INTRODUCTION TO OPEN QUANTUM SYSTEMS

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ABSTRACT

INTRODUCTION TO OPEN QUANTUM SYSTEMS

The purpose of this study is to introduce open quantum systems to the reader. The study should not be taken as a comprehensive resource on the subject. In the study, some of the fundamental concepts of quantum mechanics have been given with a different point of view; thus, it is intended to take the reader beyond the orthodox attitude of standard quantum mechanics. Here, we will start with the introduction of basic tools to understand the open quantum systems. In the study, system-plus-reservoir model, which is the most fundamental model to understand the nature of open systems, will be introduced. Then, quantum evolutions will be generalized and how non-unitary evolution will be defined will be shown. In this study, only markovian open systems will be introduced. Finally, the example of quantum harmonic oscillator, which is isomorphic to most quantum systems, will be given as a physical example for explaining the dynamics of open systems.

ÖZET

AÇIK KUANTUM SİSTEMLERE GİRİŞ

Bu tezin amacı açık kuantum sistemlerle okuyucuyu tanıştırmaktır. Tez konuyla ilgili kapsamlı bir kaynak olma iddiasında değildir. Tezde kuantum mekaniğinin bazı temel kavramları farklı bir bakış açısıyla verilmiştir, bununla okuyucunun standart kuantum mekaniğinin ortodoks tavrının ötesine geçmesi hedeflenmiştir. Burada açık kuantum sistemlerin anlaşılması için temel araçları tanıtarak başlayacağız. Tezde açık sistemlerin doğasını anlamada kullanılan en temel model olan sistem-artı-ısı banyosu modeli tanıtılacaktır. Ardından kuantum evrimleşmeler genelleştirilerek üniter olmayan evrimleşmelerin nasıl tanımlanacağı gösterilecektir. Tezde, sadece markof ozelliklerine sahip açık sistemler incelenmiştir. Son olarak açık sistemlerin dinamiğinin anlaşılması için, fiziksel örnek olarak bir çok kuantum sisteme izomorfik olan kuantum harmonik salınıcı örneği verilecektir.

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LIST OF SYMBOLS

$\{\lvert 0\rangle, \lvert 1\rangle\}$	A basis of TLS
$\mathbb{1}$	Identity operator
a	Annihilation operator
$a^\dagger$	Creation operator
$\{\lvert a\rangle\}$	An orthonormal basis set
$\lvert a\rangle\langle a\rvert$	Projection onto $\lvert a\rangle$
A^{-1}	Inverse of A
$A^\dagger$	Hermitian adjoint of A
$\tilde{A}(t)$	Interaction operator
$A_H(t)$	Heisenberg operator
A_k	Kraus operator
$[A, B]$	The operator commutator
$\{A, B\}$	The operator anticommutator
$\mathbb{C}$	Set of Complex Numbers
E_n	n$_{th}$ energy eigenvalue
$g(\omega)$	Density of states
H	Hamiltonian
$\mathcal{H}$	Hilbert Space
$\mathcal{H}^*$	Dual of $\mathcal{H}$
H_I	Interaction Hamiltonian
$\hat{n}$	Number operator
$\mathbb{P}$	Projection operator
$P(.)$	A probability function
Q	System of interest
R	Reservoir
T	The chronological operator
T_C	The correlation time
T_Q	The characteristic life time of a system Q

$tr(.)$	Trace operation
$tr_{\omega}(.)$	Partial-trace operation
U	Unitary operator
$U(t)$	Unitary time evolution operator
$X(t)$	Stochastic variable
Z	Partition function
Z^*	Complex conjugate of Z
δt	A coarse graining time
κ_{α}	Coupling constant
ρ	Density operator
σ_x	Pauli-X operator
σ_y	Pauli-Y operator
σ_z	Pauli-Z operator
Φ	Superoperator
$\vert\phi\rangle\langle\psi\vert$	Outer product
$\vert\psi\rangle$	Ket vector
$\langle\psi\vert$	Bra vector
$\langle\psi\vert\phi\rangle$	Inner product
$\Vert \, \vert\psi\rangle \, \Vert$	Norm of $\vert\psi\rangle$
$\vert\psi\rangle_H$	Heisenberg state
$\vert\tilde{\psi}\rangle(t)$	Interaction state vector
$\otimes$	Tensor product

LIST OF ACRONYMS/ABBREVIATIONS

h.c.	Hermitic conjugate
TLS	Two Level System

1. INTRODUCTION

No system in the universe -maybe neither the universe itself- is closed. All systems are in interaction with the rest of the universe. These systems are called open systems. For example, the coffee in your cup is a good example of an open system. However, in theoretical physics the notion of closed system is very important; the rudimentary versions of physics laws are verified very well by systems of this type. If you want to estimate for how long your coffee will remain warm, i.e. for how long you are going to enjoy drinking it, considering it as a subsystem of a closed system on which physics rules will make very good estimations, will ease your job.

For example, you can take the coffee you drink in your warm room on a cold winter day, as the subsystem of the closed room system. If you go out with it for a walk, the only place your coffee will be a subsystem of and that may be accepted as closed will be a cold city. But note that you are just interested in your coffee. In order to get the most accurate information on your coffee's temperature, you have to eliminate as much as possible the degrees of freedom of the system you accept as closed. For an open system, we have provided the definition of "a system that interacts with the rest in the universe"; however, if you are outside you can start with eliminating the world except for the street you are in. The better you isolate your coffee, the slower its condition will change. In order to enjoy your coffee longer, you should prefer a thermos mug, which is quite stylish nowadays. It may be considered as a good approximation to accept the coffee as totally closed for 1 minute in a mug that can keep your coffee warm for 1 hour, and as a much better approximation to accept it closed for 1 second.

In physics, we can define a closed system as one that is completely under our control. In fact, in the example above, we have maximized our control by putting our coffee in an insulated mug, thus we have minimized the environmental impact. We can clearly state that, in classical physics, taking the system as a closed one provides you with highly accurate information on the system. However, taking the quantum systems, which live in the microscopic world that is too small for us to experience, completely under control is impossible even with the most advanced experiment techniques. This is why all quantum systems should be considered as open systems. This type of open systems comprises the subject of this study. Let's mention the closed quantum system assumption in order to better understand the open quantum systems. Standard "Introduction to Quantum Mechanics" lessons are largely concerned with closed systems. The most important assumption regarding the closed systems is about the environment outside of the system. The environment of the closed system is another system that does not exhibit quantum-mechanical properties. To put it more simply, we consider the environment of the closed quantum system as a classic system, which we take under complete control. The closed system is fully defined by the corresponding Hamiltonian. In case of an open system, our assumption about the environment is completely different. The environment of the open quantum system is another system that exhibit quantum-mechanical properties. The result of the interaction between a quantum system and another one is very different than the result of the interaction between two classic systems. When quantum systems interact, we come across an exceptional phenomenon called entanglement. Two quantum systems become entangled when they interact with each other dynamically. Entanglement is the heart of quantum computing.

Quantum information theory has begun to develop quickly in the last 30 years. In fact, quantum information theory was the inevitable result of the fact that quantum mechanics is a probabilistic theory. Open quantum systems have a key role in quantum information theory and quantum computing. When the system interacts with the environment, its quantum-mechanical properties are erased by the environment. In case of entanglement, the environment's quantum-mechanical properties will have been also erased by the system of interest. We call this phenomenon "decoherence". In the future, when the open systems are better understood, we will control the quantum systems better to prevent decoherence and be able to develop quantum computer. Let's remember the coffee example: what we were interested in was only the condition of the coffee. In the open systems theory the basic idea is to define an open system picture in which we exclude the dynamic properties of the environment as much as possible. In addition, it is important for you to know that the theory of open systems is based firmly on standard quantum mechanics. We were looking for a composite system that can be accepted as closed, to examine the state of the coffee. In order to extend the standard mechanics to the open systems, we define a larger closed system, which the open system is a subsystem of. Such a system can always be found with a good approximation.

In this study, it is intended to make a quick and simple start to quantum open systems. It is targeted to close the gap between standard quantum mechanics and open quantum systems theory, this way. Furthermore, some concepts of quantum mechanics have been re-discussed and attributed new meanings. Here, markovian open systems, which are the simplest example of open systems, have been introduced. This is important for providing a pedagogic introduction. In the second chapter of the study, mathematical tools that are important for the open systems theory have been given and physical considerations have been developed. In the third chapter, an introduction to the open systems has been presented in its most common form. In addition, the evolution of quantum system are defined with superoperator formalism in its most common form, has been shown. In the fourth chapter, Stochastic processes and a short introduction to the classical markov processes have been presented and markovian master equations that describe the dynamics of the open systems have been derived through two different ways. Then, in the chapter five, an example of open quantum harmonic oscillator has been solved explicitly for explaining the physical foundations of markovian approach.

2. TOOLS FOR OPEN QUANTUM SYSTEMS

2.1. Basics of Quantum Mechanics

In this first section, I wish to provide a rapid review of the formalism of quantum mechanics.

2.1.1. Hilbert Spaces

The formalism of quantum mechanics is built on the theory of Hilbert Spaces. Identical quantum systems are associated with the same type of Hilbert Space. A Hilbert Space $\mathcal{H}$ is a complex vector space which is equipped with the inner product [1]. In the language of Dirac, elements of $\mathcal{H}$ are called "ket" vectors, and are denoted by the "ket" symbol $| \ \rangle$, i.e. $|\psi\rangle$. The inner product of two vectors $|\psi\rangle$ and $|\phi\rangle$ in $\mathcal{H}$ is denoted $\langle\psi|\phi\rangle$, where $\langle\psi|$ is called "bra" vector. The bra vectors belong to the dual space of $\mathcal{H}$, which is denoted by $\mathcal{H}^*$.

The most important properties of a Hilbert Space consisting of a set of vectors $|\psi\rangle$,$|\phi\rangle$, $|\chi\rangle$,...and a set of complex numbers a,b,c,... is given in the list below [2]:

- Each ket vector $|\psi\rangle$ corresponds to a unique bra vector $\langle\psi|$ and vice versa:

$$(|\psi\rangle)^* = \langle\psi| \tag{2.1}$$

- Kets obey the principle of the superposition:
  (i)

$$|\psi\rangle + |\phi\rangle = |\phi\rangle + |\psi\rangle \in \mathcal{H} \tag{2.2}$$

(ii)

$$a|\psi\rangle + b|\phi\rangle \in \mathcal{H} \tag{2.3}$$

- The inner product is a map $\langle,|,\rangle$:$\mathcal{H}^*$x$\mathcal{H} \rightarrow \mathbb{C}$ satisfying the following properties:

  (i)

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* \tag{2.4}$$

  (ii)

$$\langle a\psi|\phi\rangle = a^*\langle\phi|\psi\rangle \tag{2.5}$$

  (iii)

$$\langle\psi(a|\phi + b|\chi\rangle) = a\langle\psi|\phi\rangle + b\langle\psi|\chi\rangle \tag{2.6}$$

  (iv) $\langle\psi|$ and $\langle\phi|$ are orthogonal if

$$\langle\psi|\phi\rangle = 0 \tag{2.7}$$

  (v)

$$\langle\psi|\psi\rangle \geq 0 \tag{2.8}$$

- The norm of $\langle\psi|$ is written by

$$\parallel \langle\psi| \parallel = \sqrt{\langle\psi|\psi\rangle} \tag{2.9}$$

  if $\langle\psi|\psi\rangle$ =1, $|\psi\rangle$ is called a normalised (unit) vector.

- In N-dimensional Hilbert Space $\mathcal{H}$, a set of vectors $\{|a_1\rangle, ...|a_N\rangle\}$ is said to be an orthonormal basis if

$$\langle a_i | a_j \rangle = \delta_{ij} \tag{2.10}$$

is satisfied.

From now on, I will assume that the term "basis" means an orthonormal basis.

- By the above, for any $|\psi\rangle \in \mathcal{H}$ can be decomposed as

$$|\psi\rangle = \sum_a C_a |a\rangle \tag{2.11}$$

where C$_a$ are called the components of the vector $|\psi\rangle$ in the basis $\{|a\rangle\}$. For a more general discussion of the properties of Hilbert Spaces, see [1,3].

2.1.2. State Vectors

Consider a closed system, described by the Hilbert Space $\mathcal{H}$. If we have complete information about the system, the state of the system is represented by a normalized state vector. Such a state is referred as a pure state. Suppose the system is in the state $|\psi\rangle$. Then we have the following,

$$\langle \psi | \psi \rangle = 1 \tag{2.12}$$

This is called the normalization condition.

The state vector $|\psi\rangle$ can be written as a linear combination of the basis. Let $\{|a\rangle\}$ be basis of $\mathcal{H}$, then we have,

$$|\psi\rangle = \sum_a C_a |a\rangle \tag{2.13}$$

where C$_a$'s are called the probability amplitudes of the superposition state $|\psi\rangle$ in the basis.

2.1.3. Operators

A basic formalism of quantum operators can be summarized in the list below:

- For any $|\psi\rangle$, $|\phi\rangle \in \mathcal{H}$ and a,b $\in \mathbb{C}$, an operator A is said to be linear if,

$$A(a|\psi\rangle + |\phi\rangle) = aA|\psi\rangle + bA|\phi\rangle \tag{2.14}$$

- Identity operator is denoted by $\mathbb{1}$, which transforms each vector into itself:

$$\mathbb{1}|\psi\rangle = |\psi\rangle \tag{2.15}$$

- In general, the matrix multiplication of two operators is not commutative:

$$AB \neq BA \tag{2.16}$$

- Inverse of the operator A is denoted by A^{-1} which satisfies the following relation:

$$AA^{-1} = A^{-1}A = \mathbb{1} \tag{2.17}$$

- Matrix multiplication between a ket $\mid \psi \rangle$ and a bra $\langle \phi \mid$ respectively is called the "outer product" which is denoted by $\mid \psi \rangle \langle \phi \mid$. The outer product acts as an operator.
- A state vector $\mid \psi \rangle$ is said to be an eigenstate of an operator A if,

$$A|\psi\rangle = a|\psi\rangle \tag{2.18}$$

where a $\in \mathbb{C}$ is called an eigenvalue of A. This equation is called the eigenvalue equation.

- The Hermitian adjoint of an operator A is denoted by A† and is defined as

$$\langle\psi|A|\phi\rangle^* = \langle\phi|A^\dagger|\psi\rangle \tag{2.19}$$

- An operator A is said to be Hermitian if

$$A = A^\dagger \tag{2.20}$$

- An Hermitian operator A is said to be positive if, for any $|\psi\rangle$,

$$\langle\psi|A|\psi\rangle \geq 0 \tag{2.21}$$

- The operator $\mathbb{P}$ is said to be a projection operator if,

$$\mathbb{P}^\dagger = \mathbb{P}, \mathbb{P}^2 = \mathbb{P} \tag{2.22}$$

  Consider a basis $\{|a\rangle\}$. The operator

$$\mathbb{P}_a = |a\rangle\langle a| \tag{2.23}$$

  is called the projection operator onto the basis vector $|a\rangle$.

- The commutator [A,B] is defined as,

$$[A, B] = AB - BA \tag{2.24}$$

- The anticommutator {A,B} is defined as

$$\{A, B\} = AB + BA \tag{2.25}$$

- An operator U is said to be a unitary operator if,

$$UU^\dagger = U^\dagger U = \mathbb{1} \tag{2.26}$$

- Consider a basis $\{|a\rangle\}$ of an infinite dimensional Hilbert Space. The "completeness relation" for this basis is given by

$$\sum_{a=1}^{\infty} |a\rangle\langle a| = \mathbb{1} \tag{2.27}$$

2.1.4. Observables

In quantum mechanics, every measurable quantity is represented by a Hermitian operator on $\mathcal{H}$. Consider an observable A.

- All eigenvalues of A are real.
- Eigenstates of an operator A form a basis.
- If a system is in the state $|\psi\rangle$, the expectation value of A is given by the "Born Rule":

$$\langle A\rangle = \langle\psi|A|\psi\rangle \tag{2.28}$$

- For any $|\psi\rangle \in \mathcal{H}$ and $\{|a\rangle\}$ is a basis of $\mathcal{H}$, the probability of finding the system in a_{th} basis state is given by

$$P(a) = |\langle a|\psi\rangle|^2 \tag{2.29}$$

where $P(a)$ satisfies the usual probability laws:

$$0 \leq P(a) \leq 1 \tag{2.30}$$

$$\sum_a P(a) = 1 \tag{2.31}$$

- The operators x and p, corresponding to the position and momentum of a particle, satisfy the well-known commutation relation

$$[x, p] = i\hbar \tag{2.32}$$

2.2. Time-Evolution in Closed Quantum Systems

In the present section, I will touch a few aspects of the quantum dynamics from the perspective of the main subject of thesis.

2.2.1. Unitary Time-Evolution

We begin with the concept of a closed quantum system, that is necessary to understand the unitary time evolution and, also the main text of this thesis.

A closed quantum system is the one that cannot access any physical information from the rest of the universe and vice versa. Thus, information confined to a system is conserved while a state moves in time [4]:

- Describing with physics language, a pure state evolves into another pure state.
- Mathematically speaking, the normalization condition is conserved in time evolution of a closed system.
- Closed quantum systems obey the unitary time evolution.

Consider that a closed system starts out in state $|\psi(0)\rangle$ at time zero. Let $|\psi(t)\rangle$

be the state at time t>0. We can write

$$\begin{aligned}\langle\psi(t)|\psi(t)\rangle &= \langle\psi(0)|\psi(0)\rangle \\ &= \langle\psi(0)|\mathbb{1}|\psi(0)\rangle \\ &= \langle\psi(0)|U^\dagger(t)U(t)|\psi(0)\rangle\end{aligned} \tag{2.33}$$

where we have used Equation 2.26 Then, we write

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \tag{2.34}$$

where U(t) is called a unitary time evolution operator. More generally, for given initial state $|\psi(t_0)\rangle$, a state $|\psi(t)\rangle$ at any time t>t_0 is written as,

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \tag{2.35}$$

U(t) satisfies the following properties [5]:

$$U^\dagger(t,t_0)U(t,t_0) = U(t,t_0)U^\dagger(t,t_0) = \mathbb{1} \tag{2.36}$$

$$U(t,t) = \mathbb{1} \tag{2.37}$$

$$U^\dagger(t,t_0) = U^{-1}(t,t_0) = U(t_0,t) \tag{2.38}$$

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3) \tag{2.39}$$

where t_1>t_2>t_3.

2.2.2. The Schrödinger Picture

The Schrödinger picture is useful for problems with time independent Hamiltonian. In this picture, states evolve in time while the operators are constant. The time evolution of the Schrödinger state is governed by the Schrödinger equation:

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle \tag{2.40}$$

where H is time-independent Hamiltonian. In this picture, the Schrödinger equation for U(t) can be written as

$$i\hbar\frac{dU(t)}{dt} = HU(t) \tag{2.41}$$

For time-independent Hamiltonian H, the solution of Equation 2.41 can be given as

$$U(t) = e^{-i\frac{Ht}{\hbar}} \tag{2.42}$$

2.2.3. The Heisenberg Picture

In this picture the states are constant and the operators are time dependent. The Heisenberg states can be thought as completely frozen in time [2]. They are often denoted by $|\psi\rangle_H$, such that,

$$|\psi(t)\rangle = U^{\dagger}(t)|\psi(t)\rangle = |\psi(0)\rangle \tag{2.43}$$

Since the Heisenberg states are constant, the equation of motion is derived from the Heisenberg operators. The observable A is denoted by A$_H$(t) in the Heisenberg picture.

The Heisenberg operator A$_H$(t) is decomposed in terms of U(t) and A:

$$A_H(t) = U^\dagger(t)AU(t) \tag{2.44}$$

where A corresponds to the observable in the Schrödinger picture. The Heisenberg equation of motion can be given as

$$\frac{dA_H(t)}{dt} = \frac{1}{i\hbar}[A_H(t), H] \tag{2.45}$$

where we have assumed that A does not depend explicitly on time and H is time independent [6].

If H is time dependent, the solution of Equation 2.41 can be given under two headings [5]:

- If [H(t),H(t')] = for all t,t',

$$U(t, t_0) = exp\{-\frac{i}{\hbar}\int_0^t H(t')dt'\} \tag{2.46}$$

- If [H(t),H(t')]≠0 for t≠t'

$$U(t, t_0) = Texp\{-\frac{i}{\hbar}\int_{t_0}^t H(t')dt'\} \tag{2.47}$$

where T is the chronological operator. This solution can be given more explicitly as

$$U(t, t_0) = \mathbb{1} + \sum_{n=1}^{\infty}(-\frac{i}{\hbar})^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 ... \int_{t_0}^{t_{n-1}} dt_n H(t_1)...H(t_n) \tag{2.48}$$

2.2.4. Interaction Picture

In the interaction picture, both states and operators evolve in time. This picture is particularly useful, when the Hamiltonian includes a weak, time dependent perturbation term. Suppose that the Hamiltonian can be decomposed into two parts as

$$H(t) = H_0 + H_I(t) \tag{2.49}$$

where H_0 is time independent, and H_I(t) is often called the interaction Hamiltonian. Interaction state vectors are defined as

$$|\tilde{\psi}(t)\rangle = e^{\frac{iH_0t}{\hbar}}|\psi(t)\rangle \tag{2.50}$$

Also, the interaction operators are obtained as follows:

$$\tilde{A}(t) = e^{\frac{iH_0t}{\hbar}} A e^{-\frac{iH_0t}{\hbar}} \tag{2.51}$$

where $\tilde{A}$(t) is the interaction operator, and A is the well-known Schrödinger operator. Similarly, interaction Hamiltonian H_0(t) can be written in the interaction picture as

$$\tilde{H}_I(t) = e^{\frac{iH_0t}{\hbar}} H_I(t) e^{-\frac{iH_0t}{\hbar}} \tag{2.52}$$

The time evolution of $|\tilde{\psi}(t)\rangle$ is governed by the following equation [2]:

$$i\hbar\frac{d\tilde{\psi}(t)\rangle}{dt} = \tilde{H}_I(t)|\tilde{\psi}(t)\rangle \tag{2.53}$$

The operator $\tilde{A}$(t) evolves in time as

$$\frac{d\tilde{A}(t)}{dt} = \frac{1}{i\hbar}[\tilde{A}(t), H_0] \tag{2.54}$$

where we have assumed $\frac{dA}{dt}$=0.

It is sufficient to make the following observation

- The state $|\tilde{\psi}(t)\rangle$ has a time dependence and is determined by $\tilde{H}_I(t)$ Equation 2.53.
- An operator $\tilde{A}(t)$ has a time dependence and is determined by H_0(Equation 2.54).

2.3. Tensor Product and Trace

In this section, I wish to provide a mathematical formalism which we need to formulate the physics of the open systems.

2.3.1. Tensor Product

A tensor product is a way of combining two or more Hilbert spaces, whose dimensions does not need to be the same, to form larger Hilbert spaces.

Let $\mathcal{H}^{(A)}$ and $\mathcal{H}^{(B)}$ be two Hilbert spaces whose dimensions N and M respectively. The tensor product of $\mathcal{H}^{(A)}$ and $\mathcal{H}^{(B)}$ is denoted by

$$\mathcal{H}^{(AB)} = \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)} \tag{2.55}$$

where $\mathcal{H}^{(AB)}$ is called product Hilbert space with dimension NM. If $\{|a\rangle\}$ is a basis for $\mathcal{H}^{(A)}$ and $\{|b\rangle\}$ is a basis for $\mathcal{H}^{(B)}$ then $\{|a\rangle \otimes |b\rangle\}$ forms a basis for $\mathcal{H}^{(AB)}$. Then one can see that, for any $|\psi^{(A)}\rangle \in \mathcal{H}^{(A)}$, $|\phi^{(B)}\rangle \in \mathcal{H}^{(B)}$, there is a product vector $|\psi^{(A)}\rangle \otimes |\phi^{(B)}\rangle \in \mathcal{H}^{(AB)}$, which can be written in different ways [7].

$$|\psi^{(A)}\rangle \otimes |\phi^{(B)}\rangle = |\psi^{(A)}\rangle|\phi^{(B)}\rangle = |\psi^{(A)}, \phi^{(B)}\rangle = |\psi, \phi\rangle \tag{2.56}$$

The tensor product satisfies following properties:

- for any $|\psi \in \mathcal{H}^{(A)}\rangle, |\phi \in \mathcal{H}^{(B)}\rangle$ and c$\in \mathbb{C}$,

$$c(|\psi\rangle \otimes |\phi\rangle = (c|\psi\rangle) \otimes |\phi\rangle = |\psi\rangle \otimes (c|\phi\rangle); \tag{2.57}$$

- for any $|\psi_1\rangle$,$|\psi_2\rangle \in \mathcal{H}^{(A)}$ and $|\phi\rangle \in \mathcal{H}^{(B)}$,

$$(|\psi_1\rangle + |\psi_2\rangle) \otimes |\phi\rangle = |\psi_1\rangle \otimes |\phi\rangle + |\psi_2\rangle \otimes |\phi\rangle; \tag{2.58}$$

- for any $|\psi\rangle \in \mathcal{H}^{(A)}$ and $|\phi_1\rangle$, $|\phi_2\rangle \in \mathcal{H}^{(B)}$,

$$|\psi\rangle \otimes (|\phi_1\rangle + |\phi_2\rangle) = |\psi\rangle \otimes |\phi_1\rangle + |\psi\rangle \otimes |\phi_2\rangle \tag{2.59}$$

We can define the inner product on $\mathcal{H}^{(AB)}$=$\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$ as follows:

- for any $|\psi^{(AB)} = |\psi_1^{(A)}\rangle, |\phi_1^B\rangle$,$|\phi^{(AB)}\rangle$=$|\psi_2^{(A)}\rangle, |\phi_2^B\rangle \in \mathcal{H}^{(AB)}$ an inner product

$$\langle\psi^{(AB)}|\phi^{(AB)}\rangle = \langle\psi_1^{(A)}|\psi_2^{(A)}\rangle\langle\phi_1^{(B)}|\phi_2^{(B)}\rangle \in \mathbb{C} \tag{2.60}$$

- for any $|\psi_1^{(A)}\rangle$, $|\psi_2^{(A)}\rangle \in \mathcal{H}^{(A)}$ and $|\phi^{(AB)}\rangle$=$|\psi_2^A, \phi^{(B)}\rangle \in \mathcal{H}^{(AB)}$

$$\langle\psi_1^{(A)}|\phi^{(AB)}\rangle = \langle\psi_1^{(A)}|\psi_2^{(A)}\rangle|\phi^{(B)}\rangle \in \mathcal{H}^{(B)} \tag{2.61}$$

which is called a partial inner product on $\mathcal{H}^{(AB)}$.

We may also define the tensor product of two operators acting on different Hilbert spaces. Let X$^{(A)}$ and Y$^{(B)}$ be two operators acting on $\mathcal{H}^{(A)}$ and $\mathcal{H}^{(B)}$ respectively. The tensor product X$^{(A)} \otimes Y^{(B)}$ acts ”space by space”,

$$(X^{(A)} \otimes Y^{(B)})(|\psi^{(A)}\rangle \otimes |\phi^{(B)}\rangle) = X^{(A)}|\psi^{(A)}\rangle \otimes Y^{(B)}|\phi^{(B)}\rangle \tag{2.62}$$

Finally, some useful relations are given in the list below

$$X^{(A)} \otimes Y^{(B)} = (X^{(A)} \otimes \mathbb{1}^{(B)})(\mathbb{1}^{(A)} \otimes Y^{(B)}) \tag{2.63}$$

$$X^{(A)} = X^{(A)} \otimes \mathbb{1}^{(B)} \tag{2.64}$$

$$Y^{(B)} = \mathbb{1}^{(A)} \otimes Y^{(B)} \tag{2.65}$$

$$[X^{(A)}, Y^{(B)}] = 0 \tag{2.66}$$

- for any $|\psi_1^{(A)}\rangle$, $|\psi_2^{(A)}\rangle \in \mathcal{H}^{(A)}$ and $|\phi_1^{(B)}\rangle$, $|\phi_2^{(B)}\rangle \in \mathcal{H}^{(B)}$, then

$$|\psi_1^{(A)}\rangle\langle\psi_2^{(A)}| \otimes |\phi_1^{(B)}\rangle\langle\phi_2^{(B)}| = |\psi_1^{(A)}, \phi_1^{(B)}\rangle\langle\psi_2^{(A)}, \phi_2^{(B)}| \tag{2.67}$$

is an operator on $\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$. For an extended discussion about tensor products and tensor product spaces, see [8].

2.3.2. Trace

The trace operation is a linear functional which acts on operators. The definition of trace can be given under two headings [9,10]:

- the trace transforms the outer product into the inner product,

$$tr(|\alpha\rangle\langle\beta|) = \langle\beta|\alpha\rangle \tag{2.68}$$

- If $\{|n\rangle\}$ is a basis for $\mathcal{H}$ where A acts on, the trace of A is given by

$$tr(A) = \sum_n \langle n|A|n\rangle = \sum_n A_{nn} \tag{2.69}$$

The trace operation satisfies the following properties:

- A and B on $\mathcal{H}$, and a,b$\in \mathbb{C}$,

$$tr(aA + bB) = atr(A) + btr(B) \quad (linearity); \tag{2.70}$$

$$tr(AB) = tr(BA) \quad (cyclic\ \ property). \tag{2.71}$$

The cyclic property of trace can be generalize to the product of more than two operators.

- By above,

$$\begin{aligned} tr[A, B] &= tr(AB - BA) \\ &= tr(AB) - tr(BA) \\ &= tr(AB) - tr(AB) \\ &= 0 \end{aligned} \tag{2.72}$$

$$tr(A|\psi\rangle\langle\phi|) = tr(|\psi\rangle\langle\phi|A) = \langle\phi|\ A|\psi\rangle \tag{2.73}$$

$$tr(A|\psi\rangle\langle\phi|B) = \langle\phi|BA|\psi\rangle \tag{2.74}$$

2.3.3. Partial Trace

The partial trace is an operator valued function, which acts on product operators. Let X$^{(A)}$ and Y$^{(B)}$ be operators on $\mathcal{H}^{(A)}$, $\mathcal{H}^{(B)}$ respectively. The partial trace tr$_{(A)}$ is a linear map that is determined by

$$tr_{(A)}(X^{(A)} \otimes Y^{(B)}) = Y^{(B)} tr(X) \tag{2.75}$$

If $\{|n^{(B)}\rangle\}$ is a basis for $\mathcal{H}^{(B)}$, tr$_{(B)}$ is determined by

$$\begin{aligned} tr_{(B)}(X^{(A)} \otimes Y^{(B)}) &= \sum_n \langle n^{(B)}| X^{(A)} \otimes Y^{(B)} |n^{(B)}\rangle \\ &= \sum_n \langle n|Y|n\rangle X^{(A)} \\ &= X^{(A)} \sum_n Y_{nn} \end{aligned} \tag{2.76}$$

2.4. Composite Systems

As in classical physics, also quantum systems often have an internal structure [7]. Such a system is referred to as a quantum composite system. We always assume that composite systems are closed systems consisting of two or more individually detectable subsystems.

This section is devoted to introduce a formalism whereby the state of a composite system is expressed, in terms of states of its subsystems. Without loss of generality we will restrict our attention to the composite systems consisting of only two subsystems, which is called a "bipartite system". Once this case is understood, it will be easy to generalize composite systems including more than two subsystems.

2.4.1. Product States

The state of a composite system consisting if uncorrelated subsystems can be represented as the tensor product of states of the subsystems. Such a state is called the product state.

Consider that we have a composite system AB consisting of subsystems A and B, described by the Hilbert spaces $\mathcal{H}^{(A)}$ and $\mathcal{H}^{(B)}$. Let $|\psi^{(AB)}\rangle$ be the state of AB. If A and B have been prepared independently in states $|\psi^{(A)}\rangle$ and $|\psi^{(B)}\rangle$, then the state $|\psi^{(AB)}\rangle$ can be written as

$$|\psi^{(AB)}\rangle = |\psi^{(A)}\rangle \otimes |\phi^{(B)}\rangle \tag{2.77}$$

where $|\psi^{(AB)}\rangle \in \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$.

A product state is also called an entangled or separable state.

2.4.2. Entangled States

We consider again the system AB with the subsystems A and B. If A and B interact sometime in the past, it is not possible to define state vectors to the individual subsystems in the future [9]. Therefore, the state of AB cannot be written as the tensor product of states of subsystems, such a state is called entangled state. This phenomena is called the entanglement, which was introduced by Erwin Schrödinger in 1935 [11].

It is important to realize that the interacted systems A and B erase quantum mechanical properties of each other while the system AB remains in purely quantum mechanical. Mathematically speaking, the state of AB cannot be written in the form of Equation 2.77:

$$|\psi^{(AB)}\rangle \neq |\psi^{(A)}\rangle \otimes |\phi^{(B)}\rangle \tag{2.78}$$

To illustrate this more rigorously let us look at the following abstract model constructed step by step :

- consider two systems A, B described by $\mathcal{H}^{(A)}$ and $\mathcal{H}^{(B)}$ whose bases $\{|a_k^{(A)}\rangle\}$ and $\{|b_l^{(B)}\rangle\}$ (k,l=1,2) respectively.
- at time t=0, the composite system AB is in the product state given by

  $$|\psi_0^{(AB)}\rangle = |\psi_0^{(A)}\rangle \otimes |\phi_0^{(B)}\rangle \in \mathcal{H}^{(AB)} \tag{2.79}$$

  where

  $$|\psi_0^{(A)}\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle + |a_2\rangle) \tag{2.80}$$

  and

  $$|\phi_0^{(B)}\rangle = |b_1\rangle \tag{2.81}$$

- at a time t=t_1 >0, A and B interacts each other. Since the system AB is closed, this interaction corresponds to a unitary operator on $\mathcal{H}^{(AB)}$. Suppose the unitary operator described by

  $$U^{(AB)} = |a_1\rangle\langle a_1| \otimes 1| + |a_2, b_2\rangle\langle a_2, b_1| + |a_2, b_1\rangle\langle a_2, b_2| \tag{2.82}$$

then the state of AB at any time t$>t_1$ can be calculated as:

$$\begin{aligned} U^{(AB)}|\psi_0^{(AB)}\rangle &= (|a_1\rangle\langle a_1| \otimes 1| + |a_2, b_2\rangle\langle a_2, b_1| + |a_2, b_1\rangle\langle a_2, b_2|)|\psi_0, \phi_0\rangle \\ &= \langle a_1|\psi_0\rangle|a_1, \phi_0\rangle + \langle a_2|\psi_b\rangle\langle b_1|\phi_0\rangle|a_2, b_2\rangle \\ &+ \langle a_2|\psi_0\rangle\langle b_2|\phi_0\rangle|a_2, b_1\rangle \\ &= \frac{1}{\sqrt{2}}|a_1\rangle \otimes |b_1\rangle + \frac{1}{\sqrt{2}}|a_2\rangle \otimes |b_2\rangle \end{aligned} \tag{2.83}$$

Let be $|\psi^{(AB)}\rangle$ denotes the state of AB at time t, then we obtain

$$|\psi^{(AB)}\rangle = \frac{1}{\sqrt{2}}(|a_1, b_1\rangle + |a_2, b_2\rangle) \tag{2.84}$$

The state $|\psi^{(AB)}\rangle$ is an entangled state of AB, where A and B are perfectly correlated. Obviously, in this entangled state, neither the subsystem A nor the subsystem B is in a pure state. Since the states of two subsystems are statistically dependent on each other, learning the state of A or B changes the probability of the states B or A.

2.5. A Basic Formalism of the Density Operator

Density operator formalism is fundamental to understand the main text of the thesis. In this section, I wish to give a brief introduction to the formalism. For a more comprehensive definition of the formalism, see [12].

2.5.1. Mixed States

A quantum system is said to be in a "mixed state" if we don't have complete knowledge about a system. In mixed case, a state of a system may not be known explicitly, but may be known statistically.

By the definition, we can say that the accessible information about a system contains some classical indeterminism. Indeed, the advantage of using density operators lies in the fact that the notion of a state vector breaks down to illustration of some information arising from classical uncertainty.

In general, this incomplete information can be presented in the following way:

Consider an ensemble whose identical members can be found in one of the n different pure states, $\{|\psi_\alpha\rangle\}$ with corresponding probabilities $\{|P_\alpha\rangle\}$ (α = 1,...,n). Obviously, we have

$$\sum_{\alpha=1}^{n} P_\alpha = 1 \tag{2.85}$$

Now, let's construct the density operator describing the state of an ensemble. Remember that the accessible information can be extracted by using appropriate measurements on the state. We measure the state using A. Let $\langle A\rangle_\alpha$ denotes the expectation value A measured over the state $|\psi_\alpha\rangle$.

$$\langle A\rangle_\alpha = \langle\psi_\alpha|A|\psi_\alpha\rangle \tag{2.86}$$

By analogy with statistical mechanics, one can write the expectation value of A in the state of an ensemble as a convex sum of $\langle A\rangle_\alpha$:

$$\begin{aligned}\langle A\rangle &= \sum_\alpha P_\alpha\langle\psi_\alpha|A|\psi_\alpha\rangle \\ &= \sum_\alpha P_\alpha tr(|\psi_\alpha\rangle\langle\psi_\alpha|A) \\ &= tr\{\sum_\alpha P_\alpha|\psi_\alpha\rangle\langle\psi_\alpha|A\}\end{aligned} \tag{2.87}$$

We can now define the density operator ρ to be

$$\rho = \sum_\alpha P_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| \tag{2.88}$$

Equation 2.87 can be rewritten in terms of ρ as

$$\langle A \rangle = tr(\rho A) \tag{2.89}$$

Obviously, ρ is independent of the observable, determined by the construction of an ensemble. Note that, the pure states $\{|\psi_\alpha\rangle\}$ should be normalized, but not necessarily orthogonal to each other. If each pure state $|\psi_\alpha\rangle$ can be represented by the same vector, the sate of an ensemble is called a pure state.

Basic properties of density operators can be given in the following ways [12]:

- ρ is a Hermitian

$$\rho = \rho^\dagger \tag{2.90}$$

- ρ satisfies the normalization condition

$$tr(\rho) = 1 \tag{2.91}$$

- ρ is a positive operator

$$\rho \geq 0 \tag{2.92}$$

- The criterion for determining whether a state is pure or mixed:

$$tr(\rho^2) \leq 1 \tag{2.93}$$

where the equality holds only in pure case.

2.5.2. The Reduced Density Operator

The most important application of density operator is in the description of subsystems of composite systems. Let's begin by the reformulation of composite systems in terms of density operators:

- Consider a composite system AB consisting of uncorrelated subsystems A and B, in the $\rho^{(A)}$ and $\rho^{(B)}$ respectively. Thus, we can write the state of AB as

$$\rho^{(AB)} = \rho^{(A)} \otimes \rho^{(B)} \tag{2.94}$$

  This is also called a product state.
- When AB is in an entangled state $\rho^{(AB)}$. The system cannot be described by the tensor product of density operators corresponding individual subsystems. Then, we have

$$\rho^{(AB)} \neq \rho^{(A)} \otimes \rho^{(B)} \tag{2.95}$$

As mentioned in 2.4.2, there is no way to define state vectors to describe the states of A and B in entangled case. However, we can write convenient density operators describing A and B as follows by using a partial trace operation as

$$\rho^{(A)} = tr_{(B)}\rho^{(AB)} \tag{2.96}$$

$$\rho^{(B)} = tr_{(A)}\rho^{(AB)} \tag{2.97}$$

where $\rho^{(A)}$ and $\rho^{(B)}$ are often referred as the reduced density operators.

For any pure state $\rho^{(AB)}$, the criterion to determine whether the state is entangled or product is;

$$\begin{aligned} &tr_{(A)}(\rho^{(A)})^2 \leq 1 \qquad or \\ &tr_{(B)}(\rho^{(B)})^2 \leq 1 \end{aligned} \tag{2.98}$$

where equalities hold in product case.

2.5.3. Time Evolution of Density Operators

Consider a quantum system is in the state $|\psi(t)\rangle$. Let $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ denotes the state of a system, and H is the Hamiltonian of the system. Time evolution of ρ is written by [9]

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H, \rho] \tag{2.99}$$

This called the von Neumann Equation.

If H includes a weak time dependent perturbation, such that H $=$ H$_0$ $+$ H$_I(t)$, the von Neumann equation can be written in interaction picture as [6]

$$\frac{d\tilde{\rho}(t)}{dt} = \frac{1}{i\hbar}[\tilde{H}_I(t), \tilde{\rho}(t)] \tag{2.100}$$

where

$$\tilde{\rho}(t) = e^{i\frac{H_0(t)}{\hbar}}\rho(t)e^{-i\frac{H_0(t)}{\hbar}} \tag{2.101}$$

$$\tilde{H}(t) = e^{i\frac{H_0(t)}{\hbar}}\rho(t)e^{-i\frac{H_0(t)}{\hbar}} \tag{2.102}$$

The solution of Equation 2.98 is given by

$$\tilde{\rho}(t) = \rho(0) - \frac{i}{\hbar}\int_0^t [\tilde{H}_I(t'), \tilde{\rho}(t')]dt' \tag{2.103}$$

2.6. The Two Level System and Harmonic Oscillator

In this section, I will give a brief description of two important quantum models: the two level system (TLS) and harmonic oscillator. They are often used to illustrate the behavior of open quantum systems. Extended discussion about these models can be found in standard quantum mechanics books.

2.6.1. The Two Level System

A two level system can be found in two different states. These states form a basis of 2-dimensional Hilbert space which describes the system. The basis is often denoted by $\{|0\rangle, |1\rangle\}$ and $\{|g\rangle, |e\rangle\}$:

$$|g\rangle = |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.104}$$

$$|e\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.105}$$

In quantum information theory, TLS's are called a qubit. An arbitrary qubit representation can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{2.106}$$

with $\alpha, \beta \in \mathbb{C}$.

The operators acting on the space of TLS are represented by 2x2 matrix or a summation of outer products of the basis. Some well-known TLS operators, are given below:

- Projection operator $|e\rangle$

$$\mathbb{P}_e = |e\rangle\langle e| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.107}$$

- Projection operator on $|g\rangle$

$$\mathbb{P}_g = |g\rangle\langle g| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.108}$$

- Creation TLS operator

$$\sigma_+ = |e\rangle\langle g| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.109}$$

- Destruction TLS operator

$$\sigma_- = |g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.110}$$

- Identity operator

$$\mathbb{1} = \mathbb{P}_g + \mathbb{P}_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.111}$$

- Pauli-X operator

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.112}$$

- Pauli-Y operator

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.113}$$

- Pauli-Z operator

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.114}$$

TLS operators are also called a qubit gate.

2.6.2. The Harmonic Oscillator

The Hamiltonian of the harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{2.115}$$

where x is a position operator, p is momentum operator. These operators satisfy the well-known commutation relation:

$$[x, p] = i\hbar\mathbb{1} \tag{2.116}$$

The key aspects of the harmonic oscillator formalism, which the reader should already be familiar with, can be given as follows [13]:

- annihilation operator a

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega}) \tag{2.117}$$

- creation operator at

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega}) \tag{2.118}$$

$$[a, a^\dagger] = \mathbb{1} \tag{2.119}$$

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{2.120}$$

- the number operator

$$\hat{n} = a^\dagger a \tag{2.121}$$

- The basis set of the harmonic oscillator is often denoted by $\{|n\rangle\}$.

$$H|n\rangle = E_n|n\rangle \tag{2.122}$$

where E_n denotes the n_{th} energy eigenvalue.

$$E_n = \hbar\omega(n + \frac{1}{2}) \tag{2.123}$$

$$\hat{n}|n\rangle = n|n\rangle, \quad (n = 1, 2, 3, ...) \tag{2.124}$$

$$\langle n|m\rangle = \delta_{nm} \tag{2.125}$$

$$a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \tag{2.126}$$

$$a|0\rangle = 0 \tag{2.127}$$

$$|n\rangle = \frac{(a^{\dagger})^n}{\sqrt{n!}}|0\rangle \tag{2.128}$$

$$\sum_{n=0}^{\infty} = |n\rangle\langle n| = \mathbb{1} \tag{2.129}$$

3. OPEN QUANTUM SYSTEMS

In nature, a closed quantum system does not really exist. Namely, each quantum system interacts with its environment. Thus, the notion of a closed quantum system can be assumed as an approximation only. This approximation may be physically reasonable in some cases, where the interaction between a system and an environment is so weak, which does not induce significant changes in the system during the typical experiment time. In such cases, the influence of the environment can be ignored and the system can be considered as a closed system.

On the other hand, this approximation is completely unrealistic in many cases, where the system of interest being strongly coupled to its environment. In such situation, the systems must be regarded as an open quantum system.

This chapter is devoted as a rigorous introduction in the subject of open quantum systems. We first set up our fundamental approach to open quantum systems. Then, we focus on generalized evolutions of open systems in terms of the dynamical maps.

3.1. Open-System Approach

In this section, I wish to provide a convenient representation that clarifies what an open system really means. As we know, an open system is the one that interacts with its environment. Loosely speaking, the term "environment" means the rest of the universe. In many cases, an environment can be taken as a reservoir, which is directly in contact with the system of interest rather than vast space and galaxies to high degree of accuracy. As in the literature, this concept will be our fundamental approach to the description of open quantum systems called a "system-plus-reservoir" model.

To be more precise about the description, let's define the system of interest Q and a reservoir R to which Q is coupled.

Also we say that Q and R are associated with Hilbert spaces $\mathcal{H}^{(Q)}$ and $\mathcal{H}^{(R)}$ respectively. To extend the standard quantum mechanics to open quantum systems, we need to have a closed system consisting of the system of interest Q and the reservoir R. Thus, we combine Q and R to make a composite system QR whose Hilbert space is $\mathcal{H}^{(QR)} = \mathcal{H}^{(Q)} \otimes \mathcal{H}^{(R)}$.

In most general case, the Hamiltonian of QR is written in the form

$$H^{(QR)}(t) = H^{(Q)} \otimes \mathbb{1}^{(R)} + \mathbb{1}^{Q} \otimes H^{(R)} + H_I^{(QR)}(t) \tag{3.1}$$

where

- $H^{(Q)}$ is a Hamiltonian for Q.
- $H^{(R)}$ is a Hamiltonian for R.
- $H_I^{(QR)}$(t) is a time dependent Hamiltonian describing an interaction between Q and R.

Figure 3.1 shows a schematic picture of an open quantum system.

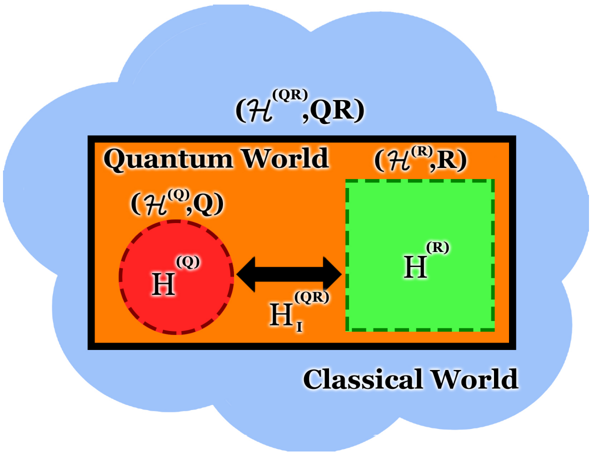

Figure 3.1. Schematic picture of an open quantum system.

Lets look at the reduced dynamics of Q, taking into account the influence of R.

Let $\rho^{(QR)}(0)$ be the density operator for QR at time zero. Since QR is closed, its state moves in the time unitarily such that

$$\rho^{(QR)}(t) = U^{(QR)}(t)\rho^{(QR)}(0)U^{(QR)t}(t) \tag{3.2}$$

where U$^{(QR)}(t)$ is a unitary time evolution operator on $\mathcal{H}^{(QR)}$. By a similar argument, we can express a unitary time evolution via the von Neumann Equation:

$$\frac{d\rho^{(QR)}}{dt} = \frac{1}{i\hbar}[H^{(QR)}(t), \rho^{(QR)}(t)] \tag{3.3}$$

where we have used Equation 3.1. We are now in a position to show the reduced dynamics of Q. By analogy with the reduced density operator, we can write the reduced dynamics of Q in the following ways:

$$\rho^{Q}(t) = tr_{(R)}\{U^{(QR)}(t)\rho^{(QR)}(0)U^{(QR)}(t)\} \tag{3.4}$$

$$\frac{d\rho^{(Q)}}{dt} = \frac{1}{i\hbar}tr_{(R)}[H^{(QR)}(t), \rho^{(QR)}(t)] \tag{3.5}$$

Since these equations still include $\rho^{(QR)}$, they are nor very applicable. From now on, we begin to present same approximations that lead us to dynamical equations which are only consisting of the reduced density operator $\rho^{(Q)}$.

3.2. Quantum Operations

In the present section, we briefly sketch the quantum operations and their relations to the theory of open quantum systems.

3.2.1. Superoperators

Consider again an open system Q and a reservoir R which Q is coupled. Even QR evolves unitarily, Q in general does not obey a unitary evolution due to the presence of R. Our goal is to construct an operation that leads us to the description of generalized quantum evolutions.

Remember that both pure and mixed states can be represented by density operators. Thus, all quantum evolutions can be regarded as a map connecting an initial density operator to a final one. We denote the map with Φ.

Let ρ and ρ' denote an initial state and an evolved state respectively. In this case, write the quantum evolution in the map form:

$$\Phi : \rho \longrightarrow \rho' \tag{3.6}$$

This relation can also be written in the quantum operation form

$$\Phi(\rho) = \rho' \tag{3.7}$$

where Φ is called a quantum operation [14].

Obviously Φ is an operator acting on operators, such operators are called superoperators.

All possible quantum operations Φ must satisfy the following properties [15]:

- Suppose a system is in a mixed state $\rho = \sum_\alpha P_\alpha \rho_\alpha$. Φ must be linear in ρ such that

$$\Phi(\rho) = \Phi\{\sum_\alpha P_\alpha \rho_\alpha\} = \sum_\alpha P_\alpha \Phi(\rho_\alpha) \tag{3.8}$$

- Φ must be trace-preserving:

$$tr\Phi(\rho) = tr(\rho) = 1 \tag{3.9}$$

- Φ must be positive:

$$\Phi(\rho) \geq 0 \tag{3.10}$$

- Φ must be completely positive:

$$\Phi(\rho \otimes \mathbb{1}) \geq 0 \tag{3.11}$$

3.2.2. Kraus Decomposition

We now construct a representation to describe the reduced dynamics of Q shown by Equation 3.4 in the form known as Kraus decomposition. This construction is motivated from Schumacher [10]. To get Equation 3.4 in this form, we need the following assumptions:

- Q and R are uncorrelated (unentangled) at time t = 0.
- Q is initially in a pure state $\rho_0^{(Q)} = |\phi^{(Q)}\rangle\langle\phi^{(Q)}|$.
- R is initially in a pure state $\rho_0^{(R)} = |O^{(R)}\rangle\langle O^{(R)}|$ and $\{|K^{(R)}\rangle\}$ is a basis for $\mathcal{H}^{(R)}$.
- The interaction between Q and R starts at time t = 0. We say that $\mathrm{U}^{(QR)}(t)$ is the unitary time evolution operator for QR.

Under the above assumptions, we can then write Equation 3.4 as

$$\begin{aligned}
\rho^{(Q)}(t) &= tr_R\{U^{(QR)}(t)[|\phi^{(Q)}\rangle\langle\phi^{(Q)}| \otimes |O^{(R)}\rangle\langle O^{(R)}|]U^{(QR)t}(t)\} \\
&= tr_R\{U^{(QR}|\phi^{(Q)}, O^{(R)}\rangle\langle\phi^{(Q)}, O^{(R)}|U^{(QR)t}\} \\
&= \sum_k \langle k^{(R)}|U^{(QR)}|\phi^{(Q)}, O^{(R)}\rangle\langle\phi^{(Q)}, O^{(R)}|U^{(QR)t}|k^{(R)}\rangle \\
&= \sum_k \langle k^{(R)}|U^{(QR)}(t)|O^{(R)}\rangle\rho_0^{(Q)}\langle O^{(R)}|U^{(QR)t}(t)|k^{(R)}\rangle
\end{aligned} \tag{3.12}$$

Taking $A_k^{(Q)}(t) = \langle k^{(R)}|U^{(QR)}(t)|O^{(R)}\rangle$, we obtain the Kraus form of the time evolution

$$\rho^{(Q)}(t) = \sum_k A_k^{(Q)}(t)\rho^{(Q)}(0)A_k^{(Q)t}(t) \tag{3.13}$$

where $A_k^{(Q)}(t)$ are known as Kraus operators. Then we denote a quantum operation $\Phi(\rho_0)$ as [9]:

$$\Phi(\rho_0) = \sum_k A_k\rho_0A_k^\dagger \tag{3.14}$$

with

$$\sum_k A_k^\dagger A_k = \mathbb{1} \tag{3.15}$$

As we have described here the most general description of quantum evolutions can be represented by a superoperator Φ. For a more comprehensive discussion about Φ, see [16].

4. Markovian Description of Open Systems

The aim of this chapter is to investigate the markovian description of open systems. We start with a presentation of some preliminaries, (Section 4.1). Section 4.2 is intended to motivate our investigation of markovian quantum open systems. In Section 4.3, we derive a markovian master equation for open quantum systems in two different ways. Our first derivation is based on dynamical maps satisfying semigroup property. Then we introduce microscopic derivation of markovian master equation. For more extended discussion of the subject see [16-18].

4.1. Classical Markov Processes

4.1.1. Stochastic Processes

A stochastic process is simply a chain of events occurring in time whose outcomes are unpredictable and uncertain. Formally, a stochastic process can be regarded as an ensemble of random variables X indexed by t on probability space [16]:

$$\{X(t), t \in [0, \infty)\} \tag{4.1}$$

It should be noted that a random variable X always must be taken as a member of an ensemble whose statistical properties change in time. Thus, for a better understanding of the dynamics of stochastic processes, it is useful to picture the dynamics as probabilistic transitions.

Let $P(X_n|X_{n-1}, X_{n-2}, ..., X_1)$ be a probability density of finding $X(t = t_n) = X_n$, when $X(t_{n-1}), X(t_{n-2}), ..., X_1$ are given. The transition probability $P(X_n, X_{n-1}, ..., X_1)$ is defined as [19]:

$$P(X_n, X_{n-1}, ..., X_1) = P(X_n|X_{n-1}, ..., X_1)P(X_{n-1}, ..., X_1) \tag{4.2}$$

$$\sum_{X_k} P(..., X_{k+1}, X_k, X_{k-1}, ...) = P(..., X_{k+1}, X_{k-1}, ...) \tag{4.3}$$

(t_n,> t_{n-1} > ...> t_1), (k=2,...,n-1).

If the value of X at time t is completely independent of its values in the past and the future, the transition probability $P(X_n, ..., X_1)$ can be factorized as

$$P(X_n, ..., X_1) = P(X_n)P(X_{n-1})...P(X_1) \tag{4.4}$$

This stochastic process is purely random.

4.1.2. Markovian Evolutions

We begin with the formal definition of a markov process. A stochastic process $\{X(t_n) = X_n, n \in \mathbb{N}\}$ is said to be a markov process, if

$$P(X_n|X_{n-1}, X_{n-2}, ..., X_1) = P(X_n|X_{n-1}) \tag{4.5}$$

($t_n > t_{n-1} >$ ...> $t_2 > t_1$).

The relation shows a markov property, which means the values of X in the future depends only on the value of X at the present. Loosely speaking, a markov process is a memoryless stochastic process where the system removes its own history in each definable state transition, such a system is referred as a markovian system.

From now on, we consider only markovian processes in continuous space and time. Let $\{X(t)\}$ be a markov process. We say that a value of X at time t+dt depends only on a value of X at time t:

$$P(X(t + dt)|X(t), X(t - dt), ..., X(dt), X(0)) = P(X(t + dt)|X(t)) \tag{4.6}$$

Our problem is now to construct the representation of a markov property via the markovian evolution operator `V` whose matrix elements are the probability densities [20], such that

$$\langle X_n|\mathtt{V}(t_n, t_{(n-1)})|X_{n-1}\rangle = P(X_n|X_{n-1}) \tag{4.7}$$

Let's begin by writing Equation 4.2 for n=1 and n=3,

$$P(X_2, X_1) = P(X_2|X_1)P(X_1) \tag{4.8}$$

and

$$\begin{aligned} P(X_3, X_2, X_1) &= P(X_3|X_2, X_1)P(X_2, X_1) \\ &= P(X_3|X_2)P(X_2, X_1) \end{aligned} \tag{4.9}$$

where we have used a markov property in Equation 4.9. Substituting Equation 4.8 into Equation 4.9 yields

$$P(X_3, X_2, X_1) = P(X_3|X_2)P(X_2, X_1)P(X_1) \tag{4.10}$$

An argument similar to the one used in Equation 4.3 enables us to write

$$P(X_3, X_1) = \int_{-\infty}^{+\infty} dX_2 P(X_3|X_2)P(X_2, X_1)P(X_1)dX_2 \tag{4.11}$$

Taking $P(X_3, X_1) = P(X_3|X_1)P(X_1)$ gives

$$P(X_3|X_1) = \int_{-\infty}^{+\infty} dX_2 P(X_3|X_1)P(X_2|X_1) \tag{4.12}$$

This is called the Chapman-Kolmogorov Equation. By the definition of `V`, Equation

4.2 can be rewritten as

$$\begin{aligned}\langle X_3|\mathtt{V}(t_3,t_1)|X_1\rangle &= \int_{-\infty}^{+\infty} dX_2\langle X_3|\mathtt{V}(t_3,t_2)|X_2\rangle\langle X_2|\mathtt{V}(t_3,t_2)|X_1\rangle \\ &= \langle X_3|\mathtt{V}(t_3,t_2)\{\int_{-\infty}^{+\infty} dX_2|X_2\rangle\langle X_2|\}\mathtt{V}(t_2,t_1)|X_1\rangle \\ &= \langle X_3|\mathtt{V}(t_3,t_2)\mathbb{1}\mathtt{V}(t_2,t_1)|X_1\rangle \\ &= \langle X_3|\mathtt{V}(t_3,t_2)\mathtt{V}(t_2,t_1)|X_1\rangle\end{aligned} \tag{4.13}$$

where we have used a completeness relation $\int_{-\infty}^{+\infty} dX|X\rangle\langle X| = \mathbb{1}$. It suggests that

$$\mathtt{V}(t_3,t_1) = \mathtt{V}(t_3,t_2)\mathtt{V}(t_2,t_1) \tag{4.14}$$

which denotes a markov property in time.

From Equation 2.39, it is easily seen that a unitary time evolution operator U satisfies markov property in time. A time evolution satisfying Equation 4.14 is called a markovian evolution. For this reason, each quantum closed system can be regarded as a markovian system.

4.2. Quantum Master Equations

In Section 2.5, we identified the von Neumann Equation (Equation 2.97) as a convenient picture for describing a unitary time evolution of quantum systems. In the same manner, to study the dynamics of open systems one needs a first-order differential equation to describe the non-unitary evolutions in continuous time. Such an equation is called a quantum master equation.

For an appropriate master equation to exist, the evolution of the system must be markovian [21]. In this section, our goal is to construct a quantum master equation, which only depends on variables of the open systems.

4.2.1. Markov Approximation

Consider again a composite system QR, consisting of an open system Q and a reservoir R. To derive a master equation for the system of interest Q, we make the following assumptions [22]:

- a "coarse-graining" time δt much smaller than the characteristic life time of the system Q T_Q:

$$\delta t \ll T_Q \tag{4.15}$$

- a "coarse-graining" time δt much larger than the correlation time T_C:

$$T_C \ll \delta t \tag{4.16}$$

Roughly speaking, the reservoir R should forget about the system at the T_C time scale. Thus, the overall density operator ρ^{QR} can be taken approximately as a product state

$$\rho^{(QR)} \approx \rho^{(Q)} \otimes \rho^{(R)} \tag{4.17}$$

Under the above assumptions the superoperator Φ satisfies the markov property:

$$\Phi(t_3, 0) = \Phi(t_3, t_2)\Phi(t_2, t_1) \tag{4.18}$$

This relation is also called the semigroup property.

4.2.2. The Lindblad Equation

Keeping in mind above assumptions, let us first give the main ideas of the derivation of Lindblad Equation [22]:

- By assumption, we write

$$\rho(\delta t) = \rho(0) + \mathcal{O}(\delta t) = \sum_k A_k(\delta t)\rho(0)A_k^{-1}(\delta t) \tag{4.19}$$

to satisfy this, we take a first order Kaust operator close to unity, such that

$$A_0(\delta t) = \mathbb{1} - iK\delta t \tag{4.20}$$

and all others take the form $A_k = \sqrt{\delta t} L_k$.

- Let's decompose K into hermitian and anti-hermitian parts:

$$K = \frac{H}{\hbar} - iJ \tag{4.21}$$

where

$$H = \hbar(\frac{K + K^\dagger}{2}) \quad and \quad J = i(\frac{K - K^\dagger}{2}) \tag{4.22}$$

- Then, we calculate

$$\begin{aligned} A_0\rho A_0 &= \{\mathbb{1} + (\frac{H}{i\hbar} - J)\delta t\}\rho\{\mathbb{1} + (-\frac{H}{i\hbar} - J)\delta t\} \\ &= \rho + (\frac{H}{i\hbar} - J)\rho\delta t - \rho(\frac{H}{i\hbar} + J)\delta t + \mathcal{O}(\delta t^2) \\ &= \rho + \frac{1}{i\hbar}[H, \rho]\delta t - \{J, \rho\}\delta t + \mathcal{O}(\delta t^2) \end{aligned} \tag{4.23}$$

and

$$A_k\rho A_k^\dagger = L_k\rho L_k^\dagger \delta t + \mathcal{O}(\delta t^2) \tag{4.24}$$

Combining these gives,

$$\Phi_{\delta t}(\rho) = \rho + \frac{1}{i\hbar}[H, \rho]\delta t - \{J, \rho\}\delta t + \sum_{k>0} L_k\rho L_k^\dagger \delta t \tag{4.25}$$

We are now in a position to show the Lindblad Equation. From Equation 4.18 we can write

$$\frac{d\rho}{dt} = \frac{\Phi_{\delta t}(\rho(t) - \rho(t))}{\delta t} \tag{4.26}$$

Substituting eq. 4.25 into Equation 4.26 yields

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H, \rho] - \{J, \rho\} + \sum_k L_k \rho L_k^\dagger \tag{4.27}$$

Since $tr\frac{d\rho}{dt} = \frac{d}{dt}tr\rho = 0$, we can write

$$\begin{aligned} 0 &= \frac{1}{i\hbar}tr[H, \rho] - tr\{J, \rho\} + tr\sum_k L_k \rho L_k^\dagger \\ &= -tr(\rho 2J) + tr(\rho \sum_k L_k^\dagger L_k) \end{aligned} \tag{4.28}$$

By above, J is written in terms of $L_{\mathcal{K}}$ as

$$J = \frac{1}{2}\sum_k L_k^\dagger L_k \tag{4.29}$$

Substituting Equation 4.29 into Equation 4.27, we obtain

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H, \rho] + \sum_{k\neq 0}(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}) \tag{4.30}$$

which is called the Lindblad Equation [23].

4.2.3. Microscopic Derivation of Master Equation

In the previous section we introduced Lindblad Equation. In the derivation procedure physical meanings of approximations are not obvious. Now we will see these approximations will form microscopic master equation with a better physical meaning.

Once again, we consider the system of interest Q in interaction with a reservoir R. For abbreviation, we use the following shorthand notations:

$$H^{(QR)} = H^{(Q)} \otimes \mathbb{1}^{(R)} + \mathbb{1}^{(Q)} \otimes H^{(R)} + H^{(QR)}_{(I)} = H_0 + H_I \tag{4.31}$$

$$\rho^{(QR)}(t) = \chi(t), \quad \rho^{(Q)}(t) = \rho(t) \tag{4.32}$$

where $H_0 = H^{(Q)} + H^{(R)}$. We assume the interaction is weak. We will take the interaction Hamiltonian acting as a small perturbation term. In the interaction picture this Hamiltonian is written as,

$$\tilde{H}_I(t) = e^{+i\frac{H_0 t}{\hbar}} H_I e^{-i\frac{H_0 t}{\hbar}} \tag{4.33}$$

From Equation 2.100, we get,

$$\dot{\tilde{\chi}}(t) = \frac{1}{i\hbar}[\tilde{H}_I(t), \tilde{\chi}(t)] \tag{4.34}$$

From Equation 2.103, we get,

$$\tilde{\chi}(t) = \chi(0) + \frac{1}{i\hbar}\int_0^t [\tilde{H}_I(t'), \tilde{\chi}(t')]dt' \tag{4.35}$$

Now we will insert Equation 4.35 into Equation 4.34 to get the perturbative result as

$$\dot{\tilde{\chi}}(t) = \frac{1}{i\hbar}[\tilde{H}_I(t), \chi(0)] - \frac{1}{\hbar^2}\int_0^t dt'[\tilde{H}_I(t), [\tilde{H}_I(t'), \tilde{\chi}(t')]] \tag{4.36}$$

Taking partial trace over R yields

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2}\int_0^t dt'[\tilde{H}_I(t), [\tilde{H}_I(t'), \tilde{\chi}(t')]] \tag{4.37}$$

where we have assumed $\frac{1}{i\hbar} tr_R[H_I(t), \chi(0)] = 0$ [24]. Under the assumptions in previous part, the reduced dynamics of the system Q can be written as

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2}\int_0^t dt' tr_R\{[\tilde{H}_I(t), [\tilde{H}_I(t'), \tilde{\rho}(t') \otimes \rho^R]]\} \tag{4.38}$$

Since $\tilde{\rho}(t')$ represents the memory effects, the equation is non-markovian (0< t$'$ <t). When the reservoir is a canonical ensemble which is much larger than the system and is in thermal equilibrium, system does not cause a significant change on the environment. In such a case, we can consider the direction of information flow only from environment to system. By writing $\tilde{\rho}(t') \approx \tilde{\rho}(t)$ we can make the equation markovian

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2}\int_0^t dt' tr_R\{[\tilde{H}_I(t), [\tilde{H}(t'), \tilde{\rho}(t) \otimes \rho^{(R)}]]\} \tag{4.39}$$

We expand the interaction Hamiltonian $H_I^{(QR)}(t)$ as

$$\tilde{H_I}^{(QR)}(t) = \hbar\sum_\alpha \tilde{S}_\alpha(t) \otimes \tilde{\Gamma}_\alpha(t) \tag{4.40}$$

where

$$\tilde{S}_\alpha(t) = e^{i\frac{H^{(Q)}t}{\hbar}} S_\alpha(t) e^{-i\frac{H^{(Q)}t}{\hbar}} \tag{4.41}$$

$$\tilde{\Gamma}_\alpha(t) = e^{i\frac{H^{(R)}t}{\hbar}} \Gamma_\alpha(t) e^{-i\frac{H^{(R)}t}{\hbar}} \tag{4.42}$$

To understand the physics underlying the markov approximation, let's begin from non-markovian case Equation 4.38 [25]:

$$\begin{aligned}
\dot{\tilde{\rho}}(t) &= -\sum_{\alpha}\sum_{\beta}\int_0^t dt' tr_{(R)}\{[\tilde{S}_\alpha(t)\otimes\tilde{\Gamma}_\alpha(t),[\tilde{S}_\beta(t')\otimes\tilde{\Gamma}_\beta(t'),\tilde{\rho}(t')\otimes\rho^{(R)}]]\} \\
&= -\sum_{\alpha}\sum_{\beta}\int_0^t dt' tr_{(R)}\{(\tilde{S}_\alpha(t)\tilde{S}_\beta(t')\tilde{\rho}(t')\otimes\tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rho^{(R)} \\
&- \tilde{S}_\beta(t')\tilde{\rho}(t')\tilde{S}_\alpha(t)\otimes\tilde{\Gamma}_\beta(t')\rho^{(R)}\tilde{\Gamma}(t)) - (\tilde{S}_\alpha(t)\tilde{\rho}(t')\tilde{S}_\beta(t')\otimes\tilde{\Gamma}_\alpha(t)\rho^{(R)}\tilde{\Gamma}_\beta(t') \\
&- \tilde{\rho}(t')\tilde{S}_\beta(t')\tilde{S}_\alpha(t)\otimes\rho^{(R)}\tilde{\Gamma}_\beta(t')\tilde{\Gamma}_\alpha(t))\} \\
&= -\sum_{\alpha}\sum_{\beta}\int_0^t dt'(\tilde{S}_\alpha(t)\tilde{S}_\beta(t')\tilde{\rho}(t')tr(\tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rho^{(R)}) \\
&- \tilde{S}_\beta(t')\tilde{\rho}(t')\tilde{S}_\alpha(t)tr(\tilde{\Gamma}_\beta(t')\rho^{(R)}\tilde{\Gamma}(t)) - \tilde{S}_\alpha(t)\tilde{\rho}(t')\tilde{S}_\beta(t')tr(\tilde{\Gamma}_\alpha(t)\rho^{(R)}\tilde{\Gamma}_\beta(t')) \\
&- \tilde{\rho}(t')\tilde{S}_\beta(t')\tilde{S}_\alpha(t)tr(\rho^{(R)}\tilde{\Gamma}_\beta(t')\tilde{\Gamma}_\alpha(t))) \\
&= -\sum_{\alpha}\sum_{\beta}\int_0^t dt'(\tilde{S}_\alpha(t)\tilde{S}_\beta(t')\tilde{\rho}(t') - \tilde{S}_\beta(t')\tilde{\rho}(t')\tilde{S}_\alpha(t))tr(\tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rho^{(R)}) \\
&+ (\tilde{\rho}(t')\tilde{S}_\beta(t')\tilde{S}_\alpha(t) - \tilde{S}(t)\tilde{\rho}(t')\tilde{S}_\beta(t'))tr(\tilde{\Gamma}_\beta(t')\tilde{\Gamma}_\alpha(t)\rho^{(R)}) \\
&= -\sum_{\alpha}\sum_{\beta}\int_0^t dt'[\tilde{S}_\alpha(t),\tilde{S}_\beta(t')\tilde{\rho}(t')]tr(\tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rho^{(R)}) \\
&- [\tilde{S}_\alpha(t),\tilde{\rho}(t')\tilde{S}_\beta(t')]tr(\tilde{\Gamma}_\beta(t')\tilde{\Gamma}_\alpha(t)\rho^{(R)}) \\
&= -\sum_{\alpha}\sum_{\beta}\int_0^t dt'\{[\tilde{S}_\alpha(t),\tilde{S}_\beta(t')\tilde{\rho}(t')]tr(\tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rho^{(R)}) \\
&- [\tilde{S}_\alpha(t),\tilde{\rho}(t')\tilde{S}_\beta(t')]tr(\tilde{\Gamma}_\beta(t')\tilde{\Gamma}_\alpha(t)\rho^{(R)})\}
\end{aligned} \tag{4.43}$$

In the same manner with the expectation value of an operator $\langle A\rangle = tr(\rho A)$ correlation functions can be defined as the expectation value of interactions over the reservoir

$$tr(\tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rho^{(R)}) = \langle\tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rangle_{(R)} \tag{4.44}$$

$$tr(\tilde{\Gamma}_\beta(t')\tilde{\Gamma}_\alpha(t)\rho^{(R)}) = \langle\tilde{\Gamma}_\alpha(t')\tilde{\Gamma}_\beta(t)\rangle_{(R)} \tag{4.45}$$

We now assume that reservoir correlation functions are memoryless:

$$\langle \tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rangle_{(R)} = \tilde{\gamma}_{\alpha\beta}\delta(t-t') \tag{4.46}$$

This implies that

$$\dot{\tilde{\rho}}(t) = -\sum_{\alpha,\beta}\frac{\tilde{\gamma}_{\alpha\beta}}{2}\{\tilde{S}_\alpha(t)\tilde{S}_\beta(t)\tilde{\rho}(t) - \tilde{S}_\beta(t)\tilde{\rho}(t)\tilde{S}_\alpha(t) - \tilde{S}_\beta(t)\tilde{\rho}(t)\tilde{S}_\alpha(t) + \tilde{\rho}(t)\tilde{S}_\alpha(t)\tilde{S}_\beta(t)\} \tag{4.47}$$

This relation also in Schrödinger picture can be written as [26]:

$$\dot{\rho} = \frac{1}{i\hbar}[H^{(Q)},\rho] + \sum_\alpha\sum_\beta \frac{\gamma_{\alpha\beta}}{2}\{[S_\alpha,\rho S_\beta] + [S_\alpha\rho, S_\beta]\} \tag{4.48}$$

is called a master equation.

5. OPEN QUANTUM HARMONIC OSCILLATOR

To illustrate the master equation as a physical example, we consider the open harmonic oscillator Q, interacting with the reservoir R. The reservoir is also an ensemble of harmonic oscillators. The quantized field often can be modelled as such an ensemble, i.e. electromagnetic field. The convenient system-plus-reservoir model can be described by the following Hamiltonians [25]:

- The Hamiltonian of the system Q is

$$H^{(Q)} = \hbar\omega_0 a^\dagger a \tag{5.1}$$

- The Hamiltonian of the Reservoir R is

$$H^{(R)} = \hbar \sum_\alpha \omega_\alpha b_\alpha^\dagger b_\alpha \tag{5.2}$$

  with $[b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}\mathbb{1}$.
- The interaction between the system and the reservoir described by the Hamiltonian

$$H_I^{(QR)} = \hbar \sum_\alpha (a \otimes \kappa_\alpha^* b_\alpha^\dagger + a^\dagger \otimes \kappa_\alpha b_\alpha) \tag{5.3}$$

  where κ_α are coupling constants.

For abbreviation, we follow the shorthand notations:

$$\Gamma_1 = \Gamma_2^\dagger = \sum_\alpha \kappa_\alpha^* b_\alpha^\dagger \tag{5.4}$$

$$\Gamma_2 = \Gamma_1^\dagger = \sum_\alpha \kappa_\alpha b_\alpha \tag{5.5}$$

$$a = a_1, \quad a^\dagger = a_2 \tag{5.6}$$

Thus Equation 5.3 can be written as

$$H^{(QR)} = \hbar \sum_{\alpha=1}^{2} a_\alpha \otimes \Gamma_\alpha \tag{5.7}$$

Consider the case where the reservoir behaves like a canonical ensemble in thermal equilibrium at temperature T. Thus, to obtain a convenient derivation we need a little quantum statistical mechanics. First we note that for a density operator of an ensemble [27]:

$$\rho = \frac{e^{-\beta H}}{Z} \tag{5.8}$$

where

$$Z = tr(e^{-\beta H}) \tag{5.9}$$

$(\beta = \frac{1}{k_B T})$. Let be $H_\alpha^{(R)} = \hbar\omega_\alpha b_\alpha^\dagger b_\alpha$ and $Z_\alpha = tr(e^{-\beta H_\alpha^{(R)}})$. Similarly, we define

$$\rho_\alpha^{(R)} = \frac{1}{Z_\alpha} e^{-\beta H_\alpha^{(R)}} \tag{5.10}$$

$$\hat{n} = b_\alpha^\dagger b_\alpha \tag{5.11}$$

Since the density operator is Hermitian, it can be written in diagonal form. This allows us to write following relations:

$$\rho_\alpha^{(R)} = \frac{1}{Z_\alpha} e^{-\beta\hbar\omega_\alpha n_\alpha} |n_\alpha\rangle\langle n_\alpha| \tag{5.12}$$

$$Z_\alpha = tr(e^{-\beta\hbar\omega_\alpha \hat{n}_\alpha}) \tag{5.13}$$

We assume that the interaction is weak. To separate the fast intrinsic dynamics of a reservoir from slow dynamics Q, we again use the interaction picture. In order to write Hamiltonians in interaction picture, we need the "Baker-Campbell-Hausdorff Formula" [28]:

$$e^{\lambda B} A e^{-\lambda B} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n \tag{5.14}$$

where $C_0 = A$, $C_1 = [B, C_0]$, $C_2 = [B, C_1]$ and so on. We give explicit derivations only for $\tilde{a}_1(t)$ and $\tilde{\Gamma}_2(t)$.Let's calculate first $\tilde{a}_1(t)$:

$$\tilde{a}_1(t) = e^{i\omega_0 t a^\dagger a} a e^{-i\omega_0 t a^\dagger a} \tag{5.15}$$

From Equation 5.14, we obtain the recursion relation

$$\begin{aligned} C_0 &= a_1 = a \\ C_1 &= [a^\dagger a, a] = -a \\ &\vdots \\ C_n &= (-1)^n a \end{aligned} \tag{5.16}$$

Thus,

$$\tilde{a}_1(t) = \sum_{n=0}^{\infty} \frac{(i\omega_0 t)^n}{n!} (-1)^n a_1 \tag{5.17}$$

It is easy to show

$$\tilde{a}_1(t) = a_1 e^{-i\omega_0 t} \tag{5.18}$$

$$\tilde{a}_2(t) = a_2 e^{i\omega_0 t} \tag{5.19}$$

Our task is now to calculate $\tilde{\Gamma}_2(t)$,

$$\begin{aligned}
\tilde{\Gamma}_2(t) = &= e^{it\sum_\alpha \omega_\alpha b_\alpha^\dagger b_\alpha}\Gamma_2 e^{-it\sum_\beta \omega_\beta b_\beta^\dagger b_\beta} \\
&= \prod_\alpha \prod_\beta \sum_\gamma e^{it\omega_\alpha b_\alpha^\dagger b_\alpha}\kappa_\gamma b_\gamma e^{-it\omega_\beta b_\beta^\dagger b_\beta} \\
&= \prod_\alpha \prod_\beta \sum_\gamma e^{it\omega_\alpha b_\alpha^\dagger b_\alpha}\kappa_\gamma b_\gamma \delta_{\alpha\beta} e^{-it\omega_\alpha b_\alpha^\dagger b_\alpha} \\
&= \sum_\gamma \prod_\alpha \kappa_\gamma e^{it\omega_\alpha b_\alpha^\dagger b_\alpha} b_\gamma e^{-it\omega_\alpha b_\alpha^\dagger b_\alpha} \\
&= \sum_\gamma \prod_\alpha \frac{(it\omega_\alpha)^n}{n!}\kappa_\gamma C_n
\end{aligned} \tag{5.20}$$

From Equation 5.14, we obtain the recursion relation

$$\begin{aligned}
&C_0 = b_\alpha \delta_{\alpha\beta} \\
&C_1 = [b_\alpha^\dagger b_\gamma, b_\alpha] = -\delta_{\alpha\gamma} b_\alpha \\
&\vdots \\
&C_n = \delta_{\alpha\gamma}(-1)^n b_\alpha
\end{aligned} \tag{5.21}$$

It is easy to show

$$\tilde{\Gamma}_2(t) = \sum_\alpha \kappa_\alpha b_\alpha e^{-it\omega_\alpha}, \tag{5.22}$$

$$\tilde{\Gamma}_1(t) = \sum_\alpha \kappa_\alpha^* b_\alpha^\dagger e^{it\omega_\alpha} \tag{5.23}$$

Similarly, $\tilde{H}_I^{(QR)}$ is decomposed as

$$\tilde{H}_I^{(QR)} = \hbar \sum_{\alpha=1}^{2} \tilde{a}_\alpha(t) \otimes \tilde{\Gamma}_\alpha(t) \tag{5.24}$$

Using this expression together with Equation 4.38, we obtain

$$\begin{aligned}
\dot{\tilde{\rho}} &= -\sum_\alpha \sum_\beta \int_0^t dt' tr_{(R)}\{[\tilde{a}_\alpha(t) \otimes \tilde{\Gamma}_\alpha(t), [\tilde{a}_\beta(t') \otimes \tilde{\beta}(t'), \tilde{\rho}(t') \otimes \rho^{(R)}]]\} \\
&= -\sum_\alpha \sum_\beta \int_0^t dt' tr_{(R)}\{(\tilde{a}_\alpha(t)\tilde{a}_\beta(t')\tilde{\rho}(t') \otimes \tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rho^{(R)} \\
&- \tilde{a}_\beta(t')\tilde{\rho}(t')a_\alpha(t') \otimes \tilde{\Gamma}_\beta(t')\rho^{(R)}\tilde{\Gamma}_\alpha(t)) - (\tilde{a}_\alpha(t)\tilde{\rho}(t')\tilde{a}_\beta(t') \otimes \tilde{\Gamma}_\alpha(t)\rho^{(R)}\tilde{\Gamma}_\beta(t') \\
&- \tilde{\rho}(t')\tilde{a}_\beta(t')\tilde{a}_\alpha(t) \otimes \rho^{(R)}\tilde{\Gamma}_\beta(t')\tilde{\Gamma}_\alpha(t))\} \\
&= -\sum_\alpha \sum_\beta \int_0^t dt' tr_{(R)}\{(\tilde{a}_\alpha(t)\tilde{a}_\beta(t')\tilde{\rho}(t') - \tilde{a}_\beta(t')\tilde{\rho}(t')\tilde{a}_\alpha(t))tr(\tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\beta(t')\rho^{(R)}) \\
&+ (\tilde{\rho}(t')\tilde{a}_\beta(t')\tilde{a}_\alpha(t) - \tilde{a}_\alpha(t)\tilde{\rho}(t')\tilde{a}_\beta(t'))tr(\tilde{\Gamma}_\beta(t')\tilde{\Gamma}_\alpha(t)\rho^{(R)})\} \\
&= -\sum_{\alpha=1}^{2} \int_0^t dt'([\tilde{a}_\alpha(t), \tilde{a}_\alpha^\dagger(t)\tilde{\rho}(t')]\langle\tilde{\Gamma}_\alpha(t)\tilde{\Gamma}_\alpha^\dagger(t')\rangle - [\tilde{a}_\alpha(t), \tilde{\rho}_\alpha(t')\tilde{a}_\alpha^\dagger(t')]\langle\tilde{\Gamma}_\alpha^\dagger(t')\tilde{\Gamma}_\alpha(t)\rangle) \\
&= \int_0^t dt'\{[\tilde{a}_1(t), \tilde{a}_2(t')\tilde{\rho}(t')]\langle\tilde{\Gamma}_1(t)\tilde{\Gamma}_2(t')\rangle - [\tilde{a}_1(t), \tilde{\rho}(t')\tilde{a}_2(t')]\langle\tilde{\Gamma}_2(t')\tilde{\Gamma}_1(t)\rangle \\
&+ [\tilde{a}_2(t), \tilde{a}_1(t')\tilde{\rho}(t')]\langle\tilde{\Gamma}_2(t)\tilde{\Gamma}_1(t')\rangle - [\tilde{a}_2(t), \tilde{\rho}(t')\tilde{a}_1(t')]\langle\tilde{\Gamma}_1(t')\tilde{\Gamma}_2(t)\rangle\} \\
&= \int_0^t dt'\{[a, a^\dagger\tilde{\rho}(t')]e^{-i\omega_0(t-t')}(\sum_\alpha |\kappa_\alpha|^2 e_\alpha^{-i\omega}(t'-t)\langle\hat{n}_\alpha\rangle) \\
&- [a\tilde{\rho}(t')a^\dagger]e^{-i\omega_0(t-t')}(\sum_\alpha |\kappa_\alpha|^2 e_\alpha^{-i\omega}(t'-t)\langle\hat{n}_\alpha + 1\rangle) \\
&+ [a^\dagger, a\tilde{\rho}(t')]e^{-i\omega_0(t'-t)}(\sum_\alpha |\kappa_\alpha|^2 e^{-i\omega_\alpha(t'-t)}\langle\hat{n}_\alpha + 1\rangle) \\
&- [a^\dagger, \tilde{\rho}(t')a]e^{-i\omega_0(t'-t)}(\sum_\alpha |\kappa_\alpha|^2 e^{-i\omega_\alpha(t-t')}\langle\hat{n}_\alpha\rangle)\}
\end{aligned} \tag{5.25}$$

making the change of variable $\tau = t - t'$ allows us

$$\begin{aligned}\dot{\tilde{\rho}} = -\int_0^t d\tau\{&e^{-i\omega_0\tau}([a, a^\dagger\tilde{\rho}]\sum_\alpha |\kappa_\alpha|^2 e^{i\omega_\alpha\tau} n(\omega_\alpha)) \\ &- e^{-i\omega_0\tau}([a, \tilde{\rho}a^\dagger]\sum_\alpha |\kappa_\alpha|^2 e^{i\omega_\alpha\tau} n(\omega_\alpha) + 1) \\ &+ e^{-i\omega_0\tau}([a^\dagger, a\tilde{\rho}]\sum_\alpha |\kappa_\alpha|^2 e^{-i\omega_\alpha\tau} n(\omega_\alpha) + 1) \\ &- e^{-i\omega_0\tau}([a^\dagger, \tilde{\rho}a]\sum_\alpha |\kappa_\alpha|^2 e^{-i\omega_\alpha\tau} n(\omega_\alpha))\}\end{aligned} \tag{5.26}$$

with

$$n(\omega_\alpha) = tr(\rho^{(R)} b_\alpha^\dagger b_\alpha) = \frac{e^{-\hbar\omega_\alpha\beta}}{1 - e^{-\hbar\omega_\alpha\beta}} \tag{5.27}$$

Now we assume that the reservoir is a canonical ensemble of photons at temperature T. Then, we make the following approximations:

$$\omega_\alpha \sim \omega \tag{5.28}$$

$$\sum_\alpha |\kappa_\alpha|^2 \sim \int_0^\infty |\kappa(\omega)|^2 g(\omega) d\omega \tag{5.29}$$

where $g(\omega)$ is the density of states [25]. This allows us to write

$$\begin{aligned}\dot{\tilde{\rho}} = &[a\tilde{\rho}, a^\dagger]\int_0^t d\tau \int_0^\infty |\kappa(\omega)|^2 g(\omega) e^{-i\tau(\omega-\omega_0)} d\omega \\ &+ ([a\tilde{\rho}, a^\dagger] + [a^\dagger, \tilde{\rho}a])\int_0^t d\tau \int_0^\infty |\kappa(\omega)|^2 g(\omega) e^{-i\tau(\omega-\omega_0)} n(\omega) d\omega + h.c.\end{aligned} \tag{5.30}$$

where we also made the Born-Markov approximation $\tilde{\rho}(t') = \tilde{\rho}(t)$.

We now apply the time separation argument discussed in 4.2.1, thus, in Equation

5.30, τ intergration becomes more dominated than ω integration. We arrive at

$$\begin{aligned}
\dot{\tilde{\rho}} &= [a\tilde{\rho}, a^\dagger] \int_0^\infty d\omega \{ \lim_{t\to\infty} \int_0^t d\tau e^{-i\tau(\omega-\omega_0)} \} |\kappa(\omega)|^2 g(\omega) \\
&+ ([a\tilde{\rho}, a^\dagger] + [a^\dagger, \tilde{\rho}a]) \int_0^\infty d\omega \{ \lim_{t\to\infty} \int_0^t d\tau e^{-i\tau(\omega-\omega_0)} \} |\kappa(\omega)|^2 g(\omega) n(\omega) + h.c. \\
&= [a\tilde{\rho}, a^\dagger] \int_0^\infty d\omega \pi \delta(\omega-\omega_0) |\kappa(\omega)|^2 g(\omega) \\
&+ ([a\tilde{\rho}, a^\dagger] + [a^\dagger, \tilde{\rho}a]) \int_0^\infty d\omega \pi \delta(\omega-\omega_0) |\kappa(\omega)|^2 g(\omega) n(\omega) + h.c. \\
&= \pi |\kappa(\omega)|^2 g(\omega_0) [a\tilde{\rho}, a^\dagger] + \pi |\kappa(\omega)|^2 g(\omega_0) n(\omega_0) ([a\tilde{\rho}, a^\dagger] + [a^\dagger, \tilde{\rho}a]) + h.c.
\end{aligned} \tag{5.31}$$

After defining

$$\gamma = 2\pi g(\omega_0) |\kappa(\omega_0)|^2 \tag{5.32}$$

$$n = n(\omega) \tag{5.33}$$

we obtain from Equation 5.31

$$\begin{aligned}
\dot{\tilde{\rho}} &= \frac{\gamma}{2}[a\tilde{\rho}, a^\dagger + \frac{\gamma n}{2}([a\tilde{\rho}, a^\dagger] + [a^\dagger, \tilde{\rho}, a]) + h.c. \\
&= \frac{\gamma}{2}(a\tilde{\rho}a^\dagger - a^\dagger a\tilde{\rho} + a\tilde{\rho}a^\dagger - \tilde{\rho}a^\dagger a) \\
&+ \frac{\gamma n}{2}(a\tilde{\rho}a^\dagger - a^\dagger a\tilde{\rho} + a\tilde{\rho}a^\dagger - \tilde{\rho}a^\dagger a + a^\dagger \tilde{\rho}a - \tilde{\rho}aa^\dagger + a^\dagger \tilde{\rho}a - aa^\dagger \tilde{\rho}) \\
&= \frac{\gamma}{2}(2a\tilde{\rho}a^\dagger - a^\dagger a\tilde{\rho} - \tilde{\rho}a^\dagger a) \\
&+ \frac{\gamma n}{2}(2a\tilde{\rho}a^\dagger + 2a^\dagger \tilde{\rho}a - 2a^\dagger a\tilde{\rho} - 2\tilde{\rho}aa^\dagger) \\
&= \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\tilde{\rho} - \rho a^\dagger a) + \gamma n(a\tilde{\rho}a^\dagger + a^\dagger \tilde{\rho}a - a^\dagger a\tilde{\rho} - \tilde{\rho}aa^\dagger)
\end{aligned} \tag{5.34}$$

Let's transform this to the Schrödinger picture. To make this, we need the following calculations:

First note that

$$U = e^{-i\omega_0 a^\dagger a t} \tag{5.35}$$

$$\dot{\tilde{\rho}} = \frac{d}{dt}(U^\dagger \rho U) = \dot{U}^\dagger \rho U - U^\dagger \rho \dot{U} + U^\dagger \dot{\rho} U \tag{5.36}$$

then, we obtain

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho] + U\dot{\tilde{\rho}}U^\dagger \tag{5.37}$$

with

$$U\dot{\tilde{\rho}}U^\dagger = \dot{\tilde{\rho}} \tag{5.38}$$

Using this expression together with Equation 5.34, we can write

$$\dot{\tilde{\rho}} = \frac{1}{i\hbar}[H, \rho] + \frac{\gamma}{2}([a, \rho a^\dagger] + [a\rho, a^\dagger] + \frac{\gamma}{2}n([a\rho, a^\dagger] + [a^\dagger, \rho a]) \tag{5.39}$$

This equation is the master equation for the damped harmonic oscillator.

6. CONCLUSION

In this study, we tried to present a simple introduction to quantum open systems theory. We have seen the first concept of this theory in Section 2.4.2.: entanglement is clearly a result of the dynamic interaction of the system with its environment. Here, it is important to notice that the subsystems of the composite system are in mixed state. We have come across the other key concept of the subject in Section 2.5.2. At this point, we have shown how to exclude the information from the environment, using the partial trace operation. This operation enabled us to obtain the reduced density operator, which defines the subsystem only. In Section 3.1, we have learned how to picture an open system in the most common form. This was the system-plus-reservoir model. Here, it is important to notice that the basic goal of open quantum systems theory is to exclude the information from environmental dynamics. Then, we have developed the quantum operation formalism that will define the non-unitary evolution. In Section 3.2.2, we have defined the non-unitary evolution of the open system's state by Kraus decomposition. The important thing was the fact that an external environment, which would define the system's dynamics, could always be found out of the system. In Chapter 4, we have examined the dynamics of open systems under markov approximation. At this point, we have made the assumption that information only flows from the environment to the system. This was very necessary for the evolution of the open system to be markovian, i.e. memoryless. Then, we have derived the master equations through two separate ways. Although the first way was an elegant one, the physical responses to our assumptions were not very clear. These assumptions have become clearer in microscopic derivation. In Chapter 5, we have derived the markov equation that defines the dynamics of the open harmonic oscillator. This is a good example for quantum master equations.

Thus, a reader who is not yet familiar with the open systems theory has been provided with quickly accessible information on the subject. However, in my opinion, it will be useful to point out that the reader's adventure with the quantum open systems has just begun.

REFERENCES

1. Debnath, L., and P. Mikusinski, *Hilbert Spaces with Application,* Elsevier, Oxford, UK., 2005.

2. Zettili, N., *Quantum Mechanics: Concepts and Applications,* John Wiley & Sons., Boston, USA., 2009.

3. Heinosaari, T., and M. Ziman, *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement,* Cambridge University Press., Cambridge, UK., 2012.

4. Susskind, L., and A. Friedman, *Quantum Mechanics: The Theoretical Minimum,* Basic Books, New York, USA., 2014.

5. Zwiebach, B., *Quantum Phyics II,* 2013, http://ocw.mit.edu/courses/physics, [Accessed May 2015].

6. Steck, D. A., *Quantum and Atom Optics,* 2014, http://atomoptics-nas.uoregon.edu, [Accessed May 2015].

7. Audretsch, J., *Entangled Systems: New Directions in Quantum Physics,* Wiley-VCH., Germany., 2007.

8. Jeevanjee, N., *An Introduction to Tensors and Group Theory for Physicists,* Birkhäuser., Boston, USA., 2011.

9. Schumacher, B., and M. Westmoreland, *Quantum Processes Systems, and Information,* Cambridge University Press., Cambridge, U.K., 2010.

10. Schumacher, B., *Quantum Theory,* 2010, http://pirsa.org/C10028, [Accessed May 2015].

11. Schrödinger, E., 1935, "Discussion of Probability Relations Between Separated Systems", *Mathematical Proceedings of Cambridge Philosophical Society* Vol. 31, 555–563.

12. Blum, K., *Springer Series on Atomic, Optical, and Plasma Physics 64, Density Matrix Theory and Applications,* Springer-Verlag., Berlin, Germany., 2012.

13. Schieve, W.C., and L.P. Horwitz, *Quantum Statistical Mechanics,* Cambridge University Press., Cambridge, U.K., 2009.

14. Nielsen, M.A., and I.L. Chuang, *Quantum Computation and Quantum Information,* Cambridge University Press., Cambridge, U.K., 2010.

15. Schumacher, B., *Interaction and Information Flow Between Quantum Systems Part 1A,* 2007, http://pirsa.org/07080052, [Accessed May 2015].

16. Breuer, H.P., and F. Petruccione, *The Theory of Open Quantum Systems,* Oxford University Press., New York, USA., 2002.

17. Gardiner, C.W., *Springer Series in Synergetics Handbook of Stochastic Methods,* Springer., Berlin, Germany., 1996.

18. Gardiner, C.W., and P. Zoller, *Springer Series in Synergetics Quantum Noise: A of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics,* Springer., Berlin, Germany., 2004.

19. Huang, K., *Introduction to Statistical Physics,* Chapman and Hall/CRC, USA., 2009.

20. Zinn-Justin, J., *Path Integrals in Quantum Mechanics,* Oxford University Press., USA., 2005.

21. Preskill, J., *Quantum Computation,* 2001, http://www.theory.caltech.edu., [Accessed May 2015].

22. Haroche, S., and J.M. Raimond, *Exploring the Quantum: Atoms, Cavities, and Photons,* Oxford University Press., USA., 2013.

23. Lindblad, G., 1976, "On the Generators of Quantum Dynamical Semigroups", *Comm. Math. Phys.* Vol. 48, 119–130.

24. Schaller G., *Open Quantum Systems Far from Equilibrium,* Springer, Berlin, Germany., 2014.

25. Carmichael, H. J., *Texts and Monographs in Physics Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations,* Springer-Verlag, Berlin, Germany., 1999.

26. Benenti, G., G. Casati, and G. Strini, *Principles of Quantum Computation and Information - Volume II: Basic Tools and Special Topics [2],* World Scientifific, Singapore., 2004.

27. David, F., *Lecture Notes in Physics 893 The Formalisms of Quantum Mechanics: An Introduction,* Springer, Switzerland., 2015.

28. Riley, K. F., M. P. Hobson, and S. J. Bence , *Mathematical Methods for Physics and Engineering: A Comprehensive Guide,* Cambridge University Press., Cambridge, UK., 2006.