MODELLING EFFECTS OF CORRELATION IN PRICE COMPETITION

by

Engin Yıldız

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ABSTRACT

MODELLING EFFECTS OF CORRELATION IN PRICE COMPETITION

Firms try to sell their products or services with a predetermined scheme to earn the most profit possible. For this, they first should know the demand characteristics of their product or service. Commonly, the market structure of a product or service is not a monopoly, that is, there are several firms competing on the sale of a typical product or service. In such environments, it is possible that customers decide on their choice considering the whole market instead of focusing on a particular firm. Having this in mind, one should consider the correlation between the demands of different firms' product or services to better analyze the system. In this study, we examine a price competition under the assumption of a simple demand model which utilizes willingness to pay distributions of the customers in the market. The model allows that the demands of the firms may be correlated. This way, the demand model takes into account the dependencies between the firms' services. We present the characteristics of the model and the equilibrium behavior of the competition. After, we investigate the dynamics of the competition under the assumption of Gumbel's bivariate exponential distribution (so-called Gumbel's Type 2) for the demands for the firms. We prove the existence of a unique Nash equilibrium for the price decisions of the firms. It turns out that under certain conditions, the equilibrium price of the firms may be higher than the monopoly case optimum price. Besides, some comparative statics for centralization and decentralization are presented. Numerical examples are presented for different cases.

ÖZET

FİYAT REKABETİNDE KORELASYONUN ETKİLERİNİN MODELLENMESİ

Firmalar mümkün olan en fazla karı elde etmek için ürünlerini veya hizmetlerini önceden belirlenmiş bir planla satmaya çalışırlar. Bunun için öncelikle ürün veya hizmetlerinin talep özelliklerini bilmelidirler. Genellikle, bir ürünün veya hizmetin pazar yapısı bir tekel değildir, yani, tipik bir ürün veya hizmetin satışında rekabet eden birkaç şirket vardır. Bu tür ortamlarda, müşterilerin belirli bir firmaya odaklanmak yerine tüm pazarı göz önünde bulundurarak seçimlerine karar vermeleri mümkündür. Bu göz önüne alındığında, sistemi daha iyi analiz etmek için farklı firmaların ürün veya hizmetlerinin talepleri arasındaki ilişki göz önünde bulundurulmalıdır. Bu çalışmada, piyasada müşterilerin ödeme istekliliği dağılımlarını kullanan basit bir talep modelinin varsayımı altında fiyat rekabeti incelendi. Model firmaların talepleri arasında korelasyon olmasına olanak vermektedir. Bu şekilde, talep modeli, firmaların hizmetleri arasındaki bağımlılıkları dikkate almaktadır. Öncelikle, modelin özellikleri ve rekabetin denge davranışları sunuldu. Ardından, firmaların talepleri için Gumbel'in iki değişkenli üstel dağılım (2. tipi) varsayımı altında rekabetin dinamiklerini araştırıldı. Firmaların fiyat kararları için benzersiz bir Nash dengesinin varlığını kanıtlandı. Belli koşullar altında, firmaların denge fiyatının tekel durumundaki optimum fiyattan daha yüksek olabileceği ortaya çıkmıştır. Ayrıca, merkezileşme ve ademi merkeziyete yönelik bazı karşılaştırmalı istatistikler sunulmaktadır. Sayısal örnekler farklı durumlar için sunulmuştur.

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LIST OF SYMBOLS

$\mathcal{A}$	Best response region
p_i	Price of product i
R_i	The distribution of willigness to pay amount of the customer pool for product i
α	Correlation parameter of Gumbel's exponential distribution
λ_i	Rate parameter of the exponentially distributed wtp of the customers for firm i
π_i	Expected revenue of firm i
ρ	Correlation of a distribution

LIST OF ACRONYMS/ABBREVIATIONS

BVE	Bivarite Exponential Distribution
cdf	Cumulative Distribution Function
GFR	Generalized Failure Rate
IGFR	Increasing Generalized Failure Rate
MNL	Multinomial Logit
pdf	Probability Density Function
wtp	Willingness to Pay

1. INTRODUCTION

Pricing could be defined as the effort to set the price of a product or service. On the customer side, there are many factors affecting the decision of the customers about the worth of the product or the service. The quality, availability, suitability of a product or service could be some of those factors. Customers observe those characteristics and the price of the product or service, assess their needs, and give a decision of purchasing or not. In fact, customers set their decision of purchasing considering the price of the item, because all other factors can be thought of as used in forming a decision regarding the shopping budget. That is, a customer could determine a maximum monetary value that she is willing to pay for a product. For example, in general, if a customer wants a high-quality product, she should allocate higher budget for this shopping activity. Therefore, we could reduce multiple factors affecting the decision of the customers into a single determinant: the price. We could comfortably say that customers have some monetary amounts in their mind which is the maximum that they are willing to pay for the product of interest. This brings us to the concept of willingness to pay (wtp). As mentioned, wtp could be explained as the maximum monetary amount that a customer intends to pay for a product or service [1, 2]. For the analysis of a market, wtp bears all factors affecting the purchase decision of a customer. Although it is a parameter determined by the customer, it could reflect all aspects of the product or service.

On the other side, firms are in need of pricing their product or service in order to get the most revenue or profit possible. For this, generally, they try to plan their pricing scheme according to some predefined consumer behavior they think their potential customers exhibit. However, the characteristics of their potential customers may change in time so that updates may be needed for the pricing activity. If a firm surely knows how the customers react to the prices set in the market, it could position itself in order to gain highest possible revenue from the market. Actually, our effort in this study is to investigate the strategies of the firms in a market where customers' reactions to the prices are known in advance.

If we consider today's business environment, the wtp of the possible customer pool could be obtained thanks to some technological advances. Of course, in a traditional marketing or shopping, we could only observe if an individual would buy a product/service or she rejects it. Besides, in such market environment, it is a very rare event that a customer asks the price of a product/service multiple times and decides accordingly unless the concern is a high-price product/service like a car or a real estate. However, nowadays, online shopping and marketing let societies access the features and the prices of all kinds of products/services in seconds. Besides, the internet technologies could provide the activities someone does while exploring the products/services on the internet. All kinds of movements someone performs while on the web such as clicks, scrolls, or just moves of the mouse cursor could be logged and processed. Such information is usually referred as clickstream data. This could provide a lot of information about the behaviors, expectations or decisions of the possible customers. For example, one could easily extract how many times a customer visits a product's page and what kind of movements she made in those visits. This could provide if a customer accepts a price or not if she compares two particular items or the features she observed while investigating the product. Besides, the movements could be used to estimate the possibility of a purchase [3]. Then, firms could easily analyze what to focus on the sales of a particular product. Resultingly, one could analyze a particular product's demand with respect to the price which is actually giving the willingness to pay (wtp) of the customer pool in the market.

Consider a hotel chain trying to enter a new market where there are local housing service providers already. This chain should position its new facility according to the needs of the region and also the characteristics of potential customers. Naturally, there are many parameters which contribute to the determination of the market structure of the region. For example, it is a fundamental determinant whether the region is a touristic region or a commercial hub. If it is a touristic one, is it a seasonal tourism center like Antalya of Turkey or a year-round center of attention like Paris of France? Based on the customer profile, it may be advantageous to have a boutique hotel instead of a five-star grand facility serving all-inclusive holidays. Those characteristics could make a huge difference on the decision of the potential investor. Once the facility is established, the question of how to decide on the price of the services arises. Again, there are countless determinants of the price for any service. However, a simplistic approach could overcome this situation by considering the single determinant of the price which takes all other factors into account: the customers in the market. As stated, the market structure and profitable actions may vary extremely from time to time. This is mainly because of the customer preferences in the market. Although the customer preferences may be manipulated through serial actions such as commercials or promotions, long-term characteristics are true indicators of what the market looks like. Therefore, to be able to reveal the market structure, one should investigate the customer preferences of this market. Assuming all information about all alternatives that may influence the choice of customers is available to the whole market, we could have all customers with a willingness to pay amount for the alternatives, that is, the maximum monetary value that a customer intends to pay for a service. In such a case, customer preferences may or may not depend on the existing alternatives of the same product or service. Assuming customers have their own valuation of all alternatives, which may be correlated to the other alternatives or not, and there is no capacity or product availability restrictions, we could model a market where alternative products or services are offered to the customers who determine a willingness to pay amount for them.

Such a model assumes that customer preferences are solely based on their valuation and the price of the alternatives, therefore, all characteristics of the alternatives are assumed to be reflected to their willingness to pay amount set by the customer pool, which may imply that all information about the service or the product is available to all of the customers.

The model we propose involves the prices set by the firms in competition and the wtp distribution of the customers in the market. The firms try to maximize their expected revenue, which is market share multiplied by the price set, by setting an optimum price for their product. The model allows correlation between the wtp of the customers for the products of the two firms. Resultingly, the firms in competition build a strategy in response to the price set by their counterparts. Based on this model, we present analytical findings in a competition and also in a centralized market under the assumption of Gumbel's exponential distribution for the wtp of the customers. We first investigate the competition between two firms and make parametric analysis, then we study the centralized market under the same distribution. After that, we present the numerical results for all the cases investigated. The sections in the thesis are organized as the following: a summary of the related literature is presented in the next section, it is followed by the description of the model we propose, then in section 4, the analytical findings under the assumption of bivariate exponential distribution for wtp of the customers are presented, numerical results are shared in section 5, the conclusion in section 6 finalizes the thesis.

2. LITERATURE REVIEW

Our problem setting is actually studied under various topics in the literature. The objective of the problem is pricing, the environment is a revenue management competition, and the main concern is actually the modelling of the demand in the market, where willingness to pay of the customers is utilized. Pricing is studied in different settings in which one of the main differentiation is the existence of an inventory. In general, while the only objective is to set the price in a pricing without inventory problem, there are other concerns in the pricing with inventory problems such as the replenishment policy and the inventory level. Besides, the problem fundamentally changes if there are multiple firms in the market. If there is a monopoly, the problem usually comes down to meet a demand in the market. However, if there is a competition, the problem becomes more complicated. This angle that the literature is interested about our problem setting is competitive revenue management where there is usually a game theoretic competition among several firms with concerns such as pricing, inventory management, etc. Here, the modelling of the problem requires more sophisticated approaches. At this point, the concern of modelling the demand arises for researchers. There are several approaches to modelling the demand in revenue management games, among which are linear model and attraction model. Although all mentioned topics are interrelated, we try to outline the literature that concerns our problem setting. Besides, in modelling, we benefit from the concept of willingness to pay. We give a brief account of recent methods used in measuring the wtp of the customers in the market.

2.1. Pricing

The problem we are interested in is basically a pricing problem. In pricing, there are different categories studied in the literature. Philips [4] gives alternative approaches to pricing as cost-plus, market-based, and value-based pricing. Cost-plus is the oldest approach that proposes a profit margin on top of the total cost of a product or service as the price of it. Market-based pricing offers a pricing scheme which compromises with the prices of the existing alternatives in the market. Although there may be other meanings, value-based pricing defines pricing as the determination of the prices from the value created for the customers. In our context, the pricing we deal with is somewhere in between the market-based and value-based pricing. Plus, the pricing problem fundamentally changes when we include inventory in the problem context. Therefore, the related literature could be reviewed in two different setting: pricing with inventory and pricing without inventory.

2.1.1. Pricing with Inventory

In this problem setting, the decisions about the price and inventory level are usually made simultaneously. Kocabıyıkoğlu and Popescu [5] presents new optimality conditions for coordinated and uncoordinated pricing and inventory problems under stochastic demand similar to Lariviere and Porteus [6], which proposes optimality conditions for pricing problem without inventory. The paper introduces a measure of elasticity, which is called elasticity of lost sales rate, which provides optimality conditions for pricing in newsvendor problem.

Federgruen and Heching [7] presents a single product simultaneous pricing and inventory replenishment problem in a multiperiod setting. The objective of the problem is to maximize total expected time average profit. The paper gives optimal policies for finite and infinite horizon problems. Petruzzi and Dada [8] introduces a similar setting in which the inventory level is also a decision variable under a single period environment.

Gallego and van Ryzin [9] investigates optimal dynamic pricing under a fixed inventory to be consumed in a predetermined period which we could find the examples in seasonal markets such as fashion goods and tourism sectors. It provides optimal pricing for a family of exponentially distributed demand functions.

Goyal and Giri [10] gives an outline of the advances in demand models with a deteriorating inventory which spans the inventory policies for the goods with shelf-life. Although the main categorization is on the characteristics of the shelf-life of the product, subcategorizations include deterministic and stochastic demand variations with several assumed demand distributions.

Chan *et al.* [11] presents a classification of the studies in pricing under inventory considerations. The survey includes different pricing schemes, such as fixed and dynamic pricing, and different period settings, single and multiple periods, as well as different product quantity models, single and multiple item models. Besides, the applications of different strategies in the industry are also presented.

Elmaghraby and Keskinocak [12] presents and classifies the literature and the current practices only in dynamic pricing with inventory. It provides the dynamic pricing studies in literature depending on different categorizations. The categorizations include allowance of replenishment (replenishment or no replenishment) and customer attitudes (myopic or strategic customers). Our study includes only the myopic customers since the problem is single period and the customers' decisions are one-time decisions.

2.1.2. Pricing without Inventory

In this problem type, it is usually assumed that there is no capacity or inventory restrictions in the problem. The concern is generally maximizing the market share or the expected market share in a stochastic demand environment. Actually, in such setting, an initial and fixed inventory may be included since it does not require any replenishment policy. If there is a fixed initial inventory, the problem could be easily turned into a without inventory type problem by normalizing the quantities so that initial inventory becomes unity. Therefore, in this setting, we could include the problems without inventory replenishment. Actually, the pricing problem without inventory has been mostly studied under economics literature, after the introduction of competition with quantity and price decisions [13, 14].

Chou and Parlar [15] presents the solution to a multi-period single product pricing problem with a fixed initial inventory where linear demand model is assumed. A dynamic pricing approach is used for determining the optimum price at each period.

Chen and Riordan [16] investigates how pricing evolves in a competition, and when the existence of market share effect and price sensitivity effect dominates in a market. The study empirically shows that there may be a competition where the equilibrium price is higher compared to a monopoly optimum price.

Allon *et al.* [17] provides upper bound for the prices in a price competition under the assumption of multinomial logit demand functions and it shows the existence of a unique equilibrium within the price bounds.

Mizuno [18] provides the proof of the existence of a unique equilibrium in a price competition under product differentiation for a special type of demand functions which include logit and nested logit models.

2.2. Competitive Revenue Management

Revenue management under service competition has been widely studied in the literature. It could be defined as the effort to make strategic, tactical, or operational level decisions to achieve a particular objective or set of objectives. Those decisions include pricing, capacity allocation, service level, etc. In the literature, this type of revenue management has been generally seen from a game-theoretic viewpoint as there is usually a game in which firms try to set their decisions in response to the decisions of the others. Cachon and Netessine [19] presents the use of game-theoretic view in studying supply chain competitions and gives the methods to demonstrate the existence and uniqueness of the equilibrium in these competitions.

While there have been competitive revenue management studies focusing on single factors such as capacity, service level, or price [6, 20–22]; there also have been studies which consider multiple factors included in the decisions [7].

Lariviere and Porteus [6] proposes a simple supply chain contract where the contract parameter is the wholesale price of the product. The problem it presents for the manufacturer side presents the IGFR property and its benefit in the solution. We benefit from the same property while making analytical proofs under the assumption of the model proposed. Lariviere [23] provides a simplified characterization of IGFR property to ease verification of the existence of such property. Banciu and Mirchandani [24] gives extensive information on probability distributions with IGFR property.

Martinez-de-Albéniz and Talluri [20] studies a dynamic price competition with fixed capacities under stochastic demand environment. The existence of a unique equilibrium in a multi-stage Bertrand competition is shown.

Netessine and Rudi [25] presents a supply chain competition where the competition is on the inventory held by internet retailers and the wholesaler. The choices of holding the inventory and direct shipment from the supplier create a game theoretic competition among the retailers and their supplier.

Netessine and Shumsky [21] studied an airline revenue management problem under vertical and horizontal competition in which former refers to the competition for the same flight leg and the latter refers to the competition for the sales of the flights in a multi-leg itinerary. It studies the competition as the inventory management problem which corresponds to the number of seats allocated to each fare classes.

Gallego and Hu [22] studies a price competition where there are fixed selling capacities of the competitors who are to set dynamic prices over a finite horizon. It provides existence and uniqueness of the equilibrium for linear and multinomial logit demand rate functions.

Of course, those price, capacity or service level decisions are highly dependent on the market conditions. For example, a seasonal market may require different pricing schemes in different periods in a year, or there may be peak times in a year that particular service should have a higher capacity than usual [26]. Also, there may be different constraints or considerations while those decisions are made. Besides, there may be multiple firms competing or there may be a monopoly market. All in all, revenue management could be handled by considering various factors or assumptions depending on the context.

2.3. Modelling Demand

For revenue management activities, modelling the demand is central. Constructing the model for the demand becomes the determinant of the decisions to be made. There have been various approaches to demand modelling in literature. The models for operational decision making are extensively investigated by Huang *et al.* [27], and although there are models considering factors like rebate, lead time, etc., most commonly used models concern about price and quality of the product or service. In fact, most of the models are constructed dependent on the price of the product as the price is seen as the dominant factor shaping the decisions of customers. Of course, another major determinant for developing the model is the market type. If there are more than a single firm, the operational decisions become more complicated as the decisions may affect or be affected by the decisions of other firms. In short, multiple firms may create a competition in their decisions. In fact, there are widely-used demand models in revenue management literature. If there is a monopoly, there are several demand models used frequently in literature. Linear model [8], exponential model [5, 28], power model [5], and logit model [4] are most widely used single firm demand models. If a competition is to be modeled, most of the models in literature are linear demand models or attraction demand models [27]. The linear model in a competition assumes a linear relationship between the price of the product and associated demand [29, 30], and utility obtainable from a product is used to explain the likelihood of a purchase by a customer. Attraction model, usually a multinomial logit model (MNL), assumes an attractiveness for the choices and utilizes the probability of a purchase from the attractiveness of the alternatives. Generally, in the use of this model, no-purchase option is assumed to have an attractiveness, too, otherwise, it is assumed not to have a no-purchase option.

It seems that Whitin [31] is the first study investigating the pricing decision in the newsvendor problem. As the model is built for a monopoly and for simplicity, the demand function is assumed to be a linear function. We could see this work as an example use of linear model on pricing. Petruzzi and Dada [8] also assumes a linear model to study pricing on the same problem. Lus and Muriel [32] uses a linear model to investigate product substitutability under monopoly and oligopoly.

Chen *et al.* [33] studies a pricing problem where the inventory is reviewed periodically. As in Kocabıyıkoğlu and Popescu [5], the study assumes an exponential demand model for a monopoly.

Bernstein and Federgruen [34] uses an attraction demand model to develop a stochastic general equilibrium inventory model for an oligopoly. Gallego *et al.* [35] also assumes an attraction demand model and shows the existence of a unique Nash equilibrium in a price competition similar to our study. Bitran and Ferrer [36] uses the attraction model to optimize the pricing and the composition of bundles offered to the customers.

Price-dependent demand models are widely used in literature to explain the behavior of the customers in purchasing decisions. As stated, some of those models are constructed in order to take into account additional factors such as quantity and quality. For example, Xiao and Yang [37], Bernstein and Federgruen [34] examine a competition model where price and service are both in concern. Van *et al.* [38] includes capacity decision together with price and quantity decisions in the model.

The models dependent on the price may include different prices than the current price of the product. Güler *et al.* [39] studies a demand model with dynamic inventory control and pricing decisions in which the reference effect, which is the effect of the former prices on the demand of the product, is particularly of interest.

Our model differs from the existing demand models in several ways. We use willingness to pay (wtp) concept that represents customers' attitude towards the whole market, which is an extension of Kalish [40]. To the best of our knowledge, this concept in the literature is used only for monopoly models. Existing commonly used competition models usually assume a utility from a product or service which could be defined as the benefit a customer possesses by purchasing the item. This requires considering different aspects of the product-customer relationship in order to assume this utility. Also, one could not propose a direct way of assuming such utility as it could not be communicated clearly. However, a customer's wtp could be obtained directly since almost every potential customer could give a maximum price s/he is willing to pay for a service or product, for example, in a survey. Another aspect of our model is that existing literature using wtp in demand modelling deal with monopoly cases, however, our model is applicable for competitions via correlated wtp distributions. Besides, to keep the model simple, we do not include any capacity restrictions and normalize the market size, we make the analysis based on the fact that one representative customer is in the analysis and we calculate the market shares as the probabilities of this customer's purchase behavior.

2.4. Willingness to pay

We use the concept of willingness to pay in our demand model to explain customer purchase behavior. Advances in technologies related to marketing and retail businesses allow more reliable measurements about the wtp of the customers in a market. Several methods for measuring wtp is proposed in the literature. Generally, the real purchase data and surveys are used to estimate wtp of the customers. However, there have been studies revealing the problems using such methods for wtp estimation [41, 42].

Breidert *et al.* [43] presents the systematic categorization of the methods used to estimate wtp. Potential problems with the methods are discussed. Comparison of the methods is also presented.

Miller *et al.* [41] compares the common methods of measuring wtp of the customers. Real purchase data and open-ended question format are among the methods the study compares. It is argued that although there may be hypothetical bias in estimations, some methods, e.g. open-ended question survey, may result in right demand distributions.

Chan *et al.* [44] studies estimation of wtp from internet auctions. The study benefits from data from notebook auctions to reveal customers' wtp.

Wetenbroch and Skeira [42] presents the result of wtp estimates from incentive-based approaches, which incentivize the customers to reveal their true wtp, and non-incentive-based approaches. It concludes that incentive-based approaches yield lower wtp estimations than the others.

3. MODEL DESCRIPTION

In a typical market, customers try to decide on their choice based on the available information about the product or service and the budget they allocate. If the market is a monopoly, e.g. if there is only one firm providing the product or service in the market, the process is relatively simple for the customers: they only decide whether they want to buy the product or not. However, if there are multiple firms in the market, the need for comparison between the firms arises which complexifies the process for customers as well as for market analysis. Such activities in a market could be modeled in various ways. For such modelling, one first needs to decide on the factors influencing the decisions of the customers. Huang *et al.* [27] classifies the commonly used demand models in the literature based on the factors that the models include. It is stated that although there are models considering the rebate, advertising, lead time, and available space, most of the models use price and the quality of the product or service as the determinants in the market. In fact, this is because the customers in the market are believed to decide on their choice basing their decisions on those attributes. After deciding on the factors, the problem of in what way those factors should affect the choices of the customers arises, for example, the effect of the attributes on the purchase may be deterministic or stochastic. In literature, there are models where the determinants shape a utility for the choices, e.g. attraction demand model, while there are other models where the determinants directly influence the probability of a purchase for an individual, e.g. the linear model. The downside of the models where there is a utility attributed to every choice is that the utility is not a feature that is directly quantifiable for individuals. On the other hand, the models where the determinants directly influence the choices may be applicable in some cases where there is a similarity with the real life. For example, we could use the exponential model where there is a monopoly with customer valuations following an exponential distribution. However, in this setting, building a model where there are multiple firms competing may be problematic, since it usually requires to include dependencies between the firms which possibly influences the choices of the customers in the market. We could overcome the difficulties by considering a single determinant in shaping the decisions of the customers: the price. In fact, it is not an unreliable assumption if we say that the customers make their choice based on the price of the alternatives if they possess all information about them. This assumption shifts all difficulties from the modeller to the customers in the market, as they are the decision makers who determine the influence of all other factors on the price of a product. Actually, in literature, there is a term which explains this situation: willingness to pay (wtp) [40]. Wtp of a customer is the maximum monetary value that she accepts paying for a particular product. In modelling, we could benefit from this attribute of the customers. Actually, the wtp concept has been used in the examination of monopoly cases where the market analysis is relatively simple [4, 45]. However, to the best of our knowledge, this concept has not been utilized in a multi-firm competition environment.

Based on this view, we build a competition model that is simple and comprehensive with respect to customer preferences and existing alternatives. The model is applied on a market share basis, that is, the objective of a firm is to maximize its market share by setting an optimum price. Here, by market share, we mean the proportion of all possible customers in the market where the population who may not purchase a product is also included. If we represent a duopoly in our model, we introduce the market share of a firm's product or services as:

$$Pr\{R_1 \geq p_1, R_1 - p_1 \geq R_2 - p_2\}$$

Here, R_i stands for the random variable representing the willingness to pay amount of the customer population for the product or service provided by firm i, while p_i stands for the price set by the firm i for the product or service. As seen, to capture a customer, a firm should have a price that not only attracts the customer but also attracts the customer more than the other alternative in terms of its price. In the economics literature, the positive difference between the wtp of a customer and the price is called consumer surplus [2, 46]. Therefore, the model is based on maximizing the consumer surplus from the customer side, while it is based on maximizing the expected revenue from the firm side.

As stated, within this model, it is possible for a customer not to purchase from any of the alternatives, therefore, the sum of the market shares of all competitors does not have to reflect the total demand potential of the market.

The model does not include any cost figure. We could consider the case as if the cost structures of the competitors are quite similar so that the concern is for the revenue generation. This makes the model easier to analyze. Besides, this is not an unrealistic approach as competitive firms approximately incur similar fixed or variable costs in their economic activities. Therefore, the terms revenue and profit could be used interchangeably without any information loss.

The model assumes that all views and expectations of the customers about the product or the service are reflected to the willingness to pay (wtp) of the customers in the market. Therefore, for example, it is very obvious that if a customer has the same wtp for all alternatives, she chooses to purchase the cheapest one only if its price is lower than her wtp. In fact, it is a reasonable view when we think of a customer coming to a touristic region with no reservation. Assuming all information is available to her and there are two equivalent alternatives available, she would choose the cheaper one only if she could afford it.

Under the assumption of such model, we could investigate three different cases, namely, the monopoly case, the competition, and the centralized market.

3.1. The Monopoly Model

The first environment is the simplest case where there is only one firm and it tries to maximize its expected revenue from the market. The rules are as stated before: the customer purchase the product if she has a consumer surplus, otherwise she does not purchase. Since there is no competitor, there is no need to compare consumer surpluses of the firms. In this model, we could show the market share of the firm as the following:

$$Pr\{R \geq p\}.$$

Then firm's problem is to maximize expected revenue, the following:

$$pPr\{R \geq p\}.$$

Since there is no competition, the problem is rather simple. From the properties of the probability, the expected revenue is a nonincreasing function of the price. Therefore, the solution could be found from the first order condition:

$$\frac{\partial}{\partial p}\Pi(p^*) = \frac{\partial}{\partial p} p^* Pr\{R \geq p^*\} = Pr\{R \geq p^*\} + p^* \frac{\partial}{\partial p} Pr\{R \geq p^*\} = 0,$$

$$p^* = -\frac{Pr\{R \geq p^*\}}{\frac{\partial}{\partial p} Pr\{R \geq p^*\}}.$$

This case is actually investigated by Lariviere and Porteus [6] where there is a special definition for the distributions resulting in a unique p value: increasing generalized failure rate (IGFR). Generalized failure rate is defined as

$$g(x) = x\frac{\phi(x)}{1 - \Phi(x)}$$

where $\phi(x)$ is the pdf and $\Phi(x)$ is the cdf of the distribution that random variable X has. Then, we could write the condition above as follows:

$$\frac{\partial}{\partial p}\Pi(p^*) = D(p^*) + p^* D'(p^*) = \bar{\Phi}(p^*)(1 - g(p^*)) = 0$$

where $D(p^*)$, demand function, corresponds to $\bar{\Phi}(p^*)$, and $D'(p^*)$, the first derivative of the demand function, corresponds to $-\phi(p^*)$. If the distribution possesses the IGFR property, then it leads to a unique p value in this context, since an increasing generalized failure rate can only be equal to 1 at one point which is the optimum price.

3.2. The Competition between two Firms

In this setting, there are two firms competing with each other to maximize their expected revenue. Here, they only need to decide on the price of their product observing their counterparts' price decision. As stated before, the firms could define their strategy in response to every possible price of their competitors from the beginning, since there is no time dimension to the problem. That is, it is not of a question whether a firm first decides on the price or it waits for the competitor's decision. Here, the condition for a customer to choose one of the alternatives is that the price of the product should lead to maximum positive consumer surplus. That is, the product's price should be below the wtp of the customer and also the consumer surplus this product provides should be the maximum among the alternatives. In such model for a duopoly, assuming a distribution for the willingness to pay amount of the customers, the expected revenue of firm 1 becomes:

$$
\begin{aligned}
\Pi_1(p_1, p_2) &= p_1 Pr\{R_1 \geq p_1, R_1 - p_1 \geq R_2 - p_2\} \\
&= p_1 Pr\{R_1 - max\{0, R_2 - p_2\} \geq p_1\} \\
&= p_1 Pr\{Y_1(p_2) \geq p_1\} \\
&= p_1 \bar{H}_1(p_1, p_2)
\end{aligned}
$$

Here, we could define $Y_1(p_2)$ as the net valuation for firm 1's product when the other party quotes price p_2. Under this definition, the optimal strategy could be obtained with a similar technique as in single product revenue management problem:

$$
\begin{aligned}
\frac{\partial \Pi_1(p_1, p_2)}{\partial p_1} &= \bar{H}_1(p_1, p_2) - p_1 h_1(p_1, p_2) \\
&= \bar{H}_1(p_1, p_2) \left(1 - \frac{p_1 h_1(p_1, p_2)}{\bar{H}_1(p_1, p_2)}\right) \\
&= \bar{H}_1(p_1, p_2) \left(1 - g_1(p_1, p_2)\right)
\end{aligned}
\tag{3.1}
$$

Since $g_1(0, p_2) = 0$, if $g_1(p_1, p_2)$ is increasing in p_1 for a given value of p_2, the best response to price p_2 can be obtained as

$$\frac{p_1 h_1(p_1, p_2)}{\bar{H}_1(p_1, p_2)} = g_1(p_1, p_2) = 1$$

This is a two-product extension of IGFR property of Lariviere and Porteus [6].

$$\frac{E_{R_2}[f_1(x + (R_2 - p_2)^+ | R_2)]x}{E_{R_2}[\bar{F}_1(x + (R_2 - p_2)^+ | R_2)]}$$

increases in x for all $p_2 \geq 0$. We could consider the case as there is a firm which tries to maximize its expected revenue under an assumption of a wtp distribution of the customer pool which is conditional on the price set by the competitor. Actually the conditional distribution bears the same properties that the single variate distribution has. This is mainly because the conditional distribution is also a single variate distribution and multivariate distribution is only the collection of single variate conditional distributions. As a result, we expect an optimum price in response to all possible prices that the competitor could set for its product.

If there is no correlation between the wtp distributions of the customers for the two firms, the case becomes two distinct monopoly as in the first setting. Then, we should see the same results in both models when there is no correlation.

3.3. Centralized Model

In this model, it is assumed that there are two firms in the market whose owners are the same. Therefore, there is no more competition but there is a cooperation between the firms to maximize the total revenue generated. Certainly, if there is no correlation between the wtp distributions of the customers for the two firms, the model becomes two separate monopoly model discussed in the first section. Therefore, again, we expect the same result in case there is no correlation between the wtp distributions of the customers for the firms' product. The market share the firms hold together could be expressed as follow the following:

$$Pr\{R_1 \geq p_1, R_1 - p_1 \geq R_2 - p_2\} + Pr\{R_2 \geq p_2, R_2 - p_2 \geq R_1 - p_1\}.$$

The total expected revenue they try to maximize becomes

$$\begin{aligned}\Pi(p_1, p_2) &= \Pi_1(p_1, p_2) + \Pi_2(p_1, p_2) \\ &= p_1 Pr\{R_1 \geq p_1, R_1 - p_1 \geq R_2 - p_2\} + p_2 Pr\{R_2 \geq p_2, R_2 - p_2 \geq R_1 - p_1\}.\end{aligned}$$

To be able to obtain the optimum price levels, we need to take the derivative with respect to both prices p_1 and p_2 and make it equal to zero:

$$\begin{aligned}\frac{\partial^2 \Pi(p_1, p_2)}{\partial p_1 \partial p_2} &= \frac{\partial^2 \Pi_1(p_1, p_2)}{\partial p_1 \partial p_2} + \frac{\partial^2 \Pi_2(p_1, p_2)}{\partial p_1 \partial p_2} \\ &= \frac{\partial}{\partial p_2} \bar{H}_1(p_1, p_2)\,(1 - g_1(p_1, p_2)) + \frac{\partial}{\partial p_1} \bar{H}_2(p_1, p_2)\,(1 - g_2(p_1, p_2)) = 0\end{aligned}$$

from equation 3.1. We could present the portion of the customers that firms could make a sale as follows:

$$1 - Pr\{R_1 \leq p_1, R_2 \leq p_2\}.$$

Actually, in this case, the portion of the customers the firms are not able to capture is the population that has a wtp which is lower than both firms' prices.

4. ANALYTICAL FINDINGS: BIVARIATE EXPONENTIAL DISTRIBUTION AS JOINT WILLINGNESS TO PAY DISTRIBUTION

To be able to analyze the competition model, a joint willingness to pay distribution which allows correlation is needed. Also, it should be analytically tractable. Besides, the effect of correlation is to be investigated, therefore the distribution should allow both positive and negative correlation. One of the distributions providing those necessities is Gumbel's type II exponential distribution [47]. This distribution has the following joint pdf and cdf:

$$f(r_1, r_2) = e^{-\lambda_1 r_1 - \lambda_2 r_2}(1 + \alpha(2e^{-\lambda_1 r_1} - 1)(2e^{-\lambda_2 r_2} - 1)),$$

$$F(r_1, r_2) = (1 - e^{-\lambda_1 r_1})(1 - e^{-\lambda_2 r_2})[1 + \alpha e^{-\lambda_1 r_1} e^{-\lambda_2 r_2}].$$

The distribution allows negative correlation, yet in a limited range: $Corr(R_1, R_2) = \rho = \alpha/4$. Expected wtp of the customers: $E[R_i] = 1/\lambda_i$, and $\alpha \in [-1, 1]$. As a result, conditional expected value of one variable becomes:

$$E[R_1|R_2 = r] = \frac{2+\alpha}{2\lambda_1} - \frac{\alpha}{\lambda_1}e^{-\lambda_2 r}.$$

Let us see the monopoly case under the assumption of single variate exponential distribution as the wtp distribution of the customers. Actually, in this case, the demand model is exponential model frequently used in literature [5, 28]. Under such assumed model, the firm's expected demand becomes:

$$Pr\{R \geq p\} = 1 - F(p) = e^{-\lambda p}.$$

Then, the expected revenue of the firm is

$$\Pi(p) = pPr\{R \geq p\} = pe^{-\lambda p}.$$

To find the optimum price, we usually apply the first order condition, the case that the first derivative is equal to zero. However, for this, one needs to make sure that the profit function is unimodal. At this point, Lariviere and Porteus [6] introduces the concept of generalized failure rate (GFR). GFR of a probability distribution is defined as:

$$g(\xi) = \xi h(\xi)$$

where $h(\xi)$ is the classical failure rate defined by Barlow *et al.* [48]:

$$h(\xi) = \frac{\phi(\xi)}{1 - \Phi(\xi)}.$$

Here, ϕ and Φ refers to pdf and cdf of the distribution, respectively. The distributions with increasing generalized failure rate is named as IGFR distributions [23]. IGFR distributions has some useful properties from which pricing and revenue management applications benefit. If a distribution is IGFR, then a profit function with customer valuations with this distribution is unimodal and has single optimum price [23]. Banciu and Mirchandani [24] presents common discrete and continuous IGFR distributions which involves exponential distribution. As a result, we could find the optimum price that the firm sets to maximize its profit by:

$$\begin{aligned}
\frac{\partial \Pi(p^*)}{\partial p} = \frac{\partial}{\partial p} p^* e^{-\lambda p^*} &= 0 \\
e^{-\lambda p^*}(1 - \lambda p^*) &= 0 \\
\Rightarrow p^* &= \frac{1}{\lambda}.
\end{aligned}$$

This shows that the solution of the monopoly case is the mean of the exponential distribution assumed for the wtp distribution of the customers in the market.

Now, let us see the second case under the assumption of Gumbel's bivariate exponential distribution as the wtp distribution of the customers for the two firms. If we use this distribution as customers' wtp distribution in our demand model for a duopoly, the expected demand function for the product of firm 1, the first and second derivatives of its complement becomes:

$$\begin{aligned} \bar{H}_1(p_1,p_2) &= a_1 - a_2 + a_3 + a_4 - a_5, \\ h_1(p_1,p_2) &= \frac{\partial H_1(p_1,p_2)}{\partial p_1} = \lambda_1(a_1 - a_2 + 2a_3 + a_4 - 2a_5), \\ h_1'(p_1,p_2) &= \frac{\partial h_1(p_1,p_2)}{\partial p_1} = -\lambda_1^2(a_1 - a_2 + 4a_3 + a_4 - 4a_5) \end{aligned} \tag{4.1}$$

respectively, where we define the following for the simplicity:

$$\begin{aligned} a_1 &= e^{-\lambda_1 p_1}, \\ a_2 &= \frac{(1+\alpha)\lambda_1}{\lambda_1+\lambda_2} e^{-\lambda_1 p_1 - \lambda_2 p_2}, \\ a_3 &= \frac{2\alpha\lambda_1}{2\lambda_1+\lambda_2} e^{-2\lambda_1 p_1 - \lambda_2 p_2}, \\ a_4 &= \frac{\alpha\lambda_1}{\lambda_1+2\lambda_2} e^{-\lambda_1 p_1 - 2\lambda_2 p_2}, \\ a_5 &= \frac{\alpha\lambda_1}{\lambda_1+\lambda_2} e^{-2\lambda_1 p_1 - 2\lambda_2 p_2}. \end{aligned} \tag{4.2}$$

The detailed derivation of these equations could be found in Appendix A. For example, if customer valuations (wtp) for the two firms are uncorrelated ($\alpha = 0$), expected demand function becomes the following:

$$\bar{H}_1(p_1,p_2) = e^{-\lambda_1 p_1}\left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-\lambda_2 p_2}\right).$$

We could think of this as follows: The expected demand for the first firm is fully considered as in a monopoly case ($e^{-\lambda_1 p_1}$), from this total expected demand, our firm looses some of the customers whose consumer surplus is greater ($R_2 - p_2 \geq R_1 - p_1$) if they purchase second firm's product or service. Actually, the following holds when there is no correlation between the wtp distributions (R_1 and R_2) of the customers:

$$\begin{aligned}\bar{H}_1(p_1,p_2) =& Pr\{R_1 \geq p_1 + (R_2 - p_2)^+\}\\ =& Pr\{R_1 \geq p_1\}Pr\{R_1 \geq (R_2 - p_2)^+\}\end{aligned}$$

where

$$(R_2 - p_2)^+ = max\{0,(R_2 - p_2)\}.$$

Since multivariate distributions are collection of conditional single variate distributions, we could make use of IGFR property for multivariate distributions. Actually we could say that IGFR property leads to an optimum price for every single price set by the competitor. Single variate IGFR wtp distributions lead to a single optimum price, while bivariate IGFR wtp distributions lead to a best response function which maps all possible price levels of the competitor to the optimum prices for the firm.

Proposition 4.1. *For Gumbel's bivariate exponential distribution, we have the IGFR property as*

$$g_1(p_1,p_2) = \frac{p_1 h_1(p_1,p_2)}{\bar{H}_1(p_1,p_2)}$$

is increasing in p_1 for every given $p_2 \geq 0$.

Proof. For simplicity, define

$$\begin{aligned}U_1 =& \bar{H}_1(p_1,p_2) = a_1 - a_2 + a_3 + a_4 - a_5 \geq 0 \ \forall\alpha,\\ U_2 =& a_3 - a_5 \begin{cases} < 0 & \text{if } \alpha < 0 \\ > 0 & \text{if } \alpha > 0 \end{cases},\\ U_1 - U_2 =& a_1 - a_2 + a_4 \geq 0 \ \forall\alpha.\end{aligned}$$

The detailed proofs of these equations could be found in Appendix A. Observe that

$$p_1^*(p_2) = \frac{\bar{H}_1(p_1^*, p_2)}{h_1(p_1^*, p_2)} = \frac{U_1}{\lambda_1(U_1+U_2)} \begin{cases} < 1/\lambda_1 \text{ if } \alpha > 0 \\ = 1/\lambda_1 \text{ if } \alpha = 0 \\ > 1/\lambda_1 \text{ if } \alpha < 0 \end{cases} \tag{4.3}$$

from equations 4.1. Then, the generalized failure rate could be expressed as follows:

$$g_1(p_1^*, p_2) = \frac{\lambda_1 p_1^*(a_1 - a_2 + 2a_3 + a_4 - 2a_5)}{(a_1 - a_2 + a_3 + a_4 - a_5)}$$

and the following holds:

$$\frac{\partial g_1(p_1^*, p_2)}{\partial p_1^*} = \frac{\lambda_1 U_1(U_1+U_2) - \lambda_1^2 p_1^* U_1 U_2 + \lambda_1^2 p_1^* U_2^2}{U_1^2} \begin{cases} > 0 \text{ if } \alpha < 0 \text{ since } U_2 < 0 \\ > 0 \text{ if } \alpha \geq 0 \text{ since } p_1^*\lambda_1 \leq 1 \end{cases}.$$

Therefore, this distribution is IGFR for a given competitor's price level p_2. □

Now, we have a distribution whose demand function is unimodal on $[0, \infty)$ meaning that for a given price level of the competitor p_2, firm 1 has a sinlge profit-maximizing price level $p_1^*(p_2)$. This allows us to declare that the response of firm 1 is a function $p_1^*(p_2)$. As seen in equation 4.3, the duopoly generates a price-increasing competition if we have $\alpha < 0$. That is, any equilibrium point results in an equilibrium price higher than the price in the monopoly case. This case is numerically investigated in Chen and Riordan [16]. We may illustrate this situation by considering an extreme case example: Suppose that there are two firms whose products are substitutable and the firms are in a price competition. Also, the customer environment is such that people are very fanatic about their product choice. That is, any customer in the market favors one of the product and sets her wtp amount accordingly: she loves one of the products that she is willing to pay a high amount for her favorite product while she "insults" the other product by setting her wtp very low amount. In this market, since firms try to maximize their profit and it is very hard or impossible to attract the fans of the other product, they set a higher price for their own product where their profit is maximized.

Now, some properties of this model are introduced considering the parameters of the model.

4.1. Price Competition under a Duopoly

The analysis is made on best responses and corresponding expected profits of the firms. For this, the following definition helps on derivations:

$$g_1(p_1^*(p_2), p_2) = \frac{p_1 h_1(p_1^*(p_2), p_2)}{\bar{H}_1(p_1^*(p_2), p_2)} = 1,$$

$$K_1(p_1^*, p_2) = p_1^* h_1(p_1^*(p_2), p_2) - \bar{H}_1(p_1^*(p_2), p_2) = 0.$$

This K_1 function helps us to make some derivations by the help of implicit differentiation.

4.1.1. Analysis of the best response region

There is a unique response to any price set by the competitor. Those responses may construct a region in the space that all possible actions lie on, which could be called as best response region.

Proposition 4.2. *Best response region can be confined to:*

$$\mathcal{A} = \begin{cases} \{(p_i^*, p_j) : 1/\lambda_i \leq p_i^* \leq 2/\lambda_i, \;\; i, j = 1, 2, i \neq j\} & \textit{if } \alpha < 0, \\ \{(p_i^*, p_j) : 1/2\lambda_i \leq p_i^* \leq 1/\lambda_i, \;\; i, j = 1, 2, i \neq j\} & \textit{if } \alpha > 0. \end{cases} \quad (4.4)$$

Proof. Since our profit function is unimodal, that is, any response below the best response yields a positive slope for the profit function while any reponse above the best response yields a negative slope for the profit function, we could check the sign of the slope at the boundaries:

$$\frac{\partial \pi_1(p_1^*, p_2)}{\partial p_1^*} = \bar{H}_1(p_1^*, p_2)(1 - g_1(p_1^*, p_2)). \tag{4.5}$$

Since $\bar{H}_1(p_1^*, p_2) \geq 0 \; \forall \alpha$, we could go with the rest:

$$\begin{aligned} 1 - g_1(p_1, p_2|p_2)_{|_{p_1=2/\lambda_1}} &= \frac{-a_1 + a_2 - 3a_3 - a_4 + 3a_5}{a_1 - a_2 + a_3 + a_4 - a_5} < 0 \\ \Rightarrow p_1^*(p_2) < \frac{2}{\lambda_1} \; \forall \alpha, \end{aligned}$$

$$\begin{aligned} 1 - g_1(p_1, p_2|p_2)_{|_{p_1=1/2\lambda_1}} &= \frac{1}{2} \frac{a_1 - a_2 + a_4}{a_1 - a_2 + a_3 + a_4 - a_5} > 0 \\ \Rightarrow p_1^*(p_2) > \frac{1}{2\lambda_1} \; \forall \alpha. \end{aligned}$$

The details of the inequalities could be found in Appendix A. As a result, we could say that our best response varies between $1/2\lambda$ and $2/\lambda$. □

Figure 4.1 illustrates how the confined best response region looks. Any firm responds to the other with a single price. Also, we have shown that those responses could lie in the confined best response region we just introduced. Therefore, it is enough for any analysis related to the best responses to be justified within this best response region.Now, we show how this response behaves as other parameters in the model changes. For simplicity, the following parameters are used in calculation of the numerical results: $\lambda_1 = 0.1, \lambda_2 = 0.3$.

4.1.2. Behavior of the response with respect to competitor's price p_2

Now, we could investigate how the response changes as the competitor's price p_2 changes within the confined best response region. This is actually the sudden reaction the firm makes for a price change of the competitor.

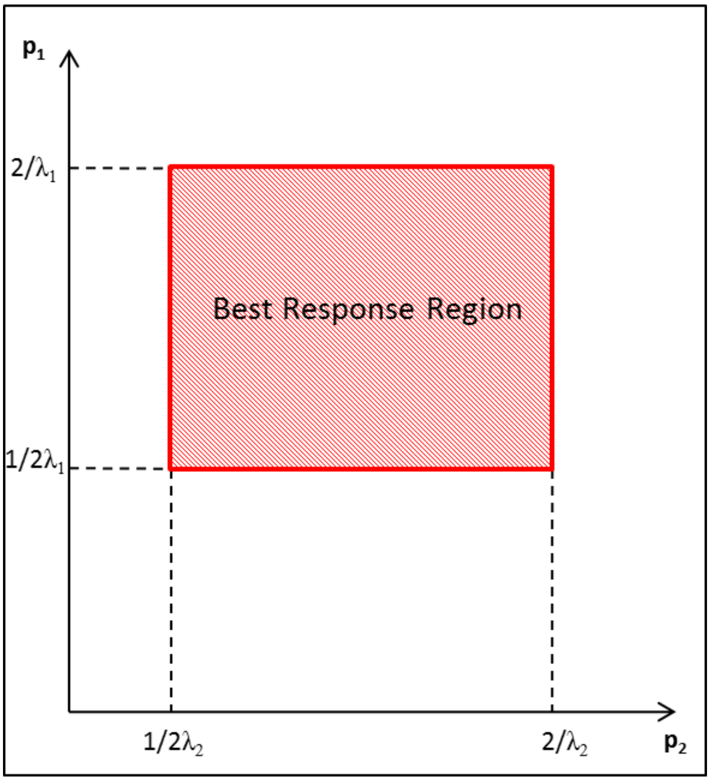

Figure 4.1. The Region where a Possible Best Response Occurs

Proposition 4.3.

$$\frac{\partial p_1^*}{\partial p_2}\begin{cases} < 0 \ \textit{if} \ \alpha < 0 \\ > 0 \ \textit{if} \ \alpha > 0 \end{cases}. \tag{4.6}$$

In words, the best response of a firm increases in response to a price increase of the other firm if there is a positive correlation between the wtp distributions of the customers for the firms, the relation is reversed if the correlation is negative.

Proof. From implicit differentiation, we could write the following:

$$\frac{\partial p_1^*}{\partial p_2} = -\frac{\frac{\partial K_1}{\partial p_2}}{\frac{\partial K_1}{\partial p_1^*}} = \frac{\lambda_2((a_2 - a_3 - 2a_4 + 2a_5) - (p_1^*\lambda_1)(a_2 - 2a_3 - 2a_4 + 4a_5))}{\lambda_1(2(a_1 - a_2 + 2a_3 + a_4 - 2a_5) - (p_1^*\lambda_1)(a_1 - a_2 + 4a_3 + a_4 - 4a_5))}.$$

Let's start with the denominator:

$$\begin{aligned}\frac{\partial K_1}{\partial p_1^*} &= 2h_1(p_1^*(p_2), p_2) + h_1'(p_1^*(p_2), p_2) \\ &= \lambda_1(2(a_1 - a_2 + 2a_3 + a_4 - 2a_5) - (p_1^*\lambda_1)(a_1 - a_2 + 4a_3 + a_4 - 4a_5)) \\ &= \lambda_1((2 - p_1^*\lambda_1)(a_1 - a_2 + a_4) + 4(1 - p_1^*\lambda_1)(a_3 - a_5)) \\ &\geq 0 \ \forall \alpha.\end{aligned} \tag{4.7}$$

Then, the numerator:

$$\begin{aligned}-\frac{\partial K_1}{\partial p_2} =& \lambda_2((p_1^*\lambda_1)(a_2 - 2a_3 - 2a_4 + 4a_5) - (a_2 - a_3 - 2a_4 + 2a_5)) \\ =& \lambda_2((1 - p_1^*\lambda_1)(a_2 - 2a_4) + (2p_1^*\lambda_1 - 1)(a_3 - 2a_5)) \\ =& \lambda_2(\frac{U_2}{U_1 + U_2}(a_2 - 2a_4) + \frac{U_1 - U_2}{U_1 + U_2}(a_3 - 2a_5)) \\ =& \lambda_2\left(\frac{a_5(a_2 - 2a_1) + a_3(a_1 - a_4)}{U_1 + U_2}\right)\begin{cases} < 0 \text{ if } \alpha < 0 \\ > 0 \text{ if } \alpha > 0 \end{cases}.\end{aligned}$$

The proof of this expression is given in Appendix A. Then,

$$\Rightarrow \frac{\partial p_1^*}{\partial p_2}\begin{cases} < 0 \text{ if } \alpha < 0 \\ > 0 \text{ if } \alpha > 0 \end{cases}. \tag{4.8}$$

□

That is, increase in p_2 causes the optimum price of the firm 1 to get closer to the monopoly-case optimum price $\frac{1}{\lambda_1}$. This could be an expected consequence. In general, as the competitor sets a higher price, which results in a decrease in the expected demand amount of the competitor, firm's market gets closer to the monopoly market, which implies an optimal strategy which is closer to the optimum strategy in a monopoly market.

4.1.3. Behavior of the response with respect to correlation coefficient

In this section, the relationship between the best response and the correlation coefficient is examined. The relationship is important since the market is highly dependent on the correlation between the wtp distributions of the customers.

Proposition 4.4.

$$\frac{\partial p_1^*}{\partial \alpha} < 0 \ \forall \alpha.$$

In words, as the correlation between the wtp distributions of the customers for the two firms increases, their optimal prices decreases.

Proof. To prove the structure of the relationship, we start with the implicit differentiation:

$$\frac{\partial p_1^*}{\partial \alpha} = -\frac{\frac{\partial K_1}{\partial \alpha}}{\frac{\partial K_1}{\partial p_1^*}} = -\frac{(2p_1^*\lambda_1 - 1)(\frac{a_3-a_5}{\alpha}) + (p_1^*\lambda_1 - 1)(\frac{a_4}{\alpha} - \frac{a_2}{1+\alpha})}{\lambda_1(2(a_1 - a_2 + 2a_3 + a_4 - 2a_5) - (p_1\lambda_1)(a_1 - a_2 + 4a_3 + a_4 - 4a_5))}.$$

Only thing to look for is the numerator, as the denominator is positive from Equation 4.7:

$$\begin{aligned}
-\frac{\partial R_1}{\partial \alpha} &= (2p_1^*\lambda_1 - 1)\left(\frac{a_3 - a_5}{\alpha}\right) + (p_1^*\lambda_1 - 1)\left(\frac{a_4}{\alpha} - \frac{a_2}{1+\alpha}\right) \\
&= -\left(\frac{U_1 - U_2}{U_1 + U_2}\right)\left(\frac{a_3 - a_5}{\alpha}\right) + \left(\frac{a_3 - a_5}{U_1 + U_2}\right)\left(\frac{a_4}{\alpha} - \frac{a_2}{1+\alpha}\right) \\
&= -\left(\frac{a_3 - a_5}{U_1 + U_2}\right)\left[\frac{a_1 - a_2 + a_4}{\alpha} - \frac{a_4}{\alpha} + \frac{a_2}{1+\alpha}\right] \\
&= -\left(\frac{a_3 - a_5}{U_1 + U_2}\right)\left(\frac{a_1}{\alpha} - \frac{a_2}{\alpha(1+\alpha)}\right) \\
&= -\left(\frac{a_3 - a_5}{U_1 + U_2}\right)\left(\frac{e^{-\lambda_1 p_1^*}}{\alpha}\right)\left(1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_2 p_2}\right) \\
&\leq 0
\end{aligned}$$

since

$$a_3 - a_5 \begin{cases} \leq 0 \text{ if } \alpha \leq 0 \\ \geq 0 \text{ if } \alpha \geq 0 \end{cases}.$$

$$\therefore -\frac{\partial R_1}{\partial \alpha} \leq 0 \Rightarrow \frac{\partial p_1^*}{\partial \alpha} < 0 \ \forall \alpha.$$

□

In such a market, the customers whose only concern is the price desire to have similar alternatives in terms of the perception of the population towards them which causes lower prices which is very realistic. For example, if there is a duopoly where the two firms could not make solid differentiations from each other, the optimum prices they charge decreases as the level of their similarity increases.

4.1.4. Behavior of the expected profit with respect to correlation coefficient

In this section, the relationship between the expected profit of the firms and the level of correlation between the wtp distributions of the customers for the firms is investigated.

Proposition 4.5.

$$\frac{\partial \Pi_1(p_1^*(p_2), p_2)}{\partial \alpha} \leq 0, \ for\ \alpha \leq 0.$$

In words, expected profit of a firm decreases if the correlation between the demands increases when the correlation is negative.

Proof. Let

$$u_1 = e^{-\lambda_1 p_1^*},$$
$$u_2 = e^{-\lambda_2 p_2}.$$

Then, the following holds:

$$\begin{aligned}
\frac{\partial \Pi_1(p_1^*, p_2)}{\partial \alpha} &= \frac{\partial p_1^*}{\partial \alpha}\bar{H}_1(p_1^*, p_2) + p_1^* \frac{\partial \bar{H}_1(p_1^*, p_2)}{\partial \alpha} \\
&= \frac{\partial p_1^*}{\partial \alpha}\bar{H}_1(p_1^*, p_2) + p_1^* \left(-h_1(p_1^*, p_2)\frac{\partial p_1^*}{\partial \alpha} + \left(\frac{-a_2}{1+\alpha} + \frac{a_3 + a_4 - a_5}{\alpha}\right)\right) \\
&= \frac{\partial p_1^*}{\partial \alpha}\left(\bar{H}_1 - p_1^* h_1\right) + p_1^* \left(\frac{-a_2}{1+\alpha} + \frac{a_3 + a_4 - a_5}{\alpha}\right) \\
&= p_1^* \left(\frac{-a_2}{1+\alpha} + \frac{a_3 + a_4 - a_5}{\alpha}\right) \\
&= p_1^* \lambda_1 u_1 u_2 \left(-\frac{1}{\lambda_1 + \lambda_2} + \frac{2u_1}{2\lambda_1 + \lambda_2} + \frac{u_2}{\lambda_1 + 2\lambda_2} - \frac{u_1 u_2}{\lambda_1 + \lambda_2}\right) \\
&\leq \frac{\lambda_1(-1 + u_1 + u_2 - u_1 u_2) + \lambda_2(-1 + 2u_1 + \frac{1}{2}u_2 - u_1 u_2)}{(\lambda_1 + \lambda_2)^2} \\
&\leq \frac{1}{(\lambda_1 + \lambda_2)^2}\left(\lambda_1((u_1 - 1)(1 - u_2)) + \lambda_2((u_1 - \frac{1}{2})(2 - u_2))\right) \\
&\leq 0 \text{ for } \alpha \leq 0
\end{aligned}$$

since $0 \leq u_2 \leq 1$ and $0 \leq u_1 \leq e^{-1}$ for $\alpha \leq 0$. □

Since $e^{-1} \leq u_1 \leq e^{-1/2}$ for $\alpha \geq 0$, we could not make sure the relation for nonnegative α case.

4.1.5. Existence and Uniqueness of the Equilibrium

Now, we could proceed with the investigation of the equilibrium price. In such a case, the firms respond to each other with a single price. Cachon and Netessine [19] presents contraction mapping theorem as the most widely used method in existence and uniqueness proofs.

Definition: Mapping $f(x)$ is a contraction if and only if $||f(x_1)-f(x_2)|| \leq \alpha||x_1-x_2||$, $\forall x_1, x_2, \alpha < 1$.

Then, the existence of a unique equilibrium is ensured by the related theorem.

Theorem 4.6. *If the best response mapping is a contraction on the entire strategy space, there is a unique Nash Equilibrium in the game.*

As a result, if we could prove that the best response is a contraction mapping, then we could easily say that a unique equilibrium exists. However, Hogan [49] states that if the function $p_1^*(p_2)$ is a quasi-contraction mapping, then this means the equilibrium price exists and is unique in this game. This relaxes the previous theorem and make it easier to prove the existence of a unique equilibrium. Hogan [49] states the following definition and theorem:

Definition: If there exists $\beta \in$ (0,1) such that
$f(x) - f(y) \leq \beta(x - y) \forall x, y \in X$ and if f is continuous on X, then f is a quasi-contraction mapping.

Theorem 4.7. *If X is a closed, connected subset of R and f is a quasi-contraction mapping, then there exists a unique fixed point, x*∈X, such that f(x*)=x*.*

Since we previously restrict the best response region, we surely know that any best response lies in this region. Therefore, the existence of a unique equilibrium within this region also assures the existence of global unique equilibrium.

Proposition 4.8. *The best response function $p_1^*(p_2)$ is a quasi-contraction mapping within the confined best response region.*

Proof. By implicit differentiation, we could write

$$\frac{\partial p_1^*(p_2)}{\partial p_2} = -\frac{\frac{\partial Q_1(p_1^*, p_2)}{\partial p_2}}{\frac{\partial Q_1(p_1^*, p_2)}{\partial p_1^*}}, \tag{4.9}$$

where $Q_1(p_1^*(p_2), p_2)$ is defined as:

$$Q_1(p_1^*(p_2), p_2) = 1 - \frac{p_1^* h_1(p_1^*(p_2), p_2)}{\bar{H}_1(p_1^*(p_2), p_2)} = 0$$

at optimality of p_1. From equation 4.6 we know that

$$\frac{\partial p_1^*}{\partial p_2} = \begin{cases} < 0 \text{ if } \alpha < 0 \\ > 0 \text{ if } \alpha > 0 \end{cases} .$$

Then, only thing to show is that $\frac{\partial p_1^*}{\partial p_2} < 1$ when $\alpha > 0$. Note that from equation 4.9, if $\frac{\partial Q_1(p_1^*, p_2)}{\partial p_2} + \frac{\partial Q_1(p_1^*, p_2)}{\partial p_1^*} \geq 0$, then the proof is complete. Considering the definitions in Equation 4.2, let

$$\begin{aligned}
A_1 =& (2 - p_1\lambda_1)\lambda_1, \\
A_2 =& (p_1\lambda_1 - 2)\lambda_1 + (p_1\lambda_1 - 1)\lambda_2, \\
A_3 =& 4(1 - p_1\lambda_1)\lambda_1 + (1 - 2p_1\lambda_1)\lambda_2, \\
A_4 =& (2 - p_1\lambda_1)\lambda_1 + 2(1 - p_1\lambda_1)\lambda_2, \\
A_5 =& 4(p_1\lambda_1 - 1)\lambda_1 + 2(2p_1\lambda_1 - 1)\lambda_2.
\end{aligned}$$

Then, the expression we are interested becomes:

$$\begin{aligned}
\frac{\partial Q_1(p_1^*, p_2)}{\partial p_2} + \frac{\partial Q_1(p_1^*, p_2)}{\partial p_1^*} &= a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 + a_5A_5 \\
&> 0.
\end{aligned}$$

The detailed proof of this inequality is given in the Appendix. As a result:

$$\therefore \frac{\partial p_1^*}{\partial p_2} < 1 \text{ when } \alpha > 0 \Rightarrow \frac{\partial p_1^*(p_2)}{\partial p_2} < 1.$$

□

To conclude, $p_1^*(p_2)$ is a quasi-contraction mapping, that is, the best response function does exists and also is unique for a given $1/2\lambda_2 < p_2 < 2/\lambda_2$.

4.2. Examination of Centralized Market and Comparison with Decentralized Market

We could go one step further by considering the cases where the market in concern is a centralized or decentralized market. That is, it is possible that the two firms in the market are owned by a single holding, or they are separate firms. Of course, if the market is centralized, the things change as it is no longer the objective to maximize the expected profits of the firms individually, but the sum of the two firms' expected profits is to be maximized. We could expect certain behaviors in such environment. For example, it could be expected that the equilibrium prices for the services increase compared to the decentralized case since the firms no longer try to offset the prices of each other. They enjoy the benefit of acting as a single entity instead of having a price war in between. As another consequence, we could expect higher expected profits in the centralized market. They no longer try to steal customers from each other, but try to cover as large percentage of the customer population as possible with as high profit as possible.

We simplify the model by assuming the wtp distribution with the same parameter λ for both of the firms, which yields to a symmetric environment. Considering the monopoly case where the two firms are owned by a single owner, we could analyze the market. In this setting, the following redefines the previously obtained definitions:

$$\begin{aligned}\Pi_0(p) &= p(1 - Pr\{R_1 \leq p, R_2 \leq p\}) \\ &= p(1 - Pr\{R_{max} \leq p\}) \\ &= p(1 - F_{max}(p)),\end{aligned}$$

$$F_{max}(p) = F(p,p) = 1 - 2e^{-\lambda p} + (1+\alpha)e^{-2\lambda p} - 2\alpha e^{-3\lambda p} + \alpha e^{-4\lambda p},$$
$$\bar{F}_{max}(p) = 1 - F_{max}(p) = 2e^{-\lambda p} - (1+\alpha)e^{-2\lambda p} + 2\alpha e^{-3\lambda p} - \alpha e^{-4\lambda p},$$
$$f_{max}(p) = 2\lambda e^{-\lambda p} - 2(1+\alpha)\lambda e^{-2\lambda p} + 6\lambda\alpha e^{-3\lambda p} - 4\lambda\alpha e^{-4\lambda p}.$$

Then, let us define the following for simplicity:

$$a_1 = 2e^{-\lambda p},$$
$$a_2 = (1+\alpha)e^{-2\lambda p},$$
$$a_3 = 2\alpha e^{-3\lambda p},$$
$$a_4 = \alpha e^{-4\lambda p}.$$

As a result:

$$\begin{aligned} \bar{F}_{max}(p) &= a_1 - a_2 + a_3 - a_4, \\ f_{max}(p) &= \lambda(a_1 - 2a_2 + 3a_3 - 4a_4), \\ f'_{max}(p) &= -\lambda^2(a_1 - 4a_2 + 9a_3 - 16a_4). \end{aligned} \tag{4.10}$$

Also let:

$$\begin{aligned} &U_1 = a_1 - a_2 + a_3 - a_4 = \bar{F}_{max} \geq 0, \\ &U_2 = -a_2 + 2a_3 - 3a_4 \leq 0 \ \forall\alpha, \\ &\Rightarrow f_{max}(p) = \lambda(U_1 + U_2). \end{aligned} \tag{4.11}$$

The detailed proof of these expressions is given in Appendix.

Proposition 4.9. *The centralized symmetric market always has an optimum price higher than the monopoly optimum price.*

$$p_c^* \geq \frac{1}{\lambda}.$$

Proof.

$$p^* = \frac{\bar{F}_{max}(p)}{f_{max}(p)} = \frac{1}{\lambda}\frac{U_1}{U_1 + U_2} \geq \frac{1}{\lambda}. \tag{4.12}$$

from Equation 4.11 □

Proposition 4.10. *The symmetric centralized case satisfies IGFR property.*

Proof. We know, from Equation 4.12, that $p^* \geq \frac{1}{\lambda}$. That is, any optimal price is above the single-firm monopoly case price $\frac{1}{\lambda}$. Now let's check the following for $g(p) = \frac{pf_{max}}{\bar{F}_{max}}$:

$$\begin{aligned}
\frac{\partial g(p)}{\partial p} &= \frac{(f_{max} + pf'_{max})\bar{F}_{max} - (f_{max})^2 p}{(\bar{F}_{max})^2} \\
&= \frac{(\lambda(U_1 + U_2)U_1 - \lambda^2 pU_1(U_1 + 3U_2 + 2a_3 - 6a_4) + p(U_1 + U_2)^2\lambda^2)}{U_1^2} \\
&= \frac{\lambda U_1(U_1 + U_2) - \lambda^2 pU_1U_2 - \lambda^2 pU_1(2a_3 - 6a_4) + \lambda^2 pU_2^2}{U_1^2} \\
&\geq 0 \text{ for } p \in [1/\lambda, \infty).
\end{aligned}$$

The detailed proof of the expressions is given in Appendix A. This means $g(p)$ is increasing in p over $p \in [1/\lambda, \infty)$ as we have shown that any optimum price in this setting is greater than $1/\lambda$. □

Then, we have proven that IGFR property is still satisfied for the symmetric centralized case. Again, this leads us to conclude that there is a single profit-maximizing price for the firms.

We could compare the prices in centralized and decentralized environments. Intuitively, we could claim that the prices in the centralized market would be greater than those in a decentralized market. This is mainly because of the effect of collective action of the firms in the centralized environment. They, in fact, do not try to offset each other's price, but they try to maximize sum of their revenues.

Proposition 4.11. *The optimal price in a centralized market is greater than the optimal price in a decentralized market.*

$$p_c^* \geq p_d^*.$$

Proof. We could write the expected profit of a firm under a decentralized market as the following:

$$\Pi_d(p_1, p_2) = p_1 \bar{H}_1(p_1, p_2)$$

where $\bar{H}_1(p_1, p_2)$ is given in Equation 4.1. Since the market is symmetric, that is, the demand parameters for the firms are the same, the optimal prices of the firms should be the same. Then, the profit maximizing price could be obtained by

$$\bar{H}_1(p_d, p_d)(1 - \frac{p_d h_1(p_d, p_d)}{\bar{H}(p_d, p_d)}) = 0 \tag{4.13}$$

where

$$\bar{H}_1(p,p) = e^{-\lambda p} - \frac{(1+\alpha)}{2}e^{-2\lambda p} + \alpha e^{-3\lambda p} - \frac{\alpha}{2}e^{-4\lambda p},$$
$$h_1(p,p) = \lambda e^{-\lambda p} - \lambda\frac{(1+\alpha)}{2}e^{-2\lambda p} + \frac{5}{3}\lambda\alpha e^{-3\lambda p} - 2\lambda\alpha e^{-4\lambda p},$$

and p_d is the price in a decentralized market, as the optimum prices are the same in such environment. In this point, from Equation 4.10, note that

$$\bar{F}_{max}(p) = 2\bar{H}_1(p,p) \tag{4.14}$$

Let p_c be the maximizer of

$$\Pi_c(p) = p\bar{F}_{max}(p)$$

with

$$\bar{F}_{max}(p_c)(1 - \frac{p_c f_{max}(p_c)}{\bar{F}_{max}(p_c)}) = 0. \tag{4.15}$$

If proposition is true, then we know that the optimum price is higher in a centralized environment compared to the decentralized environment. Then, the optimality condition in Equation 4.13 should give a negative result if it is evaluated at centralized case optimum price. This is because we know that the profit function in decentralized case is unimodal, that is, any price above the decentralized optimum price leads to a negative slope in profit function. As a result, if the slope of profit function in decentralized case at centralized optimum price is negative, then the proof is complete. From Equation 4.5, Equation 4.14, and Equation 4.15 we know the following:

$$1 - g(p_c, p_c) = (1 - \frac{p_c h_1(p_c, p_c)}{\bar{H}_1(p_c, p_c)}) = (1 - \frac{p_c h_1(p_c, p_c)}{\frac{1}{2} p_c f_{max}(p_c)}).$$

Note that

$$\begin{aligned} 2h_1(p_c, p_c) - f_{max}(p_c) &= (1+\alpha)\lambda e^{-2\lambda p_c} - \frac{8}{3}\lambda\alpha e^{-3\lambda p_c} + 2\lambda\alpha e^{-4\lambda p_c} \\ &= e^{-2\lambda p_c}\lambda\left(2\alpha e^{-2\lambda p_c} - \frac{8}{3}\alpha e^{-\lambda p_c} + (1+\alpha)\right). \end{aligned}$$

Recall that $\lambda p_c \geq 1 \Rightarrow e^{-\lambda p_c} \leq e^{-1}$. Then, letting

$$e^{-\lambda p_c} = x$$

and

$$\varphi(x) = \left(2\alpha e^{-2\lambda p_c} - \frac{8}{3}\alpha e^{-\lambda p_c} + (1+\alpha)\right) = 2\alpha x^2 - \frac{8}{3}\alpha x + (1+\alpha),$$

$$\varphi(x) > 0 \; \forall x \in (0, e^{-1}).$$

Hence,

$$(1 - \frac{2h_1(p_c, p_c)}{f_{max}(p_c)}) \leq 0,$$
$$\Rightarrow p_c \geq p_d.$$

□

Now, let us consider the behavior of p_c with respect to the changes in α. So far, we have proven that the correlation has a negative effect on the optimal price in a decentralized environment. As the customer pool becomes a population of similar purchasing interests, the competitors try to lower their prices to maximize their profit. However, the behavior of the firms may change in a centralized market since there is no competition but there is a concern of maximizing the total profit. Let us first show that $p_c \geq \frac{1.1}{\lambda}$ which is a strengthened version of previously found $p_c\lambda \geq 1$. Consider

$$g(p) = \frac{pf_{max}(p)}{\bar{F}_{max}(p)}$$

at $p = \frac{1.1}{\lambda}$:

$$\begin{aligned} g(\frac{1.1}{\lambda}) &= \frac{\frac{1.1}{\lambda}(\lambda e^{-1.1} - (1+\alpha)\lambda e^{-2.2} + 3\lambda\alpha e^{-3.3} - 2\lambda\alpha e^{-4.4})}{e^{-1.1} - \frac{(1+\alpha)}{2}e^{-2.2} + \alpha e^{-3.3} - \frac{\alpha}{2}e^{-4.4}} \leq 1 \\ &\Rightarrow \frac{\partial \Pi_c(p)}{\partial p}_{|_{p=\frac{1.1}{\lambda}}} > 0 \\ &\Rightarrow p_c \geq \frac{1.1}{\lambda}. \end{aligned} \tag{4.16}$$

In words, any optimum price in a centralized market is greater than $\frac{1.1}{\lambda}$.

Proposition 4.12. *As the correlation between the wtp of the customers for the two firms increases, the optimal price increases in a centralized market.*

Proof. Pick $\alpha_2 > \alpha_1$ with $p_c^\alpha, f_{max}^\alpha, \bar{F}_{max}^\alpha$ such that

$$\frac{p_c^\alpha f_{max}^\alpha(p_c^\alpha)}{\bar{F}_{max}^\alpha} = 1$$

for α_1 and α_2 with respective values. Then, evaluate the first order condition for $p_c^{\alpha_2}$ at $p = p_c^{\alpha_1}$:

$$\begin{aligned}\frac{\partial \Pi_c}{\partial p}_{|_{p=p_c^{\alpha_1}}} &= \bar{F}_{max}^{\alpha_2}(p_c^{\alpha_1})\left(1 - \frac{p_c^{\alpha_1} f_{max}^{\alpha_2}(p_c^{\alpha_1})}{\bar{F}_{max}^{\alpha_2}(p_c^{\alpha_1})}\right)\\ &= \bar{F}_{max}^{\alpha_2}(p_c^{\alpha_1})\left(1 - \frac{\bar{F}_{max}^{\alpha_1}(p_c^{\alpha_1}) f_{max}^{\alpha_2}(p_c^{\alpha_1})}{f_{max}^{\alpha_1}(p_c^{\alpha_1})\bar{F}_{max}^{\alpha_2}(p_c^{\alpha_1})}\right),\end{aligned}$$

since

$$p_c^{\alpha_1} = \frac{\bar{F}_{max}^{\alpha_1}(p_c^{\alpha_1})}{f_{max}^{\alpha_1}(p_c^{\alpha_1})}.$$

This time, we know from Equation 4.16 that $p \in [1.1/\lambda, \infty)$,

$$\begin{aligned}\bar{F}_{max}^{\alpha_2}(p) f_{max}^{\alpha_1}(p) - \bar{F}_{max}^{\alpha_1}(p) f_{max}^{\alpha_2}(p) &= (b - \alpha_2 d_1)(\lambda b - \alpha_1 d_2) - (b - \alpha_1 d_1)(\lambda b - \alpha_2 d_2)\\ &= (\alpha_2 - \alpha_1) b (d_2 - \lambda d_1) \qquad ,\\ &> 0\end{aligned}$$

where

$$\begin{aligned}b &= 2e^{-\lambda p}(1 - e^{-\lambda p}),\\ d_1 &= e^{-2\lambda p} - 2e^{-3\lambda p} + e^{-4\lambda p},\\ d_2 &= 2\lambda e^{-2\lambda p} - 6\lambda e^{-3\lambda p} + 4\lambda e^{-4\lambda p},\\ f_{max} &= \lambda b - \alpha d_2,\\ \bar{F}_{max} &= b - \alpha d_1.\end{aligned}$$

Then, since

$$\frac{\partial \Pi_c(p)}{\partial p}_{|_{p=p_c^{\alpha_1}}} > 0,$$

$$p_c^{\alpha_1} < p_c^{\alpha_2}.$$

□

Next, we could study the effect of correlation on the potential demand in the market. This may be beneficial in explaining the behavior of the firms in response to the correlation change in the market.

Proposition 4.13. *In centralized market, as the correlation increases, total expected demand for the firms decreases. That is:*

$$\frac{\partial \bar{F}_{max}(p)}{\partial \alpha} \leq 0.$$

Proof. We could write the following from Equation 4.10:

$$\begin{aligned}\frac{\partial \bar{F}_{max}(p)}{\partial \alpha} &= -\, e^{-2\lambda p} + 2e^{-3\lambda p} - e^{-4\lambda p} \\ &= -\, e^{-2\lambda p}(1 - e^{-\lambda p})^2 \\ &\leq 0\end{aligned}$$

□

Resultingly, we could say that the central owner desire to have differentiated firms in terms of customers' perception, as differentiation increases the total expected demand.

4.3. Summary of the Results

In this part, we have analyzed our model under the assumption of Gumbel's bivariate exponential distribution as the wtp distribution of the customers in the market. We start by introducing the monopoly model under such assumption and present the optimum price in this market. The optimum price turns out to be the mean of the exponential distribution assumed for the wtp distribution of the customers. Next, we study the competition model where there are two firms trying to maximize their profit in the market. For this model, we first prove the existence and uniqueness of the optimum price. It differs as the correlation changes and if there is no correlation, the optimum price becomes the mean of the distribution as in the monopoly case. If there is a positive correlation, the optimum price is lower than the monopoly case optimum price and if there is a negative correlation, it is greater than the monopoly case optimum price. Then we investigate the behavior of the optimum price and the expected profit in response to the parametric changes in the model. Here, we study the behavior of the best response with respect to competitor's price and correlation coefficient. The relation of the best response with competitor's price depends on the correlation sign. Actually, the relation has the same sign with the correlation, that is, if the correlation is negative the increase in the competitor's price cause a decrease in optimum price, and if it is positive, it causes an increase in the optimum price. The relation between the expected profit and the correlation coefficient is negative when the correlation is negative. Then, we prove the existence and the uniqueness of the equilibrium in this model with the help of quasi-contraction mapping theorem. Finally, we examine the centralized model where there are two firms whose owners are the same. In this case, the relation of the optimum price and the correlation is positive. That is, increase in the correlation causes an increase in the optimum price. We also compare some characteristics of centralized and decentralized (competition) models. Here, it is proven that any optimum price in a centralized market is greater than the optimum price in a decentralized market. Besides, it is shown that the total expected demand for the firms decreases with an increase in correlation.

5. NUMERICAL RESULTS

In this chapter, we investigate the dynamics of our competition model with exemplary parameters. We first present the relations between the optimum price and the parameters of the model. Then, the relations between the expected profit and the parameters of the model are shared. Finally, centralized and decentralized market comparison is presented with numerical examples. Here, we investigate not only the relations proved analytically in the previous chapter, but also the other types of relations. Numerical examples are studied in order to support the analytical results found in the previous chapter and also to observe the other relations in the model. The model is simple, that is, there are no intense calculations, therefore, we are able to test the model with different parameter settings. However, for the sake of simplicity, only the results of some parameter settings are shared. In calculations, R programming language is used and the codes related to calculations are given in Appendix B. As stated, the model does not require long cpu time, therefore, it is easy to test the model with parameters with the help of these codes.

5.1. Behavior of the response with respect to competitor's price p_2

In this section, we analyze the competition model in terms of the relation between the best response and the competitor's price. Here, $\lambda_1 = 0.1$ and $\lambda_2 = 0.3$ are used as the parameters. Several levels of correlation are used and the results are presented in Figure 5.1. As shown in Figure 5.1, the best response converges to the monopoly-case optimum price ($p_1^* = \frac{1}{\lambda_1} = 10$) as the competitor sets high prices. We could infer this as follows: as the competitor sets a high price, its expected demand becomes smaller and smaller which allows our firm to set a price as if there is a monopolistic market. Therefore, we see the same profit-maximizer price for the first firm as p_2 gets higher. Besides, we see a single optimum price for $\alpha = 0$ case. In this case, it seems that there is no motivation to diverge from the monopoly case optimum price since the wtp distributions of the customers for the two firms are not correlated, that is, there is no possibility to draw more customers by changing the price in response

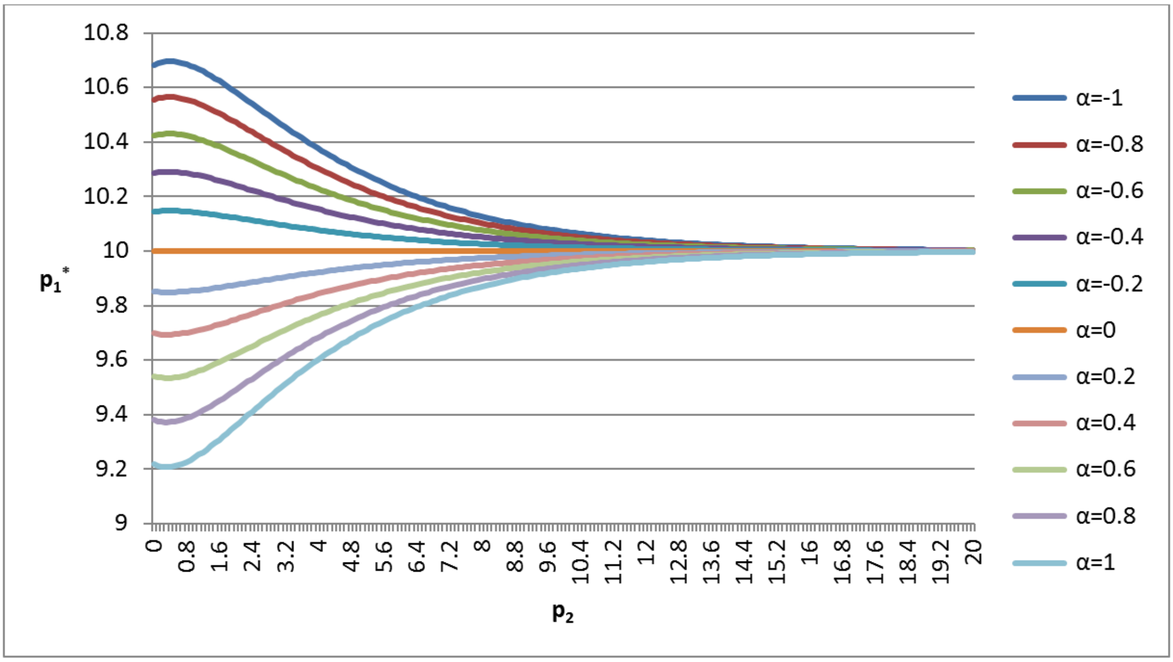

Figure 5.1. The Best Responses with respect to Different α Values

to the changes in the price of the competitor's product. Actually, these results are in line with Proposition 4.3. We should remind that the analytical findings are the results of the studies where the best response region is confined as in Equation 4.4. Observe in the figure that there is a small region near the y-axis that the relation is reverse of what is found in analytical proofs. These regions are out of the confined best response region so that we make the proofs without any problem. Actually, since we are investigating the nature of the equilibrium conditions in the market in general, and those regions are not part of the region that the equilibrium occurs in any setting, they do not cause a conflict with the findings in this study. Particularly, we could clearly say that the findings are in line with the figure within the confined best response region $1/2\lambda_2 \leq p_2 \leq 2/\lambda_2$, e.g. $1/0.6 \leq p_2 \leq 2/0.3$.

5.2. Behavior of the response with respect to α

Here, the relation between the best response and the correlation coefficient is presented. In Figure 5.1, the best response function gets smaller values as α gets higher. Those results verify Proposition 4.4 in the previous chapter. It could be said that as the correlation gets higher, the proportion of the customers with similar opinions about the two firms increases, which causes the demand structure for the two firms becomes similar. Then, firms try to capture higher demand by lowering their price.

5.3. Behavior of the response with respect to λ_2

In this section, we investigate the relationship between the best response and the competitor's wtp distribution's rate parameter. Figure 5.2 shows how the best response behaves as λ_2 and α changes in the competition. From the numerical results, we see that increase in λ_2 causes the optimum prices to draw near the monopoly-case price ($\frac{1}{\lambda_1}$). While the optimum prices decrease in response to increase in λ_2 with negative α, they do the opposite with positive α figures. As a result, a profit-maximizer firm should draw its price towards the monopoly-case optimum price in the case of a decrease in the wtp of the customers for the competitor's product. We could look at this result from another angle. We could think of the effect of the increase in p_2 and the effect of the increase in λ_2 are similar to the strategy of the firm 1 since both changes result in a lower competitor's demand which could be interpreted as the market gets closer to the monopoly case for firm 1. Then it becomes reasonable to see optimum prices closer to the monopoly case optimum price.

5.4. Behavior of the expected profit with respect to competitor's price p_2

In this section, the relation between the expected profit of a firm and the competitor's price is presented. Figure 5.3 shows how maximized expected profit changes as p_2 changes with respect to different α values. As shown in the figure, the expected profit converges to the same value as the competitor's price p_2 increases, regardless of α value. We could be familiar to this value from the monopoly case.

$$\pi(p) = p\,\Pr\{R \geq p\} = pe^{-\lambda p} \Rightarrow p^* = \frac{1}{\lambda}$$
$$\Rightarrow \text{if } \lambda = 0.1, \text{then } \pi(p^*) = \frac{1}{0.1}e^{-1} = 3.678794$$

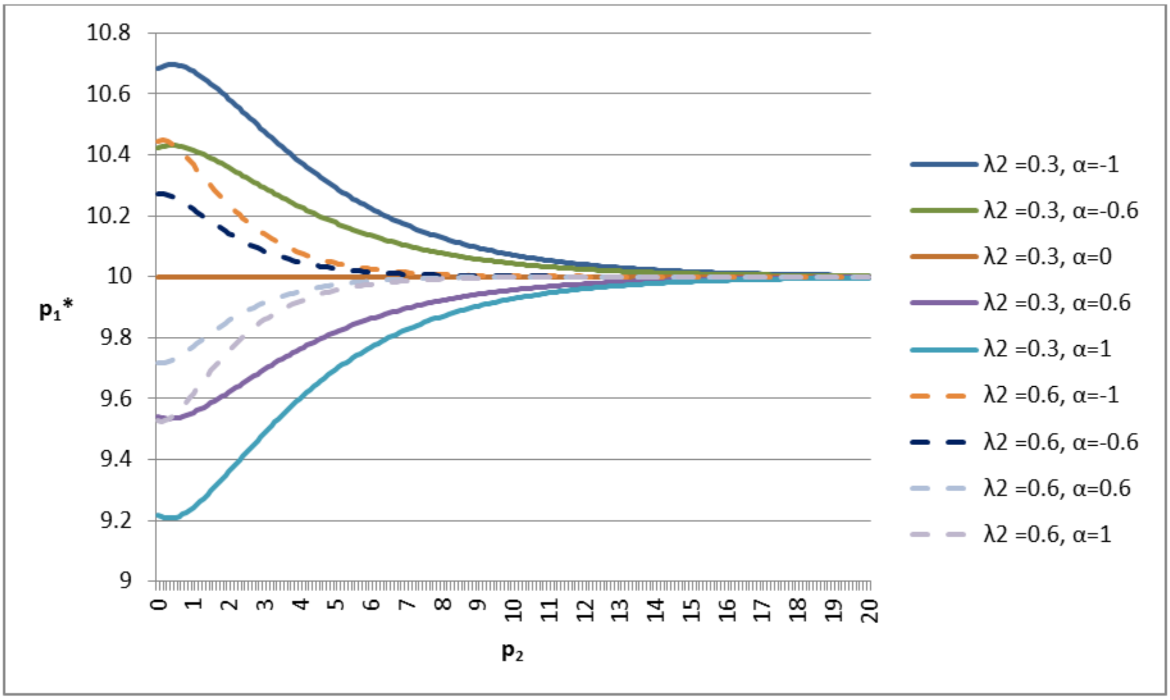

Figure 5.2. The Best Responses with respect to Different α and λ_2 Values

Since p_2 increases to the values for which almost nobody intends to buy firm 2's product, the market structure becomes like a monopoly as there is only one product which is "affordable". Then, firm 1's expected profit gets closer to the monopoly case expected profit. Higher correlation causes the market to consist of customers with similar purchasing interests. This results in lower expected profit for a firm at the same p_2. Another observation is that the convergence to the monopoly case expected profit is realized at higher p_2 values as correlation increases. We could make the conclusion that higher correlation causes lower expected profits. Of course, this is an undesirable market characteristic for the firms. Recall that customers desire to have similar alternatives if we assume that only concern for the customers is the price since higher correlation leads to decrease in equilibrium price in the competition. This analysis completes the picture: if the firms get similar to each other, this is beneficial for the customers; on the other hand, if the firms diverge from each other in terms of their product or service in the eyes of the customers, this becomes beneficial for the firms as their expected profit become higher.

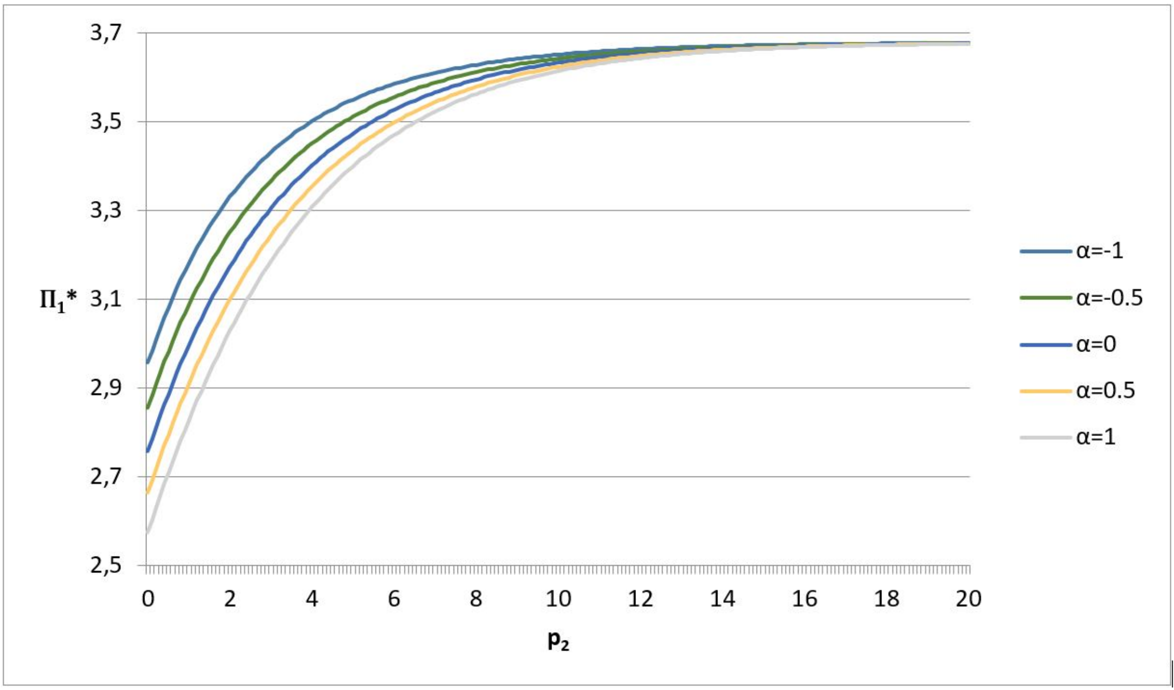

Figure 5.3. Maximum Expected Profits with respect to Different α Values

5.5. Behavior of the expected profit with respect to α

In this section, the relation between the expected profit and the correlation coefficient is presented. In Figure 5.3 we can see that as the correlation between wtp distributions of the customers for the firms increases, maximized expected profit decreases. We could think of this situation as follows: as the correlation increases, the two firms appeal to the similar customers more. In fact, the exact definition of the distributions explains the situation by itself: the distributions in the model are used for wtp amount of the customers in the market. That is, high correlation means the individuals in the market assess the two firms' products in the same way that they are willing to pay similar amounts for both of the products. As a result, the firms begin to share the customer population with similar purchasing interest as correlation increases. Then, expected profit of the firms decreases.

5.6. Behavior of the expected profit with respect to λ_2

Here, we examine the relation between the expected profit and the rate parameter of the wtp distribution of the competitor. As seen in Figure 5.4, increase in λ_2 results in an increase in the maximum expected profit of the firm 1. Also, with lower λ_2 values, the convergence to the monopoly-case expected profit is realized at higher p_2 figures.

5.7. Results for Centralized and Decentralized Market Structures

In this section, the centralized and decentralized market examples are presented together so that a comparison could be made. In Table 5.1, our first observation is on the comparison of the optimum prices in the centralized and decentralized market. The optimal prices for the two products in the centralized market are greater than those in the decentralized market. We could think of this situation as follows: the central decision maker can manage the market by herself, which gives her more room to boost the prices compared to the owners of the individual firms in a decentralized market. The firms no longer cut off profits of each other.

Another inference is that, in the symmetric scenario, where the wtp distributions of the customers for the firms are the same, the optimal price decreases as the correlation increases in the decentralized case, while it increases in the centralized environment. The case could be explained by the effect of "market shrinkage". In the decentralized environment, as the correlation increases, the firms start to face with a population of similar purchasing interest, that is, their potential customer population gets similar to each other. This causes them to lower their prices to capture more customers from this market. However, in the centralized case, increase in correlation means a decrease in the total potential customer pool, since both firms are owned by a single holding. Actually, this is proven in Proposition 4.13. Then, for the centralized owner, boosting the prices becomes the way out. On the other hand, in the asymmetric scenario, higher price follows a flat path as the correlation increases, while lower price constantly increases.

Lastly, the total expected profit of the firms is greater in the centralized market, which is an expected result of the fact that the firms experience no competition.

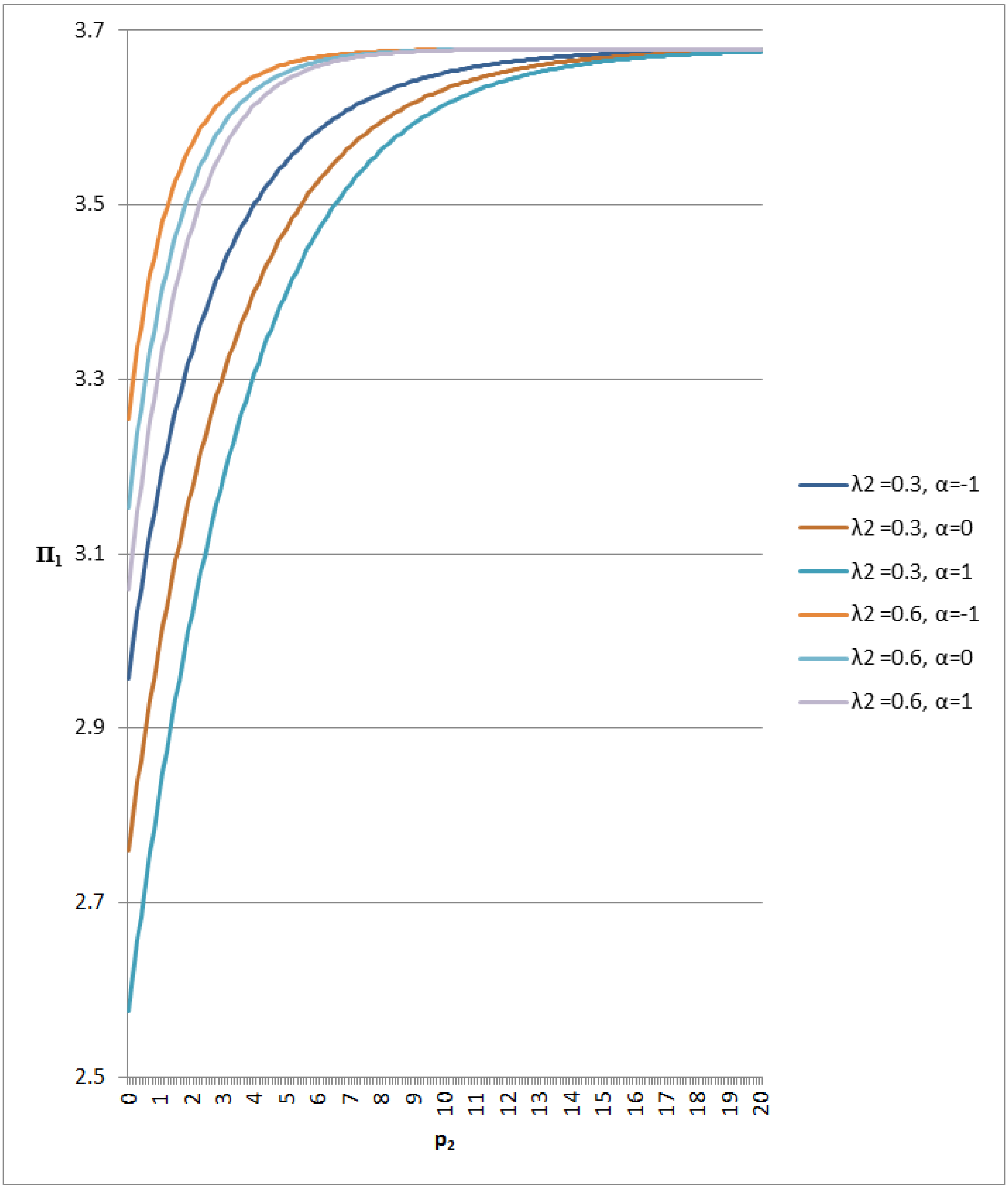

Figure 5.4. Maximum Expected Profits with respect to Different α and λ_2 Values

Table 5.1. Equilibrium Prices in Centralized and Decentralized Markets with respect to Different α Values

α	λ_1=0.1,λ_2=0.3							λ_1=λ_2=0.1			
	Decentralized				Centralized			Decentralized		Centralized	
	p_1^*	p_2^*	Π_1^*	Π_2^*	p_1^*	p_2^*	Π_0^*	$p_{1,2}^*$	$\Pi_{1,2}^*$	$p_{1,2}^*$	Π_0^*
-1	10.41	3.63	3.48	1.04	10.96	4.30	4.55	10.69	3.31	11.85	6.67
-0.9	10.37	3.61	3.47	1.03	10.96	4.32	4.52	10.63	3.28	11.88	6.62
-0.8	10.34	3.58	3.45	1.01	10.96	4.34	4.50	10.57	3.25	11.90	6.57
-0.7	10.30	3.55	3.44	1.00	10.95	4.37	4.47	10.50	3.22	11.92	6.51
-0.6	10.26	3.53	3.43	0.98	10.95	4.39	4.45	10.44	3.19	11.95	6.46
-0.5	10.22	3.50	3.41	0.97	10.94	4.41	4.42	10.37	3.16	11.97	6.41
-0.4	10.18	3.47	3.40	0.95	10.94	4.44	4.40	10.30	3.13	12.00	6.35
-0.3	10.13	3.44	3.38	0.94	10.94	4.47	4.37	10.23	3.10	12.03	6.30
-0.2	10.09	3.40	3.37	0.92	10.94	4.50	4.35	10.16	3.07	12.06	6.25
-0.1	10.05	3.37	3.36	0.90	10.93	4.53	4.32	10.08	3.03	12.09	6.19
0	10.00	3.33	3.34	0.89	10.93	4.56	4.30	10.00	3.00	12.12	6.14
0.1	9.95	3.30	3.33	0.87	10.93	4.59	4.28	9.92	2.97	12.15	6.09
0.2	9.91	3.26	3.31	0.86	10.93	4.62	4.25	9.83	2.94	12.19	6.03
0.3	9.86	3.22	3.29	0.84	10.93	4.66	4.23	9.75	2.90	12.22	5.98
0.4	9.81	3.18	3.28	0.82	10.92	4.70	4.20	9.66	2.87	12.26	5.93
0.5	9.76	3.13	3.26	0.81	10.92	4.74	4.18	9.57	2.84	12.30	5.88
0.6	9.70	3.09	3.24	0.79	10.92	4.79	4.15	9.47	2.80	12.34	5.82
0.7	9.65	3.04	3.23	0.78	10.92	4.84	4.13	9.37	2.77	12.38	5.77
0.8	9.59	2.99	3.21	0.76	10.93	4.89	4.10	9.27	2.73	12.42	5.72
0.9	9.53	2.94	3.19	0.75	10.93	4.94	4.08	9.16	2.69	12.47	5.67
1	9.47	2.89	3.17	0.73	10.93	5.00	4.06	9.05	2.66	12.51	5.61

6. CONCLUSION

In this thesis, we studied a model for a price competition where the correlation between the demands for the competitor firms is allowed. We introduced a simple demand model where the decision of the customers depends on their valuation of the alternatives and the prices of those alternatives. The choice of an individual is made based on the valuation of the person and the price of the alternatives where the benefit that the person could obtain is to be maximized. Here, by benefit, we mean the difference between the customer wtp and the price of the alternative. This is actually the consumer surplus in the economics literature. Under this model, the distribution of the wtp of the customers for the alternatives is assumed to be bivariate exponential distribution introduced by Gumbel [47]. This distribution allows correlation between the variates which is suitable to our demand model. Then, under such assumptions, some analytical and numerical results are introduced where the characteristics of the model are explained. All analysis is supported by numerical examples and some of the results are proven analytically. One of the interesting results of the analysis is that the competition leads to a higher optimal price than the price in a monopoly when there is a negative correlation between the demands for the firms. Then, the competition is investigated under a duopoly environment. The existence of a unique equilibrium price is shown with the help of quasi-contraction mapping and implicit differentiation. Naturally, when there is a negative correlation between the demands for the firms, the equilibrium occurs at prices higher than the monopoly case optimum prices. Finally, the effect of centralization is considered and the comparison of the centralized and decentralized case is made. Numerical results, together with some analytical findings, are presented.

One of the further research directions would be the investigation of the competition by assuming several other distributions which allow correlation. Of course, all analysis fully depend on the distribution assumed for the demand for the firms. Changing the distribution type could change the whole game. Investigation of the cases where there are more than two firms may be another research direction. There may be other dynamics in such environment. For example, it may be an interesting study where there is a competition between a single firm and a separate owner having two or more firms. However, it should be noted that analytical study may be very hard in these cases. Such study may require utilization of simulation. Studying the effect of centralization and the worth of it for the firms could be another topic to be analyzed. Since the game fundamentally changes with the centralization, it may be fruitful to investigate the importance of centralization to the individual firms. For example, such analysis could allow the valuation of a merger of two competitors in a market where merger consideration could be one of the topics to be discovered within the analysis. Another research direction may result from a change in the objective function. In our setting, the objective is maximizing the expected profit. However, maximizing the market share may be another objective function to optimize. Finally, there may be additions to the problem such as capacity or inventory considerations.

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APPENDIX A: DERIVATIONS WITH BVE AS WTP DISTRIBUTION

Throughout this chapter, for simplicity, the following definitions are used:

$$v_1 = e^{-\lambda_1 r_1}$$

$$v_2 = e^{-\lambda_2 r_2}$$

$$v = e^{-\lambda r}$$

$$u_1 = e^{-\lambda_1 p_1}$$

$$u_2 = e^{-\lambda_2 p_2}$$

$$u = e^{-\lambda p}$$

A.1. The Expected Demand Function

We could write the complement of the expected demand as the following:

$$H_1(p_1, p_2) = \int_{r_2=0}^{p_2} \int_{r_1=0}^{p_1} f(r_1, r_2) dr_1 dr_2 + \int_{r_2=p_2}^{\infty} \int_{r_1=0}^{r_2-p_2+p_1} f(r_1|r_2) f_2(r_2) dr_1 dr_2.$$

Let us derive the two integrals separately:

$$\begin{aligned} V_1(p_1, p_2) &= \int_{r_2=0}^{p_2} \int_{r_1=0}^{p_1} f(r_1, r_2) dr_1 dr_2 \\ &= \int_{r_2=0}^{p_2} \int_{r_1=0}^{p_1} (\lambda_1 v_1)(\lambda_2 v_2)(1 + \alpha(2v_1 - 1)(2v_2 - 1)) dr_1 dr_2 \\ &= 1 - u_1 - u_2 + u_1 u_2 (1 + \alpha) - \alpha u_1 u_2 (u_1 + u_2 - u_1 u_2), \end{aligned}$$

$$V_2(p_1, p_2) = \int_{r_2=p_2}^{\infty} \int_{r_1=0}^{r_2-p_2+p_1} f(r_1|r_2) f_2(r_2) dr_1 dr_2,$$

$$W_2(p_1,p_2) = \int_{r_1=0}^{r_2-p_2+p_1} f(r_1|r_2)f_2(r_2)dr_1$$
$$=\lambda_2 v_2[(1+\alpha)-(1+\alpha)e^{-\lambda_1 r_2}u_1 e^{\lambda_1 p_2} - 2\alpha v_2 + 2\alpha e^{-r_2\lambda_1}v_2 u_1 e^{\lambda_1 p_2}]$$
$$-\lambda_2 v_2 \alpha[1-e^{-2\lambda_1 r_2}u_1^2 e^{2\lambda_1 p_2} - 2v_2 + 2v_2 e^{-r_2 2\lambda_1}u_1^2 e^{2\lambda_1 p_2}],$$

$$V_2(p_1,p_2) = \int_{r_2=p_2}^{\infty} W_2 dr_2$$
$$= u_2 - (1+\alpha)\frac{\lambda_2}{\lambda_1+\lambda_2}u_1u_2 + \frac{2\alpha\lambda_2}{\lambda_1+2\lambda_2}u_1u_2^2 + \frac{\alpha\lambda_2}{2\lambda_1+\lambda_2}u_1^2u_2 - \frac{\alpha\lambda_2}{\lambda_1+\lambda_2}u_1^2u_2^2,$$

then, the resulting expression becomes as follows:

$$H_1(p_1,p_2) = V_1(p_1,p_2) + V_2(p_1,p_2)$$
$$=1-u_1+u_1u_2\left(\frac{(1+\alpha)\lambda_1}{\lambda_1+\lambda_2}\right) - \left(\frac{2\alpha\lambda_1}{2\lambda_1+\lambda_2}\right)u_1^2u_2 - \left(\frac{\alpha\lambda_1}{\lambda_1+2\lambda_2}\right)u_1u_2^2$$
$$+\left(\frac{\alpha\lambda_1}{\lambda_1+\lambda_2}\right)u_1^2u_2^2.$$

We could write the expected demand function from its complement:

$$\bar{H}_1(p_1,p_2) = 1 - H_1(p_1,p_2)$$
$$=u_1 - u_1u_2\left(\frac{(1+\alpha)\lambda_1}{\lambda_1+\lambda_2}\right) + \left(\frac{2\alpha\lambda_1}{2\lambda_1+\lambda_2}\right)u_1^2u_2 + \left(\frac{\alpha\lambda_1}{\lambda_1+2\lambda_2}\right)u_1u_2^2$$
$$-\left(\frac{\alpha\lambda_1}{\lambda_1+\lambda_2}\right)u_1^2u_2^2.$$

A.2. The First Derivative of Expected Demand Function

The following gives the first derivative of the expected demand function:

$$h_1(p_1,p_2) = \frac{\partial}{\partial p_1}H_1(p_1,p_2)$$
$$=\lambda_1\left[u_1 - (1+\alpha)u_1u_2\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right) + 2\alpha\left(\frac{2\lambda_1}{2\lambda_1+\lambda_2}\right)u_1^2u_2\right.$$
$$\left.+\alpha\left(\frac{\lambda_1}{\lambda_1+2\lambda_2}\right)u_1u_2^2 - 2\alpha\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)u_1^2u_2^2\right].$$

A.3. The Second Derivative of Expected Demand Function

The second derivative of the expected demand function is as follows:

$$
\begin{aligned}
h_1'(p_1,p_2) =& \frac{\partial}{\partial p_1} H_1(p_1,p_2) \\
=& -\lambda_1^2 \Bigg[u_1 - (1+\alpha)u_1u_2 \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right) + 4\alpha \left(\frac{2\lambda_1}{2\lambda_1+\lambda_2}\right) u_1^2u_2 \\
& +\alpha \left(\frac{\lambda_1}{\lambda_1+2\lambda_2}\right) u_1u_2^2 - 4\alpha \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right) u_1^2u_2^2 \Bigg]
\end{aligned}
$$

A.4. Some Useful Expressions used in the Proofs Related to IGFR Property

As given in the text, for simplicity, we make the following definitions:

$$
\begin{aligned}
a_1 =& e^{-\lambda_1 p_1} = u_1, \\
a_2 =& \frac{(1+\alpha)\lambda_1}{\lambda_1+\lambda_2} u_1u_2, \\
a_3 =& \frac{2\alpha\lambda_1}{2\lambda_1+\lambda_2} u_1^2u_2, \\
a_4 =& \frac{\alpha\lambda_1}{\lambda_1+2\lambda_2} u_1u_2^2, \\
a_5 =& \frac{\alpha\lambda_1}{\lambda_1+\lambda_2} u_1^2u_2^2.
\end{aligned} \tag{A.1}
$$

We also make the following definitions:

$$U_1 = \bar{H}_1(p_1,p_2) = a_1 - a_2 + a_3 + a_4 - a_5$$

since $\bar{H}_1(p_1,p_2) = 1 - H_1(p_1,p_2)$ is a demand function, it is certainly nonnegative. Let us check U_2:

$$
\begin{aligned}
U_2 = a_3 - a_5 =& \frac{2\alpha\lambda_1}{2\lambda_1+\lambda_2} u_1^2u_2 - \frac{\alpha\lambda_1}{\lambda_1+\lambda_2} u_1^2u_2^2 \\
=& \left(\alpha\lambda_1u_1^2u_2\right) \left[\frac{2}{2\lambda_1+\lambda_2} - \frac{u_2}{\lambda_1+\lambda_2}\right],
\end{aligned}
$$

$$\frac{2}{2\lambda_1+\lambda_2}-\frac{e^{-\lambda_2 p_2}}{\lambda_1+\lambda_2}\geq 0.$$

$$\alpha\lambda_1 e^{-2\lambda_1 p_1-\lambda_2 p_2}\begin{cases}<0 & \text{if } \alpha<0\\ >0 & \text{if } \alpha>0\end{cases}\Rightarrow U_2\begin{cases}<0 & \text{if } \alpha<0\\ >0 & \text{if } \alpha>0\end{cases}.$$

Resultingly, U_2 has the same sign with α. Next, let us see the difference U_1-U_2:

$$\begin{aligned}U_1-U_2 =& a_1-a_2+a_4\\ =& u_1-\frac{(1+\alpha)\lambda_1}{\lambda_1+\lambda_2}u_1u_2+\frac{\alpha\lambda_1}{\lambda_1+2\lambda_2}u_1u_2^2\\ =& u_1\left(1-\frac{(1+\alpha)\lambda_1}{\lambda_1+\lambda_2}u_2+\frac{\alpha\lambda_1}{\lambda_1+2\lambda_2}u_2^2\right)\\ =& \frac{u_1\left(2\lambda_2^2+\lambda_1^2(1-\alpha u_2)(1-u_2)+\lambda_1\lambda_2((2-\alpha u_2)(2-u_2)-1)\right)}{(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)}\\ \geq& 0.\end{aligned}$$

Now, let us check the following:

$$\begin{aligned}U_3 =& -a_1+a_2-3a_3-a_4+3a_5\\ =& -e^{-\lambda_1 p_1}\left[(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)-(1+\alpha)\lambda_1 e^{-\lambda_2 p_2}(2\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)\right.\\ &+\alpha\lambda_1 e^{-2\lambda_2 p_2}(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2)+6\alpha\lambda_1 e^{-\lambda_1 p_1-\lambda_2 p_2}(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)\\ &\left.-3\alpha\lambda_1 e^{-\lambda_1 p_1-2\lambda_2 p_2}(2\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)\right]\end{aligned}$$

Then,

$$\begin{aligned}U_3 = -e^{-\lambda_1 p_1}&\left[\lambda_1^3(2-2(1+\alpha)u_2+2\alpha u_2^2+6\alpha u_1u_2-6\alpha u_1u_2^2)+\lambda_2^3(2)\right.\\ &+\lambda_1^2\lambda_2(7-5(1+\alpha)u_2+3\alpha u_2^2+18\alpha u_1u_2-15\alpha u_1u_2^2)\\ &\left.+\lambda_1\lambda_2^2(7-2(1+\alpha)u_2+\alpha u_2^2+12\alpha u_1u_2-6\alpha u_1u_2^2)\right]\end{aligned}$$

It is enough if we could show this expression is negative when $\alpha>0$ since we use this expression to show $p_1^*\lambda_1<2$. In such case, $0\leq u_1\leq e^{-1}$ and $0\leq u_2\leq 1$.

Then, investigating the coeeficients of every λ_i, we could say that all coeeficients are nonnegative in such conditions above. As a result, we could say that $U_3 \leq 0$.

A.5. Proofs of the Expressions used in the Proofs Related to Duopoly Case Analytical Findings

$$e^{-1/2} \geq u_1, u_2 \geq e^{-2}$$

as we confine the best response region. Then, let

$$\begin{aligned} A_1 &= 2(u_2 - 1)^2, \\ A_2 &= (5 + \alpha)u_2^2 - 14u_2 + 8, \\ A_3 &= 2u_2^2 - 14u_2 + 10, \\ A_4 &= -4u_2 + 4. \end{aligned}$$

Then, the derivative becomes:

$$\begin{aligned} -\frac{\partial K_1}{\partial p_2} =& \lambda_2 \left(\frac{a_5(a_2 - 2a_1) + a_3(a_1 - a_4)}{U_1 + U_2} \right) \\ =& \frac{\alpha \lambda_1 \lambda_2 e^{-3\lambda_1 p_1 - \lambda_2 p_2}}{U_1 + U_2} \left(\frac{\lambda_1^3 A_1 + \lambda_1^2 \lambda_2 A_2 + \lambda_1 \lambda_2^2 A_3 + \lambda_2^3 A_4}{(\lambda_1 + \lambda_2)^2 (2\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2)} \right). \end{aligned}$$

As a result, the following holds:

$$\therefore -\frac{\partial K_1}{\partial p_2} \begin{cases} < 0 \text{ if } \alpha < 0 \\ > 0 \text{ if } \alpha > 0 \end{cases}.$$

A.6. Proofs of the Expressions used in the Proofs Related to Centralized Case Analytical Findings

In this part, we change our definitions for:

$$a_1 = 2e^{-\lambda p},$$
$$a_2 = (1+\alpha)e^{-2\lambda p},$$
$$a_3 = 2\alpha e^{-3\lambda p},$$
$$a_4 = \alpha e^{-4\lambda p}.$$

and also for:

$$U_1 = a_1 - a_2 + a_3 - a_4 = \bar{F}_{max} \geq 0,$$
$$U_2 = -a_2 + 2a_3 - 3a_4 \leq 0 \ \forall \alpha,$$
$$\Rightarrow f_{max}(p) = \lambda(U_1 + U_2).$$

Then, for derivative of GFR with respect to price:

$$\begin{aligned}\frac{\partial g(p)}{\partial p} &= \frac{(f_{max} + pf'_{max})\bar{F}_{max} - (f_{max})^2 p}{(\bar{F}_{max})^2} \\ &= \frac{(\lambda(U_1+U_2)U_1 - \lambda^2 pU_1(U_1 + 3U_2 + 2a_3 - 6a_4) + p(U_1+U_2)^2\lambda^2)}{U_1^2} \\ &= \frac{\lambda U_1(U_1+U_2) - \lambda^2 pU_1U_2 - \lambda^2 pU_1(2a_3 - 6a_4) + \lambda^2 pU_2^2}{U_1^2}.\end{aligned}$$

Let us focus on the numerator, as denominator is nonnegative. Without any doubt, we could say the following:

$$\lambda U_1(U_1+U_2) \geq 0,$$
$$\lambda^2 pU_2^2 \geq 0.$$

If we could show the remaining terms are positive in total, then the proof is complete.

$$
\begin{aligned}
-\lambda^2 pU_1U_2 - \lambda^2 pU_1(2a_3 - 6a_4) &= -\lambda^2 pU_1(U_2 + 2a_3 - 6a_4) \\
&= -\lambda^2 pU_1(-a_2 + 4a_3 - 9a_4).
\end{aligned}
$$

Let $e^{-\lambda p} = m$

$$-a_2 + 4a_3 - 9a_4 = m^2(-(1+\alpha) + 8\alpha m - 9\alpha m^2).$$

Now, we need to check if the polynomial inside the paranthesis is negative for any value of m. If $\alpha \leq 0$:

$$-9\alpha m^2 + 8\alpha m - (1+\alpha) \leq 0 \ \forall m \in [0, e^{-1}].$$

If $\alpha \geq 0$:

$$-9\alpha m^2 + 8\alpha m - (1+\alpha) \leq 0 \ \forall m \in [0, 1].$$

Then,

$$\frac{\partial g(p)}{\partial p} \geq 0 \ \forall p \in [1/\lambda, \infty)$$

APPENDIX B: R FUNCTIONS USED FOR CALCULATION OF BVE

CDF of Bivariate Exponential Distribution

$$F < -function(p1, p2, lambda1, lambda2, alpha)\{$$
$$u1 = exp(-lambda1 * p1)$$
$$u2 = exp(-lambda2 * p2)$$
$$(1 - u1) * (1 - u2) * (1 + alpha * u1 * u2)\}$$

PDF of Bivariate Exponential Distribution

$$f < -function(p1, p2, lambda1, lambda2, alpha)\{$$
$$u1 = exp(-lambda1 * p1)$$
$$u2 = exp(-lambda2 * p2)$$
$$(lambda1 * u1) * (lambda2 * u2) * (1 + alpha * (2 * u1 - 1)(2 * u2 - 1))\}$$

H_1 function of Bivariate Exponential Distribution

$$H1 < -function(p1, p2, lambda1, lambda2, alpha)\{$$
$$u1 = exp(-lambda1 * p1)$$
$$u2 = exp(-lambda2 * p2)$$
$$(1 - u1) * (1 - u2) * (1 + alpha * (u1 * u2))$$
$$+ ((-lambda2/(lambda1 + lambda2)) * u1 * u2)$$
$$- alpha * ((lambda2 * u1 * u2) * ((u1 * u2/(lambda1 + lambda2))$$
$$- (2 * u2/(lambda1 + 2 * lambda2)) - (u1/(2 * lambda1 + lambda2))$$
$$+ (1/(lambda1 + lambda2)))) + u2\}$$

h_1 function of Bivariate Exponential Distribution

$$h1 < -function(p1, p2, lambda1, lambda2, alpha)\{$$
$$u1 = exp(-lambda1 * p1)$$
$$u2 = exp(-lambda2 * p2)$$
$$((lambda1 * u1) * (-u2 + alpha * (2 * u1 - 1) * (-u2^2 + u2) + 1))$$
$$+ (lambda1 * lambda2 * u1 * u2 * ((1/(lambda1 + lambda2))$$
$$- alpha * ((-2 * u1 * u2/(lambda1 + lambda2))$$
$$+ (2 * u1/(2 * lambda1 + lambda2))$$
$$+ (2 * u2/(lambda1 + 2 * lambda2)) - (1/(lambda1 + lambda2)))))\}$$

h_1' function of Bivariate Exponential Distribution

$$h1\_prime < -function(p1, p2, lambda1, lambda2, alpha)\{$$
$$u1 = exp(-lambda1 * p1)$$
$$u2 = exp(-lambda2 * p2)$$
$$(lambda1 * u1) * (((-lambda1) * (-u2 + alpha * (2 * u1 - 1) * (-u2^2 + u2) + 1))$$
$$+ ((-u2^2 + u2) * alpha * (-2 * lambda1 * u1)) + (lambda2 * u2)$$
$$* ((-lambda1) * ((1/(lambda1 + lambda2)) - alpha$$
$$* ((-2 * u1 * u2/(lambda1 + lambda2)) + (2 * u1/(2 * lambda1 + lambda2))$$
$$+ (2 * u2/(lambda1 + 2 * lambda2)) - (1/(lambda1 + lambda2))))$$
$$+ (-alpha * ((2 * lambda1 * u1 * u2/(lambda1 + lambda2))$$
$$- (2 * lambda1 * u1/(2 * lambda1 + lambda2))))))\}$$

Expected profit function of Bivariate Exponential Distribution

$$Pi < -function(p1, p2, lambda1, lambda2, alpha)\{$$
$$p1 * (1 - H1(p1, p2, lambda1, lambda2, alpha))\}$$

Optimum price function of Bivariate Exponential Distribution for decentralized environment

$p1finder <- function(p2, lambda1, lambda2, alpha)\{$

$optim(0, Pi, lambda1 = lambda1, lambda2 = lambda2, alpha = alpha, p2 = p2,$

$control = list(fnscale = -1), method = "Brent", lower = 0, upper = 50)\$par\}$

Total expected profit function of Bivariate Exponential Distribution for decentralized environment

$Totalpi <- function(prices, lambda1, lambda2, alpha)\{$

$p1 <- prices[1]$

$p2 <- prices[2]$

$p1 * (1 - H1(p1, p2, lambda1, lambda2, alpha))$

$+ p2 * (1 - H1(p2, p1, lambda2, lambda1, alpha))\}$

Optimum price function of Bivariate Exponential Distribution for centralized environment

$p1p2finder <- function(lambda1, lambda2, alpha)\{$

$optim(c(0, 0), Totalpi, lambda1 = lambda1, lambda2 = lambda2, alpha = alpha,$

$control = list(fnscale = -1))\$par\}$

Function for search of optimum prices of Bivariate Exponential Distribution for decentralized environment

$decentralizedequilibriumexpon <- function(initp1 = 0, initp2 = 0, lambda1,$

$lambda2, alpha)\{$

$while(p1! = initp1)\{$

$p1 <- p1finder(initp2, lambda1, lambda2, alpha)$

$p2 <- p1finder(p1, lambda2, lambda1, alpha)$

$initp1 <- p1finder(p2, lambda1, lambda2, alpha)$

$initp2 <- p1finder(initp1, lambda2, lambda1, alpha)$

$if(abs(p2 - initp2) < 0.00000001)\{break\}$

$\}$

$print(c(p1, p2))\}$