MODIFIED THEORIES OF GRAVITY VIA ALTERNATIVE COUPLINGS AND THEIR COSMOLOGICAL ANALYSES

by
Mehmet KAVUK
B.S., Physics, Middle East Technical University, 2002
M.S., Physics, Middle East Technical University, 2005

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ABSTRACT

MODIFIED THEORIES OF GRAVITY VIA ALTERNATIVE COUPLINGS AND THEIR COSMOLOGICAL ANALYSES

We propose alternative gravitational couplings, fields and design some models to explain some cosmological problems. We investigate a cosmological model in which the Stueckelberg fields describing a massive photon are non-minimally coupled to the scalar curvature in a gauge invariant manner. We present not only a solution that can be considered in the context of the late time acceleration of the universe but also a solution compatible with the inflationary cosmology. The mass mechanism that gains to the vector field in a gauge invariant manner, designed by Stueckelberg, and via coupling to gravitation is used as a first in cosmology and we investigate cosmological effects. While Stueckelberg mechanism is generally used in particle physics , distinct behaviors of the scalar and vector fields together with the real valued mass gained by the Stueckelberg mechanism lead the universe to go through the two different accelerated expansion phases with a decelerated expansion phase between them. On the other hand, in the solutions we present, if the photon mass is null then the universe is either static or exhibits a simple power law expansion due to the vector field potential. In other work in this thesis, we propose a new model of gravity where the Ricci scalar (R) in Einstein-Hilbert action is replaced by an arbitrary function of R and of the norm of energy-momentum tensor i.e., $f(R, T^{\mu\nu}T_{\mu\nu})$ in metric formalism. We find that the equation of motion of massive test particles is non-geodesic and these test particles are acted upon by a extra force which is orthogonal to the four-velocity of the particles. We also find the Newtonian limit of the model to calculate the extra acceleration which can affect the perihelion of Mercury. There is a deviation from the general relativistic(GR) result unless the energy density of the fluid is constant. Arranging α parameter gives an opportunity to cure the inconsistency between the observational values for the abundance of light elements and the standard Big Bang Nucleosynthesis results. Even the dust dominated universe undergoes an accelerated expansion without using a cosmological constant in Model II. With this specific choice of $f(R, T^{\mu\nu}T_{\mu\nu})$ we get the so-called Cardassian-like expansion in which standard Friedmann equation is modified as $H^2 = A\rho + B\rho^n$ in an ad hoc way in the literature.

ÖZET

GRAVİTASYONUN ALTERNATİF KUPLAJLAR KULLANARAK MODİFİKASYONU VE BU MODELLERİN KOZMOLOJİK ANALİZLERİ

Gravitasyona alternatif kuplajlar ve farklı alanlar konularak halihazırdaki kozmolojik sorunları çözmek adına birtakım kozmolojik modeller tasarlamaktayız. Stueckelberg tarafından bulunan ve foton alanına ayar değişmezliği altında kütle kazandıran mekanizmayı ilk kez kozmolojik olarak kullanıp gravitasyonla etkileşmesine izin vererek elde edilen modelin kozmolojik etkilerini inceledik. Bu mekanizma genellikle parçacık fiziğinde kullanılırken, reel kütleye sahip skaler ve vektör alanının farklı davranışları birbirinden farklı iki hızlanarak genişleme periyodunun arasında bir yavaşlama dönemi de içermektedir. Öte yandan çözümlerde kütlenin sıfır olduğu durumlarda evren statik hale gelmekte, vektör potansiyeline bağlı olarak basit üstel genişleme vermektedir. Bu tezdeki diger bir çalışmada ise, Einstein-Hilbert aksiyonundaki Ricci skalerinin yerine, Ricci skalerinin ve enerji momentum tensörünün normunun bir fonksiyonunu koyarak $f(R, T^{\mu\nu}T_{\mu\nu})$ yeni bir model geliştirdik. Kütleli test parçacıklarının jeodezikler üzerinde gitmediğini ve bu parçacıkların dörtlü hızlarına dik ekstra bir kuvvete maruz kaldıklarını gördük. Merkür'ün perihelion hareketini etkileyebilecek bu ek kuvveti hesaplamak için sistemin Newton limitini hesapladık. Sistemdeki akışkanın enerji yoğunlugu sabit olmadığı sürece genel grelilik'ten hesaplanan değerden sapma oldugunu tespit ettik. Model parametresi olan α'yı Büyük Patlama nükleosentezi ve hafif elementlerin oluşumundaki gözlemsel verileri kullanarak belirleme imkanımızda bulunmakta idi. Model II'de madde baskın evrende kozmolojik sabit olmasa da hızlanarak genişleme vermektedir. α'nın belli bir değeri için literatürde keyfi olarak $H^2 = A\rho + B\rho^n$ şeklinde modifiye edilen Friedmann denklemi ile önerilen Cardassian tipi genişlemeyi elde ettik.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

A_μ	The vector field
$T_{\mu\nu}$	Energy momentum tensor
$F_{\mu\nu}$	Field strength tensor of the electromagnetic field
$g_{\mu\nu}$	The second rank metric tensor
R	The Ricci curvature scalar for the four-dimensional metric
ϕ	Scalar Field

LIST OF ACRONYMS/ABBREVIATIONS

BAO	Baryon acoustic oscillations
BBN	Big Bang Nucleosynthesis
CMBR	Cosmic microwave background radiation
COBE	The cosmic background explorer satellite
DM	Dark matter
dS	DeSitter
EFE	The Einstein Field Equations
EoS	Equation of state
FRW	Friedmann Robertson Walker
G	The Newtonian gravitational coupling constant
GR	General Relativity
GeV	Gigaelectronvolt
JBD	Jordan-Brans-Dicke
ΛCDM	Lambda Cold Dark Matter

1. INTRODUCTION

1.1. Standard Cosmology

From the observational data of Supernovae Type Ia (SN Ia) in 1998, Riess et. al. [1] in the High-redshift Supernovae Search Team and Perlmutter et. al. [2] in the Supernovae Cosmology Project Team reported that the present universe is accelerating. From then on during the past fifteen years physical cosmology has progressed a lot and led to a standard cosmological model called ΛCDM which is in agreement with all available data. Nevertheless, it requires two unknowns; namely dark matter and dark energy. Dark matter is accounted for the most of the matter in the universe whereas dark energy is thought to be responsible for the acceleration of the universe. Despite the success of ΛCDM cosmology we will see that it requires a mechanism called inflation to solve the horizon problem and flatness problem.

1.1.1. Reference Model: ΛCDM

ΛCDM model assumes that the universe is spatially isotropic and homogeneous which lead to a Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime and is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\sigma^2 \tag{1.1}$$

where $g_{\mu\nu}$ is a metric tensor, $a(t)$ is a scale factor with cosmic time t, and $d\sigma^2$ is the time-independent metric of the 3-dimensional space with constant curvature k:

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.2}$$

Here $k = +1, -1, 0$ correspond to closed, open, and flat geometries, respectively. Metric, curvature and the Einstein tensor calculations of FLRW metric are given in Appendix A.2. Moreover, ΛCDM model uses general relativity as the theory of gravity whose action is given as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\mathbf{\Lambda}) + \sum_{standard\ model+\mathbf{CDM}} S_{matter}[\phi_i, g_{\mu\nu}] \tag{1.3}$$

where $g = det(g_{\mu\nu})$, G is the Newtonian gravitational constant, Λ is the cosmological constant, R is Ricci curvature scalar and ϕ_i's are all known matter fields. Note that two unknown components are written in bold face[3]. It should be noted here that all matter fields are minimally coupled to the metric tensor $g_{\mu\nu}$ which guarantees the universality of free fall which has been tested at the 10^{-13} level[4].

By varying the action in Equation 1.3 with respect to the metric one obtains the Einstein's Field Equations as (with $c = 1$)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.4}$$

or

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.5}$$

where Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{1.6}$$

and

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} \tag{1.7}$$

If we take the covariant divergence of the Equation 1.5 we obtain

$$\nabla^\mu G_{\mu\nu} = 8\pi G \nabla^\mu T_{\mu\nu} \tag{1.8}$$

On purely geometrical grounds the left hand side of the above equation is zero[5].

Therefore we have the matter conservation equation

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{1.9}$$

In the FRLW spacetime the energy-momentum tensor of the background matter is described by the prefect fluid whose form is :

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu}, \tag{1.10}$$

where $u^{\mu} = (1, 0, 0, 0)$ is the four-velocity of the fluid in comoving coordinates, and ρ and p are functions of t. From Equation 1.9 ($\nu = 0$) we get

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{1.11}$$

which is the continuity equation.

It had been realized that the farther away the galaxies are, the faster they are flying outward. The wavelength and the scale factor relation is obtained as

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)}, \tag{1.12}$$

where t_o is the observed instant time, t_e is the emission time of light from the object and z is the redshift.

There is a linear relationship between the distance to galaxies and their recessional velocities and this was discovered by Hubble in 1929 [6, 7]. Hubble constant is achieved by slope of the velocity versus distance graph. The ratio of the rate of change of the scale factor to the value of the scale factor is called as Hubble parameter

$$H = \frac{\dot{a}}{a} \tag{1.13}$$

where $a(t)$ is the scale factor of the universe.

Using the expressions for the Einstein tensor and the form in Equation 1.10 of the energy-momentum tensor, we can easily obtain the Friedmann equations as

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{1.14}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{1.15}$$

Note that without the cosmological constant the condition

$$\rho + 3p < 0 \tag{1.16}$$

also gives an accelerated expansion, $\ddot{a} > 0$.

From Equations 1.14 and 1.15 we can easily obtain the conservation equation given in Equation 1.11

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{1.17}$$

Hence we have two independent equations for three unknowns; scale factor, the energy density and the pressure. Therefore we need more informations to solve the system. We choose an equation of state for matter in the form

$$p = w\rho. \tag{1.18}$$

For pressureless matter $w = 0$ whereas for radiation $w = \frac{1}{3}$. For the cosmological constant, Λ, which corresponds to a constant energy density, Equation 1.11 implies $p = -\rho$ and thus $w = -1$. Solution of Equation 1.17 implies that for any constant w,

$$\rho \sim a^{-3(1+w)} \tag{1.19}$$

It is worth to note that when the matter density is dominated by the cosmological constant space has an accelerated expansion and it is called the de-Sitter space. That is for ($p = -\rho$) and k=0 we get

$$a \sim e^{Ht} \tag{1.20}$$

Because of the solution in Equation 1.20, the standard paradigm to explain the acceleration of the cosmic expansion is to postulate the existence of a diffuse form of dark energy described by an exotic equation of state ($w = -1$) and amounting to roughly 70 percent of the critical energy density. The cosmological constant is the most natural candidate for this dark fluid, although its tiny value (as inferred by cosmological observations) clashes with the value of vacuum energy as inferred from particle physics.

Dark energy differs from the other components of the universe such as baryonic matter and radiation, in the sense that it has a negative pressure. It is this negative pressure which creates a gravitational repulsion to suppress the gravitational attraction and hence resulting in accelerated expansion. Despite the success of the cosmological constant as dark energy, it has a flaw which shows itself in explaining its value [9, 10]. According to the observations, the energy density of cosmological constant must be of the order of $\rho_\Lambda \simeq 10^{-47}$ GeV^4. However from the perspective of a particle physicist, the origin of the cosmological constant must be found in the vacuum energy density whose value is estimated to be $\rho_{\text{vac}} \simeq 10^{74}$ GeV^4. This huge discrepancy between the two values must be explained. Although the cosmological constant is also thought as a non-zero vacuum expectation value of a scalar field, there is a nearly 120 orders of magnitude inconsistency between the cosmological and the PP predicted constant [8, 9, 11, 12].

1.1.2. Inflation

ΛCDM model has some flaws in it such as horizon and flatness problems and it needs a mechanism called inflation to cure them. Inflation is called for the accelerated expansion era in the very early universe ($\sim 10^{-35}$ seconds) at energy scales $\sim 10^{16}$ GeV. Inflationary mechanism both resolves the problems of standard Big Bang scenario mentioned as horizon and flatness problems, and generates large scale fluctuations in the cosmic microwave background (CMB) (see [18] for a recent review). In the following subsections I will mostly follow the arguments given in [13].

1.1.2.1. Flatness Problem. Let us recall Equation 1.14 and ignore the cosmological constant as in the standard big-bang theory. Then we have,

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}. \tag{1.21}$$

If we rewrite this as

$$\Omega - 1 = \frac{k}{a^2 H^2}, \tag{1.22}$$

where $\Omega \equiv \frac{\rho}{\rho_c}$,with $\rho_c \equiv \frac{3H^2}{8\pi G}$. Here the density parameter Ω is the ratio of the energy density to the critical density. In the standard big-bang theory with $\ddot{a} < 0$, the a^2H^2term in Equation 1.22 decreases meaning that it tends to deviate from unity as time goes on. However from the observations we know that Ω is very close to one. In order to be compatible with the observations Ω must be so small in the past that it is now very close to one. For instance, it needs to be $|\Omega - 1| < \mathcal{O}(10^{-16})$ at the epoch of nucleosynthesis and $|\Omega - 1| < \mathcal{O}(10^{-64})$ at the Planck epoch which indicates that it requires extreme fine tuning. Otherwise the structure formation cannot be realized.

1.1.2.2. Horizon Problem. Another problem that the ΛCDM model must handle is the horizon problem. Consider a comoving wavelength, λ. Its value with respect to the coordinate system in use does not change. If we multiply it with the scale factor we get physical wavelength, $a\lambda$. Standard big-bang cosmology gives the behaviour of the scale factor as $a \sim t^p$ where $0 < p < 1$. Then the physical wavelength changes as $a\lambda \sim t^p$, but the Hubble radius changes as $H^{-1} \sim t$. Hence the physical wavelength gradually becomes smaller than the Hubble radius as time passes. This result says that the dimension of causally connected regions is very small[13]. However CMB observations tell us that the photons we see when we look at the sky have the same temperature which means that the photons in the past somehow were in touch to thermalize in all regions.

The inflationary mechanism is the way to solve these problems. Inflation is driven by a scalar field(s) that is/(are) called as inflaton, introduced in an ad hoc way. The basic inflationary model includes an homogeneous scalar field ϕ with a potential energy $V(\phi)$ whose Lagrangian can be written as $L = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)$. Using Equation 1.7

we find

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{1.23}$$

Substituting Equation 1.23 for Equations 1.14 and 1.17 while ignoring the cosmological constant we obtain

$$H^2 = \frac{8\pi G}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right). \tag{1.24}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi) \tag{1.25}$$

where we ignored the curvature term. During the inflation, the condition in Equation 1.16 yields $\dot{\phi}^2 < V(\phi)$, which implies that the potential energy of the inflaton dominates over the kinetic energy of it.

If the scalar field ϕ initially was large, the Hubble parameter H was large too, according to the second equation. This means that the friction term $3H\dot{\phi}$ was very large, and therefore the scalar field was moving very slowly, as a ball in a viscous liquid. Therefore at this stage the energy density of the scalar field, unlike the density of ordinary matter, remained almost constant, and expansion of the universe continued with a much greater speed than in the old cosmological theory.

Slow Roll inflation conditions may simplify the system of equations, Slow Roll means that the ϕ field was large initially and Equation 1.24 predicts a large Hubble constant as a result, so the friction term dominated over the others. Due to the rapid growth of the scale of the universe and a slow motion of the field ϕ, soon after the beginning of this regime one has $\ddot{\phi} \ll 3H\dot{\phi}$ and $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ so Equations 1.24 and 1.25 are given as

$$H^2 \simeq \frac{8\pi G}{3}V(\phi), \qquad 3H\dot{\phi} \simeq -V'(\phi). \tag{1.26}$$

Defining the slow-roll parameters

$$\epsilon \equiv \frac{1}{16\pi G}\Big(\frac{V'}{V}\Big)^2, \quad \eta \equiv \frac{1}{8\pi G}\frac{V''}{V}. \tag{1.27}$$

Slow-roll approximations are valid when $\epsilon \ll 1, \quad |\eta| \ll 1$. Amount of inflation is the number of e-foldings, defined by

$$N \equiv \ln(\frac{a_f}{a_i}) = \int_{t_i}^{t_f} H dt. \tag{1.28}$$

In order to solve the flatness problem, Ω is required to be $|\Omega_f - 1| \leq 10^{-60}$ after the end of inflation. The ratio $|\Omega - 1|$ between the initial and the final phase of the inflation is given by

$$\frac{|\Omega_f - 1|}{|\Omega_i - 1|} \simeq \Big(\frac{a_i}{a_f}\Big)^2 = e^{-2N}. \tag{1.29}$$

Number of e-foldings is required to be $N \geq 70$ to solve the flatness problem if $|\Omega_i - 1|$ is of order unity. We have the same e-foldings to solve the horizon problem.

One may see [19] for a comprehensive list of scalar fields considered in the context of inflation.

With independent studies [1, 21–23] it is established that the current universe is evolving with an accelerated expansion that started approximately 6 Gyr ago. There are excellent reviews regarding the accelerated expansion[10, 24, 25]. The timeline of universe is given in Figure 1.1. The latest data from the Planck CMB experiment, whose major goal is to test this model to high precision and identify areas of tension, shows a remarkable consistency with the predictions of the base ΛCDM model. However, it reveals also a number of intriguing features of the data that might be ascribed to the cosmological constant assumption of the model; for instance, it is found that the data alone is compatible with Λ assumption, but a dark energy component yielding a

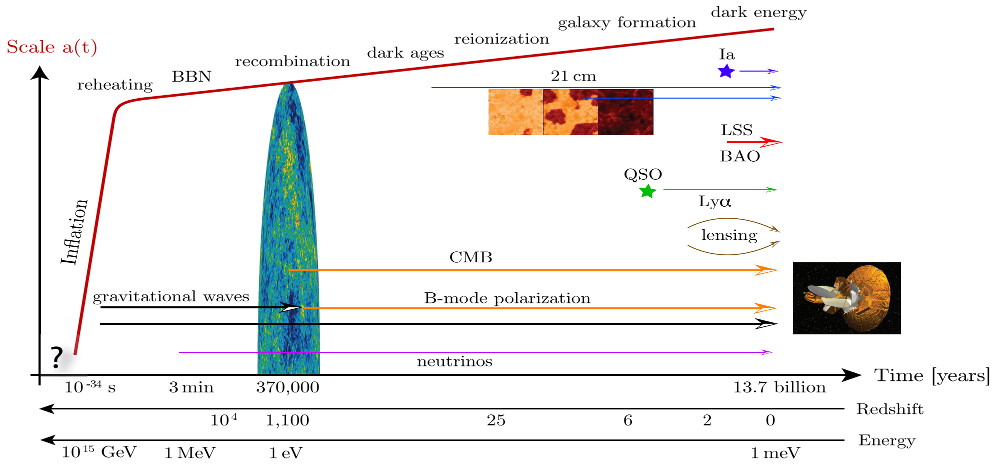

Figure 1.1. The time line of the universe [27].

time varying equation of state (EoS) parameter is favored when the astrophysical data is also taken into account [23]. This is in line with the idea of describing dark energy as a scalar field that was first considered to alleviate the theoretical problems related with Λ. However, the scalar field models of dark energy are also mostly ad hoc and/or considered phenomenologically rather than being derived from a fundamental theory (see [25, 28] for comprehensive reviews on dark energy). Scalar fields are in fact ubiquitous in theories beyond the standard model such as string theory and super-symmetry. After the discovery of Higgs boson [29, 30] with a mass 125 GeV, the existence of the Higgs scalar field and so the possible existence of scalar fields with a mass consistent with the cosmological scalar fields (the inflaton and dark energy fields) become more interesting.

If the origin of dark energy is not the cosmological constant and if we cannot explain the inflation by some fields within the context of GR then one must seek some alternative models to explain these phenomena. There are two approaches to construct models of dark energy or inflaton. Either we can modify the matter sector,right hand side of Equation 1.5, by adding an exotic field brings an effective negative pressure, for instance quintessence or we can alter the left hand side of Equations 1.5, that is, modify gravity. Scalar-tensor theories belong to this class and they are the most established and well studied modified theories and Brans-Dicke (BD) theory of gravity [31] is the prototype of these theories. Moreover, f(R) theories of gravity where f is a function of Ricci scalar are also representative models of this class[32–34]. However the two approaches are not fundamentally different in the framework of classical General Relativity. One can always rephrase one into the other by defining a suitable conserved energy-momentum tensor that equals the Einstein tensor[35].

Units and conventions

Throughout the thesis we use units such that $c = \hbar = 1$, where c is the speed of light, $\hbar$ is reduced Planck's constant. We reinsert these symbols when they are needed. In the second chapter of the thesis we adopt the metric signature $(+,-,-,-)$ whereas in the third chapter the signature of the metric is taken to be $(-,+,+,+)$. Greek indices refer to space-time components and we use Latin indices to refer to space components.

2. STUECKELBERG MECHANISM AND ITS APPLICATION TO COSMOLOGY

2.1. Stueckelberg Mechanism- Generating Mass in a gauge invariant way

Apart from the discussion about the presence of scalar fields in nature, we know that the only long range interaction which could be relevant on cosmological scales apart from gravity is the electromagnetic field, which is a vector field. A vector field based inflationary cosmological model was also suggested in 1989 [36] but it started to receive keen attention only a decade ago. In recent years, on the other hand, vector fields have been discussed and considered with an increasing interest not only as an alternative to the scalar field models of inflaton but also that of dark energy[37–49]. The primary reason behind this increased interest is the efforts to explain some of the anomalies found in the large-scale CMB temperature in the WMAP data [50]. These anomalies have also been confirmed by the recent high precision Planck data [23, 51, 52]. However, vector field models that give an accelerated expansion usually suffer from ghost instabilities [53–55] due to imaginary (tachyonic) mass. In particular, such inflationary models require huge mass for the vector field, and hence a huge amount of tachyonic mass which makes the issue even worse.

Motivated by the above discussion, in this study we investigate a cosmological model where the Stueckelberg fields couple directly to the scalar curvature in a particular way. The reason being that Stueckelberg action [56, 57] involves both scalar and vector fields, and also such actions arise naturally in compactifications of higher-dimensional string theory [58, 59]. Vector field actions with a mass term usually spoil the gauge invariance as in the Proca action that gives Maxwell's equations when the mass is set to zero. Stueckelberg [56, 57], on the other hand, described a massive photon by maintaining gauge invariance by introducing a scalar field B that mixes with the electromagnetic field A_μ under gauge transformations. The scalar field arises from the extra degrees of freedom and corresponds to the longitudinal mode of the photon polarization [58, 59]. Extending this idea (i.e. stueckelberging the electromagnetic $U(1)$ and thus giving a mass to the physical photon) to cosmological scales and investigating cosmological solutions by constraining the mass term to positive real values is quite appealing. For instance, in a recent study [60] it is shown that the Stueckelberg fields can play the role of dark energy since they can give an effective cosmological constant on large scales.

The force mediated by a massive particle is given by the Yukawa type behavior $\sim \frac{e^{-mr}}{r}$, where m denotes the mass. The laboratory bound on the photon mass is 10^{-14} eV, derived from the measurements of deviations from the Coulomb law (i.e. $m = 0$) [61] potential, and is far above the bounds obtained from the astronomical and cosmological tests. The bound on m is $\sim 10^{-15}$ eV from the measurements of Earth's magnetic field [62] and Pioneer-10 measurements of Jupiter's magnetic field [63], and is 10^{-27} eV from the galactic magnetic fields [64, 65] (see [66] for a review).

2.2. Accelerated expansion of the Universe a la the Stueckelberg mechanism

In this study, we investigate a cosmological model in which the Stueckelberg fields are non-minimally coupled to the scalar curvature in a gauge invariant way. We present not only a solution that can be considered in the context of the late time acceleration of the universe but also a solution compatible with the inflationary cosmology. Distinct behaviors of the scalar and vector fields together with the real valued mass gained by the Stueckelberg mechanism lead the universe to go through the two different accelerated expansion phases with a decelerated expansion phase between them. On the other hand, in the solutions we present, if the mass is null then the universe is either static or exhibits a simple power law expansion due to the vector field potential.

We are particularly interested in the background expansion history of the universe and hence for convenience we consider spatially maximally symmetric and flat Robertson-Walker space-time. In accordance with this, assuming the universe is electrically neutral we consider only the temporal electromagnetic field i.e. the electric potential of the vector field. Temporal electromagnetic field [67] and vector fields [41, 68, 69] are considered in the cosmological context. We follow the same approach to construct the gravitational action and treat the scalar field as the Jordan-Brans-Dicke (JBD) scalar [70, 71]. We propose new type of gauge invariant coupling to the scalar curvature applying a particle physics approach and investigate its cosmological solutions.

One may note that the higher the scale the tighter the bounds, which demonstrates also that even an extremely small value of the photon mass can have a considerable effect on the evolution of the universe. There are various applications of the Stueckelberg mechanism in the context of cosmology; for instance, it has been used as a natural source to account for some sort of dark matter related to the gauge-group parameter in [72] and as a mechanism for giving a mass to graviton in the context of massive gravity in [73, 74]. In this study, on the other hand, we show that a massive non-minimally coupled photon that gains its mass by the Stueckelberg mechanism in curved spacetime may give rise to interesting expansion histories for the universe, even a history that can be considered in the context of inflation (including a switch-off mechanism) in the early universe and to the current acceleration of the universe.

The action we propose is

$$\begin{aligned} S \ &= \ \int \mathrm{d}^4x\sqrt{-g}\Big[-\frac{1}{8\omega m^2}(mB+\nabla_\mu A^\mu)^2R-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}\left(\nabla_\mu B-mA_\mu\right)\left(\nabla^\mu B-mA^\mu\right) \\ &\quad -\frac{1}{2}\left(mB+\nabla_\mu A^\mu\right)^2\Big]+S_{\mathrm{M}}, \end{aligned} \tag{2.1}$$

where ω is a dimensionless coupling constant, R is the scalar curvature of the spacetime metric g, $F^{\mu\nu}$ is the electromagnetic field strength tensor, A_μ and B are vector and scalar fields, respectively. Here the constant m is the mass of the Stueckelberg fields and is defined as a real valued positive number, so that we also avoid an imaginary (tachyonic) mass for the vector field that leads to a ghost instability [53–55]. The action S_{M} stands for the matter source. We use natural units with $\hbar = c = 1$ and hence the reduced Planck mass is given by $M_{\mathrm{pl}} = 1/\sqrt{8\pi G}$, where G is the gravitational coupling. We note that the Stueckelberg action, given in Equation 2.1, preserves gauge invariance under

$$A_\mu \to A_\mu + \nabla_\mu \lambda \quad \text{and} \quad B \to B + m\lambda, \tag{2.2}$$

transformations provided that λ satisfies

$$(\Box + m^2)\lambda = 0. \tag{2.3}$$

Neglecting gravity and investigating in Minkowski spacetime, the action under consideration reduces to the free Stueckelberg action. For free Stueckelberg theory, i.e., for Stueckelberg photon interacting with fermions, the Stueckelberg scalar field B satisfies the free wave equation so that the gauge function λ which also satisfies the free wave equation can be used to choose a gauge where B is zero. This is the Proca limit of the Stueckelberg mechanism. However, for our action, in curved spacetime B field does not satisfy the free wave equation so it cannot be set to zero with a gauge transformation. Now denoting

$$f = mB + \nabla_\mu A^\mu, \tag{2.4}$$

simplifies the action in Equation 2.1 and varying this we have

$$\begin{aligned}\delta S = \int \mathrm{d}^4x \Big[\quad & \delta(\sqrt{-g}) \left(-\frac{Rf^2}{8\omega m^2} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} \left(\nabla_\mu B - mA_\mu\right) \left(\nabla^\mu B - mA^\mu\right) - \frac{f^2}{2} \right) \\ & + \sqrt{-g} \Bigg(-\frac{R}{4\omega m^2} f\delta f - \frac{f^2}{8\omega m^2} g^{\mu\nu}\delta R_{\mu\nu} - \frac{f^2}{8\omega m^2}\delta g^{\mu\nu} R_{\mu\nu} - \frac{1}{4}\delta\left(F_{\mu\nu}F^{\mu\nu}\right) \\ & + \frac{1}{2}\delta\left(\left(\nabla_\mu B - mA_\mu\right)\left(\nabla^\mu B - mA^\mu\right)\right) - f\delta f \Bigg) \Big] + \delta S_{\mathrm{M}},\end{aligned} \tag{2.5}$$

supplemented by

$$\delta f = \delta g^{\mu\nu}\nabla_\nu A_\mu + \nabla_\mu(\delta g^{\mu\nu})A_\nu - \frac{1}{2}(\nabla^\alpha \delta g^{\mu\nu})A_\alpha g_{\mu\nu}. \tag{2.6}$$

Details of the variation are given in Appendix B. The variations of Equation 2.1 with respect to the inverse metric give the Einstein field equations

$$\begin{aligned}& \frac{f^2}{4\omega m^2}G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f^2 + \frac{1}{4\omega m^2}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f^2 \\ & + \left(\frac{R}{4\omega m^2} + 1\right)(-\nabla_\mu f A_\nu - \nabla_\nu f A_\mu + \nabla^\alpha A_\alpha g_{\mu\nu} f + g_{\mu\nu}\nabla^\alpha f A_\alpha) \\ & - \frac{f}{4\omega m^2}(A_\nu\nabla_\mu R + A_\mu\nabla_\nu R) + \frac{1}{4\omega m^2} f g_{\mu\nu} A_\alpha \nabla^\alpha R + F^a_\mu F_{\nu a} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \\ & - (\partial_\mu B - mA_\mu)(\partial_\nu B - mA_\nu) + \frac{1}{2}g_{\mu\nu}(\partial_\alpha B - mA_\alpha)^2 = T_{\mu\nu},\end{aligned} \tag{2.7}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter source. The variation of Equation 2.1 with respect to the vector field A_μ yield the vector field equation

$$\frac{1}{4\omega m^2} f \nabla^\mu R + \frac{R}{4\omega m^2}\nabla^\mu f + \nabla_\alpha F^{\alpha\mu} + \nabla^\mu(\nabla_\alpha A^\alpha) + m^2 A^\mu = 0. \tag{2.8}$$

Finally we obtain the scalar field equation from the variation of Equation 2.1 with respect to the scalar field B,

$$\left(\Box+\frac{R}{4\omega}+m^2\right)B+\frac{R}{4\omega m}\nabla_\mu A^\mu=0. \tag{2.9}$$

We note here that gravitational gauge invariance under general coordinate transformations is also preserved. We consider the spatially flat Robertson-Walker (RW) metric with a maximally symmetric spatial section

$$\mathrm{d}s^2=\mathrm{d}t^2-a(t)^2[\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2], \tag{2.10}$$

where $a(t)$ is the scale factor and t is the cosmic time. The non-zero components of the Ricci tensor and the Ricci scalar are given in Appendix A.1. Consistently with the spatially isotropic and homogeneous RW metric, we represent the energy-momentum tensor of the matter source with

$$T^\mu_\nu=\mathrm{diag}\left[\rho,-p,-p,-p\right], \tag{2.11}$$

where ρ and p are the energy density and pressure respectively and are only cosmic time t dependent, then we consider spatially homogeneous scalar field

$$B=B(t) \tag{2.12}$$

and finally consider only the scalar potential of the vector field, i.e., spatial part of the vector field is null, as follows:

$$A_0=A(t)\quad\text{and}\quad A_\alpha=0. \tag{2.13}$$

We thus end up with a system of ordinary differential equations given below to be solved:

$$\frac{3f^2}{4\omega m^2}\left(\frac{\dot{a}^2}{a^2}\right)-\frac{f^2}{2}+\left(\frac{R}{4\omega m^2}+1\right)\left(-\dot{f}A+f\dot{A}+3fA\frac{\dot{a}}{a}\right)+\frac{3f\dot{f}}{2\omega m^2}\frac{\dot{a}}{a}-\frac{fA\dot{R}}{4\omega m^2}$$
$$-\frac{1}{2}(\dot{B}-mA)^2=\rho, \tag{2.14}$$

$$-\frac{f^2}{4\omega m^2}\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)+\frac{f^2}{2}-\frac{1}{4\omega m^2}\left(2\dot{f}^2+2f\ddot{f}+4f\dot{f}\frac{\dot{a}}{a}\right)-\left(\frac{R}{4\omega m^2}+1\right)(\dot{f}A+f\dot{A}$$
$$+3fA\frac{\dot{a}}{a})-\frac{fA\dot{R}}{4\omega m^2}-\frac{1}{2}(\dot{B}-mA)^2=p, \tag{2.15}$$

$$\ddot{A}+3\dot{A}\frac{\dot{a}}{a}+3A\left(\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)+m^2A+\frac{\dot{R}f+\dot{f}R}{4\omega m^2}=0, \tag{2.16}$$

$$\ddot{B}+3\dot{B}\frac{\dot{a}}{a}+m^2B+\frac{Rf}{4\omega m}=0, \tag{2.17}$$

where

$$f=mB+\dot{A}+3A\frac{\dot{a}}{a}. \tag{2.18}$$

We would like to note at this point that the massive Jordan-Brans-Dicke limit cannot be achieved from the action we consider relying on the Stueckelberg theory. At first sight it seems that at $A\to 0$ limit, in the action only the massive scalar field remains and the ω becomes the JBD coupling parameter. However, it is well known that substituting $A=0$ in the action is not the same with substituting $A=0$ in the equations of motion. Indeed, one may check that Equation 2.16 brings an additional constraint on the system as

$$\frac{\dot{R}}{R}+\frac{\dot{B}}{B}=0 \tag{2.19}$$

for $A\to 0$ case, hence the solutions that would be obtained with $A\to 0$ will be different than the massive JBD solutions.

This system is consist of four linearly independent ordinary differential Equations 2.14-2.17 that should be satisfied by five unknown functions ρ, p, A, B, a and therefore is not fully determined. The customary way of determining the system fully at this stage is to introduce an equation of state (EoS) that characterizes the internal properties of the matter source

$$p = w\rho, \tag{2.20}$$

where w is the EoS parameter of the matter source, which is not necessarily constant, but is a constant for the most commonly considered sources in cosmology; namely, takes values 0, $\frac{1}{3}$ and -1 for dust, radiation and cosmological constant respectively. However, the system is far too complicated to be solved analytically and its general solution cannot be obtained even under the assumption of a matter source with a constant EoS parameter. On the other hand, in what follows we shall give various solutions following a strategy moving on from the relation between f and the effective gravitational coupling G that gives us opportunity to investigate some properties of the model that might be of interest from the cosmological point of view.

2.2.1. The Cosmological Solutions

In comparison with Einstein-Hilbert action of general relativity, the term f in front of the scalar curvature R can be related to the gravitational coupling as follows:

$$\frac{f^2}{8\omega m^2} = \frac{1}{16\pi G}. \tag{2.21}$$

We note that, however, in our model f can be time dependent hence it can give rise to a time dependent effective gravitational coupling. Therefore the investigation of our model may be done by considering this property of our model and the constrains on the possible time variation of the effective gravitational coupling utilizing the following relation:

$$\frac{\dot{f}}{f} = -\frac{1}{2}\frac{\dot{G}}{G} \tag{2.22}$$

that follows Equation 2.21. The constraints on the the rate of change of the gravitational coupling $|\dot{G}/G|$ from various observations (big bang nucleosynthesis, pulsar timing and etc.) can be given as $10^{-10} - 10^{-12}\,\mathrm{yr}^{-1}$. For instance, in a recent study [75] it is given as $\approx -1.8 \times 10^{-10}\,\mathrm{yr}^{-1}$ from pulsating white dwarfs. One may see [76] for a comprehensive and recent review on the possible time variation of the effective gravitational coupling. We restrict our study in this paper with the cosmological solutions for which the function f and hence the effective gravitational coupling G are time independent, although it may be possible to obtain solutions with time varying f (hence G) consistent with these constraints. However, we do not ignore the possibility of varying effective gravitational coupling and give two sets of solutions: We shall first give solutions for which f is a non-zero constant in subsection 2.2.1.1. We then give solutions for which f is zero, which corresponds to infinitely large G, in subsection 2.2.1.2. We discuss that this extreme case may be considered in the context of very early universe by giving a solution that is compatible with inflationary cosmology.

2.2.1.1. Case I: $f = \text{constant} \neq 0$. In this case, we assume that the effective gravitational coupling G is a finite positive constant as in general relativity, and hence $w > 0$ from Equation 2.21 and f is a finite valued non-zero constant as

$$f = mB + \dot{A} + 3A\frac{\dot{a}}{a} = \text{constant} \neq 0. \tag{2.23}$$

According to this assumption B, A and a can still be dynamical but such that f will be yielding a constant value, and the system of Equations 2.14-2.17 to be solved reduces to

$$\frac{3f^2}{4\omega m^2}\left(\frac{\dot{a}^2}{a^2}\right) - \frac{f^2}{2} + \left(\frac{R}{4\omega m^2} + 1\right)\left(f\dot{A} + 3fA\frac{\dot{a}}{a}\right) - \frac{fA\dot{R}}{4\omega m^2} - \frac{1}{2}(\dot{B} - mA)^2 = \rho, \tag{2.24}$$

$$-\frac{f^2}{4\omega m^2}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + \frac{f^2}{2} - \left(\frac{R}{4\omega m^2} + 1\right)\left(f\dot{A} + 3fA\frac{\dot{a}}{a}\right) - \frac{fA\dot{R}}{4\omega m^2} - \frac{1}{2}(\dot{B} - mA)^2 = p, \tag{2.25}$$

$$\ddot{A} + 3\dot{A}\frac{\dot{a}}{a} + 3A\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + m^2A + \frac{\dot{R}f}{4\omega m^2} = 0, \tag{2.26}$$

$$\ddot{B} + 3\dot{B}\frac{\dot{a}}{a} + m^2 B + \frac{Rf}{4\omega m} = 0, \tag{2.27}$$

supplemented by Equation 2.23. We obtain two different solutions of the system that could be of interest in cosmology.

- Solution I In this solution the universe exhibits a de Sitter expansion; the scale factor a, Hubble parameter H and the deceleration parameter q of the universe are given as follows:

  $$a = a_1 e^{\sqrt{\frac{\omega}{3}} mt}, \quad H = \frac{\dot{a}}{a} = \sqrt{\frac{\omega}{3}} m \quad \text{and} \quad q = -\frac{\ddot{a}a}{\dot{a}^2} = -1, \tag{2.28}$$

  where a_1 is the integration constant. We find that the scalar field is a constant and the vector field is null

  $$B = \frac{f}{m} \quad \text{and} \quad A = 0. \tag{2.29}$$

  The energy density and pressure of the matter source are found to be constant as follows

  $$p = -\rho = \frac{f^2}{4}. \tag{2.30}$$

  The universe expands exponentially with a rate directly proportional to m, and is static for $m = 0$. This is a result in line with our expectation that there may be a connection between the accelerated expansion of the universe and the small but non-zero mass term of the Stueckelberg fields. The matter source predicted in this solution in Equation 2.30, on the other hand, yields an EoS in the form of a cosmological constant and a negative energy density with a particular value. A negative energy density is allowed only if it is in the form of vacuum energy. Even though negative energy density violates the dominant energy condition, null dominant energy condition allows negative vacuum energy as long as $p = -\rho$ [77]. Accordingly, adding a bare cosmological constant $\bar{\Lambda}$ to the action in Equation 2.1 as

  $$S \to S - \bar{\Lambda} \int \sqrt{-g}\, \mathrm{d}^4 x, \tag{2.31}$$

the energy density and pressure of the matter source given in Equations 2.14 and 2.15, and hence given in Equations 2.24 and 2.25, will be shifted as

$$\rho \to \rho + \bar{\Lambda} \quad \text{and} \quad p \to p - \bar{\Lambda}, \tag{2.32}$$

while the equations of the vector and scalar fields in Equations 2.16 and 2.17, and hence Equations 2.26 and 2.27, are unchanged. Therefore, the energy density and pressure of the matter source given in Equation 2.30 can now be elevated to zero,

$$p = 0 = \rho, \tag{2.33}$$

by choosing

$$\bar{\Lambda} = -\frac{f^2}{4}, \tag{2.34}$$

which is always negative since f is a non-zero real number. We note that $\bar{\Lambda}$ indeed corresponds to the energy density of the vacuum, i.e. $\bar{\Lambda} = \rho_{\rm vac}$ and in this sense it is not the cosmological constant defined by $\Lambda = 8\pi G\rho_{\rm vac}$. Negative vacuum energies, appear in string theory (and other models of quantum gravity), supersymmetry, super gravity and etc. and have been largely studied for addressing the cosmological constant problem [10, 78]. For instance, in exact supergravity the lowest energy state of the theory, generically has negative energy density [78] and string theory, the most prominent candidate for a consistent theory of quantum gravity, naturally predicts the existence of negative energy vacua [79]. This introduction of a negative vacuum energy with a particular energy density for elevating the energy density of the matter source to zero will particularly be very useful in the investigation of the following solution.

- Solution II In this solution the universe starts expanding with a decelerated expansion rate and then starts to accelerate at a certain time; setting $a = 0$ at $t = 0$, we obtain the scale factor, Hubble parameter and deceleration parameter as follows:

$$\begin{aligned} a &= a_1 \sinh^{1/2}\left(2\sqrt{\frac{w}{3}}mt\right), \\ H &= \sqrt{\frac{\omega}{3}}m\coth\left(2\sqrt{\frac{w}{3}}mt\right) \\ q &= 8\cosh^2\left(2\sqrt{\frac{w}{3}}mt\right)-1, \end{aligned} \tag{2.35}$$

where a_1 is the integration constant. We find that the scalar field is a constant and the vector field is null

$$B = \frac{f}{m} \quad \text{and} \quad A = 0. \tag{2.36}$$

The energy density and pressure of the matter source are obtained as follows:

$$\rho = -\frac{f^2}{4}+f^2\sinh^{-2}\left(2\sqrt{\frac{w}{3}}mt\right) \quad \text{and} \quad p = \frac{f^2}{4}+\frac{f^2}{3}\sinh^{-2}\left(2\sqrt{\frac{w}{3}}mt\right), \tag{2.37}$$

that yield the following EoS parameter

$$w = \frac{3+4\sinh^{-2}\left(2\sqrt{\frac{w}{3}}mt\right)}{-3+12\sinh^{-2}\left(2\sqrt{\frac{w}{3}}mt\right)}. \tag{2.38}$$

We note first that the scale factor has a similar behavior with the ΛCDM model with the difference that they have different powers; it is $\frac{1}{2}$ in this solution while it is $\frac{2}{3}$ in the ΛCDM model. In the ΛCDM model, which is based on GR, the universe evolves from pressure-less matter ($w=0$) dominated universe to Λ dominated universe (de Sitter universe), such that $q\sim\frac{1}{2}$ at $t\sim 0$ and $q\rightarrow -1$ as $t\rightarrow\infty$. One may check that, on the other hand, solving field equations in GR in the presence of Λ and radiation/relativistic fluid, which can be described with an EoS parameter $w=1/3$, instead of pressure-less matter, one would obtain the same behavior we obtained for the scale factor in Equation 2.35 in this solution, which yields $q\sim 1$ at $t\sim 0$ and $q\rightarrow -1$ as $t\rightarrow\infty$. In GR, the value $q=1$ corresponds to the value of the deceleration parameter in the radiation dominated universe that can describe the early universe, e.g., the time when primordial nucleosynthesis took place. We note that the fluid we obtained in this solution also has the EoS parameter equal $\frac{1}{3}$ at $t=0$ but exhibits a bizarre behavior later on; it reaches infinitely large positive values at $t_{\rm c} = \frac{1}{m}\sqrt{\frac{3}{2w}}\ln(3+2\sqrt{2})$, and then starts with an infinitely large negative value at $t_{\rm c}$ and approaches monotonically to -1 as $t \to \infty$. The reason being that its energy density becomes zero and changes sign at $t_{\rm c}$ and then approaches a negative constant equal to $-\frac{f^2}{4}$ as $t \to \infty$, all the while the pressure decreases too but at a slower rate and approaches a positive constant equal to $\frac{f^2}{4}$ as $t \to \infty$. In fact, one may check that, as $t \to \infty$, this solution approaches the solution we gave above in section 2.2.1.1, where we elevate the energy density of the matter source to zero by introducing a negative vacuum energy density with a value equal to $-\frac{f^2}{4}$. Let us now apply the same procedure to Equation 2.31; using Equations 2.32 and 2.34, namely introduce a vacuum energy with an energy density equal to $-\frac{f^2}{4}$, the energy density and pressure of the matter source given in Equation 2.37 can now be written as follows:

$$\rho = f^2 \sinh^{-2}\left(2\sqrt{\frac{w}{3}}mt\right) \quad \text{and} \quad p = \frac{f^2}{3}\sinh^{-2}\left(2\sqrt{\frac{w}{3}}mt\right) \tag{2.39}$$

yielding the following properties

$$\rho \propto a^{-4} \quad \text{and} \quad w = \frac{1}{3}, \tag{2.40}$$

which is exactly the EoS that describes radiation/relativistic fluid. It is interesting that this solution obtained by assuming that f is constant, hence effective gravitational coupling is constant, doesn't predict an unknown kind of matter source but a radiation/relativistic fluid provided that the negative vacuum energy density is isolated appropriately.

2.2.1.2. Case II: $f = 0$. As we mentioned before, we restrict the investigation of the model with the cases for which f is constant that gives rise to a time independent effective gravitational constant. Now, in this subsection, we investigate an extreme case for a constant f solution such that

$$f = mB + \dot{A} + 3A\frac{\dot{a}}{a} = 0, \tag{2.41}$$

which corresponds to an infinitely large effective gravitational coupling limit. Although such an extreme case may not be advocated as a physically viable case, an investigation of the solution under this assumption may give us an idea about the behavior of our model in case of very large values of the effective gravitational coupling. Although there are strong constraints on the possible time variation of the gravitational coupling in the observable past of the universe, our understanding on the very early universe, strictly speaking the time scales between the Planck time scale 10^{-43} s and SUSY breaking time scale $< 10^{-10}$ s, is still quite speculative. Indeed there is no fundamental theory of physics that assures the constancy of the gravitational coupling at the energy scales that correspond to the time scales close to the Planck time scales. Hence, there is a room for the solutions that are obtained in this extreme case, such that they maybe considered in the context of the dynamics of the very early universe, for instance, in the context of inflation that is believed to took place at time scales $\sim 10^{-35}$ s with the corresponding energy scales $\sim 10^{15}$ GeV. It is also noteworthy to point out here that setting f equal to zero in our theory described by the action given in Equation 2.1 is in fact not the same as setting a constant of a theory to zero, namely, as setting inverse of the gravitational coupling constant $1/G$ to zero in GR described by EH action: f is in fact not a true constant of our model/the action in Equation 2.1 but a dynamical parameter consisting of three additive terms that are dynamical too (see Equation 2.18). Hence, the investigation of a solution under the assumption $f = 0$ should be understood as the investigation of the behavior of our model in the period of time when the constituents of f possibly evolve such that f vanishes.

In this case, i.e., choosing Equation 2.41, Equations 2.14-2.17 reduce to the following

$$-\frac{1}{2}(\dot{B} - mA)^2 = \rho, \tag{2.42}$$

$$-\frac{1}{2}(\dot{B} - mA)^2 = p, \tag{2.43}$$

$$\ddot{A} + 3\dot{A}\frac{\dot{a}}{a} + 3A\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + m^2 A = 0, \tag{2.44}$$

$$\ddot{B} + 3\dot{B}\frac{\dot{a}}{a} + m^2 B = 0, \tag{2.45}$$

supplemented by Equation 2.41. It is important to note here that this reduced system Equations 2.41-2.45 is not fully determined since there are five unknown functions but only four linearly independent equations in this case: Differentiating Equation 2.41

once and using the result in Equation 2.44 we find

$$mA = \dot{B}, \tag{2.46}$$

and substituting this back into Equation 2.41 we get Equation 2.45, which means that we lost one equation and hence one additional constraint is required to fully determine the system. We first give the solution of this undetermined system in terms of the ratio of the vector and scalar fields denoted as

$$F = \frac{A}{B}, \tag{2.47}$$

which will provide us with an insight for choosing a useful and reasonable function for the additional constraint rather than an arbitrary function. Now using Equation 2.41 we obtain the scale factor as

$$a = a_1 e^{-\frac{1}{3}\int m\frac{B}{A}+\frac{\dot{A}}{A}\,\mathrm{dt}}, \tag{2.48}$$

where a_1 is an integration constant. Next using Equation 2.47 with Equations 2.46 and 2.48 we find that the scale factor a, the scalar field B and the vector field A can be written in terms of F as

$$a = a_1 e^{-\frac{1}{3}\int \frac{m}{F}+\frac{\dot{F}}{F}+mF\,\mathrm{dt}}, \quad B = B_1 e^{\int mF\,\mathrm{dt}} \quad \text{and} \quad A = F B_1 e^{\int mF\,\mathrm{dt}} \tag{2.49}$$

where B_1 is an integration constant. The energy density and pressure, on the other hand, are always null as can immediately be seen upon substituting Equation 2.46 in Equations 2.42 and 2.43

$$\rho = 0 \quad \text{and} \quad p = 0, \tag{2.50}$$

i.e., there is nothing in the universe other than the vector and scalar fields, which is plausible since the presence of a matter source in this extreme case would be fatal.

We note that the scale factor given in Equation 2.49 possesses some interesting properties. To make this more clear, we give also the Hubble and deceleration parameters in terms of F:

$$3H = -\frac{m}{F} - \frac{\dot{F}}{F} - mF \quad \text{and} \quad q = 3\frac{-m\dot{F} + F\ddot{F} - \dot{F}^2 + m\dot{F}F^2}{(m + \dot{F} + mF^2)^2} - 1. \tag{2.51}$$

We note that the Hubble parameter consists of three additive terms: Two terms that contribute to the Hubble parameter positively if the vector and scalar fields yield opposite signs ($F < 0$). These are directly proportional to the mass term m, and while one of them is directly proportional to F, the other is inversely proportional to F. And another term ($-\dot{F}/F$) that contributes to the Hubble parameter positively/negatively if the rate of change of the vector field is less/higher than that of the scalar field. In contrast to the other two, this term is independent of the mass term and arises only if the ratio between the scalar and vector fields is not constant. Accordingly, the expansion of the universe, viz. the Hubble parameter, is not only contributed by the distinct behaviors of the vector and scalar fields ($-\dot{F}/F$), but also, interestingly, by the ratio of these two fields in a non-trivial way ($-m(1/F + F)$) if there is a non-zero mass term. It is apparent that the presence of a non-zero mass term can lead to an intricate expansion history of the universe, even if the evolution of the ratio between the scalar and vector fields obeys a simple function. On the other hand, if m is null and/or F is constant then we obtain the following simple cases:

- If the mass term is non-zero and the ratio between the scalar and vector fields is constant then we have:

  $$H = -\frac{m}{3F} - \frac{mF}{3} \quad \text{and} \quad A \propto B \propto \exp(mFt) \quad (m > 0 \text{ and } F = \text{const.}). \tag{2.52}$$

  If $F < 0$, the universe exhibits de Sitter expansion and the scalar and vector fields decrease exponentially as t increases.
- If the mass term is null and the ratio between the scalar and vector fields is not constant then we have

  $$a \propto A^{-\frac{1}{3}} \quad \text{and} \quad B = \text{const.} \qquad (m = 0 \text{ and } F \neq \text{const.}). \tag{2.53}$$

- If the mass term is null and the ratio between the scalar and vector fields is a constant then we have a static universe:

$$a = \text{const.}, \quad B = \text{const.} \quad \text{and} \quad A = 0 \quad (m = 0 \text{ and } F = \text{const.}). \tag{2.54}$$

In the light of the above discussion, let us now determine the equations given in Equation 2.49 by making a plausible assumption on the time evolution of the ratio between the scalar and vector fields, i.e., F. We demand (i) the universe to start from a singularity at $t = 0$, namely, $H \to +\infty$ and $a \to 0$ as $t \to 0$, which can be achieved if either $F \to 0$ as $t \to 0$ or $F \to -\infty$ as $t \to 0$, as can be seen from Equation 2.51, (ii) the assumed function for F to yield minimum number of free parameters, namely only one, but yet can realize the simple cases given above as particular cases as well as various cases depending on the value of the free parameter, (iii) the model to be able to approximate a power-law expansion (i.e., $H \propto t^{-1}$) for a certain period of time, which maybe achieved due to the term $\frac{\dot{F}}{F}$ in Equation 2.51. The simplest function that can be utilized in accordance with all our demands is maybe a power-law relation given as follows

$$F = \frac{A_0}{B_0}\left(\frac{t}{t_0}\right)^{-k}, \tag{2.55}$$

where k is a constant whose sign will determine whether the vector field will be dominant over the scalar field at the earlier times or the later times. Finally, solving Equation 2.48 using this assumption Equation 2.55 we obtain the scale factor as
for $|k| \neq 1$

$$a = a_0 t^{\frac{k}{3}} \times \exp\left[\frac{A_0}{B_0}\frac{mt_0}{3(k-1)}\left(\frac{t}{t_0}\right)^{-k+1}\right] \times \exp\left[-\frac{B_0}{A_0}\frac{mt_0}{3(k+1)}\left(\frac{t}{t_0}\right)^{k+1}\right], \tag{2.56a}$$

for $k = -1$,

$$a = a_0 t^{-\frac{B_0}{A_0}\frac{mt_0}{3}-\frac{1}{3}} \times \exp\left[-\frac{A_0}{B_0}\frac{mt_0}{6}\left(\frac{t}{t_0}\right)^{2}\right], \tag{2.56b}$$

for $k = 1$

$$a = a_0 t^{-\frac{A_0}{B_0}\frac{mt_0}{3}+\frac{1}{3}} \times \exp\left[-\frac{B_0}{A_0}\frac{mt_0}{6}\left(\frac{t}{t_0}\right)^2\right]. \tag{2.56c}$$

On the other hand, one may check that the Hubble and deceleration parameters can be given uniquely for arbitrary values of k as

$$H = -\frac{A_0}{B_0}\frac{m}{3}\left(\frac{t}{t_0}\right)^{-k} + \frac{k}{3}t^{-1} - \frac{B_0}{A_0}\frac{m}{3}\left(\frac{t}{t_0}\right)^{k}, \tag{2.57}$$

$$q = 3k{t_0}^k t^{k-1}\frac{A_0}{B_0}\frac{\frac{A_0}{B_0}{t_0}^k t^{k-1} + m(t^{2k} - \frac{{A_0}^2}{{B_0}^2}{t_0}^{2k})}{\left[k\frac{A_0}{B_0}{t_0}^k t^{k-1} - m(t^{2k} + \frac{{A_0}^2}{{B_0}^2}{t_0}^{2k})\right]^2} - 1. \tag{2.58}$$

We will show that the three terms in Equation 2.57 dominate in different eras giving an inflationary phase followed by a deceleration phase followed by an acceleration phase provided that the values of the constants are chosen appropriately. We obtain the scalar and vector fields as follows:
for $k \neq 1$,

$$B = B_0 e^{-\frac{A_0}{B_0}\frac{mt_0}{k-1}\left(\frac{t}{t_0}\right)^{-k+1}} \quad \text{and} \quad A = A_0\left(\frac{t}{t_0}\right)^{-k} e^{-\frac{A_0}{B_0}\frac{mt_0}{k-1}\left(\frac{t}{t_0}\right)^{-k+1}} \tag{2.59a}$$

for $k = 1$

$$B = B_0\left(\frac{t}{t_0}\right)^{\frac{A_0}{B_0}mt_0} \quad \text{and} \quad A = A_0\left(\frac{t}{t_0}\right)^{\frac{A_0}{B_0}mt_0-1}. \tag{2.59b}$$

We note that the evolution of the scale factor in Equation 2.56 is characterized by the mass term $m > 0$ (in particular, according to whether it is null or non-null) and the constant k that determines the relative rate of change of the scalar and vector fields with respect to time in Equation 2.55. Hence, in what follows, we shall carry out a detailed discussion considering the cases $m = 0$ and $m \neq 0$ separately.

- The case $m = 0$: Power-law expansion

  We note that the choice $m = 0$ sets the exponential terms to unity and leads to a simple power-law expansion/contraction as

$$a = a_0 t^{\frac{k}{3}}, \quad H = \frac{k}{3} t^{-1} \quad \text{and} \quad q = \frac{3}{k} - 1 \tag{2.60}$$

with a constant scalar field but a vector field yielding a power-law evolution in time

$$B = B_0 \quad \text{and} \quad A = A_0 \left(\frac{t}{t_0} \right)^{-k}. \tag{2.61}$$

The vector field is inversely proportional to the volume of the universe $A \propto a^{-3}$, and the universe expands at an accelerating rate if $k > 3$ and at a decelerating rate if $0 < k < 3$ while the universe contracts if $k < 0$. The case $k = 0$ is a special case for which the universe becomes static and both scalar and vector fields are also constant.

- The case $m \neq 0$: Inflation with a switch-off mechanism
  We showed, in the previous subsection, that the case with zero mass $m = 0$ leads to a simple power-law behavior of the scale factor and that the further choice $k = 0$ leads to a static universe. We note that the static universe arises since, in Equation 2.56, the choice $m = 0$ sets the exponential terms to unity while the choice $k = 0$ sets the power term to unity. Hence, in this subsection, we shall first consider the case $m \neq 0$ but $k = 0$ and then discuss the case $m \neq 0$ and $k \neq 0$ that can give rise to an evolution that might be considered in the context of the inflation mechanism. We observe that, setting

$$k = 0, \tag{2.62}$$

the universe exhibits exponential behavior as

$$a = a_0 e^{-\left(\frac{A_0}{B_0} + \frac{B_0}{A_0}\right)\frac{m}{3} t}, \quad H = -\frac{m}{3} \left(\frac{A_0}{B_0} + \frac{B_0}{A_0} \right) \quad \text{and} \quad q = -1, \tag{2.63}$$

and that the scalar and vector fields evolve with the same rate as

$$B = B_0 e^{\frac{A_0}{B_0} m t} \quad \text{and} \quad A = A_0 e^{\frac{A_0}{B_0} m t}. \tag{2.64}$$

The universe expands exponentially for $m > 0$ and $A_0/B_0 < 0$ and the value of the Hubble parameter is proportional with the mass term, and hence a static universe is obtained when $m = 0$ as expected. This is because the choice $k = 0$ in Equation 2.56 sets the power term to unity and the exponents of the two exponential terms identically to t. On the other hand, the values $k \neq 0$ not only give rise to a power term, but also cause the exponents of the two exponential terms to differ from each other and therefore the power term (dependent on k only) and the two exponential terms (which arise when the mass term is non-zero and are dependent on k in distinct ways) all together give rise to a non-trivial evolution that can even be related with the inflation model.

One may check that the model can give rise to various behaviors depending on the choice of the parameters. However, we are particularly interested in whether the model can give rise to a behavior that is compatible with the inflationary cosmology. Looking at the Hubble parameter in Equation 2.57 and the scale factor in Equation 2.56, it can be easily seen that choosing the values of the parameters appropriately under the assumption $A_0/B_0 < 0$ and $k > 1$ the universe starts expanding at $t = 0$ and will always expand passing through three different stages respectively;

$$a \sim \exp\left[\frac{A_0}{B_0}\frac{mt_0}{3(k-1)}\left(\frac{t}{t_0}\right)^{-k+1}\right],$$

$$H \sim -\frac{A_0}{B_0}\frac{m}{3}\left(\frac{t}{t_0}\right)^{-k}$$

$$q \sim -\frac{3k}{mt_0}\frac{B_0}{A_0}\left(\frac{t}{t_0}\right)^{(k-1)} - 1 \quad \text{at} \quad t \simeq 0, \tag{2.65}$$

then

$$a \sim t^{\frac{k}{3}}, \quad H \sim \frac{k}{3}t^{-1} \quad \text{and} \quad q \sim \frac{3}{k} - 1 \quad \text{at} \quad t \gtrsim 0, \tag{2.66}$$

and finally at later times

$$a \sim \exp\left[-\frac{B_0}{A_0}\frac{mt_0}{3(k+1)}\left(\frac{t}{t_0}\right)^{k+1}\right], \tag{2.67}$$

$$H \sim -\frac{B_0}{A_0}\frac{m}{3}\left(\frac{t}{t_0}\right)^{k} \tag{2.68}$$

$$q \sim \frac{3k}{mt_0}\frac{A_0}{B_0}\left(\frac{t}{t_0}\right)^{-k-1} - 1 \quad \text{at} \quad t \gg 0. \tag{2.69}$$

In the first stage given by in Equation 2.65, the universe begins with an accelerating expansion rate, such that $a \to 0$, $H \to \infty$ and $q \to -1$ as $t \to 0$. After a while the power term will become dominant over the two exponential terms in Equation 2.56 and the second stage in which the evolution of the universe can be described by Equation 2.66 will start. Accordingly, one may check from Equation 2.66 that if $1 < k < 3$ then the accelerated expansion achieved in the previous stage will end and the universe will enter into a decelerated expansion phase, otherwise, i.e. if $k > 3$, it will keep on accelerated expansion accordingly in Equation 2.66. Eventually, the exponential term on the right will be dominant over the exponential term at the middle and the power term in Equation 2.56 and the third stage, in which the universe will be described by Equation 2.67, will start. Accordingly, the universe will first evolve into a super-accelerated phase ($q < -1$) and eventually will start to approach monotonically to an expansion rate with a deceleration parameter equal -1, $a \to \infty$ and $q \to -1$ as $t \to \infty$. In this picture, the case $A_0/B_0 < 0$ with $1 < k < 3$ is of particular interest since it can give rise to a behavior compatible with inflationary cosmology, such that the expansion of the universe starts with an accelerated expansion that will be switched off and the universe will enter into a decelerated expansion phase. Moreover, interestingly, this decelerated expansion phase will be followed by an another accelerated expansion phase that may be related with the late time acceleration of the universe. Such a behavior, two different accelerated expansion phases with a decelerated expansion phase between them is consistent with the current paradigm in cosmology (ΛCDM cosmology supplemented by inflationary cosmology).

We demonstrate the evolution of the universe in this solution by giving some suitable values to the parameters. To do so, we first choose $k = \frac{3}{2}$ so that in the decelerated expansion phase that follows the first accelerated expansion phase the value of the deceleration parameter will be $q = 1$, which is the value of the deceleration parameter when the primordial nucleosynthesis took place ($\sim 10^2$ seconds after the Big Bang) in the standard cosmology based on GR. We choose $t_0 = 14$ Gyr, $H_0 = 10^{-32}$ eV and $\frac{A_0}{B_0} = -10^{-13}$ for the present universe, and choose $m = 10^{-45}$ eV, which is a value almost 20 orders of magnitude less than the most strict upper limits given for the photon mass. Using these values we find that $q \simeq -1$, $H \simeq 10^{39}$ s^{-1} and $A/B \simeq -10^{70}$ at $t = 10^{-38}$ s (inflation), $q \simeq 0$, $H \simeq 10^{35}$ s^{-1} and $A/B = -10^{66}$ at $t = 10^{-35}$ s (inflation ends). In a short while following the end of the inflationary phase, the universe achieves an expansion rate with a deceleration parameter equal to unity and preserves this value for a long time: $q \simeq 1$, $H \simeq 10^{31}$ s^{-1} and $A/B \simeq -10^{61}$ at $t \simeq 10^{-32}$ s^{-1}, $q \cong 1$, $H \simeq 0.5$ s^{-1} and $A/B \simeq -10^{13}$ at $t \simeq 1$ s, $q \cong 1$, $H \simeq 0.005$ s and $A/B \simeq -10^{10}$ at $t \simeq 100$ s. The value of the deceleration parameter does not deviate from the value $q \cong 1$ till the age of the universe reaches $t \simeq 10^{17}$ s. Because the value of the effective gravitational coupling is infinitely large in this solution extending the model for large t values may not be reliable. On the other hand, because there is no matter source ($\rho = 0$) in this solution, extending the model to large t values will still be consistent within the model itself. Interestingly, we find that $q \sim -0.4$, $H \simeq 10^{-18}$ s^{-1} and $A/B \sim -10^{-13}$ at $t \sim 10$ Gyr and the values of the deceleration and Hubble parameters here are consistent with the observations. We note that the universe in our model begins already with an accelerated expansion rate. Therefore we are not able to calculate the e-fold of the size of the universe between the switch-on and -off of the inflation as in the usual inflationary models. However we calculate in our model that the size of the universe (a) goes through 50 e-folds from $t = 10^{-38}$ s to the end of inflation at $t = 10^{-35}$ s. As the final remark, we note that the vector field is dominant over the scalar field in the early times, namely, $|A/B| > 10^{66}$ when the inflation took place $t < 10^{-35}$ s and $|A/B| \sim 10^{10}$ at $t \sim 100$ s, while the scalar field is dominant over the vector field at the times of the late time acceleration, namely, $|A/B| \sim 10^{-13}$ at $t = 10$ Gyr. This tells us that it is the vector field who is responsible for the inflationary phase in the early universe while it is the scalar field who is responsible for the late time acceleration of the universe.

3. $f(R)$ THEORIES OF GRAVITY

As an alternative to the ΛCDM model, it has been proposed that infrared modifications of gravity may be the explanation for the late time acceleration in expansion. In this context, $f(R)$ modified gravities have a long history [96] and have been explored as infrared corrections to GR. The action for $f(R)$ gravity reads

$$\mathcal{S} = \int \sqrt{-g} d^4x \left[\frac{1}{16\pi G} f(R) \right] + \mathcal{S}_m \tag{3.1}$$

where R is replaced by $f(R)$ in the Einstein-Hilbert action and $\mathcal{S}_m$ is the usual matter field action. The variation is popularly known as the metric $f(R)$ gravity as opposed to the Palatini formulation where the variation is carried out with respect to both the metric and the affine connections. A variation with respect to the metric, yields the field equations as

$$\Sigma_{\mu\nu} \equiv F(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) F(R) = 8\pi G T^{(m)}_{\mu\nu}, \tag{3.2}$$

where subscript $F(R) = \frac{\partial f}{\partial R}$ denotes differentiation with respect to the R and $T^{(m)}_{\mu\nu}$ represents the energy momentum tensor of matter fields defined by

$$T^m_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}, \tag{3.3}$$

which satisfies the continuity equation

$$\nabla^\mu T^{(m)}_{\mu\nu} = 0, \tag{3.4}$$

as well as $\Sigma_{\mu\nu}$, i.e., $\nabla^\mu \Sigma_{\mu\nu} = 0$.

If we take the trace of Equation 3.2 we get

$$3\Box F(R) + F(R)R - 2f(R) = 8\pi G\, T, \tag{3.5}$$

where $T = g^{\mu\nu}T^{(m)}_{\mu\nu}$. In standard Einstein gravity, without the cosmological constant, one has F(R)= constant, so that the term $\Box F(R)$ in Equation 3.5 vanishes. However in modified gravity it does not vanish, which means that there is a propagating scalar degree of freedom[97]. When we consider the vacuum solution at which Ricci scalar is constant from Equation 3.5 we obtain

$$F(R)R - 2f(R) = 0, \tag{3.6}$$

since $\Box F(R) = 0$ at this point. The $f(R) \propto R^2$ type of theories satisfy this condition and successfully generate inflationary universe scenario for the early universe [14, 84]. In the model $f(R) = R + \alpha R^2$, because of the linear term in R, the inflationary expansion ends when the term αR^2 becomes smaller than the linear term R[97]. As the negligibly small curvature of the present epoch makes the term αR^2 smaller than R, this model is not suitable for the present time acceleration. Models of the type $f(R) = R - \alpha/R^n$ were proposed as a candidate for dark energy(for the detailed reviews, see [86, 92]). The conditions of the viable cosmological models corresponds to f(R) gravity can be found in [98–118], and constraints obtained from the classical tests of GR for the Solar System scale seem to rule out most of the models proposed so far[119–124, 126]. However some models passing Solar System tests can be obtained [125, 128–133].

Recently $f(R, T)$ gravity theory, where gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and of the trace of the energy-momentum tensor T, received some attention [134]. Actually this model can be seen as the application of more general theory where L_g is given by an arbitrary function of Ricci scalar and of the matter Lagrangian, i.e., $f(R, L_m)$[135]. Furthermore, $f(R, T, R_{\mu\nu}T^{\mu\nu})$ was proposed in [136] which received much attention. The peculiar feature related to this type of theories is that either the matter Lagrangian directly couples to the Ricci scalar or energy-momentum tensor of matter or of a field is seen in the action. Conventionally we expect energy-momentum tensor to be appear after the variation of the matter action with respect to the metric. Moreover in these models the most remarkable part is that as matter is non-minimally coupled to the Ricci scalar, matter-geometry coupling, hence the motion of particles is non-geodesic and the particles are acted upon by a force which is orthogonal to their four-velocity.

3.1. $f(R, T_{\mu\nu}T^{\mu\nu})$ gravity

We propose a new model of gravity where R in EH action is replaced by an arbitrary function of R and of the norm of energy-momentum tensor (the contraction of the energy-momentum tensor with itself) i.e., $f(R, T_{\mu\nu}T^{\mu\nu})$. In our gravity model, the energy-momentum tensor $T_{\mu\nu}$ is seen at the level of gravitational action. Even though our gravity theory shares some resemblances with the above mentioned theories, in our gravity model we have matter-matter coupling as opposed to the other cases. We derived the field equations in the metric formalism. We will follow almost the same line of reasoning as in [134] and [135] to discuss our model's predictions. We find that the equation of motion of massive test particles is non-geodesic and these test particles are acted upon by a force which is orthogonal to the four-velocity of the particles. We also find the Newtonian limit of the model to calculate the extra acceleration which can affect the perihelion of Mercury.

We explicitly show that energy in general is not conserved in this gravity theory, it is either created or destroyed according to the sign of our model parameter α. However we show that by imposing the energy condition we get a regime where energy conservation law emerges. There is a deviation from the GR result unless the energy density of fluid is constant. Arranging α parameter gives an opportunity to cure the inconsistency between the observational values for the abundance of light elements and the standard Big Bang Nucleosynthesis results. Even the dust dominated universe undergoes an accelerated expansion without using a cosmological constant in Model II. With this specific choice of $f(R, T_{\mu\nu}T^{\mu\nu})$, we get the a Cardassian-like expansion.

In section III we choose two different specific functions $f(R, T_{\mu\nu}T^{\mu\nu})$ to study its cosmological implications and in section IV we use observational Helium and Deuterium abundances to put a constraint on the model parameter α. In the last section we find the Newtonian limit for both models.

The action for the $f(R, T_{\mu\nu}T^{\mu\nu})$ gravity is considered as

$$S = \int \sqrt{-g} d^4 x f(R, T_{\mu\nu}T^{\mu\nu}) + S_M, \tag{3.7}$$

As an alternative $T_{\mu\nu}$ can be taken as a different matter which we do not consider here.

Varying the action with respect to the inverse metric we get

$$\delta S = \int \left(f_R \delta R + f_{T^2} \delta(T_{\mu\nu}T^{\mu\nu}) - \frac{1}{2} g_{\mu\nu} f \delta g^{\mu\nu} + \frac{1}{\sqrt{-g}} \delta(\sqrt{-g} L_m) \right) \sqrt{-g} d^4x, \tag{3.8}$$

where

$$f_R(R, T_{\mu\nu}T^{\mu\nu}) = \frac{\partial f}{\partial R}, \quad f_{T^2}(R, T_{\mu\nu}T^{\mu\nu}) = \frac{\partial f}{\partial (T_{\mu\nu}T^{\mu\nu})}. \tag{3.9}$$

The energy momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}} \tag{3.10}$$

If the Lagrangian density of matter solely depends on the metric components and not on their derivatives then we have

$$T_{\mu\nu} = g_{\mu\nu} L_m - 2 \frac{\partial L_m}{\partial g^{\mu\nu}}. \tag{3.11}$$

Field equations derived from Equation 3.8 is

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + (g_{\mu\nu} \nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu) f_R = \frac{1}{2} T_{\mu\nu} - f_{T^2} \theta_{\mu\nu}, \tag{3.12}$$

where

$$\theta_{\mu\nu} = \frac{\delta(T_{\alpha\beta}T^{\alpha\beta})}{\delta g^{\mu\nu}}, \tag{3.13}$$

$$\theta_{\mu\nu} = -2L_m \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) - T T_{\mu\nu} + 2 T^\alpha_\mu T_{\nu\alpha} - 4 T^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}. \tag{3.14}$$

As is seen from the equation above $\theta_{\mu\nu}$ depends on matter Lagrangian explicitly. In what follows we will only use the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{3.15}$$

where ρ is the energy density and p is the thermodynamic pressure. As is known that the definition of matter Lagrangian giving the perfect fluid energy-momentum tensor in Equation 3.15 is not unique, for consistency $L_m = p$ is assumed, and so the second variation of the matter Lagrangian in Equation 3.20 is null [137]. Thus we have

$$\theta_{\mu\nu} = -2L_m\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) - TT_{\mu\nu} + 2T^{\alpha}_{\mu}T_{\nu\alpha}. \tag{3.16}$$

3.1.1. Non-conservation of energy

The first thing to note is that in this model the continuity equation

$$\dot{\rho} + 3H(\rho + p) \neq 0, \tag{3.17}$$

is not satisfied. The covariant divergence of Equation 3.12 is

$$\nabla^{\mu}T_{\mu\nu} = -f_{T^2}g_{\mu\nu}\nabla^{\mu}(T_{\alpha\beta}T^{\alpha\beta}) + 2\nabla^{\mu}(f_{T^2}\theta_{\mu\nu}). \tag{3.18}$$

With the aim of obtaining the modified form of continuity equation, we contract the above equation with the four velocity u^{μ} of test particles and easily get

$$\dot{\rho} + 3H(\rho + p) = f_{T^2}\nabla_0(T_{\alpha\beta}T^{\alpha\beta}) - 2u^{\nu}\nabla^{\mu}(f_{T^2}\theta_{\mu\nu}). \tag{3.19}$$

As explicitly seen from the above equation the RHS terms act as a source for the matter content of the universe and so the energy is not conserved. However in the next section we see that for a specific choice of function $f(R, T_{\mu\nu}T^{\mu\nu})$, there is a regime where total energy is conserved. Moreover, for different choices of $f(R, T_{\mu\nu}T^{\mu\nu})$, we will investigate Equation 3.19 in the next section.

3.1.2. Conservation of total energy: Determination of special choice of function $f(R, T_{\mu\nu}T^{\mu\nu})$

To obtain the conservative models, RHS of the Equation 3.19 must be equal to zero. As we need $\theta_{\mu\nu}$, we insert Equation 3.15 into Equation 3.16 and obtain

$$\theta_{\mu\nu} = (-\rho^2 - 4\rho p - 3p^2)u_\mu u_\nu. \tag{3.20}$$

Following the argument given in [138] we assume that the general function $f(R, T_{\mu\nu}T^{\mu\nu})$ has the form

$$f(R, T_{\mu\nu}T^{\mu\nu}) = f_1(R) + f_2(T_{\mu\nu}T^{\mu\nu}) \tag{3.21}$$

and demand that the RHS of Equation 3.19 to be equal to zero. Then we get

$$f_{2T^2}(+3w^2 + 5w) + 2T^2 f_{2T^2T^2}(1 + 4w + 3w^2) = 0, \tag{3.22}$$

where

$$f_{2T^2} = \frac{df_2}{d(T_{\mu\nu}T^{\mu\nu})} \tag{3.23}$$

and $f_2 = f_2(T_{\mu\nu}T^{\mu\nu})$. The general solution of this differential equation is $(w \neq -1)$

$$f_2(T_{\mu\nu}T^{\mu\nu}) = c_1(T_{\mu\nu}T^{\mu\nu})^{\frac{2+3w+3w^2}{2(1+4w+3w^2)}} + c_2, \tag{3.24}$$

where c_1 and c_2 are integration constants. For example, for equation of state $w = 0$, i.e., dust, Equation 3.24 becomes

$$f_2(T_{\mu\nu}T^{\mu\nu}) = c_1(T_{\mu\nu}T^{\mu\nu}) + c_2 \tag{3.25}$$

If we choose our function as $f(R, T_{\mu\nu}T^{\mu\nu}) = \frac{1}{16\pi G}(R + \alpha(T_{\mu\nu}T^{\mu\nu}))$ then we have a conserved model. However the constant α has the dimension of $[M]^{-6}$ which is not natural. Hence in the next section we will choose models where we need dimensionless constants.

3.1.3. Cosmological applications of $f(R, T_{\mu\nu}T^{\mu\nu})$ gravity

In this section, we will analyze some cosmological solutions of the theory by choosing appropriate function $f(R, T_{\mu\nu}T^{\mu\nu})$. The geometry of space-time is described by the FRW metric, and for flat space-like sections given by

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{3.26}$$

where $a(t)$ is the scale factor of the universe. The non-zero components of the Ricci tensor and the Ricci scalar are given in Appendix A.2. Now we will analyze different cases and their cosmological implications by fixing the function $f(R, T_{\mu\nu}T^{\mu\nu})$.

3.1.3.1. Model I. $f(R, T_{\mu\nu}T^{\mu\nu}) = \frac{R}{16\pi G} + \alpha\sqrt{T_{\mu\nu}T^{\mu\nu}}$

This choice of function makes α a dimensionless coupling parameter. With this consideration, Equation 3.12 becomes

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} + 8\pi G\alpha g_{\mu\nu}\sqrt{T_{\alpha\beta}T^{\alpha\beta}} - 8\pi G\alpha\frac{\theta_{\mu\nu}}{\sqrt{T_{\alpha\beta}T^{\alpha\beta}}}, \tag{3.27}$$

By using Equation 3.20 and the metric given in Equation A.7, Equation 3.27 becomes

$$3H^2 = 8\pi G\rho\left(1 + \alpha\frac{4w}{\sqrt{1+3w^2}}\right) = \rho_{\text{eff}} \tag{3.28}$$

$$2\dot{H} + 3H^2 = 8\pi G\rho\left(-w - \alpha\sqrt{1+3w^2}\right) = -p_{\text{eff}} \tag{3.29}$$

As $\alpha = 0$ case corresponds to GR we will call the RHS of Equation 3.29, as effective density and pressure respectively to distinguish the difference from the GR. Inserting energy density given in Equation 3.29 into pressure, we obtain

$$2\dot{H} + 3H^2 = C(\alpha, w)3H^2, \tag{3.30}$$

where

$$C(\alpha, w) = \frac{-w - \alpha\sqrt{1+3w^2}}{1 + \alpha\frac{4w}{\sqrt{1+3w^2}}}. \tag{3.31}$$

The modification vanishes at $\alpha = 0$, and we consider the effective EoS is parameterized as $w_{\text{eff}} = -C$ thus we relate w_{eff} and w as,

$$w_{\text{eff}} = \frac{w + \alpha\sqrt{1+3w^2}}{1+\alpha\frac{4w}{\sqrt{1+3w^2}}}. \tag{3.32}$$

In order to obtain the Hubble parameter, Equation 3.30 is integrated

$$H = \frac{2}{3(1+w_{\text{eff}})t}. \tag{3.33}$$

Now if we use the energy-momentum tensor of perfect fluid given in Equation 3.15 then Equation 3.19 becomes

$$\dot{\rho} + 3H\gamma\rho = 0, \tag{3.34}$$

where

$$\gamma = \frac{1 + w + \alpha\frac{1+4w+3w^2}{\sqrt{1+3w^2}}}{1+\alpha\frac{4w}{\sqrt{1+3w^2}}}. \tag{3.35}$$

The energy density varies depending on α as

$$\rho = \rho_0(\frac{a}{a_0})^{-3\gamma}. \tag{3.36}$$

The dependence of the effective EoS in our proposed model on α for an interval $\alpha = [-0.5, 0.5]$ in Figure 3.1. For example, for matter dominated universe $p = 0$, thus $w = 0$ then $\gamma = 1 + \alpha$ giving $\rho_m = \rho_0 a^{-3(1+\alpha)}$ which shows that the depending of the sign of α the matter in the universe will more slowly (if α is negative) or more quickly (if α is positive) dilute away than the expected evolution of energy density $\rho = \rho_0(\frac{a}{a_0})^{-3}$. In the same manner $w = -1,\ \gamma = 0,\ \rho_m =$const. meaning that the behavior of the density of dark energy doesn't change in this model and w_{eff} is independent of α. For radiation, we still have the same behavior as for matter since $w = \frac{1}{3}$ gives $\gamma = \frac{\frac{4}{3}+\alpha\frac{4\sqrt{3}}{3}}{1+\alpha\frac{2\sqrt{3}}{3}}$ and $\rho = \rho_0(\frac{a}{a_0})^{-4-\frac{4\alpha}{2\alpha+\sqrt{3}}}$ showing that the evolution of the density of radiation depends on the sign and the magnitude of α. We will also put a constraint on α as it changes the behavior of energy densities from the expected energy behavior of ΛCDM.

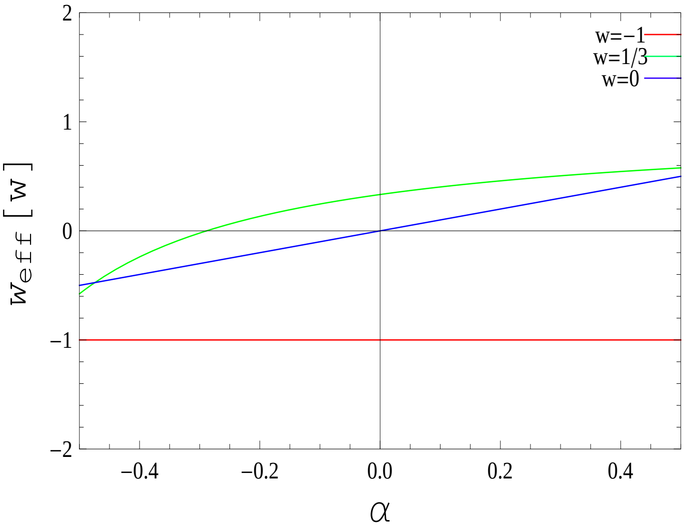

Figure 3.1. The dependence of effective (EoS) on α, the coupling of the $\sqrt{T_{\mu\nu}T^{\mu\nu}}$ for different EoS in standard cosmology.

3.1.3.2. Model II. $f(R, T_{\mu\nu}T^{\mu\nu}) = \frac{1}{16\pi G}\Bigg(R + \alpha(T_{\mu\nu}T^{\mu\nu})^{1/4}\Bigg)$

First thing to be noted here is that the parameter α is still dimensionless as in Model I but G also couples the additional term as a difference from that of Model I. With this choice Equation 3.12 yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} + \frac{\alpha}{2}g_{\mu\nu}(T_{\alpha\beta}T^{\alpha\beta})^{1/4} - \frac{\alpha}{4}\frac{\theta_{\mu\nu}}{(T_{\alpha\beta}T^{\alpha\beta})^{3/4}}. \tag{3.37}$$

For the FRW metric flat space-like sections, Friedmann equations of motion becomes

$$3H^2 = 8\pi G\rho + \alpha\frac{\sqrt{\rho}}{2}\Bigg(\frac{-1+4w-3w^2}{(1+3w^2)^{3/4}}\Bigg), \tag{3.38}$$

$$2\dot{H} + 3H^2 = -8\pi G w\rho - \alpha\frac{\sqrt{\rho}}{2}(1+3w^2)^{1/4}. \tag{3.39}$$

Equation 3.38 has the same form with

$$H^2 = A\rho + B\rho^n, \tag{3.40}$$

where $n = 1/2$ in our case. Equation 3.40 is given in [139] in which authors explained the late time acceleration without the need of DE. As is seen for early times as the second term on the right hand side of Equation 3.40 can be ignored, conventional cosmology can be obtained whereas for late times the second term in Equation 3.40 dominates giving the scale factor evolution as $a(t) = t^{\frac{2}{3n}}$ which gives an expansion for $n \leq 2/3$. It needs to be mentioned that for $\alpha(T_{\mu\nu}T^{\mu\nu})^{\beta}$ we still get the Cardassian term for the general case and the relation $n = 2\beta$ is obtained. Theory predicts that the parameter α has units of $M^{2-8\beta}$ or M^{2-4n} and it is easily realized that other choices except $\beta = \frac{1}{4}$ for model II will need dimensionful coefficients α in Equation 3.37 and B in Equation 3.40 which makes the theory less natural. Considering the relation $n = 2\beta$ and the accelerated expansion requirement ($n \leq 2/3$), the condition $\beta \leq \frac{1}{3}$ should be satisfied. We show that $T_{\mu\nu}T^{\mu\nu}$ coupling naturally gives Cardassian-like accelerated expansion.

We also note that in Cardassian-type expansion scenario they assumed that the continuity equation holds and the matter density evolves as $\rho = \rho_0(\frac{a}{a_0})^{-3}$ whereas in our case this situation is not valid. Still as will be seen, our model gives similar results although we do not use any hypothetical fluid in an ad hoc way. The deviation from standard GR is identified by the second term of Equation 3.40 and the best fit values for the Cardassian model parameters are given as $n = -1.33$, $z_{eq} = 0.43$ and $\omega_M = 0.076$ using supernova magnitude versus redshift measurements [140] which showed that the best fitted values for the two parameters (n, z_{eq}) of the Cardassian model gives lower matter density than the current value derived from the measurements of the cosmic microwave background anisotropy and galaxy clusters. Due to the violation in the conservation of the total energy, standard matter EoS parameter is not valid for our case and there arises a deviation from $w_m = 0$.

When conservation of energy momentum tensor is imposed as done in Section IIB, $w_m = 0$ can be safely used for the dust dominated case. For the matter dominated case, the conservation requires the special choice of function $f(T_{\mu\nu}T^{\mu\nu}) = T_{\mu\nu}T^{\mu\nu}$. Using the relation $n = 2\beta$ between the two parameters n and β which correlates the Cardassian model with this model. Actually the fitting procedure should be followed for our proposal, the handicap is not to solve the system to find $H(z)$ function, and to use limiting behavior of the system instead. This is not the scope of this paper. Equations 3.38 and 3.39 can be put into the form

$$\dot{H}(t) = K(w,\alpha) + L(w,\alpha)(1 + H^2) + M(w,\alpha)\sqrt{1 + H^2}. \tag{3.41}$$

With the substitution $H = \sinh u$ where u is just a parameter we obtain the following integral

$$t = \int \frac{\cosh u \; du}{K + M\cosh u + L\cosh^2 u} + c. \tag{3.42}$$

Equation 3.42 can be solved but its solution is too complicated to analyze and interpret. Instead in the following we consider limit cases for our purpose. It is evident that one may choose the terms such that the energy density is low and neglect the first terms on the rhs of Equations 3.38-3.39 compared with the second ones in the late time universe

$$8\pi G w\sqrt{\rho} \ll \frac{\alpha(1+3w^2)^{1/4}}{2}, \tag{3.43}$$

where α and w are order of one and the relation gives the $\rho \ll M_p^2$ condition satisfied. Substituting this approximation, given in Equation 3.43, on the solution of Equations 3.38-3.39, we obtain the Hubble parameter as follows:

$$H(t) = \int \frac{-\sqrt{\rho}\alpha w}{(1+3w^2)^{3/4}} dt + c_1. \tag{3.44}$$

It is interesting remark that $w = 0$ case gives the constant Hubble parameter that signs the de Sitter expansion phase. In what follows, we make the analogy of Equation 3.30 and obtain the effective EoS parameter as:

$$w_{eff} = \frac{(1+3w^2)}{-3w^2+4w-1}. \tag{3.45}$$

As explicitly seen, w_{eff} is independent of α as a basic difference from that of the model I, given in Equation 3.32. In other words in this model even the dust dominated universe ($w = 0$) undergoes an accelerated expansion ($w_{\text{eff}} = -1$). For the late time we don't need any hypothetical substance whose EoS parameter is $w = -1$ or cosmological constant. Here, in model II, an accelerating cosmological solution can be searched for the early time limit. Then the second term should be negligible when compared to the first in Equations 3.38-3.39 as $t \to 0$. One may check that this case corresponds to standard GR and $w = 0$ case gives rise to $w_{eff} = 0$. Hence inflation is not obtained in this case. Likewise, in model I, Equations 3.32 and 3.33 demonstrate the deviation from standard GR. It can be deduced that the rapid early expansion is not included in Model I, too.

3.1.4. Constraint from Big Bang Nucleosynthesis (BBN)

We will now find how the abundances of light elements changes for Model I in terms of parameter α and put a constraint on it in order to check whether our model can be in accordance with the observations or not. To do this, we will use formulae given in [141, 142] and follow the arguments of [143, 144].

3.1.4.1. ^{4}He abundance. In *GR*, Friedmann equations ,when $\alpha = 0$ in Equation 3.29 yield the Hubble parameter as

$$H = \frac{2}{3(1+w_{\rm rad})t}, \tag{3.46}$$

where $w_{\rm rad} = \frac{1}{3}$. On the other hand in our model this equation becomes

$$H = \frac{2}{3(1+w_{\rm eff(rad)})t}. \tag{3.47}$$

To find that how our model changes the abundances of primordial light elements during the Nucleosynthesis, we are interested in the ratio of our model's Hubble parameter to the Hubble parameter of GR during the early radiation dominated era. Putting Equation 3.32 into Equation 3.47, S parameter is obtained as

$$S = \frac{H_a}{H_{\rm SBBN}} = \frac{1+w_{\rm rad}}{1+\left(\frac{w_{\rm rad}+\alpha\sqrt{1+3w_{\rm rad}^2}}{1+\alpha\frac{4w_{\rm rad}}{\sqrt{1+3w_{\rm rad}^2}}}\right)}, \tag{3.48}$$

where SBBN is abbreviation for the Standard Big Bang Nucleosynthesis. The primordial abundances of the light elements (primordial D, ^{3}He, ^{4}He, ^{7}Li, T) depend on the baryon density and the expansion rate of the universe [141, 142]. The baryon density parameter is given by [141]

$$\eta_{10} \equiv 10^{10}\eta_B \equiv 10^{10}\frac{n_B}{n_\gamma}, \tag{3.49}$$

where η_B gives the baryon to photon ratio and we can take $\eta_{10} \simeq 6$ [145]. Here we will use the expression for the ^{4}He abundance given in [146, 147]

$$Y_p = 0.2485 \pm 0.0006 + 0.0016[(\eta_{10} - 6) + 100(S - 1)]. \tag{3.50}$$

As is seen from the equation above, for $S = 1$ we get the SBBN helium fraction which is $Y_p^{SBBN} = 0.2485 \pm 0.0006$. We will choose the parameter α of our model to fit the observed abundances of Helium,

$$0.2561 \pm 0.0108 = 0.2485 \pm 0.0006 + 0.0016((\eta_{10} - 6) + 100(S - 1)). \tag{3.51}$$

Since $\eta_{10} \simeq 6$, we find that the desired value of S must be $S = 1.0475 \pm 0.1050$ in order the Helium abundances to fit the observation. Moreover during the primordial Nucleosynthesis the universe was in radiation dominated era where the value of equation of state parameter is $w = \frac{1}{3}$. From Equation 3.32 we have $w_{eff} = \frac{1}{3} + 0.77\alpha + O(\alpha^2)$ and using this in Equation 3.51 we found the limit for $\alpha = -0.0785$. This model provides a great opportunity to solve the the problem about the difference between the observation (slightly greater than SBBN predictions) and SBBN predictions without proposing a new neutrino species or new physics (NP). Fine tuned α slightly decrease the Helium abundance and also gives a better deuterium abundance predictions without changing the standard system drastically.

3.1.4.2. Deuterium and Lithium-7 abundances. In order to calculate the Deuterium abundance we will use the expression based on a numerical best fit given in [141]

$$y_{Dp} = 2.60\,(1 \pm 0.06)\left(\frac{6}{\eta_{10} - 6(S - 1)}\right)^{1.6}. \tag{3.52}$$

To find the SBBN value of y_{Dp} we put $S = 1$ and $\eta_{10} \simeq 6$ which gives the value of y_{Dp} as $y_{Dp}^{SBBN} = 2.60 \pm 0.16$. By the same line of reasoning we have in the previous section we will find the value of S for which the Deuterium abundance fits the observation. From the Table I we see that the observed value of Deuterium abundance is $y_{Dp} = 2.88 \pm 0.22$.

Models, Data, Abundances	Y_p	y_{Dp}	y_{Lip}
Observational data:	0.2561 ± 0.0108[148]	2.88 ± 0.22[149]	$1.1 - 1.5$[150]
SBBN model:	0.2485 ± 0.0006	2.60 ± 0.16	4.82 ± 0.48
Model I($\alpha = -0.08$)	0.2562 ± 0.0083	2.81 ± 0.17	4.59 ± 0.44

Table 3.1. The abundances He-4, Deuterium and Li-7 for different models.

Using Equation 3.52 and by equating it with the observed value

$$2.88 \pm 0.22 = 2.60\,(1 \pm 0.06)\left(\frac{6}{\eta_{10} - 6(S-1)}\right)^{1.6} \tag{3.53}$$

we find $S = 1.062 \pm 0.444$ and using Equation 3.48 our parameter turns out to be $\alpha = -0.10109$. Now we will fix the value of parameter of our model as $\alpha = -0.08$ and calculate the abundances by using this new value which can be seen in Table I. For y_{Lip}, we will use the expression based on a numerical best fit given in [141] as

$$y_{Lip} = 4.82(1 \pm 0.10)(\frac{\eta_{10} - 3(S-1)}{6})^2. \tag{3.54}$$

It can be clearly seen from the Table I, Lithium-7 abundance remains still a problem for this model. Although both SBBN and our predictions are far from the observations, y_{Lip} is found slightly better fit to observations.

3.1.5. Equation of Motion of Test Particles for both Models I and II

We will define projection tensor as $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$ which is orthogonal to the four-velocity of the test particles $h_{\mu\nu}u^\mu = 0$. We will use the energy-momentum tensor of the perfect fluid for our massive test particles given by Equation 3.15,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{3.55}$$

where u^μ is the four-velocity of the massive test particles satisfying $u_\mu u^\mu = -1$. If we now take the covariant divergence of Equation 3.55 we get,

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu(\rho+p)u_\mu u_\nu + (\rho+p)u_\mu \nabla^\mu u_\nu + (\rho+p)u_\nu \nabla^\mu u_\mu + \nabla^\mu p g_{\mu\nu}$$

Multiplying the above equation with $h^{\nu\lambda}$ and using $h_{\mu\nu}u^\mu = 0$ one finds

$$h^{\nu\lambda}\nabla^\mu T_{\mu\nu} = (\rho+p)h^{\nu\lambda}u_\mu \nabla^\mu u_\nu + h^\lambda_\mu \nabla^\mu p.$$

Contracting the above equation with $g_{\lambda\alpha}$ we get,

$$g_{\lambda\alpha}h^{\nu\lambda}\nabla^\mu T_{\mu\nu} = (\rho+p)u_\mu \nabla^\mu u_\alpha + \nabla^\mu p h_{\alpha\mu}$$

where we have used the condition $u_\nu \nabla^\mu u^\nu = 0$ which can be obtained from the covariant divergence of $u_\mu u^\mu = -1$.

3.1.5.1. Model I. If we take the covariant divergence of Equation 3.27 and apply the same operations that we did above we get

$$g_{\lambda\alpha}h^{\nu\lambda}\nabla^\mu T_{\mu\nu} = -\alpha h_{\alpha\mu}\nabla^\mu(\sqrt{\rho^2+3p^2}) - \alpha\frac{\rho^2+4\rho p+3p^2}{\sqrt{\rho^2+3p^2}}u_\mu \nabla^\mu u_\alpha.$$

Thus we have,

$$u^\mu \nabla_\mu u^\alpha = -\frac{\alpha\nabla_\mu(\sqrt{\rho^2+3p^2}) + \nabla_\mu p}{\rho+p+\alpha\frac{(\rho+3p)(\rho+p)}{\sqrt{\rho^2+3p^2}}}h^{\mu\alpha} = f^\alpha \tag{3.56}$$

As can be seen from the above equation, there is an extra force f^α acting on them which is orthogonal to their four-velocity $f^\alpha u_\alpha = 0$ causing non-geodesic motion. When the parameter α of the model vanishes, this extra force reduces to the form of the standard general relativistic fluid motion, i.e.,

$$f^\alpha = -\frac{\nabla_\mu p}{\rho+p}h^{\mu\alpha}.$$

3.1.5.2. Model II. Now if we take the covariant divergence of Equation 3.37 and apply the same operations that we did for model I we get

$$u^\mu \nabla_\mu u^\alpha = -\frac{2\alpha\nabla_\mu(\rho^2+3p^2)^{1/4} + 32\pi G\nabla_\mu p}{32\pi G(\rho+p) + \alpha\frac{(\rho^2+4\rho p+3p^2)}{(\rho^2+3p^2)^{1/4}}} h^{\mu\alpha} = f^\alpha. \tag{3.57}$$

Note that the resulting four-force is orthogonal to the four-velocity of the particles as in Model I and it again becomes

$$f^\alpha = -\frac{\nabla_\mu p}{\rho+p} h^{\mu\alpha}.$$

which is the standart general relativistic fluid motion when the parameter α goes to zero.

3.1.6. Action for a Free Particle

In General Relativity dynamics of a free test particle can be determined by varying the action given by

$$S = -m\int d\tau = -m\int\sqrt{-g_{\mu\nu}dx^\mu dx^\nu}, \tag{3.58}$$

where m is the mass of the test particle. Variation of the above equation gives us the usual geodesic equation

$$u^\alpha\nabla_\alpha u^\mu = 0. \tag{3.59}$$

In what follows we will first make an assumption that the action governing the dynamics of test particles in a spacetime governed by our gravity model is

$$S = -m\int\sqrt{Q}\sqrt{-g_{\mu\nu}u^\mu u^\nu}d\tau, \tag{3.60}$$

where proper time τ is used as a parametrization variable. Then we will show that the variation of Equation 3.60 gives us the same form with that of Equation 3.56 [151]. To prove this result we start with the Lagrange equations corresponding to the action given in Equation 3.60 and demand that

$$\delta S = \delta \int \sqrt{Q}\sqrt{-g_{\mu\nu}u^{\mu}u^{\nu}}d\tau = 0. \tag{3.61}$$

Hence

$$\delta S = \int \left(\frac{Q_{,\alpha}}{2\sqrt{Q}}\delta x^{\alpha} + \frac{1}{2}\sqrt{Q}\big(- g_{\mu\nu,\alpha}\,\delta x^{\alpha}u^{\mu}u^{\nu} - g_{\mu\nu}\frac{d}{d\tau}(\delta x^{\mu})u^{\nu} - g_{\mu\nu}\frac{d}{d\tau}(\delta x^{\nu})u^{\mu}\big) \right) d\tau = 0 \tag{3.62}$$

After integration by parts we obtain the equations of motion of the particle as

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\lambda}u^{\nu}u^{\lambda} + (u^{\mu}u^{\nu} + g^{\mu\nu})\,\nabla_{\nu}\ln\sqrt{Q} = 0, \tag{3.63}$$

or

$$u^{\alpha}\nabla_{\alpha}u^{\mu} = -(u^{\mu}u^{\nu} + g^{\mu\nu})\nabla_{\nu}\ln\sqrt{Q}, \tag{3.64}$$

which has the same form as that of Equation 3.56.

3.1.6.1. Model I. Identifying Equation 3.64 with Equation 3.56 we have,

$$\nabla_{\mu}\ln\sqrt{Q} = \frac{\alpha\nabla_{\mu}\sqrt{\rho^2+3p^2} + \nabla_{\mu}p}{\rho + p + \alpha\frac{(\rho+3p)(\rho+p)}{\sqrt{\rho^2+3p^2}}} \tag{3.65}$$

We will use the EoS of the form $p = w\rho$ for the pressure of the fluid where w satisfies the condition $w \ll 1$. Therefore the conditions such as $\rho + p \sim \rho$ and $\rho^2 + p^2 \sim \rho^2$ are considered and when these assumptions are used in RHS of Equation 3.65,

$$\frac{\alpha\nabla_{\mu}\rho + w\nabla_{\mu}\rho}{\rho + \alpha\rho} = \frac{\nabla_{\mu}\rho}{\rho}\left(\frac{\alpha + w}{1+\alpha}\right) = \nabla_{\mu}ln\rho^{\frac{\alpha+w}{1+\alpha}} \tag{3.66}$$

is obtained. Comparing with Equation 3.65 we obtain

$$\sqrt{Q} = (c\rho)^{\frac{\alpha+w}{1+\alpha}}, \tag{3.67}$$

where c is an arbitrary constant of integration. The above equation can be written as $\sqrt{Q} = 1 + (\frac{\alpha+w}{1+\alpha})ln(c\rho) = 1 + U(\rho)$. Thus $U(\rho)$ is determined as

$$U(\rho) = (\frac{\alpha + w}{1 + \alpha})ln(c\rho). \tag{3.68}$$

We use this to find the extra term appearing in the Newtonian limit of the model. Starting from the usual line element in GR, we know that

$$ds = \sqrt{-g_{\mu\nu}dx^{\mu}dx^{\nu}} = \sqrt{-g_{00}dt^2 - 2g_{0i}dtdx^i - g_{ij}dx^i dx^j}. \tag{3.69}$$

In the weak field approximation the metric components are $g_{00} = -(1 + 2\phi)$ and $g_{ij} = (1 - 2\phi)$. Thus the above equation to the first order becomes,

$$ds \sim (1 + 2\phi - \vec{v}^2)^{1/2}dt \sim (1 + \phi - \vec{v}^2/2)dt, \tag{3.70}$$

where $\vec{v}$ is the velocity of the fluid. For the action given in Equation 3.60, using the weak field approximation corresponding equation becomes

$$S = \int \sqrt{Q}\sqrt{g_{\mu\nu}u^{\mu}u^{\nu}} \sim \int (1 + U(\rho) + \phi - \frac{\vec{v}^2}{2})dt \tag{3.71}$$

whose variation gives us the equation of motion of the fluid to the first order approximation

$$\delta \int (1 + U(\rho) + \phi - \frac{v^2}{2})dt = 0. \tag{3.72}$$

The total acceleration of the system, $\vec{a}$, is given as

$$\vec{a} = -\vec{\nabla}\phi - \vec{\nabla}U(\rho) = \vec{a}_N + \vec{a}_p + \vec{a}_E, \tag{3.73}$$

where $\vec{a}_N = -\vec{\nabla}\phi$ is the Newtonian acceleration, $\vec{a}_p$ is the hydrodynamical acceleration and $\vec{a}_E$ is the supplementary acceleration induced by the matter-matter coupling. Using

Equation 3.68 we get

$$\vec{a}_p + \vec{a}_E = -\vec{\nabla} U(\rho) = -c\frac{\nabla p}{(1+\alpha)\rho} - c\frac{\alpha}{1+\alpha}\frac{\nabla \rho}{\rho}, \tag{3.74}$$

where the constant c can be chosen as $1+\alpha$ to have the hydrodynamical acceleration. With this choice, we get

$$\vec{a}_E = -\alpha\frac{\vec{\nabla}\rho}{\rho}, \tag{3.75}$$

which shows that if the energy density of the fluid is constant then the extra acceleration $\vec{a}_E$ is zero. So our predictions will not differ from the standard GR and we do not put into constraint on the model parameter α.

<u>3.1.6.2. Model II.</u> Likewise if we identify Equation 3.64 with Equation 3.57 we get

$$\nabla_\mu \ln\sqrt{Q} = \frac{2\alpha\nabla_\mu(\rho^2+3p^2)^{1/4} + 32\pi G\nabla_\mu p}{32\pi G(\rho+p) + \alpha\frac{(\rho^2+4\rho p+3p^2)}{(\rho^2+3p^2)^{1/4}}}. \tag{3.76}$$

With the same assumptions that we considered in the previous part, i.e., $w \ll 1$, $\rho + p \sim \rho$, and $\rho^2 + p^2 \sim \rho^2$, Equation 3.76 becomes

$$\nabla_\mu \ln\sqrt{Q} \sim \frac{2\alpha\nabla_\mu\sqrt{\rho} + 32\pi G w\nabla_\mu \rho}{32\pi G\rho + \alpha\rho^{3/2}}. \tag{3.77}$$

Following the same procedure as for Model I we get the function $U(\rho)$ as

$$U(\rho) = 2(-1+w)\ln(\alpha + 32\pi G\sqrt{\rho}) + \ln\rho. \tag{3.78}$$

from which we obtain the extra acceleration

$$\vec{a}_E = -c(\rho_0)\frac{\alpha}{\alpha\rho + 32\pi G\rho^{3/2}}\vec{\nabla}\rho \tag{3.79}$$

where $c(\rho_0)$ is a constant and ρ_0 is a fixed value of density around which we made the expansion to get the correct form of hydrodynamical accelaration $\vec{a_p} = -\frac{\vec{\nabla} p}{\rho}$. We again see from Equation 3.79 that the extra acceleration is zero as the energy density of the fluid is constant. Hence in the Newtonian limit the predictions of this model are not different than that of GR.

4. CONCLUSIONS

In this thesis we search for the cosmological effects of different coupling terms to the gravitation. The results of thesis have been published as two independent papers [153, 154]. In the first study, we couple the gauge invariant term to the scalar curvature. Doing so photon gains a barely small mass in a gauge invariant way and the coupling of photon with gravity has been first analyzed in Stueckelberg formalism.

Introducing a mass for a vector field, e.g., a mass to the photon, requires reorganization of the degrees of freedom. The mechanism to achieve this by preserving the gauge symmetry is known as the Stueckelberg mechanism. Proposing an extra scalar field as an extra degree of freedom seems similar to the usage of the JBD field in cosmology, we have introduced an action by extending this idea and constructed a cosmological model in Robertson-Walker spacetime. We have also showed that this model does not reduce to cosmological model in massive JBD theory for zero vector field.

The effective gravitational coupling in the model is determined by three dynamical parameters; the scalar and vector fields as well as the expansion rate of the universe. We have given expanding universe solutions under the assumption that the effective gravitational coupling is constant, which implies that the scalar and vector fields can be dynamical but subject to the invariability of the effective gravitational coupling. We have given two sets of solutions: the case f is constant and nonzero, which is the case similar to GR with constant gravitational coupling, and the case $f = 0$, which is an extreme case that corresponds to infinitely large effective gravitational coupling.

In the case $f = \text{constant} \neq 0$, the universe is static if the mass term of the Stueckelberg fields is null. If the mass term has a positive real value, then the universe exhibits either a de Sitter expansion or a ΛCDM type expansion but with a different power. We showed that these two solutions predict a certain amount of negative vacuum energy, and that while the former solution is matter source-free, the latter solution involves radiation/relativistic fluid. In the case $f = 0$, we have found a matter source free solution which can yield a behavior compatible with the inflationary cosmology (including the switch-off mechanism) provided that the mass term is positive valued, unless otherwise it gives nothing but a simple power law expansion. In particular, we obtain a universe going through a deceleration phase sandwiched by two different accelerated expansion phases provided that the vector field decays faster than the scalar field as the universe expands, which in turn implies that essentially the vector field A drives the inflation while the scalar field B gives rise to a late time acceleration. Moreover the solution allows us to set the value of the deceleration parameter to a value required for a successful primordial nucleosynthesis and the decelerating expansion phase can last for long enough time. However, although this solution gives very interesting dynamics for the universe, the effective gravitational coupling yielding infinitely large values stands as an important issue to be faced. We think that solutions giving rise to such an interesting behavior of the universe but not suffering from this issue may be obtained by allowing the effective gravitational coupling ($G = wm^2/2\pi f^2$) to be a particular function of time such that it will start with infinitely large values but will then approach to a non-zero value by changing slowly enough after the end of inflation consistently with the observational constraints. We are working for such solutions as the extension of this work and our results will be reported elsewhere.

In the second work, we proposed a new gravity model where matter is non-minimally coupled to geometry via the contraction of the energy-momentum tensor with itself. We derived the field equations in metric formalism and for some cases we examined the cosmological implications. We had two considerations, in Model II, G also couples the additional term as a difference from that of Model I and we concluded that only matter gives rise to the accelerated expansion without any need for the cosmological constant or hypothetical fluid. This case is similar to the Cardassian expansion model and we realized that the addition of the norm of energy-momentum tensor to the action automatically gives the expansion in an accelerated way. The condition that satisfies the energy conservation in our proposed model requires β to be 1. We checked whether model I can be in accordance with the observations or not. By using the observational values of primordial abundances of light elements during the BBN we have put a constraint on the parameter of Model I and by fixing it we recalculated the Helium and Deuterium abundances. This means that we are free to fine-tune α to Helium and Deuterium abundance without proposing New Physics (NP). We derived the equation of motion of massive test particles and showed that a force is acted upon them resulting in non-geodesic motion. By finding the Newtonian limit we showed that our models can not be tested by the precession of the perihelion of Mercury and there is no new limit on the model parameter α.

APPENDIX A: THE FRIEDMANN-ROBERTSON-WALKER METRIC WITH DIFFERENT SIGN CONVENTIONS

A.1. (+,-,-,-)

In the second part the geometry of space-time is described by the FRW metric, and for flat space-like sections given by

$$ds^2 = +dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{A.1}$$

where $a(t)$ is the scale factor of the universe. For this sign convention non-zero Christoffel symbols are

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a\dot{a}, \qquad \Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}. \tag{A.2}$$

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{11} = R_{22} = R_{33} = a\ddot{a} + 2\dot{a}^2. \tag{A.3}$$

The Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$ and we obtain

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{A.4}$$

Finally, the components of the Einstein tensors are calculated via

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{A.5}$$

and the non-zero diagonal components are written as

$$G_{00} = 3\frac{\dot{a}^2}{a^2}, \quad G_{11} = G_{22} = G_{33} = -\dot{a}^2 - 2a\ddot{a}. \tag{A.6}$$

A.2. (-,+,+,+)

In the first and the third parts the geometry of space-time is described by the FRW metric, and for flat space-like sections given by

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{A.7}$$

where $a(t)$ is the scale factor of the universe. For this sign convention non-zero Christoffel symbols are

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a\dot{a}, \quad \Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}. \tag{A.8}$$

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{11} = R_{22} = R_{33} = a\ddot{a} + 2\dot{a}^2. \tag{A.9}$$

The Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$ and we obtain

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{A.10}$$

Finally, the components of the Einstein tensors are calculated via

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{A.11}$$

and the non-zero diagonal components are written as

$$G_{00} = 3\frac{\dot{a}^2}{a^2}, \quad G_{11} = G_{22} = G_{33} = -\dot{a}^2 - 2a\ddot{a}. \tag{A.12}$$

APPENDIX B: VARIATION OF GRAVITY COUPLED STUECKELBERG ACTION

$$
\begin{aligned}
S = \int \mathrm{d}^4x\sqrt{-g}\Big[-\frac{1}{8\omega m^2}(mB+\nabla_\mu A^\mu)^2 R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\
+\frac{1}{2}\left(\nabla_\mu B - mA_\mu\right)\left(\nabla^\mu B - mA^\mu\right) - \frac{1}{2}\left(mB+\nabla_\mu A^\mu\right)^2\Big] + S_{\mathrm{M}}.
\end{aligned} \tag{B.1}
$$

Denoting

$$
f = mB + \nabla_\mu A^\mu \tag{B.2}
$$

simplifies the action in Equation B.1 and by varying this we have

$$
\begin{aligned}
\delta S = \int \mathrm{d}^4x \Big[\delta(\sqrt{-g})\Big(-\frac{Rf^2}{8\omega m^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\nabla_\mu B - mA_\mu\right)\left(\nabla^\mu B - mA^\mu\right) \\
-\frac{f^2}{2}\Big) + \sqrt{-g}\Big(-\frac{R}{4\omega m^2} f\delta f - \frac{f^2}{8\omega m^2} g^{\mu\nu}\delta R_{\mu\nu} - \frac{f^2}{8\omega m^2}\delta g^{\mu\nu} R_{\mu\nu} \\
-\frac{1}{4}\delta\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{2}\delta\left(\left(\nabla_\mu B - mA_\mu\right)\left(\nabla^\mu B - mA^\mu\right)\right) - f\delta f\Big)\Big] + \delta S_{\mathrm{M}}.
\end{aligned} \tag{B.3}
$$

where

$$
\delta f = \delta g^{\mu\nu}\nabla_\nu A_\mu + \nabla_\mu(\delta g^{\mu\nu})A_\nu - \frac{1}{2}(\nabla^\alpha \delta g^{\mu\nu})A_\alpha g_{\mu\nu}. \tag{B.4}
$$

Three terms must be calculated carefully in action variation given in Equation B.3.

i)

$$\begin{aligned}\int d^4x\sqrt{-g}f\delta fR &= \int d^4x\sqrt{-g}fR\Big(\delta g^{\mu\nu}\nabla_\nu A_\mu + \nabla_\mu(\delta g^{\mu\nu})A_\nu \\ &\qquad -\frac{1}{2}\nabla^\alpha(\delta g^{\mu\nu})A_\alpha g_{\mu\nu}\Big) \\ &= \int d^4x\sqrt{-g}\Big(\delta g^{\mu\nu}fR\nabla_\nu A_\mu - \nabla_\mu(fRA_\nu)\delta g^{\mu\nu} \\ &\qquad +\frac{1}{2}\nabla^\alpha(fRA_\alpha)\delta g^{\mu\nu}g_{\mu\nu}\Big) + \text{boundary terms} \\ = \int d^4x\sqrt{-g}\delta g^{\mu\nu}&\Big(fR(\nabla_\nu A_\mu - \nabla_\mu A_\nu) - A_\nu(\partial_\mu f)R - A_\nu f(\partial_\mu R) \\ &+\frac{1}{2}g_{\mu\nu}\nabla^\alpha A_\alpha fR + \frac{1}{2}g_{\mu\nu}(\partial^\alpha f)A_\alpha R + \frac{1}{2}g_{\mu\nu}f(\partial^\alpha R)A_\alpha\Big)\end{aligned} \tag{B.5}$$

ii)

$$\begin{aligned}&\int d^4x\sqrt{-g}f^2\nabla_\sigma\Big[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla^\lambda(\delta g^{\sigma\lambda})\Big] \\ &= \int d^4x\sqrt{-g}\Big[-(\nabla_\sigma f^2)(g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla^\lambda(\delta g^{\sigma\lambda})\Big] \\ &= \int d^4x\sqrt{-g}\Big[\Box f^2 g_{\mu\nu}\delta g^{\mu\nu} \\ &-\nabla_\lambda\nabla_\sigma f^2\delta g^{\sigma\lambda}\Big] = \int d^4x\sqrt{-g}\delta g^{\mu\nu}\Big[(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f^2\Big] \\ &\qquad + \text{boundary terms}\end{aligned} \tag{B.6}$$

iii)

$$\begin{aligned}&\int d^4x\sqrt{-g}f\Big(\delta g^{\mu\nu}\nabla_\nu A_\mu + \nabla_\mu(\delta g^{\mu\nu})A_\nu - \frac{1}{2}(\nabla^\alpha\delta g^{\mu\nu})A_\alpha g_{\mu\nu}\Big) \\ &= \int d^4x\sqrt{-g}\Big(\delta g^{\mu\nu}f\nabla_\nu A_\mu - (\nabla_\mu f)\delta g^{\mu\nu}A_\nu \\ &\qquad -f\nabla_\mu A_\nu\delta g^{\mu\nu} + \frac{1}{2}f\nabla^\alpha A_\alpha\delta g^{\mu\nu}g_{\mu\nu} \\ &\qquad +\frac{1}{2}A_\alpha\delta g^{\mu\nu}g_{\mu\nu}\partial^\alpha f\Big)\end{aligned} \tag{B.7}$$

When these three terms are put into Equation B.3 we get

$$\begin{aligned}
\delta S &= \int d^4x\sqrt{-g}\delta g^{\mu\nu}[\frac{1}{16\omega m^2}g_{\mu\nu}f^2R + \frac{1}{8}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \\
&-\frac{1}{4}(\partial_\alpha B - mA_\alpha)^2 g_{\mu\nu} + \frac{1}{4}g_{\mu\nu}f^2 \\
&-\frac{1}{4\omega m^2}\Big(- A_\nu(\partial_\mu f)R - A_\nu f(\partial_\mu R) \\
&+\frac{1}{2}g_{\mu\nu}\nabla^\alpha A_\alpha fR + \frac{1}{2}g_{\mu\nu}(\partial^\alpha f)A_\alpha R + \frac{1}{2}g_{\mu\nu}f(\partial^\alpha R)A_\alpha\Big) - \frac{f^2}{8\omega m^2}R_{\mu\nu} \\
&-\frac{1}{8\omega m^2}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f^2 - \frac{1}{2}(\partial_\mu B - mA_\mu)(\partial_\nu B - mA_\nu) \\
&-f(\nabla_\mu A_\nu - \nabla_\nu A_\mu) + A_\nu\nabla_\mu f - \frac{1}{2}fg_{\mu\nu}\nabla^\alpha A_\alpha - \frac{1}{2}A_\alpha g_{\mu\nu}\partial^\alpha f] \\
&+\delta S_M
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
\delta S &= -\frac{1}{2}\Bigg[\int d^4x\sqrt{-g}\delta g^{\mu\nu}\Bigg[\frac{f^2}{4\omega m^2}G_{\mu\nu} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + \frac{1}{2}(\partial_\alpha B - mA_\alpha)^2 g_{\mu\nu} \\
&-\frac{1}{2}g_{\mu\nu}f^2 + \frac{R}{2\omega m^2}(-A_\nu\partial_\mu f + \frac{1}{2}g_{\mu\nu}\nabla^\alpha A_\alpha f + \frac{1}{2}g_{\mu\nu}(\partial^\alpha f)A_\alpha) \\
&+\frac{1}{2\omega m^2}(-A_\nu f\partial_\mu R + \frac{1}{2}g_{\mu\nu}fA_\alpha\partial^\alpha R) + \frac{1}{4\omega m^2}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f^2 \\
&+F_\nu^\beta F_{\beta\mu} - (\partial_\mu B - mA_\mu)(\partial_\nu B - mA_\nu) - 2A_\nu\nabla_\mu f + g_{\mu\nu}f\nabla^\alpha A_\alpha \\
&+g_{\mu\nu}\partial^\alpha fA_\alpha\Bigg] - 2\delta S_M\Bigg]
\end{aligned} \tag{B.9}$$

Following this way Equation 2.7 is obtained.

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