum is the following: the students will be able to: reason deductively and inductively, express their mathematical thinking and reasoning while solving problems, and use mathematical language and symbols correctly in order to communicate their mathematical thinking (MEB, 2005).

Considering the importance of proof in mathematics education, emphasized both by research and mathematics curriculum, this study aims to investigate the proving practices of mathematics and teaching mathematics majors in Boğaziçi University, İstanbul. By involving freshmen students who are at the very beginning of their programs, the researcher also aimed to see what kind of experiences regarding proof students bring from high school.

The details of the study and its results are reported in the following manner: Section 2 starts with introducing relevant concepts then goes on to explore some of the research done regarding students' beliefs and conceptions about proof, and their ways of evaluating and constructing proof, as well as role of proof in mathematics education. The research questions and operational definitions of variables are given in Sections 3 and 4.

Three instruments were developed for this study, in order to collect data regarding students' views and beliefs about proof, their own proof constructions and their evaluations of other peoples' proofs. These instruments are introduced and explained in Section 5, and how data were collected using these instruments is reported in Section 6.

Findings of the research are given in Section 7 and conclusions and implications are discussed in Section 8.

The instruments used and rubrics developed for scoring these instruments are given in the appendices. Detailed results of the pilot studies conducted to develop and test the instruments are also reported in this section.

2. LITERATURE REVIEW

Various aspects of mathematical proof have been reported by numerous researches in recent decades. The scope of this section requires focusing mainly on literature about the role of proof in mathematics education, and students' ability of constructing and understanding of proof across different grades. Such studies are reported here following the definition of proof in the context of this study and a brief introduction of relevant mathematical concepts.

2.1. Relevant Concepts

2.1.1. Meanings and Functions of Proof

Proof process begins by making an observation. This observation remains as a *conjecture* until the person who made the observation becomes certain of its truth. Therefore proving is the "process employed by an individual to remove or create doubts about the truth of an observation" (Harel and Sowder 1998). Hanna (1991) considers mathematical proof as "an argument needed to validate a statement, an argument that may assume several different forms as long as it is convincing". Selden and Selden (2003) refer to *proofs* as "texts that establish the truth of theorems" and define *validation of proofs* as "readings of, and reflections on proofs to determine their correctness". Proving involves "constructing a deductive argument using valid rules of inference, axioms, definitions and previously proven conclusions" (Morris, 2002). According to Stylianides and Stylianides (2009), an argument for the truth of a statement that is "general, valid and accessible to the members of the community" qualifies as proof.

Proof process includes both inductive and deductive reasoning. *Reasoning* is a type of thinking that involves inference; the process where "one proposition (conclusion) is arrived at and accepted on the basis of other propositions (premises) that were originally accepted" (Overton, 1990) . In *deductive reasoning* inference process leads from general to particular and premises provide necessary evidence for the truth of the conclusions. In *inductive reasoning*, inference process leads from particular to general, and premises provide probable, but not necessary evidence for conclusions (Morris, 2007; Overton, 1990).

An *argument*, in logic, is defined as “a sequence of sentences or propositions of which one (conclusion) is said to follow from others (premises), and the premises are said to provide evidence for the truth of the conclusion” (Overton, 1990). A deductive argument is *valid* (correct) when it is impossible to have true premises and a false conclusion in the argument. Inductive arguments cannot be assigned as valid or invalid because the premises are only probable evidence for the conclusion.

Forms of reasoning such as pragmatic reasoning (based on context), statistical reasoning (based on probability) and modal reasoning (based on possibilities and necessities) involve inductive inferences (Overton, 1990). While the final formal mathematical proof is based on deductive arguments, in the process of conjecturing or arriving at a solution (before the formal proof is given) mathematicians use inductive reasoning as well. Lakatos (1976, 1978) recognizes three stages of mathematical reasoning used in proofs: First, mathematicians use induction to discover conjectures worth trying to prove. At the next stage, they develop and criticize *informal proofs* of their conjectures. Finally, they formalize the informal theory so that theorems are deducible by formal transformations of the axioms. According to Lakatos mathematics grows through continuous “improvement of guesses by speculation and criticism, by the logic of proof and refutation” (Lakatos, 1976). Hence proof is never finished; its improvement and gaining acceptance depends on negotiation of meaning, which is a social process.

Bell (1976) suggest there are three senses of mathematical meaning of proof: *verification* or *justification* which is concerned with the truth of a proposition, *illumination* where a good proof should give an insight as to why the proposition is true, and *systematization*, organization of results into a deductive system of axioms, major concepts and theorems. de Villiers (1999), expended Bell’s (1976) list of functions of proof and proposed the following model:

- verification (concerned with the truth of a statement)
- explanation (providing insight into why it is true)

- systematization (the organization of various results into a deductive system of axioms, major concepts and theorems)
- discovery (the discovery or invention of new results)
- communication (the transmission of mathematical knowledge)
- intellectual challenge (the self-realization/fulfillment derived from constructing a proof)

Hanna (2000) added the following to the first 5 items of de Villier's (1999) list above: *construction* of an empirical theory, e*xploration* of the meaning of a definition or the consequences of an assumption, *incorporation* of a well known fact into a new framework and thus viewing it from a fresh perspective.

Hanna (2000) also points out the importance of the explanatory aspect of proof, especially in educational domain. According to Hanna, best proofs help to understand the meaning of a theorem; to see why it is true. Almeida (2003) indicates that a mathematical proof is used to verify a result, communicate and persuade others, discover a result, and to systemize a result into a deductive system.

Recio and Godino (2001) emphasizes that proof have different meanings in different (institutional) contexts, such as *daily life* (informal, subjective and does not necessarily produce truth), *empirical sciences* (intention of validity, theories are experimentally validated), *professional mathematics* (argumentative process that mathematicians develop to justify the truth of mathematical propositions), and *logic and foundation of mathematics* (pure deductive argumentations that take place in axiomatic and formal systems).

In this study, the term *proof* or *mathematical proof* is referred as a deductive, valid argument that establishes the truth of a mathematical proposition.

2.1.2. Proof Methods

A truth of a proposition of the form of logical implication P => Q (P implies Q or if P then Q, where P is the premise and Q is the conclusion) can be obtained using various methods. *Direct proof* is to assume that P is true and arrive at the conclusion that Q must be true, using axioms and previously proven results.

The statement “if P is true then Q is true” is logically equivalent to the statement “if Q is false then P is false”, which is called the *contra-positive* of the statement P => Q. The *contra-positive proof* method is to assume that Q is false and conclude that P is also false. It should also be noted that the statement P => Q is not logically equivalent to its *converse*; Q => P, therefore proving Q => P is true is not the same thing as proving P => Q.

The proposition P => Q is accepted as false only when P is true and Q is false. *Proof by contradiction* is the method where the statement is assumed to be false (i.e. P is true and Q is false) and then a contradicting result is obtained. Hence the assumption has to be false; the statement is proved to be true.

Proof by cases (or *proof by exhaustion*) is dividing all possible situations into (finitely many) cases and then checking that the statement is true for each case.

One can *disprove* or *refute* a mathematical statement by providing a *counter example*, a case which makes the statement false, therefore refuting the claim that the statement is always true.

Proof by mathematical induction is a method generally used to prove that a statement P(n) regarding a natural number n, is true for all values of $n \geq n_0$. The basis step is to show that the statement is true for the smallest value of n, which is n_0. Then the inductive step is to show that if the statement is true for one arbitrary value of n, then it is also true for the next value. In other words, to prove the implication P(k) => P(k + 1). Proving these two steps guarantee that P(n) is true for all n; since it is true for the smallest value n_0 (basis step), it will be true for $n_0 + 1$ (inductive step), and since it is true for $n_0 + 1$, it will be true for n_0 + 2 and so on. Mathematical induction should not be confused with inductive reasoning mentioned above; it is actually a process that uses deductive reasoning.

Pigeonhole principle basically states that if there are *n* pigeonholes and *n* + 1 pigeons, no matter how you place the pigeons into the holes, there will be at least one hole with at least two pigeons. In other words, it is impossible to have exactly one pigeon in every hole, because the number of pigeons is more than the number of holes. A more general form of this principle states that if there *n* pigeonholes and *kn* + 1 pigeons, there will be at least one hole with at least *k* + 1 pigeons (because if there are at most *k* pigeons in every hole, then total number of pigeons will be at most *kn*). This simple observation is used to prove (some quite complex) results in various fields of mathematics, especially in combinatorics.

After this introduction of the mathematical concepts related to this study, in the next section, details of the studies analyzing students' proof schemes are reported.

2.2. Proof Schemes

A variety of classifications of proof exists in literature. For example, Balacheff (1988) classified proofs in the following manner:

Pragmatic proofs: "Those having resource to actual action or showings". Types of pragmatic proofs are naïve empiricism and crucial experiment. *Intellectual proofs*: "Those which do not involve action and rest on formulations of the properties in question and relations between them". Types of intellectual proofs are generic example and thought experiment. *Demonstration*: "Which requires a specific status of knowledge that must be organized in a theory and recognized as such in community; the validity of definitions, theorems and deductive rules is socially shared" (Balaceff, 1988, p. 217).

Levels of proof Miyazaki established on the basis of Balacheff's (1988) ideas are steps from inductive proof to algebraic demonstration. In this context, *demonstration* refers to "human activities to reason a proposition from assumptions deductively and to present the reasoning with formal language". The objective to learn demonstration is given as follows: “A student can show others the reason why a proposition is true for himself or herself”. Prerequisites to achieve this objective is specified in terms of *content* (what it shows), and *representation* (how does it show). In terms of content, the students need to deduce the proposition from assumptions that are true for themselves. In terms representation, they need to use a language called *functional language of demonstration* which consists of: (i) symbols and rules of their arrangement to represent objects, their properties, and relations between them, (ii) terms and rules of their arrangement to represent a proposition, (iii) sentences and rules of their arrangement and abbreviation to represent a chain of propositions.

Miyazaki distinguishes between *proof* and *demonstration* in lower secondary school mathematics. In terms of content in a proof the students are expected to logically reason (using induction, deduction, analogy and so on) from assumptions true to themselves. In terms of representation, they are expected to use language, drawings and manipulable objects.

Miyazaki uses three axes to establish the levels of proof: Contents, representation and students’ thinking. There are two categories in each axis. Content consists of ‘inductive reasoning’ and ‘deductive reasoning’; representation consists of ‘functional language of demonstration’ and ‘language other than functional language of demonstration, drawings, and/or manipulable objects’; and students thinking is divided into ‘concrete operations’ and ‘formal operations’. Combining two categories in the first two axes, the following four basic levels, shown in Table 2.1, are established. When the third axis is considered, Miyazaki concludes that only students of level A can be in formal operational stage, and for levels C and D students in concrete and formal operational stages cannot be distinguished, therefore only level B is divided into formal and operational stages, which makes a total of 6 levels on three axes.

In this classification, proof A is the most advanced category in both axes, therefore it is considered as the most advanced level of proof; which is equivalent to a *demonstration*. Proof C is the lowest level, since it has the lowest categories in both axes. Prof B and proof D are intermediate levels.

Table 2.1. Miyazaki's proof levels

Representation	Contents	
	Inductive reasoning	Deductive reasoning
Functional language of demonstration	*Proof D*	*Proof A*
Other language, drawings, and/or manipulable objects	*Proof C*	*Proof B*

According to Harel and Sowder (1998), *ascertaining* and *persuading* are two sub-processes of the process of proving. *Ascertaining* refers to removing a person's own doubts about the truth of an observation, while *persuading* is the process of removing other people's doubts about the truth of an observation. Therefore, a person's *proof scheme* "consists of what constitutes ascertaining and persuading for that person". The authors assert that these processes are subjective; they may be based on logical and deductive arguments, empirical evidence, intuitions, personal beliefs, an authority, social conventions and so on.

Harel and Sowder's (1998, 2003) taxonomy of proof schemes is given below.

External conviction

- Authoritarian (teacher or a book)
- Ritual (appearance of the argument)
- Non-referential symbolic (symbol manipulations without any potential coherent system of referents)

Empirical

- Inductive (evidence from examples)
- Perceptual

Deductive

- Transformational
    - Generality (goal is to justify for all)
    - Operational Thought (forming goals and sub-goals, attempting to anticipate their outcomes)
    - Logical Inference (use of logical inference rules)
- Axiomatic (proof process starts from accepted principles)

Stylianides and Stylianides (2009) use the following definitions regarding levels of proof. *General argument* is “a sequence of assertions that refer to all cases of the domain of the statement”. A *valid* argument is “deductive and provides conclusive evidence for the truth of the statement”. The authors note that while *proof* is a valid general argument, a valid general argument may not necessarily be regarded as proof; for example, further justification of a step may be required. *Invalid general argument* is a general argument that has some flaw in its logic. *Empirical argument* is “an invalid argument that provides inconclusive evidence for the truth of a statement by verifying its truth in a proper subset of the cases in the domain of the statement”.

Recio and Godino (2001) conducted a study on university students who were at the start of their first year. Their responses to two problems (one from algebra, one from geometry) were classified to reveal four types of proof schemes. *Explanatory argumentative schemes* refer to type of answers where students try to explain the meaning of the proposition by providing specific examples and there is no intention to validate the proposition or check the truth for all cases. Answers where empirical-inductive procedures are used and specific examples are given to verify the proposition are called e*mpirical inductive proof schemes*. If informal logical approaches are observed with the use of analogies, graphs etc, it is classified as *informal deductive proof schemes*. Formal *deductive proof schemes* refer to answers that use a formal approach, more in agreement with the transformation rules of mathematical language.

In their investigation of proof constructs and strategies of first year collage students, Coe and Ruthven (1994) aimed to classify the types of proof used as *empirical proof* (a conjecture explicitly stated, supported by confirming examples or pattern of results, without explanation), *weak deductive proof* (some attempt to suggest an underlying reason) and *strong deductive proof* (an attempt at a clear, logical argument with explicit link between assumptions and clearly defied conclusion). They concluded that the strategies the students used were mainly empirical, few students concerned to explain why the rule or patterns occurred, students' main concern was to validate conjectured rules and patterns where most of them tested these against a few examples, they mainly attempted to use routinely the investigation techniques from their textbook, which were designed for numeric data analysis.

Martin and Harel (1989) asked the participants of their study, who were sophomore level pre-service elementary school teachers, to judge verifications of two mathematical generalizations. The verifications consisted of four kinds of inductive arguments and three related to deductive arguments. Types of inductive arguments used in the instrument were *examples* (two particular instances involving small numbers), *pattern* (a sequence of twelve instances-one can as many examples as wanted to support the general statement), *big number* (a particular instance involving large numbers – if the statement is true for an arbitrary large number, it is probably true for all numbers), *example and non-example* (an example supporting the statement, an a non example- if the condition does not hold, neither does the conclusion or if the conclusion does not hold, neither does the condition). Deductive arguments used were in the form of *general proof*, *false proof* and *particular proof* (correct proof of the statement, where particular numbers were substituted for each of the variables).

Martin and Harel (1989) observed that many of the students who correctly accepted the general proof did not reject the false proof; which suggested that they were influenced by the appearance of the argument rather than the truth of it. The authors indicate that such students have *syntactic-level deductive frame* where "a verification of a statement is evaluated according to ritualistic, surface features". Relatively few students have a *conceptual-level deductive frame*, meaning "a judgment is made according to causality and purpose of the argument". Students who accepted the general proof also had high levels of acceptance of the particular proof, suggesting that students were replaying the general argument in the particular case. Thus conceptual-level deductive frame was divided into two sub-frames: *generalized results sub-frame* and *generalized process sub-frame*. In addition, many students simultaneously accepted inductive and deductive arguments as proofs; leading the authors to conclude that *inductive* and *deductive proof frames* exist simultaneously in many students.

2.3. Beliefs, Conceptions and Views about Mathematical Proof

The studies mentioned above focuses mostly on the cognitive aspects of proof. This section discusses studies that pay more attention to affective domain; how the view proof, their belief and conceptions about the nature and role of proof in school mathematics.

In a study designed to investigate the student experience of proof at the university level, Jones (2000) examined the conceptions of proof of 75 secondary mathematics teacher trainees during the middle of their one-year graduate course. Concept map was used as a methodological tool. The students were expected to generate a map of their conception of mathematical proof. Concept maps were analyzed by looking at the use of key terms (how many were used and which ones were included and the specified) and relationship between key terms (how many are specified, how are they specified, whether there are cross-links or multiple relationships). Results showed that richer concept maps were produced by graduates with a better classification of honors degree. When concept maps of recent graduates were considered, results suggested that the least well qualified graduates have the poorest understanding of mathematical proof; however the most highly qualified graduates did not necessarily have the richest type of subject matter knowledge required for most effective teaching.

Moralı et al. (2006) conducted a study about the views and beliefs of pre-service mathematics teachers' about proof . The sample consisted of 182 freshmen and 155 senior students. A likert type scale was used as an instrument. Factor analysis resulted in seven components: competence in making proofs, views about the importance of proof, views about the effect of proof in understanding a theorem, self concept in making proofs, general views about proof, points of view regarding examples and theorems, and relationship between problem solving and proof. Findings show that although the pre-service teachers think proof is essential for theoretical mathematics, many believe if the truth of a mathematical result is obvious, proof is not necessary. Most of the subjects were unsure about their ability to make proofs. It is also interesting that there were no significant differences between the views of freshmen and senior students. Another conclusion was that even the pre-service teachers with high ability of producing mathematical proofs do not have positive views about making proofs.

Another study about conceptions of proof focused on 17 experienced secondary school teachers (Knuth, 2002). Interviews were conducted with the participants, intending to explore teachers’ conceptions about the nature and role of proof in school and what they expected from their students in terms of proof. Different sets of arguments for a number of mathematics statements were presented to the teachers who were asked to evaluate them in terms of appropriateness (whether they would use it to convince the students of the statement’s truth). The arguments varied in terms of their validity as proofs and their explanatory power.

Teachers viewed proof as “an argument that conclusively demonstrates the truth of a statement”. In the context of secondary school mathematics, Knuth categorizes teachers’ descriptions in terms of formality: *formal proofs*, *less formal proofs* and *informal proofs*. Teachers’ descriptions of *formal proof* rely on prescribed formats and/or use of particular language. *Less formal proofs* are considered as “proofs which do not necessarily have a rigidly defined structure or are not perceived as being mathematically rigorous”. Teachers regarded them as valid proofs. They defined these proofs in terms of whether the argument established the truth of its premise rather than in terms of the strictness involved in its presentation. Teachers described *informal proofs* as “explanations and empirically based arguments”, which are not considered valid because they do not prove the general case.

The majority of the teachers thought formal proofs and less formal proofs were not central in secondary school mathematics. They were not sure of its appropriateness for every student. These considered formal proofs suitable for students taking advanced mathematics classes and who are likely to be pursuing mathematics-related fields in college. On the other hand, all teachers indicated informal proofs to be a central idea throughout secondary school mathematics; which is appropriate for every student and must be included in all classes and said they would accept empirically based arguments as proof from their students in lower level mathematics classes.

Regarding the role of proof in school mathematics, the following categories emerged in Knuth's study: Developing logical thinking skills, communicating mathematics, displaying thinking, explaining why, creating mathematical knowledge.

Baştürk (2010) conducted a study on first year secondary school mathematics education students, in order to identify their conceptions of proof and proving in mathematics and mathematics education, via questionnaires and interviews. It is revealed that majority of the students believe that the role of proof in mathematics is important and emphasize the explanatory function of proof, however, they also have difficulties in understanding and doing proofs and indicated that the main reason of this is the difference between mathematics education in high school and university. Lack of proof in high school practices results in an abrupt introduction to proof in university level.

The participants in Mingus and Grassl's (1999) study were asked the question "what constitutes a proof?" Responses revealed the following categories: making sense of data, showing relationship between concepts; a way of explaining why something works; something made up of other simpler proofs; an argument that is convincing to its audience; like a map from A to B, with directions for each step via different paths. When asked about the role of proof in mathematics, a majority of participants' responses related to "how proofs explain, why concepts work the way they do in mathematics, and how constructing proof help students understand the mathematics they are doing". When asked what exposure to proof is appropriate in K-12 to pre-service teachers, 69 per cent of participants wanted proofs to be introduced before taking 10^{th} grade geometry.

2.4. Relationship between Students' Views and Conceptions about Proof and Their Proof Schemes

Naturally, there is an intersection between cognitive and affective domains and students beliefs of proof may affect their proof performances. This section explores studies that take both aspects into account.

Almeida (2000) conducted a study on mathematics undergraduates in UK about their declared perceptions of proof (by means of a questionnaire), and their actual proof perceptions and their proof practices (via interviews and a proof preference questionnaire in which they are asked to choose most convincing, least convincing and incorrect arguments). The author concludes that while students publicly declare their agreement with the notions of formal mathematical proof, they appear to prefer or do not reject informal/visual methods of proving. Almeida suggests that this may arise from the difference between the way mathematics is practiced by mathematicians and the way mathematics is thought; while in the former the path followed is *intuition → trial and error → speculation → conjecture → proof*; the latter is only concerned with end point of the flow.

In order to gain insight about student views of what comprised as proofs, its role and its generality, as well as to look for an indication of students' competence in constructing proofs, Healy and Hoyles (2000) conducted a with 14-15 year old high attaining students. Students' written descriptions about proof and its purpose were collected. Then they were presented with mathematical conjectures and a set of arguments supporting them. Students indicated which argument was nearest to their own approach and which argument they believed would get the best mark from the teacher. They were also asked to assess the validity and explanatory power of the arguments. Finally students were given two conjectures to construct their own proofs. Results indicate that majority of students were unable to construct their own proofs, they valued general and explanatory arguments. Furthermore, while most students used empirical arguments in their proofs, they recognized that it would not receive high marks from their teachers. They were aware that a valid proof must be general. Another finding is that students preferred arguments presented in words as choices of their own approaches and found them explanatory, whereas arguments containing algebra were less popular and found to be hard to understand. Students who used narrative form in their own constructions were more successful than students who attempted to use algebra. It is concluded that students held two conceptions of proof simultaneously: those about arguments they thought to would receive the best mark (containing algebra) and those about arguments they would use themselves (narrative from).

Students' views about proof and its purpose were coded into four categories: *truth* (verification, validity and providing evidence), *explanation* (explanation, reason, communicating to others), *discovery* (discovering or systemizing new theories and ideas) and *none/other*. 50 per cent of students indicated that proof is used to establish truth and 35 per cent mentioned their explanatory power. Follow-up interviews suggested even more emphasis on the explanatory power.

In Selden and Selden's (2003) study, eight mathematics and secondary mathematics majors were each given four student-generated arguments that are supposed proofs of the following theorem: "For any positive integer n, if n^2 is a multiple of 3, then n is a multiple of 3". One of the four arguments was a proof of this theorem, the other three were not. Semi- structured interviews were conducted with each participant. They were first given the statement of the theorem and asked to think about what it meant, to give examples and how they would prove it. Then they were shown the four 'proofs', and asked to think out loud while they decided whether the given argument was actually a proof of the theorem. If not, they would point out which parts of the proof they thought was problematic. Last part of the interview consisted of eight general questions about proofs and how the students read, understand and validate them. Seven of the eight students maintained or increased their number of correct judgments over time (during the course of the interview), however the authors conclude that students tended to focus on surface features and they were very limited in their ability to decide whether the arguments were proofs.

Morris (2002) points out that many adolescents and adults do not sincerely believe general mathematical statements. The author also states that there is some indication that many adolescents and adults consider mathematical objects as unpredictable, believing that there may be exceptions or unusual occurrences. This suggests that they may not find deductive conclusions logically necessary. Thirty undergraduate pre-service teachers participated in the study. Ninety minute interviews were conducted with each participant. They were asked to give a written solution for the following problem: "The set of counting numbers consists of the numbers 1, 2, 3, 4, 5 … Prove that for every counting number n, the expression $n^2 + n$ will always be even." Then they were given four arguments for this problem. Two of them were valid deductive arguments, while the other two were invalid inductive solutions. Participants were required to answer the following question: "Do any or all of the arguments prove the conclusion is true for each and every counting number? In other words, can you be absolutely certain that the conclusion is true for each and every counting number because of the argument? Why or why not?"

Understanding of mathematical content was measured by the written solutions to the problem and ability to distinguish necessary and independent arguments was measured by their responses to the above question. Then whole process was repeated with a second problem: "A prime number is a counting number that has exactly two factors- itself and 1. For example number 2 has exactly two factors: 2 and 1. However, 4 is not a prime number since it has more than two factors: 1, 2, and 4. The primes 2 and 3 are two consecutive counting numbers. Is there another pair of consecutive primes? Prove your answer is correct." After that, five more questions were asked in order to understand the participants' point of view on deductive and inductive arguments.

Results show that 30 per cent of the participants distinguished deductive and inductive forms of argument; found deductively derived conclusions necessary and inductively derived conclusions uncertain. 40 per cent could not distinguish between deductive and inductive forms; found deductively and inductively derived conclusions necessary. 30 per cent distinguished between deductive and inductive forms but found deductively and inductively derived conclusions uncertain. It is concluded that adults' thoughts on the necessity of inductive and deductive conclusions depends on a complex coordination of ability to attend to abstract premise-conclusion relations and beliefs about mathematical objects and regularities.

In a later study, Morris (2007) examined the factors affecting pre-service mathematics teachers' evaluations of students' mathematical arguments. Specifically, she examined their ability to distinguish logical deductive arguments from other forms of arguments.

Pre-service teachers were given a transcript of a classroom lesson where the students tried to prove a mathematical generalization, and were asked to evaluate the validity of the students' arguments. Since it was hypothesized that the participants' evaluations may change across different classroom contexts, two conditions were employed. In one condition the transcript included a valid argument by a student that proofs the generalization. In the other condition this part was omitted in the transcript, and there were no other valid arguments.

Participants were thirty four undergraduates in a K-8 teacher preparation program. There were 17 participants in each condition. Ninety minute interviews were conducted with each participant. First part of the interview consisted of the participants' evaluation of the students' arguments, and second part examined the participants' own understandings about logical and non-logical mathematical arguments.

The transcript was of a real third grade mathematics lesson, where the students were forming arguments to prove the followings: the sum of two odd counting numbers is always an even counting number and the sum of two even counting numbers is always an even counting number. The participants are asked to: (i) rank the three best arguments (the ones that they would like their students to make) and explain why they think it is best, (ii) determine whether any of the arguments made are valid, (iii) determine whether any of the students make correct or incorrect statements, and (iv) determine whether any students understand *why* the sum of two odd counting numbers is always an even counting number or the sum of two even counting numbers is always an even counting number. For the second part of the interview, the participants were given five arguments (written by the interviewer) for the following statement: "For every counting number n, $n^2 + n$ will always be even."

Participants were asked whether they thought any of the arguments proved the conclusion for every counting number. Some of the findings are as follows: Pre-service teachers' evaluations of students' inductive arguments differed dramatically across conditions, they rarely used logical validity as a criterion for evaluating arguments, and exhibited a wide variety of conceptions about the relationships among mathematical proof, explaining why something is true in mathematics, and inductive arguments; and these conceptions affected their evaluations of students' arguments. Many of the participants were able to distinguish between student responses that did and did not explain why a generalization was true. However, they used their own knowledge to fill in holes in students' arguments which led to inappropriate evaluations of students' arguments and understanding.

The studies mentioned until now show that students across have difficulties with proof. The following section focuses more onto how proof should take place in the classroom so that the aforementioned difficulties can be overcome and meaningful learning can be improved.

2.5. Role of Proof in Mathematics Education

The role of proof in mathematics education and its importance in the curriculum is discussed in Hanna's (2000) overview. Hanna says that the fact that leading journals of mathematics education have published over a hundred research papers on proof and there is a website (launched in 1997 and active to date) called *International Newsletter on the Teaching and Learning of Mathematical Proof* that posts information about research on proof, is an indication that this topic is an important issue in mathematics education. Here, it is stated that while proof is obviously an important part of mathematics, its key role in the classroom is the promotion of mathematical understanding.

The author suggests that a student should start with the fundamental functions verification and explanation. However, in the educational domain proof should be viewed primarily as explanation and proofs that best help to explain should be valued most. On the other hand, the author also accepts the fact that one cannot find explanatory proofs to every theorem and in many mathematical subjects some theorems need to be proved with other methods such as mathematical induction or contradiction.

Selden and Selden (1995) found out that, none of the participants in their study, who mostly consists of third and fourth year mathematics and secondary mathematics education students familiar with predicate calculus, could consistently associate informally written mathematical statements with equivalent formal versions using logical symbols. Authors conclude that these students could not be able to construct or validate proofs of the informally stated theorems, because they would not be able to relate them with the high level logical structure of their proofs.

Epp (2003) talks about the difficulties mathematics majors have in writing proofs. Over the years working with many students at the university level, she observed that very few of them had intuitive understanding of the reasoning principles that mathematicians take for granted. Epp suggests several explanations for this lack of understanding. One of the reasons why students struggle with formal mathematical reasoning is the different use of some statements in everyday language and mathematical language. Especially 'if-then' statements, quantified statements and their negations can be interpreted in various ways in ordinary language. Another reason can be the influence of previous mathematical instruction where the emphasis has been on narrow problem solving strategies rather than focusing on general principles. Some teachers omit proofs of theorems and rely on examples for justification. This can lead to the misconception that empirical evidence is enough to prove mathematical statements. She notes that a course that focuses on mathematical reasoning and proof should start with a few weeks of elementary logic and have the students practice logical connectives and quantifiers. Teaching logic in a mechanical way may not improve reasoning abilities. Some advice on teaching logic is given as follows:

- Exploiting similarities between formal and everyday language: introduce logical principles by giving examples whose everyday interpretation coincides with the formal one.
- Translation exercises: practice on translating back and forth between formal and informal expressions. Exercises that mix logic, language and mathematics.
- Use of truth tables: Truth tables are useful to make summarizations and deciding validity of arguments but must be used carefully and mechanical interpretations should be avoided.
- Dealing with transfer issues: Continue to refer to logical principles as they come up naturally in mathematical contexts.

The role of everyday language in reasoning is also the focus of the study of Schliemann and Carraher (2002), where they contrasted formal mathematics instruction with informal mathematical practices and conceptions children develop by themselves out-of-school contexts. They point out that cognitive performance can differ significantly across contexts. They cite research that show children may perform differently in multicasual reasoning and syllogistic reasoning. They report from their own studies that similar results are found in case of mathematical reasoning. Different relations, representations and approaches are evoked by contexts, social goals and values related to activities the children are involved. As they continue school, tensions arise between their everyday experience and problems they come across in mathematics classroom. Authors suggest that, over time, they learn to overlook realistic concerns and concentrate on mathematical relations. It is stated that, in order to understand how mathematical reasoning develops, one needs to analyze "how children learn as they participate in cultural practices, as they interact with teachers and peers in the classroom, as they become familiar with mathematical symbols and tools, as they deal with mathematics across a variety of situations". Taking these points into account, authors indicate that design of classroom activities should require:

- Considering children's previous understanding and intuitive ways of making sense and representing relationships between physical quantities and between mathematical objects,
- Providing opportunities for children to participate in novel activities that will allow them to explore and to present mathematical relations they would otherwise not encounter in everyday environments,
- Exploring multiple, conventional and non-conventional ways to represent mathematical relations,
- Constantly exploring matches and mismatches between rich contexts and the mathematical structures being death with.

Finally the authors add that, in classrooms that they used such activities, the discourse was closer to an everyday problem solving situation rather than a traditional mathematics lesson based on transmission and application of basic rules. But the students still had access to the new representations that the instructors introduced.

Holvikivi (2007) also mentions that reasoning is very affected by context and content; that it is difficult for humans to separate a reasoning task from surroundings and focus on only the given premises. The author also refers deductive reasoning as "unnatural" and is difficult process because one has to "ignore most thinking". The study analyses the responses to four logic questions, asked as a part of a larger survey conducted to engineering students and teaching staff in international and national degree programs in a university in Finland. The four questions were syllogisms taken from a study of D'Andre (1995, as cited in Holvikivi, 2007). A syllogism is "an inference in which one proposition (the conclusion) follows of necessity from two others (known as premises)". The statements were not of mathematical content. The results showed that the participants often applied pragmatic reasoning although they had formal training in mathematical logic. Pragmatic reasoning refers to arriving at plausible conclusions that does not necessarily follow from the premises.

As a teacher-researcher, Zack (1999) examined the discussion in a fifth year mathematics classroom, where three students were trying to convince another two that their position is could not be true. The focus is the interplay between everyday and mathematical knowing and speaking. It was observed that the students were talking as if something was obvious without putting justification into words. While describing the culture of the classroom, the author the students are part of a "problem solving culture", where they are accustomed to solving cognitively demanding tasks throughout the curriculum. They have been constantly encouraged to engage in conversation about ideas. In this classroom, different mathematical meanings of proof emerge. One is "an assertion that an answer is correct and can be shown to be correct"; while another is "a pattern is correct and will continue as such forever". Patterns to the solution of the problem emerged from inductive reasoning and informal deductive reasoning. Students used these patterns to evaluate evidence, to explain and to convince. Daily and emerging academic discourse co-existed in their talks; they went back and forth between 'I'll bet you' and the act of proving. The conversation topic centered and closely linked to propositions and challenges. The logical connectives 'if…then' and 'but' were not used explicitly but they were implied. Author adds that the children are at ease when they are arguing and trying to convince another, and sometimes it sounds like they are bantering in the schoolyard but there is actually a complex mathematical structure in their conversations. She concludes with saying that she aims to attain the balance between promoting students' informal ways of thinking and speaking and helping them to develop their formal mathematical competence. She believes it is important to explicitly encourage them to keep in touch with their personal meanings as they become more competent in doing formal ways of doing mathematics.

The gap between formal and informal proof can be overcome using some technological aids. Visualization can be effective tool for teaching proof and logical reasoning. There are some studies that focus on teaching proof and logical reasoning using software technology to high school students (Öner, 2008a, 2008b) and to university students (Eysink *et al.*, 2002; Huertas, 2007; Perez-Lancho *et al.*, 2007).

Mathematical logic and proof are important in higher education, not only for mathematics majors, but also for computer science and engineering students. An example from higher education is the study of Roberts (2003), who proposes a way to motivate engineering students to do mathematical proofs. He states that even though engineering students take quite a lot of mathematics courses, they "do not learn the art of doing mathematical proofs", which will help them to have a deeper understanding of the course material, develop creative solutions, and be more aware of the errors. He proposes to present them a case study where the result can be proven in a number of ways using different techniques. This enables the students to compare different approaches and see their advantages and limitations. As an example, the author gives twenty different proofs of the known identity $1 + 2 + 3 + \dots + n = n (n + 1) / 2$. The students are invited to prove a similar identity by modifying as many of the given proofs as they can.

As the above review demonstrates, students across all grades have difficulties in understanding and constructing proofs, and struggle with switching between formal logical reasoning and everyday informal reasoning. The following sections describe and report the results of the study conducted to investigate the situation in a state university in İstanbul, Turkey, regarding proof in mathematics education. Specifically, the aim is to observe students' proof schemes when they enter the teaching mathematics and mathematics programs as high school graduates, and when they graduate as candidates to teach mathematics to future generations.

3. STATEMENT OF THE PROBLEM

Importance of proof, justification and reasoning in school mathematics is emphasized both in the current high school curriculum and various recent research studies (e.g. Hanna, 2000) as mentioned in the previous section. Therefore, high school graduates, prospective mathematics teachers and prospective mathematicians should be sufficiently competent in this regard. The aim of this study is to explore this competency in a particular setting. More precisely, to investigate the following:

- Freshmen and senior Mathematics, Primary and Secondary Education Teaching Mathematics students' attitudes and beliefs regarding proof in school mathematics, and types of proof methods and reasoning they use while proving mathematical arguments (their proof construction practices),
- Senior students' evaluation process of given mathematical arguments generated by freshmen students (their proof evaluation practices).

The target population consists of the students from Primary Education (Teaching Mathematics Program), Secondary School Science and Mathematics Education (Teaching Mathematics Program) and Mathematics Departments in Boğaziçi University, İstanbul.

The reason for conducting the study in this group of students is that, they are prospective mathematics teachers and mathematicians; therefore important figures who will shape high school and university students' conceptualizations related to mathematical concepts in the future. Participants of the study were newly graduated from various high schools -freshmen students- and students who were about to graduate from university to become mathematicians and mathematics teachers. These characteristics give an idea about the basic tendencies of conceptualizations on proof in high school and university graduates. Therefore, this study is an important step for understanding and comparing mathematicians' and prospective mathematics teachers' proof patterns at the time of starting the program and finishing it. Clarification of these proof patterns will be helpful in developing instructional implications for teaching mathematics programs as well as being helpful for high school mathematics teachers and instructors of freshmen mathematics courses in terms of showing some of the tendencies proof patterns seen in high school graduates.

4. RESEARCH QUESTIONS AND OPERATIONAL DEFINITIONS

The research questions that have been posed in relation with the aims of the study and the operational definitions of the research variables are listed in the following sections.

4.1. Research Questions

The questions related with the study listed below are grouped into four themes: students’ attitudes and beliefs, students’ proof construction practices, students’ proof evaluation practices, and relationships between them.

Research Question 1:

- What are freshmen Mathematics, Primary and Secondary Education Teaching Mathematics students’ attitudes and beliefs regarding proof in school mathematics?
- What are senior Mathematics, Primary and Secondary Education Teaching Mathematics students’ attitudes and beliefs regarding proof in school mathematics?
- Are there any significant differences between freshmen and senior Mathematics, Primary and Secondary Education Teaching Mathematics students’ attitudes and beliefs regarding proof in school mathematics?
- Are there any significant differences between students from Mathematics, Secondary Education Teaching Mathematics and Primary Education Teaching Mathematics programs, with respect to their attitudes and beliefs regarding proof?

Research Question 2:

- What are the proof practices of freshmen Mathematics, Primary and Secondary Education Teaching Mathematics students, when they are asked to prove mathematical statements?

- What are the proof practices of senior Mathematics, Primary and Secondary Education Teaching Mathematics students, when they are asked to prove mathematical statements?
- Are there any differences between freshmen and senior students’ proof practices that are observed, when they are asked to prove mathematical statements?
- Are there any differences between students from Mathematics, Secondary Education and Primary Education Teaching Mathematics Programs, with respect to the proof practices that are observed, when they are asked to prove mathematical statements?

Research Qusetion 3:

- How do senior Mathematics, Primary and Secondary Education Teaching Mathematics students decide what constitutes a mathematical proof, when they are asked to evaluate freshmen students’ mathematical arguments?
- Are there any differences between senior students from Mathematics, Secondary Education and Primary Education Teaching Mathematics Programs, with respect to their proof evaluation practices?

Research Question 4:

- Are there any relationships between freshmen and senior Mathematics, Primary and Secondary Education Teaching Mathematics students' proof construction practices and their attitudes and beliefs regarding proof?
- Are there any relationships between senior mathematics Mathematics, Primary and Secondary Education Teaching Mathematics students' proof construction practices and their evaluation of freshmen students’ arguments?
- Are there any relationships between senior Mathematics, Primary and Secondary Education Teaching Mathematics students' attitudes and beliefs regarding proof and their evaluation of freshmen students’ arguments?

4.2. Operational Definition of Variables

As mentioned before, in this study, the term *proof* or *mathematical proof* refers to a deductive, valid argument that establishes the truth of a mathematical proposition. Variables used in this study are as follows:

Attitudes and beliefs regarding proof: Scores obtained from *Attitudes and Beliefs Scale* (ABS), which is a 25 item likert type scale with four sub dimensions: *background, attitude, self efficacy* and *beliefs*. Average scores for each sub dimension range from 1 to 5, where total maximum score is 20.

Proof construction practice: Students' proof construction practices consist of:

- The score obtained from *Proof Exam* (PE), which contains four items measuring students proof construction abilities, each having a score between 0 and 3. Therefore score range for this item is between 0 and 12.
- Categorization of participants' responses to PE, with respect to the proof methods they attmpt to use.

Proof evaluation practice: Score obtained from Proof Evaluation Exam (PEE), which contains arguments that are claimed to prove (or disprove) the mathematical statements given in PE. There are five arguments each for statements 1 and 2, and four arguments each for items 3 and 4; and all items are scored between 0 and 3. Hence the total score for PEE range from 0 to 54.

More detailed explanation about each instrument and their scoring is given in Section 5.3. (See Page 31.)

These variables are analyzed by using *grade* (freshmen and senior students) and *department* (Mathematics, Primary Education Teaching Mathematics and Secondary Education Teaching Mathematics Programs) as grouping variables.

5. DESIGN OF THE STUDY

5.1. Sample

There are two groups of participants. These groups consist of freshmen and senior students from the following departments: Primary Education (Program of Teaching Mathematics - PRED), Secondary School Science and Mathematics Education (Program of Teaching Mathematics - SCED) and Department of Mathematics - MATH. Numbers of participants in each group are given in Table 5.1.

Table 5.1. Sample characteristics

		MATH	PRED	SCED	Total
Freshmen	Female	22	23	18	63
	Male	17	8	5	30
Seniors	Female	13	15	12	40
	Male	10	15	17	42
Total		62	61	52	175

The reasons for selecting these groups are; to be able to compare and analyze the mathematical reasoning skills and conceptions about the nature and role of proof in school mathematics of students who have just finished high school (freshmen) and students who are about to become mathematics teachers and mathematicians (seniors), and to use the data gathered from freshmen students for instrument development (*Proof Evaluation Exam*) in order to collect data from senior students.

Mathematics program is a four year program leading to a bachelor's degree in mathematics, which is designed to prepare students for graduate study in mathematics or in related areas of the natural or social sciences or engineering. The program provides a foundation for those who wish to pursue careers in related areas of science, technology, business, or government where mathematics is important. Primary Education Teaching

Mathematics Program is a four year program leading to a bachelor's degree in mathematics education, where the graduates teach mathematics in primary schools grades 5 through 8. Secondary Education Teaching Mathematics Program is a five year program that leads to a master's degree without thesis (M. Ed.), where graduates teach years 9 through 12. Details of all three programs are given in Appendix D. (See Page 189.)

5.2. Procedure

The study is conducted in three phases: Instrument development (Beliefs and Attitudes Scale, *Proof Exam*); data collection from freshmen students and further instrument development (*Proof Evaluation Exam*); data collection from senior students and data analysis.

5.2.1. Phase One

For the first phase of the study, two instruments were developed. *Attitudes and Beliefs Scale* (ABS) consists of items asking about the participants' background and views about proof in school mathematics. In *Proof Exam* (PE), which is the second instrument, the participants are asked to prove some mathematical statements.

In the 2008-2009 spring semester, a pilot study was conducted to develop and test these two instruments. Another study for further developing ABS was conducted before the instruments took their last form. Details about the instruments and pilot studies are explained in Section 5.3. (See Page 31.)

5.2.2. Phase Two

After the first instrument development phase was completed, ABS and PE were finally used in order to collect data from freshmen students in the first week of 2009-2010 fall semester; in order to ensure that the participants gave their responses according to their high school knowledge and experiences. Instruments were conducted in paper-pencil format. The reason for conducting the instrument in paper-pencil format is to be able to select students that use different types of reasoning and proof techniques, in order to obtain enough material for the development the third instrument, *Proof Evaluation Exam* (PEE). Different types of freshmen's responses to PE were selected to form items for this instrument. The purpose of this instrument is to see how seniors evaluate and categorize these student-generated proofs.

5.2.3. Phase Three

Third phase of the study was to collect data from senior students. In 2009-2010 fall and spring semesters, all three instruments were administered to Senior Mathematics, Primary and Secondary Education Teaching Mathematics students.

Finally, all data gathered from freshmen and seniors were analyzed in accordance with the research questions. Details of this process can be found in Section 7. (See Page 43.)

5.3. Instrument Development

Three instruments were developed for this study. *Attitudes and Beliefs Scale* (ABS) consists of 25 likert type items and one open ended question which measures participants' attitudes, beliefs, and background regarding mathematical proof. Second instrument is *Proof Exam* (PE), in which the participants are asked to prove some simple mathematical statements, in order to understand their proof construction practices. *Proof Evaluation Exam* (PEE) is an instrument where only senior participants are asked to evaluate mathematical arguments (student responses from PE). Details of how each of these instruments was developed are explained below.

5.3.1. Attitudes and Beliefs Scale (ABS)

This instrument was developed in order to collect information about: (i) how much the students think they were exposed to proof in high school mathematics lessons, (ii) their opinion about importance and incorporation of proof in mathematics lessons, (iii) their personal experiences and feelings towards proof, (iv) their understanding of proof. First version of this scale was prepared as 15 open ended questions, but before conducting the pilot study, (upon expert suggestion) the instrument was redesigned as a likert-type scale with 16 items in order to make data collection more efficient. The item “What does mathematical proof mean to you? Explain briefly” was asked separately as an open ended question. The five response categories for the scale ranged and scored as follows: *Strongly Disagree* (1), *Disagree* (2), *Neutral* (3), *Agree* (4), and *Strongly Agree* (5). Items reflecting negative attitude were scored in reverse order. Expert opinion was taken to make adjustments to wording of the items.

The following were considered during the development of items in the instrument:

- Literature: Instruments used in the studies that investigate participants’ beliefs and attitudes towards proof were examined (Almeida, 2000; Knuth, 2002; Jones, 2000; Morris, 2002; Healy and Hoyles, 2000, Mingus and Grasl, 1999).

- Expert opinion: Opinions and suggestions a professor and an associate professor from faculty of education, working in the field of measurement and evaluation were taken into consideration during item development process, in order to ensure content validity.

As a pilot study, this version of the scale was administered to two groups in spring 2009 semester: freshmen students from Department of Primary Education Teaching Science and Mathematics program (n=40) and seniors from Department of Secondary Education Teaching Mathematics program (n=19). Data were collected during lecture session with the presence of the instructors and the researcher. Freshmen were in the middle of the second semester of their program; therefore they had taken math courses in university. Most of the seniors students were about to graduate at the end of that semester. None of the students who took part in this pilot study were included in the sample of the main study.

Analysis of the responses revealed that reliability of the scale was moderate ($\alpha = 0.60$). Bartlett's test of sphericity yielded a score of 214.74 (p=0.00 < 0.05). KMO measure of sampling adequacy was 0.63. This measure is represented by an index ranging from 0 to 1. A value less than 0.50 indicates that factor analysis will not reveal a meaningful result (Spicer, 2005). Therefore it can be assumed that the sample was suitable for factor analysis.

Thus, factor analysis over 16 items were performed. As a result, five factors were extracted, four of which were interpretable (one of the components had only one item). Table 5.2 shows component means, standard deviations and alpha coefficients. Details of the pilot studies can be found in Appendix C. (See Page 171.)

Table 5.2. Mean, standard deviation and reliability coefficients of the extracted factors, first pilot study

Component	Mean	Standard Deviation	Cronbach Alpha	Number of Items
Background	2.73	0.92	0.76	4
Importance	3.25	0.81	0.73	3
Belief and Experience	3.35	0.64	0.63	5
HighSchool/University	3.63	0.70	0.31	3

In view of the results of the first pilot study, alterations were made to the scale. Expert opinion was taken and some items were discarded or rewritten while new items were added to the scale to ensure a more accurate measurement. This revised version had 25 likert type items and one open ended question, and was tested on a sample of freshmen students ($n = 94$), enrolled in a calculus course in Bilgi University (second pilot study). Reliability analysis showed a high Cronbach's alpha score ($\alpha = 0.88$). KMO measure of sampling adequacy was 0.80 and Bartlett's test of sphericity yielded a significant result ($p=0.00 < 0.05$); confirming that factor analysis can be performed. Factor analysis resulted in seven components, which did not yield an interpretable result. There were items appearing in more than one component and a component had less than three items. Since the pilot study resulted in four factor components, and the scale was expected to measure four sub dimensions, factor analysis was repeated with the restriction of four components. Results can be found in Appendix C.1. (See Page 171.)

Final version of ABS had 25 open likert type items and one open ended question, which is given in Appendix A. (See Page 108.) Response categories for the scale ranged from strongly disagree to strongly agree and scored as follows: *Strongly Disagree* (1), *Disagree* (2), *Neutral* (3), *Agree* (4), and *Strongly Agree* (5). Items reflecting negative attitude were scored in reverse order. For the open ended item in the attitude scale, the responses are analyzed qualitatively to form categories.

Reliability analysis showed a high Cronbach's alpha score (α = 0.85). KMO measure of sampling adequacy was 0.81 and Bartlett's test of sphericity yielded a significant result (p=0.00 < 0.05); confirming that factor analysis can be performed. Six components were obtained from the initial factor analysis; however two of the components had less than three items with low reliability, therefore they were not interpretable. Factor analysis was repeated with the restriction to four components. This time one item did not appear in any components, hence discarded. Another item was discarded because it did not fit in with rest of the items in the component, as seen below.

The first component, labeled *background*, consists of items about how students perceive proof content in high school mathematics. Items are as follows:

- In our high school mathematics textbook, there were exercises about proofs
- In high school, we had proofs in mathematics lessons
- In high school, I was expected to do simple proofs in mathematics lessons and exams.
- In high school, our mathematics teachers encouraged us to do proofs.
- In high school, our mathematics teachers never talked about the importance of proof.
- I think knowledge and skills I gained in high school mathematics lessons is/will be useful for me in university.

Items in the second component are about whether participants think that proofs are enjoyable, boring, difficult, and important; hence this component is labeled *attitudes*. Items in this component are listed below:

- Proofs make mathematics enjoyable.
- Proofs are important only for mathematicians.
- Incorporating proofs in high school mathematics lessons may make it difficult for the students.
- In high school, proof should be used to explain a mathematical concept.
- A high school student should be expected to do mathematical proofs.
- In mathematics, proofs are usually confusing.
- I find dealing with proofs boring.

Next component consists of items regarding participants' beliefs about the nature of proof and its place in mathematics, and it is labeled as *beliefs*. Contributing items are as follows:

- I feel confident that I can do mathematical proofs.
- I can use mathematical language efficiently while doing proofs.
- I believe that my mathematical knowledge is adequate for doing simple mathematical proofs.
- I usually have difficulty in understanding proofs.
- I do not have too much experience in doing proofs.

Finally, the component including items regarding participants' perceptions about their competency in doing proof is labeled as *self efficacy*. Perceived self efficacy is defined as "people's judgments of their capabilities to organize and execute courses of action required to attain designated type of performances" (Bandura, 1986). Bandura distinguishes between judgments of personal efficacy and response-outcome expectations. While *self efficacy* is judgment of one's capability to accomplish a certain level of performance, outcome expectation is the judgment of likely consequence of that behavior. Items contributing to this component are as follows:

- In order to comprehend a mathematical statement, I try to understand its proof.
- In order to decide whether a mathematical statement is true, I have to check that it is true for all cases.
- Proof is a vital part of mathematics.
- It is not compulsory to be able to do proofs to be successful in mathematics lessons.
- We do not need to know the proof of a mathematical result in order to understand why that mathematical result is true.

The following items are discarded because they did not fit into any components:

- More emphasis should be given to proofs in university mathematics lectures than in high school.
- I think mathematics instruction is/will be different in lectures in the university than in high school.

Factor loadings for the items in each component can be seen in Table 5.3. Table 5.4 shows the reliability coefficients and the number of contributing items for each subscale.

Table 5.3. Factor loadings for four components

Item no	Component			
	Background	Attitude	Self efficacy	Beliefs
8	**0.84**	0.10	0.08	-0.06
9	**0.83**	0.19	0.02	0.00
11	**0.82**	0.10	0.05	0.05
10	**0.76**	-0.08	-0.00	0.08
7	**0.73**	-0.09	0.06	-0.19
13	**0.60**	0.11	-0.00	0.20
21	0.01	**0.81**	0.12	0.14
15	0.15	**0.71**	0.21	0.10
24	0.03	**0.66**	0.01	0.07
23	0.06	**0.66**	0.10	0.16
22	0.05	**0.64**	0.26	0.14
4	-0.01	0.07	**0.69**	0.07
16	-0.01	0.33	**0.68**	0.07
6	0.02	0.31	**0.67**	0.00
5	0.03	0.12	**0.64**	0.27
2	0.18	0.32	**0.62**	0.36
3	0.04	-0.14	**0.61**	0.04
19	0.08	0.40	**0.52**	0.35
18	0.07	0.24	0.14	**0.74**
17	0.12	0.03	0.00	**0.72**
20	-0.14	0.04	0.17	**0.71**
25	-0.07	0.40	0.21	**0.56**
1	0.18	0.31	0.34	**0.51**

Table 5.4. Reliability coefficients and contributing items

Scale	Cronbach's Alpha	Number of items
Background	0.86	6
Attitude	0.81	7
Self Efficacy	0.77	5
Beliefs	0.68	5

From Table 5.4, it is seen that, Crohanbach's alpha coefficients of the subscales indicate good reliability of internal consistency.

5.3.2. Proof Exam (PE)

This instrument was designed to collect information about participants' proof construction practices, types proof techniques they use, and how efficiently they can use it. The students were asked to prove the given mathematical statements. Data were collected in paper-pencil form.

To develop the PE, initially, 13 mathematical statements were produced / selected. The selection process for the items is described in detail below. These items were examined by experts to determine which are most suitable for the target population (level of students) and aim of the study. Adjustments were made accordingly. The following were considered during the development of items in the instrument:

- Literature: Instruments used in the studies that investigate participants' mathematical reasoning and proof techniques were examined (Almeida, 2000, 2003; Miyazaki, 2000; Morris, 2002; Özer and Arıkan, 2002; Selden and Selden, 2003; Stylianides and Al-Murani, 2010; Stylianides et al., 2004, 2007; Recio and Godino, 2001; Healy and Hoyles, 2000).

- Books on mathematical proof: Typical examples that can be found in books about methods of mathematical proof (Cupillari, 2001; D'Angelo and West, 2000; Solow, 2005) were considered in item development.
- Mathematical Content: Content knowledge required for the items were aimed to be kept at minimum, so that the participants' reasoning is not obstructed by the lack of knowledge in a certain mathematical subject.
- Curriculum: Content covered by the items is included in the high school curriculum (MEB, 2005): properties of natural numbers and integers, divisibility (grade 9, subject: algebra/numbers). All types of proof methods mentioned in the curriculum for grade 9, subject of logic (MEB, 2005) are covered by the items. No other proof method is needed, though may be used by the participant.
- Alternative solutions: All items can be proved in several ways using alternative proof methods.
- Expert opinion: To ensure content validity, opinions and suggestions of several people including a high school teacher, two instructors from Department of Mathematics, an instructor and a graduate student from Program of Teaching Mathematics, were taken into consideration.
- Difficulty level: Items that have different levels of difficulty have been selected in order to ensure a more accurate idea about participants' reasoning skills.

In the pilot study the students were asked to prove some mathematical statements. There were ten items in total asked to three groups. Freshmen students were divided into two groups (group 1A and group 1B) and asked different items, since there were time constraints and the main purpose was to collect as diverse data as possible for further development of the instrument. Group 2 consisted of senior students. Each item was asked to at least two groups: For group 1A and 1B, six questions were asked and they were expected to answer four (3 + Choose 1 from 3). Students in group 2 were asked all ten items, expected to answer 6 (5 + Choose 1 from 5).

Results of the pilot study were used to develop the rubric for PE and to develop PEE. Rubric was developed in the following manner: First, responses were categorized with respect to the method (type of proof) used. In addition to the researcher, two graduate students (one from teaching mathematics and one from mathematics programs) coded the data from PE (graduate students were given a sample of student responses). Results from these three coders were organized to form the final categorization. For each item, scores between 0 and 3 were given according to the following criteria:

- Incoherent response, no basis for a valid proof construction, no attempt at generalization: 0 points
- Attempt at generalization; complete use of known formulas and information without any justification; correct idea with insufficient explanation; presenting a valid general argument that does not prove the given statement: 1 point
- Presenting a valid general argument but missing steps, needs more clarification or some justification; some use of mathematical language and symbols: 2 points
- Presenting a valid general argument with sufficient explanation and clarity; good use of mathematical language and symbols: 3 points

More explanation regarding the scoring process can be found in Section 7. (See Page 43.)

PE administered to freshmen and senior prospective secondary school teachers were identical. In order to collect data more efficiently, senior prospective primary school teachers and prospective mathematicians were given a shortened version. This version of PE has only the items that were selected for PEE. Data analysis was carried out on items common to all versions of PE. Versions of PE conducted to freshmen and seniors are given in Appendices A.1 (Page 114) and A.2 (Page 116).

5.3.3. Proof Evaluation Exam (PEE)

This instrument was developed in order to collect information about senior students' proof evaluation practices. Items (mathematical statements) of PE with most diverse responses provided by freshmen students were selected and alternative arguments for each of these items were included in PEE. These arguments range from empirical-inductive to formal-deductive forms, similar to the selection process of Healy and Hoyles (2000). The approach taken during item development was to use student generated arguments similar to the study of Selden and Selden (2003) rather than partly or all expert generated ones, as used in the studies of Stylianides et al. (2004) and Healy and Hoyles (2000); because student generated arguments are more authentic, they better represent the type of arguments the participants will have to make sense of as mathematics teachers/instructors in the future.

For each alternative argument (proof), participants were asked to choose one of the following and explain the reason for their choice: “A. The proof shows the statement is true for in some cases”, “B. The proof shows the statement is always true”, “C. The proof is false”, “D. I have no opinion”.

First version of the instrument, administered to prospective secondary school teachers, was the long version. In order to collect data from senior prospective primary school teachers and prospective mathematicians more efficiently, the PEE was shortened in the following way: while the statements in the instrument remained the same, number of alternative proofs provided for each statement was decreased by eliminating proofs that were similar in a way to another proof given, or the proofs that majority of the prospective secondary school teachers categorized the same way. In order to form the rubric, first the instrument was given to two teaching assistants and an instructor working in mathematics department. Their responses were used for the development of the rubric. The following criteria were used in scoring:

- Wrong choice (A or C) without any explanation or incorrect explanation: 0 points
- Wrong choice but reasonable explanation or correctly indicates a mistake or a missing step: 1 or 2 points
- Correct choice without any explanation: 1 point (for A and C), 3 point (for B, if the given response is a full proof)
- Correct choice but insufficient or irrelevant explanation: 1 or 2 points
- Correct choice with sufficient explanation: 3 points

Final version of the rubric was examined and approved by an associate professor from mathematics department. Details of how the rubrics are developed can be found in Sections 7.2 (Page 49) and 7.3 (Page 63). Rubrics for PE and PEE are given in Appendices B1 (Page 124) and B2 (Page 133).

6. DATA COLLECTION

For the main study, data were collected from the freshmen (from all departments) and senior Secondary Education Teaching Mathematics students in the first week of 2009-2010 fall semester. ABS and PE were conducted to freshmen students during lecture sessions of the course named “Introduction to Mathematical Structures”, which is mostly about introduction to doing and writing mathematical proofs. Students from Mathematics, Primary and Secondary Education Teaching Mathematics Programs all take this course together. Data were collected especially in first week of fall semester, so that freshmen’s responses would solely reflect their high school knowledge and experiences. During data collection, researcher and the instructor of the course were present.

Data were collected from senior Secondary Education Teaching Mathematics students in 2009-2010 fall semester, during lecture sessions of the course “Teaching Methods in Mathematics” in the following manner: First ABS and PE were administered (identical to the instruments given to freshmen). After that, students were given the PEE. All data were collected with the presence of the researcher.

In 2009-2010 spring semester, ABS and shortened versions of PE and PEE were administered to Primary Education Teaching Mathematics and Mathematics students during lecture hours, with the presence of the researcher and the instructor. Because of time constraints, PEE was given in a separate session the following week.

7. DATA ANALYSIS AND RESULTS

Results of the analysis of data collected by the three instruments are reported in the following sections. Sample sizes in some sub-groups are small, so Shapiro - Wilk normality test have been conducted for each subgroup. In cases where normal distributions could not be assumed, non-parametric tests were carried out instead of parametric tests to ensure that interpretable results could be obtained.

7.1. Student Beliefs and Attitudes Regarding Proof in School Mathematics

Data collected by ABS were analyzed to examine student beliefs about and attitudes towards proof, in order to answer research question RQ1: “What are freshmen and senior Mathematics and Teaching Mathematics students’ attitudes and beliefs regarding proof in school mathematics and are there any significant differences between students’ attitudes and beliefs regarding proof in school mathematics, with respect to grade (freshmen and seniors) or department?”. Response categories were scored as follows: *Strongly Disagree* (1), *Disagree* (2), *Neutral* (3), *Agree* (4), and *Strongly Agree* (5). Items reflecting negative attitude were scored in reverse order. Mean values and standard deviations for scores of the four subscales, *Background*, *Attitude*, *Self Efficacy* and *Belief* are given in Tables 7.1, 7.2, 7.3 and 7.4 respectively.

Table 7.1. Means and standard deviations, background subscale (number of items: 6)

		PRED	SCED	MATH
Freshmen	Mean	3.13	3.42	3.11
	Std.Dev.	0.87	0.78	0.84
	n	31	22	39
Seniors	Mean	2.88	2.70	3.03
	Std.Dev.	0.87	0.87	1.09
	n	28	29	15

Table 7.2. Means and standard deviations, attitude subscale (number of items: 7)

		PRED	SCED	MATH
Freshmen	Mean	3.30	3.21	3.27
	Std.Dev.	0.77	0.55	0.73
	n	27	22	39
Seniors	Mean	3.78	3.61	3.59
	Std.Dev.	0.55	0.63	0.51
	n	28	27	15

Table 7.3. Means and standard deviations, self efficacy subscale (number of items: 5)

		PRED	SCED	MATH
Freshmen	Mean	3.01	2.90	2.99
	Std.Dev.	0.53	0.54	0.60
	n	28	21	39
Seniors	Mean	3.76	3.33	3.81
	Std.Dev.	0.68	0.59	0.71
	n	28	29	14

Table 7.4. Means and standard deviations, beliefs subscale (number of items: 5)

		PRED	SCED	MATH
Freshmen	Mean	3.59	3.73	3.68
	Std.Dev.	0.66	0.50	0.61
	n	31	22	39
Seniors	Mean	3.69	3.78	4.29
	Std.Dev.	0.71	0.64	0.45
	n	28	29	15

It can be observed from these means that, while freshmen students from all departments have higher *background* scores than seniors, the situation is reverse for *attitude, self efficacy* and *beliefs* scores; in all departments, seniors have higher means than freshmen.

Shapiro - Wilk test conducted to check normality yielded significant results for *attitude* scores of freshmen Mathematics students ($W = 0.93$, $p = 0.03 < 0.05$), as well as senior prospective secondary school teachers' *background* ($W = 0.91$, $p = 0.03 < 0.05$) and *beliefs* ($W = 0.92$, $p = 0.04 < 0.05$) scores. *Background* scores of freshmen ($W = 0.97$, $p = 0.04 < 0.05$); *beliefs* scores of freshmen ($W = 0.96$, $p = 0.01 < 0.05$) and seniors ($W = 0.95$, $p = 0.01 < 0.05$) were also significant. Therefore, for these groups normal distribution cannot be assumed and corresponding non parametric tests should be carried out. Since some of the groups were normally distributed, comparisons among them can be done using necessary parametric tests. Results for both parametric and corresponding non parametric tests for all subgroups are reported.

GLM analysis has been conducted to determine whether there are any significant differences between means of the subscales with respect to grade and department. Results of multivariate test indicate that there are significant differences between subscale scores with respect to grade, $F(4, 143) = 16.71$, $p = 0.00 < 0.05$ and department, $F(8, 286) = 2.70$, $p = 0.01$. Tests of between subject effects show that mean scores differ significantly among freshmen and seniors in the following subscales: *attitude* ($p = 0.00 < 0.05$), *self efficacy* ($p = 0.00 < 0.05$) and *beliefs* ($p = 0.02 < 0.05$).

In order to further investigate the mean differences with respect to departments; One way ANOVA was performed for freshmen and seniors separately. Results show that there are no significance differences between departments in any subscales among freshmen students, however, significant mean differences are observed between senior prospective primary and secondary school teachers' *self efficacy* scores ($p = 0.04 < 0.05$). It is also observed that senior prospective mathematicians' *belief* score means significantly differ from prospective primary school ($p = 0.03 < 0.05$) and secondary school ($p = 0.01 < 0.05$) teachers.

Corresponding non parametric Kruskal Wallis test also indicates no significant differences between departments among freshmen students, in any subscale. Among senior students, significant results occur in *self efficacy* and *beliefs* subscales: $\chi^2(2, N=71) = 7.71$, $p = 0.02 < 0.05$ and $\chi^2(2, N=72) = 9.72$, $p = 0.01 < 0.05$ respectively. Pair wise comparisons among departments were done using Mann-Whitney *U* test, which revealed that these significant differences of mean ranks are among *beliefs* scores of prospective primary school teachers and mathematicians ($U = 98.50$, $p = 0.00 < 0.05$, $r = 0.44$) and prospective secondary school teachers and mathematicians ($U = 108.50$, $p = 0.01 < 0.05$, $r = 0.41$), both in favor of mathematicians. In addition, significant results were obtained in *self efficacy* subscale between prospective mathematicians and secondary school teachers ($U = 123$, $p = 0.04 < 0.05$, $r = 0.32$), in favor of mathematicians and between prospective primary and secondary school teachers ($U = 250$, $p = 0.01 < 0.05$, $r = 0.33$), in favor of prospective primary school teachers.

Significant differences are also observed between freshmen and senior students in all subscales: *background* $\chi^2(1, N=164) = 6.98$, $p = 0.01 < 0.05$, *attitude* $\chi^2(1, N=158) = 13.58$, $p = 0.00 < 0.05$, *self efficacy* $\chi^2(1, N=159) = 31.35$, $p = 0.00 < 0.05$ and *beliefs* $\chi^2(1, N=164) = 5.48$, $p = 0.02 < 0.05$. For *background* scores, significant results are in favor of freshmen students, while for the *self efficacy*, *beliefs* and *attitude* subscales, significant results are in favor of seniors.

ABS also included the open ended question “What does proof mean to you? Explain briefly”. Responses were categorized as follows:

(a) To verify a statement is true, remove doubt, believable, convincing, acceptance.

(b) Generality of the results, showing for all cases, proof process, use of symbolic language, consistency.

(c) Explain where the facts (information, theorems) we know and use comes from, deeper understanding, easier to remember information, no need for memorization, makes you think.

(d) Essential part of mathematics, we must know it to understand our profession,

(e) Affect: Boring, exciting, interesting, enjoyable, fun, scary, hard to understand, difficult, complicated, requires hard work, it is important, necessary.

(f) Self concept: I do not know anything about proof, I cannot do it, I do not know how to do proofs, I am not successful at doing proofs.

(g) Helps to solve related problems, discover new theorems: Warrant to use information (we cannot use it unless we know for sure it is true).

(h) High school background: We did not do any proof-we did very little proof, the teacher wrote proofs on the board, but did not explain.

(i) How proof should be thought: In a way that would be exciting to understand it (in algebra course)

- Proofs should be motioned, repeated more often
- Variety of proof types (from all directions) should be thought
- The system in high school depends on memorization; no student will be interested in proof
- I can understand [proof] if it is thought well
- It is introduced to the students in a very late stage (in Turkey)

Number and percentage of freshmen and senior students who responded to this item, in each department (response rates) are shown in Table 7.5, while Table 7.6 shows the frequency and percentage distributions of responses in each category. Since multiple coding was used, sum of percentages may exceed 100.

Table 7.5. Response rates for the open ended item in ABS

	Freshmen	Seniors			
		PRED	SCED	MATH	Total
n	42	3	19	12	34
Percentage	40.8	10.1	67.9	63.2	47.1

Table 7.6. Frequency and percentage distributions of open ended item responses, with respect to categories

Categories			a	b	c	d	e	f	g	h	i
Freshmen		n	12	4	21	1	18	2	3	6	5
		percentage	28.6	9.5	50	2.4	42.9	4.8	7.1	14.3	11.9
Seniors	PRED	n	0	0	3	0	1	0	1	0	0
		percentage	0	0	100	0	33.3	0	33.3	0	0
	SCED	n	7	4	12	2	6	2	1	0	1
		percentage	36.8	21	63.2	10.5	31.6	10.5	5.3	0	5.3
	MATH	n	3	4	1	6	1	0	2	0	0
		percentage	25	33.3	8.3	50	8.3	0	16.7	0	0

It is observed from the responses that the majority of participants (freshmen and seniors alike), mentioned the explanatory aspect of proof: that it helps them to understand why the statement is true which leads to better understanding. Next popular response was, from mostly freshmen students, that proofs are used to verify a statement is true. While equal number of freshmen and seniors found proofs enjoyable and fun, participants who found it difficult to understand were mostly freshmen students.

7.2. Proof Construction Practices

Data collected by PE were analyzed to answer research question RQ2: “What are the proof construction practices of freshmen and senior Mathematics and Teaching Mathematics students, when they are asked to prove mathematical statements and are there any differences in students’ proof construction practices that are observed when they are asked to prove mathematical statements, with respect to grade (freshmen and seniors) and department?”.

In order to examine participants’ responses to PE, a rubric was developed in the following manner: First, responses were categorized with respect to the method (type of proof) used. In addition to the researcher, two graduate students (one from teaching mathematics and one from mathematics programs) coded the data from PE (graduate students were given a sample of student responses). Results from these three coders were organized to form the final categorization. Then each response was given a score between 0 and 3 according to the following criteria (similar to the scorings of Regio and Godino, 2001; Healy and Hoyles, 2000; Stylianides and Stylianides, 2009):

- Incoherent response, no basis for a valid proof construction, no attempt at generalization: 0 points
- Attempt at generalization; complete use of known formulas and information without any justification; correct idea with insufficient explanation; presenting a valid general argument that does not prove the given statement: 1 point
- Presenting a valid general argument but missing steps, needs more clarification or some justification; some use of mathematical language and symbols: 2 points
- Presenting a valid general argument with sufficient explanation and clarity; good use of mathematical language and symbols: 3 points

In addition, types of proof methods that freshmen and seniors provided for the PE were categorized. For each item, letters A, B, C, D, E and G were assigned to different types of proof (type of proof these letters represent vary from item to item). Through all items, letter F is assigned if the participant attempted the proof but could not provide a meaningful argument, and NA stands for no attempt. For each item, tables displaying frequencies for types of proofs and PE scores are given (Tables 7.7 through 7.14). In addition, two figures per item (Figures 7.1 through 7.8) show frequency distributions of scores for freshmen and seniors. For scoring and examples of each type of response, see the rubric for PE in Appendix B. (See Page 118.)

Participants' responses to the first item of PE; "Prove that the statement is true: If the square of a natural number is even, then that number must be even" were categorized, with respect to the proof methods they attempted to use, as follows:

- Proof 1A: If n was odd, then its square would be odd (proof by contrapositive).
- Proof 1B: If n is even then its square is even (this proofs the converse of the given statement; not equivalent to the original statement).
- Proof 1C: Assume that n is odd but n^2 is even. If n is odd then n^2 will be odd (proof by contradiction).
- Proof 1D: Assume n^2 is even ...then n must be even (direct proof).
- Proof 1E: The square of an even number is even, the square of an odd number is odd. Hence, if the square of a number is even, then that number should be even (proof by cases).

Most attempted proof types by freshmen students for this item are direct proof (20.4 per cent) and proof by cases (22.6 per cent). 21.5 per cent of freshmen students attempted to prove the converse of this statement: "if n is even then its square must be even". Even though it is a true proposition, it does not prove the given statement. More interestingly, senior prospective secondary school teachers (20.7 per cent) and prospective primary school teachers (46.7 per cent) also made the same mistake. No senior prospective mathematicians provided this type of response.

Table 7.7. Frequencies for PE response type, item 1

			Proof Type Item 1							Total
			A	B	C	D	E	F	NA	
Freshman	SCED	n	3	4	0	4	7	0	5	23
		percentage	13.0	17.4	0	17.4	30.4	0	21.7	100
	PRED	n	3	5	2	6	7	5	3	31
		percentage	9.7	16.1	6.5	19.4	22.6	16.1	9.7	100
	MATH	n	2	11	0	9	7	2	8	39
		percentage	5.1	28.2	0	23.1	17.9	5.1	20.5	100
	Total	n	8	20	2	19	21	7	16	93
		percentage	8.6	21.5	2.2	20.4	22.6	7.5	17.2	100
Senior	SCED	n	6	6	1	7	9	0	0	29
		percentage	20.7	20.7	3.5	24.2	31.0	0	0	100
	PRED	n	4	13	3	2	4	2	0	28
		percentage	14.3	46.4	10.7	7.1	14.3	7.1	0	100
	MATH	n	3	0	10	2	0	0	0	15
		percentage	20.0	0	66.7	13.3	0	0	0	100
	Total	n	13	19	14	11	13	2	0	72
		percentage	18.1	26.4	19.4	15.3	18.1	2.8	0	100

When scores for the first item are examined, it is seen that 42 per cent of freshmen students did not receive any points and only 7.5 per cent were given maximum points. Amount of seniors who received minimum and maximum points are 13.9 per cent and 34.3 per cent respectively. Figures 7.1 and 7.2 show the distribution of scores for freshmen and senior students by departments.

Table 7.8. Frequencies for PE score, item 1

		Proof Score Item1				Total
		0	1	2	3	
Freshman	SCED	10	5	7	1	23
	PRED	12	6	10	3	31
	MATH	20	8	8	3	39
	Total	42	19	25	7	93
Senior	SCED	4	7	11	7	29
	PRED	6	11	6	5	28
	MATH	0	1	2	12	15
	Total	10	19	19	24	72

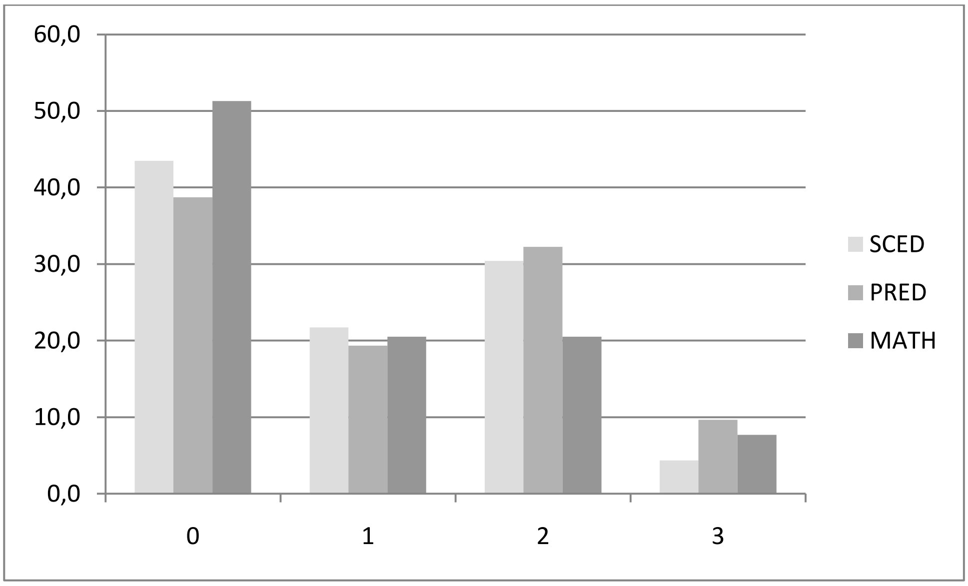

Figure 7.1. Percentage frequencies of freshmen scores for PE, item 1

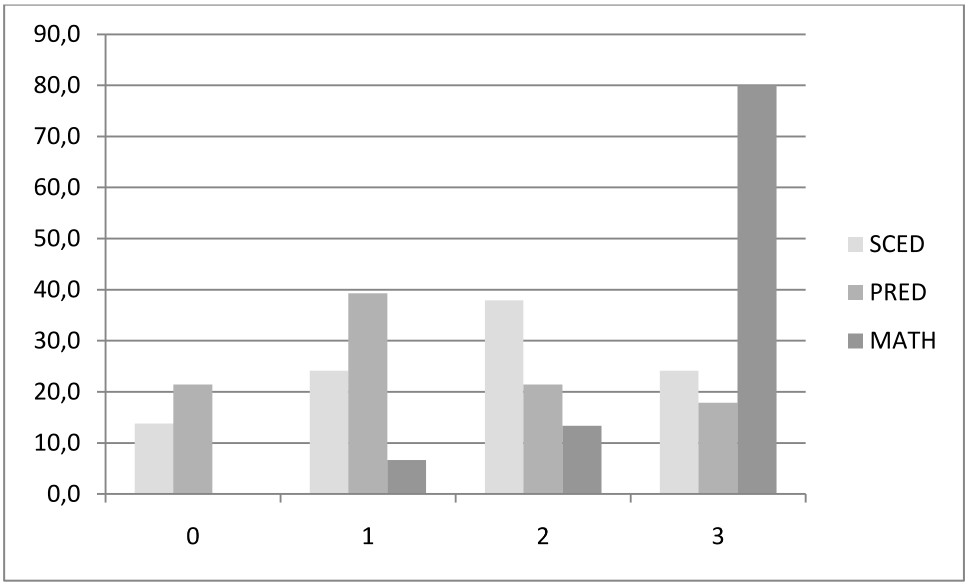

Figure 7.2. Percentage frequencies of senior scores for PE, item 1

Participants' responses to item 2a; "Prove or disprove: the equality 1+ 3 + 5 + ... + 2n-1 = n^2 is true for all integers $n \geq 1$" were categorized as follows:

- Proof 2a_A: By formula: Sum = number of terms x (last term + first term)/2.
- Proof 2a_B: Using Gauss' method (writing the same sum in reverse and adding up the terms).
- Proof 2a_C: By induction.
- Proof 2a_D: Using the equality 1+2+3+…+n = n (n+1)/2.
- Proof 2a_E: By giving numerical examples.

Majority (67.7 per cent) of freshmen either did not attempt this item or failed to provide a coherent response. Among the rest, most commonly observed (9.7 per cent) response was to use a known general formula (without justification) which verifies that the statement is true.

This statement is one of the common examples used explaining proof by mathematical induction. While 66.7 per cent of seniors used mathematical induction, only 4.3 per cent of freshmen attempted to prove the statement with this method.

Table 7.9. Frequencies for PE response type, item 2a

			Proof Type Item 2a							Total
			A	B	C	D	E	F	NA	
Freshman	SCED	n	1	1	0	0	4	4	13	23
		percentage	4.3	4.3	0	0	17.4	17.4	56.5	100.0
	PRED	n	4	0	3	2	2	3	17	31
		percentage	12.9	0	9.7	6.5	6.5	9.7	54.8	100.0
	MATH	n	4	4	1	2	2	1	25	39
		percentage	10.3	10.3	2.6	5.1	5.1	2.6	64.1	100.0
	Total	n	9	5	4	4	8	8	55	93
		percentage	9.7	5.4	4.3	4.3	8.6	8.6	59.1	100.0
Senior	SCED	n	2	3	19	2	1	1	1	29
		percentage	6.9	10.3	65.5	6.9	3.4	3.4	3.4	100.0
	PRED	n	0	2	17	2	4	0	3	28
		percentage	0	7.1	60.7	7.1	14.3	0	10.7	100.0
	MATH	n	0	2	12	1	0	0	0	15
		percentage	0	13.3	80.0	6.7	0	0	0	100.0
	Total	n	2	7	48	5	5	1	4	72
		percentage	2.8	9.7	66.7	6.9	6.9	1.4	5.6	100.0

According to the frequencies in table 7.10, the scores for this item indicate that 78.5 per cent of freshmen and 13.9 per cent of seniors received minimum score. Maximum score was received by 5.4 per cent of freshmen and 54.2 per cent of seniors. Distributions of scores for freshmen and seniors are given in Figure 7.3 and Figure 7.4 respectively.

Table 7.10. Frequencies for PE score, item 2a

		Proof Score Item 2a				Total
		0	1	2	3	
Freshman	SCED	21	1	0	1	23
	PRED	23	3	4	1	31
	MATH	29	4	3	3	39
	Total	73	8	7	5	93
Senior	SCED	3	3	6	17	29
	PRED	7	3	10	8	28
	MATH	0	0	1	14	15
	Total	10	6	17	39	72

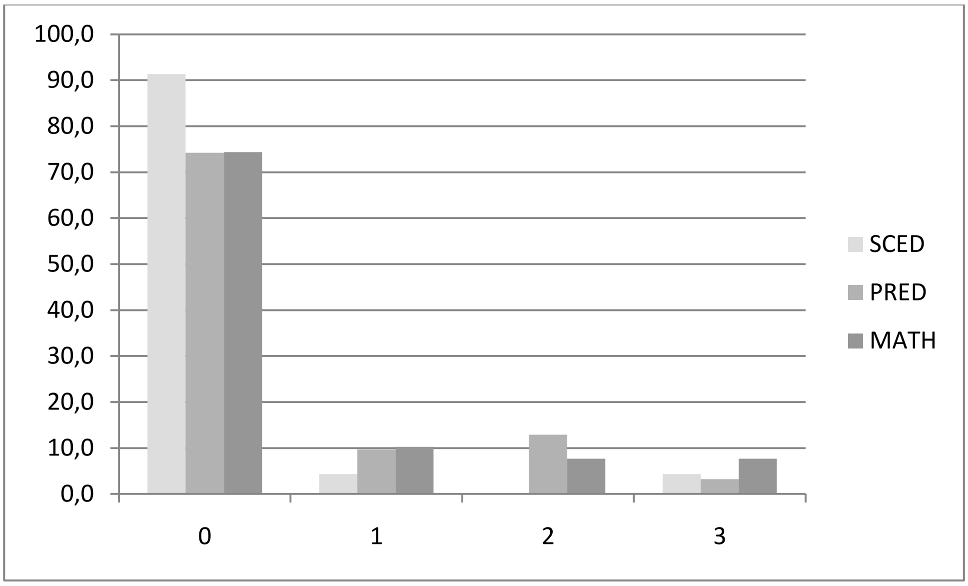

Figure 7.3. Percentage frequencies of freshmen scores for PE, item 2a

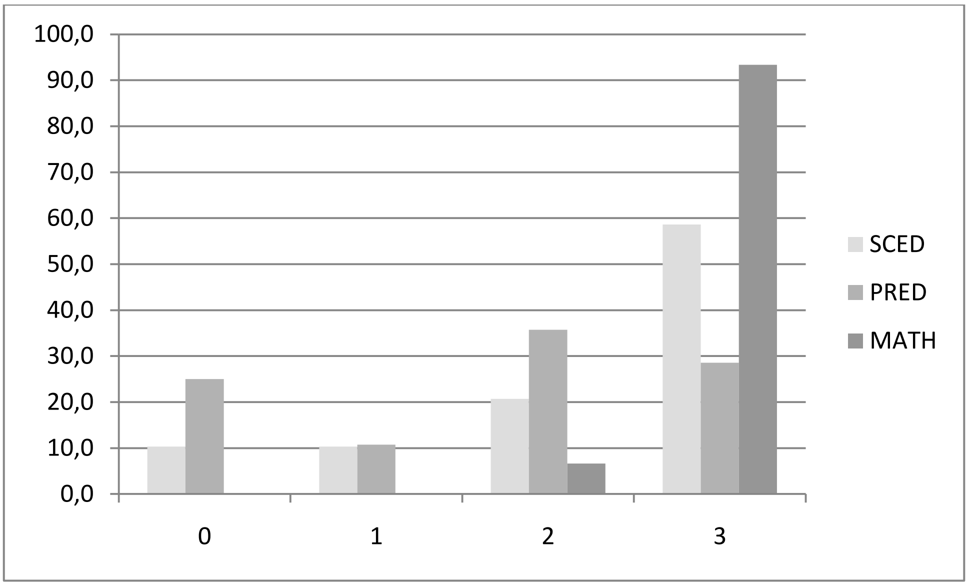

Figure 7.4. Percentage frequencies of senior scores for PE, item 2a

Categorization of the responses to item 2b of PE; "Prove or disprove: Given any three consecutive integers, one of them is always divisible by three" is given below:

- Proof 2b_A: Direct proof
- Proof 2b_B: Let a, $a + 1$, $a + 2$ be three consecutive integers. If $a = 3k$, then a is divisible by three. If $a = 3k + 1$ then $a + 2$ is divisible by three. If $a = 3k + 2$ then $a + 1$ is divisible by three. In any case, one of them will be divisible by three (proof by cases).
- Proof 2b_C: Showing that the sum is divisible by 3. (This does not prove the statement is true: if the sum of three numbers is divisible by 3, it cannot be concluded that one of them is divisible by 3.)
- Proof 2b_D: Giving counter-example to show that the statement is false: "0 is not divisible by 3" (the statement is not false, the counter-example is not valid because 0 is divisible by 3).
- Proof 2b_E: By giving numerical examples
- Proof 2b_G: Proof by contradiction

Table 7.11. Frequencies for PE response type, item 2b

			Proof Type Item 2b								Total
			A	B	C	D	E	F	G	NA	
Freshman	SCED	n	3	2	4	2	3	0	0	9	23
		percentage	13.0	8.7	17.4	8.7	13.0	0	0	39.1	100.0
	PRED	n	5	1	4	3	5	5	0	8	31
		percentage	16.1	3.2	12.9	9.7	16.1	16.1	0	25.8	100.0
	MATH	n	3	2	2	3	4	1	0	24	39
		percentage	7.7	5.1	5.1	7.7	10.3	2.6	0	61.5	100.0
	Total	n	11	5	10	8	12	6	0	41	93
		percentage	11.8	5.4	10.8	8.6	12.9	6.5	0	44.1	100.0
Senior	SCED	n	5	16	1	1	1	1	2	2	29
		percentage	17.2	55.2	3.4	3.4	3.4	3.4	6.9	6.9	100.0
	PRED	n	0	8	8	0	2	5	0	5	28
		percentage	0	28.6	28.6	0	7.1	17.9	0	17.9	100.0
	MATH	n	0	12	0	1	0	0	1	1	15
		percentage	0	80.0	0	6.7	0	0	6.7	6.7	100.0
	Total	n	5	36	9	2	3	6	3	8	72
		percentage	6.9	50.0	12.5	2.8	4.2	8.3	4.2	11.1	100.0

As seen from Table 7.11, 50.6 per cent of freshmen did not provide a meaningful response. Most common response type was to show the statement holds by giving numerical examples (no generalization). One commonly made mistake by freshmen (8.6 per cent) was to falsely assume that 0 cannot be divided by 3, hence concluding that the statement is false. Showing that the sum of 3 consecutive integers is divisible by 3 was another common mistake (10.8 per cent freshmen, 28.6 per cent senior prospective primary school teachers). This is a true proposition but it cannot be used to prove the statement.

Table 7.12. Frequencies for PE score, item 2b

		Proof Score Q2b				Total
		0	1	2	3	
Freshman	SCED	16	2	4	1	23
	PRED	23	5	3	0	31
	MATH	31	3	3	2	39
	Total	70	10	10	3	93
Senior	SCED	8	2	8	11	29
	PRED	20	0	2	6	28
	MATH	1	1	2	11	15
	Total	29	3	13	28	72

Scores for this item indicate that, according to frequencies in Table 7.12, 75.3 per cent of freshmen and 40.3 per cent of seniors (most of which are prospective primary school teachers) received no points. Maximum points were taken by 3.2 per cent of freshmen and 38.9 per cent of seniors. Figures 7.5 and 7.6 below show the distribution of scores received by freshmen and seniors.

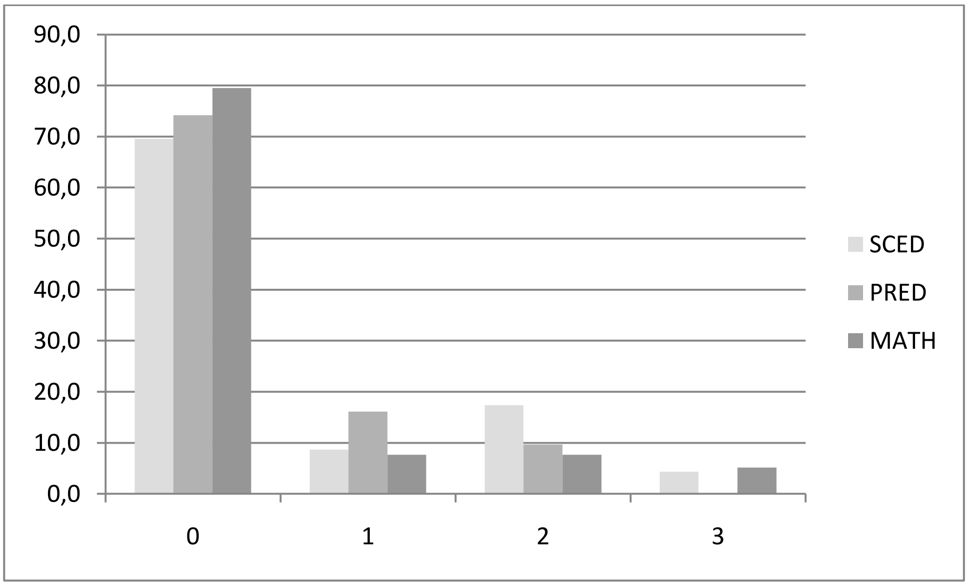

Figure 7.5. Percentage frequencies of freshmen scores for PE, item 2b

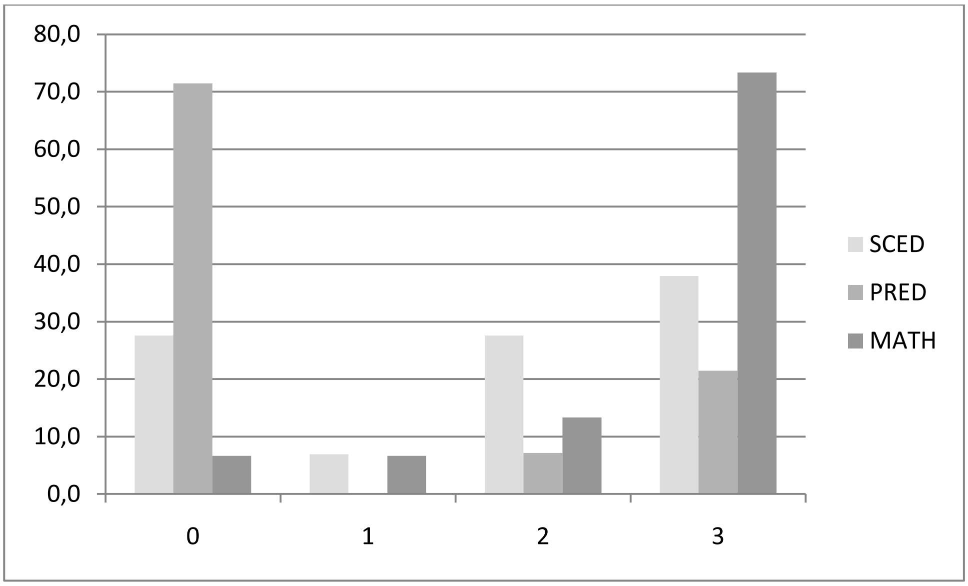

Figure 7.6. Percentage frequencies of senior scores for PE, item 2b

The third item of PE is: “Show that, in a party of n people ($n \geq 2$), we can always find at least two people with the same number of friends”. The responses are categorized as follows:

- Proof 3_A: Using pigeonhole principle.
- Proof 3_B: Using mathematical induction.
- Proof 3_C: Graph theory (trying to represent the problem as a graph).
- Proof 3_D: Trying all possibilities for small n (no generalizations).

Table 7.13. Frequencies for PE response type, item 3

			Proof Type Item 3						Total
			A	B	C	D	F	NA	
Freshman	SCED	n	0	0	0	3	1	19	23
		percentage	0	0	0	13.0	4.3	82.6	100.0
	PRED	n	2	0	0	5	4	20	31
		percentage	6.5	0	0	16.1	12.9	64.5	100.0
	MATH	n	1	0	0	2	1	35	39
		percentage	2.6	0	0	5.1	2.6	89.7	100.0
	Total	n	3	0	0	10	6	74	93
		percentage	3.2	0	0	10.8	6.5	79.6	100.0
Senior	SCED	n	11	0	1	2	2	13	29
		percentage	37.9	0	3.4	6.9	6.9	44.8	100.0
	PRED	n	6	2	0	4	2	14	28
		percentage	21.4	7.1	0	14.3	7.1	50.0	100.0
	MATH	n	7	1	1	0	1	5	15
		percentage	46.7	6.7	6.7	0	6.7	33.3	100.0
	Total	n	24	3	2	6	5	32	72
		percentage	33.3	4.2	2.8	8.3	6.9	44.4	100.0

This was the least attempted item among all participants (no attempt by 79.6 per cent freshmen, 44.4 per cent seniors). This statement can be proven by using pigeonhole principle. Mathematical induction can also be used, although it is difficult to apply in this situation. Trial for small *n* values (without generalization) were the most common response type among freshmen (10.8 per cent), while 33.3 per cent of seniors used pigeonhole principle.

Table 7.14. Frequencies for PE score, item 3

		Proof Score Q3				Total
		0	1	2	3	
Freshman	SCED	22	1	0	0	23
	PRED	27	2	0	2	31
	MATH	37	1	1	0	39
	Total	86	4	1	2	93
Senior	SCED	18	2	3	6	29
	PRED	24	2	1	1	28
	MATH	8	1	1	5	15
	Total	50	5	5	12	72

Most participants (92.5 per cent of freshmen and 69.4 per cent of seniors) did not receive any points for this item. Maximum score was received by 2.2 per cent of freshmen and 16.7 per cent of seniors. Distribution of freshmen scores are given in Figure 7.7. Figure 7.8 shows the seniors' score distribution for the third item.

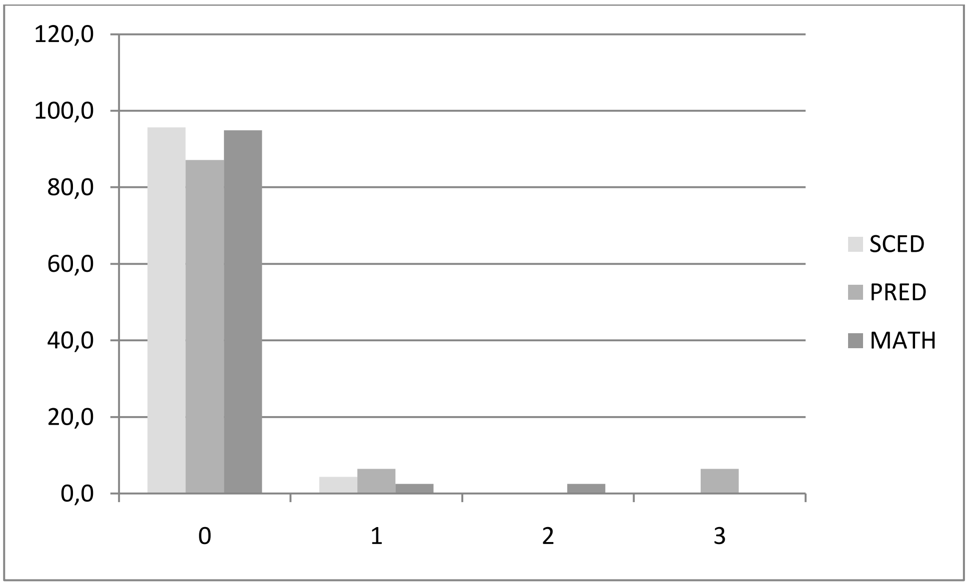

Figure 7.7. Percentage frequencies of freshmen scores for PE item 3

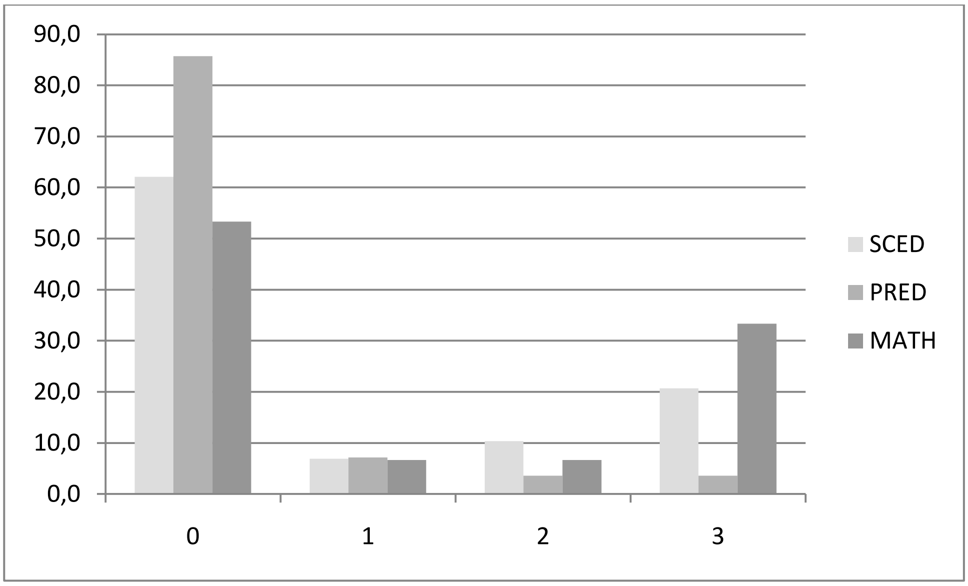

Figure 7.8. Percentage frequencies of seniors score for PE item 3

Means for the total PE score with respect to grade and department are given below in Table 7. 15. Maximum score for PE is 12, while minimum score is 0.

Table 7.15. Means and standard deviations of PE total score

		PRED	SCED	MATH
Freshmen	Mean	2.19	1.74	1.79
	Std.Dev.	2.32	1.84	2.09
	N	31	23	39
Seniors	Mean	4.07	6.66	9.40
	Std.Dev.	2.75	2.88	1.77
	N	28	29	15

Shapiro - Wilk test was conducted to check normality and yielded significant results for PE scores of freshmen prospective mathematicians (W = 0.82, p = 0.00 < 0.05), prospective primary school (W = 0.84, p = 0.00 < 0.05) and secondary school teachers

($W = 0.85$, $p = 0.00 < 0.05$). Significant results were also observed for senior prospective mathematicians ($W = 0.88$, $p = 0.048 < 0.05$) and freshmen students as a whole ($W = 0.86$, $p = 0.00 < 0.05$). Therefore, since normal distribution cannot be assumed for most subgroups, non parametric tests are carried out to see whether there are significant differences between total PE scores with respect to grade and department.

Kruskal-Wallis test conducted on PE scores, revealed no significant results among freshmen prospective mathematicians, primary and secondary school teachers. However, significant results were observed among seniors: $\chi^2(2, N{=}72) = 27.42$, $p = 0.00 < 0.05$). Mann-Whitney tests were conducted to make pair wise comparisons among departments. Tests yielded significant results between prospective primary and secondary school teachers ($U = 210$, $p = 0.02 < 0.05$, $r = 0.42$), in favor of prospective secondary school teachers; between prospective mathematicians and secondary school teachers ($U = 92.50$, $p = 0.02 < 0.05$, $r = 0.47$), and prospective mathematicians and primary school teachers ($U = 23$, $p = 0.00 < 0.05$, $r = 0.73$), both in favor of prospective mathematicians. PE scores also differ significantly among freshmen and seniors ($U = 964.50$, $p = 0.00 < 0.05$, $r = 0.62$) and they are in favor of seniors.

7.3. Proof Evaluation

In order to examine how senior students evaluated student generated proofs, PEE was conducted. Data collected were analyzed to answer research question RQ3: “How do senior Mathematics and Teaching Mathematics students decide what constitutes as a mathematical proof, when they are asked to evaluate freshmen students’ mathematical arguments? Are there any differences between senior students from Mathematics, Secondary and Primary Education Teaching Mathematics Programs, with respect to their proof evaluation practices?”

In this instrument, there are four mathematical statements with alternative proofs (student responses). For each alternative proof, students were asked to choose one of the following and explain the reason for their choice: “A. The proof shows the statement is true for in some cases”, “B. The proof shows the statement is always true”, “C. The proof is false", "D. I have no opinion". Full responses and scoring are presented in the rubric for PEE, which can be found in Appendix B. (See Page 118.)

In order to form the rubric, first the instrument was given to two teaching assistants and an instructor working in mathematics department. Their responses were used for the development of the rubric. The following criteria were used in scoring:

- Wrong choice (A or C) without any explanation or incorrect explanation: 0 points
- Wrong choice but reasonable explanation or correctly indicates a mistake or a missing step: 1 or 2 points
- Correct choice without any explanation: 1 point (for A and C), 3 point (for B, if the given response is a full proof)
- Correct choice but insufficient or irrelevant explanation: 1 or 2 points
- Correct choice with sufficient explanation: 3 points

Final version of the rubric was examined and approved by an associate professor from mathematics department. Items, alternative proofs and responses are given below. Tables 7.16 through 7.55 show response and score frequencies for each alternative proof as well as mean scores for each item.

The first item of PEE was: "If the square of a natural number is even, then that number must be even". The participants were asked to evaluate five alternative proof attempts (arguments provided by freshmen students) for this statement.

Proof 1A was the first argument to evaluate: "If n is odd, $n = 2k + 1$, then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ is odd (even + even + odd). If n is even, $n = 2k$, then $n^2 = (2k)^2 = 4k^2 + 4k$ even (even + even). Since n^2 is even, n must also be even." This argument correctly proves the statement (proof by cases); there is one calculation mistake which does not affect the generality of the argument. Therefore, the correct choice here is B. While most students indicated the correct choice (63.6 per cent), some students concluded that the proof is false (23.6 per cent) or only shows the statement is true for some cases (12.7 per cent) because of the calculation mistake or claiming that the argument shows the converse of the statement. As a result, 60 per cent of participants received maximum score.

Table 7.16. PEE response types, item 1A

	Response 1A					Total
	A	B	C	D	NA	
SCED	5	17	4	0	0	26
PRED	2	7	6	0	0	15
MATH	0	11	3	0	0	14
Total	7	35	13	0	0	55

Table 7.17. PEE score, item 1A

	Score 1A				Total
	0	1	2	3	
SCED	5	4	2	15	26
PRED	7	1	0	7	15
MATH	3	0	0	11	14
Total	15	5	2	33	55

Second argument to evaluate was proof 1B: "Assume *n* is odd. $(2k+1)^2 = 2m$, $4k^2 + 4k + 1 = 2m$. Left hand side is odd, right hand side is even. Contradiction. This means *n* must be even." This is an attempt a proof by contradiction. While the argument proves the statement, the wording can be a bit confusing, it could have been clearer. Again the correct choice is B and 61.8 per cent of students correctly identified it.

Table 7.18. PEE response types, item 1B

	Response 1B					Total
	A	B	C	D	NA	
SCED	2	15	8	1	0	26
PRED	2	8	4	0	1	15
MATH	1	11	2	0	0	14
Total	5	34	14	1	1	55

Table 7.19. PEE sores, item 1B

	Score 1B				Total
	0	1	2	3	
SCED	9	1	1	15	26
PRED	2	3	3	7	15
MATH	0	3	0	11	14
Total	11	7	4	33	55

Third alternative proof attempt was proof 1C: "$n^2 = n \cdot n = 2k$. Here k must be even because $2k$ is a whole square: $k = 2m, n^2 = 4m$, $\sqrt{n^2} = \sqrt{4m}$, $n = 2\sqrt{m}$, hence n is even." There are missing steps in this argument; the premise "k must be even because $2k$ is a whole square" should be justified because it is the essence of the proof. It would also explain why $\sqrt{m}$ must be a whole square. 18.2 per cent of the students pointed out this missing step (choice A of B) and received full points.

Table 7.20. PEE response types, item 1C

	Response 1C					Total
	A	B	C	D	NA	
SCED	7	8	6	4	1	26
PRED	4	5	5	1	0	15
MATH	4	8	2	0	0	14
Total	15	21	13	5	1	55

Table 7.21. PEE scores, item 1C

	Score 1C				Total
	0	1	2	3	
SCED	11	10	2	3	26
PRED	5	4	5	1	15
MATH	1	6	1	6	14
Total	17	20	8	10	55

Next argument was proof 1D: “Assume $n = 2k$. Then $n^2 = 4k^2$, which is even.” This argument proves the converse of the statement. The mistake here is proving the truth of the implication $q \rightarrow p$ instead of $p \rightarrow q$. These two propositions are not equivalent. Therefore the correct choice is C. Another correct interpretation observed in responses is that the proof is incomplete; the case where n is odd should also be checked (with choice A). Then it would be valid proof (proof by cases). Both responses received full points.

Table 7.22. PEE response types, item 1D

	Response 1D					Total
	A	B	C	D	NA	
SCED	6	6	14	0	0	26
PRED	2	9	4	0	0	15
MATH	3	0	11	0	0	14
Total	11	15	29	0	0	55

Table 7.23. PEE scores, item 1D

	Score 1D				Total
	0	1	2	3	
SCED	5	4	2	15	26
PRED	10	3	0	2	15
MATH	0	0	0	14	14
Total	15	7	2	31	55

Last proof attempt for the first item was proof 1E: "Even = {2, 4, 6, 8 ...}. If $n^2 = 4$ then $n = 2$, $n^2 = 16$ then $n = 4$, if $n^2 = 36$ then $n = 6$ .... $n^2 = 114$ then $n = 12$." Here, the truth of the statement is verified for only a couple of values of *n*. Therefore the correct choice is A. Since there is no generalization, this cannot be accepted as a valid proof. Students who stated that giving examples is not a proof (choice C) also received full points (69.1 per cent).

Table 7.24. PEE response types, item 1E

	Response 1E					Total
	A	B	C	D	NA	
SCED	10	3	13	0	0	26
PRED	8	0	7	0	0	15
MATH	7	0	7	0	0	14
Total	25	3	27	0	0	55

Table 7.25. PEE score, item 1E

	Score 1E				Total
	0	1	2	3	
SCED	4	0	4	18	26
PRED	3	3	0	9	15
MATH	0	3	0	11	14
Total	7	6	4	38	55

The percentage frequency distributions of evaluation scores for proofs 1A through 1E are given in Figure 7.9.

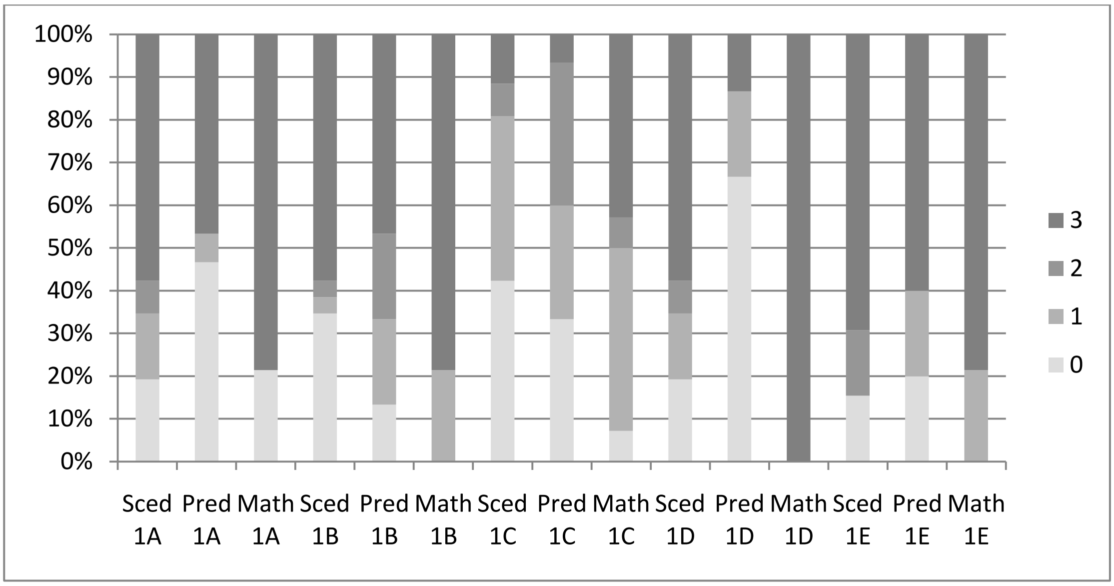

Figure 7.9. Percentage frequencies of scores for item 1

Mean scores for Item 1 for each department is given in Table 7.26 below. It can be observed from the table that, from proof 1A through 1E, prospective mathematicians have highest mean scores, prospective primary school teachers have lowest mean scores in proofs 1A, 1D, 1E and for 1B and 1C, the mean scores of prospective secondary school teachers are the lowest.

Table 7.26. Mean scores and standard deviations, PEE item 1

Item		1A	1B	1C	1D	1E
SCED	Mean	2.04	1.85	0.88	2.04	2.38
	Std.Dev.	1.25	1.43	0.99	1.25	1.10
PRED	Mean	1.47	2.00	1.13	0.60	2.00
	Std.Dev.	1.51	1.13	0.99	1.06	1.31
MATH	Mean	2.36	2.57	1.86	3.00	2.57
	Std.Dev.	1.28	0.85	1.10	0.00	0.85

For the second item in PEE; “1 + 3 + 5 + ... + 2n-1 = n^2 holds for all integers $n \geq 1$”, five alternative proof attempts were given. The arguments and participants’ responses are given below.

Proof 2A was the first argument to evaluate: “Let *ST* denote the last term, *IT* the first term, and Δm the increment. Then $\frac{ST - \dot{I}T}{\Delta m} + 1$ gives the number of terms. According to this, the sum will be: $\frac{ST + \dot{I}T}{2} \cdot \left(\frac{ST - \dot{I}T}{\Delta m} + 1 \right) = \frac{2n - 1 + 1}{2} \cdot \frac{2n - 1 - 1}{2} + 1 = n^2$.” This is a general formula to find sums, and it is correctly used here to verify the statement is true. However, there is no explanation about why this formula is true or why it is applicable in this case. Students who stated that the proof would be valid if the formula was also proved received full points.

Table 7.27. PEE response types, item 2A

	Response 2A					Total
	A	B	C	D	NA	
SCED	4	15	2	5	0	26
PRED	3	5	1	5	1	15
MATH	1	9	3	1	0	14
Total	8	29	6	11	1	55

Table 7.28. PEE scores, item 2A

	Score 2A				Total
	0	1	2	3	
SCED	4	13	7	2	26
PRED	7	7	1	0	15
MATH	2	7	4	1	14
Total	13	27	12	3	55

Second argument was proof 2B: "Since 1 + 2 + 3 + ...+ 2*n*-2 + 2*n* -1 = $\frac{(2n-1)\cdot 2n}{2}$, the required sum is: $\frac{(2n-1)\cdot 2n}{2} - 2(1+2+...+\frac{2n-2}{2}) = 2n^2 - n - (n-1)n = n^2$."

Here, first the sum from 1 to 2*n*-1 is calculated, and then sum of even numbers in this range is subtracted from the total to find the sum of odd numbers. This shows the statement is true for all cases, however, the fact that sum of integers from 1 to *n* is *n* (*n* +1) / 2 is used without proof. 50.1 per cent of the participants gave this explanation.

Table 7.29. PEE response types, item 2B

	Response 2B					Total
	A	B	C	D	NA	
SCED	2	16	2	4	2	26
PRED	2	4	4	4	1	15
MATH	0	8	4	2	0	14
Total	4	28	10	10	3	55

Table 7.30. PEE scores, item 2B

	Score 2B				Total
	0	1	2	3	
SCED	6	2	2	16	26
PRED	11	0	0	4	15
MATH	4	2	0	8	14
Total	21	4	2	28	55

Proof 2C was the third argument: “S(n) = 1 + 3 + 5 +...+ 2n-1 = n^2, S(n+1) = 1 + 3 + 5 +...+ 2n-1 + 2n+1 = n^2 + 2n +1 S(n) + 2n+1 = $(n+1)^2$, which means the statement is true.” This is an attempt at proof by mathematical induction. The missing step is the induction basis: It would prove the statement is true for all cases if it was checked that the equality holds for n = 1. But since it is missing, it cannot be proved that the statement is true for any n. Hence the correct choice is C. 30.1 per cent of the students pointed out the missing step but failed to give the correct choice. Only 12.7 per cent of the students concluded that the missing step would make the proof invalid.

Table 7.31. PEE response types, item 2C

	Response 2C					Total
	A	B	C	D	NA	
SCED	6	15	3	0	1	26
PRED	3	7	3	2	0	15
MATH	2	5	7	0	0	14
Total	11	27	13	2	1	55

Table 7.32. PEE scores, item 2C

	Score 2C				Total
	0	1	2	3	
SCED	4	9	9	3	26
PRED	6	7	2	0	15
MATH	1	2	6	5	14
Total	11	18	17	8	55

Next proof attempt was proof 2D: "$\sum_{k=1}^{n} 2k-1=n^2$, $2\sum_{k=1}^{n} k-\sum_{k=1}^{n} 1=n^2$, $2\sum_{k=1}^{n} k-\sum_{k=1}^{n} 1=n^2$, $2\cdot\frac{n\cdot(n+1)}{2}-n=n^2+n-n=n^2$" Here, 1 is subtracted from each even number from 2 to 2*n*, which gives us the terms of the desired sum. But, again it should be noted that in order to find the sum of even numbers, the fact that sum of integers from 1 to *n* is *n* (*n* +1) / 2 is used without proof. Another point to mention is that the notation used is misleading; it seems as though the proof starts with the assuming equality that is required to be proved is true. Still, the correct choice is B and 61.8 per cent of the students received full points for this response.

Table 7.33. PEE response types, item 2D

	Response 2D					Total
	A	B	C	D	NA	
SCED	2	19	1	2	2	26
PRED	0	8	0	4	3	15
MATH	0	7	4	3	0	14
Total	2	34	5	9	5	55

Table 7.34. PEE scores, item 2D

	Score 2D				Total
	0	1	2	3	
SCED	4	1	2	19	26
PRED	7	0	0	8	15
MATH	2	4	1	7	14
Total	13	5	3	34	55

Last argument for the second item, proof 2E is given as follows: "Consider the sum:

$1 + 3 + 5 + ... + 2n-1 = S$

$\underline{+ 2n-1 + 2n-3 + 2n-5 + ... + 1 = S}$

$2n + 2n + ... + 2n = 2S$ There are n terms: $n.2n = 2S$, $S = n^2$." This is a simple, valid proof which does not use any previously known formulas or facts; the terms of the sum are written in reverse order and the first term is added to the last, second term is added to the second one from the last etc. Each these sums are equal to $2n$, and if we add them all up we get $2n^2$, which is twice the sum we are looking for.

Table 7.35. PEE response types, item 2E

	Response 2E					Total
	A	B	C	D	NA	
SCED	1	18	4	2	1	26
PRED	3	9	0	3	3	15
MATH	2	10	2	0	2	14
Total	6	37	6	5	6	55

Table 7.36. PEE scores, item 2E

	Score 2E				Total
	0	1	2	3	
SCED	5	1	1	18	26
PRED	5	0	1	9	15
MATH	2	0	2	10	14
Total	12	1	4	37	55

Figure 7.10 shows percentage frequency distributions of the scores for proofs 2A through 2E.

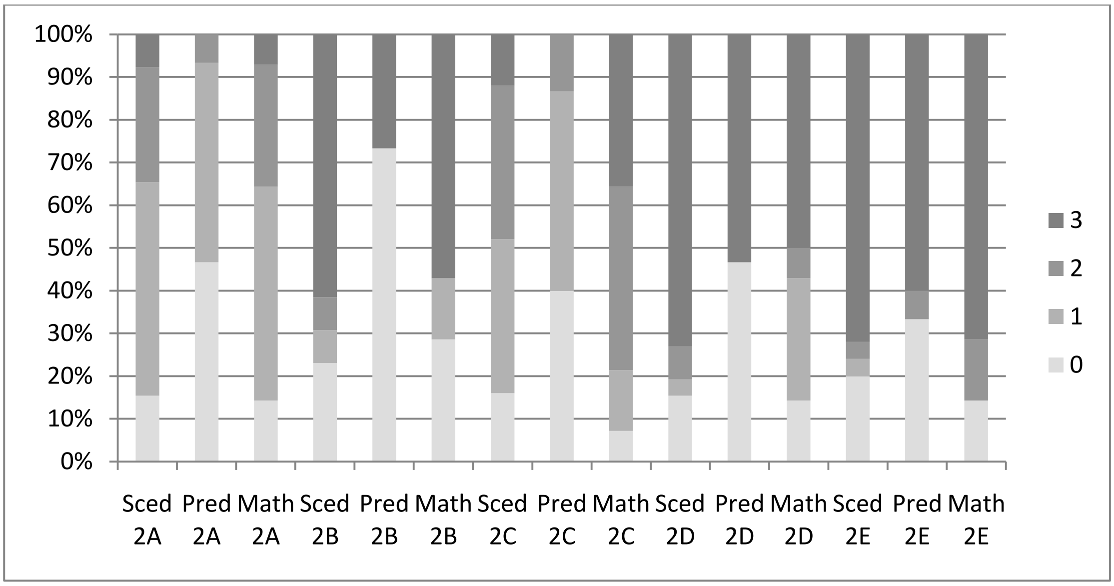

Figure 7.10. Percentage frequencies of scores of item 2

When mean scores for item 2 in Table 7.37 are examined, it is seen that prospective primary school teachers have the lowest score from proof 2A through 2E. Highest mean scores are observed by prospective mathematicians in proofs 2A, 2C and 2E, and by prospective secondary school teachers in proofs 2B and 2D.

Table 7.37. Mean scores and standard deviations, PEE item 2

Item		2A	2B	2C	2D	2E
SCED	Mean	1.27	2.08	1.44	2.38	2.28
	Std.Dev.	0.83	1.30	0.92	1.10	1.20
PRED	Mean	0.60	0.80	0.73	1.60	1.93
	Std.Dev.	0.63	1.37	0.71	1.55	1.44
MATH	Mean	1.29	1.86	2.07	1.93	2.43
	Std.Dev.	0.83	1.41	0.92	1.21	1.10

The third item of PEE; “Given any three consecutive integers, one of them must be divisible by three” had four alternative proofs.

The first argument was proof 3A: “$n+ n+1+ n+2 = 3n +3 = 3(n+1)$. Since their sum is divisible by three, one of them must be divisible by three.” The sum of three integers being divisible by three does not imply that one of them is divisible by three. Therefore, even though it is true that the sum of three consecutive integers is divisible by three, it cannot be used in the proof of this statement. This argument does not prove the statement; the correct choice here is C. 55.5 per cent of the students pointed out this fact and received full points.

Table 7.38. PEE response types, item 3A

	Response 3A					Total
	A	B	C	D	NA	
SCED	6	2	14	3	0	25
PRED	4	6	4	1	0	15
MATH	1	0	12	1	0	14
Total	11	8	30	5	0	54

Table 7.39. PEE scores, item 3A

	Score 3A				Total
	0	1	2	3	
SCED	2	9	2	12	25
PRED	6	3	0	6	15
MATH	2	0	0	12	14
Total	10	12	2	30	54

Next argument is proof 3B: “The statement is false: none of the numbers -1, 0, 1 is divisible by three.” While giving counter example is a valid method to disprove a statement, it is not applicable here. The counter example is inappropriate because zero is actually divisible by three. The argument does not disprove the statement as intended, as 74.1 per cent of the students indicated.

Table 7.40. PEE response types, item 3B

	Response 3B					Total
	A	B	C	D	NA	
SCED	2	0	20	1	2	25
PRED	1	3	11	0	0	15
MATH	0	0	14	0	0	14
Total	3	3	45	1	2	54

Table 7.41. PEE scores, item 3B

	Score 3B				Total
	0	1	2	3	
SCED	4	2	0	19	25
PRED	4	3	0	8	15
MATH	0	1	0	13	14
Total	8	6	0	40	54

Proof 3C was the third argument for this item: “Multiples of three are obtained by adding three to the previous one. When we write down consecutive integers, we see that there are two numbers between multiples of three: ***n*** <u>*n*+1 *n*+2 ***n*+3**</u> *n*+4 *n*+5 *n*+6, where *n*+1 *n*+2 ***n*+3** are any three numbers.” This is a valid argument but it needs to be expressed in a clearer manner. From the notation it is understood that *n* is assumed to be divisible by three, which makes it true for only some cases.

Table 7.42. PEE response types, item 3C

	Response 3C					Total
	A	B	C	D	NA	
SCED	2	13	5	5	0	25
PRED	6	4	3	2	0	15
MATH	7	5	1	1	0	14
Total	15	22	9	8	0	54

Table 7.43. PEE scores, item 3C

	Score 3C				Total
	0	1	2	3	
SCED	7	11	7	0	25
PRED	6	4	5	0	15
MATH	1	4	7	2	14
Total	14	19	19	2	54

Last argument is proof 3D: “Let x, x+1, x+2 be three consecutive integers. When x is divided by three, the remainder is 0, 1 or 2. If the remainder is 0, then x is divisible by three, if it is 1 then x + 2 is divisible by three, and if it is 2 then x +1 is divisible by three.” This argument proves the given statement, the correct choice is B and 79.6 per cent of the students gave the correct response.

Table 7.44. PEE response types, item 3D

	Response 3D					Total
	A	B	C	D	NA	
SCED	2	22	1	0	0	25
PRED	6	8	1	0	0	15
MATH	1	13	0	0	0	14
Total	9	43	2	0	0	54

Table 7.45. PEE scores, item 3D

	Score 3D				Total
	0	1	2	3	
SCED	3	0	0	22	25
PRED	7	0	0	8	15
MATH	0	0	1	13	14
Total	10	0	1	43	54

Figure 7.11 below shows the percentage frequency distributions for proofs 3A through 3D for item 3.

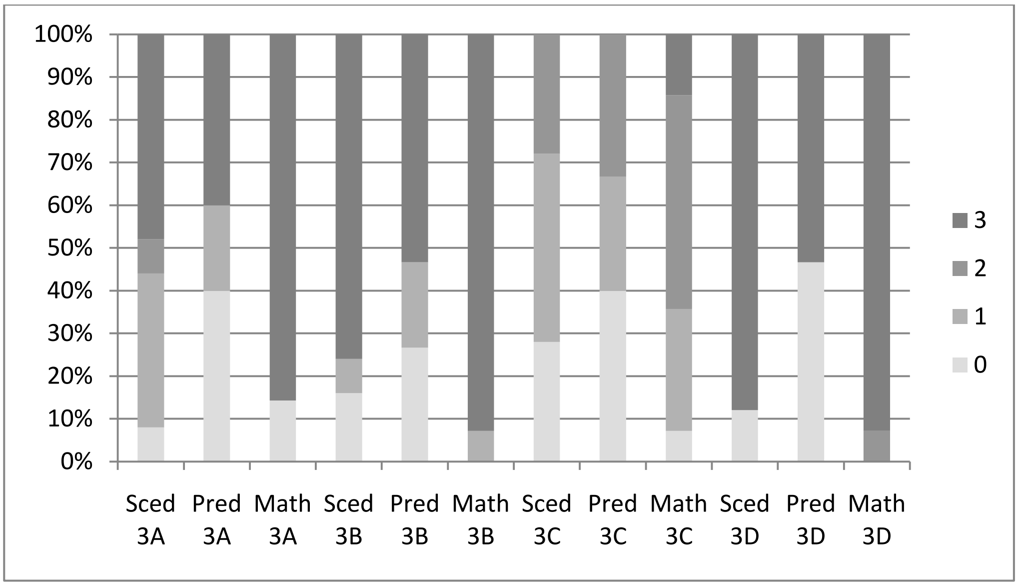

Figure 7.11. Percentage frequencies of scores for item 3

From Table 7.46 below it can be observed that prospective primary school teachers have the lowest score means and prospective mathematicians have highest score means for item 3 in all cases.

Table 7.46. Mean scores and standard deviations, PEE item 3

Item		3A	3B	3C	3D
SCED	Mean	1.96	2.36	1.00	2.64
	Std.Dev.	1.10	1.20	0.76	0.99
PRED	Mean	1.40	1.80	0.93	1.60
	Std.Dev.	1.40	1.37	0.88	1.55
MATH	Mean	2.57	2.86	1.71	2.93
	Std.Dev.	1.09	0.54	0.83	0.27

Last item of PEE is: “Suppose there are n people at a party ($n \geq 2$). Prove that there are at least two people who have the same number of friends in this party.” Four alternative proof attempts are given to the participants to evaluate.

First proof attempt proof 4A: “Consider two people who have no friends but each other. Say they saw the party by coincidence while walking down the street. This way, at least two people (each other) have one friend, which means the same number of friends.” This is only one case. There is no generalization here and it cannot as be accepted as a proof. Students who indicated that the proof is incomplete because it deals with only one possible situation and who said that the argument does not prove the statement received the maximum score (64.8 per cent).

Table 7.47. PEE response types, item 4A

	Response 4A					Total
	A	B	C	D	NA	
SCED	9	0	13	2	1	25
PRED	8	2	3	1	1	15
MATH	3	0	10	0	1	14
Total	20	2	26	3	3	54

Table 7.48. PEE scores, item 4A

	Score 4A				Total
	0	1	2	3	
SCED	3	0	2	20	25
PRED	4	0	4	7	15
MATH	1	0	5	8	14
Total	8	0	11	35	54

Second argument was proof 4B: “If n = 2 either they know each other or they are strangers. In both cases, they have the same number of friends. If n = 3, either no one knows each other or everybody knows each other or there are two people who know each other. In all cases, there are at least two people with the same number of friends. Assume the statement is true for n-1 people and one more person attends the party. If this person does not know anybody in the party, the statement is still true because there are at least two people who know each other. We can go on like this.” Proof by mathematical induction is attempted but could not be completed here. Since the possibilities are checked for some values of n, the correct choice is A and 25.9 per cent of students got maximum scores.

Table 7.49. PEE response types, item 4B

	Response 4B					Total
	A	B	C	D	NA	
SCED	5	12	4	3	1	25
PRED	6	2	1	4	2	15
MATH	7	4	1	1	1	14
Total	18	18	6	8	4	54

Table 7.50. PEE scores, item 4B

	Score 4B				Total
	0	1	2	3	
SCED	7	9	4	5	25
PRED	7	4	2	2	15
MATH	3	3	1	7	14
Total	17	16	7	14	54

Proof 4C was the next argument: "If $n > 2$ and n is even, we can find two people with the same number of friends. For $n = 3$ it is not true." This is not a valid argument. It says the statement is true for even values of n, but no justification is provided. In addition, the statement is not false for $n = 3$.

Table 7.51. PEE response types, item 4C

	Response 4C					Total
	A	B	C	D	NA	
SCED	1	0	22	1	1	25
PRED	0	1	8	5	1	15
MATH	0	0	12	1	1	14
Total	1	1	42	7	3	54

Table 7.52. PEE scores, item 4C

	Score 4C				Total
	0	1	2	3	
SCED	2	1	5	17	25
PRED	7	0	6	2	15
MATH	2	0	0	12	14
Total	11	1	11	31	54

Proof 4D was the last argument given for the fourth item: "If we consider that everyone has at least one friend and let 1, 2, 3, ...,n denote the people in the party. First person can have 1, second person can have 2, third person can have 3, … , $(n\text{-}1)^{th}$ person can have n-1 friends but n^{th} person can have at most n-1 friends (we exclude himself/herself). Hence n^{th} person must have the same number of friends with one of the other n-1 people." This is a direct proof using the pigeonhole principle, the correct choice is B. Some students indicated that the case where a person has no friends in the party must also be examined, which can be shown using exactly the same reasoning as explained above. 55 .6 per cent of the students got maximum score for this item. Figure 7.12 shows the percentage frequency distribution of scores for item 4.

Table 7.53. PEE response types, item 4D

	Response 4D					Total
	A	B	C	D	NA	
SCED	2	19	1	2	1	25
PRED	1	4	5	4	1	15
MATH	3	6	1	3	1	14
Total	6	29	7	9	3	54

Table 7.54. PEE scores, item 4D

	Score 4D				Total
	0	1	2	3	
SCED	3	2	1	19	25
PRED	10	0	0	5	15
MATH	4	1	3	6	14
Total	17	3	4	30	54

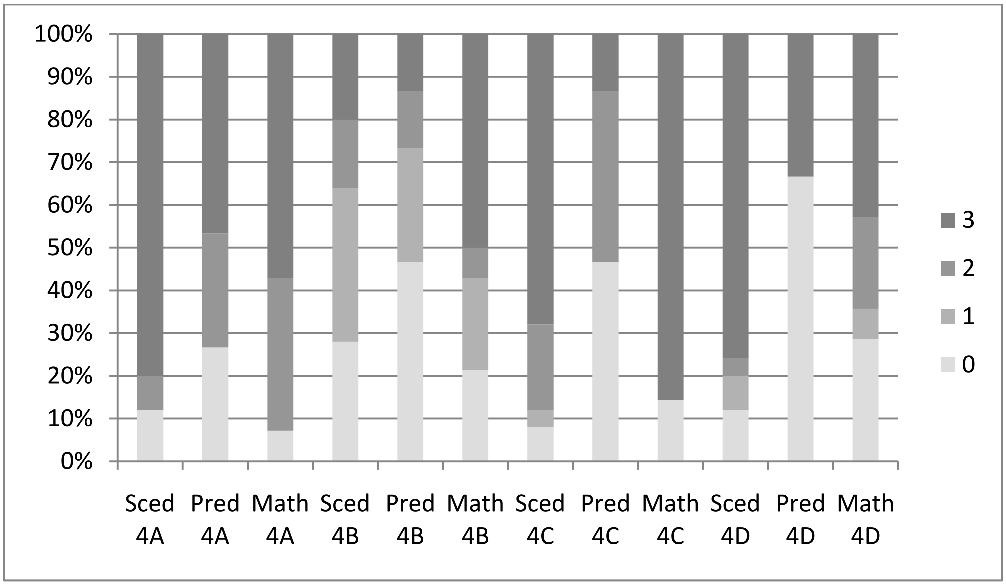

Figure 7.12. Percentage frequency of scores for item 4

Examining the mean scores displayed in table 7.55, one can see that once again, prospective primary school teachers have the lowest scores in all cases. Prospective secondary school teachers have highest means for proofs 4A and 4D, while for proofs 4B and 4C, prospective mathematicians have the highest mean scores.

Table 7.55. Mean scores and standard deviations, PEE item 4

Item		4A	4B	4C	4D
SCED	Mean	2.56	1.28	2.48	2.44
	Std.Dev.	1.00	1.10	0.92	1.09
PRED	Mean	1.93	0.93	1.20	1.00
	Std.Dev.	1.28	1.10	1.21	1.46
MATH	Mean	2.43	1.86	2.57	1.79
	Std.Dev.	0.85	1.29	1.09	1.31

Table 7.56 shows the means of total scores for each item and total PEE score. Maximum possible scores for item 1 and item 2 are 15, for item 3 and item 4 are 12. Total maximum possible score is 54.

Table 7.56. Mean scores and standard deviations for each PEE item

Scores		Item1 (15)	Item 2 (15)	Item 3 (12)	Item 4 (12)	Total Score (54)
SCED	Mean	9.19	9.64	7.96	8.76	35.56
	Std.Dev.	2.73	2.72	2.56	2.85	6.84
PRED	Mean	7.20	5.67	5.73	5.13	23.73
	Std.Dev.	3.36	3.52	3.35	3.27	8.55
MATH	Mean	12.36	9.57	10.07	8.64	40.64
	Std.Dev.	2.34	3.78	1.77	3.50	8.12

Shapiro - Wilk test did not reveal any significant results for any sub group, therefore normal distributions can be assumed and parametric tests are carried out.

One way ANOVA was performed for seniors' total PEE score. Results show that there are significant differences between mean scores with respect to department: $F(2, 51) = 19.11$, $p = 0.00 < 0.05$. Post hoc analysis revealed that prospective primary school teachers have significantly lower mean score than prospective secondary school teachers ($p = 0.00 < 0.05$) and mathematicians ($p = 0.00 < 0.05$). Mean score difference between prospective mathematicians and secondary school teachers is not significant.

7.4. Relationships between Attitudes and Beliefs Regarding Proof, Proof Construction and Evaluation

In order to answer the research question RQ4: "Are there any relationships between: freshmen and senior Mathematics and Teaching Mathematics students' proof construction practices and their attitudes and beliefs regarding proof; senior Mathematics and Teaching Mathematics students' proof construction practices and their evaluation of freshmen students' arguments; and senior Mathematics and Teaching Mathematics students' attitudes and beliefs regarding proof and their evaluation of freshmen students' arguments?", Pearson Correlation test was conducted to see whether there are significant correlations between students ABS, PE and PEE scores. Results indicate significant positive correlations between ABS and PE scores (r = 0.28, p = 0.00 < 0.01), and between PE and PEE scores (r = 0.52, p = 0.00 < 0.01). No significant correlation is found between ABS and PEE scores (r = 0.15, p = 0.35 > 0.05). Meaning that positive significant correlations are found between students proof construction practices and their attitudes and beliefs. Also significant is the positive correlation between proof construction and proof evaluation practices of seniors. However, seniors' attitudes and beliefs do not seem to be related with their proof evaluation practices.

When subscales of ABS are taken into consideration, positive significant correlations are observed between PE score and subscales *attitude* (r = 0.27, p = 0.00 < 0.01), *self efficacy* (r = 0.40, p = 0.00 < 0.01) and *belief* (r = 0.29, p = 0.00 < 0.01). Correlation between *background* and PE was not significant (r = -0.11, p = 0.18 > 0.05).

When correlations for freshmen and seniors are considered separately, it is seen that, for freshmen, there is a positive significant correlation between ABS and PE scores (r = 0.24, p = 0.03 < 0.05). When subscales of ABS are taken into consideration, significant relationships are observed between PE score and subscales *attitude* (r = 0.27, p = 0.01 < 0.05), *self efficacy* (r = 0.30, p = 0.005 < 0.01) and *belief* (r = 0.31, p = 0.00 < 0.01). Relationship between *background* and PE was not significant (r = 0.03, p = 0.80 > 0.05).

For seniors, a positive significant correlation is observed between PE and PEE scores ($r = 0.52$, $p = 0.00 < 0.01$). No significant relationship is found between ABS and PEE scores: *background* ($r = 0.07$, $p = 0.66 > 0.05$), *attitude* ($r = -0.07$, $p = 0.66 > 0.05$), *self efficacy* ($r = 0.11$, $p = 0.47 > 0.05$) and *belief* ($r = 0.24$, $p = 0.12 > 0.05$).

In this section; results of data analysis with respect to the research questions about freshmen and senior students coming from three different departments, related with their attitudes and beliefs regarding proof, their proof construction practices and senior students proof evaluation practices are reported. These results are discussed in the following section in detail, with conclusions and comments given regarding implications for teaching and further research.

8. CONCLUSION

The role and importance of proof in mathematics education has been discussed in many recent studies (e.g. Mariotti, 2006; Brown *et al.*, 2008; Hanna and Barbeu, 2008). Janke (2007) points out that findings of studies suggest "many school and university students and even teachers of mathematics have only superficial ideas on the nature of proof". To investigate the situation in a particular setting, this study was conducted with the aim of examining freshmen and senior students' beliefs, attitudes towards proof, and their proof construction tendencies as well as senior students' proof evaluation practices. The sample consisted of students coming from Mathematics, Primary and Secondary Education Teaching Mathematics Programs in Boğaziçi University. Instruments developed for the study were administered to freshmen students at the very beginning of their programs; hence their responses reflect their high school knowledge and experiences. Hence the findings of this study give an idea about high school graduates' and prospective mathematicians' and mathematics teachers' conceptualizations and practices regarding proof.

Results of the study indicate that there are no significant differences between departments among freshmen regarding their attitudes and beliefs, as well as their proof construction abilities. Considering that the instruments were conducted to freshmen at the very beginning of their first semester in the university, which means their responses reflect their high school knowledge and experiences, this is an expected result; it can be assumed that students have more or less the same exposure in high school regarding proof. *Attitudes and Beliefs Scale* has four sub dimensions labeled *background*, *attitude*, *self efficacy* and *beliefs*. Significant differences are observed between freshmen and seniors in all subscales. An interesting result is that freshmen have higher *background* scores than seniors while in all the other subscales, seniors' scores are higher. One interpretation for freshmen having higher *background* scores is that, since they have not yet been exposed to formal proof as seniors have; what they perceive as proof may be more empirical-inductive. Seniors might have responded to these items comparing their high school and university experiences. This interpretation is backed with the result that freshmen predominantly responded to the question "what does proof mean to you" by saying "explaining where a result comes from", indicating a broader sense of justification. In addition, they could not provide valid general arguments in most cases. Another interpretation is that recent changes made in the high school curriculum and the central university entrance exam might be the cause of this difference, but further research is needed to back this claim. Seniors having higher scores in other subscales suggests that they developed a more positive attitude towards proof throughout their university education. Highest scores are observed in *belief* subscale (close to 4, out of 5), both for freshmen and seniors. In this subscale, there are items highlighting the importance of proof in mathematics lessons and its explanatory aspect, which was also emphasized in their responses to the open ended item in the *Attitudes and Beliefs Scale*.

When responses to the open ended item "*What does proof mean to you? Explain briefly*" are examined, it is seen that 50 per cent freshmen and 46 per cent of seniors mention the explanatory aspect of proof: that it helps them to understand why the statement is true, which leads to better understanding. They report that knowing where a result comes from prevents memorization. Next most observed response type for freshmen is the affect category (42.9 per cent). While there are students who find it enjoyable and fun, most responses in this category declare that proofs are complicated, difficult and require hard work. Participants also mention the importance of proof and report that it is necessary in mathematics. Second most observed response for senior students (29 per cent) mention conviction: that proof verifies the statement is true and removes any doubt that it may be false. 28.6 per cent of freshmen responses are also in this category. When we look at departments, it is observed that 50 per cent of mathematics seniors emphasize that the proof is an essential part of mathematics, therefore understanding it is important for their profession. 33.3 per cent of Mathematics students also mentioned the process and the generality of results. Most common response for seniors from Primary and Secondary Teaching Mathematics Programs is again the explanatory aspect of proof (100 per cent and 63 per cent respectively).

From these responses it can be seen that the explanatory aspect of proof is highlighted by both freshmen and seniors. While freshmen students also focus on the affective aspects, seniors mention that proof removes doubt and verifies a statement is true (what proof is) and explain the proof process (how it is done). Freshmen participants, as people who chose mathematics and teaching mathematics as future professions, are aware of the importance of proof in mathematics, and that they will dealing with proofs in university. However, they do not have much experience with proof yet (as can be seen from their responses to *Proof Exam*) and find it difficult and complicated. Senior students, who are more experienced, focus more on technical aspects of proof and have a better idea of what proof is and how it is done. Mathematics students emphasize that proof is an essential part of mathematics and therefore important for their profession, but rarely mention the explanatory aspect. One explanation for this may be the following: while other participants see proofs as an aid to understand mathematics or be successful in their mathematics courses, Mathematics students see it as a part of their everyday practice.

Senior prospective secondary school mathematics teachers have significantly lower *self efficacy* scores than prospective primary school mathematics teachers and prospective mathematicians, while *beliefs* scores of prospective mathematicians were significantly higher than prospective primary secondary school mathematics teachers. Mathematics students have more experience with high level proofs than the students in teaching mathematics programs. Students from Secondary School Teaching Mathematics Program take more mathematics courses than the students from Primary School Teaching Mathematics Program, like algebra and complex number theory, which are more theoretical; but still they do not have the experience with proofs as much as Mathematics students. This may be the reason for their relatively lower self efficacy score.

Examining the scores of *Proof Exam*, which was developed to investigate participants' proof construction practives, it is observed that freshmen have an average score of 1.92, where maximum possible score is 12. No significant differences are observed between departments among freshmen. Seniors' average scores are 4.07, 6.66, and 9.40 for prospective primary and secondary school teachers, and mathematicians respectively. These results are significantly higher than freshmen, but not sufficient; when it is considered that the items in the instrument consist of high school level problems. Mean differences between all departments are significant. Here, again we see that while there are no differences among freshmen students, seniors' scores differ significantly with respect to department. Regardless of their scores, seniors were able to recognize the need to use certain proving methods, such as proof by mathematical induction, proof by contradiction, direct proof and contra-positive method. To look at the proof constructing practices of the students, responses to the specific items below are examined.

The first item in the *Proof Exam*, “if n^2 is even then n is even”, can be proven by using various methods, and the responses of both freshmen and seniors reflect that. As mentioned in Section 7.2, among the most attempted proof types by freshmen students are proof by cases (22.6 per cent) and direct proof (20.4 per cent). Most mathematics seniors (66.7 per cent) used proof by contradiction, and 20 per cent used proof by contrapositive. Among senior prospective secondary school teachers, most preferred proof method was proof by cases (31 per cent), followed by direct proof (24.2 per cent) and proof by contrapositive (20.7 per cent). 14.3 per cent and 10.7 per cent of senior prospective primary school teachers attempted proof by contrapositive and contradiction respectively.

This item is one of the common examples of proof by contradiction or contrapositive, and these indirect proof methods were mostly used by mathematics seniors and some of the seniors from Teaching Mathematics Programs. With the exception of mathematics seniors, most participants preferred direct approaches such as direct proof and proof by cases. According to Antonini and Mariotti (2008) studies regading indirect proof report that “students’ difficulties with indirect proof seem to greater than those related with direct proof”, and assuming that what needs to be proved is false may be mentally demanding and false hypotheses and contradictions make it harder to follow the deductive steps of the proof.

One important observation is that 21.5 per cent of freshmen, 20.7 per cent of senior prospective secondary school mathematics teachers and 46.7 per cent of senior prospective primary school mathematics teachers proved the converse of this statement: “if n is even then its square must be even”. While this statement is true, it is not logically equivalent to the original statement. No senior mathematics student provided this type of response. Inability to distinguish between a statement and its converse indicates a poor understanding of logical implication. Such difficulties were also reported in the study of Hoyles and Küchemann (2002). When scores for this item are examined, it is seen that 42 per cent of freshmen students did not receive any points and only 7.5 per cent were given maximum points. Amount of seniors who received minimum and maximum points are 13.9 per cent and 34.3 per cent respectively. When we look at departments, it is seen that 80 per cent of mathematics, 24.1 per cent of secondary education and 17.9 per cent of prospective primary school teachers were given maximum points.

Even though freshmen's proof scores are low, they produced a more variety of approaches than seniors, which was especially apparent in their responses to the item "prove or disprove: the equality $1 + 2 + \ldots + 2n + 1 = n^2$ is true for all integers $n \geq 1$". This equality is one of the classic examples of proof by mathematical induction, and expectedly, majority of seniors attempted to use this method. Freshmen however, attempted other methods which could be considered as more creative. Mingus and Grassl (1999) found a similar result. In addition, among the students who attempted mathematical induction, a common mistake observed was to omit the basis step of induction, which was also observed by Stylianides, et al. (2007). While 66.7 per cent of seniors used mathematical induction, only 4.3 per cent of freshmen attempted to prove the statement with this method. Mathematical induction was the most commonly attempted proof method by seniors from all departments (60.7 per cent from Primary Education Teaching Mathematics, 65.5 per cent from Secondary Education Teaching Mathematics and 80 per cent from Mathematics programs). The scores for this item indicate that 78.5 per cent of freshmen and 13.9 per cent of seniors received minimum score. Maximum score was received by 5.4 per cent of freshmen and 54.2 per cent of seniors (93.3 per cent of prospective mathematicians, 58.6 per cent of prospective secondary school and 28.6 per cent of prospective primary school teachers).

Next item in the *Proof Exam* was: "prove or disprove: given any three consecutive integers, one of them must be divisible by three". 50.6 per cent of freshmen did not provide a meaningful response for this item. Most common response type for freshmen was to show the statement holds by giving numerical examples with no generalization (12.8 per cent) and direct proof (11.8 per cent). One common mistake made by freshmen (8.6 per cent) was to falsely assume that zero cannot be divided by three, hence concluding that the statement is false. Showing that the sum of three consecutive integers is divisible by three was another common mistake done (by 10.8 per cent of freshmen, 28.6 per cent of senior prospective primary school teachers). This is a true proposition but it cannot be used to prove the statement. When we look at seniors' responses, it is observed that most attempted proof type was proof by cases (80 per cent of prospective mathematicians, 55.2

per cent of prospective secondary school and 28.6 per cent of prospective primary school teachers). Second common response type for prospective secondary school teachers is direct proof, but no prospective mathematicians or primary school teachers gave this response. Scores for this item indicate that 75.3 per cent of freshmen and 40.3 per cent of seniors (most of which are prospective primary school teachers) received no points. Maximum points were taken by 3.2 per cent of freshmen and 38.9 per cent of seniors (73.3 per cent of prospective mathematicians, 37.9 per cent of prospective secondary school and 21.4 per cent of prospective primary school teachers).

The item “in a party of *n* people show that at least two people have the same number of friends”, was the most challenging both for freshmen and seniors. It can be proved using “pigeonhole principle” which was explained in Section 2.1.2. While it can be expected for freshmen to be unfamiliar with this principle, it is a simple observation that they could have come up with themselves. Seniors also had difficulties with this item, even though it can be assumed that they are familiar with the concept. The reason for these difficulties probably arises from the nature of the problem. While it has a simple solution which can be easily understood by high school graduates, it may not look like a typical mathematical problem they’ve come across in their high school mathematics or calculus courses.

This was the least attempted item among all participants (no attempt by 79.6 per cent freshmen and 44.4 per cent seniors). Trial for small *n* values (without generalization) were the most common response type among freshmen (10.8 per cent), while 33.3 per cent of seniors used pigeonhole principle (46.7 per cent of prospective mathematicians, 37.9 per cent of prospective secondary and 21.4 per cent of prospective primary school teachers). Most participants (92.5 per cent of freshmen and 69.4 per cent of seniors) did not receive any points for this item. Maximum score was received by 2.2 per cent of freshmen and 16.7 per cent of seniors (33.3 per cent of prospective mathematicians, 20.7 per cent prospective secondary and 3.6 per cent of prospective primary school teachers).

To summarize, it can be observed from participants’ responses to *Proof Exam* that empirical methods are usually preferred by freshmen students. Seniors attempt to use general arguments, but they (mostly prospective primary and secondary school teachers) have difficulty in distinguishing the difference between a proving statement and its converse, prefer direct proof approaches eventhough indirect approaches could have been conveniently applied, and omit the basis step in the mathematical induction method. All participants had difficulty in proving a statement that was relatively unfamiliar to most participants.

In addition to freshmen and senior students' proof construction practices, proof evaluation practices of senior students were also examined in this study. For this purpose, *Proof Evaluation Exam* was administered to senior students, where participants were given alternative proof attempts related to each of the items in the *Proof Exam*, and were asked to evaluate these "proofs". For each alternative proof, participants were asked to choose one of the following and explain the reason for their choice: "A. The proof shows the statement is true for in some cases", "B. The proof shows the statement is always true", "C. The proof is false", and "D. I have no opinion". The results of data analysis of *Proof Evaluation Exam* reveal that most senior students were successful at differentiating between inductive and deductive arguments and stated that giving specific examples cannot be accepted as proof. They also were good at for each alternate proof pointing out the arguments that did not check the truth of the statement for all cases. Nonetheless, proof evaluation scores of seniors showed significant differences between primary and secondary education students, and primary education and prospective mathematicians. To get a better understanding of the results of *Proof Evaluation Exam*, responses are examined item by item in the following paragraphs.

For the item "if n^2 is even then n is even", seniors were asked to evaluate five alternative attempts. Proof 1A was an example of proof by cases, with minor calculation mistake, which did not affect the generality of the result. Therefore, the correct choice would be B: the proof shows the statement is true for all cases. 78.6 per cent of prospective mathematicians, 57.7 per cent of prospective secondary school, and 46.7 per cent prospective primary school teachers received maximum points for this item. Some students concluded that the proof is false (23.6 per cent) or only shows the statement is true for some cases (12.7 per cent) because of the calculation mistake or claiming that the argument shows the converse of the statement.

Proof 1B was constructed using the contradiction method, even though it could have been expressed better. Therefore, it showed the proof is true for all cases and 78.6 per cent of mathematicians, 57.7 per cent of prospective secondary school and 46.7 per cent of prospective primary school teachers received maximum score for the evaluation of this argument.

An attempt at direct proof was given in proof 1C, with a missing justification. Students who correctly identified the missing step (42.9 per cent from mathematics, 11.5 per cent from secondary school teaching mathematics and 6.7 per cent from primary school teaching mathematics programs) received full points. As these percentages indicate, this argument was harder to evaluate because no obvious mistakes stood out.

The proof of converse statement “if n is even then n^2 is even” was given in proof 1D. As mentioned before, this is a valid argument but does not prove the given statement. Correct choice should be C. Another interpretation emerged is that the proof is incomplete, and the case where n is odd should also have been examined (choice A), then it would be proof by cases. Both interpretations were given full points. Percentages of students who received maximum points are 100 per cent, 57.69 per cent and 13.34 per cent for mathematics, secondary and primary school teaching mathematics programs respectively. It is also worth noting here that 64.29 per cent of prospective primary school teachers thought the argument proves the given statement for all cases. As mentioned above, majority of prospective primary school teachers provided this type of proof for the corresponding item in the *Proof Exam*.

Finally in proof 1E, truth of the statement is verified for only a couple of values of n. Therefore the correct choice is A. Since there is no generalization, this cannot be accepted as a valid proof. Students who stated that giving examples is not a proof (with choice C) also received full points. As a result, 78.6 per cent of prospective mathematicians, 69.23 per cent of prospective secondary and 60 per cent of prospective primary school teachers got full points. 18.3 per cent of seniors thought the argument showed the statement is true for all cases. While majority of students correctly detected that this cannot be accepted as a proof, one would expect that the percentages would have been higher, since this argument is the most apparent example in the *Proof Evaluation Exam* where the statement is not proven for all cases.

When mean scores for each argument related to this item are examined (see Table 7.26), it is observed that, from proof 1A through 1E, prospective mathematicians have highest mean scores. Prospective primary school teachers have lowest mean scores in proofs 1A, 1D, 1E; and for proofs 1B and 1C, the mean scores of prospective secondary school teachers are the lowest. While prospective secondary school teachers and mathematicians have lowest means for proof 1C (0.88 and 1.86 respectively), prospective primary school teachers' lowest mean (0.60) is for proof 1D. Mathematicians' highest mean score (3.00) is achieved in proof 1D. Prospective secondary (2.38) and primary school (2.00) teachers' highest scores come from proof 1E. To summarize, it can be said that prospective mathematicians were best at correctly distinguishing between a statement and its converse, while prospective primary school teachers had the most difficulty with it. Prospective secondary and primary teachers were best in recognizing that giving a finite number of numerical examples cannot be accepted as a valid proof (where the domain of discourse is infinite). Prospective secondary school teachers and mathematicians had the most difficulty with proof 1C, where there was a crucial step needed to be justified.

There were also five arguments in the *Proof Evaluation Exam* for the item "prove or disprove: the equality $1 + 2 + \ldots + 2n +1 = n^2$ is true for all integers $n \geq 1$". In proof 2A, a general formula to find sums is correctly used to verify the statement is true. However, no explanation about why this formula is true or why it can be used in this particular case is given. Students who stated that the proof would be valid if the formula was also proved received full points. 52.73 per cent of students chose B. 7.69 per cent of prospective secondary school teachers and 7.14 per cent of prospective mathematicians received full marks. No prospective primary school teachers received maximum points.

In proof 2B, first the sum from 1 to $2n$-1 is calculated, and then sum of even numbers in this range is subtracted from the total to find the sum of odd numbers. This shows the statement is true for all cases, however, the fact that sum of integers from 1 to n is calculated by the formula $n\ (n\ +1)\ /\ 2$ is used without proof. 50.1 per cent of the participants gave this explanation. Looking at departments, it is seen that 61.5 per cent of prospective secondary school teachers, 57.1 per cent of prospective mathematicians and 26.7 per cent of prospective primary school teachers received full marks.

Proof 2C is an attempt at proof by mathematical induction. The missing step is the induction basis: Truth of the statement for all cases would be shown if it was also checked that the equality holds for $n = 1$. But since it is missing, it cannot be proved that the statement is true for any n. Hence the correct choice is C. 30.1 per cent of the participants pointed out the missing step but failed to give the correct choice. Only 12.7 per cent of the participants concluded that the missing step would make the proof invalid. As a result, 35.7 per cent of prospective mathematicians and 11.5 per cent of prospective secondary school teachers and no prospective primary school teachers received maximum points. When compared with the responses of *Proof Exam*, it is seen that majority of senior students preferred induction, and much higher percentage of them received maximum points. This result indicates that while most students did not make this mistake in their own proofs, they do not consider omitting the basis step of induction as a major mistake. One reason for this can be that usually checking that the smallest number satisfies the condition is trivial but showing that if the statement is true for n, then it would also be true for $n + 1$ is the challenging part of the proof.

The argument presented in proof 2D, shows that if 1 is subtracted from each even number from 2 to $2n$, the resulting numbers give the terms of the desired sum. But, again it should be noted that in order to find the sum of even numbers, the formula $n\,(n + 1) / 2$, which gives the sum of integers from 1 to n is used without proof. Another point to mention is that the notation used is misleading; it seems as though the proof starts with the assuming equality that is required to be proved is true. Still, the correct choice is B and 73.1 per cent of prospective secondary school teachers, 53.3 per cent of prospective primary school teachers and 50 per cent of prospective mathematicians received full points for this response. Some prospective mathematicians were skeptical about the notation, therefore preferred choice C.

Proof 2E is a valid proof which does not use any previously known formulas or facts. Here the terms of the sum are written in reverse order and the first term is added to the last, second term is added to the second one from the last etc. Each these sums are equal to $2n$, and if we add them all up we get $2n^2$, which is twice the sum we are looking for. 71.4 per cent, of mathematics, 69.2 per cent of prospective secondary school teachers and 60 per cent of prospective primary school teachers received maximum points.

When mean scores for item 2 of *Proof Evaluation Exam* (which were given in table 7.37) are examined, it is seen that prospective primary school teachers have the lowest score from proof 2A through 2E. Highest mean scores are observed by prospective mathematicians in proofs 2A, 2C and 2E, and by prospective secondary school teachers in proofs 2B and 2D. Lowest scores for all departments are achieved in proof 2A (mathematics 1.29 prospective secondary school teachers 1.27 and prospective primary school teachers 0.60). Highest scores for prospective primary school teachers (1.93) and mathematics (2.43) are seen in their responses to proof 2E. Highest mean score for prospective secondary school teachers (2.38) is achieved from proof 2D. This means, students from all departments had difficulties evaluating the argument where the result is obtained by using a formula, which should not have been used here without proof. Prospective primary school teachers and prospective mathematicians were most successful in evacuating the proof where the result is obtained simply by adding the terms of the required sum in reverse order, and prospective secondary school teachers were most successful in evaluating the argument where the terms of the sum is obtained by subtracting one from each even number from 2 to $2n$.

Next item in *Proof Evaluation Exam* is “prove or disprove: given any three consecutive integers, one of them must be divisible by three”. Four arguments were given to seniors for evaluation. Proof 3A is the claim that since the sum of three consecutive integers is divisible by three, one of them must be divisible by three. This argument does not prove the statement; the correct choice here is C. 55.5 per cent of the students pointed out this fact and received full points. Percentages of maximum scores with respect to departments are as follows: mathematics 85.7 per cent, prospective secondary school teachers 48 per cent, and prospective primary school teachers 40 per cent.

Proof 3B claims the statement is false by giving a counter example: -1, 0, 1 are three consecutive integers none of which are divisible by three. While giving counter example is a valid method to disprove a statement, it is not applicable here because since 0 is actually divisible by three. The argument does not disprove the statement as intended, as 74.1 per cent of the students indicated. Percentages of students with maximum scores are 92.9 per cent from Mathematics, 76 per cent from Secondary Education and 53.3 per cent from Primary Education Teaching Mathematics Programs.

The argument given in proof 3C is a valid argument which needs to be expressed in a clearer manner. It was written using a notation such that it is understood that *n* is assumed to be divisible by three, which makes it true for only some cases. 40.7 per cent participants chose B, and 27.8 per cent chose A. Only 14.3 per cent of prospective mathematicians received maximum scores.

Proof 3D is valid argument using proof by cases examining the remainders when the integers are divided by three. The correct choice is B and 79.6 per cent of the students gave the correct response. Maximum scores with respect to departments are as follows: mathematics 92.9 per cent, prospective secondary school teachers 88 per cent and prospective primary school teachers 53.3 per cent.

Mean scores for proofs 3A, 3B, 3C and 3D were given in Table 7.46. Examining these scores, it can be observed that prospective primary school teachers have the lowest and prospective mathematicians have the highest means in all cases. Proof 3C have the lowest means for all departments (0.93 for prospective primary school teachers, 1.00 for prospective secondary school teachers and 1.71 for prospective mathematicians). Highest means for prospective secondary school teachers (2.64) and mathematics (2.93) are achieved in proof 3D. For prospective primary school teachers, highest mean (1.80) is observed in proof 3B.

The last item in *Proof Evaluation Exam* is “in a party of *n* people show that at least two people have the same number of friends”. Four arguments are presented to the participants for evaluation. Proof 4A is a response which deals with only one case. There is no generalization here and it cannot as be considered as proof. Students who indicated that the proof is incomplete because it deals with only one possible situation and who said that the argument does not prove the statement received the maximum score (prospective secondary school teachers 80 per cent, mathematics 57.1 per cent and prospective primary school teachers 46.7 per cent).

Proof 4B is an argument that attempts mathematical induction, but it could not be completed. Since the possibilities are checked for some values of *n*, the correct choice is A and 25.9 per cent of students got maximum scores. Looking at departments, it is seen that 50 per cent of mathematics, 20 per cent of prospective secondary school teachers and 13.3 per cent of prospective primary school teachers received maximum scores.

The argument of proof 4C is not valid. It says the statement is true for even values of *n*, but no justification is provided. In addition, it is claimed that the statement is false for $n = 3$, which is not true. 77.8 per cent of participants chose C. Maximum score distribution is as follows: prospective mathematicians 85.7 per cent, prospective secondary school teachers 68 per cent, and prospective primary school teachers 13.3 per cent.

Final argument for this item is proof 4D. This is a direct proof using the pigeonhole principle; hence the B is the correct choice, as 53.70 per cent of participants indicated. Some students (mostly from mathematics department) indicated that the case where a person has no friends in the party must also be examined, which can be shown using exactly the same reasoning as explained above. 76 per cent of prospective secondary school teachers, 42.9 per cent of mathematicians and 33.3 per cent of prospective primary school teachers received maximum points.

Mean scores for proofs 4A, 4B, 4C and 4D were given in Table 7.55. From these scores it can be seen that prospective primary school teachers have the lowest scores in all cases. Prospective secondary school teachers have highest means for proofs 4A and 4D, while for proofs 4B and 4C, prospective mathematicians have the highest mean scores. Lowest mean scores for all departments are observed in proof 4B (0.93, 1.28 and 1.86 for prospective primary school teachers, prospective secondary school teachers and prospective mathematicians respectively). Highest scores are observed in primary (1.93) and prospective secondary school teachers (2.56) in proof 4A, and for prospective mathematicians (2.57) in proof 4C. This item was the most challenging for freshmen and seniors alike in *Proof Exam*, and they also had difficulties evaluating the related arguments probably since they did not have too many ideas on how to prove the statement themselves. Proof 4B is an attempt at proof by induction, and it looks like a valid argument but it is unfinished. Since the argument obviously uses a “legitimate” proof method, students failed to accurately evaluate it.

It can be seen from the results summarized above that, students were better at accurately evaluating arguments that prove the statement is true for all cases, or arguments that clearly do no prove the statement, like the incorrect counter-example in proof 3B or giving numerical examples instead of a general proof, like in proof 1E. They do have difficulties in evaluation when there is not an obvious mistake in the argument, but some steps are missing or a crucial piece of information is given without justification.

Findings of the study indicate that while there are no significant differences between departments among freshmen regarding their attitudes, beliefs and proof construction practices; significant differences among departments were observed in the case of seniors. This suggests the differences occur as a result of their university education. It is seen from the findings that prospective mathematicians have the highest scores, especially in *Proof Exam* and *Proof Evaluation Exam*, and prospective primary school teachers’ scores are the lowest in most cases. One explanation for this situation is that prospective primary school teachers do not take as many math courses as prospective secondary school teachers and prospective mathematicians. Students in education departments enroll to mathematics courses given by the Mathematics Department, together with Mathematics students. Hence, their content knowledge is formed by the courses that they take from Mathematics Department. Comparing Primary and Secondary Teaching Mathematics Programs, it is seen that the additional courses prospective secondary school teachers take are Discrete Mathematics, Linear Algebra and Introduction to Complex Analysis. For both prospective primary and secondary school teachers, the compulsory mathematics courses taken are first and second year courses. Secondary School Teaching Mathematics Program has additional six elective mathematics courses, which can be of third or fourth year. However, available elective mathematics courses differ from year to year and not all students choose the same elective courses. Prospective primary school teachers also have elective course options in their program, where they may choose mathematics courses, but it is not compulsory. Details of departmental programs and descriptions of mathematics courses can be found in

Appendix D. (See Page 189.) While it can be argued that higher level content knowledge is required for prospective secondary school teachers, more exposure to university level mathematics may increase students' ability to construct and evaluate proofs.

Results of the study revealed significant positive correlation between freshmen students' beliefs and attitudes towards proof and their *Proof Exam* score. Correlations between affective constructs and achievement were found in numerous studies, some of which are reported in Schoenfeld (1989). It is interesting to note that, while significant positive correlations were observed between *Proof Exam* scores and *beliefs*, *attitude* and *self efficacy* subscales, relationship between *background* and *Proof Exam* scores were not significant for either freshmen or seniors. This may be because of the result that even though freshmen have significantly higher background scores than senior students; their proof scores are significantly lower, as stated above. Hence freshmen's perceived background regarding proof is not backed up by their performance in constructing proofs. In addition, since *background* items were related to students' high school experiences; it makes sense that seniors' proof constructing performance is not related with their *background* scores. *Beliefs* and *attitude* scores of senior students were not significantly related to their *Proof Exam* scores either. As prospective mathematicians and mathematics teachers, seniors are aware of the importance and role of proof in mathematics, as can be seen from their responses to *Attitudes and Beliefs Scale*, but they still have some difficulties in constructing proofs. A significant positive relationship is observed however, between *Proof Exam* and *Proof Evaluation Exam* scores of seniors. The item by item examination of responses to these two instruments above also reveals the parallels between them.

As can also be seen from the results of this study, senior students, while being aware of the necessity of generalization of their arguments, still resort to inductive methods when they cannot think of any other way, as seen in the third item of *Proof Exam*. Martin and Harel (1989) reported that inductive and deductive proof schemes exist simultaneously in the student. Healy and Hoyles' (2000) study revealed that while students chose deductive arguments as the ones their teacher would give the best mark, they chose inductive ones as the arguments which they would adopt as their own approach. This suggests while the students know the proofs need a formal deductive approach, it does not come naturally to them. Baştürk's (2010) study revealed that lack of proof in high school practices results in an abrupt introduction to proof in university level, which is the prominent cause students' difficulty in proof at the university level. Epps (2003) and Carraher (2002), also emphasize the difficulty of students' transition from informal reasoning to formal deductive reasoning and use of symbolic language. Designing specifically introductory mathematics courses in the university considering the points mentioned above will help students to be better acquainted with proof. Hanna (1991) states that a teaching activity that includes formal and informal reasoning is valued to the degree that it supports higher understanding. Starting point of a naïve mathematical idea should be everyday experience. Then this idea is developed and made clear. Hanna also asserts that clarification needs a degree of formalism. Creation of a language, definitions of symbols, specification of rules of manipulation is required in order to achieve greater generality. Almeida (2003) suggests that the flow *theorem → proof → examples* generally used in advance level mathematics instruction should be replaced by either with *examples → theorem → proof* or *examples → proof → theorem*; while carefully selecting examples that illustrates the theorem.

As prospective mathematics teachers, the courses that these students take from mathematics department are aimed to cover their content knowledge. The question of how to teach proof is a matter of pedagogical content knowledge, hence the responsibility of education departments. Hence it is essential to design the bridge courses in these departments in a way that facilitates creating learning environments that involves proof activities. Mathematicians' proof processes involve inductive reasoning; intuition, trial and error lead to a conjecture and then formal proof practices requiring deductive methods follows. As mentioned above, most participants mentioned explanatory aspect of proof; how proofs can help them understand why a result is true, which leads to a deeper understanding. In addition, since one function of proof is convincing oneself and others, including the conjecture development in the proof process in classrooms, allowing the students to form their own conjectures and urging them to convince others that their conjecture is true may result in a deeper understanding of the subject and decrease the risk of students seeing proof as a topic to be learned, instead of as a process that is in the essence of mathematics.

To overcome gap between formal and informal proof, some technological aids can be used. Visualization can be effective tool for teaching proof and logical reasoning in mathematics courses for all levels. Dynamic geometry software, for example, enables students to take active participation in doing proofs in geometry, by facilitating experimenting and conjecturing so that students can reason about the generality of their hyphotheses (Öner, 2008a, 2008b; Eysink *et al.*, 2002; Huertas, 2007; Perez-Lancho *et al.*, 2007).

It must also be kept in mind that, mathematical statements used in the study were chosen so that they could be proven using high school knowledge and experience; these students come across more complex mathematical tasks involving proof in their university courses. Future studies can focus on students' difficulties regarding proof used in specific mathematical topics and designing appropriate proof activities. In addition, these statements in the instruments were typical examples used, when the notion of proof is introduced in mathematics courses. So the senior participants were familiar with these problems, probably with the exception of the last item in *Proof Exam*, as the responses suggest. In further studies, presenting students with proof problems that they are not familiar with can give an idea whether they can transform their knowledge into unfamiliar situations.

As a result, findings in this study give an idea of the conceptions and practices regarding proof of high school graduates and prospective mathematicians and mathematics teachers in Boğaziçi University. High school graduates (freshmen) who have participated to this study are students who chose to continue their careers as mathematicians or mathematics teachers and they succeeded in a central national test to be able to enter a prestigious state university in Turkey. While they may not accurately represent typical high school graduates, their poor performance in constructing proofs in this study gives an idea about the situation regarding proof in high schools in general. One must keep in mind that the questions in the *Proof Exam* were chosen from high school curriculum. Not only high school graduates achieved a very low average score, seniors' scores, while being significantly higher than freshmen, are not as high as one would expect. Considering the difficulties that teaching mathematics majors, especially prospective primary school teachers have regarding proof suggests that being introduced to proof at this late stage in their educations causes them to struggle in university. Further studies should be conducted to investigate the situation in high schools. Attitudes and practices of high school teachers and textbooks and curriculum materials can be examined.

The instruments developed for this study, which are given together with their rubrics in the appendices, can be used in future studies involving both high school and university students’ conceptions regarding proof. Teachers in high school and instructors of introductory courses in university can also use them in classrooms as assessment tools. A toolkit for mathematics school teachers and university instructors can be developed to help them evaluate arguments students have generated, as well as to help students evaltuate their own proofs.

APPENDIX A: INSTRUMENTS

A1. Attitudes and Belief Scale (Tutum ve İnanç Ölçeği)

Sevgili Öğrencimiz,
Bu ölçek Boğaziçi Üniversitesi Orta Öğretim Fen ve Matematik Alanları Eğitimi Bölümünde yürütülmekte olan, matematiksel ifadelerin ispatına yönelik yaklaşımları konu alan bir araştırma kapsamında hazırlanmıştır. Ölçeğe vereceğiniz yanıtlar araştırmaya katkı sağlayacak, veriler yalnızca araştırma amacı ile kullanılacaktır.
Ölçek iki bölümden oluşmaktadır. Birinci bölümde matematiksel ispat ve matematik derslerinde ispatın yeri ve önemi ile ilgili düşünceleriniz sorulmaktadır. İkinci bölümde ise bazı matematiksel ifadelerin ispatını yapmanız beklenmektedir. Birinci bölümde işaretleyeceğiniz seçeneğin gerçeği yansıtması, ikinci bölümde ise cevaplarınızı mümkün olduğu kadar açıklamanız araştırma sonuçlarının güvenilirliği bakımından önemlidir. İsminiz araştırmacıda saklı kalıp, araştırmanın hiçbir aşamasında kullanılmayacaktır. Gerek görüşler, gerekse ispata verdiğiniz yanıtlar doğru veya yanlış olarak değerlendirilmeyecektir. Cevaplarınız sadece bu konudaki görüş ve yaklaşımlarınızı yansıtması açısından önemlidir.
Değerli katkılarınız için teşekkür ederiz.

Bölüm 1: Aşağıda matematiksel ispata yönelik tavrınız ve deneyimleriniz ile ilgili ifadelere yer verilmiştir. Her ifade için beş seçenekten kendinize en uygun olanını işaretleyiniz.	Hiçbir zaman katılmıyorum	Katılmıyorum	Kararsızım	Katılıyorum	Her zaman katılıyorum
1. Matematiksel bir ifadeyi (teorem, özellik, eşitlik vs) kavrayabilmek için ispatını anlamaya çalışırım.					
2. İspatlar matematiği zevkli hale getirir.					
3. İspat sadece matematikçiler için önemlidir.					
4. Lise düzeyinde matematik derslerinde ispatlara yer verilmesi öğrencileri zorlayabilir.					
5. Lisede matematiksel bir kavramı açıklarken ispattan faydalanılmalıdır.					
6. Bir lise öğrencisinden matematiksel ispat yapabilmesi beklenmelidir.					

7. Lise matematik kitabımızda ispata yönelik alıştırmalar vardı.					
8. Mezun olduğum lisede matematik derslerinde ispata yer veriliyordu.					
9. Lisede matematik derslerinde veya sınavlarda basit ispatlar yapmam bekleniyordu.					
10. Lisede matematik öğretmenlerimiz ispatın öneminden hiç söz etmiyordu.					
11. Lisede matematik öğretmenlerimiz bizi ispat yapmaya teşvik ediyordu.					
12. Üniversitedeki matematik derslerinin işleniş tarzı açısından lise matematik derslerinden farkları olacağını düşünüyorum.					
13. Lisede matematik derslerinde öğrendiğim bilgilerin ve edindiğim becerilerin üniversitede faydalı olacağını düşünüyorum.					
14. Matematiksel bir ifadenin doğru olduğuna karar vermek için ifadenin bütün durumlarda geçerli olduğunu kontrol etmem gerekir.					
15. Matematiksel ispat yapma konusunda kendime güveniyorum.					
16. Matematikte ispatlar genelde kafa karıştırıcıdır.					
17. Matematiksel bir sonucun neden doğru olduğunu anlamak için ispatını bilmemiz gerekmez.					
18. İspat matematiğin ayrılmaz bir parçasıdır.					
19. İspatla uğraşmak bana sıkıcı geliyor.					
20. Matematik dersinde başarılı olmak için ispat yapabilmek şart değildir.					
21. İspat yaparken matematiksel dili etkili bir şekilde kullanabilirim.					
22. İspatları anlamada genellikle zorlanıyorum.					
23. İspat yapma konusunda fazla deneyimim yok.					
24. Matematik bilgimin basit ispatlar yapmak için yeterli olduğuna inanıyorum.					
25. Üniversitede matematik derslerinde ispata lisedekinden daha çok yer verilmelidir.					

Matematiksel ispat sizin için ne ifade ediyor? Kısaca açıklayınız.

A2. Proof Exam (İspat Sınavı)

A2. 1. Freshmen

Bölüm 2: *Bu bölümde bazı matematiksel ifadeleri ispatlamanız istenmektedir. Cevaplarınızı mümkün olduğu kadar açıklamanız araştırmanın amacı için önemlidir.*

1. "Bir doğal sayının karesi çift sayı ise kendisi de bir çift sayıdır" ifadesinin doğruluğunu ispatlayınız.

2. Aşağıdaki ifadelerin doğruluğunu veya yanlışlığını ispatlayınız.

a. $1 + 3 + 5 + ... + 2n\text{-}1 = n^2$ eşitliği, $n \geq 1$ tamsayıları için doğrudur.

b. Herhangi 3 tane ardışık sayı içerisinde her zaman üçe bölünebilen bir sayı vardır.

c. $n^2 + (n + 1)^2 = (n + 2)^2$ eşitliği tüm doğal sayılar için doğrudur.

d. $n^2 + (n + 1)^2 = (n + 2)^2$ eşitliği tüm doğal sayılar için yanlıştır.

3. n kişinin katıldığı bir parti düşünelim ($n \geq 2$). Bu partide aynı sayıda arkadaşı olan en az iki kişi bulunacağını ispatlayınız.

4. Aşağıdaki özelliklerden/teoremlerden hangilerini biliyorsunuz? **İşaretleyiniz.** Bildikleriniz içinden **bir tanesini** ispatlayınız.

a. ( ) Bir üçgenin iç açıları toplamı 180 derecedir.

b. ( ) Düzlemde birbirlerine dik iki doğrunun eğimlerinin çarpımı -1'dir.

c. ( ) Bir dik üçgende, dik kenarların uzunluğu a ve b, hipotenüsün uzunluğu c ile gösterilsin. O zaman $a^2 + b^2 = c^2$ dir (Pisagor teoremi).

d. ( ) A, B ve C herhangi üç küme olmak üzere,

A ∩ (B U C) = (A ∩ B) U (A ∩ C).

e. ( ) (a, b) ve (c, d) düzlemde herhangi iki nokta olsun. Bu iki nokta arasındaki uzaklık $\sqrt{(a-c)^2+(b-d)^2}$ dir.

A2. 2. Seniors

Bölüm 2: *Bu bölümde bazı matematiksel ifadeleri ispatlamanız istenmektedir. Cevaplarınızı mümkün olduğu kadar açıklamanız araştırmanın amacı için önemlidir.*

1. "Bir doğal sayının karesi çift sayı ise kendisi de bir çift sayıdır" ifadesinin doğruluğunu ispatlayınız.

2. Aşağıdaki ifadelerin doğruluğunu veya yanlışlığını ispatlayınız.

a. $1 + 3 + 5 + ... + 2n-1 = n^2$ eşitliği, $n \geq 1$ tamsayıları için doğrudur.

b. Herhangi 3 tane ardışık sayı içerisinde her zaman üçe bölünebilen bir sayı vardır.

3. n kişinin katıldığı bir parti düşünelim ($n \geq 2$). Bu partide aynı sayıda arkadaşı olan en az iki kişi bulunacağını ispatlayınız.

A.3. Proof Evaluation Exam (İspat Değerlendirme Sınavı)

Aşağıda bazı matematiksel ifadeler ve onların değişik ispatları verilmiştir. İspatlarda bazı hatalar olabilir. Lütfen ispatların her biri için A, B, C, D seçeneklerinden birini seçip, kısaca seçiminizin nedenini açıklayınız.

A. İfadenin bazı durumlar için doğru olduğunu gösterir C. İspat yanlıştır
B. İfadenin tüm durumlar için doğru olduğunu gösterir D. Bir fikrim yok

İfade 1: *Bir doğal sayının karesi çift sayı ise kendisi de bir çift sayıdır.*

İspat 1A: n tek sayı ise: $n = 2k + 1$, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ tek sayı (çift+çift+tek) n çift sayı ise: $n = 2k$, $n^2 = (2k)^2 = 4k^2 + 4k$ çift sayı (çift+çift). n^2 çift olduğuna göre n de çift olmak zorunda.

Seçim:

Neden:

İspat 1B: n tek sayı olsun. $(2k+1)^2 = 2m$, $4k^2 + 4k + 1 = 2m$. Sol taraf tek sayı, sağ taraf çift. Çelişki elde ettik. Demek ki n çift olmalı.

Seçim:

Neden:

İspat 1C: $n^2 = n \cdot n = 2k$. Burada k çift sayı olmak zorunda çünkü $2k$ tam kare: $k = 2m$, $n^2 = 4m$, $\sqrt{n^2} = \sqrt{4m}$, $n = 2\sqrt{m}$, dolayısı ile n çift olur.

Seçim:

Neden:

İspat 1D: $n = 2k$ olsun. O zaman $n^2 = 4k^2$ çift.

Seçim:

Neden:

İspat 1E: Ç = {2, 4, 6, 8, ...}

n^2 = 4 ise n = 2, n^2 = 16 ise n = 4, n^2 = 36 ise n = 6 .... n^2 = 114 ise n =12.

Seçim:

Neden:

1. İfade	İspat No
En çok açıklayıcı / ikna edici ispat(lar)	
En az açıklayıcı / ikna edici ispat(lar)	

İfade 2: *1 + 3 + 5 + ... + 2n-1 = n^2 eşitliği, n ≥1 tamsayıları için doğrudur.*

İspat 2A: *ST*: son terim, *İT*: ilk terim, Δm: artış miktarı ise $\frac{ST - İT}{\Delta m} + 1$ terim sayısı.

Buna göre toplam $\frac{ST + İT}{2} \cdot \left(\frac{ST - İT}{\Delta m} + 1 \right) = \frac{2n - 1 + 1}{2} \cdot \left(\frac{2n - 1 - 1}{2} + 1 \right) = n^2$ olur.

Seçim:

Neden:

İspat 2B: 1 + 2 + 3 + ...+ 2n-2 + 2n -1 = $\frac{(2n-1) \cdot 2n}{2}$ dir , o halde istenen toplam da:

$\frac{(2n-1) \cdot 2n}{2} - 2(1 + 2 + ... + \frac{2n-2}{2}) = 2n^2 - n - (n-1)n = n^2$ olur.

Seçim:

Neden:

İspat 2C: S(n) = 1 + 3 + 5 +...+ 2n-1 = n^2, S(n+1) = 1 + 3 + 5 +...+ 2n-1 + 2n+1 = n^2 + 2n +1, S(n) + 2n+1 = $(n+1)^2$, yani ifade doğrudur.

Seçim:

Neden:

İspat 2D: $\sum_{k=1}^{n} 2k - 1 = n^2$, $\sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 = n^2$, $2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n^2$

$$2 \cdot \frac{n \cdot (n+1)}{2} - n = n^2 + n - n = n^2$$

Seçim:

Neden:

İspat 2E: 1 + 3 + 5 + … + 2*n*-1 = S

+ 2*n*-1 + 2*n*-3 + 2*n*-5+ … + 1 = S

2*n* + 2*n* + ... + 2*n* = 2S   *n* tane terim var: *n*.2*n* = 2S, S = n^2

Seçim:

Neden:

2. İfade	İspat No
En çok açıklayıcı / ikna edici ispat(lar)	
En az açıklayıcı / ikna edici ispat(lar)	

İfade 3: *Herhangi 3 tane ardışık sayı içerisinde her zaman üçe bölünebilen bir sayı vardır.*

İspat 3A: *n*+ *n*+1+ *n*+2 = 3*n* +3 = 3 (*n*+1). Toplamları üçe bölündüğüne göre aralarında mutlaka üçe bölünebilen bir sayı olmalı.

Seçim:

Neden:

İspat 3B: İfade yanlıştır. -1, 0 ,1 sayılarının hiçbiri üçe bölünmez.

Seçim:

Neden:

İspat 3C: Üçün katları olan sayılar öncekine 3 eklenerek bulunur. Ardışık sayıları yazdığımızda üçe bölünenler arasında iki sayı kalacağını görürüz. *n* <u>*n*+1 *n*+2 *n*+3</u> *n*+4 *n*+5 *n*+6. *n*+1 *n*+2 *n*+3 herhangi üç sayı.

Seçim:

Neden:

İspat 3D: x, x+1, x+2 üç ardışık sayı olsun. x üçe bölündüğünde kalan 0, 1 veya 2 olur. Kalan 0 ise x üçe bölünür, 1 ise x + 2 üçe bölünür, 2 ise x +1 üçe bölünür.

Seçim:

Neden:

3. İfade	İspat No
En çok açıklayıcı / ikna edici ispat(lar)	
En az açıklayıcı / ikna edici ispat(lar)	

İfade 4: *n* kişinin katıldığı bir parti düşünelim (n ≥ 2). Bu partide aynı sayıda arkadaşı olan en az iki kişi bulunacağını ispatlayınız.

İspat 4A: Mesela iki kişi düşünelim ve birbirlerinden başka arkadaşları olmasın. Tesadüfen yoldan geçerken partiyi gördüklerini düşünelim. Bu şekilde en az iki kişinin birer arkadaşı (birbirleri) yani aynı sayıda arkadaşı vardır.

Seçim:

Neden:

İspat 4B: *n* = 2 ise iki kişi ya birbirini tanıyor ya da tanımıyor. İki durumda da arkadaş sayıları eşit. *n* = 3 ise ya kimse birbirini tanımıyordur, ya herkes birbirini tanıyordur, ya da birbirini tanıyan iki kişi vardır. Her durumda arkadaş sayısı aynı olan en az iki kişi var. İfadenin *n*-1 kişi için doğru olduğunu düşünelim ve son bir kişi partiye gelsin. Eğer kimseyi tanımıyorsa hala doğru olacak çünkü birbirini tanıyan en az iki kişi var. Bu şekilde devam edebiliriz.

Seçim:

Neden:

İspat 4C: Eğer $n>2$ ve n bir çift sayı ise aynı sayıda arkadaşı olan iki kişi bulunabilir. $n = 3$ durumunda doğru olmaz.

Seçim:

Neden:

İspat 4D: Bir kişinin en azından bir tane arkadaşı olduğunu düşünürsek ve kişileri 1, 2, 3, ...,n ile gösterirsek birinci kişinin 1, ikinci kişinin 2, üçüncü kişinin 3, ..., n-1. kişinin n-1 arkadaşı olabilir ama n. kişinin en fazla n-1 arkadaşı olabilir (kendisini çıkartırız). Dolayısı ile n. kişinin arkadaş sayısı diğer n-1 kişiden birininkiyle aynı olmak zorundadır.

Seçim:

Neden:

4. İfade	İspat No
En çok açıklayıcı / ikna edici ispat(lar)	
En az açıklayıcı / ikna edici ispat(lar)	

APPENDIX B: RUBRICS

B.1. Rubric for PE

Tüm maddeler için, öğrencinin soruya yanıt verdiği ama anlamlı bir argüman ortaya koyamadığı durumlar F harfi ile temsil edilmiştir.

1. "*Bir doğal sayının karesi çift sayı ise kendisi de bir çift sayıdır" ifadesinin doğruluğunu ispatlayınız.*

İspat 1A: n tek olsaydı karesi de tek olurdu. (Olmayana ergi/Proof by contrapositive)

Puan: 1

- Tek sayının karesi çift olamaz (açıklama yok)
- Eğer x = T , mesela: 3, $3^2 = 9$ çift değil.

Puan: 2

- a'nın tek sayı olduğunu varsayalım. O zaman a'nın karesini alırsak a^2 de tek sayıdır. Çünkü iki tek sayının çarpımı daima tek sayıdır. (Tersten giderek.)

Puan: 3

- $n^2 = 2k \rightarrow n = 2m \quad n = 2k + 1$, olsaydı $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ olurdu (çift + çift + tek = tek)

İspat 1B: n çift sayı ise karesi de çifttir. (Bu doğru bir önermedir fakat verilen önermenin tersini ispatlar, ona denk değildir.)

Puan: 0

- İki çift sayının çarpımı çift olacağından karesi her zaman çifttir. Daha fazla işlemle açıklayamam.

- Eğer karesi çift olmayan bir çift sayı bulursam ifadeyi çürütmüş olurum. Böyle bir değer de olmadığı için bu ifade doğrudur.
- $a^2 = 4n^2$ olsun. O zaman $a = 2n$'dir.

Puan: 1

- $n = 2k$ olsun. O zaman $n^2 = 4k^2$ çift.
- $n= 2(k+1)$ olsun. ...$n^2 = 2\ (2k^2+4k+2)$ çift. İfade doğrudur.

İspat 1C: n'in tek sayı ve n^2 'nin çift sayı olduğunu varsayalım. Ama n tek ise karesi de tektir, bu da varsayımımızla çelişir. Demek ki baştaki varsayımımız yanlış; n çift olmalı. (Çelişki metodu ile ispat/proof by contradiction)

Puan: 2

- a^2 nin çift sayı olduğunu kabul edelim ve *a*'nın tek sayı olduğu durumu inceleyelim. *a* tek ise, *a* x *a* = T x T tek sayı olması gerekirdi. Bulduğumuz bu sonuç kabul ettiğimiz doğru ile çeliştiği için *a* çift sayı olmalı.

Puan: 3

- Let $a \in$ N such that $a^2 \equiv 0$ mod 2. Then $a \equiv 0$ mod 2 or $a \equiv 1$ mod 2. Assume $a \equiv 1$ mod 2. Then $a^2 = a.a \equiv\ 1.1 \equiv 1$ mod 2 , which is a contradiction.
- Eğer *a* bir doğal sayı ve $a.a = 2.k$ (*k* doğal sayı), yani *a.a* çift ise, $a = 2t$, *t* doğal sayı. Çünkü 2 asal sayı olduğundan, eğer *a*'nın asal çarpanlara ayrılmış hali $a = p_1\ p_2 \ldots p_n$ için $p_i \neq 2$, $\forall$ i = 1, 2, …, *n* ise $a^2 = p_1^2\ p_2^2 \ldots p_n^2$ için hiçbir $p_i^2 \neq 2\ \rightarrow a^2 \neq 2k$. Bu bir çelişkidir$\rightarrow a = 2t$.

İspat1D: n^2 'yi çift kabul edip *n*'nin çift olduğunu göstermek. (Doğrudan ispat/ direct proof).

Puan: 0

- $n^2 = n \cdot n = 2k$, $n^2/\ k = 2$ that is *n.n* is a power of 2, so *n* is even.
- $y/2 = n \rightarrow x/2 = m$. $2n = (2m)^2$ ...$y = 4m^2$, y = $(2m)^2$, $\sqrt{y} = 2m = x$ (ifadenin doğru olduğunu var sayarak başlamış).

- Ç = {2, 4, 6, 8, ...} $n^2 = 4$ ise $n = 2$, $n^2 = 16$ ise $n = 4$, $n^2 = 36$ ise $n = 6$ .... $n^2 = 144$ ise $n = 12$.
- $n^2 = 2x$ ise $n = \sqrt{2x}$. $2x$'in çift olmasını sağlayan her x değeri için $\sqrt{2x}$ çifttir.

Puan: 1

- Bir sayının çift olabilmesi için n'in içinde 2 çarpanı olmalı. Yani karenin çift olabilmesi için n'de de iki çarpanı olmalı. Bu yüzden n mutlaka çifttir.

Puan: 2

- Doğal sayılarda çarpmanın sonucunun çift olması için çarpanların en az biri çift olmalıdır. Bir sayının karesinin çarpanları o sayının kendisi ve çarpanlarıdır. Dolayısı ile sayı çift olmalıdır ki çarpanları çift olsun. (Karesini alırken aynı sayıyı çarptığımıza göre x^2 çift ise x de çift olur.)
- n bir doğal sayı olsun. $n^2 = n \cdot n = 2k$. Burada k çift sayı olmak zorunda çünkü $2k$ tam kare: $k = 2m$, $n^2 = 4m$, $\sqrt{n^2} = \sqrt{4m}$, $n = 2\sqrt{m}$, dolayısı ile n çift olur.

İspat 1E: n ya tek ta ya çifttir. n tek ise karesi tek, çift ise karesi çift olur. Bu durumda karesi çift olan bir sayı çift olmalıdır. (Durum analizi/proof by case analysis)

Puan: 0

- $x = 2 \rightarrow x^2 = 4$ (sağladı), $x = 4 \rightarrow x^2 = 16$ (sağladı) , $x = 6 \rightarrow x^2 = 36$ (sağladı), $x = 3$ $x^2 = 9$ (X)
- $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $5^2 = 25$, $8^2 = 64$. Aksine bir örnek bulamadığımıza göre doğrudur.

Puan: 1

- Ç x Ç = Ç : 2 x 2 =4, 4 x 4 = 16 ; T x T = T 3 x 3 = 9

Puan: 2

- Tek sayı x tek sayı = daima tek sayı, çift sayı x çift sayı = daima çift sayı. Tüm doğal sayılar için geçerlidir. Öyleyse iki çift sayının çarpımı da daima bir çift sayı olur.

Puan: 3

- n tek sayı ise: $n = 2k + 1$, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ tek sayı (çift+çift+tek)

  n çift sayı ise: $n = 2k$, $n^2 = (2k)^2 = 4k^2 + 4k$ çift sayı (çift+çift)

  n^2 çift olduğuna göre n de çift olmak zorunda.

2.a) *1 + 3 + 5 + ... + 2n-1 =* n^2 *eşitliği, n ≥1 tamsayıları için doğrudur.*

İspat 2a_A: Formül kullanarak:

Puan: 1

- *ST*: son terim, *İT*: ilk terim, Δm: artış miktarı ise $\frac{ST - İT}{\Delta m} + 1$ terim sayısı.

  Buna göre toplam $\frac{ST + İT}{2} \cdot \left(\frac{ST - İT}{\Delta m} + 1 \right) = \frac{2n-1+1}{2} \cdot \frac{2n-1-1}{2} + 1 = n^2$ olur.

İspat 2a_B: Gauss' metodunu kullanarak:

Puan: 2

- 1 + 2n-1, 3 + 2n-3, ... bu ikililerin toplamı 2n dir. n terim var, o zaman bu ikililerin sayısı $\frac{n}{2}$ dir. O halde toplam $2n \cdot \frac{n}{2} = n^2$ dir.

Puan: 3

- 1 + 3 + 5 +...+ 2n-1 = S

  \+ 2n-1 + 2n-3 + 2n-5+... +1 = S

  2n + 2n + ...+ 2n = 2S n tane terim var: n.2n = 2S , S = n^2

İspat 2a_C: Tümevarım yöntemi

Puan: 1

- $1+3+5+\ldots+2(n+1)-1 = (n+1)^2$

  $\underline{-1+3+5+\ldots+\ 2n-1\ \ =\ \ n^2}$

  $2(n+1)-1= (n+1)^2 - n^2$

  …

  0 = 0 Sağladığından doğrudur

Puan: 2

- $S(n) = 1 + 3 + 5 +...+ 2n-1 = n^2$, $S(n+1) = 1 + 3 + 5 +...+ 2n-1 + 2n+1 = n^2 + 2n +1$

  $S(n) + 2n+1 = (n+1)^2$, yani ifade doğrudur.

Puan: 3

- Temel adım: $n = 1,\ 1 = 1^2$. P(n) doğru olsun. P(n+1)' in doğru olduğunu gösterelim:

  $P(n) = 1 + 3 + 5 +...+ 2n-1 = n^2$,

  $P(n+1) = 1 + 3 + 5 +...+ 2n-1 + 2n+1 = n^2 + 2n +1 = (n+1)^2$ doğru.

İspat 2a_D: $1+2+3+\ldots+n = n\,(n+1)/2$ eşitliğini kullanarak

Puan: 2

- $\sum_{k=1}^{n} 2k-1 = n^2$, $2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n^2$, $2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n^2$

  $$2\cdot\frac{n\cdot(n+1)}{2} - n = n^2 + n - n = n^2$$

- $1\ +\ 3\ +\ 5 +...+ 2n-1 = n^2$ öte yandan $2 + 4 + 6 + ...+ 2n = 2(1 + 2 + 3 + ...+n)$

  $\underline{+\ 1 + 1\ +\ 1\ +...+\ \ 1\ \ = n}$ $= n.(n+1) = n^2 + n$

  $2 + 4 + 6 + ...+ 2n = n^2 + n$

- $2-1 + 4-1 + 6-1 + \ldots+ 2n-1 = 2 + 4 + 6 +\ldots+ 2n - n.1 = n\,(n+1) - n = n\,(n+1-1) = n^2$

- 1 + 1 + 2 + 1 + 2 + 3 + 3 + 4 +…+2n-1 = n^2;   2 (1 + 2 +…+ n) = n^2 + 1

  $2 \cdot \frac{n \cdot (n+1)}{2} = n^2 + 1$, n^2 + 1= n^2 + 1
- 1 + 2 + 3 + 4 + 5 + …+ 2n-2 + 2n-1 = $\frac{(2n-1) \cdot 2n}{2}$

  1 + 3 + 5 +...+2n-1 = $\frac{(2n-1) \cdot 2n}{2}$ - 2(1 + 2 + …+ $\frac{2n-2}{2}$)

  $= 2n^2 - n - 2\frac{(n-1) \cdot n}{2} = n^2$

İspat2a_E: Sayısal örnek vererek:

Puan: 0

- n =1 ise 1 = 1^2 doğru. n = 2 ise 1 + 3 = 4 doğru n = 5 ise 1 + 3 + 5 + 7 + 9 = 25 doğru. O halde ifade doğrudur.
- 1 + 3 + 5 + 7 + 9 + 11 = 36 = n^2 ; 11 = 2n-1 , n = 6, n^2 = 36

2.*b) Herhangi 3 tane ardışık sayı içerisinde her zaman üçe bölünebilen bir sayı vardır.*

İspat 2b_A: Doğrudan ispat

Puan: 2

- n n+1 n+2. Üçe bölünen sayılar her üç sayıda bir tekrar eder. Üç ardışık sayıda mutlaka üçe bölünen bir sayı olmalıdır.
- Tüm tamsayıları üçerli gruplara ayırdığımız zaman her grupta üçün katı vardır çünkü üçe bölünen her iki ardışık sayı arasında iki sayı vardır. 3 4 5 6 gibi.
- Üçün katları olan sayılar öncekine üç eklenerek bulunur. Ardışık sayıları yazdığımızda üçe bölünenler arasında iki sayı kalacağını görürüz.

  n n+1 n+2 n+3 n+4 n+5 n+6; n+1 n+2 n+3 herhangi üç sayı.

İspat2b_B: a, a+1, a+2 üç ardışık tam sayı olsun. a = 3k ise a üçe bölünür. a= 3k+1 ise a+2 üçe bölünür. a=3k+2 ise a+1 üçe bölünür. Her durumda üçe bölünen bir sayı vardır. (Durum analizi)

Puan: 2

- Doğrudur. Eğer ilk doğal sayı bölünmüyorsa kendine 1 veya 2 eklediğinde üçün katı olmak zorundadır ve bölünür.

Puan: 3

- x x+1 x+2 üç ardışık sayı olsun. x üçe bölündüğünde kalan 0, 1 veya 2 olur. Kalan 0 ise x üçe bölünür, 1 ise x + 2 üçe bölünür, 2 ise x +1 üçe bölünür.
- Üç durum var:

  $x \equiv 0$ mod 3 ise ifade doğrudur,

  $x \equiv 1$ mod 3 ise $x + 2 \equiv 0$, doğru.

  $x \equiv 2$ mod 3 ise $x+1 \equiv 0$, doğru. Her durumda ifade doğrudur.

İspat 2b_C: Toplamın üçe bölündüğünü göstermek. (Önermenin doğru olduğunu ispatlamaz.)

Puan: 0

- $n + n+1 + n+2 = 3n +3 = 3 (n+1)$. Toplamları üçe bölündüğüne göre aralarında mutlaka üçe bölünebilen bir sayı olmalı.

İspat 2b_D: Önermenin yanlış olduğunu göstermek için karşıt örnek vermek.

Puan: 1

- İfade yanlıştır. -1, 0 ,1 sayılarının hiçbiri üçe bölünmez.

İspat 2b_E: Sayısal örnekle ispat.

Puan: 0

- 0, 1, 2 ise 0/3 = 0 ; 1, 2, 3 ise 3/3 = 1 ; 8, 9, 10 ise 9/3 =3 doğru.
- a, a +1, a +2. a' ya değer versen bile kanıtlanır :)

İspat 2b_G: Çelişki yöntemini kullanmak

Puan: 3

- Aksini varsayalım ve sayılarımız *x*-1, *x*, *x* + 1 olsun. O zaman iki tanesi üçe bölününce aynı kalanı verecek ve farkları da üçe bölünecek. Halbuki ikişer ikişer farkları 1 veya 2.

3. *n kişinin katıldığı bir parti düşünelim (n ≥ 2). Bu partide aynı sayıda arkadaşı olan en az iki kişi bulunacağını ispatlayınız.*

İspat 3_A: Güvercin yuvası prensibini kullanmak

Puan: 2

- *n* kişiden herhangi biri en fazla *n*-1 kişi, en az 2 kişi tanıyordur. Kişi sayısı *n*, tanışma kümesi elemanı *n*-3. Herkes farklı sayıda olamaz Çünkü *n* kişi olmasına rağmen *n*-3 tanışma sayısı var.

Puan: 3

- *n* kişiden herhangi birinin en fazla *n* -1 arkadaşı olabilir. Şimdi herkesin arkadaş sayısının diğerlerinden farklı olduğunu varsayalım.

1.kişi	2.kişi …	*n*. kişi
1 arkadaş	2 arkadaş	*n* arkadaş

  Herhangi birinin en fazla *n* -1 arkadaşı olacağı için varsayımımız yanlıştır. Böylece aynı sayıda arkadaşı olan en az iki kişi bulunabileceğini güvercin yuvası prensibi ile ispatladık.
- Bir kişinin en azından bir tane arkadaşı olduğunu düşünürsek ve kişileri 1, 2, 3, ...,*n* ile gösterirsek birinci kişinin 1, ikinci kişinin 2, üçüncü kişinin 3, ..., *n*-1. kişinin *n*-1 arkadaşı olabilir ama *n*. kişinin en fazla *n*-1 arkadaşı olabilir (kendisini çıkartırız). Dolayısı ile *n*. kişinin arkadaş sayısı diğer *n*-1 kişiden birininkiyle aynı olmak zorundadır.

İspat 3_B: Tümevarım yöntemi

Puan: 2

- *n* = 2 ise iki kişi ya birbirini tanıyor ya da tanımıyor. İki durumda da arkadaş sayıları eşit. *n* = 3 ise ya kimse birbirini tanımıyordur, ya herkes birbirini tanıyordur, ya da birbirini tanıyan iki kişi vardır. Her durumda arkadaş sayısı aynı olan en az iki kişi var. İfadenin *n*-1 kişi için doğru olduğunu düşünelim ve son bir kişi partiye gelsin. Eğer kimseyi tanımıyorsa hala doğru olacak çünkü birbirini tanıyan en az iki kişi var. Bu şekilde devam edebiliriz.

İspat 3_C: Graf (çizge) teorisi kullanarak (problemi bir graf ile temsil ederek):

Puan: 1

- Her noktayı farklı sayıda bağlantıyla rastgele bir noktaya götüren kapalı bir ağ tasarlanamaz.

İspat 3_D: Genelleme yapmadan küçük *n*'ler için denemek.

Puan: 0

- Mesela iki kişi düşünelim ve birbirlerinden başka arkadaşları olmasın. Tesadüfen yoldan geçerken partiyi gördüklerini düşünelim. Bu şekilde en az iki kişinin birer arkadaşı (birbirleri) yani aynı sayıda arkadaşı vardır.
- Eğer *n*>2 ve *n* bir çift sayı ise aynı sayıda arkadaşı olan iki kişi bulunabilir. *n* = 3 durumunda doğru olmaz.

Puan: 1

- *n* = 2 ise bu iki kişi birbirini tanıyordur.
  n = 3 ise bunlardan biri parti sahibi diğerleri ya sadece parti sahibini ya da birbirlerini biliyorlardır...

B.2. Rubric for PEE

Not: *Yanıtların yanında parantez içindeki sayılar öğrenci numaralarını göstermektedir.*

İfade 1: *Bir doğal sayının karesi çift sayı ise kendisi de bir çift sayıdır.*

İSPAT 1A: n tek sayı ise: $n = 2k + 1$, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ tek sayı (çift + çift + tek) n çift sayı ise: $n = 2k$, $n^2 = (2k)^2 = 4k^2 + 4k$ çift sayı (çift+çift) n^2 çift olduğuna göre n de çift olmak zorunda.

Seçim A (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (178)
- n=2k çifti doğruladı. n=2k+2 için de yapmalıydı. (156)
- Bize sorulan soruda n^2 çift ise n'in çift olduğudur. Ama ispatta n çift ise n^2 çifttir ispatlanmış. (6)

Puan: 1

- İspatın birinci kısmı doğru çünkü p→q ile ¬q→¬p aynı şeydir. Yani onu göstermek yeterlidir (n tek ile başlayarak). Ama n'i ele alarak n^2'ye ulaşıp test etmesi her zaman doğruyu söylemez. (12)
- Önce n'i tek sayı olarak seçmiş p→q olarak sorulanı ¬q→¬p olarak bulmuş yani doğru. n'i çift olarak seçmiş doğru bir ispat seçimi olmuyor, zaten ispatlamaya çalıştığı n'in çift olduğu. (14)
- n tek sayı diye başladığı yer yeterlidir. n'i çift sayı olarak kabul ederek ispata başlayamayız bu soru için. (26)

Puan: 2

- Her iki durumda da doğrudur. Tek ve çift kombinasyonunda başladığı için sonuç çifttir ama k'nın Z'den bir sayı olduğunu belirtmesi gerekir. (21)

<u>*Seçim B*</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 1

- İfade bu soru için doğru açıklanmış fakat yapılan ispat *n* tek ise n^2 tektir, *n* çift ise n^2 çifttir'in ispatıdır. (13)

Puan: 2

- İspatta hem olayın kendisinin doğru olduğu, hem de tersinin doğru olduğu gösterilmiştir. (15)

Puan: 3

- Açıklama yok. (151, 174, 132, 182)
- İkna oldum. (143)
- Doğruluğunu direk gösteriyor. (22)
- *n* ile sayının farklı değerler alabileceği ve tüm durumlarda geçerli olduğu gösterilmiş. (154)
- İspat doğru ve genel terimlerle yapılmış. (163)
- İfadede genel terimler kullanıldığı için ve bütün sayıları sağladığını kabul ettiği için. (170)
- Tüm durumları incelemiş. *n* tek ise çelişki bulacağımızı da göstermiş. (138)
- *n* tek ya da çift olabilir, iki durum da incelenmiş. (149)
- Bir doğal sayı ya tek, ya çift olduğundan, tek sayıların karesinin tek, çift sayıların karesinin çift olduğunu göstermek soru için yeterlidir. Sadece çiftin karesi çift bir sayıdır. (186)
- Tüm durumları incelemiş. (181, 185)
- Durum analizi doğru yapılmış. (183)
- Her iki case'i de içermiş, daha matematiksel. Diğerleri zaten *x* çifttir diye başlıyor. (184)
- Tüm durumları incelemiş. (10)
- Tek sayının karesinin çift olamayacağını göstermiş. Bütün sayılar ya tektir, ya çifttir. Sanki her sayıya tek tek bakmış gibi. (7)
- Tüm durumları itina ile incelemiş (8)

- İki durum için de incelemiş. Sonuçta n^2 çift ise n hakkında teklik/çiftlik belirtmesi gerekiyordu. Bunu yapmış. Sadece tek olduğu durum bile yeterli. (p→q = ⌐q→⌐p) (9)
- n sayısını her iki durum için de incelemiş. (11)
- Çok açıklayıcı tek tek neden böyle olduğu göstermiş. (1)
- Bir doğal sayı ya tektir ya çifttir. 3. Seçenek yok fakat A→B ddemek B→A demek değil her zaman fakat bu soruda B→A. (18)
- Bir sayı tek ya da çift olabilir. Burada iki yolun sonuna gidip sonucu göstermiş. (19)
- Çünkü case analysis yapılmış ve olabilecek tüm durumlar denendiğinde cevap bulunmuştur. (20)
- Her iki durum da değerlendirilmiş. (23)
- n tek veya çift sayıdır. Her iki durumu da ele almış. (24)
- n’i tek sayı seçtiğimizde n^2 sadece tek olabiliyor. n'i çift sayı seçtiğimizde n^2 sadece çift olabiliyor. Bu nedenle n çift olmak zorunda. (25)
- p→q ifadeler (difrect proof) p’yi kabul edip q’nun doğruluğunu ispatlarız. Ya da ⌐q’yu assume edip ⌐p’yi de gösterebiliriz. Bu iki durum gayet iyi yapılmış. (17)
- Çift sayı x çift sayı/tek sayı=çift sayı. (179)
- Yapılan ispatta bir hata yok. Sadece $(2k)^2 = 4k^2 + 4k$ da $4k$ fazla ama bu sonucu değiştirmiyor. n^2 çift → n çift oluyor. (173)
- Fakat $(2k)^2 = 4k^2$ olacak, bir yanlış yapmış. (135)
- k’nın nasıl bir sayı olduğu belirtilmemiş $k \in$? gerisi doğru. (4)

<u>*Seçim C*</u> (ispat yanlıştır)

Nedenler:

Puan: 0

- $(2k)^2 = 4k^2 + 4k$ yazılmış. $(2k)^2 = 4k^2$ olmalıydı. (161,162,164,176)
- $(2k)^2 = 4k^2$ dir. Bu aşamada hata yapılmıştır. (2, 5, 16)
- İkinci kısımdaki $4k$ fazla. (167)
- $(2k)^2 = 4k^2$ olmalı. $+4k$ nerden geldi? (187)
- Aslında tersten düşünse doğru olabilirmiş ama işlem hatası da var $(2k)^2 \neq 4k^2 + 4k$. (3)

- İspatta eksiklik var. k tanımlanmamış. k tamsayı ise doğru diyebiliriz. k=1/100 için çift tek denilemez. $k \in Z$ demeliydi. (177)
- n^2 çift ise n'in çift olduğunu değil, n çift ise n^2 nin çift olduğunu göstermiş. Her zaman doğru olma mecburiyeti yok. Yanlış. (145)
- İspatta n sayısını tek ve çift olarak ayırmamalı, ispata "n^2 çift ise" diye başlamalı. (180)

<u>İSPAT 1B:</u> n tek sayı olsun. $(2k+1)^2 = 2m$, $4k^2 + 4k + 1 = 2m$. Sol taraf tek sayı, sağ taraf çift. Çelişki elde ettik. Demek ki n çift olmalı.

<u>*Seçim A*</u> (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan 0:

- İspat eksiktir, n çift seçilerek doğruluğu gösterilmelidir. (15)

Puan 1:

- Bu soru çelişki yöntemi ile yapılmaz. (184)

Puan 2:

- Proof by contradiction'dan ispatlamış tersinin doğru olabileceğini söyleyip çelişki elde etmiş. Doğrudur ama m ve k'nın tamsayı olduğunu belirtmesi gerekir. (21)
- İspatta eksiklik var. k, m tanımlanmamış. $k, m \in Z$ gibi bir tanımlama olmalıydı. Bu ispat çelişki elde edilerek yapılıyor. Ama çelişki elde ettik, demek ki n tek sayı olamaz kısmı doğru. (177)
- Her durum için doğrudur. (156)

<u>*Seçim B*</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 2

- Yapılan ispatta yanlışlık yok. "Counter example" vererek kanıtlamaya çalışmış. (173)

Puan: 3

- Açıklama yok. (174, 178, 179, 132, 135, 182)
- İfadede genel terimler kullanıldığı için ve bütün sayıları sağladığını kabul ettiği için. (170)
- İkna oldum. (143)
- Doğru gibi geliyor. (16)
- Doğru (esasen *k*, *m*'nin doğal sayı olduğu belirtilmeli). (187)
- Kullanılan yol ve yapılan işlem doğru ama *k* hala eksik. (4)
- *n* için iki durum söz konusu; ya tek ya çift. Tek sayıyı sağlamadığına göre (işlemde), *n* çifttir. (5)
- İspat tekniği uygulamış. (11)
- Bir çelişki bulunarak ilk durumun yanlış olduğu gösteriliyor. Böylelikle diğer durumun doğruluğu ortaya çıkıyor. (163)
- Tersi kabul edilip çelişki elde edilmiş. İspat doğrudur. (164)
- Çelişki doğru bulunmuş. (176)
- Çelişki yolu ile ispat yapılmış, gerçi başlangıcı daha düzgün yazılabilirdi. Ayrıca eğer *n* çift olsaydı durumun doğru olacağı da belirtilmeliydi. (138)
- Çelişki yoluyla doğru ispat yapılmış. (145)
- Çelişki elde etmek, matematiksel ispatlarda sıkça kullanılan doğru ve pratik bir yöntemdir. (149)
- Proof by contradiction (186, 180)
- Contraposition yapmak istemiş, ama contradictionla bitirmiş. Metodları karıştırmış. (185)
- Çelişki ile ispatlamış 1A ile benzerlik gösteriyor. (10)
- Tersini kabul ettiğimizde sonuç yanlış çıkarsa asıl ifademiz doğru olur. (2)
- Bu ispatta tek sayının karesinin çift olamayacağı ispatlanmış. Yani bir sayının karesi çiftse kendisi de çift olmak zorundadır. (6)
- Bu bir prof yöntemi (contradiction). (8)
- Çelişki yolu ile ispatlamış ama biraz açıklaması eksik (başta ve sonda). (9)
- Burada aslında çelişki elde etmiş. Yani başta assume etmesi gerekirdi (*n* tek ve onun karesi çift diye). (12)

- n ya tektir ya çifttir. Tek olamayacağına göre çifttir şeklinde açıklamış. Doğrudan (13)
- p→q sorulurken ¬q (assume etti) kabul etti çelişki buldu. Assume edilen şey yanlıştır dedi ispat doğru. (14)
- Çünkü zıttının doğru olduğunu varsaydık fakat sonuç yanlış çıktı onun için ¬(¬A)=A değilinin değili yani kendi (baştaki) önerme doğru. (18)
- Çelişki yöntemi de uygun. (22)
- Çelişki yöntemi ile ispatlanmış. (23)

Seçim C (ispat yanlıştır)

Nedenler:

Puan: 0

- Açıklama yok. (7, 17, 20, 25)
- Tek sayının karesinin çift olduğu kabul edilmiş. Bu yanlış. (161)
- Tek sayının karesinin tek sayı olduğunu ispat ediyor ama karesi çift sayı olan sayının çift sayı olduğunu ispat etmiyor. (3)
- Bir önceki ispattan yararlanırsak ispatın veya temel iddianın yanlış olduğunu bilir. (19)
- *n*'nin tek olduğu durumu almış. Buradan çifte geçemeyiz. (24)

Puan: 1

- Çelişki yöntemi ile ispat tam olarak bu değildir. (26)
- İfadeler tam olarak iyi tanımlanmamış. (183)
- Tek sayı olarak seçilen *n* hiç kullanılmamış. (151)
- Kabul ettiğimiz '*n* tek sayı olsun ifadesi ispatın hiçbir yerinde kullanılmamış. (162)
- *n*'den bahsediyorsunuz, *m*'e geçiyorsunuz. Anlamadım. (167)
- Yazımda hata var. *n*'in ne olduğunu anlamak çok güç. (181)

Seçim D (bir fikrim yok)

Nedenler:

Puan : 0

- *n*'i nerede kullandığını anlamadım. (1)

<u>İSPAT 1C:</u> $n^2 = n \cdot n = 2k$. Burada k çift sayı olmak zorunda çünkü $2k$ tam kare: $k = 2m,\ n^2 = 4m,\ \sqrt{n^2} = \sqrt{4m}\ ,\ n = 2\sqrt{m}$, dolayısı ile n çift olur.

<u>Seçim A</u> (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Olabilir, neden olmasın? (4)
- k=2m için doğru. k=2m+2 için de denemeliydi. (156)
- Çünkü 2k'nın tam kare olmasına gerek yok. (17)
- n'in tek olduğu durum gösterilmemiş. (23)

Puan: 1

- Açıklama yok. (154)
- k bazı durumlarda tek sayı da olabilir. Çünkü çift x tek çifttir. (163)
- m'in tamsayı olduğunu söylemesi gerekirdi. (13)
- n=0 ? (182)

Puan: 3

- m=1/81 için doğru değil. Ayrıca m pozitif şekilde tanımlanmamış ve kareden çıkabilmesi için tam kare olması gerekiyor. m tam kare ve pozitif ise doğru. (177)
- $4 \mid n^2 \rightarrow 2 \mid n$ dese daha iyi olur. (132)
- k çift sayı olmak zorunda kısmı yeterince açık değil, o zaman neden $\sqrt{m}$? (149)
- Eksik. "Burada k çift sayı olmak zorunda çünkü 2k tam kare" cümlesi bir kanıta dayandırılmamış. (181)
- $n = 2\sqrt{m}$ de m tam kare olmayabilir, dolayısı ile n irrasyonel olabilir. (5)
- Çünkü n^2 nin çift olmadığı, tek oldğu durumlarda $n^2 = 2k$'ya eşit olmaz n^2 çift olduğu zaman bu ispat doğru fakat birşeyler eksik. (18)
- $\sqrt{m}$ sayısı tam olmayabilir, o zaman n doğal sayı olmaz. (25)

<u>*Seçim B*</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Negatif sonuç çıkabilir. (22)

Puan: 1

- OK. n = 0 durumu ayrı düşünülmeli sanki. (187)
- n, m, k sıfır olabilir. Sıfır çift kabul edilirse doğrudur. (184)
- Zor yolu tercih etmiş. (185)
- k bir çift sayı olmak zorunda ifadesi doğru değil, ama bunun dışında ispat doğru. (3)
- Benim diğer kâğıtta anlattığım ispat. (21)

Puan: 2

- Açıklama yok. (12, 16, 145, 161, 164)
- Doğrudan ispatlamış. (24)
- Direct proof. (186)
- Mantıklı bir açıklama. p'yi kabul ediyor q'ya ulaşıyor. (14)
- Teori doğru ama yöntem yanlış bence. Çünkü m=3 olursa iş sakata gelir. (19)
- Yapılan ispatta bir sorun yok. 2k tam kare olabilmesi için k'nın çift olması gerekiyor. (173)
- 2 ile çarpılması gerektiği sonucun çift olduğunu gösteriyor. (176)
- n tek ise ≠ 2k çift sayı dolayısıyla n çift olmak zorunda n çift ise çift x çift = 2 x çift olur. (179)

Puan: 3

- m'in tam kare olması açıkça ifade edilmemiş. (135)
- Fakat m ile ilgili bir şey söylenmemiş. (138)
- Güzel, belki 2 $\sqrt{m}$ ' in tamsayı olacağı belirtilse daha güzel olurdu. (143)

<u>*Seçim C*</u> (ispat yanlıştır)

Nedenler:

- Açıklama yok. (20, 178)
- Kabulleri yanlış almış. (183)
- 3^2 =3.3=9, 9=2k, k=9/2 çift sayı değil. (167)
- k çift olmak zorunda değil. (174)

- Tam olarak yapmak istediğini anlamadım ama $2k$'nın tam kare olması k'nın çift olmasını gerektirmez. (9)
- k'nın çift oluşu m'in çift oluşuna bağlı ki böyle bir bilgimiz yok. (15)
- "m" ne olduğunu belirtmemiş. (151)
- k çift sayı olmak zorunda değildir, Çünkü $2k$ çift ise k tek olabilir. (162)
- $2\sqrt{m}$ gibi bir ifade her zaman çift olmaz. (6)
- k çift olmak zorunda mı? Zaten soru o değil mi bir yerde? (7)
- Burada m'nin tam kare olup olmadığı hakkında fikrimiz yok. m eğer tam kare değilse n doğal sayı olmaz. (26)
- $n^2 = 2k$ çift ise k çift olmak zorunda değildir. Çünkü k tek ise de n^2 çifttir. (180)

Seçim D (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (8, 11, 170)
- Ne yapmaya çalıştığını anlamadım. (10)

Puan: 1

- k'nın neden çift olmak zorunda olduğunu anlamadım. (1)

İSPAT 1D: $n = 2k$ olsun. O zaman $n^2 = 4k^2$ çift.

Seçim A (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- $n \in Z$ olmak zorunda diye bir şey dememiş. (156)

Puan: 1

- İkna edici değil. (22)

Puan: 2

- Çift sayıların karesinin de çift olduğunu göstermiş, bu doğru değildir. (7)

Puan: 3

- *n* sayısının tek olması durumu ihmal edilmiştir. (11)
- İspat eksik, n^2 nin çift olduğu ama *n*'in tek olduğu durumların olmadığı ispatlanmalı. (15)
- n=2k+1 durumu da incelenmelidir. (5)
- Bizim istediğimiz tam tersini bulmak. Yani n^2 çiftse *n* çifttir. (6)
- *n*'in tek olduğu durumlar incelenmemiş. (149)
- $2k^2$ de çift sayı olacaktır ve bir tek sayının karesi olabilir. Bu durum incelenmemiş. (164)
- Sadece çiftse doğru olduğunu gösterir. (183)
- *n* tek sayı olabilir. (185)

Seçim B (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (2, 151, 161, 167, 179)
- Her zaman doğru. (154)
- Kabul edilen ile sonuç tutarlı. (162)
- Çift x Çift = Çift olduğu için doğru. (163)
- $4k^2$ ifadesi, *k* çift ya da tek olsun her zaman çifttir. (173)
- 4 ile çarpıldığı çıkmış. (176)
- *k* değerleri için doğrudur. (1)
- Doğrudan ispat $k \in$? Gerisi idare eder. (4)
- Doğru gibi geliyor. (16)
- Tabii ilk ispattan yararlanarak var olan iki yoldan gidersek doğru olur her zaman. (19)

Puan: 1

- İspatı sonlandıramamış ama yol doğru bence. (3)

<u>*Seçim C*</u> (ispat yanlıştır)

Nedenler:

Puan: 1

- Açıklama yok. (20)
- Böyle bir ispat yolu yok. (8)
- Tam olarak bir ispat olmadığını düşünüyorum. (170)
- Yine k tanımlanmamış. İspat değil gösterim yapılmış. (177)
- Genellemesi mümkün değil. (178)

Puan: 2

- İspat eksik. (23)

Puan: 3

- Karesi çift sayı ise kendisinin de çift olduğunu göstereceğiz. Bu ispat çift sayının karesi çifttir sonucunu gösterir. (174)
- İspat yapmayı bilmiyor. (132)
- İfadenin kendisini değil, mantıksal olarak tersini göstermiş. (135)
- Bu bir ispat olmamış. Sandığımız durumun gösterilmesi gereken kısmını kabulle başlamış. Fakat herhangi bir açıklayıcı bir ifade yok. (138)
- Yani aslında C seçeneği uygun değildi, ama en yakını oydu. Bu ispat değil. (143)
- 1A'daki gibi hata. (145)
- $\sim p \rightarrow \sim q \neq p \rightarrow q$ (denk değildir). (186)
- Önermenin tersini ispatladı. $q \rightarrow p$ nin doğru oluşu $p \rightarrow q$ yu gerektirmez ki! (187)
- n sayısını çift kabul etmiş, halbuki bunu bulmak istiyoruz. (180)
- Bu ifade çift sayının karesinin çift olduğunu gösteriyor. Karesi çift olan sayı çifttir ifadesini değil. (181)
- İfade tersten zorlanıyor. (182)
- It is not a proof. (184)
- n çiftse n^2 çifttir diyemesin. n çift değilse n^2 de çift değildir diye ispatlarsan olur. (9)
- Bu ispat 'n çift olduğunda n^2 de çifttir' in ispatıdır. (13)
- Zaten n'in $2k$ olduğunu ispat etmeye çalışıyoruz. Onu $2k$ olsun diye varsayamayız. (10)
- Bu onun ispatı değildir. Contrapositive kullanılmadı yanlış oldu. (12)

- Soruda sorulan zaten *n*'in çift olduğunu ispatlamak. p→q soruluyor bunu q→p diye çeviremeyiz. Böyle yapabilseydi soru şöyle olurdu: p↔q (14)
- Çünkü p→q ifadesi ispatlanırken sadece p'yi assume edip q'yu gösterebiliriz. Tersini göstermek için ¬q'yu assume edip ¬p'yi gösteririz. (17)
- Çünkü n=2k+1 olduğu zamanda n^2 çift midir değil midir sorgulamıyor çünkü karesi çift olan sayı tek de olabilir sorgulamak lazım. (18)
- İspatta assume edeceğimiz kısım karesinin çift olduğuyla başlamamız gerekiyor. (21)
- p→q önermesi q→p diye ispatlanamaz. (24)
- *n* sayısının neden tek olamayacağını göstermesi gerekir. (25)
- *n*'nin çift olduğunu kabul ederek ispat yapamayız. n^2' in çift olduğunu kabul edebiliriz. (26)

İSPAT 1E: Ç = {2, 4, 6, 8, ...}
n^2 = 4 ise n = 2, n^2 = 16 ise n = 4, n^2 = 36 ise n = 6 .... n^2 = 114 ise n =12.

Seçim A (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- n^2 =144 yerine n^2 = 114 yazılmış. $n \neq 12$. (154)
- Tek sayılarda da incelenmelidir. (5)

Puan: 1

- n^2 = 114 ise n = 12 yazılmış. n^2 = 144 olmalıydı. Ancak diğer örnekler için doğru. (161)
- Ç kümesi için doğru ama negatif tam sayılar kümesini içermiyor. (163)
- *n* negatif de olabilir. (179)
- Göstermiş. (143)
- Doğru ancak tam değil. (145)

Puan: 3

- Zaten bir ispat değil. Bazı sayılar için doğru olduğunu gösterir. (151)
- Bu sayı için sağlanmış ama her durum için doğruluğuna bakmak gerekir. Genelleme yapılmamış. (170)
- Her durum için doğru olabilmesi için bütün sayıların denenmesi lazım. Bu da imkânsız gibi duruyor. (173)
- Doğru olduğu bir durum gösterilmiş. (178)
- Tüm çift sayılar için göstererek genelleme yapması gerekirdi. Ona da bu şekilde deneyerek ulaşamaz. (2)
- Bütün küme için ispat yapılmamış. (3)
- Tüm *n*'ler için göstermek gerekir. (4)
- Bazı kare sayıları kareköklerinin de çift olabileceğini görüyoruz. (7)
- Tüm sayılar incelenmemiş. (8)
- İspat değil bu, genelleme yapamaz. (12)
- Bu da sadece verilen ifadeyi algılamaya yarayan gözlemlerdir. (17)
- Yine burada tüm doğal sayılar hakkında bir fikir elde edemeyiz. (26)
- Sadece üç tam sayı için sonucu kontrol etmiş. (135)
- Tümevarım yapmaya çalışmış ancak yöntemi bilmediğinden sanırım olmamış. (138)
- Her çift sayı için doğruluğundan emin olmalıyız. (186)
- Sadece dört sayı için denemiş. Genellendiremez. Tümevarım da yapmamış. (187)
- Söylenen durumlar için doğru olduğunu gösterir. Ama 1468…. Gibi bir sayı için hiçbir şey söylemez. (181)

Seçim B (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Tümevarım kullanılmış. (1)
- Bir çift sayının karesi çifttir. (16)
- Eveet. Çünkü çözüm kümesi doğal sayıları içeriyor. (19)

<u>Seçim C</u> (ispat yanlıştır)

Nedenler:

Puan: 0

- n^2 = 114 ise n= $\sqrt{114}$ ≠12. 12^2 = 144. (156)
- İspatta hem n^2=2 ifadesi yer almıyor hem de n^2=114 ise n=12 olmaz, n^2=144 ise n=12 olur. (162)

Puan: 1

- n^2 =2? (182)

Puan: 2

- Açıklama yok. (20, 22, 23)
- Bu ispatta da *n* çift ise n^2 çifttir ispatı yapılmış. (13)

Puan: 3

- Bu bir ispat değildir. Farklı durumlar incelenmiş ancak tüm durumlar incelenmemiş. (164)
- İspat böyle olmaz ki. (167)
- Her sayı için doğru mu? Anlayamayız. (174)
- Genel bir ispat değil. Örnek üstünde durulmuş. (176)
- İspata bu şekilde devam edip genelleme yapılmalıydı. n=15 için nedir? Ayrıca n^2=4 ise n= ±2 olur. (177)
- Yalnızca birkaç durum. (9)
- Yine sadece birkaç durum değerlendirilmiş. Tüm sayıları tek tek incelemeli, ki imkansız. O yüzden ispat doğru değil. (10)
- Değer vererek ispatlanamaz. Tüm doğal sayılar göz önüne alınmalı. (11)
- Sonsuz sayıda çift sayı var, genelleme yapmamış. (14)
- Tüm sayılar için doğruluğu ispatlanmadıkça ki $n \to \infty$ olduğundan yapılamaz, doğru değil. (15)
- n^2 =114'den sonraki kareli ifadelerin karekökünün tek olduğunu göstermiyor, bu tarz bir ispatın olması için kümedeki bütün elemanlar için göstermemiz lazım. (18)
- İspatta örneğe yer yok. (21)

- 1D ile aynı. (24)
- Örnek vererek ispat yapılmaz. (25)
- Böyle ispat olmaz. (132)
- Örneklerden genelleme yapılmaz. (149)
- Bir sonuca ulaşılmamıştır. (180)
- İspat değil bu. (183)
- It is not a proof. (184)
- Örnek verilerek ispat olmaz. (185)

İfade 2: 1 + 3 + 5 + ... + 2n-1 = n^2 eşitliği, n ≥1 tamsayıları için doğrudur.

İSPAT 2A: *ST*: son terim, *İT*: ilk terim, Δm: artış miktarı ise $\frac{ST-İT}{\Delta m}+1$ terim sayısı.
Buna göre toplam $\frac{ST+İT}{2}\cdot\left(\frac{ST-İT}{\Delta m}+1\right)=\frac{2n-1+1}{2}\cdot\frac{2n-1-1}{2}+1=n^2$ olur.

Seçim A (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (174)
- Genellenebilir. (178)

Puan: 1

- Daha matematiksel diğerlerine göre. Böyle yapmaya çalışırdım. Lisede de böyle öğrenmiştim. (184)

Puan: 2

- Eğer öğrenci üniversite öğrencisi olsaydı ispat yanlış derdim. Ama lise öğrencisi tümevarım metodunu bilmiyorsa bu tür bir yol izlemesi doğaldır. (20)
- Formülle ispat çözülmez. Formül (zaten?) ispatın sonucundan çıkan bir sonuçtur. (21)
- Formülle ispat olmaz. (22)
- Formül kullanılmamalıydı, açıklayıcı değil. (25)

<u>Seçim B</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 1

- Açıklama yok. (26, 154, 162, 164, 170, 179, 187, 180, 185)
- İspat genel kavramlar kullanılarak doğru olarak yapılmış. (163)
- Metot kullanmış yapmış. (138)
- Olmuş. (143)
- Sıkça kullanılan bir toplam formülü kullanılmış. (149)
- Olabilir, neden olmasın? (4)
- Bütün koşullarda sağladığı için doğrudur. (5)
- Formülle çözüm, doğru. (10)
- Sorunun sorduğu şey ispatlanmıştır. (11)
- Ortalama . terim sayısı olarak ifade etmiş, doğrudur. (13)
- Çocuk ona öğretilen en genel yoldan gitmeye çalışmış. (14)
- Çünkü $\frac{ST+İT}{2}$ ortalama bir terim demektir. $\frac{ST-İT}{\Delta m}+1$ kaç terimin olduğunu gösterir. $\frac{ST+İT}{2}\cdot\left(\frac{ST-İT}{\Delta m}+1\right)$ de genel toplamı verir. (17)
- Çünkü kullanılan formül doğruluğu ispatlanmış bir teorem. Teorem kullanarak ispat yapılabilir. Uygulandığında doğru olduğu ortaya çıkıyor. (18)
- Genel bir formülü kabulle yapmış ama olsun. (19)
- Formül kullanarak ispat tam sonucu vermiş. (23)

Puan: 2

- B verdim ama ispattan ziyade formül uygulaması yapmış. (132)
- Toplamın neden $\frac{ST+İT}{2}\cdot\left(\frac{ST-İT}{\Delta m}+1\right)$ ifadesiyle verildiği belirtilmese de doğru bir kanıt. (135)
- Ama $\frac{ST-İT}{\Delta m}+1$ in ispatı verilmiş ise doğru. (181)
- Formüller üzerine kurulu ispat açıklayıcı olmayabilir. (2)
- Lisede öğrendiği formülü uygulamış. (8)

Puan: 3

- Formül bilgisini kabul edersek işlem doğru ama işlem yapabilme yeteneği ölçülmüş oluyor, ispat yapabilme değil. (7)
- Yaptığı şey formül çıkarmak aslında ispat değil. (24)

Seçim C (ispat yanlıştır)

Nedenler:

Puan: 0

- Doğru değil. (145)
- $n\,(n\text{-}1) \neq n^2$. Different and wrong. (16)

Puan: 2

- Kabul ettiği şeyin nerden geldiğini söylememiş. (151)
- İspat gerektiren paketler kullanmış. $\left(\frac{ST - \dot{I}T}{\Delta m} + 1\right)$? (186)
- İspat yok bildiklerini kabul etmiş, genel formül yazmış. (183)
- Sonuç olarak varılan sayı $n\,(n+1) \neq n^2$. Ayrıca formülün ispatı yok. (3)

Seçim D (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (15, 156, 161, 176, 182)
- Formülün toplam değeri verip vermediğini çıkartamadım. (173)
- $\frac{ST + \dot{I}T}{\Delta m}$ neyi ifade ediyor anlayamadım. (1)
- Çok karmaşık geldi bu şekilde ispatlamak. (11)

Puan: 1

- Sadece yapmış ama niye yapmış hiç açıklama yok. (6)
- Formülünü anlamadım. Zaten bu yolla ispat yapmak doğru değil. (9)
- Formül kullanılarak yapılmış. Yalnız bu formül kuraldan önce mi sonra mı emin değilim. Kontrol etmem lazım. Temel ispat ilkelerine göre değil. (177)

<u>İSPAT 2B</u>: 1 + 2 + 3 + ...+ 2n-2 + 2n -1 = $\frac{(2n-1)\cdot 2n}{2}$ dir , o halde istenen toplam da:

$$\frac{(2n-1)\cdot 2n}{2} - 2(1+2+...+\frac{2n-2}{2}) = 2n^2 - n - (n-1)n = n^2 \text{ olur.}$$

<u>Seçim A</u> (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (170,178)

Puan: 2

- Formülün nereden çıktığını da bilmek gerekir. (22)
- Formül kullanılmış, açıklayıcı değil. (25)

<u>Seçim B</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 3

- Açıklama yok. (161, 164, 174, 187, 180, 182)
- Yalnız yine n∈ Z denilmemiş. Ama artış miktarından anlaşılabiliyor. Diğer kısımları doğru bir ispat. (177)
- Süper ispat. (132)
- Kanıt doğru fakat daha açık ifade edilebilir. (135)
- Doğru tüm toplamdan çiftlerin toplamını çıkarmış. (138)
- Aynı ispatın [2A] daha basit ifade edilişli gibi zaten. (143)
- Sıkça kullanılan toplam formüllerinin varyasyonları kullanılmış. (149)
- Açıklama yok. (2, 11, 26)
- Doğruluğu gösteriyor ama ilk başta eşitliğe nereden ulaşıldığı yok. (3)
- Sade matematik işlemlerle doğru sonuç elde edilmiş. (5)
- Genel toplamdan çift sayıların toplamını çıkararak tek sayıların toplamını bulmuş. Ama yine de çift sayıların toplamını yazarken işlemlerini biraz daha açıklamalı. (6)
- 2A yine: işlem yapabilme! (7)

- Bildiği bir formülü kullanarak yapmış ama iyi bir yöntem değil. (9)
- Tüm durumlardan çiftleri çıkarmış, doğru. (10)
- $2n$-1'e kadar olan sayılardan çiftleri çıkarmış. (13)
- Tüm sayıların toplamından çift sayıları çıkarmış. Bildiği yöntemi kullanmış. (14)
- Tüm sayılar için doğrudur. (15)
- Çünkü tüm sayılardan çift sayıların toplamını çıkarıyor. Bu da bize tek sayıların toplamını verir. (18)
- Formül kullanılarak tam sonuç elde edilmiş. (23)
- Yine formül çıkartmışlar. (24)

Seçim C (ispat yanlıştır)

Nedenler:

Puan: 0

- Açıklama yok. (20, 154, 185)
- Açıklama yok. (154)
- Terimler toplamı yanlış uygulanmış. (156)
- $(1+2+...+\frac{2n-2}{2})$ Yani bu ifade son terimden önceki sayıların toplamını çıkarmak için kullanılır. Sonuç negatif bir sayı çıkar bu durumda. (162)
- Artış miktarı 1 olduğu için. (163)
- Bu proof bile değil. (184)

Puan: 1

- Yine formül kullanılarak yapılmış. (21)
- $\frac{n(n+1)}{2} = 1 + 2 + \cdots + n$ olduğunu bil(m)iyoruz. (186)
- Başka bir kabullenmeden faydalanmış. (183)

Seçim D (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (145, 181, 151, 176, 179)
- Nasıl yapıldığını anlamadım. (173)

- Ben bile anlamadım. (4)
- Fikrim yok. (8)
- İspatı anlamadım. (19)

Puan: 1

- $2(1+2+...+\frac{2n-2}{2})$ olan bölümü neden yaptığını anlatmıyor. (1)

<u>İSPAT 2C</u>: S(n) = 1 + 3 + 5 +...+ 2n-1 = n^2, S(n+1) = 1 + 3 + 5 +...+ 2n-1 + 2n+1 = n^2 + 2n +1 S(n) + 2n+1 = $(n+1)^2$, yani ifade doğrudur.

<u>Seçim A</u> (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- S(n+1) için de demeli. (156)

Puan: 1

- Açıklama yok. (170,174)
- Açıklama yetersiz. (22)
- Belli sayılar için doğruluğunu göstermiş. (151)
- Temel basamağı yapması gerekirdi. (185)

Puan: 2

- n=1 (induction basis) kontrol edilmemiş. (149)
- Induction yapmaya çalışmış. Olabilir. (184)
- Sadece (n+1) için de doğruluğunu gösteriyor. (1)
- Eksik ifadeler var, bütün koşullar değerlendirilmemiş. (5)
- Eksiktir, n=1 için doğru olduğunu da göstermeli. (15)
- İlk başta 1 için 2 için de sağlayıp sağlamadığına baktıktan sonra S(n)'e baksaydı tam doğru olacaktı. (20)
- İspatın biraz daha geniş olması gerekmekte. Örneğin önce 1 için doğru mu değil mi ona bakmamız gerekmektedir. (26)

<u>*Seçim B*</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (2, 161, 162, 164)
- İfadenin doğruluğunu kanıtlamak için ifadeyi kullanmış. (3)
- OK. (145)
- Detaylı bir şekilde durumlar özetlenmiş. (176)

Puan: 1

- İlk durumu koruyup ikinci durumda da doğruluğu kanıtlanmış. (163)
- Bir ifade doğruysa dizideki başka bir terim için de doğru olmalı. (177)
- Induction- tümevarım. (179)
- T.V. [Tümevarım] (132)
- Matematiksel bir yolla açıklamış. (24)
- *n* için doğruluğunu kabul ederek *n*+1 için doğruluğunu göstermiş. (6)
- Ben de böyle yapmıştım (induction method). (8)
- Tümeverım kullanılmıştır. (11)
- Burada induction yapılmış ve doğrudur. (12)
- Tümevarım ile yapmaya çalışmış, Hoş, güzel. (19)
- Tümevarım yöntemi ile ispat yapılmıştır. (23)
- Tümevarım'ı kullanmış, doğru. (25)

Puan: 2

- Tümevarım bana sorunlu geliyor. Her ne kadar bu şekilde ispat gösteriliyorsa da! Kısmen doğruluğunu gösterir bence, daha çok. (B- -A) (7)
- Ama açıklaması biraz eksik, ben de kağıdımda bu yolla ispatlamaya çalışmıştım. (9)
- Tümevarımla ispatlamış ama başında 1 için de doğru olduğunu ispatlamalıydı. (13)
- İspat doğru ama P(1) için de ispatın doğru olduğunu göstermeli. (14)
- Tümevarım yöntemiyle ispatlanmış (temel adım eksik). (17)
- Tümevarım kullanmış. Tek eksiklik hipotezin doğruluğunu kabul ettiğini söylemesi ve *n*=1 için doğruluğunu göstermemiş olması. (138)
- Induction kullanmış. Ama 1 için de göstermeliydi. Yazım eksik. (181)

- Tümevarım, ilk step eksik. (182)

<u>Seçim C</u> (ispat yanlıştır)

Nedenler:

Puan: 2

- Açıklama yok. (154,178, 187, 180)

Puan: 3

- Induction'a benziyor ama eksiklik var. Daha iyi ifade edilmeli. Basic step'i yok. (10)
- B eksik ispat. Onun için seçim C. *n* için doğru olduğunu kabul eder, *n*+1 için de doğru olduğunu gösterir isek ispat olur. Fakat en küçük eleman olan 1 için doğru olduğunu göstermek gerekiyor (basis step). *n*→*n*+1 gösterdiğimiz için 1'in doğru olması 2 için doğru, 2→3, 3→ sonsuza kadar gider. (18)
- İnduction'a benziyor ama ilk ana stepi yapılmamış. 1 için doğru olduğunu göstermemiş. (21)
- Aslında kanıt yanlış değil eksiktir. Tümevarım adımı kapalı bir şekilde ispatlanmış fakat 'induction basis' e değinilmemiş. (135)
- 1 için doğru olduğunu göstermeliymiş. Burada *n* için doğru ise *n*+1 için de doğrudur diyoruz. (143)
- Induction basis gösterilmemiş. (186)
- Tümevarım hatası. Baz adım açıklanmamış. (183)

<u>İSPAT 2D</u>: $\sum_{k=1}^{n} 2k - 1 = n^2$, $\sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 = n^2$, $2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n^2$

$$2 \cdot \frac{n \cdot (n+1)}{2} - n = n^2 + n - n = n^2$$

<u>Seçim A</u> (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 2

- Formül ispatı matematikte pek ispat sayılmaz. (22)
- Formül kullanmış, açıklayıcı değil. (25)

<u>*Seçim B*</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 3

- Açıklama yok. (2, 6, 11, 26, 132, 149, 187, 180, 181, 154, 162, 164, 170, 174)
- Doğru. (156)
- Güzel, mantıklı bir ispat. (177)
- Toplam sembolü ile gayet açıklayıcı. (179)
- “,” ler [virgüller] ↔ olarak görülürse doğrudan bir ispat verilmiş. (135)
- Bu eşitlik soruluyor, kabul ederek başlayamayız. (182)
- İşlemleri açık açık yaparak sonucunun neden öyle çıktığını gösteriyor. (1)
- Σ formüllerini hatırlamıyorum ama ifade doğru. (3)
- Olur. (4)
- Farklı ama doğru yöntemle (toplam işareti) doğru işlem. (5)
- İşlem yapabilme. (7)
- Bu yöntem de doğru. (9)
- Doğru. (10)
- Böyle bir ispat düşünmemiştim ama mantıklı göründü. Doğru olmalı. (12)
- Toplam formülünü doğru kullanmış. (13)
- Mantıklı. (14)
- Tüm şartlarda doğrudur. (15)
- Straightforward, do not see anything wrong. (16)
- Toplam sembolü ve kuralları kullanılarak toplam doğrulanmış. (17)
- Yapılan yöntem doğru, ispat geçerli yapılan işlemler matematikte kullandığımız işlemler. (18)
- Yine bir hoş yol. Yalnız toplama işareti bazen kafa karıştırıcı olabilir. (19)

<u>*Seçim C*</u> (ispat yanlıştır)

Nedenler:

Puan: 1

- Olması istenen duruma uygun koşullar yazıp toplamış. Ancak bir ispat yöntemi değil. (138)
- Göstermek istediğimiz kullanılmış. (186)

- Başka genel formülleri kullanmış. (183)
- $\sum k = \frac{n(n+1)}{2}$ olduğunu nereden biliyoruz? (185)
- Yine formül kullanmış. (21)

Seçim D (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (20, 24, 145, 161,163)
- Fakat eğer yapılan ispat doğru ise, en ikna edici "proof" budur derim. (173)
- Parçalara ayrılarak işlemler detaylandırılmış. (176)
- Σ işaretini bilmeden böyle çözülmez. Kısıtlı bir kesim bunu anlar. Bunu seçmezdim. (184)

Puan: 2

- Aslında ispat var ama baştan eşitlik doğruymuş gibi bir notasyon kullanmak onu matematiksel olarak yanlış hale getirmiş. (143)

İSPAT 2E:

$$\begin{array}{r} 1 + 3 + 5 + \ldots + 2n\text{-}1 = S \\ \underline{+\ 2n\text{-}1 + 2n\text{-}3 + 2n\text{-}5 + \ldots + 1 = S} \\ 2n + 2n + \ldots + 2n = 2S \end{array}$$

n tane terim var: $n.2n = 2S$, $S = n^2$

Seçim A (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (154)
- Sadece iki ifadenin toplamı bulunarak bir genelleme yapılmaz diye düşünüyorum. (173)

Puan: 2

- *n* tane terim olduğunu göstermeli. (22, 180)
- Gauss gibi yapmaya çalışmış. (184)
- Doğru. (156)

<u>*Seçim B*</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 3

- Açıklama yok. (2, 7, 11, 15, 19, 26, 161, 162, 164, 170, 174, 132, 149, 187, 181)
- İki ifadenin toplamından yola çıkılarak birinin toplamına ulaşılmış. (163)
- Gauss ispatı. (176)
- Hiç aklıma gelmeyen, mantıkla açıklanabilen bir ispat. (177)
- *n* tane terim için doğru. (179)
- Bütün sayılar için ispatlıyor. (3)
- Basit aritmetik işlemlerle doğru sonuca ulaşılmış. (5)
- Çok mantıklı. İspat 2B'de de benzer durum söz konusu. Sadece anlatımı farklı yapmışlar. (8)
- Gayet anlaşılır ve açık. (9)
- Gayet mantıklı bir şekilde açıklamış. (12)
- 2B'deki ispatın hemen hemen aynısı. (13)
- Good and simple proof. (16)
- Gauss'un asıl formül bulduğu mantıkla çözülmüş. Modern olmayan ispatın ta kendisidir. (17)
- Tüm *n*'ler için geçerli, mantık hatası (yanlış işlem) yok. (18)
- Güzel bir şekilde açıklamış. (21)
- 2B ile aynı. (24)
- Açıklayıcı, doğru. Formül yok. (25)
- Basit bir kanıt. (135)
- Bravo. (138)
- Çok güzel. (143)
- Açıkça gösterilmiş. Gauss'un yaptığı gibi. (186)
- Doğru, açık ve öz şekilde verilmiş. (183)

- İspat doğru. (185)

<u>*Seçim D*</u> (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (151)
- Fazla karışık, açıklayıcı değil. (1)
- Ne yapmaya çalıştığını anlamadım. (10)
- Ne yapmak istediğini anlamadım. (14)
- Açıklama yok. (145)
- =S [ikinci satır] Anlamadım? (182)

Puan: 1

- *n* tane terim olduğu açıklanmalı. (6)

İfade 3: Herhangi 3 tane ardışık sayı içerisinde her zaman üçe bölünebilen bir sayı vardır.

<u>İSPAT 3A</u>: $n+\ n+1+\ n+2\ =\ 3n\ +3\ =\ 3\ (n+1)$. Toplamları üçe böldüğüne göre aralarında mutlaka üçe bölünebilen bir sayı olmalı.

<u>*Seçim A*</u> (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan:0

- Süper. Bunu yapmaya çalışırdım. Ardışık sayıları *n*, *n*+1, *n*+2 şeklinde yazıp toplamak mantıklı. (184)

Puan: 1

- Açıklama yok. (178)
- Her zaman doğru olmayabilir. (5)
- Yetersiz açıklama. (22)

Puan: 3

- Soru işareti bırakıyor, ikna edici değil, toplamlarının 3’e bölünmesi o sayıların bölünmesini gerektirmez. (161)
- Toplamların üçe bölünmesi içinde en az bir tane üçe bölünen sayı olacağını göstermez. (2+2+2).
- -1, 0, 1 için olmaz. (179)
- Ardışık iki sayının toplamı 3’e bölünseydi bu iki sayıdan üçe mi bölünecekti? (2)
- ‘Toplamları üçe bölündüğü için üçe bölünürler’ sonucu, açıkça, doğru değil. (7)
- Yapılan çıkarım yanlış. 1+4+7 = 12 3’e bölünür ama bu sayılardan hiçbiri üçe bölünmez. (13)
- Toplamı üçe bölünen her sayının içinden 3’e bölünen bir sayı çıkmaz. (19)

Seçim B (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (14, 16)
- 3(n+1) de, 3 sayısı 3’e bölünebildiği için doğrudur. (163)
- Aslında ardışık “doğal sayı desek doğru olur, sadece sayı dersek olmaz. (167)
- n çift ya da tek olsun, aynı tür ifade elde edileceğinden doğrudur. (173)
- 3’ün katı olması her durum için geçerli. (176)

Puan: 1

- n=-1 aldığında durum gerçekleşmez. (154)

Seçim C (ispat yanlıştır)

Nedenler:

Puan: 1

- Açıklama yok. (3, 17, 20, 24)
- İspat eksik. (15)
- Açıklayıcı değil. (25)

Puan: 3

- İstenilen cevabı değil, başka bir sorunun cevabını vermiş. (151)
- *n*, *n*+1, *n*+2'den biri 3'e bölünmek zorunda değil. (156)
- İspat değil. (170)
- Ne alakası var anlamadım. (174)
- *n*'in tanımlanması eksik. $n \in Z$. Böyle bir şeyle ispatlayamayız. (177)
- Saçmalamış sanki. Böyle bir kural yok ki! (8)
- "Toplamları üçe bölündüğüne göre aralarında mutlaka üçe bölünebilen bir sayı olmalı." Bu cümle yanlış. 1+1+1=3 (bu da ters örneği). (9)
- 'Toplamları 3'e bölünüyor diye mutlaka 3'e bölünebilen bir sayı olmalı' ifadesi ne kadar doğru? (10)
- Toplamlarının üçe bölünüyor olması aralarından bir tanesinin üçe bölündüğü anlamına gelmez. (11, 18)
- Toplamları bölünebilir ama kendisi önemli. Bizim için bağıntı olmadığını düşünüyorum. 2+2+2'de 3'e bölünür ama 2 bölünmez. (21)
- Dayanaksız bir iddia ortaya atmış. (23)
- Toplamları üçe bölünen sayıların içinde 3'e bölünen bir sayı bulunmak zorunda değildir. (26)
- Saçma. (132)
- Yanlış bir varsayım. Ör: $1+1+1=3$ toplamları üçe bölünen hiçbiri 3'e bölünmeyen üç sayı. (135)
- Yazdığı cümleye karşı örnek verilebilir. 1+1+1=3 ama 3 [bölmez] 1. Fakat cümleye ardışık 3 sayının toplamı üçe bölündüğüne göre … diye başlasaydı böyle bir örnek bulamazdık. (138)
- Karşı Örnek: 1+1+1=3 ama 3 biri bölmez. (143)
- 4+2 = 6→ 4, 2 bölünmez 6 bölünür, ifada yanlış. (145)
- Üçe bölünmeyen bir a sayısı için $3a$ üçe bölünür ama bu a'nın üçe bölündüğünü göstermez. Ayrıca a-2, a-1, a+1, a+2, … gibi sayılar hakkında bir şey söylemez. (149)
- $3|3$ ama $\frac{1+1+1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ 1 (bölmez) 3. Ardışıklık ifadesi kullanılmamış. (186)
- "Toplamları üçe bölündüğüne göre aralarında mutlaka üçe bölünebilen bir sayı olmalı" ifadesi yanlıştır. (180)

- Zaten bu cümle ispat bekliyor. (181)
- 2+2+2=6 3 (bölmez) 2. (182)
- Toplamın bölünmesi üçe bölünen bir sayı içerdiğini söylemez. (183)
- Toplamın üçe bölünmesi bir şey ifade etmez. (185)

Seçim D (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (162, 187)

Puan: 1

- Bu her zaman doğruluğu göstermez. (12)

Puan: 2

- Toplamlarının bölünmesi, aralarında bölünebilen bir sayı olduğu anlamına geliyor mu bilmiyorum. 1+2 = 3 ama 1 de 2 de 3’e bölünmez. (1)
- Teori içinden teori çıkarmış. Toplamları üçe bölünüyor diye sayılardan biri de üçe bölünmek zorunda değil. (6)

İSPAT 3B: İfade yanlıştır. -1, 0 ,1 sayılarının hiçbiri üçe bölünmez.

Seçim A (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (154)

Puan: 1

- Bu örnek için, ifadenin yanlış olduğunu gösteriyor. (1)
- 0, üçe tam bölünür çünkü kalan 0’dır. 0/3=0 (10)

<u>Seçim B</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Örnek verilerek ispatlanmış. (163)
- Açıklama yok. (174,179)

<u>Seçim C</u> (ispat yanlıştır)

Nedenler:

Puan: 0

- Toplamları bölünebilmeli, soru yanlış anlaşılmış. (15)

Puan: 1

- İnsaf ya. (132)
- İspat değil. (170)
- İspat yapılmamış, direkt yargı var. (176)
- Açıklama yok. (20, 178)
- Yanlış. (138, 145)

Puan: 3

- Sıfır üçe bölünür. (2, 5, 7, 8, 17, 21, 23, 25, 151, 162, 164, 173, 177, 135, 143, 149, 186, 187, 180, 182, 184, 185)
- 0/3 = 0. (156, 161,167)
- 0'ın bölünebilirliği yanlış. (181)
- Karşı örnek vererek doğru olmadığını söylemek istemiş ama 0 kötü bir seçim. (183)
- İfadenin $n \geq 1$ olması gerektiğini gösteriyor. Ayrıca 0/3=0. (3)
- 0 üçe bölünür (tüm sayılara bölünür). (9)
- Karşıt örnek verilmiştir. Ancak "0" 3'e bölünür. (11)
- 0 üçe tam bölünebilen bir sayıdır. (13)
- 0/3=0 (16, 22)
- 0 üçe bölünür. Önerme yanlış. (18)
- 0 bölünür gibi gözüküyo :) (19)
- Örnekleme yöntemi sonunda doğru kullanılmış ama 0 üçe bölünür. (24)

- Bu sayılardan sıfır üçe bölünür. (26)

<u>*Seçim D*</u> (bir fikrim yok)

Nedenler:

Puan: 0

- Doğal sayılar için ele almamış. (12)

<u>İSPAT 3C</u>: Üçün katları olan sayılar öncekine 3 eklenerek bulunur. Ardışık sayıları yazdığımızda üçe bölünenler arasında iki sayı kalacağını görürüz. ***n*** <u>*n*+1 *n*+2 ***n*+3**</u> *n*+4 *n*+5 *n*+6; *n*+1 *n*+2 *n*+3 herhangi üç sayı.

<u>*Seçim A*</u> (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- 3B'deki ifadeyi açıklamıyor. (163)
- Sadece pozitif sayılar için doğrudur. (1)

Puan: 1

- Açıklama yok. (154, 184)

Puan: 2

- Diğer durumları da göstermesi gerekirdi. (151)
- Genelleme yapılmamış, her durum için geçerli değil. (156)
- Her durum için geçerli değil. (161)
- Önceki sayı ne? 3'ün katı mı acaba? (176)
- Yani az çok doğru. (132)
- Aslında A şıkkı da değil. Açıklamada eksikler var. (138)
- Pek olmamış. (145)
- Tam olarak iyi ifade edilmemiş. (183)
- Tüm sayılar için gösteremiyoruz. (26)

Puan: 3

- *n* her zaman üçe bölünür mü? Dizi nereden başlıyor? (182)

- Mantık var, açıklama yetersiz bence. Eğer *n* ve *n*+3’ koyulaştırarak 3’e bölünenin *n* olduğunu ima etmeseymiş daha güzel olurmuş. (143)

Seçim B (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 1

- Açıklama yok. (2, 3, 7, 14, 164, 177, 185)
- Doğru. (179)
- Aferin. (187)
- 3D ispatı ile aynı. (6)
- İspat 3A ile aynı. (8)
- Doğrudur. (5)
- Bilemiyorum. Her zaman doğru ama ispat çok sıkıcı. (19)
- Güzel bir ispat. Herhangi *n* sayısı yaptığı için tutarlı bir ispat. (21)
- Mantıklı. (22)

Puan: 2

- Tam olarak bir ispat değil. (170)
- Yeteri kadar açık olmayan bir kanıt. (135)
- Bütün *n*’ler için doğru. *n*’in üçün katı olduğunu iddia etmiş. *n*+3’ün de üçün katı olduğunu daha iyi açıklayabilirdi. (149)
- Ama ifadeden ne dediği anlaşılmıyor hemen. (181)
- Düşüncesi doğru ama tam olarak anlatamamış. (9)
- Doğru mantık. (10)
- 3D’ye benzer bir açıklama ama ikna edici. Genellemiş. Sadece örnek vermemiş. (12)

Seçim C (ispat yanlıştır)

Nedenler:

Puan: 0

- Açıklama yok. (20, 174)
- 3’ün katları ile ilgili ispat istenilmiyor. Ardışık sayılar soruluyor. (162)
- 3’ün katları üç eklenerek bulunmaz. 5+3 =8 üçün katı değil. (167)

Puan: 1

- Ortada ispat yok. 3D'dekinden bir farkı yok. (13)
- Eksik. (15)
- [n+1 n+2 n+3 herhangi üç sayı] Bu assumption ilk başta yapılmalı. (186)

Puan: 2

- n+1 n+2 n+3 seçmek zorunda değilim. Bunlar 3'e bölüneceği kesin değil. (18)
- Eksik bilgi var, ispatın daha genel olması gerek. (23)

Seçim D (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (11, 16, 24, 178, 180)
- Hiçbir fikrim yok. (173)
- Fikrim yok. (17)
- Ne yazmak istediği bile anlaşılmamış. (25)

İSPAT 3D: x, x+1, x+2 üç ardışık sayı olsun. x üçe bölündüğünde kalan 0, 1 veya 2 olur. Kalan 0 ise x üçe bölünür, 1 ise $x + 2$ üçe bölünür, 2 ise $x + 1$ üçe bölünür.

Seçim A (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (154,178)
- Her durum için geçerli değil. (156)
- 3B'deki ifadeyi açıklamıyor. (163)
- x'i bilmememiz lazım. (176)
- x=2 için doğru olmaz mesela. (177)
- Sadece pozitif sayılar için doğrudur bu açıklama. (1)
- Bütün koşullarda sağladığı gösterilmedi. (5)

Puan: 2

- Modüler aritmetik, her sayı için çalışır. (184)

<u>*Seçim B*</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 3

- Açıklama yok. (2,3,7,9,10,11,24,26,132, 180, 182, 185, 151,161,162,164)
- İspat aslında çok zekice ama ispat gibi yazılmamış. (167)
- Genelleme yapılmış. (170)
- Temel matematik ispatı yok ama mantıkta yanlışlık yapılmamış. (173)
- Doğrudur. (179)
- Durum incelemesi kullanan doğru bir kanıt. (135)
- Doğru. (138, 145)
- Tamamdır. (143)
- B/A. İddia doğru ama yeterince açık değil. (149)
- Tüm durumlar açıklanmış. (186)
- Bravo vallahi, ben bile düşünemedim. (187)
- Doğru bir ispat. Olacak durumları açıkça belirtmiş. (181)
- Doğru. Durum analizi ispatı gerçekliyor. (183)
- 3E ile aynı. (6)
- x'i genel olarak aslında mod'u kullanarak çözmüş. (8)
- 3E'ye benzer bir açıklama, sadece yazıyla açıklamış. (12)
- 3E'dekinin aynısı. (13, 16)
- Gayet açıklayıcı. (14)
- Metematiksel olarak söylenenin yazılması gerekir. (17)
- Tüm durumları incelemiş, hepsi için doğru. (18)
- 3E ispatın aynısı gibi geldi bana. (19)
- Case analysis iyi incelenmiş. (20)
- Benim yaptığım mod mantığıyla 3E'ye benziyor. Case'lere ayırarak bulmuş. (21)
- Mantıklı. (22)
- Her üç durum da gözönüne alınmış. (23)
- Açıklayıcı. (25)

<u>Seçim C</u> (ispat yanlıştır)

Nedenler:

Puan: 0

- Açıklama yok. (174)
- Eksik. (15)

İfade 4: n kişinin katıldığı bir parti düşünelim ($n \geq 2$). Bu partide aynı sayıda arkadaşı olan en az iki kişi bulunacağını ispatlayınız.

<u>İSPAT 4A:</u> Mesela iki kişi düşünelim ve birbirlerinden başka arkadaşları olmasın. Tesadüfen yoldan geçerken partiyi gördüklerini düşünelim. Bu şekilde en az iki kişinin birer arkadaşı (birbirleri) yani aynı sayıda arkadaşı vardır.

Seçim A (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 2

- Açıklama yok. (16, 177)
- Yetersiz bilgi. (5)

Puan:3

- Genelleme yok. (156)
- İki kişinin birbirlerinden başka arkadaşları olduğu durumları açıklamıyor. (163)
- Genelleme yapılmamıştır. (164)
- İki kişi için doğru mantık. Çünkü birbirlerini tanıyorlarsa her ikisinin bir, tanımıyorlarsa her ikisinin de partide 0 arkadaşı vardır. Ama sayı arttırıldığı zaman aynı mantık işler mi bilemeyeceğim. (173)
- Genellemiyor (tek örnek). (176)
- Doğru bir durum örneği vermiş. (178)
- Herkes için yani hiç böyle kimse bulunmayabilir. (179)
- Genelleme yapmalıydı. (2)
- Örnek vermiş sadece ama kesinlik yok. (6)
- Ya tek başına geçen bir kişi olursa? (7)

- 2’den fazla arkadaş için geçerli olmalı. (15)
- Garip bir yaklaşım. İfade güzel ama bu her $n \geq 2$ için sağlandığı anlamına gelmez. (20)
- Kısmen doğru, genelleme yapılmamış. (22)
- İki kişi değil de teasdüfen yoldan geçerken partiyi gören bir kişi olduğunu düşündüğümüzde aynı sonuca ulaşamayız. (26)
- Sadece iki kişi için göstermiş, anlatmış durumu. Aslında burada bir ispat yok. (138)
- “Tesadüfen yoldan geçerken partiyi gördüklerini düşünelim ” hikâye kısmı. Bu ispat değil. (143)
- Örnek vermiş. (185)

Seçim B (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Örneklerle güzel açıklamış. (154)
- Açıklama yok. (161)

Seçim C (ispat yanlıştır)

Nedenler:

Puan: 2

- Açıklama yok. (149, 182, 151, 170, 174)
- Çok komik. (132)
- Absürt. (135)
- Bariz. (145)

Puan: 3

- Örnekle proof olmaz. (178)
- Kabul yapmış kafadan, kabulü yanlış olabilir. Bir durum sadece. (179)
- Bu böyle olabilir diyor. Ama bulunması kesin demiyor ki. Yoldan geçerken girmedilerse, bu ifade yanlış olmuş oluyor. (181)
- Argüman hayali tahminlere dayanmamalı. (183)
- Böyle tesadüfen proof olmaz. (184)
- Hiçbir matematiksel anlatım yok. İspat değildir. (1)

- İspat değil. (3)
- Partinin içindeki insanlardan bahsederken biraz tuhaflaştırmış soruyu. Gerçi ben hiç yorumlamadım bu soruyu. Fikir beyan etmesi de güzel gene de. (8)
- Sorunun sorduğu şeyi ispatlamamıştır. (11)
- Sadece bir durum için incelemiş. (13)
- Zorlamış ispatı. Sorunun doğru olduğuna inandığı için sadece sonucu uydurmuş. (14)
- Assume edilen durum yanlış olabilir. Bunlar dışardan arkadaş olmayabilir. (17)
- Tesadüf. Her durumu temsil etmiyor. (18)
- İşi şansa bırakmış. (19)
- Örnekle ispat olmaz. (21)
- Tesadüfler üzerinde ispat yapamayız. 2 için doğru olabilir ama daha genel çözüm gerek. (23)
- Ya böyle birileri varsa partide? Örnekle ispat olmaz. (24)
- Açıklayıcı değil. (25)

Seçim D (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (10, 162)
- Çok anlayamadım. (12)

İSPAT 4B: n = 2 ise iki kişi ya birbirini tanıyor ya da tanımıyor. İki durumda da arkadaş sayıları eşit. n = 3 ise ya kimse birbirini tanımıyordur, ya herkes birbirini tanıyordur, ya da birbirini tanıyan iki kişi vardır. Her durumda arkadaş sayısı aynı olan en az iki kişi var. İfadenin n-1 kişi için doğru olduğunu düşünelim ve son bir kişi partiye gelsin. Eğer kimseyi tanımıyorsa hala doğru olacak çünkü birbirini tanıyan en az iki kişi var. Bu şekilde devam edebiliriz.

Seçim A (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 1

- Açıklama yok. (145, 154, 161)

Puan: 2

- Genelleme yapamayız. (164)
- Aynı şekilde [3A ile] mantık doğru ama matematiksel bir ispat yok. (173)
- Yaklaşmış. (132)

Puan: 3

- *n* bulup oradan *n*+1'de göstermesi gerekir. (151)
- Tek durum genellemeler için yeterli veri yok. (176)
- n sayıda kişiye kadar incelemesi gerekir. 2 ve 3 kişide doğru olduğu bize *n* kişide de doğru olacağı anlamını vermez. (11)
- İfade sadece verilen sayılar için doğru olabilir. (17)
- Bu da olmaz. Dışarıdan gelen kişi ya sayısı eşit olanlardan birini tanıyorsa bu durumda o iki kişi artık eşit olmaz. (24)
- Eksik. Tümevarımı tamamlamalı. (25)
- İspat yarım kalmış durumda. (26)
- Durumların hepsi düşünülmemiş, ayrıca sona erdirilmemiş. (138)
- Ya birden fazla kişi tanıyorsa? Ayrıca *n*=3 durumunda bir kişi iki kişiyi tanıyıp [cümle tamamlanmamış]. (143)
- A, C. *n*'e gelene kadar doğru, *n* kısmı doğru incelenmemiş. (149)
- Belirli sayıda kişiler için incelemiş. Inductionın amacı bu zaten. (186)
- [Eğer kimseyi tanımıyorsa] →bazılarını tanıyorsa? (182)

Seçim B (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (2, 14, 162)
- Doğru sanki. (187)

Puan: 1

- Tümevarım yöntemi iyi kullanılmış. (183)
- Güzel bir proof olmuş. Bütün case'leri içeriyor. Induction gibi gitmiş. Hoş. (184)
- Evet, tümevarım yoluyla ispatlamış. (179)

- Tatmin edici ispat. (5)
- Induction kullanarak açıklamış ve doğru. (12)
- Induction. (16)
- Tümevarım prensibi işlenmeye çalışılmış ve metod doğru. (20)
- Yöntemi doğru, tümevarım yapmış. (22)
- Tümevarım. (23)

Puan: 2

- Tüm durumları incelemiş anlatmış. (8)
- Her durum incelenmiş. (10)
- Doğru yolda ama tümevarımı tamamlayamamış. (13)
- Bu yol doğru ama pek açıklayıcı değil. (19)
- Induction yapmış. Biraz daha açıklaması gerekirdi ama. (181)

Seçim C (ispat yanlıştır)

Nedenler:

Puan: 0

- Açıklama yok. (174)
- İspat değil. (3)
- Pek anlamadım. (6)

Puan: 2

- Önermeyi ilk elden doğru kabul ederek düşünmüş. Sondan ikinci cümlesi tamamen yanlış. (7)
- "Her durumda arkadaş sayısı aynı olan en az iki kişi var." İspata muhtaç, doğru olduğunu nerden biliyoruz. (18)

Puan: 3

- Tümevarım kullanılmaya çalışılmış. Fakat n-1'den n'e geçiş doğru yapılmamış. (135)

<u>Seçim D</u> (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (15, 21, 156, 163, 170, 178, 185)

Puan: 1

- Tüm değerler için doğru mu bilinmez. Bence bu ispat değildir. (1)

<u>İSPAT 4C</u>: Eğer *n*>2 ve *n* bir çift sayı ise aynı sayıda arkadaşı olan iki kişi bulunabilir. *n* = 3 durumunda doğru olmaz.

<u>Seçim A</u> (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 1

- Yetersiz bilgi. (5)

<u>Seçim B</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 0

- Açıklama yok. (10, 154)

<u>Seçim C</u> (ispat yanlıştır)

Nedenler:

Puan: 2

- Açıklama yok. (7, 15, 19, 20, 24, 151, 156, 170, 174, 177, 181)
- İfadede en az iki kişi dediği için yanlıştır. (163)

Puan: 3

- *n*=3 durumu açıklanmamış. (164)
- *n*=3 olduğunda herkes birbirini tanıyor olabilir mesela (handikap var). (176)
- Yanlış. (132)
- Hem yanlış hem de açıklamasız bir ifade. (135)

- Herhangi bir ispat yok aslında. (138)
- Açıklama yok. n = 3 ile ilgili yorum da doğru değil. (143)
- Böyle olmak zorunda değil. (145)
- Yanlış bir iddiada bulunmuş ve açıklanmamış. (149)
- Proof değil. (186)
- Yanlış. Üçü de arkadaştır mesela. (187)
- İspat değil. (182)
- Kısmi doğru ama ispat değil. (183)
- n tek ise ifade yanlıştır. (184)
- n = 3 te de doğrudur çünkü 3. kişi 2 kişinin de arkadaşı olabilir. (1)
- Biri için geçerli olan arkadaş sayısı arkadaşı için de geçerli olabilir. (2)
- İspat değil. (3)
- $n \geq 2$ ise bütün n'ler için doğrudur. (6)
- Nasıl bir ilişki kurulmaya çalışılmış anlayamadım. (8)
- Eksik bir ispat. (10)
- İspat tamamlanmamıştır. (11)
- Diğer durumları düşünmemiş. Yanlıştır. (12)
- Herkesin 1'er tanıdığı olabilir. (n = 3 için) (13)
- Kendi ispatını açıklamamış, yanlış zaten. (14)
- Sayının çift ya da tek olmasına bağlı değildir. (17)
- Varsayım bunlar, ispatı gerekli doğruluğu kanıtlanmamış. (18)
- 3'ü göstermek yetmiyor. İspatta sayı olmaz. (21)
- İspat 4A'da açıklamış. (22)
- Spesifik örnek ve yanlış ispat. (23)
- n = 3 iken de doğrudur. Yanlış. (25)
- Bu soru için ispat olmaya çok uzak bir açıklama. (26)

Seçim D (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (16, 161, 162, 178, 185)
- Mantığı anlamadım. Kurgulayamadım. (173, 179)

<u>İSPAT 4D</u>: Bir kişinin en azından bir tane arkadaşı olduğunu düşünürsek ve kişileri 1, 2, 3, ...,*n* ile gösterirsek birinci kişinin 1, ikinci kişinin 2, üçüncü kişinin 3, ..., *n*-1. kişinin *n*-1 arkadaşı olabilir ama *n*. kişinin en fazla *n*-1 arkadaşı olabilir (kendisini çıkartırız). Dolayısı ile *n*. kişinin arkadaş sayısı diğer *n*-1 kişiden birininkiyle aynı olmak zorundadır.

<u>*Seçim A*</u> (ifadenin bazı durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 1

- Emin olamamakla beraber doğru olduğunu düşünüyorum. (8)

Puan: 2

- Diğer durum, 1 kişinin hiç arkadaşı olmaması incelenmemiş belki o durumda sağlamayabilir. Her zaman doğru olduğunu göstermez. (18)

<u>*Seçim B*</u> (ifadenin tüm durumlar için doğru olduğunu gösterir)

Nedenler:

Puan: 3

- Açıklama yok. (2, 3, 6, 10, 22, 26, 132, 145, 154, 170)
- *n*. Kişinin en fazla *n*-1 arkadaşı olması durumu (*n*-1). kişiyle uyuşuyor. (163)
- Durum güzelce anlatılmış. (164)
- Böyle bir simetri olabilir, sonuçta birinin arkadaşı varsa, o da tanıdığının arkadaşıdır. (179)
- Bazı kişilerin hiç arkadaşı olmadığı durum da incelenmelidir. (135)
- “Pigeonhole Principle” kullanılarak ispatlanmış, daha açık yazılabilirdi ama zaman sıkıntısı vs. doğru bir ispat. (149)
- Pigeonhole principle. (186)
- Eksik ama doğru gibi. (181)
- Matematiksel olarak açıklamış ve ikna edici. (1)
- Tam ispat, her koşulda sağlar. (5)
- Bence doğru her ne kadar arkadaş sayıları sorunlu gözüküyorsa da (hiç arkadaşı olmayan bir kişi katarsak gruba sorunlu olabilir). (7) B-A

- Ben de böyle yazıyordum kağıdıma ancak tam olarak ne soruyor anlamadım. O yüzden bu kategoriyi boş bırakıyorum. (9)
- Pigeonhole kullanılmıştır. (11)
- Gayet mantıklı bir açıklama. Benim kafamdan geçen ama açıklayamadığım ispat. (12)
- Güvercin yuvası prensibini uygulamış. (13)
- Doğru açıklayıcı olmuş. (14)
- Pigeonhole principle kullanılarak açıklanmış. (17)
- İşte ispat budur. (19)
- Pigeonhole prensibi doğru bir şekilde işlenmiş. (20)
- Case'lerle düşünmüş en baştan zaten hepsini farklı düşünmüş. (21)
- Güzel bir ispat olmuş. Anlaşılır ve basit. (24)
- Doğru. (25)

Seçim C (ispat yanlıştır)

Nedenler:

Puan: 0

- Açıklama yok. (151,161,174)
- Zorunluluk olması için tersi bir durum olması gerekir. Farklı sayıda arkadaşı olabilir. (176)
- (n-1). kişi için de kendisini (n-2) çıkarırız. (177)

Puan: 1

- 0 arkadaş durumunu hesaba katmamış. En kötü durumu düşünerek gitmiş ama bu ispat değil. (143)
- Kimse birbirini tanımıyor olabilir. Ya da 1. kişinin n-1 arkadaşı olabilir. (23)

Seçim D (bir fikrim yok)

Nedenler:

Puan: 0

- Açıklama yok. (15, 16, 156, 162, 178)
- Eğer doğru ise en matematiksel ispattır. (173)

- Düşünmem lazım ama üşendim. (187)
- Bir fikrim yok. (183)
- Açıklama yok. (185)

APPENDIX C: PILOT STUDIES

C.1. Pilot study for Attitudes and Beliefs Scale

The following are the detailed of the results of the first pilot study, as mentioned in Section 5.3. In this version of ABS, there were 16 items. Factor analysis yielded four interpretable components. Table C.1. shows the factor loadings for each component.

Table C.1. Factor loadings for the first version of ABS,

Item no	Component				
	1	2	3	4	5
12	**0.81**	-0.13	0.06	-0.01	-0.07
11	**0.81**	0.11	-0.01	-0.09	-0.05
10	**0.70**	-0.06	-0.05	0.02	0.03
9	**0.65**	-0.01	0.12	0.46	0.06
6	0.07	**0.83**	0.19	0.04	-0.03
5	0.06	**0.81**	0.14	0.05	-0.05
8	-0.21	**0.68**	0.04	-0.07	0.08
4	-0.33	-0.05	**0.72**	-0.03	0.17
15	0.17	0.26	**0.65**	-0.22	-0.07
16	0.08	0.06	**0.57**	0.47	0.06
1	-0.08	0.49	**0.52**	-0.15	0.38
2	0.08	0.20	**0.45**	0.10	-0.07
3	-0.03	0.00	-0.06	0.88	0.00
14	0.28	-0.32	0.15	0.09	**0.72**
13	-0.15	0.11	-0.45	0.15	**0.65**
7	-0.21	0.24	0.11	-0.31	**0.51**

Labels**,** means, standard deviations and reliability coefficients of the components, as well as the corresponding item numbers are given in Table C.2.

Table C.2. Mean, standard deviation and reliability coefficients of the extracted factors

Component	Mean	Standard Deviation	Cronbach Alpha	Number of Items
Background	2.73	0.92	0.76	4
Importance	3.25	0.81	0.73	3
Belief and Experience	3.35	0.64	0.63	5
High School/University	3.63	0.70	0.31	3

First component, labeled as *background*, consists of items about background information; how much the students were exposed to the concept of proof in high school. Contributing items are as follows:

- In high school, we had proofs in mathematics lessons.
- In high school, I was expected to do simple proofs in mathematics lessons and exams.
- In high school, our mathematics teachers never talked about the importance of proof.
- In high school, our mathematics teachers encouraged us to do proofs.

The items in the second component, labeled as *importance*, are about the importance and use of proof in high school mathematics:

- Incorporating proofs in high school mathematics lessons may make it difficult for the students.
- In high school, proof should be used to explain a mathematical concept.
- A high school student should be expected to do mathematical proofs.

Third component consists of items about students' personal beliefs and experiences regarding proof and it is labeled as *Beliefs and Experience*:

- In order to comprehend a mathematical statement, I try to understand its proof.
- In order to decide whether a mathematical statement is true, I have to check that it is true for all cases.
- In mathematics, proofs are usually confusing.
- I am not confident that I can do proofs.
- I feel compotent that I can do proofs. (I feel adequate in doing proofs)

In the last component, *High School/Uinversity*, there are items about the differences between high school and university:

- It is sufficient for a high school student to be able to apply a mathematical property/theorem in different situations.
- I think mathematics instruction is/will be different in lectures in the university than in high school.
- I think knowledge and skills I gained in high school mathematics lessons is/will be useful for me in university.

The following item did not fit in any component, therefore discarded:

- I use examples to explain a mathematical statement to a student or to a friend.

Attitude scale also included the open ended question: "What does mathematical proof mean to you? Explain briefly". Response rates are given in Table C.3 below:

Table C.3. Response rates to open ended item

Grade		Responded	n	per cent
Freshmen	Science	6	12	50
	Math	9	26	34.6
Seniors		11	19	57.9
Total		25	57	43.9

The following categories emerged from the responses:

Definition of proof:

- Asserting that the statement is true for all cases. Everyone should reach the same conclusion under the same conditions.
- Mathematics is the language of science, explains science with its own system.
- Tool for convincing the students the theorem is valid. Proof is the point where science convinces us.
- System of logical relationships that helps us to understand when and why a statement is true.
- Consistent within itself, not contradictory.

Uses of proof : We use proof to

- Achieve deeper, better understanding, to prevent route learning.
- Learn to use information better, learn how to use concepts.
- Prepare for mathematics courses we'll take in university.

Uses of proof:

- Broadens our views and perceptions.
- Help us to think.
- Increases interest to the topic, exciting, enjoyable, it is like solving a puzzle.

Role of proof in high school mathematics:

- We cannot teach everything using proof. Using proof in every topic can be discouraging.
- Teaching some proofs can be difficult and not useful for students who are under certain level.
- In some topics teaching without proof is impossible.
- Proof should be thought in high school, otherwise we cannot learn properly in university.
- Proof is not the only way to teach/learn.

- If the students do not understand where a theorem comes from, then it will be difficult for them to apply it, they will just memorize.

Table C.4. Factor loadings for the ABS

Item no	Component						
	1	2	3	4	5	6	7
4	**0.74**	0.01	0.21	0.04	-0.06	0.03	-0.37
1	**0.71**	0.16	0.09	-0.01	0.34	0.26	0.04
5	**0.70**	-0.12	0.24	0.16	-0.10	0.01	0.22
3	**0.64**	-0.07	-0.01	0.18	0.03	0.42	0.26
6	**0.63**	0.17	0.13	-0.11	0.13	0.08	0.51
25	**0.57**	0.30	0.10	0.07	0.31	-0.08	0.20
18	**0.55**	0.36	-0.04	0.03	0.24	0.31	0.14
19	**0.54**	-0.08	0.30	**0.54**	0.20	0.12	0.04
16	**0.45**	0.29	0.09	0.23	-0.01	0.38	0.09
21	0.12	**0.82**	-0.04	-0.03	-0.19	0.02	-0.06
24	-0.07	**0.77**	0.13	0.04	0.27	0.07	-0.04
14	0.03	**0.67**	0.30	0.03	0.16	0.19	0.18
15	0.19	**0.47**	-0.15	0.40	0.45	-0.13	0.23
10	0.12	0.10	**0.83**	-0.04	-0.02	-0.05	0.06
12	0.19	0.11	**0.68**	0.28	0.00	0.30	-0.01
9	0.28	0.08	**0.67**	0.24	-0.14	0.33	0.16
23	0.00	-0.03	0.11	**0.85**	-0.09	0.07	-0.12
22	0.29	0.42	0.08	**0.57**	-0.18	-0.11	0.10
2	0.28	0.05	-0.05	-0.08	**0.69**	0.03	0.34
13	-0.06	0.21	-0.19	-0.30	**0.66**	0.17	-0.14
20	0.54	-0.21	0.25	0.16	**0.57**	-0.12	0.01
8	0.09	0.12	0.31	-0.15	0.02	**0.71**	0.06
17	0.42	0.04	-0.06	0.26	0.28	**0.57**	0.13
11	0.05	0.03	0.41	**0.46**	-0.12	**0.47**	0.15
7	0.14	0.02	0.14	0.01	0.09	0.18	**0.84**

Second version of ABS, administered to 94 freshmen calculus students in Bilgi University had 25 items. Initial factor analysis yielded seven components, as shown in Table C.4 above. Items in these components were as follows:

Component 1

- In order to comprehend a mathematical statement, I try to understand its proof.
- Proofs make mathematics enjoyable.
- Proofs are important only for mathematicians.
- Incorporating proofs in high school mathematics lessons may make it difficult for the students.
- In high school, proof should be used to explain a mathematical concept.
- In mathematics, proofs are usually confusing.
- Proof is a vital part of mathematics.
- I find dealing with proofs boring.
- More emphasis should be given to proofs in university mathematics lectures than in high school.

Component 2

- I think knowledge and skills I gained in high school mathematics lessons is/will be useful for me in university.
- I feel confident that I can do mathematical proofs.
- I can use mathematical language efficiently while doing proofs.
- I believe that my mathematical knowledge is adequate for doing simple mathematical proofs.

Component 3

- In high school, we had proofs in mathematics lessons
- In high school, I was expected to do simple proofs in mathematics lessons and exams.
- In high school, our mathematics teachers encouraged us to do proofs.

Component 4

- In high school, our mathematics teachers never talked about the importance of proof.
- I find dealing with proofs boring.
- I usually have difficulty in understanding proofs.
- I do not have too much experience in doing proofs.

Component 5

- In order to decide whether a mathematical statement is true, I have to check that it is true for all cases.
- I think mathematics instruction is/will be different in lectures in the university than in high school.
- It is not compulsory to be able to do mathematics to be successful in mathematics lessons.

Component 6

- In our high school mathematics textbook, there were exercises about proof.
- In high school, our mathematics teachers never talked about the importance of proof.
- We do not need to know the proof of a mathematical result in order to understand why a mathematical result is true.

Component 7

- A high school student should be expected to do mathematical proofs.

Seven factors did not yield an interpretable result. There were items appearing in more than one component and one component had less than three items. Since the first pilot study resulted in four factor components, and the scale was expected to measure four sub dimensions, factor analysis was repeated with the restriction of four components. Factor loadings can be seen in Table C. 5.

Table C.5. Factor loadings for four components

Item no	Component			
	1	2	3	4
1	**0.78**	0.16	0.17	-0.04
20	**0.73**	-0.04	-0.13	0.09
6	**0.69**	0.25	0.15	-0.15
3	**0.69**	0.29	-0.06	0.11
25	**0.65**	0.02	0.32	0.04
2	**0.64**	-0.12	0.17	-0.36
19	**0.63**	0.25	-0.02	0.49
18	**0.63**	0.15	0.38	-0.06
17	**0.61**	0.24	0.13	0.04
5	**0.60**	0.27	-0.17	0.28
4	**0.50**	0.14	-0.07	0.32
16	**0.45**	0.33	0.29	0.19
7	**0.39**	0.36	0.06	-0.24
9	0.26	**0.76**	0.05	0.27
12	0.20	**0.68**	0.11	0.30
8	0.16	**0.65**	0.11	-0.25
11	0.13	**0.63**	0.07	0.35
10	0.07	**0.62**	0.04	0.10
24	0.03	0.10	**0.80**	-0.07
21	-0.05	0.08	**0.75**	0.07
14	0.12	0.37	**0.68**	-0.08
15	0.43	-0.24	**0.58**	0.16
23	0.04	0.10	0.05	**0.78**
22	0.20	0.09	0.42	**0.61**
13	0.20	-0.24	0.30	-0.52

Items corresponding to each component are given below:

Component 1

- In order to comprehend a mathematical statement, I try to understand its proof.
- In order to decide whether a mathematical statement is true, I have to check that it is true for all cases.
- Proofs make mathematics enjoyable.
- Proofs are important only for mathematicians.
- Incorporating proofs in high school mathematics lessons may make it difficult for the students.
- In high school, proof should be used to explain a mathematical concept.
- A high school student should be expected to do mathematical proofs.
- In mathematics, proofs are usually confusing.
- Proof is a vital part of mathematics.
- I find dealing with proofs boring.
- More emphasis should be given to proofs in university mathematics lectures than in high school.
- It is not compulsory to be able to do mathematics to be successful in mathematics lessons.
- We do not need to know the proof of a mathematical result in order to understand why a mathematical result is true.

Component 2

- In high school, we had proofs in mathematics lessons
- In high school, I was expected to do simple proofs in mathematics lessons and exams.
- In high school, our mathematics teachers encouraged us to do proofs.
- In our high school mathematics textbook, there were exercises about proof.
- In high school, our mathematics teachers never talked about the importance of proof.

Component 3

- I think knowledge and skills I gained in high school mathematics lessons is/will be useful for me in university.
- I feel confident that I can do mathematical proofs.
- I can use mathematical language efficiently while doing proofs.
- I believe that my mathematical knowledge is adequate for doing simple mathematical proofs.

Component 4

- I usually have difficulty in understanding proofs.
- I do not have too much experience in doing proofs.

Discarded item:

- I think mathematics instruction is/will be different in lectures in the university than in high school.

C.2. Pilot Study for the Proof Exam

The results of the *Proof Exam* used in the pilot study are presented below. For each item, types of proofs that the students attempted are given.

Item 1: "If the square of a natural number is even, then that number must be even."

- Proof 1A: If n was odd, then its square would be odd. (Proof by contrapositive)
- Proof 1B: If n is even, then its square will be even. (This proofs the converse of the given statement; not equivalent.)
- Proof1C: Assume that n is odd but n^2 is even. If n is odd then n^2 is be odd. This contradicts our assumption. (Proof by contradiction)
- Proof1D: Direct proof: Assume n^2 is even ...then n must be even.
- Proof 1E: The square of an even number is even, the square of an odd number is odd. Hence, if the square of a number is even, then that number should be even. (Proof by cases)
- Proof 1G: Giving numerical examples without generalization.
- Proof 1F: There is no meaningful argument.
- NA: Not attempted.

Most used (valid) proof was proof by contradiction in both groups. Commonly made mistakes were proving that the square of an even number is even, and showing the statement is true by giving numerical examples without generalization. Frequencies of the proof types are given in Table C.6.

Table C.6. Proof types by groups, item 1

Type		NA	A	B	C	D	E	F	G	Total
Group	1B	1	9	6	1	2	3	0	3	25
	2	1	5	4	3	4	1	1	0	19
Total		2	14	10	4	6	4	1	3	44

Item 2: True or false: "The equality $1 + 3 + 5 + ... + 2n-1 = n^2$ is true for all integers $n \geq 1$."

- Proof 2_A: Using Gauss' method (writing the same sum in reverse and adding up the terms).
- Proof 2_B: By mathematical induction.
- Proof 2_C: Using the equality $1+2+3+\ldots+n = n(n+1)/2$.
- Proof F: There is no meaningful argument.
- NA: Not attempted.

Most participants in both groups attempted proof by induction. Common mistakes made here were not showing the basis step in induction (mostly seen in group 1) and calculation errors. Frequencies of the proof types are given in Table C.7.

Table C.7. Proof types by groups, item 2

Type		NA	A	B	C	F	Total
Groups	1A	8	0	7	0	0	15
	2	1	2	14	1	1	19
Total		9	2	21	1	1	34

Item3: True or false: "Given any three consecutive integers, one of them is divided by three."

- Proof 3_A: Direct proof: Explanations such as: When we group the numbers by three, there is a number in each group that is divisible by three.
- Proof 3_B: Let a, $a + 1$, $a + 2$ be three consecutive integers. If $a = 3k$, then a is divisible by three. If $a= 3k+1$ then $a+2$ is divisible by three. If $a=3k+2$ then $a+1$ is divisible by three. In any case, one of them will be divisible by three (proof by cases).

- Proof 3_C: Showing that the sum is divisible by three. (This does not prove the statement is true: if the sum of three numbers is divisible by three, it cannot be concluded that one of them is divisible by three.)
- Proof 3_D: Giving counter-example to show that the statement is false: “0 is not divisible by three” (the statement is not false, the counter-example is not valid because 0 *is* divisible by three).
- Proof 3_E: Giving numerical examples without generalization.
- Proof 3_G: Using the divisibility rule: a number is divisible by three, if and only if the sum of its digits is divisible by three.
- Proof F: There is no meaningful argument.
- NA: Not attempted.

Most common proof for this item was proof by cases (using remainders). Most participants in group 1 did not attempt this item. All participants who attempted this item in group 2 used proof by cases. Common mistakes were; showing the statement is false by claiming that 0 is not divisible by 3, showing their sum is divisible by 3, showing the statement is true by giving numerical examples without generalization. Frequencies of the proof types are given in Table C.8.

Table C.8. Proof types by groups, item 3

Type		NA	A	B	C	D	E	G	Total
Group	1B	10	2	3	2	2	5	1	25
	2	2	0	17	0	0	0	0	19
Total		12	2	20	2	2	5	1	44

Item 4: True or false: “(a) $n^2 + (n + 1)^2 = (n + 2)^2$ is true for all natural numbers (b)$n^2 + (n + 1)^2 = (n + 2)^2$ is false for all natural numbers”

- Proof 4_A: Solve for n: n = -1, n = 3 ; the statement is true for only the natural number 3.

- Proof 4_B: By giving counter-examples
- Proof 4_C: By giving counter-example for part a, concluding that part b must be true.
- NA: not attempted

This was a common item that was asked to all groups, and most correctly answered item. Two different proofs were given to this statement: Solving for *n* to find for which values of *n* this equality is true and giving counter examples for both statements. Frequencies of the proof types are given in Table C.9.

Table C.9. Proof types by groups, item 4

Type		NA	A	B	C	Total
Group	1	9	18	12	1	40
	2	1	3	15	0	19
Total		10	21	27	1	59

Item 5: "In a party of n people ($n \geq 2$), there exists at least two people who has the same number of friends."

- Proof 5_A: General case (pigeonhole principle).
- Proof 5_B: General case (induction).
- Proof 5_C: Trying possibilities for small n (no generalizations).
- Proof F: There is no meaningful argument.
- NA: Not attempted.

This was one of the least correctly responded items in all groups. The valid solution is using the Pigeonhole Principle, but even the senior students (Group 2) had problems with this item and only a few students gave a full proof. Most students who attempted to give a proof just experimented with small values of *n*, unable to generalize. Frequencies of the proof types are given in Table C.10.

Table C.10. Proof types by groups, item 5

Type		NA	A	B	C	F	Total
Group	1A	11	2	0	2	0	15
	2	5	8	3	1	2	19
Total		16	10	3	3	2	34

Item 6: Sum of the interior angles of a triangle is 180^o.

- Proof 6_A: Draw a line parallel to one of the sides; showing/using without proof that one of the exterior angles is the sum of the other two interior angles.
- Proof 6_B: Draw a circle around the triangle (circumcircle); $2\alpha+2\beta+2\gamma = 360^o$, $\alpha+\beta+\gamma = 180^o$. Length of the arc corresponding to an interior angle is twice the size of that angle.
- Proof 6_C: Draw a parallel to the base, sum of the two alternate interior angles and the third interior angle gives 180.
- Proof 6_D: Using the fact that sum of exterior angles is 360 (with or without proof).
- Proof 6_E: Proving the statement for right triangles (specific case).
- Proof F: There is no meaningful argument.
- NA: Not attempted.

This is the second item common in all groups. Common mistakes were using rules, properties which can be proved using this statement. Frequencies for the proof types are given in Table C.11.

Table C.11. Proof types by groups, item 6

Type		NA	A	B	C	D	E	F	Total
Group	1	25	4	4	4	1	1	1	40
	2	15	1	1	2	0	0	0	19
Total		40	5	5	6	1	1	1	59

Item 7: Product of the slopes of two perpendicular lines on the plane is -1.

- Proof 7_A: By giving examples

This was the least chosen item in both groups, and no one gave a valid proof. Frequencies are given in Table C.12.

Table C.12. Proof types by groups, item 7

Type		NA	A	Total
Grade	1B	22	3	25
	4	19	0	19
Total		41	3	44

Item 8: Pythagoras' theorem: In a right triangle, where *c* is the hypotenuse, and *a* and *b* are the legs, the following equality holds: $a^2 + b^2 = c^2$

- Proof 8_A: Using Cosine theorem: $c^2 = a^2 + b^2 - 2ab \cos \hat{C}$, $m(\hat{C}) = 90^o$, $\cos \hat{C} = 0$, $c^2 = a^2 + b^2$
- Proof 8_B: Using the equality $\sin^2\alpha + \cos^2\alpha = 1$
- Proof 8_C: Draw a square for each edge. Sum of the area of the two squares is equal to the area of the third square: $a^2 + b^2 = c^2$. (There is a diagram but no other explanation.)
- Proof 8_D: Draw a square with side length *a*+*b*. Place four right triangles with sides *a*, *b*, and *c* inside the square. In the middle, there is square with side length c. Sum of the areas is $c^2 + ab$, which is equal to the area of the big square: $(a+b)^2 = c^2 + 2ab$, $a^2+b^2 +2ab = c^2 + 2ab$; $a^2 + b^2 = c^2$.
- NA: Not attempted.

Common mistakes here were using rules, properties which are actually results of this statement, hence cannot be used in the proof. Frequencies of proof types are given in Table C.13.

Table C.13. Proof types by groups, item 8

Type		NA	A	B	C	D	T
Group	1B	21	2	1	1	0	25
	2	16	0	0	2	1	19
Total		37	2	1	3	1	44

Item 9: Let A, B and C be arbitrary sets. Then, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

- Proof 9_A: $x \in A \cap (B \cup C) \leftrightarrow x \in A$ and $x \in B \cup C \leftrightarrow x \in A$ and $(x \in B$ or $x \in C) \leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C) \leftrightarrow x \in (A \cap B)$ or $x \in (A \cap C) \leftrightarrow x \in (A \cap B) \cup (A \cap C) \leftrightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- Proof 9_B: Follows from distribution property. (They are *asked* to prove the distribution property.)
- NA: Not attempted.

While most people used the correct approach in this item, students in group 1 made mistakes such as using = or → (if…then) instead of ↔ (if and only if). Frequencies of proof types are given in Table C.14.

Table C.14. Proof types by groups, item 9

Type		NA	A	B	Total
Group	1A	8	5	2	15
	2	10	9	0	19
Total		18	14	2	34

Item 10: Let (*a*, *b*) and (*c*, *d*) be two points in the plane. The distance between these points is $\sqrt{(a-c)^2+(b-d)^2}$

- Proof 10_A: Using Pythagoras' theorem:

  A= (a,b), B = (b, c) (forms a right triangle with vertices A, B and C).

$$|AB|=\sqrt{|AC|^2+|CB|^2}=\sqrt{(b-d)^2+(a-c)^2}$$

All participants who attempted this item used provided this proof, which they know well from high school. Frequencies are given in Table C.15.

Table C.15. Proof types by groups, item 10

Type		NA	A	Total
Group	1B	19	6	25
	2	13	6	19
Total		32	12	44

These results were considered during the development of the final form of the *Proof Exam*, and *Proof Evaluation Exam*, as well as the preparation of the rubrics for these instruments.

APPENDIX D: DEPARTMENTAL PROGRAMS

Participants of this study are enrolled to the following programs offered by Boğaziçi University, İstanbul.

D1. Primary Education Teaching Mathematics Program

First Semester

Course Code	*Course Name*	*Credit*
MATH 101	Calculus I	4
MATH 111	Int. Mathematical Structures	4
PHYS 101	Physics I	4
ED 101	Introduction to Education	3
AE 111/- - -	Advanced English I or HSS Elective	3
		Total: 18

Second Semester

Course Code	*Course Name*	*Credit*
MATH 102	Calculus II	4
PHYS 102	Physics II	4
ENG 101	Introduction to Computers	3
PRED 154	Academic Orientation for Math and Science Teachers	3
AE 112 / - - -	Advanced English II or Complementary Elective	3
		Total: 17

Third Semester

Course Code	*Course Name*	*Credit*
MATH 201	Matrix Theory	4
PHYS 201	Physics III	4
ED 211	Educational Psychology	3
TK 221	Turkish I	2
ED 221	Fundamentals of Guidance and Counseling	3
		Total : 16

Fourth Semester

Course Code	*Course Name*	*Credit*
MATH 202	Differential Equations	4
ED 282	Principles and Methods of Instruction	3
- - - - -	Specified elective	3
- - - - -	HSS Elective	3
TK 222	Turkish II	2
		Total: 15

Fifth Semester

Course Code	*Course Name*	*Credit*
PRED 310	Special Education in Primary Math and Science Education	3
PRED 350	Teaching Geometry	3
PRED 371	Teaching Mathematics I	3
- - - - -	Complementary Elective	3
- - - - -	Unrestricted Elective	3
HTR 311	History of the Turkish Rep. I	2
		Total: 17

Sixth Semester

Course Code	*Course Name*	*Credit*
CET 360	Instructional Techniques and Material Development	3
PRED 348	Community Service	2
PRED 354	Teaching Probability and Statistics for Primary Education	3
PRED 372	Teaching Mathematics II	3
- - - - -	Complementary Elective	3
- - - - -	Complementary Elective	3
HTR 312	History of the Turkish Rep. II	2
		Total: 19

Seventh Semester

Course Code	*Course Name*	*Credit*
PRED 460	Research Methods in Math and Scince Education	4
PRED 461	Computer Assisted Mathematics Teaching	3
PRED 465	School Experience in Teaching Math and Science II	3
PRED 470	Assessment in Math and Science Education	3
ED 401	Classroom Management	3
		Total: 16

Eighth Semester

Course Code	*Course Name*	*Credit*
PRED 462	Applied Research in Math and Science Education	4
PRED 466	Practice Teaching in Mathematics	4
PRED 468	Seminar on Practice Teaching in Math	4
- - - - -	Unrestricted Elective	3
- - - - -	Complementary Elective	3
		Total: 18

Overall Total

Credits: 136

D2. Secondary Education Teaching Mathematics Program

First Semester

Course Code	*Course Name*	*Credit*
MATH 101	Calculus I	4
MATH 111	Introduction to Mathematical Structures	4
PHYS 101**	Physics I	4
SCED 103	Orientation to Science and Math. Education	1
AE 111*/---	Advanced English / HSS Elective	3
		Total: 16

Second Semester

Course Code	*Course Name*	*Credit*
MATH 102	Calculus II	4
MATH 162	Discrete Mathematics	4
PHYS 130	Thermodynamics, Waves, Optics and Modern Physics	4
ENG 101	Introduction to Computers	3
AE 112*/---	Advanced English II / HSS Elective	3
		Total: 18

Third Semester

Course Code	*Course Name*	*Credit*
ED 101	Introduction to Education	3
MATH 201	Matrix Theory	4
PHYS 201	Physics III	4
PHIL 131	Logic I	3
TK 221	Turkish I	2
		Total: 16

Fourth Semester

Course Code	*Course Name*	*Credit*
MATH 202	Differential Equations	4
MATH 224	Linear Algebra	3
-----	Area Elective	3
-----	Humanities and Social Sciences Elective	3
TK 222	Turkish II	2
		Total: 15

Fifth Semester

Course Code	*Course Name*	*Credit*
ED 213	Development and Learning	3
ED 221	Fundamentals of Guidance And Counseling	3
-----	MATH Elective	3
-----	MATH Elective	3
-----	Area Elective	3
HTR 311	Ataturk's Principles and History of Turkish Revolution	2
		Total: 17

Sixth Semester

Course Code	*Course Name*	*Credit*
MATH 232	Introduction to Complex Analysis	3
ED 262	Planning and Evaluation of Instruction	4
-----	MATH Elective	3
-----	SCED, ED, PRED or CET Elective	3
-----	MATH Elective	3
HTR 312	Ataturk's Principles and History of Turkish Revolution	2
		Total: 18

Seventh Semester

Course Code	*Course Name*	*Credit*
CET 360	Instructional Technologies and Material Development	3
SCED 420	Teaching Methods in Science and Mathematics	3
-----	MATH Elective	3
-----	MATH Elective	3
-----	Area Elective	3
		Total: 15

Eighth Semester

Course Code	*Course Name*	*Credit*
SCED 340	Science, Technology and Society	3
SCED 450	School Experience in Teaching Mathematics and Science I	3
ED 382	Classroom Management	3
-----	SCED, ED, PRED or CET Elective	3
-----	Unrestricted Elective	3
		Total: 15

Nineth Semester

Course Code	*Course Name*	*Credit*
SCED 404	Applied Research in Science Education	4
SCED 431	Teaching Methods in Mathematics	3
SCED 451	School Experience in Teaching Mathematics and Science II	3
-----	MATH Elective	3
-----	Area Elective	3
		Total: 16

Tenth Semester

Course Code	*Course Name*	*Credit*
SCED 408	Text Analysis in Science and Mathematics Education	3
SCED 416	Seminar on Practice Teaching in Mathematics	4
SCED 432	Practice Teaching in Mathematics	4
-----	Unrestricted Elective	3
-----	Philophy or History of Science Elective	3
		Total: 17

*Students who pass the preparatory year with an English proficiency level of C are required to take AE 111 and AE 112.

**Students with strong background can take "*PHYS 121-PHYS 201-PHYS 202*" instead of "*PHYS 101-PHYS 130-PHYS 201* ".

D3. Mathematics Program

First Semester

Course Code	*Course Name*	*Credit*
MATH 131	Calculus for Math Students I	4
MATH 111	Int. Mathematical Structures	4
PHYS 101**	Physics I	4
---	Unrestricted Elective	3
AE 111* / HSS	Humanities or Social Sciences	3
		Total: 18

Second Semester

Course Code	*Course Name*	*Credit*
MATH 132	Calculus for Math Students II	4
MATH 162	Discrete Mathematics	4
PHYS 130	Thermodynamics, Waves, Optics and Modern Physics	4
CMPE 150	Int. to Computing	3
AE 112* HSS	Humanities or Social Sciences	3
		Total: 18

Third Semester

Course Code	*Course Name*	*Credit*
MATH 201	Matrix Theory	4
MATH 231	Calculus for Math Students III	4
PHYS 201	Physics III	4
PHIL 131	Logic I	3
---	Unrestricted Elective	3
TK 221	Turkish	2
		Total: 20

Fourth Semester

Course Code	*Course Name*	*Credit*
MATH202	Differential Equations	4
MATH 232	Int. to Complex Analysis	3
MATH 224	Linear Algebra I	3
HSS	Humanities or Social Sciences	3
---	Science Elective	3
TK 222	Turkish	2
		Total:18

Fifth Semester

Course Code	*Course Name*	*Credit*
MATH 321	Algebra I	4
MATH 331	Real Analysis I	4
MATH 343	Probability	4
HSS	Humanities or Social Sciences	3
HTR 311	History of Turkish Revolution	2
		Total: 17

Sixth Semester

Course Code	*Course Name*	*Credit*
MATH 322	Algebra II	4
MATH 332	Real Analysis II	4
MATH 336	Numerical Analysis	4
HSS	Humanities or Social Sciences	3
HTR 312	History of Turkish Revolution	2
		Total: 17

Seventh Semester

Course Code	*Course Name*	*Credit*
MATH 431	Complex Analysis I	4
---	Math Elective	3
---	Math Elective	3
---	Unrestricted Elective	3
---	Science Elective	3
		Total: 16/17

Eighth Semester

Course Code	*Course Name*	*Credit*
---	Math Elective	4
---	Math Elective	3
---	Math Elective	3
---	Unrestricted Elective	3
HSS	Humanities or Social Sciences	3
MATH 480	Seminar	2
		Total: 17/18

Minimum Credit Hours:
141/143 Credits

*Students who pass the preparatory year with an English proficiency level of C are required to take AE 111 and AE 112.

**Students with strong background can take "*PHYS 121-PHYS 201-PHYS 202*" instead of "*PHYS 101-PHYS 130-PHYS 201* ".

D.4. Course Descriptions

The following are the descriptions of the compulsory mathematics courses that the participants of the study take, all of which are offered by Mathematics Department. Note that there are special Calculus courses for Mathematics students only. Students from other departments, including Priamry and Secondary Education Teaching Mathematics Programs, take Math 101 and Math 102. Other than those, the courses Teaching Mathematics students take are common with Mathematics students.

Math 101 *Calculus I* (4+2+0) 4

Functions, limits, continuity, differentiation and applications, integration, fundamental theorem of calculus, techniques and applications of integration, improper integrals and series, Taylor polynomials, power series, basic transcendental functions.

Math 102 *Calculus II* (4+2+0) 4

Vector calculus, functions of several variables, directional derivatives, gradient, Lagrange multipliers, multiple integrals and applications, change of variables, coordinate systems, line integrals, Green's theorem and its applications.

Math 111 *Introduction to Mathematical Structures* (4+2+0) 4

Propositional logic, quantification, methods of proof, sets, relations, functions and operations, equivalence relations, cardinality, introduction to algebraic structures.

Math 131 *Calculus for Mathematics Students I* (4+2+0) 4

Fundamental properties of real numbers, sequences and subsequences, Bolzano-Weierstrass theorem, limits of functions, continuity, intermediate and extreme value theorems, differentiation and its applications, mean value theorems.

Math 132 *Calculus for Mathematics Students II* (4+2+0) 4

Riemann integration, fundamental theorem of calculus, techniques and applications of integration, improper integrals, basic transcendental functions, infinite series, convergence tests, Taylor polynomials, power series.

Math 162 *Discrete Mathematics* (4+2+0) 4

Introduction to basic problems, sums and recurrences, elementary number theory, properties of binomial coefficients, special numbers, discrete probability theory, generating functions.

Math 201 *Matrix Theory* (4+2+0) 4

Matrix algebra, determinants, Gaussian elimination, Cramer's rule, inverses, systems of linear equations, rank, eigenvalues and eigenvectors, introduction to linear programming.

Math 202 *Differential Equations* (4+2+0) 4

First-order differential equations, linear equations, homogeneous and non-homogeneous, series solutions, the Laplace transform, systems of first-order linear equations, boundary value problems, Fourier series.

Math 224 *Linear Algebra I* (3+2+0) 3

Vector spaces, linear transformations, rank and nullity, change of basis, canonical forms, Euclidean spaces, Gram-Schmidt orthogonalization process.

Math 231 *Calculus for Mathematics Students III* (4+2+0) 4

Vector calculus, functions of several variables, directional derivatives, gradient, vector-valued functions, divergence and curl, Taylor's theorem, Lagrange multipliers, multiple integrals, change of variables, line integrals, Green's theorem.

Math 232 *Introduction to Complex Analysis* (3+2+0) 3

The field of complex numbers, the extended complex plane and its topological properties, series of complex functions, M-test, power series, analytic functions, elementary functions and their mapping properties.

Math 321 *Algebra I* (4+2+0) 4

Introduction to group theory, subgroups, Lagrange's theorem, factor groups, permutation groups, group homomorphisms, isomorphism theorems, introduction to ring theory, ideals, ring homomorphisms, divisibility, polynomial rings, field of rational functions.

Math 322 *Algebra II* (4+2+0) 4

Vector spaces over an arbitrary field, linear independence and bases, linear transformations and matrices, fields, field extensions, algebraic extensions, Kronecker's theorem, finite fields.

Math 331 *Real Analysis I* (4+2+0) 4

Metric spaces, convergence, completeness, continuity, compactness, connectedness, contraction mapping principle.

Math 332 *Real Analysis II* (4+2+0) 4

Sequences and series of functions, Arzela-Ascoli theorem, Stone-Weierstrass theorem, Fourier series, inverse and implicit function theorems, integration.

Math 336 *Numerical Analysis* (4+2+0) 4

Solutions of nonlinear equations, Newton's method, fixed points and functional iterations, LU factorization, pivoting, norms, analysis of errors, orthogonal factorization and least square problems, polynomial interpolation, spline interpolation, numerical differentiation, Richardson extrapolation, numerical integration, Gaussian quadratures, error analysis.

Math 343 *Probability* (4+2+0) 4

Sets and counting, probability and relative frequency, conditional probability, Bayes theorem, independence, discrete and continuous random variables, binomial, Poisson and normal distributions, functions of random variables, law of large numbers, generating functions, characteristic functions, moments, compound distributions, central limit theorems, Markov chains and their limiting probabilities.

Math 431 *Complex Analysis I* (4+2+0) 4

Complex differentiation, Cauchy-Riemann equations, holomorphic functions, conformal mappings, contour integration, Cauchy's theorem, Taylor and Laurent series, open mapping theorem, maximum modulus principle, applications of the residue theorem.

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