KINETIC THEORY IN k=0 FRIEDMANN-ROBERTSON-WALKER COSMOLOGY

by

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ABSTRACT

KINETIC THEORY IN k=0 FRIEDMANN-ROBERTSON-WALKER COSMOLOGY

In the thesis we calculated Boltzmann Equation in RW metric. After that in the flat space-like slices we showed that for RW metric it is not possible to find any equilibrium solutions. However we looked for the equilibrium solutions in the two limits when mass goes to zero and when mass goes to infinity. Then we applied these result to the Bose-Einstein and to the Fermi-Dirac statistics for finding the equilibrium solution in the limits. Finally in the Bose-Einstein Condensation we calculated critical temperature.

ÖZET

FRIEDMANN-ROBERSTON-WALKER k=0 KOZMOLOJİSİNDE KİNETİK TEORİ

Bu tezde Boltzmann denklemini RW metriğinde hesapladık. Daha sonra bu metrikte dengede bir dağılım fonksiyonu olmadığını gösterdik. Lakin dengede dağılım fonksiyonuna iki limit durumunda ; kütle sıfıra ve sonsuza giderken baktık. Bulduğumuz şonuçları Bose-Einstein ve Fermi-Dirac istatistiklerinde iki limit durumunda dengede dağılım fonksiyonunu hesaplamak için kullandık. Son olarak Bose-Einstein yoğunlaşmasında kritik sıcaklığı hesapladık.

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LIST OF SYMBOLS

C	Collision
f	Distribution function
f_{eq}	Equilibrium distribution function
k_B	Boltzmann Constant
$Li_s(z)$	Polylogarithm function
T_c	Critical Temperature
Tr	Trace
H	Hamiltonian
L	Liouville Operator
R	Cosmic Scale
T	Temperature
W	Collision probability
μ	Chemical potential
$\mathcal{L}$	Heat Operator

LIST OF ACRONYMS/ABBREVIATIONS

BEC	Bose-Einstein condensation
BTE	Boltzmann Transport Equation
CS	Christoffel Symbol
FRW	Friedmann-Robertson-Walker
RS	Ricci Scalar
RW	Robertson-Walker

1. INTRODUCTION

According to the standard model of cosmology at large scales the universe is homogenous and isotropic. In general relativity, this means that the the metric of the space-time is the Robertson-Walker-Friedman (FRW) metric. Our aim in this thesis is to investigate basics of statistical mechanics in such a space-time. Since FRW metric is not stationary the time dependence of the metric implies that that we deal with a non-equilibrium problem. The basic equation that governs the non-equilibrium statistical mechanics is the Boltzmann equation. Thus our aim will be the construction of the Boltzmann equation in FRW space-time. In general Boltzmann equation is composed of two parts: the Liouville part which describes the evolution of the system of particles which are not interacting with each other but which may however be in a given external field and the collision term which takes into account the particle collisions. Because of the collision term the probability density function is expected to be time dependent. If the collisions are ignored and if the external field is time independent then the natural solution to look for is a time independent static solution which describes the equilibrium. However if the given external field is time dependent no such solution may exist. Now in a cosmological background the metric of the space-time is time dependent. This implies that the Boltzmann equation is under the influence of an external time dependent field, namely the metric itself. Although we have a time dependent problem we may still hope for a quasi-equilibrium solution (what we mean by this will be explained in the text) of the Boltzmann equation. As we will see there will be a quasi-equlibrium solution only in certain regimes defined by the mass m of the particles comprising the system. We will consider only systems composed of a single type of particles. More precisely we will see that the Boltzmann equation has a quasi-static solution in the ultrarelativistic $m \to 0$ and non-relativistic $m \to \infty$ limits. We will restrict our investigations to the case where the space-like three geometry is flat. The other cases (*i.e.* spherical and hyperbolic 3-space geometries) will be studied in the future. We will show the utility of our results in statistical mechanics problems by considering non-relativistic Bose-Einstein condensation (BEC) in FRW space-time. Since the analysis of BEC in general requires the thermodynamic limit to be taken the interplay between this limit and the large scale structure of space-time is an interesting problem to consider.

We start with chapter 2 by obtaining Boltzmann Transport equation in 3 dimension.BTE describes the rate of change and the evolution of the distribution function. In principle the distribution function can be solved from it. In chapter 3 RW metric is introduced. We showed that RW metric does not have timelike killing vector. So this means in the mathematical sense it is not possible to find equilibrium solutions however we looked the solution in the two limits.In chapter 4 we studied thermal equilibrium; the entropy in thermal equilibrium and the equilibrium distribution function in the limits.

In chapter 4 we also derived Boltzmann equation in RW metric. We calculated some macroscopic quantities such as the particle current density and energy-momentum tensor.Finally we concluded the thesis by applying our results to the Bose-Einstein condensation.As for the units and dimensions we assumed $c = \hbar = 1$.

2. BOLTZMANN TRANSPORT EQUATION

Boltzmann transport equation (BTE) describes the rate of change of the density function in phase space. BTE is really important for many transport problems in plasma physics, hydrodynamics and condensed matter.

2.1. μ-space and distribution function

μ-space is a six dimensional space. A point in this space represents a state of a particle. The motion of each particle is described by momentum and position $(\overrightarrow{p}, \overrightarrow{r})$. At any instant of time, the state of the entire system of N particles is represented by N points. If we count the number of points in the cell $(volume \triangle V)$, the result is given by definition $f(\overrightarrow{r}, \overrightarrow{p}, t)$ which is called the distribution function.

2.2. Evolution of the distribution function

The gas is enclosed inside a volume V. An external field such as gravitational or electromagnetic, can effects the particles.
The evolution of distribution function can be examined in two parts:

- Streaming where the distribution function changes due to velocity and acceleration.

- Collision

The aim of kinetic theory is to find the distribution function $f(\mathbf{r},\mathbf{p},t)$. Firstly we find the equation of motion for the distribution function. The distribution function changes with time, because particles regularly enter and leave a given volume element in phase space.

2.3. Streaming

At time t, the number of particles in the volume d^3rd^3p is $f(\mathbf{r},\mathbf{p},t)d^3rd^3p$ and time t+dt $f(\mathbf{r}+\mathbf{dr},\mathbf{p}+\mathbf{dp},t+dt)d^3r^{'}d^3p^{'}$

$$\begin{aligned} &f(\mathbf{r}+\mathbf{dr},\mathbf{p}+\mathbf{dp},t+dt) = f(\mathbf{r}+\frac{d\mathbf{r}dt}{dt},\mathbf{p}+\frac{\mathbf{p}dt}{dt},t+dt) = \\ &f(\mathbf{r}+\mathbf{v}dt,\mathbf{p}+\mathbf{ma}dt,t+dt) = f(\mathbf{r}+\mathbf{v}dt,\mathbf{p}+\mathbf{F}dt,t+dt) \end{aligned} \tag{2.1}$$

where $\mathbf{F}$ is the external force acting on particle (in the absence of collisions). For example in the case of a particle with charge q in the presence of external electric and magnetic fields E(r,t) and B(r,t), $\mathbf{F}$ is the Lorentz force $q[E(r,t)+\mathbf{v}\times\mathbf{B}(\mathbf{r},\mathbf{t})]$.
We may take dt to be a truly infinitesimal quantity. Thus all the particles contained in a μ-space element d^3rd^3p at $(\mathbf{r},\mathbf{p})$ at the instant t, will all be found in an element $d^3r^{'}d^3p^{'}$ at $(\mathbf{r}+\mathbf{v}dt,\mathbf{p}+\mathbf{F}dt)$.

Hence in the absence of collisions we have

$$f(\mathbf{r}+\mathbf{v}dt,\mathbf{p}+\mathbf{F}dt)d^3r^{'}d^3p^{'} = f(\mathbf{r}+d\mathbf{r},\mathbf{p}+d\mathbf{p})d^3rd^3p \tag{2.2}$$

Examine $d^3r^{'}d^3p^{'}$; coordinate transfromation is

$$\begin{aligned} &x^{'} = x+v_xdt \quad\quad p_x^{'} = p_x+Fdt \\ &y^{'} = y+v_ydt \quad\quad p_y^{'} = p_y+Fdt \\ &z^{'} = z+v_zdt \quad\quad p_z^{'} = p_z+Fdt \\ &d^3r^{'}d^3p^{'} = |J|d^3rd^3p \quad\quad |J| = \frac{\partial(x^{'},y^{'},z^{'},p_x^{'},p_y^{'},p_z^{'})}{\partial(x,y,z,p_x,p_y,p_z)} = 1+o(dt^2) \end{aligned} \tag{2.3}$$

Therefore, up to first order in dt we have;

$$d^3r^{'}d^3p^{'} = |J|d^3rd^3p = d^3rd^3p \tag{2.4}$$

So it means

$$
\begin{aligned}
&f(\mathbf{r}+\mathbf{v}dt,\mathbf{p}+\mathbf{F}dt,t+dt)d^3r^{'}d^3p^{'} = f(\mathbf{r},\mathbf{p},t)d^3rd^3p \\
&f(\mathbf{r}+\mathbf{v}dt,\mathbf{p}+\mathbf{F}dt,t+dt) = f(\mathbf{r},\mathbf{p},t) : \text{in the absence of collisions} \\
&f(\mathbf{r}+\mathbf{v}dt,\mathbf{p}+\mathbf{F}dt,t+dt) - f(\mathbf{r},\mathbf{p},t) = (\triangle f)_{str.} = 0
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&f(\mathbf{r}+\mathbf{v}dt,\mathbf{p}+\mathbf{F}dt) - f(\mathbf{r},\mathbf{p},t) \\
&= f(\mathbf{r}+d\mathbf{r},\mathbf{p}+d\mathbf{p}) + \frac{\partial f}{\partial t} + \mathbf{v}\nabla_r f + \mathbf{F}\nabla_p f + o() - f(\mathbf{r},\mathbf{p},t) \\
&= \frac{\partial f}{\partial t} + \mathbf{v}\nabla_r f + \mathbf{F}\nabla_p f
\end{aligned}
\tag{2.6}
$$

Define D streaming operator as $D = \frac{\partial}{\partial t} + \mathbf{v}\nabla_r + \mathbf{F}\nabla_p$
Evolution equation in the absence of collisions is

$$
Df = \frac{\partial f}{\partial t} + \mathbf{v}\nabla_r f + \mathbf{F}\nabla_p f = 0 \tag{2.7}
$$

2.4. Collision

In the presence of collisions, the evolution equation of the distribution function becomes ;

$$
\frac{\partial f}{\partial t} + \mathbf{v}\nabla_r f + \mathbf{F}\nabla_p f = \frac{\partial f}{\partial t}_{coll.} \tag{2.8}
$$

Usually write the collision term in a form displaying its balance structure :

$$
(\frac{\partial f}{\partial t})_{Collision} = (\frac{\partial f}{\partial t})^{+}_{Collision} + (\frac{\partial f}{\partial t})^{-}_{Collision} \tag{2.9}
$$

$(\frac{\partial f}{\partial t})^{+}_{Collision}$ is the entering collision term and $(\frac{\partial f}{\partial t})^{+}_{Collision} d^3r d^3p dt$ represents the mean number of particles undergoing a collisions between time t and dt+t, one of these particles finding itself, after the collision in the volume element $d^3r d^3p$ around the point $(\vec{r}, \vec{p})$.
$(\frac{\partial f}{\partial t})^{-}_{Collision}$ is the leaving collision term and $(\frac{\partial f}{\partial t})^{-}_{Collision} d^3r d^3p dt$ represents the mean number of particles undergoing a collisions between time t and dt+t, one of these particles finding itself, before the collision in the volume element $d^3r d^3p$ around the point $(\vec{r}, \vec{p})$.

Assuming that gas is extremely dilute, so that we may consider only binary collisions and ignore the possibility that three or more particles may collide simultaneously.

2.4.1. Binary Collisions

Consider a collisions between two particles of respective masses m and m_1.

•$\vec{p}$ and $\vec{p'}$ the kinetic momenta of both molecules before the collision.

•ϵ and ϵ' the energies of both particles before collision.

•$\vec{p_1}$ and $\vec{p_2}$ the kinetic momenta of both particles after the collision.

•ϵ_1 and ϵ_2 the energies of both particles after collision with $\epsilon_i = \frac{p_i^2}{2m_i}$.

The collision considered as local and instantaneous. Momentum and energy conservation require that

$$\begin{aligned} &\vec{p} + \vec{p'} = \vec{p_1} + \vec{p_2} = \vec{\mathbf{P}} \quad \text{Total kinetic momentum} \\ &\epsilon + \epsilon' = \epsilon_1 + \epsilon_2 = \mathbf{E} \quad \text{Total energy} \end{aligned} \tag{2.10}$$

Introduce relative momentum

$$\vec{\Pi} = \frac{m'\vec{p} - m\vec{p'}}{m + m'} = \mu(\vec{v} - \vec{v'}) \tag{2.11}$$

with $\mu = \frac{mm'}{m+m'}$ is the reduced mass.

From these equations we have

$$\begin{aligned} \vec{p} &= \frac{m}{M}(\vec{\mathbf{P}}_{\mathbf{i}} - \vec{\Pi}_i) \\ \vec{p'} &= \frac{m'}{M}(\vec{\mathbf{P}}_{\mathbf{i}} + \vec{\Pi}_i) \end{aligned} \tag{2.12}$$

with $M = m + m'$ is the total mass.

The total energy is given by

$$E = \frac{p^2}{2m} + \frac{(p')^2}{2m'} = \frac{\mathbf{P_i}^2}{2M} + \frac{\Pi_i^2}{2\mu} = \frac{\mathbf{P_f}^2}{2M} + \frac{\Pi_f^2}{2\mu} \tag{2.13}$$

So the condition for the momentum-energy conservation becomes

$$\begin{aligned} &\mathbf{P_i} = \mathbf{P_f} \\ &|\Pi_i| = |\Pi_f| \end{aligned} \tag{2.14}$$

Only $\vec{\Pi}_i$ rotates without changing its magnitude in the collision. It is the identical problem that the scattering of a particle by a fictitious fixed center from the origin. If we look at the trajectory of one particles as if it were scattered by a fixed center of force after transforming the coordinate system to the center of mass system we can see that particle reaches to the origin with relative momentum $\vec{\Pi}_i$ and will recede from origin with the rotated momentum $\vec{\Pi}_f$.

We can regulate the collision with the knowledge of $\vec{\Pi_i}$ and $\vec{\Pi_f}$, as well as of angles(the solid angle) $\Omega(\theta, \phi)$ called scattering angles of $\vec{\Pi_f}$ with respect to $\vec{\Pi_i}$. These angles completely determines the kinematics of the collision.

As a definition $D(\Omega)$ is called the differential cross-section with $D(\Omega) = \frac{d\sigma}{d\Omega}$.

The total cross-section area $\int d\sigma = \Sigma = \int D(\Omega) d\Omega$ is the total area of beam of particles that is scattered by the target (origin). The incident flux is defined as the number of particles per unit time crossing the unit area perpendicular to the incident beam. The number of particles per unit time scattered in a direction of the solid angle element $d\Omega$ is equal to product of the incident flux(I) and $D(\Omega)d\Omega$.
Thus the transition probability per unit time is $ID(\Omega)d\Omega$. The knowledge of $\vec{p}$ and $\vec{p'}$ describes a group of collisions. We generally describes this group of collisions by imagining a beam of particles with inital relative momentum $\vec{\Pi_i}$ incident on the force center. $D(\Omega)$ specifies all the the dynamical aspects of the collision .

Consider the collision $\{\vec{p}, \vec{p'}\} \longrightarrow \{\vec{p_1}, \vec{p_2}\}$ and inverse collision $\{\vec{p_1}, \vec{p_2}\} \longrightarrow \{\vec{p}, \vec{p'}\}$.

Obtaining from the original collision by interchanging the inital and final states we define :

$$\begin{aligned} &D(\Omega) = D(\vec{p}, \vec{p'}; \vec{p_1}, \vec{p_2}) \quad \text{for the collision} \quad \{\vec{p}, \vec{p'}\} \longrightarrow \{\vec{p_1}, \vec{p_2}\} \\ &D^{-1}(\Omega) = D(\vec{p_1}, \vec{p_2}\ ; \vec{p}, \vec{p'}\ ) \quad \text{for the inverse collision} \end{aligned} \tag{2.15}$$

$$\begin{aligned} D(\Omega) &= D^{-1}(\Omega) \\ D(\vec{p}, \vec{p'}; \vec{p_1}, \vec{p_2}) &= D(\vec{p_1}, \vec{p_2}; \vec{p}, \vec{p'}) \end{aligned} \tag{2.16}$$

is called the microreversibility relation.

2.4.2. Leaving Collision Term

Consider a beam of particle incident on particle, regarded as the target. The incident flux I is defined as the number of incident particles crossing unit area per second, from viewpoint of the target.

$$I = f(\mathbf{r}, \mathbf{p}^{'}, t) d^3 p' |\mathbf{v} - \mathbf{v}^{'}| \tag{2.17}$$

So the number of collisions of the type $\{\vec{p}, \vec{p^{'}}\} \longrightarrow \{\vec{p_1}, \vec{p_2}\}$ taking place on the particle of interest between times t and t+dt is

$$f(\mathbf{r}, \mathbf{p}^{'}, t) d^3 p^{'} |\mathbf{v} - \mathbf{v}^{'}| D(\Omega) d\Omega dt \tag{2.18}$$

We get taking into account all the particles of the considered type in number $f(\mathbf{r}, \mathbf{p}, t) d^3 r d^3 p$ and integrating over $p^{'}$

$$\frac{\partial f^{(-)}}{\partial t}_{coll.} d^3 r d^3 p dt = f(\mathbf{r}, \mathbf{p}, t) d^3 r d^3 p \int d^3 p^{'} \int |\mathbf{v} - \mathbf{v}^{'}| D(\Omega) d\Omega f(\mathbf{r}, \mathbf{p}^{'}, t) dt \tag{2.19}$$

Hence the leaving collision term is

$$\frac{\partial f^{(-)}}{\partial t}_{coll.} = f(\mathbf{r}, \mathbf{p}, t) \int d^3 p^{'} \int |\mathbf{v} - \mathbf{v}^{'}| D(\Omega) d\Omega f(\mathbf{r}, \mathbf{p}^{'}, t) \tag{2.20}$$

2.4.3. The entering collision term

In order to compute the rate of increase of $f(\mathbf{r}, \mathbf{p}, t)$ due to collisions, we focus on the inverse colllisions of type $\{\vec{p_1}, \vec{p_2}\} \longrightarrow \{\vec{p}, \vec{p^{'}}\}$. Consider a given particle of momentum $\mathbf{p_1}$ and incident beam of molecules of momentum $\mathbf{p_2}$.

The corresponding incident flux is

$$I = f(\mathbf{r}, \mathbf{p_2}, t) d^3p_2 |\mathbf{v_1} - \mathbf{v_2}| \tag{2.21}$$

The number of such collisions taking place on the considered particle between times t and t+dt is

$$f(\mathbf{r}, \mathbf{p_2}, t) d^3p_2 |\mathbf{v_1} - \mathbf{v_2}| D^{-1}(\Omega) d\Omega dt \tag{2.22}$$

The rate of change $\frac{\partial f}{\partial t}^{(+)}_{coll.}$ thus verifies the equality. (Obtained by taking account all the molecules of the considered type in number $f(\mathbf{r}, \mathbf{p_1}, t) d^3r d^3p_1$ and integrating over $\mathbf{p_2}$)

$$\frac{\partial f}{\partial t}^{(+)}_{coll.} d^3p = f(\mathbf{r}, \mathbf{p_1}, t) d^3p_1 \int D^{-1}(\Omega) d\Omega |\mathbf{v_1} - \mathbf{v_2}| f(\mathbf{r}, \mathbf{p_2}, t) d^3p_2 \tag{2.23}$$

From energy-momentum conservation

$$|\mathbf{v} - \mathbf{v}^{'}| = |\mathbf{v_1} - \mathbf{v_2}| \tag{2.24}$$

and at given angles scattering angles

$$d^3p d^3p^{'} = d^3p_1 d^3p_2 \tag{2.25}$$

In conclusion we obtained

$$\frac{\partial f}{\partial t}^{(+)}_{coll.} - \frac{\partial f}{\partial t}^{(-)}_{coll.} = \int d^3p^{'} \int D(\Omega) d\Omega |\mathbf{v} - \mathbf{v}^{'}| [f_1 f_2 - f f^{'}] \tag{2.26}$$

The Boltzmann equation becomes

$$\frac{\partial f}{\partial t} + \mathbf{v} \nabla_r f + \mathbf{F} \nabla_p f = \frac{\partial f}{\partial t}^{(+)}_{coll.} - \frac{\partial f}{\partial t}^{(-)}_{coll.} = \int d^3p^{'} \int D(\Omega) d\Omega |\mathbf{v} - \mathbf{v}^{'}| [f_1 f_2 - f f^{'}] \tag{2.27}$$

Note that Boltzmann equation is not time-reversal invariant.If t is changed into -t also it must be changed the sign of the velocities.The left-hand side of the equation changes signs however the collision part is not changing the sign.Thus $f(-\mathbf{r}, -\mathbf{p}, -t)$ is not the same as $f(\mathbf{r}, \mathbf{p}, t)$.The dynamics of Boltzmann equation is irreversible.

3. ROBERTSON-WALKER METRIC

The cosmological principal says that the universe is unchanging in space from point to point; is and always had been homogeneous and isotropic. The metric for a space with homogeneous and isotropic is the maximally-symmetric Robertson-Walker metric, which can be written in the form

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right] \tag{3.1}$$

where (t, r, θ, ϕ) are coordinates, R(t) is the cosmic scale factor and k can be choosen to be 0, 1 and -1.

For Roberson-Walker Metric, metric tensor is

$$g_{\mu\nu} = diag[-1,\ \frac{R^2(t)}{1-kr^2},\ R^2(t)r^2,\ R^2(t)r^2\sin^2\theta] \tag{3.2}$$

$$\begin{aligned}
g_{00} &= 1 \\
g_{0i} &= 0 \\
g_{ij} &= R^2(t)\widetilde{g_{ij}} \\
\widetilde{g_{ij}} &= 0 \quad i \neq j \\
\widetilde{g_{rr}} &= \frac{1}{1-kr^2} \\
\widetilde{g_{\theta\theta}} &= r^2 \\
\widetilde{g_{\varphi\varphi}} &= r^2 \sin^2\theta
\end{aligned} \tag{3.3}$$

Defining Christoffel symbol as,

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\gamma}\left[\partial_\nu g_{\gamma\sigma} + \partial_\sigma g_{\gamma\nu} - \partial_\gamma g_{\nu\sigma}\right] \tag{3.4}$$

The non-zero componentes are :

$$
\begin{aligned}
&\Gamma^0_{ij} = R\dot{R}\hat{g_{ij}} \\
&\Gamma^i_{0j} = \frac{\dot{R}}{R}\delta^i_j = \Gamma^i_{0i} \\
&\Gamma^1_{11} = \frac{kr}{1-kr^2} \\
&\Gamma^1_{22} = -(1-kr^2)r \\
&\Gamma^1_{33} = -(1-kr^2)r\sin^2\theta \\
&\Gamma^2_{21} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
&\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \\
&\Gamma^2_{33} = -\sin\theta\cos\theta
\end{aligned}
\tag{3.5}
$$

We defined

$$
g = -det g_{\mu\nu} = \frac{R^2(t)r^6\sin^2\theta}{1-kr^2} \tag{3.6}
$$

and the volume element is

$$
dV = \sqrt{|g|}dx_0dx_1dx_2dx_3 \tag{3.7}
$$

Constant Curvature (Riemann-Christofell Tensor) is defined as,

$$
R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\nu}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \quad with \quad \Lambda = \frac{k+\dot{R}^2}{R^2} \tag{3.8}
$$

The general definition of $R_{\mu\nu\rho\sigma}$ is,

$$
R_{\alpha\beta\gamma\delta} = \frac{1}{2}(\frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} - \frac{\partial^2 g_{\beta\gamma}}{\partial x^\alpha \partial x^\delta} - \frac{\partial^2 g_{\alpha\delta}}{\partial x^\beta \partial x^\gamma} + \frac{\partial^2 g_{\beta\delta}}{\partial x^\alpha \partial x^\gamma}) + g_{\eta\sigma}(\Gamma^\eta_{\gamma\alpha}\Gamma^\sigma_{\beta\delta} - \Gamma^\eta_{\delta\alpha}\Gamma^\sigma_{\beta\gamma}) \tag{3.9}
$$

and Ricci Tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = g^{\lambda\sigma} R_{\sigma\mu\lambda\nu} \tag{3.10}$$

Non-zero components of Ricci Tensor are

$$\begin{aligned} R_{00} &= -3\frac{\ddot{R}}{R} \\ R_{11} &= \frac{R\ddot{R} + 2\dot{R}^2 + 2k}{1 - kr^2} \\ R_{22} &= r^2(R\ddot{R} + 2\dot{R}^2 + 2k) \\ R_{33} &= r^2(R\ddot{R} + 2\dot{R}^2 + 2k)\sin^2\theta \end{aligned} \tag{3.11}$$

Finally the Ricci Scalar is defined as,

$$\begin{aligned} \mathbf{R} &= g^{\mu\nu} R_{\mu\nu} \\ \mathbf{R} &= 6(\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R}) \end{aligned} \tag{3.12}$$

Lastly we give some other definitions :

Covariant Derivative is defined as,

$$\begin{aligned} \nabla_\mu A^\nu &= \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda = A^\nu;_\mu \\ \nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\nu_{\mu\lambda} A^\lambda = A_\nu;_\mu \end{aligned} \tag{3.13}$$

Covariant divergence of Tensor is,

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} (\sqrt{g}\, T^{\mu\nu}) + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} \tag{3.14}$$

and Einstein Tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{3.15}$$

We conclude this chapter by showing Robertson-Walker metric does not have timelike killing vectors.

Theorem says a space-time(metric is stationary) is said to be stationary if and only if it admits a timelike killing vector field. [3]
Stationary means there should be a special coordinate system in which the metric is time-independent; $\frac{\partial g_{\mu\nu}}{\partial x^0} = 0$.

If we look at the Ricci Scalar which is defined as

$$\begin{aligned} \mathbf{R} &= g^{\mu\nu} R_{\mu\nu} \\ \mathbf{R} &= 6(\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R}) \end{aligned} \tag{3.16}$$

One can easily sees that Ricci Tensor which is coordinate independent is related to R(t) so this means that it is not possible to find such a coordinate system satisfies $\frac{\partial g_{\mu\nu}}{\partial x^0} = 0$ yields that RW is stationary. So RW metric has no timelike killing vectors as a conclusion of the metric is not stationary.

4. KINETIC THEORY IN FLAT SPACE-LIKE SLICES

4.1. The Boltzmann Equation

The fundamental fact in the kinetic theory is the description of a fluid of particles in the universe. We introduce the evolution of particle's distribution function $f(x^\mu, p^\mu, t)$ is the average number of particles (fluid) with a certain momentum at each space-time point.

"The time coordinate in Robertson-Walker metric is just the proper (or clock) time measured by an observer at rest in the comoving frame, for example $(r, \theta, \varphi) = const$. Observers at rest in the comoving frame remain at rest for example (r, θ, φ) remain unchanged, and observers initally moving with respect to this frame will eventually come to rest in it. Thus, if one introduces a homogeneous,isotropic fluid initially at rest in this frame, the t=const hypersurfaces will always be orthogonal to the fluid flow, and will always coincide with the hypersurfaces of both spatial homogeneity and constant fluid density." [1]

So if we consider a spacelike surface M at constant t, the local normal surface element with k=0 is :

$$dS_\mu = (R^3(t)dx_1dx_2dx_3, 0, 0, 0) \tag{4.1}$$

The number of worldlines that penerates dM spacelike surface with four-momentum p^μ (flux of the four current through spacelike surface) is :

$$dN = -R^3(t)f(x^\mu, p^\mu)p^\mu 2\delta^{(+)}(p^2 + m^2)dp^4 dS_\mu \tag{4.2}$$

Define $dP = 2\delta^{(+)}(p^2 + m^2)dp^4$ where ;

$\delta^{(+)}(p^2 + m^2)$ is the mass-shell condition (the particles of the gas or fluid are constrained to move on the mass-shell so they satisfy the Einstein-energy momentum relation) which is

$$\begin{aligned} \mathbf{p^2} &= g_{\mu\nu}p^\mu p^\nu = R^2(t)p^2 - p_0^2 = -m^2 = \mathbf{p^2}(\mathbf{p_0}) \leqslant 0 \\ p_0^2 &= R^2(t)p^2 + m^2 \longrightarrow p_0 = \pm\sqrt{R^2(t)p^2 + m^2} \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\delta(p^2 + m^2) = \delta(g(p_0)) = \delta(R^2(t)p^2 - p_0^2 + m^2) = \delta(p_0^2 - (R^2(t)p^2 + m^2)) \\ &= \frac{\delta(p_0 + \sqrt{R^2(t)p^2 + m^2})}{2|p_0|} + \frac{\delta(p_0 - \sqrt{R^2(t)p^2 + m^2})}{2|p_0|} = \delta^{(-)}(p^2 + m^2) + \delta^{(+)}(p^2 + m^2) \end{aligned} \tag{4.4}$$

We choose $\delta^{(+)}(p^2 + m^2)$.

Looking at the particle kinematics since we are on shell, configuration of our system satisfies classical equation of motion.

So world-line of our particles x^μ obeys Lorentz-Einstein equation of motion :

$$\begin{aligned} p^\mu &= \frac{dx^\mu}{d\tau} \\ \frac{dp^\mu}{d\tau} &= -\Gamma^\mu_{\alpha\beta}p^\alpha p^\beta \qquad \tau \text{ is affine parameter} \end{aligned} \tag{4.5}$$

The equation of geodesic motion of a particle(equation of motion for one particle) is

$$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx\nu}{d\tau} = 0 \tag{4.6}$$

and Hamiltonian

$$\begin{aligned} H &= H(x^\mu, p^\mu) = \frac{1}{2m}g_{\mu\nu}(x)p^\mu p^\nu \\ \frac{dx^\mu}{d\tau} &= \frac{\partial H}{\partial p^\mu} = \frac{p^\mu}{m} \\ \frac{dp^\mu}{d\tau} &= -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2}\partial_\mu g_{\alpha\beta}p^\alpha p^\beta \end{aligned} \tag{4.7}$$

Variation of $f(x(\tau), p(\tau))$ along a worldline charaterized by an affine parameter can be written using the above identities as

$$\begin{aligned} \frac{df}{d\tau} &= \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\tau} + \frac{\partial f}{\partial p^\mu}\frac{dp^\mu}{d\tau} \\ &= p^0\frac{\partial f}{\partial t} + \vec{p}\vec{\nabla} f - \Gamma^0_{\alpha\beta}p^\alpha p^\beta\frac{\partial f}{\partial p^0} - \Gamma^i_{\alpha\beta}p^\alpha p^\beta\frac{\partial f}{\partial p^i} \end{aligned} \tag{4.8}$$

This is general for all k's.

If we look for k=0 flat space case, non-vanishing Christoffel components are

$$\begin{aligned} \Gamma^0_{ij} &= R\dot{R}\delta_{ij} \\ \Gamma^0_{0j} &= \frac{\dot{R}}{R}\delta^i_j \\ \Gamma^i_{jk} &= 0 \end{aligned} \tag{4.9}$$

So equation (3.7) becomes

$$\begin{aligned} &\Gamma^0_{\alpha\beta}p^\alpha p^\beta = R\dot{R}p^2 \\ &\Gamma^i_{\alpha\beta} = 2\Gamma^i_{0j}p^0p^j + \Gamma^i_{jk}p^jp^k = 2\frac{\dot{R}}{R}p^0p^i \\ &\frac{df}{d\tau} = p^0\frac{\partial f}{\partial t} + \vec{p}\vec{\nabla} f - 2\frac{\dot{R}}{R}p^0p^i\frac{\partial f}{\partial p^i} - R\dot{R}p^2\frac{\partial f}{\partial p^0} \\ &= \frac{\partial f}{\partial t} - 2\frac{\dot{R}}{R}p^0p^i\frac{\partial f}{\partial p^i} - R\dot{R}p^2\frac{\partial f}{\partial p^0}) \qquad \text{because of isotropy } \vec{\nabla} f = 0 \end{aligned} \tag{4.10}$$

Now we look at the variation of f on the mass-shell along affine trajectory so we calculate the integral,

$$\int \frac{df}{d\tau}2\frac{\delta(p_0 - \sqrt{R^2(t)p^2 + m^2})}{2|p_0|}dp_0 \tag{4.11}$$

$$\int \frac{df}{d\tau} 2\frac{\delta(p_0 - \sqrt{R^2(t)p^2+m^2})}{2|p_0|} dp_0 =$$
$$\int p^0 \frac{\partial f}{\partial t}\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0 - 2\frac{\dot{R}}{R}\int p^i \frac{\partial f}{\partial p^i}\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0$$
$$- R\dot{R}\int p^2 \frac{\partial f}{\partial p_0}\frac{\delta(p_0 - \sqrt{R^2(t)p^2+m^2})}{p_0}dp_0$$
$$= \int \frac{\partial f}{\partial t}\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0 - 2\frac{\dot{R}}{R}p^i\frac{\partial f}{\partial p^i}|_{p_0=\sqrt{R^2p^2+m^2}} - R\dot{R}\frac{p^2}{p_0}\frac{\partial f}{\partial p_0}|_{p_0=\sqrt{R^2p^2+m^2}} \tag{4.12}$$

Consider now the integral

$$\frac{\partial}{\partial t}\int f\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0 = \int \frac{\partial f}{\partial t}\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0 + \underbrace{\int f\frac{\partial\delta(p_0 - \sqrt{R^2(t)p^2+m^2})}{\partial t}dp_0}_{\mathrm{I}} \tag{4.13}$$

$$I = \int f\frac{\partial\delta(p_0 - \sqrt{R^2(t)p^2+m^2})}{\partial t}dp_0 = \frac{1}{2}\int f\frac{\partial p_0}{\partial t}\frac{\partial\delta(p_0 - \sqrt{R^2(t)p^2+m^2})}{\partial p_0}dp_0$$
$$= \frac{1}{2}\int \frac{\partial}{\partial p_0}\left[f\frac{\partial p_0}{\partial t}\right]\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0 = \frac{1}{2}\int \frac{\partial f}{\partial p_0}\frac{\partial p_0}{\partial t}\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0$$
$$= \int \frac{\partial f}{\partial p_0}\frac{R\dot{R}}{p_0}p^2\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0 = \frac{1}{2}\frac{R\dot{R}}{p_0}p^2\frac{\partial f}{\partial p_0}|_{p_0=\sqrt{R^2p^2+m^2}} \tag{4.14}$$

So we have

$$\frac{\partial}{\partial t}\int f\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0 = \frac{\partial f(\hat{p},t)}{\partial t}$$
$$= \int \frac{\partial f}{\partial t}\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0 - \frac{1}{2}\frac{R\dot{R}}{p_0}p^2\frac{\partial f}{\partial p_0}|_{p_0=\sqrt{R^2p^2+m^2}} \tag{4.15}$$

As a conclusion using above integral identities we found that

$$\int \frac{df}{d\tau}dP = \int \frac{\partial f}{\partial t}\delta(p_0 - \sqrt{R^2(t)p^2+m^2})dp_0 d^3p - 2\frac{\dot{R}}{R}p^i\frac{\partial f}{\partial p^i}|_{p_0=\sqrt{R^2p^2+m^2}}$$
$$-R\dot{R}\frac{p^2}{p_0}\frac{\partial f}{\partial p_0}|_{p_0=\sqrt{R^2p^2+m^2}} = \frac{\partial \hat{f}}{\partial t} - 2\frac{\dot{R}}{R}p^i\frac{\partial \hat{f}}{\partial p^i} \tag{4.16}$$

Dropping hat,

$$\frac{\partial f}{\partial t} - 2\frac{\dot{R}}{R}p^i\frac{\partial f}{\partial p^i} \qquad with \quad f(p,t) \tag{4.17}$$

Now we introduce Liouville operator L ;

$$L(f) = \frac{\partial f}{\partial t} - 2\frac{\dot{R}}{R}p^i\frac{\partial f}{\partial p^i} \tag{4.18}$$

and the collisionless Boltzmann Equation is the statement that :

$$L(f) = \int \frac{df}{d\tau}2\frac{\delta(p_0 - \sqrt{R^2(t)p^2 + m^2})}{2|p_0|}dp_0 = 0 \tag{4.19}$$

which means that there is no net variation of f on the mass-shell along affine trajectory.

"The quantity f, Boltzmann Equation ,is determined by microscopic behaviour of the system; the interparticle collisions. So it is convenient to introduce the "local momentum" ;" [4]
$R(t)\vec{p} = \underline{\vec{p}}$ and distribution functions becomes $f(\underline{\vec{p}}, t)$.
So we have ;

$$\begin{aligned} \frac{d\bar{p^i}}{d\tau} &= R\frac{dp^i}{d\tau} + p^i\frac{dR}{d\tau} = -2\dot{R}p^0p^i + p^i\frac{dR}{d\tau} \\ &= -2\dot{R}p^0p^i + p^ip^0\dot{R} = -p^ip^0\dot{R} = -\frac{\dot{R}}{R}\bar{p^i}p^0 \end{aligned} \tag{4.20}$$

in terms of local momentum Boltzmann Equation becomes ;

$$L(f) = \frac{\partial f}{\partial t} - \frac{\dot{R}}{R}\bar{p}_i\frac{\partial f}{\partial \bar{p}_i} \tag{4.21}$$

Dropping the bars the Boltzmann equation becomes ;

$$L(f) = \frac{\partial f}{\partial t} - \frac{\dot{R}}{R}p_i\frac{\partial f}{\partial p_i} \tag{4.22}$$

4.2. Collision

If we consider the incident particles have states $|p>$ with energies E(p) and distribution $f(\mathbf{p},t)$ (the occupancy of one-particle state $|p>$ at time t) and the scattered particles have states $|p_1>$ with energies $E(p_1)$ and distribution $f(\mathbf{p_1},t)$. Let $R(\mathbf{p},\mathbf{p}^{'},\mathbf{p_1},\mathbf{p_2})dt$ is the number of transitions per second due to collision(the number of collisions in dt) $\vec{p} \longrightarrow \vec{p_1}$ and $\vec{p^{'}} \longrightarrow \vec{p_2}$ and let $R(\mathbf{p^1},\mathbf{p^2},\mathbf{p},\mathbf{p}^{'})dt$ the number of inverse transitions for $\vec{p_1} \longrightarrow \vec{p}$ and $\vec{p_2} \longrightarrow \vec{p^2}$.

So full Boltzmann Equation is equal to L(f)=C(E(p)) where C(E(p)) is the change due to collision and it is equal to

$$C(E(p)) = \int [R(\mathbf{p_1},\mathbf{p_2},\mathbf{p},\mathbf{p}^{'}) - R(\mathbf{p},\mathbf{p}^{'},\mathbf{p_1},\mathbf{p_2})]d^4p_1 d^4p_2 d^4p^{'} \qquad (4.23)$$

The microscopic transition probability per second (the transition rate) $\vec{p},\vec{p^{'}} \longrightarrow \vec{p_1}\vec{p_2}$ or it is the probability density for a given particles to be in the state $|p>,|p^{'}>$ and undergoes a transition to the state $|p_1>,|p_2>$ is governed by by $Q(p,p^{'};p_1,p_2)$.
From Fermi's Golden Rule [22] [17]

$$Q(p,p^{'};p_1,p_2) = 2\pi|M(p,p^{'};p_1,p_2)|^2\delta(E(p)+E(p^{'})-E(p_1)+E(p_2)) \qquad (4.24)$$

Because of the unitarity of S-matrix we have [see appendix A]

$$\begin{aligned} &\int \delta^{(4)}(p+p^{'}-p_1-p_2)W(p_1,p_2;p,p^{'})[dp_1][dp_2] \\ =&\int \delta^{(4)}(p+p^{'}-p_1-p_2)W(p,p^{'};p_1,p_2)[dp_1][dp_2] \end{aligned} \qquad (4.25)$$

and we define collision probability as

$$\begin{aligned} &W(p,p^{'};p_1,p_2) = 2\pi|M(p,p^{'};p_1,p_2)|^2 \\ &\text{means} \\ &Q(p,p^{'};p_1,p_2) = W(p,p^{'};p_1,p_2)\delta(E(p)+E(p^{'})-E(p_1)+E(p_2)) \end{aligned} \qquad (4.26)$$

So as a conclusion

$$
\begin{aligned}
&R(\mathbf{p},\mathbf{p}^{'},\mathbf{p_1},\mathbf{p_2}) = Q(p,p^{'};p_1,p_2)\\
&= W(p,p^{'};p_1,p_2)\delta(E(p)+E(p^{'})-E(p_1)+E(p_2))
\end{aligned}
\tag{4.27}
$$

We can define collision term as

$$
\begin{aligned}
&\frac{1}{2E(p)}\int \delta\left[W(p,p^{'};p_1,p_2)f(p_1)f(p_2) - W(p_1,p_2;p,p^{'})f(p)f(p^{'})\right] dP_1dP_2dP^{'}\\
&\text{where}\\
&dP_1dP_2dP^{'} = \frac{\delta(p_{1_0}-\sqrt{R^2(t)p_1^2+m^2})}{2p_{1_0}}d^3p_1\cdots\frac{\delta(p_0^{'}-\sqrt{R^2(t)p_1^2+m^2})}{2p_0^{'}}d^3p^{'}\\
&\delta = \delta(E(p)+E(p^{'})-E(p_1)+E(p_2))
\end{aligned}
\tag{4.28}
$$

For the Fermi and Bose case the collision is defined as

$$
\frac{1}{2E}\int \delta\left[Wf(p_1)f(p_2)[1\mp f(p)][1\mp f(p')] - Wf(p)f(p^{'})[1\mp f(p_1)][1\mp f(p_2)]\right] d\mathbf{P}
\tag{4.29}
$$

So in general we can define collision term as follow

$$
\frac{1}{2E}\int \delta\left[Wf(p_1)f(p_2)[1\mp \tau f(p)][1\mp \tau f(p')] - Wf(p)f(p^{'})[1\mp \tau f(p_1)][1\mp \tau f(p_2)]\right]
\tag{4.30}
$$

with $\tau = -1, 0, 1$ [9]

From the unitarity of S-matrix we found that :

$$
\begin{aligned}
&C(E(p)) = \frac{1}{2E(p)}\int\int\int \delta(E(p)+E(p^{'})-E(p_1)+E(p_2))W(p,p^{'};p_1,p_2)\\
&\left[f(p_1)f(p_2)-f(p)f(p^{'})\right] dP_1dP_2dP^{'}\\
&= \frac{1}{2E(p)}\int \delta(E(p)+E(p^{'})-E(p_1)+E(p_2))W(p,p^{'};p_1,p_2)f(p_1)f(p_2)dP_1dP_2dP^{'}\\
&- \frac{1}{2E(p)}\int \delta(E(p)+E(p^{'})-E(p_1)+E(p_2))W(p,p^{'};p_1,p_2)f(p)f(p^{'})dP_1dP_2dP^{'}
\end{aligned}
\tag{4.31}
$$

Change of variables as p to p_1, $p^{'}$ to p_2 and integrating C(E(p)) we found that

$$\int C(E(p))dP = \int \frac{C(E(p))}{E(p)} d^3p = 0 \tag{4.32}$$

It guarantees the conservation of number current (conservation of particle number)

$$\int C(E(p))dP = \int L(f)dP = \nabla_\mu N^\mu = 0 \tag{4.33}$$

Now if we look at the integral

$$\begin{aligned} &\int C(E(p))E(p)dP \\ &= \int \delta W(p,p^{'};p_1,p_2)E(p)\left[f(p_1)f(p_2) - f(p)f(p^{'})\right] dP_1 dP_2 dP^{'} dP \\ &= \frac{1}{2}\int \delta W(p,p^{'};p_1,p_2)\left[E(p)+E(p)\right]\left[f(p_1)f(p_2) - f(p)f(p^{'})\right] dP_1 dP_2 dP^{'} dP \end{aligned} \tag{4.34}$$

change of variables p to $p^{'}$

$$= \frac{1}{2}\int \delta W(p,p^{'};p_1,p_2)\left[E(p)+E(p^{'})\right]\left[f(p_1)f(p_2) - f(p)f(p^{'})\right] dP_1 dP_2 dP^{'} dP \tag{4.35}$$

change of variables p to p_1 and $p^{'}$ to p_2

$$= \frac{1}{2}\int \delta W(p,p^{'};p_1,p_2)\left[E(p_1)+E(p_2)\right]\left[f(p)f(p^{'}) - f(p_1)f(p_2)\right] dP_1 dP_2 dP^{'} dP \tag{4.36}$$

means

$$\frac{1}{2}\int \delta W(p,p^{'};p_1,p_2)[E(p)+E(p^{'})+E(p_1)+E(p_2)][f(p)f(p^{'}) - f(p_1)f(p_2)]d\mathbf{P} = 0 \tag{4.37}$$

$$
\begin{aligned}
&\underbrace{E(p)+E(p^{'})+E(p_1)+E(p_2)}_{\neq 0 \text{ because of conservation of energy}}\\
&\underbrace{f(p)f(p^{'})-f(p_1)f(p_2)}_{\neq 0}\\
&\underbrace{\delta(E(p)+E(p^{'})-E(p_1)+E(p_2))W(p,p^{'};p_1,p_2)}_{\neq 0}
\end{aligned}
\tag{4.38}
$$

Thus

$$
\begin{aligned}
&\int \delta(E(p)+E(p^{'})-E(p_1)+E(p_2))\times\\
&W(p,p^{'};p_1,p_2)E(p)\left[f(p)f(p^{'})-f(p_1)f(p_2)\right]d\mathbf{P}=0\\
&\int \delta(E(p)+E(p^{'})-E(p_1)+E(p_2))\times\\
&W(p,p^{'};p_1,p_2)E(p^{'})\left[f(p)f(p^{'})-f(p_1)f(p_2)\right]d\mathbf{P}=0
\end{aligned}
\tag{4.39}
$$

This proves the result

$$
\int C(E(p))p_0dP=\int L(f)p_0dP=\nabla_\mu T^{\mu\nu}=0 \tag{4.40}
$$

As a conclusion we can introduce full Boltzmann equation

$$
\begin{aligned}
&L(f)=C(E(p))\\
&=p^0\frac{\partial f}{\partial t}+\vec{p}\vec{\nabla}f-\Gamma^0_{\alpha\beta}p^\alpha p^\beta\frac{\partial f}{\partial p^0}-\Gamma^i_{\alpha\beta}p^\alpha p^\beta\frac{\partial f}{\partial p^i}\\
&=\frac{1}{2E}\int \delta[Wf(p_1)f(p_2)[1\mp\tau f(p)][1\mp\tau f(p')]\\
&-Wf(p)f(p^{'})[1\mp\tau f(p_1)][1\mp\tau f(p_2)]]d\mathbf{P}
\end{aligned}
\tag{4.41}
$$

and in the flat space case (k=0)

$$
\begin{aligned}
&\frac{\partial f}{\partial t} - \frac{\dot{R}}{R}p_i\frac{\partial f}{\partial p_i} = \\
&\frac{1}{2E}\int \delta[Wf(p_1)f(p_2)[1 \mp \tau f(p)][1 \mp \tau f(p')] \\
&- Wf(p)f(p^{'})[1 \mp \tau f(p_1)][1 \mp \tau f(p_2)]]d\mathbf{P}
\end{aligned}
\tag{4.42}
$$

4.3. Some Macroscopic Quantities

Introduce some of macroscopic quantities.

The particle number flux vector (particle current density)

$$
\begin{aligned}
N^{\mu} &= \int fp^{\mu}dP = \int fp^{\mu}\frac{\delta(p_0 - \sqrt{R^2(t)p^2 + m^2})}{p_0}d^3pdp^0 \\
&= (\int f(\vec{p}, p, t)d^3p, \int f\frac{p^{\mu}}{p_0}d^3p)
\end{aligned}
\tag{4.43}
$$

because of isotropy only non vanishing component of N^{μ} is N^0.

$$
n = N^0 = \int fd^3p \quad \text{is the particle number density} \tag{4.44}
$$

Calculating covariant divergence of it we find that

$$
\begin{aligned}
&\nabla_{\mu}N^{\mu} = \frac{1}{R^3}\frac{\partial}{\partial t}(R^3\int fd^3p) \quad \text{replacing } \tfrac{\partial f}{\partial t} \text{ from Boltzmann equation} \\
&= \frac{3\dot{R}}{R}\int fd^3p + \int L(f)d^3p + \frac{\dot{R}}{R}\int p_i\frac{\partial f}{\partial p_i}d^3p \\
&\text{using integration by part and than delete terms only remaining term is} \\
&= \int L(f)d^3p
\end{aligned}
\tag{4.45}
$$

So as a conclusion we have (Condition for the conservation of total particle number)

$$\nabla_{\mu}N^{\mu} = \frac{1}{R^3}\frac{\partial}{\partial t}(R^3 n) = \int L(f)d^3p \tag{4.46}$$

The Energy- Momentum Tensor

$$T^{\mu\nu} = \int fp^{\mu}p^{\nu}dP = \int fp^{\mu}p^{\nu}\frac{\delta(p_0 - \sqrt{R^2(t)p^2 + m^2})}{p_0}d^3pdp^0 = \int \frac{fp^{\mu}p^{\nu}}{p_0}d^3p \tag{4.47}$$

and

$$\begin{aligned} T^{00} &= \int fp_0 d^3p = \rho \quad \text{is called energy density} \\ T^{ij} &= \frac{g^{?j}}{3}\int \frac{fp^2}{p_0}d^3p = p \quad \text{is called kinematic pressure} \end{aligned} \tag{4.48}$$

From energy-momentum conservation

$$\nabla_{\mu}T^{\mu\nu} = 0 \tag{4.49}$$

So calculating covariant divergence

$$\begin{aligned} &\frac{1}{R^3}\frac{\partial}{\partial x^{\mu}}\left[\int R^3\frac{fp^{\mu}p^{\nu}}{p_0}d^3p\right] = \frac{1}{R^3}\frac{\partial}{\partial t}\left[\int R^3\frac{f^{\nu}}{p}d^3p\right] \\ &= \frac{1}{R^3}\frac{\partial}{\partial t}\left[\int R^3\frac{f^{0}}{p}d^3p\right] = 3\frac{\dot{R}}{R}\int fp^0d^3p + \int\frac{\partial f}{\partial t}p^0d^3p \\ &= 3\frac{\dot{R}}{R}\int fp^0d^3p + \int L(f)p^0d^3p + \int\frac{\dot{R}}{R}p^i\frac{\partial f}{\partial p^i}p^0d^3p \\ &= 3\frac{\dot{R}}{R}\int fp^0d^3p + \int L(f)p^0d^3p + \frac{\dot{R}}{R}p^ip^0f - 3\frac{\dot{R}}{R}\int fp^0d^3p = \int L(f)p^0d^3p \end{aligned} \tag{4.50}$$

We found that

$$\nabla_{\mu}T^{\mu\nu} = \int L(f)p^0d^3p = 0 \tag{4.51}$$

As conclusion

$$\nabla_\mu T^{\mu\nu} = \int L(f) p^0 d^3p = \int C(E(p)) p^0 d^3p = 0 \tag{4.52}$$

4.4. Entropy and equilibrium solutions

The entropy density current is defined as

$$S^\mu = -\int \left[f \ln f \mp (1 \mp f) \ln(1 \pm f) \right] \frac{p^\mu}{p^0} d^3p \tag{4.53}$$

upper sign : Bose- Eisntein

lower sign : Fermi Dirac

In the classical limit 4.53 becomes ;

$$S^\mu = -\int \left[f \ln f - f) \right] \frac{p^\mu}{p^0} d^3p \tag{4.54}$$

Taking covariant divergence of entropy density (because of isotropy only non-vanishing component is S^0)

$$\begin{aligned}
\nabla_\mu S^\mu &= -\frac{1}{R^3}\frac{\partial}{\partial t}\left[R^3\int\left[f\ln f - f)\right]d^3p\right] \\
&= -\frac{1}{R^3}\frac{\partial}{\partial t}\left[R^3\int f\ln f d^3p\right] + \underbrace{\frac{1}{R^3}\frac{\partial}{\partial t}\left[R^3\int f d^3p\right]}_{=\nabla_\mu N^\mu = 0} \\
&= \frac{3\dot{R}}{R}\int f\ln f d^3p + \int \dot{f}\ln f d^3p - \int \dot{f} d^3p \\
&= \frac{3\dot{R}}{R}\int f\ln f d^3p + \int \frac{\partial f}{\partial t}\left[\ln f - 1\right]d^3p \qquad (4.55)\\
&\text{using} \quad \frac{\partial f}{\partial t} = C(E) + \frac{\dot{R}}{R}p_i\frac{\partial f}{\partial p_i} \\
&= \frac{3\dot{R}}{R}\int f\ln f d^3p - \int C(E)\left[\ln f + 1\right]d^3p - \frac{\dot{R}}{R}p_i\frac{\partial f}{\partial p_i}\left[\ln f + 1\right]d^3p \\
&\text{integrate by part and cancel terms} \\
&= -\int C(E)\ln f d^3p
\end{aligned}$$

As a conlusuion we have

$$\nabla_\mu S^\mu = -\int C(E)\ln f d^3p \qquad (4.56)$$

Now if we substitude C(E)

$$\begin{aligned}
&\nabla_\mu S^\mu = -\int C(E)\ln f d^3p \\
&= \int delta^4(p + p^{'} - p_1 - p_2)W(p, p^{'}; p_1, p_2)\left[f(p_1)f(p_2) - f(p)f(p^{'})\right]\ln f d\mathbf{P}
\end{aligned} \qquad (4.57)$$

and define

$$\begin{aligned}
&\sigma_H = \ln f(p)\left[f(p_1)f(p_2) - f(p)f(p^{'})\right] \\
&\breve{\sigma_H} = \ln f(p^{'})\left[f(p_1)f(p_2) - f(p)f(p^{'})\right] = \ln f(p)\left[f(p_1)f(p_2) - f(p)f(p^{'})\right] = \sigma_H \\
&\sigma_H + \breve{\sigma_H} = \left[\ln f(p) + \ln f(p^{'})\right]\left[f(p_1)f(p_2) - f(p)f(p^{'})\right] = 2\sigma_H
\end{aligned} \qquad (4.58)$$

So (4.48) becomes

$$\nabla_\mu S^\mu$$
$$= -\frac{1}{4}\int \sigma_H \ \delta f(p) f(p^{'}) W(p,p^{'};p_1,p_2) dP_1 dP_2 dP^{'} dP$$
$$= -\frac{1}{4}\int \delta f(p) f(p^{'}) W(p,p^{'};p_1,p_2) \ln\left[\frac{f(p)f(p^{'})}{f(p_1)f(p_2)}\right]\left[f(p_1)f(p_2) - f(p)f(p^{'})\right] d\mathbf{P} \tag{4.59}$$

since all f's ,$W(p,p^{'};p_1,p_2)$ and $\delta(E(p)+E(p^{'})-E(p_1)+E(p_2))$ are positive using the relation $(x-y)\ln(\frac{y}{x}) \leqslant 0$
we found that

$$\nabla_\mu S^\mu = \nabla_0 S^0 \geqslant 0 \tag{4.60}$$

which is also called Boltzmann H-Theorem.
Equilibrium solution (equilibrium distrubution) satisfies below conditions ;

$$\nabla_\mu S^\mu = 0 \ \ and \ \lim_{t\to+\infty} f(\vec{p}) = f_{eq}(\vec{p}) \tag{4.61}$$

from the covariant divergence of entropy
$\int \delta W(p,p^{'};p_1,p_2) \ln f_{eq}(p)\left[f_{eq}(p_1)f_{eq}(p_2) - f_{eq}(p)f_{eq}(p^{'})\right] dP_1 dP_2 dP^{'} dP = 0$
This is possible only if

$$f_{eq}(\vec{p_1}) f_{eq}(\vec{p_2}) = f_{eq}(\vec{p}) f_{eq}(\vec{p^{'}}) \tag{4.62}$$

Putting $lnf_{eq}(\mathbf{p}) = \chi(\mathbf{p})$ where ;
$\chi(\mathbf{p})$ are called "summational invariants" or "additive collision invariants." [7]
So (4.49) becomes

$$\ln f_{eq}(\vec{p_1}) + \ln f_{eq}(\vec{p_2}) = \ln f_{eq}(\vec{p}) + \ln f_{eq}(\vec{p^{'}}) \tag{4.63}$$

For "summational invariants" we have the following theorem ; [7]
Theorem (4.1) : A continuous and differentiable function of class C^2 $\chi(p^\alpha)$ is a sum-

mational invariant if and only if its given by

$$\chi(p^\alpha) = A + B_\alpha p^\alpha \tag{4.64}$$

where A is an arbitrary scalar and B_α an arbitrary four-vector that do not depend on p^α

So we found that equilibrium distribution function is

$$\begin{aligned} \ln f_{eq}(\vec{p}) &= \alpha(t) + \beta^\mu(t)p_\mu \\ f_{eq}(\vec{p}) &= \exp\left[\alpha(t) + \beta^\mu(t)p_\mu\right] \end{aligned} \tag{4.65}$$

where

$\alpha(t)$: scalar function of time

$\beta^\mu(t)$: timelike four vector (beacuse f must be bounded for arbitrary large momenta)

From equlibrium condition

$$\nabla_\mu S^\mu = -\int C(E) \ln f_{eq} dP = 0 \quad if\ and\ only\ if \quad C(E(p)) = 0 = L(f_{eq}) \tag{4.66}$$

From the definition of L(f)

$$L(f) = \left[p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^\mu}\right] f \tag{4.67}$$

the equilibrium case $L(f_{eq}) = 0$ means

$$\begin{aligned} &L(f_{eq}) = \left[p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^\mu}\right] f_{eq} = 0 \\ &\left[p^\mu \frac{\partial}{\partial x^\mu} \exp\left[\alpha(t) + \beta^\mu(t)p_\mu\right] - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^\mu}\right] \exp\left[\alpha(t) + \beta^\mu(t)p_\mu\right] = 0 \\ &p^0 \left[\dot{\alpha(t)} + \dot{\beta}^\mu(t)p_0\right] f_{eq} - \left[\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \beta^\mu(t)\right] f_{eq} = 0 \\ &p^0 \dot{\alpha(t)} + \dot{\beta}^\mu(t) p_0 p^0 - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \beta^\mu(t) = 0 \end{aligned} \tag{4.68}$$

means

$$\dot{\alpha(t)} = 0$$
$$\dot{\beta}^{\mu}(t)p_0p^0 - \beta^{\mu}(t)\Gamma^{\mu}_{\alpha\beta}p^{\alpha}p^{\beta} = 0 \tag{4.69}$$

Second equation is the killing vector equation which is

$$\nabla^{\nu}\beta_{\mu} + \nabla^{\mu}\beta_{\nu} = 0 \tag{4.70}$$

However as we showed before in RW metric there is no timelike Killing vector. So there is no β that can satisfy the second equation. So we cannot have any solution as equilibrium solution.

No evolutionary cosmological models have a timelike killing vector so we can never find equilibrium solutions to the Boltzmann Equations for these models.

So we look the equilibrium solution for the special cases in flat space with k =0
For flat space (k=0) we know that

$$L(f_{eq}) = \frac{\partial f_{eq}}{\partial t} - \frac{\dot{R}}{R}p\frac{\partial f_{eq}}{\partial p} = 0$$
$$\dot{\alpha} - \dot{\beta}E + \frac{\dot{R}}{R}\beta(t)p^2 \tag{4.71}$$
$$\frac{\dot{\alpha}}{\dot{\beta}} - E - \frac{\beta(t)}{\dot{\beta(t)}}\frac{p^2}{E}\frac{\dot{R}}{R} = 0$$

In general there is no $\alpha(t)$ and $\beta(t)$ which solves this equation because of the above discussion.
So lets look at the two limits when m $\rightarrow$ 0 and m $\rightarrow \infty$

First we look m $\rightarrow$ 0
In this case E = pc

$$
\begin{aligned}
L(f_{eq}) &= \frac{\partial f_{eq}}{\partial t} - \frac{\dot{R}}{R} p \frac{\partial f_{eq}}{\partial p} = 0 \\
\dot{\alpha} - \dot{\beta} E &+ \frac{\dot{R}}{R} \beta(t) p^2 \\
\frac{\dot{\alpha}}{\dot{\beta}} &= E - \frac{\beta(t)}{\dot{\beta}(t)} \frac{p^2}{E} \frac{\dot{R}}{R} \\
\frac{\dot{\alpha}}{\dot{\beta}} &= p - \frac{\beta(t)}{\dot{\beta}(t)} \frac{p^2}{p} \frac{\dot{R}}{R} \\
\frac{\dot{\alpha}}{\dot{\beta}} &= p \left[1 - \frac{\dot{R}}{R} \frac{\beta}{\dot{\beta}} \right]
\end{aligned} \tag{4.72}
$$

The equation has a solution if

$$
\begin{aligned}
&\bullet\ \alpha(t) = 0 \\
&\bullet\ \beta(t) = const.R(t) \longrightarrow T \sim \frac{1}{R(t)} \\
&f_{eq} = C \exp\left[-ApR(t)\right]
\end{aligned} \tag{4.73}
$$

[4] with A is constant.

Second limit is when m $\rightarrow \infty$

$$
\begin{aligned}
&E = m \left[1 + \frac{1}{2} (\frac{p}{m})^2 - \frac{1}{8} (\frac{p}{m})^4 + \cdots \right] \simeq m + \frac{p^2}{2m} \\
&\frac{\dot{\alpha}}{\dot{\beta}} = m + \frac{p^2}{2m} - \frac{\dot{R}}{R} \frac{\beta}{\dot{\beta}} \frac{p^2}{m} \\
&\frac{\dot{\alpha}}{\dot{\beta}} - m = \frac{p^2}{2m} - \frac{\dot{R}}{R} \frac{\beta}{\dot{\beta}} \frac{p^2}{m} \\
&\text{means} \\
&\bullet\ \alpha - m\beta = \text{constant} = \Omega \\
&\bullet\ \frac{\dot{R}}{R} \frac{\beta}{\dot{\beta}} = \frac{1}{2} \longrightarrow \beta = \frac{\beta(t_0)}{R^2(t_0)} R^2 \\
&\text{equilibrium function is} \\
&f_{eq} = \exp\left[\alpha - E\beta\right] = Const. \exp(-\frac{p^2}{2m} \frac{\beta(t_0) R^2(t)}{R^2(t_0)})
\end{aligned} \tag{4.74}
$$

[4]

Now if we consider entropy for bosons and fermions

$$S^{\mu} = -\int \left[f \ln f \mp (1 \mp f) \ln(1 \pm f)\right] \frac{p^{\mu}}{p^{0}} d^3p \tag{4.75}$$

if we take covariant divergence of it

$$\begin{aligned}
&\nabla_{\mu} S^{\mu} = -\frac{1}{R^3}\frac{\partial}{\partial t}\left[R^3 S^{\mu}\right] = -\frac{1}{R^3}\frac{\partial}{\partial t}\left[R^3 S^{0}\right] \\
&= -\frac{3\dot{R}}{R}\int f \ln f \mp (1 \pm f)\ln(1 \pm f) d^3p - \frac{\partial}{\partial t}\int f \ln f \mp (1 \pm f) ln(1 \pm f) d^3p \\
&= -\frac{3\dot{R}}{R}\int f \ln f \mp (1 \pm f)\ln(1 \pm f) d^3p - \int (\dot{f} \ln f + \dot{f}) d^3p \mp \int \dot{f}\ln(1 \pm f) + \dot{f} d^3p \\
&\text{Substituting} \quad \frac{\partial f}{\partial t} = C(E) + \frac{\dot{R}}{R} p_i \frac{\partial f}{\partial p_i} \\
&= \int C(E) \ln\left[\frac{1 \pm f}{f}\right] d^3p
\end{aligned} \tag{4.76}$$

So it means

$$\nabla_{\mu} S^{\mu} = \int C(E) \ln\left[\frac{1 \pm f}{f}\right] d^3p = \int C(E) \ln\left[\frac{1}{f} \pm 1\right] d^3p \tag{4.77}$$

Substituding C(E) and doing the same calculations for finding the equilibrium solutions we found that

$$\ln\left[\frac{1}{f(p)} \pm 1\right] + \ln\left[\frac{1}{f(p')} \pm 1\right] = \ln\left[\frac{1}{f(p_1)} \pm 1\right] + \ln\left[\frac{1}{f(p_2)} \pm 1\right] \tag{4.78}$$

Again putting $\ln\left[\frac{1}{f(p)} \pm 1\right] = \chi(p)$ and using theorem (4.1) we found that

$$f_{eq} = \frac{1}{\exp(\alpha(t) - \beta(t)E) \mp 1} \tag{4.79}$$

- : bosons

+ : fermions

Back to Boltzmann Equation $L(f_{eq}) = 0$

In this case substituting equilibrium function we found that

$$L(f_{eq}) = \frac{\partial f_{eq}}{\partial t} - \frac{\dot{R}}{R} p \frac{\partial f_{eq}}{\partial p} = 0 \tag{4.80}$$

So in the limits we found the solutions

For m $\rightarrow 0$:

$$f_{eq} = \frac{1}{\exp(-ApR(t)) \pm 1} \tag{4.81}$$

For m$\rightarrow \infty$:

$$f_{eq} = \frac{1}{\exp(-\frac{p^2}{2m}\frac{\beta(t_0)R^2(t)}{R^2(t_0)}) \pm 1} \tag{4.82}$$

4.4.1. Bose-Einstein Condensation

For the Bose case we found that equilibrium function has the following form ;

$$f_{eq} = \frac{1}{\exp[\alpha(t) - \beta(t)E] - 1} \tag{4.83}$$

and for the limit when m goes to infinity we have two conditions :

$\alpha = m\beta + \Omega$ with Ω is constant

$\beta(t) = \frac{\beta(t_0)}{R^2(t_0)} R^2(t)$

which means

$$f_{eq} = \frac{1}{\exp[\alpha(t) - \beta(t)E] - 1} = \frac{1}{\exp[m\beta + \Omega - \beta(t)E] - 1} = \frac{1}{\exp[\beta(m - E) + \Omega] - 1} \tag{4.84}$$

We define occupation number as

$$n_\sigma = \frac{1}{\exp[\alpha(t) - \beta(t)\epsilon_\sigma] - 1} \tag{4.85}$$

and the total number of particles is defined as

$$N = \sum n_\sigma = \sum \frac{1}{\exp[\alpha(t) - \beta(t)\epsilon_\sigma] - 1} \tag{4.86}$$

If we calculate in this limit case(when mass goes to infinity)

$$\begin{aligned} N &= \sum n_\sigma = \sum \frac{1}{\exp[\alpha(t) - \beta(t)\epsilon_\sigma] - 1} = \sum \frac{\exp[-\alpha(t) + \beta(t)\epsilon_\sigma]}{1 - \exp[-\alpha(t) + \beta(t)\epsilon_\sigma]} \\ &= \sum_k \exp[-\alpha(t)k] \sum_\sigma \exp[\beta(t)\epsilon_\sigma k] = \sum_k \exp k[-\Omega - \beta(t)m] \sum_\sigma \exp[k\beta(t)\epsilon_\sigma] \\ &= \sum_k \exp k[-\Omega - \beta(t)m] Tr[\exp(k\beta(t)H)] \\ &= \sum_k \exp k[-\Omega - \beta(t)m] \frac{R^3(t)V(2m)^{3/2}}{(4\pi)^{3/2}(\beta k)^{3/2}} \\ &= \frac{R^3(t)V(2m)^{3/2}}{(4\pi)^{3/2}\beta^{3/2}} Li_{3/2}(z) \\ n &= \frac{N}{VR^3(t)} = \frac{(2m)^{3/2}}{(4\pi)^{3/2}\beta^{3/2}} Li_{3/2}(z) \end{aligned} \tag{4.87}$$

with $z = \exp[-(\Omega + \beta m)]$ [see Appendix B]

So N is equal to

$$N = \frac{R^3(t)V(2m)^{3/2} Li_{3/2}(z)}{(4\pi)^{3/2}\beta^{3/2}} \tag{4.88}$$

We define N_0 as the number of particles when they have the energy state $\epsilon = 0$ which in this case is equal to

$$N_0 = \frac{1}{\exp[\Omega + \beta(t)m] - 1} \tag{4.89}$$

and

$$\frac{N_0}{V} = \frac{1}{V}\frac{1}{\exp[\Omega + \beta(t)m] - 1} \tag{4.90}$$

If we look at the last equation when V goes to zero the quantity $\frac{N_0}{V}$ will not go to zero so this means $\frac{1}{\exp[\Omega+\beta(t)m]-1}$ should go to infinity and this is possible if and only if $\exp[\Omega + \beta(t)m]$ goes to one. Means that $\Omega = -\beta(t)m$.
This means $z = exp - [\Omega + \beta m]k = 1$.
So we can write N when $T \leqslant T_c$ as

$$N = \frac{R^3(t)V(2m)^{3/2}Li_{3/2}(1)}{(4\pi)^{3/2}\beta^{3/2}} + N_0 \tag{4.91}$$

and now if we look N at $T \geqslant T_c$

$$\frac{N}{R^3(t)V} = n = \frac{(2m)^{3/2}Li_{3/2}(1)}{(4\pi)^{3/2}\beta^{3/2}} \tag{4.92}$$

From the last equation we can find the critical temperature T_c.

$$T_c = \frac{n^{3/2}2\pi}{mk_B\zeta^{2/3}(3/2)} \tag{4.93}$$

5. CONCLUSION

As a first step the Boltzmann equation is constructed in FRW spacetime. In particular we focused on Boltzmann equation in flat 3-space(when k=0).We searched for the equilibrium solutions in this spacetime. It is proven that in the mathematical sense it is not possible to find equilibrium solutions because FRW cosmological model doesn't have timelike killing vector. However one can be looked for the equilibrium solutions in the limits when mass goes to zero (ultrarelativistic particle) and when mass goes to infinity (nonrelativistic particle).
After we derived the equilibrium solutions in two limits in FRW spacetime, we applied our result to the non-relativistic Bose-Einstein Condensation. Finally, as a result of this application we calculated non-relativistic Bose-Einstein condensation and the critical temperature in the limit when mass goes to infinity in FRW space-time.

Further to this thesis study one can also search for the approximate equilibrium solution which means that may search for the solution which has the form $f = f_{eq} + \delta f$.

Additionally the Boltzmann equation and its consequences can be also studied in the remaining 3-geometries (*i.e.* spherical and hyperbolic 3-space geometries) which will be our future project.

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APPENDIX A: S-MATRIX

Unitary operator S is defined as

$$S = 1 + iT \quad \text{and} \quad S^{+} = 1 - iT^{+} \tag{A.1}$$

4-momentum conservation must be involved into S-matrix. So S should contain a factor like $\delta^{(4)}(p_{1f} + p_{2f} - p_{1i} - p_{2i})$. It is defined [26]

$$< p_{1f}p_{2f}|iT|p_{1i}p_{2i} >= (2\pi)^4\delta^{(4)}(p_{1f} + p_{2f} - p_{1i} - p_{2i})iM(p_{1f}, p_{2f}; p_{1i}, p_{2i}) \tag{A.2}$$

where M is invariant matrix element and it is analogous to scattering amplitude of particle. From the unitary of S-Matrix ;

$$\begin{aligned} SS^{+} &= 1 + iT - iT^{+} + TT^{+} = 1 \\ S^{+}S &= 1 + T - iT^{+} + T^{+}T = 1 \end{aligned} \tag{A.3}$$

From (B.3) we have

$$TT^{+} = T^{+}T = -i(T - T^{+}) \tag{A.4}$$

First we calculate

$$< p_{1f}p_{2f}|TT^{+}|p_{1i}p_{2i} > \tag{A.5}$$

$$
\begin{aligned}
&< p_{1f}p_{2f}|TT^{+}|p_{1i}p_{2i} >= \int < p_{1f}p_{2f}|T|p_1p_2 >< p_1p_2|T^{+}|p_{1i}p_{2i} > [dp_1][dp_2] \\
&= \int < p_{1f}p_{2f}|T|p_1p_2 >< p_{1i}p_{2i}|T|p_1p_2 >^{*} [dp_1][dp_2] \\
&= \int \delta^{(4)}(p_{1f} + p_{2f} - p_1 - p_2)M(p_{1f}, p_{2f}; p_1, p_2)\times \\
&\delta^{(4)}(p_1 + p_2 - p_{1i} - p_{2i})M^{*}(p_1, p_2; p_{1i}, p_{2i})[dp_1][dp_2]
\end{aligned} \tag{A.6}
$$

Integrating over $[dp_1]$ and $[dp_2]$

$$
\begin{aligned}
&= \int \delta^{(4)}(p_{1f} + p_{2f} - p_1 - p_2)M(p_{1f}, p_{2f}; p_1, p_2)M^{*}(p_{1f}, p_{2f}; p_1, p_2)[dp_1][dp_2] \\
&= \int \delta^{(4)}(p_{1f} + p_{2f} - p_1 - p_2)|M(p_{1f}, p_{2f}; p_1, p_2)|^2[dp_1][dp_2]
\end{aligned} \tag{A.7}
$$

As a conlusion we have

$$
< p_{1f}p_{2f}|TT^{+}|p_{1i}p_{2i} >= \int \delta^{(4)}(p_{1f} + p_{2f} - p_1 - p_2)|M(p_{1f}, p_{2f}; p_1, p_2)|^2[dp_1][dp_2] \tag{A.8}
$$

Doing the same calculation for $< p_{1f}p_{2f}|T^{+}T|p_{1i}p_{2i} >$ we found that

$$
< p_{1f}p_{2f}|T^{+}T|p_{1i}p_{2i} >= \int \delta^{(4)}(p_{1f} + p_{2f} - p_1 - p_2)M|(p_1, p_2; p_{1f}, p_{2f})|^2[dp_1][dp_2] \tag{A.9}
$$

We define

$$
W(p_{1i}, p_{2i}; p_{1f}, p_{2f}) = M|(p_{1i}, p_{2i}; p_{1f}, p_{2f})|^2 \tag{A.10}
$$

From the unitarity of S-matrix we can see that

$$
< p_{1f}p_{2f}|TT^{+}|p_{1i}p_{2i} >=< p_{1f}p_{2f}|T^{+}T|p_{1i}p_{2i} > \tag{A.11}
$$

$$
\begin{aligned}
&\int \delta^{(4)}(p_{1f} + p_{2f} - p_1 - p_2)W((p_{1f}, p_{2f}; p_1, p_2))[dp_1][dp_2] \\
&= \int \delta^{(4)}(p_{1f} + p_{2f} - p_1 - p_2)W((p_1, p_2; p_{1f}, p_{2f}))[dp_1][dp_2]
\end{aligned} \tag{A.12}
$$

APPENDIX B: HEAT KERNEL

Define Heat Operator $\mathcal{L}$ as

$$\mathcal{L} = \triangle - \frac{\partial}{\partial t} \tag{B.1}$$

and heat equation

$$\mathcal{L}u = \triangle u - \frac{\partial u}{\partial t} = 0 \tag{B.2}$$

Fundamental solution to heat equation (or heat kernel) is for $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, \infty)$ (in the flat spacetime (k=0))

$$u(x,y,t) = \left(\frac{1}{4\pi t}\right)^{n/2} exp\left(\frac{-|x-y|^2}{4t}\right) \tag{B.3}$$

$$u(x,x,t) = \left(\frac{1}{4\pi t}\right)^{3/2} \tag{B.4}$$

The heat kernel can be also written as [25]

$$u(x,y,t) = \sum_{j=1}^{\infty} exp[-\lambda_j t]\psi_j(x)\psi_j(y) \tag{B.5}$$

So if we calculate

$$\int u(x,x,t)dV = \int \sum_{j=1}^{\infty} exp[-\lambda_j t]\psi_j(x)\psi_j(y) = \sum_{j=1}^{\infty} exp[-\lambda_j t] \int \psi_j^2(x)dV$$
$$= \sum_{j=1}^{\infty} exp[-\lambda_j t] = Trexp[-Ht] \tag{B.6}$$

$with$

$H\psi_j = \lambda_j \psi_j$

In RW metric spacelike flat slice we have (using A.4 and A.6)

$$Trexp[-Ht] = \int u(x,x,t)dV = \int \left(\frac{1}{4\pi t}\right)^{3/2} dV = \frac{V}{(4\pi t)^{3/2}} \tag{B.7}$$

For the calculation of the total particle number N we used heat kernel definition and the value of Tr exp[-Ht] with $H = -\frac{R^3(t)\nabla^2}{2m}$