THE FIBERED p-BISET FUNCTOR OF THE FIBERED BURNSIDE RING

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ABSTRACT

THE FIBERED p-BISET FUNCTOR OF THE FIBERED BURNSIDE RING

We find the composition factors of the A-fibered Burnside functor kB^A of p-groups over a field k of characteristic q with $q \neq p$. We show that if A is a cyclic p-group, then kB^A is uniserial. Moreover, we also show that the simple composition factors of kB^A depends only on the prime p and the characteristic q, and not on the particular fiber group A.

ÖZET

FİBERLİ BURNSIDE HALKASININ FİBERLİ p-İKÜME İZLECİ

Bu tezde p-öbeklerinin karakteristiği $q \neq p$ olan cisimler uzerine A-fiberli Burnside izleçlerinin kB^A bileşke dizilerini bulduk. Eger fiber grubu A devirli p-grup ise, o zaman kB^A izlecinin tek sıralı olduğunu gösterdik. Ayrıca, basit bileşke dizilerinin asal sayı p'ye ve karakteristik q'ya bağlı olduğunu ve özel bir fiber grubuna bağlı olmadığını gösterdik.

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LIST OF SYMBOLS

$B(G \times H, A)$	The Burnside group of A-fibered (G,H)-bisets
$B_k(G \times H, A)$	$k \otimes B(G \times H, A)$
$\mathrm{c}^{\eta}_{G,H}$	Conjugation fibered biset
$\mathrm{Def}^{G}_{G/N}$	Deflation fibered biset
$e^{G}_{H,h}$	The primitive idempotent corresponding to the conjugacy class of A-subelement (H,h) of G
G^*	The set of group homomorphisms from G to A
Ind^{G}_{H}	Induction fibered biset
$\mathrm{Inf}^{G}_{G/N}$	Inflation fibered biset
kB^A	The A-fibered Burnside functor over $\mathcal{C}^A_k$
$\mathcal{M}_G(A)$	The set of all A-subcharacters of G
$O(G)$	The intersection of kernels of all A-characters of G
Res^{G}_{H}	Restriction fibered biset
Tw^{ϕ}_{G}	Twist fibered biset
$X \otimes_{AH} Y$	The Mackey product of fibered bisets X and Y
$\Delta(H)$	$\{(h,h) \mid h \in H\}$
$\Phi(G)$	The Frattini subgroup of G

1. INTRODUCTION

In representation theory, it is of the utmost importance to study group actions on sets. For the simplest case, one may consider the action of a finite group G on a finite set X. This action reveals the theory of Burnside rings which is introduced by Solomon in [1]. Considering the common features shared by Burnside rings and representation rings, Dress [2] and Green [3] introduced Mackey functors to give a unified treatment of these objects. The structure of Mackey functors is studied extensively by Thevénaz and Webb in [4]. Mackey functors have several important applications such as the theory of canonical induction formulae introduced by Boltje in [5]. There is also a version of Alperin's weight conjecture in terms of Mackey functors.

There are two ways to let two groups act on a set X. First, suppose that we have two groups G and H. By considering the action of G on the left and the action of H on the right, we may let $G \times H$ act on X. In this case, the set X is called a (G, H)-biset and this leads us to the theory of biset functors introduced by Bouc in [6]. One of the most important applications of biset functors, among many others, is the final determination of the structure of the Dade group by Bouc and Thevénaz [7]. In [7], Bouc and Thevénaz also studied Burnside functor of p-groups. They obtained that Burnside functor of p-groups over a field of characteristic zero has two composition factors, one of which is the torsion-free Dade group and the other one is the functor of rational representations.

As a second way of letting two groups act on a set, we may consider the action of $A \times G$ on X where G is a finite group and A is an abelian group acting on X freely. Since the A-action is free, such an action of $A \times G$ on X can be considered as G acting on the A-fibers and in this case, the set X is called an A-fibered G-set. These objects were introduced by Dress in [8] and studied by Boltje [9] and Barker [10].

Boltje and Coşkun [11] combined these two notions and introduced A-fibered (G, H)-bisets. Our aim in the present work is to extend the works of Bouc and Thevénaz on Burnside functors and to find the composition factors of the A-fibered Burnside functor kB^A of p-groups over a field k of characteristic q with $q \neq p$. To be more precise, if A is a cyclic p-group, then we show that the A-fibered p-biset functor kB^A is uniserial. Moreover, it follows from the classification of the simple fibered biset functors that the parametrizing set for the simple composition factors of kB^A depends only on the prime p and the characteristic q, and not on the particular fibre group A.

As in the case of the (ordinary) biset functor of the Burnside group, when $q = 0$, the fibered Burnside functor kB^A has only two composition factors. One of the factors is the functor A-monomial characters. The other factor has the cyclic group C_p as its minimal group, but we are unable to identify it with a natural construction.

In Chapter 2, we give some preliminary results and introduce some notation. We state the definition and some properties of fibered bisets from [11]. We also give a primitive idempotent formula for the fibered Burnside ring from Barker [10] which can also be found in [9]. In Chapter 3.1, we give the definitions of fibered biset functors and fibered Burnside functor from [11]. In chapter 3.3, we state the effects of the restriction and the induction bisets on the primitive idempotents of the fibered Burnside ring from [10]. We also find the effects of the inflation and the deflation bisets on the idempotents. In Chapter 4.1, we find the constant that comes from the effect of the deflation map, see Proposition 3.4, for some special cases. Then, using these constants, we show that for any subfunctor F of the fibered Burnside functor kB^A, the minimal group of F is an elementary abelian group of rank r where $p^{r-1} \equiv 1 \mod q$. In Chapter 4.2, we find the composition factors of fibered Burnside functors of p-groups, see Theorem 4.2.

2. PRELIMINIARIES

2.1. Fibered Bisets

First we give the definition and some properties of fibered bisets introduced by Boltje and Coşkun in [11].

Definition 2.1. *Let G and H be groups and A be an abelian group. Then, an A-fibered (G, H)-biset is an A-free $G \times H \times A$-set with finitely many A-orbits such that all three actions commute.*

Definition 2.2. *A morphism of A-fibered (G, H)-bisets is an $G \times H \times A$-equivariant map.*

For any A-fibered (G, H)-biset X, its isomorphism class is denoted by $[X]$. An A-orbit of X is called a *fibre*. For any $x \in X$, we write $[x]$ for the A-orbit of x. Note that since the A-action is free, the A-action and the (G, H)-action on X commute. Therefore, the set of fibers of X is a (G, H)-biset under the action

$$(g, h) \cdot [x] := \big[(g, h) \cdot x\big]. \tag{2.1}$$

The class of A-fibered (G, H)-bisets and their morphisms form a category which is denoted by ${}_G\mathbf{set}^A_H$. For any group G, we denote by $G^* := \mathrm{Hom}(G, A)$, the set of all group homomorphisms from G to A. Note that G^*, sometimes referred as A-characters of G, is an abelian group under point-wise addition. The pairs (K, κ), where K is a subgroup of G and $\kappa \in K^*$, are called *A-subcharacters* of G. We denote by $\mathcal{M}_G(A)$ the set of all A-subcharacters of G. For any two elements (K, κ) and (L, λ) of $\mathcal{M}_G(A)$, we have the relation $(L, \lambda) \leq (K, \kappa)$ if $L \leq K$ and $\kappa = \lambda\,|_L$. The set $\mathcal{M}_G(A)$ with this relation is a poset. Moreover, $\mathcal{M}_G(A)$ is a G-set under the conjugation action

$$g \cdot (K, \kappa) := ({}^gK, {}^g\kappa) \tag{2.2}$$

which respects the poset structure. We denote by $[K,\kappa]_G$ the G-orbit of (K,κ).

2.1.1. Transitive A-fibered (G,H)-bisets:

Let X be an A-fibered (G,H)-biset and $x \in X$ be any element. Let $S_x \leq G \times H$ be the stabilizer of the fiber $[x]$. Then for any $u \in S_x$ we have $u \cdot x = a \cdot x$ for some $a \in A$. Notice that if $y \in [x]$ is another representative of the fiber, then $y = a' \cdot x$ for some $a' \in A$. We have

$$u \cdot y = u \cdot a' \cdot x = a' \cdot u \cdot x = a' \cdot a \cdot x = a \cdot a' \cdot x = a \cdot y. \tag{2.3}$$

Thus, the element a is independent of the choice of the representative. This gives a group homomorphism $\phi_x : S_x \to A$ such that $\phi_x(u) = a$.

The pair (S_x, ϕ_x) is called the *stabilizing pair* of x. Note that we have

$$(S_{(g,h)x}, \phi_{(g,h)x}) = {}^{(g,h)}(S_x, \phi_x).$$

Since the A-action and the (G,H)-action on X commute, the set X is transitive as an A-fibered (G,H)-biset if and only if the set of fibers of X is transitive as a (G,H)-biset. Hence the fibers of X correspond to the transitive (G,H)-bisets $(G \times H)/U$ for some subgroups U of $G \times H$. The next result is from [11].

Proposition 2.1. *There is a bijective correspondence between the set of isomorphism classes of transitive A-fibered (G,H)-bisets and the set of $G \times H$-conjugacy classes of $\mathcal{M}_{G\times H}(A)$.*

If X is a transitive A-fibered (G,H)-biset and $x \in X$ is any element, then the correspondence in the proposition is given by sending X to the conjugacy class $[S_x, \phi_x]_{G\times H}$. Conversely, given $(U,\phi) \in \mathcal{M}_{G\times H}(A)$, we define the subgroup $U_\phi = \{(u, \phi^{-1}(u)) \mid u \in U\} \leq G \times H \times A$. Then, $X := (G \times H \times A)/U_\phi$ is a transitive A-fibered (G,H)-biset, with the stabilizing pair $[U,\phi]_{G\times H}$.

Given a pair $(U, \phi) \in \mathcal{M}_{G\times H}(A)$, the corresponding transitive A-fibered (G, H)-biset and its isomorphism class will be denoted by $\left(\frac{G\times H}{U,\phi}\right)$ and $\left[\frac{G\times H}{U,\phi}\right]$, respectively.

2.1.2. Fibered Double Burnside Group

Let X and Y be A-fibered (G, H)-bisets. The disjoint union, sometimes called the *coproduct* $X \cup Y$ of X and Y is again an A-fibered (G, H)-biset.

Definition 2.3. *The Burnside group $B(G \times H, A)$ of A-fibered (G, H)-bisets is the Grothendieck group of the category of A-fibered (G, H)-bisets: it is the quotient of the free abelian group on the isomorphism classes of A-fibered (G, H)-bisets by the subgroup generated by the elements of the form*

$$[X \cup Y] - [X] - [Y]. \tag{2.4}$$

The elements of the Burnside group are sometimes called virtual elements. It should be noted that transitive elements $[\frac{G\times H}{U,\phi}]$, where $[U, \phi] \in \mathcal{M}_{G\times H}(A)$ form a $\mathbb{Z}$-basis for $B(G \times H, A)$. Thus, as abelian groups, we have

$$B(G \times H, A) = \bigoplus_{[U,\phi]\in\mathcal{M}_{G\times H}(A)/G\times H} \mathbb{Z}\left[\frac{G \times H}{U, \phi}\right]. \tag{2.5}$$

Replacing $\mathbb{Z}$ by a field k, this equation is still true. We denote $k \otimes B(G \times H, A)$ by $B_k(G \times H, A)$. Then, we have

$$B_k(G \times H, A) = \bigoplus_{[U,\phi]\in\mathcal{M}_{G\times H}(A)/G\times H} k\left[\frac{G \times H}{U, \phi}\right]. \tag{2.6}$$

as k-vector spaces.

2.1.3. Mackey Product

Let X be an A-fibered (G, H)-biset and Y be an A-fibered (H, K)-biset. Let $X \times_{AH} Y$ be the set of $A \times H$ orbits of $X \times Y$ under the action

$$(a, h) \cdot (x, y) = (x \cdot (a^{-1}, h^{-1}), (a, h) \cdot y). \tag{2.7}$$

For any $(x, y) \in X \times Y$, its $A \times H$-orbit will be denoted by $[x, y]$. Notice that $X \times_{AH} Y$ is an (G, K)-biset via

$$g \cdot [x, y] \cdot k = [g \cdot x, y \cdot k]. \tag{2.8}$$

The set $X \times_{AH} Y$ is, also an A-set under the action

$$a \cdot [x, y] = [a \cdot x, y] = [x, a^{-1} \cdot y]. \tag{2.9}$$

It is important to note that the A-action and the (G, K)-action on $X \times_{AH} Y$ commute. Yet the A-action is not necessarily free. In order to obtain an A-fibered (G, K)-biset, we take the free A-orbits of $X \times Y$. First, observe that if $(g, k)(a[x, y]) = (g, k)[x, y]$, then we have $a[x, y] = [x, y]$. Therefore, the (G, K)-action permutes the free A-orbits. Now, define the A-fibered (G, K)-biset $X \otimes_{AH} Y$ as the union of the free A-orbits of $X \times_{AH} Y$. We call $X \otimes_{AH} Y$ the *Mackey product* of X and Y. If $[x, y]$ is an element of $X \otimes_{AH} Y$, it will be denoted by $x \otimes_{AH} y$.

Note that the Mackey product is distributive with respect to disjoint union. More precisely, for any A-fibered (G, H)-bisets X and X' and A-fibered (H, K)-bisets Y and Y', we have

$$(X \cup X') \otimes_{AH} Y = (X \otimes_{AH} Y) \cup (X' \otimes_{AH} Y), \tag{2.10}$$

$$X \otimes_{AH} (Y \cup Y') = (X \otimes_{AH} Y) \cup (X \otimes_{AH} Y'). \tag{2.11}$$

It is also shown in [11] that the Mackey product is associative, that is, we have an isomorphism of A-fibered (G, L)-bisets

$$(X \otimes_{AH} Y) \otimes_{AK} Z = X \otimes_{AH} (Y \otimes_{AK} Z) \tag{2.12}$$

for any $X \in {}_G\mathbf{set}_H^A$, $Y \in {}_H\mathbf{set}_K^A$ and $Z \in {}_K\mathbf{set}_L^A$.

An explicit formula for the Mackey product is given in [11]. We will give this formula, but first, we need to introduce some notation.

Let $U \leq G \times H$ be a subgroup. Then, for $i = 1, 2$, we write $p_i(U)$ for the projection of U into G and H, respectively. We also define

$$k_1(U) = \{k \in G \mid (k, 1) \in U\}$$

and

$$k_2(U) = \{k \in H \mid (1, k) \in U\}.$$

For any homomorphism $\phi \in U^*$, we define homomorphisms $\phi_i \in k_i(U)^*$ by

$$\phi \mid_{k_1(U) \times k_2(U)} = \phi_1 \times \phi_2^{-1} \tag{2.13}$$

for $i = 1, 2$.

Now, suppose $V \leq H \times K$ is a subgroup. Then we define

$$U * V = \{(g, k) \mid \text{ for some } h \in H \ (g, h) \in U \text{ and } (h, k) \in V\}.$$

Also given a homomorphism $\psi \in V^*$ satisfying the property

$$\phi_2 \mid_{k_2(U)\cap k_1(V)} = \psi_1 \mid_{k_2(U)\cap k_1(V)} \tag{2.14}$$

we define $\phi * \psi \in (U * V)^*$ by

$$(\phi * \psi)(g,k) = \phi(g,h)\psi(h,k) \tag{2.15}$$

where h is an element of H satisfying $(g,h) \in U$ and $(h,k) \in V$. Note that by the equation (2.14), $\phi * \psi$ is well-defined.

Proposition 2.2 (Corollary 2.5, [11]). *Let G, H and K be finite groups. Let also $(U,\phi) \in \mathcal{M}_{G\times H}(A)$ and $(V,\psi) \in \mathcal{M}_{H\times K}(A)$. Then, we have*

$$\left(\frac{G\times H}{U,\phi}\right) \otimes_{AH} \left(\frac{H\times K}{V,\psi}\right) \cong \bigoplus_{\substack{t\in p_2(U)\backslash H/p_1(V) \\ \phi_2|_{H_t} = {}^t\psi_1|_{H_t}}} \left(\frac{G\times K}{U * {}^tV, \phi * {}^t\psi}\right) \tag{2.16}$$

where $H_t = k_2(U) \cap {}^tk_1(V)$.

2.1.4. Decomposition of Fibered Bisets

Let G and H be finite groups and let $H_2 \le H_1 \le H$ and $G_2 \le G_1 \le G$ be subgroups. For any group homomorphism $f : G_1 \to H$, we define the subgroup $\Delta_f(G_2)$ of $G \times H$ as

$$\Delta_f(G_2) = \{(g, f(g)) \mid g \in G_2\}.$$

Similarly, for any group homomorphism $f : H_1 \to G$, we define the subgroup ${}_f\Delta(H_2)$ of $G \times H$ as

$${}_f\Delta(H_2) = \{(f(h), h) \mid h \in H_2\}.$$

Whenever f is the inclusion map, we simply write $\Delta(H_2)$ or $\Delta(G_2)$.

Let G be a group, H be a subgroup of G and N be a normal subgroup of G. Then, we define the induction, restriction, inflation and deflation fibered bisets, respectively, as

$$\mathrm{Ind}_H^G := \left(\frac{G\times H}{\Delta(H),1}\right) \in {}_G\mathbf{set}_H^A, \tag{2.17}$$
$$\mathrm{Res}_H^G := \left(\frac{H\times G}{\Delta(H),1}\right) \in {}_H\mathbf{set}_G^A, \tag{2.18}$$
$$\mathrm{Inf}_{G/N}^G := \left(\frac{G\times G/N}{\Delta_\pi(G),1}\right) \in {}_G\mathbf{set}_{G/N}^A, \tag{2.19}$$
$$\mathrm{Def}_{G/N}^G := \left(\frac{G/N\times G}{{}_\pi\Delta(G),1}\right) \in {}_{G/N}\mathbf{set}_G^A. \tag{2.20}$$

Moreover, if $\eta : G \to H$ is a group isomorphism, then we define the conjugation fibered biset as

$$\mathrm{c}_{G,H}^\eta = \left(\frac{H\times G}{\Delta(H)_\eta,1}\right) \in {}_H\mathbf{set}_G^A. \tag{2.21}$$

In [6], Bouc showed that any transitive (G,H)-biset $(G\times H)/U$ is equal to a Mackey product of these five elementary bisets. More precisely, we have

$$\left(\frac{G\times H}{U}\right) = \mathrm{Ind}_P^G \times_P \mathrm{Inf}_{P/K}^P \times_{P/K} \mathrm{c}_{P/K,Q/L}^\eta \times_{Q/L} \mathrm{Def}_{Q/L}^Q \times_Q \mathrm{Res}_Q^H \tag{2.22}$$

where $P = p_1(U)$ and $Q = p_2(U)$ are the two projections of U and $K = k_1(U)$ and $L = k_2(U)$. The groups P/K and Q/L are isomorphic and the isomorphism η is determined by U. Finally, $\times_?$ denotes the Mackey product of bisets. A similar decomposition for a transitive A-fibered (G,H)-biset is given in [11]. We have

$$\left(\frac{G\times H}{U,\phi}\right) = \mathrm{Ind}_P^G \otimes_{AP} \mathrm{Inf}_{P/\hat{K}}^P \otimes_{AP/\hat{K}} Y \otimes_{AQ/\hat{L}} \mathrm{Def}_{Q/\hat{L}}^Q \otimes_{AQ} \mathrm{Res}_Q^H \tag{2.23}$$

where $\hat{K}$ and $\hat{L}$ are kernels of ϕ_1 and ϕ_2^{-1}, respectively. Also, Y is an A-fibered

$(P/\hat{K}, Q/\hat{L})$-biset. In some cases, Y can also be decomposed (See Section 10 in [11]).

If G and H are abelian groups, and A is as in the Hypothesis A, we have a complete decomposition for transitive A-fibered (G, H)-bisets. In order to have it, we need to introduce a new basic fibered biset called a twist. Let $\phi \in G^* = \mathrm{Hom}(G, A)$ be a homomorphism from G to A. Then the A-fibered (G, G)-biset

$$\mathrm{Tw}_G^{\phi} = \Big(\frac{G \times G}{\Delta(G), \Delta(\phi)}\Big)$$

is called the *twist* by ϕ at G.

Suppose A satisfies the Hypothesis A and let $(U, \phi) \in \mathcal{M}_{G\times H}(A)$ be any element. We write $\tilde{\phi} = \tilde{\phi}_1 \times \tilde{\phi}_2$ for an extension of ϕ to $P \times Q$ which exists since the group $P \times Q$ is abelian and A is divisible by the assumption.

Theorem 2.1. *Let G, H be finite abelian groups and let A be an abelian group satisfying Hypothesis A. Let also $(U, \phi) \in \mathcal{M}_{G\times H}(A)$. Then*

$$\Big(\frac{G \times H}{U, \phi}\Big) \cong \mathrm{Ind}_P^G \mathrm{Tw}_P^{\tilde{\phi}_1} \mathrm{Inf}_{P/K}^P \mathrm{c}_{P/K,Q/L}^{\eta} \mathrm{Def}_{Q/L}^Q \mathrm{Tw}_Q^{\tilde{\phi}_2} \mathrm{Res}_Q^H.$$

Remark 2.1. Let p be a prime number and A be a finite cyclic p-group. Then the Hypothesis A is not satisfied. On the other hand, if G and H are elementary abelian groups, then the decomposition of Theorem 2.1 still holds. (see the appendix for the proof).

2.2. Fibered Burnside Ring

Assume that we have $H = \mathbf{1}$. Then, an A-fibered (G, H)-biset is called an A-fibered G-set. Dress introduced these objects in [8] and considered a more general case where the group G acts on the fiber group A. These objects are also studied by Boltje in [9]. For convenience, we will use the notation in [10]. For any group G and the fiber group A, the corresponding Burnside group will be denoted by $B(G, A)$. A transitive

A-fibered G-set $(\frac{G\times 1}{U,\phi})$ will be denoted by $(U,\phi)_G$.

Let X and Y be A-fibered G-sets. Then, $X\times Y$ is an A-set under the action

$$a\cdot(x,y)=(a\cdot x,a^{-1}\cdot y).$$

Again, we denote by $X\otimes Y$ the set of A-orbits. Now, we let $G\times A$ act on $X\otimes Y$ by

$$(a,g)\cdot(x\otimes y)=(a\cdot g\cdot x\otimes g\cdot y).$$

Note that by this multiplication, $B(G,A)$ becomes a ring. It follows from 2.5 that (cf. Remark 2.2 [10])

$$B(G,A)=\bigoplus_{[U,\phi]\in\mathcal{M}_G(A)/G}\mathbb{Z}[U,\phi]_G.$$

Moreover, for any A-subcharacters (V,ν) and (W,ω) of G, we have the Mackey formula (cf. [8], [9], Remark 2.3 [10]),

$$[V,\nu]_G\cdot[W,\omega]_G=\sum_{VgW\subseteq G}[V\cap{}^gW,\nu\cdot{}^g\omega\mid_{V\cap{}^gW}].$$

In this chapter, from now on, A will be a cyclic p-group. We recall the idempotent formula from [10] for $B_k(G,A)$, where k is a sufficiently large field of characteristic 0.

2.2.1. Subelements

Let $O(G)$ denote the intersection of kernels of the A-characters of G. Then the group G^* can be regarded as the dual of the group $G/O(G)$.

We define an A-*subelement* of G as a pair $(H,hO(H))$ where $h\in H\leq G$. An A-subelement $(H,hO(H))$ of G will be denoted by (H,h). Note that G acts on the set of A-subelements of G by ${}^g(H,h) := ({}^gH, {}^gh)$. The set of A-subelements of G is denoted by $\mathrm{el}(G,A)$.

For a given subgroup H of G, we have $|H^*| = |H/O(H)|$. Therefore, we get $|\mathcal{M}_G(A)| = |el(G,A)|$. We will see later that the primitive idempotents of $B_k(G,A)$ are parametrized by G-conjugacy classes of the G-set $\mathrm{el}(G,A)$.

2.2.2. Monomial Incidence and Mobius Functions

We recall monomial Mobius and monomial incidence functions that are used in determining idempotent formula for $B_k(G,A)$, from [10].

Let S be a proposition. Then, the Kronecker value of S is the rational integer

$$\lfloor S \rfloor = \begin{cases} 1 & \text{if S holds} \\ 0 & \text{if S fails.} \end{cases}$$

Let P be a finite poset. The function $\zeta : P \times P \to \mathbb{Z}$ such that $\zeta(x,y) = \lfloor x \leq y \rfloor$ for all $x, y \in P$ is called the *incidence function* of P. For any integer $n \geq 2$, we define the function $c_n : P \times P \to \mathbb{Z}$ by $c_{-2}(y,x) = \lfloor x = y \rfloor$ and $c_{-1}(y,x) = \lfloor x < y \rfloor$ and for any $n \geq 0$, $c_n(y,x)$ is the number of chains in P of the form $y < z_0 < \cdots < z_n < x$. The *Mobius function* of P is defined as $\mu : P \times P \to \mathbb{Z}$ such that

$$\mu(y,x) = \sum_{-2}^{\infty} (-1)^n c_n(y,x).$$

Let θ and ϕ be functions $P \to A$ where A is an abelian group. The equation

$$\theta(y) = \sum_{x \in P} \phi(x) \zeta(x,y)$$

is called the *totient equation.* The equation

$$\phi(x) = \sum_{y \in P} \theta(y)\mu(y, x)$$

is called the *inversion equation.*

The Mobius inversion principle states that the totient equation holds for all $y \in P$ if and only if the inversion equation holds for all $x \in P$. The principle can also be expressed as

$$\sum_{y \in P} \zeta(x, y)\mu(y, z) = \lfloor x = z \rfloor = \sum_{y \in P} \mu(x, y)\zeta(y, z)$$

holds for all $x, z \in P$.

Now, suppose that a group G acts on the poset P where the G-action respects the poset structure. We define the G-invariant versions of the incidence and the Mobius functions as

$$\zeta_G(x, y) = \sum_{x' \in [x]_G} \zeta(x', y), \quad \mu_G(y, x) = \sum_{y' \in [y]_G} \mu(y', x).$$

Assume that the group G acts on A trivially and the functions θ and ϕ are G-invariant. Here again, the totient equation

$$\theta(y) = \sum_{x \in_G P} \phi(x)\zeta_G(x, y)$$

holds for all $y \in P$ if and only if the inversion equation

$$\phi(x) = \sum_{y \in_G P} \theta(y)\mu_G(y, x)$$

holds for all $x \in P$.

We embed A into torsion units k^w of k. We now generalize these two functions for the poset $\mathrm{sub}(G)$. The *monomial incidence function* is defined as

$$\zeta : el(G, A) \times \mathcal{M}_G(A) \to k, \quad \zeta(H, h; V, \nu) = \nu(h)\zeta(H, V)$$

and the *monomial Mobius function* is defined as

$$\mu : \mathcal{M}_G(A) \times el(G, A) \to k, \quad \mu(V, \nu; H, h) = \nu^{-1}(V \cap hO(H))\mu(V, H)/\left|V\right|$$

where $(H, h) \in el(G, A)$ and $(V, \nu) \in \mathcal{M}_G(A)$. Here ν^{-1} is the inverse of ν in V^* and $\nu^{-1}(V \cap hO(H)) = \sum_{x \in V \cap hO(H)} \nu^{-1}(x)$.

Finally, we define the G-invariant versions of these functions:

$$\zeta_G(H, h; V, \nu) = \sum_{(H', h') \in [H, h]_G} \zeta(H', h'; V, \nu),$$
$$\mu_G(V, \nu; H, h) = \sum_{(V', \nu') \in [V, \nu]_G} \mu(V', \nu'; H, h).$$

Next result is from Barker [10].

Theorem 2.2 (Theorem 4.1, [10])**.** *Given the G-invariant functions $\theta : \mathcal{M}_G(A) \to A$ and $\phi : el(G, A) \to A$, the totient equation*

$$\theta(V, \nu) = \sum_{(H, h) \in_G el(G, A)} \phi(H, h)\zeta_G(H, h; V, \nu)$$

holds for all $(V, \nu) \in \mathcal{M}_G(A)$ if and only if the inversion equation

$$\phi(H, h) = \sum_{(V, \nu) \in_G \mathcal{M}_G(A)} \theta(V, \nu)\mu_G(V, \nu; H, h)$$

holds for all $(H, h) \in el(G, A)$.

2.2.3. Idempotents of $B_k(G,A)$

In this section, we recall, from [10], the idempotent formula in terms of the transitive A-fibered G-sets $[U,\phi]$ where $(U,\phi)\in\mathcal{M}_G(A)$.

For any A-subelement (H,h) of G, we define the *species* (algebra maps to the ground field)

$$s^G_{H,h} : B_k(G,A) \to k$$

to be the linear map such that

$$s^G_{H,h}[AX] = \sum_{Ax} \phi_x(h)$$

where Ax runs over the fibres that are stabilized by H. Note that $s^G_{H,h}$ is a k-algebra homomorphism. The following lemma is due to Dress.

Lemma 2.1 (Theorem 1(c), [8])**.** *Given A-subelements (H,h) and (K,k) of G, we have $s^G_{H,h} = s^G_{K,k}$ if and only if $(H,h) =_G (K,k)$. Moreover, every species of $B_k(G,A)$ is of this form and the species span the dual space of $B_k(G,A)$.*

By this lemma, there exists a unique element $e^G_{H,h}\in B_k(G,A)$ such that

$$s^G_{K,k}(e^G_{H,h}) = \lfloor (H,h) =_G (K,k) \rfloor \, .$$

Clearly, $e^G_{H,h}$ is a primitive idempotent by definition.

Theorem 2.3 (Theorem 5.2, [10])**.** *There is a bijective correspondence $e^G_{H,h} \leftrightarrow [H,h]_G$ between the primitive idempotents $e^G_{H,h}$ of $B_k(G,A)$ and the G-conjugacy classes $[H,h]_G$ of A-subelements of G. Moreover, we have*

$$e^G_{H,h} = \frac{1}{|N_G(H,h)|} \sum_{(V,\nu)\in_G \mathcal{M}_G(A)} |V|\mu_G(V,\nu;G,g)[V,\nu]_G.$$

Remark 2.2. The idempotent $e_{H,h}^G$ is the only non-zero idempotent of $B_k(G,A)$ such that $X \cdot e_{H,h}^G = s_{H,h}^G(X)e_{H,h}^G$ for any $X \in B_k(G,A)$. Thus, for any $X \in B_k(G,A)$, we have the coordinate decomposition

$$X = \sum_{(H,h)\in_G el(G,A)} s_{H,h}^G(X)e_{H,h}^G.$$

Remark 2.3. Let G be a p-group. The above idempotent formula still holds if we replace k with a sufficiently large field of characteristic $q \neq p$. Indeed, denumerators in the proof of the formula are p-powers which are still invertible.

3. FIBERED BURNSIDE FUNCTOR

3.1. A-fibered Biset Functors

Definition 3.1. *The A-fibered biset category of finite groups $\mathcal{C} := \mathcal{C}_k^A$ is the category defined as follows:*

- *The objects of $\mathcal{C}$ are finite groups.*
- *If G and H are finite groups, then $Hom_{\mathcal{C}}(G, H) = B_k(H \times G, A) = k \otimes B(G \times H, A)$.*
- *If K is another finite group and $x \in B_k(H \times G, A)$ and $y \in B_k(K \times H, A)$, then the composition of x and y is $y \circ x := y \cdot_H x$ where*

$$- \cdot_H - : B_k(K \times H, A) \times B_k(H \times G, A) \to B_k(K \times G, A) \tag{3.1}$$

  is the k-linear extension of the map induced by the Mackey product of A-fibered bisets.

Definition 3.2. *An A-fibered biset functor over k is a k-linear functor from $\mathcal{C}_k^A$ to the k-linear category ${}_k\boldsymbol{Mod}$ of left k-modules.*

Since ${}_k\mathbf{Mod}$ is abelian, the category of A-fibered biset functors is abelian with point-wise constructions. This allows us to define subfunctors, quotient functors, simple functors, etc.

3.2. A-fibered Burnside Functor

Definition 3.3. *The A-fibered Burnside functor over $\mathcal{C}_k^A$ is biset functor kB^A given by the following data:*

- *For any object G of $\mathcal{C}_k^A$, we have $kB^A(G) = B_k(G, A)$.*

- *For any $U \in B_k(H \times G, A)$, the map $kB^A(U) : B_k(G,A) \to B_k(H,A)$ is induced by sending any A-fibered G-set X to A-fibered H-set $U \otimes_{AG} X$.*

Note that by this definition $kB(G,A)$ is a $kB(G \times G, A)$-module.

Notation 3.1. Following the notation in $\big[$ [12], pp 32$\big]$, for any transitive A-fibered G-set $[H,\phi]_G$, we denote by $\widetilde{[H,\phi]}_G$ the A-fibered (G,G)-biset $\left(\frac{G\times G}{\Delta(H),\Delta(\phi)}\right)$.

When we extend the function linearly, we get an injective group homomorphism

$$\widetilde{\ }: kB(G,A) \hookrightarrow kB(G \times G, A).$$

The following lemma shows that it is, in fact, a ring homomorphism.

Lemma 3.1. *For any A-fibered G-sets X and Y, there is an isomorphism of A-fibered G-sets*

$$X \cdot Y \cong \widetilde{X} \otimes_{AG} Y.$$

Proof. It is sufficient to prove the equality for transitive A-fibered G-sets. So, let $X = [H,\phi]_G$ and $Y = [K,\psi]_G$ be transitive A-fibered G-sets. Then, by Remark 2.2, we have

$$X \cdot Y = [H,\phi]_G \cdot [K,\psi]_G = \sum_{HgK \subseteq G} [H \cap {}^gK, \phi \cdot {}^g\psi \mid_{H \cap {}^gK}]_G.$$

On the other hand, we have

$$\begin{aligned}\widetilde{X} \otimes_{AG} Y &= \left(\frac{G \times G}{\Delta(G), \Delta(\phi)}\right) \otimes_{AH} \left(\frac{G \times 1}{K \times 1, \psi \times 1}\right) \\ &\cong \bigoplus_{\substack{t \in p_2(\Delta(H)) \backslash G / p_1(K \times 1) \\ \Delta(\phi)_2|_{H_t} = {}^t\psi_1|_{H_t}}} \left(\frac{G \times 1}{\Delta(H) * {}^t(K \times 1), \Delta(\phi) * {}^t(\psi \times 1)}\right).\end{aligned}$$

Note that $p_2(\Delta(H)) = H$ and $p_1(K \times 1) = K$. Also, $k_2(\Delta(H)) = \{h \in G \mid (1,h) \in$

$\Delta(H)\} = \{1\}$ and $k_1(K \times 1) = \{k \in G \mid (k,1) \in (K \times 1)\} = K$. Thus, $H_t = k_2(\Delta(H)) \cap {}^t(k_1(K \times 1)) = 1$. Thus, t runs over a set of double coset representatives of H and K in G. Moreover, we have

$$\begin{aligned}\Delta(H) * {}^t(K \times 1) &= \{(g,1) \in G \times 1 \mid \exists h \in G \text{ with } (g,h) \in \Delta(H), (h,1) \in {}^t(K \times 1)\} \\ &= \{(g,1) \in G \times 1 \mid g \in H \cap {}^tK\} \\ &= (H \cap {}^tK) \times 1.\end{aligned}$$

Finally, we have

$$\begin{aligned}\Delta(\phi) * {}^t(\psi \times 1)(g,1) &= \Delta(\phi)(g,h) \cdot {}^t(\psi \times 1)(h,1) \\ &= \phi(g) \cdot {}^t\psi(g) \\ &= (\phi \cdot {}^t\psi)(g),\end{aligned}$$

as required. □

Lemma 3.2. *Let F be a subfunctor of the A-fibered biset functor kB^A. Then, for any group G, the k-module $F(G)$ is an ideal of $kB^A(G)$.*

Proof. Since F is a subfunctor, $F(G)$ is a $kB(G \times G, A)$-submodule of $kB(G,A)$. By the previous lemma, $kB(G \times G, A)$-module $kB(G,A)$ regarded as a $kB(G,A)$-module is the regular module. Therefore, $F(G)$ is an ideal $kB(G,A)$. □

This lemma shows that there is a connection between lattice of subfunctors of the Burnside functor kB^A and the structure of the Burnside ring $B_k(G,A)$. Therefore, it is natural to study the effect of elementary A-fibered biset operations on the idempotents of the $B_k(G,A)$.

Recall that our aim is to find composition factors of the fibered Burnside functor of p-groups over a field of characteristic q with $q \neq p$. Hence in the category $\mathcal{C}_k^A$, instead of taking all finite groups, we will only take the finite p-groups and the field k will be a sufficiently large field of characteristic q. The fiber group A is, again, a cyclic p-group. The Burnside functor, in this case will still be denoted by kB^A.

3.3. Biset Operations On The Idempotents

In this section, we find the effect of some elementary fibered bisets on the primitive idempotents of the fibered Burnside ring $B_k(G, A)$. The first two propositions are from [10]. We give the proof of Barker in more detail.

Proposition 3.1 (Proposition 5.5, [10]). *Given $F \leq G$ and an A-subelement (H, h) of G, then*

$$\mathrm{Res}^G_F(e^G_{H,h}) = \sum_{(J,j)} e^F_{J,j}$$

where (J, j) runs over representatives of the F-classes of the A-subelements of F such that (J, j) is G-conjugate to (H, h).

Proof. First note that for any A-fibered G-set X, we have $\mathrm{Res}^G_F(X) = X$ as sets with the action restricted to the subgroup F. So, for any A-subelement (J, j) of F, we have

$$s^F_{J,j}(\mathrm{Res}^G_F(X)) = s^G_{J,j}(X). \tag{3.2}$$

Indeed, the action of J and A and hence the fibers stabilized by J are the same in both sides. In particular, we have

$$s^F_{J,j}(\mathrm{Res}^G_F(e^G_{H,h})) = s^G_{J,j}(e^G_{H,h}) = \lfloor (J, j) =_G (H, h) \rfloor$$

and the result follows. □

Proposition 3.2 (Proposition 5.4, [10]). *Given $F \leq G$ and an A-subelement (J, j) of*

F, then

$$\mathrm{Ind}_F^G(e_{J,j}^F) = |N_G(J,j) : N_F(J,j)| e_{J,j}^G.$$

Proof. Let (H,h) be an A-subelement of G. The Mackey decomposition formula

$$\mathrm{Res}_H^G \mathrm{Ind}_F^G = \sum_{HgF \subseteq G} \mathrm{Ind}_{H \cap {}^gF}^H \mathrm{Res}_{H \cap {}^gF}^{{}^gF} \mathrm{c}_F^g$$

holds for the monomial Burnside algebras. In addition to this, observe that if $F \neq G$, then

$$e_{G,g}^G \mathrm{Ind}_F^G(e_{J,j}^F) = \mathrm{Ind}_F^G(\mathrm{Res}_F^G(e_{G,g}^G) e_{J,j}^F) = 0$$

by the previous proposition. Therefore, using (4.1) we have

$$\begin{aligned}
s^G_{H,h}(\mathrm{Ind}^G_F(e^F_{J,j})) &= s^H_{H,h}(\mathrm{Res}^G_H(\mathrm{Ind}^G_F(e^F_{J,j})) \\
&= s^H_{H,h}(\sum_{HgF\subseteq G} \mathrm{Ind}^H_{H\cap {}^gF}\mathrm{Res}^{{}^gF}_{H\cap {}^gF}\mathrm{c}^g_F(e^F_{J,j})) \\
&= \sum_{HgF\subseteq G} s^H_{H,h}(\mathrm{Res}^{{}^gF}_H \mathrm{c}^g_F(e^F_{J,j})) \\
&= \sum_{HgF\subseteq G} s^{{}^gF}_{H,h}(\mathrm{c}^g_F(e^F_{J,j})) \\
&= \sum_{HgF\subseteq G} s^{{}^gF}_{H,h}(e^{{}^gF}_{{}^gJ,{}^gj}) \\
&= \sum_{HgF\subseteq G} \lfloor (H,h) =_F {}^g(J,j) \rfloor \\
&= |\{gF \subseteq G : (H,h) =_F {}^g(J,j)\}| \\
&= \frac{1}{|F|}|\{g\in G : {}^{g^{-1}}(H,h) =_F (J,j)\}| \\
&= \frac{|F|}{|N_F(J,j)|}\frac{1}{|F|}|\{g\in G : {}^{g^{-1}}(H,h) = (J,j)\}| \\
&= \frac{1}{|N_F(J,j)|}\frac{|N_G(J,j)|}{|G|}|\{g\in G : {}^{g^{-1}}(H,h) =_G (J,j)\}| \\
&= \frac{|N_G(J,j)|}{|N_F(J,j)|}\lfloor (H,h) =_G (J,j) \rfloor.
\end{aligned}$$

□

Lemma 3.3. *Let $N \trianglelefteq G$ be a normal subgroup. Then, for any transitive A-fibered G/N-set $[H/N,\phi]_{G/N}$, we have an isomorphism of A-fibered G-sets*

$$\mathrm{Inf}^G_{G/N}[H/N,\phi]_{G/N} \cong [H,\bar{\phi}]_G$$

where $\bar{\phi}$ is the A-character of H defined by $\bar{\phi}(h) = \phi(hN)$ for any $h \in H$.

Proof. By Proposition 2.2, we have

$$\begin{aligned}\mathrm{Inf}_{G/N}^{G}[H/N,\phi]_{G/N} &= \left(\frac{G\times G/N}{\Delta_\pi(G),1}\right)\otimes_{AG/N}\left(\frac{G/N\times 1}{H/N\times 1,\phi\times 1}\right)\\ &= \coprod_{\substack{t\in p_2(\Delta_\pi(G))\backslash G/N/p_1(H/N\times 1)\\ 1|_{H_t}={}^t\phi_1|_{H_t}}}\left(\frac{G\times 1}{\Delta_\pi(G)*{}^t(H/N\times 1),1*{}^t(\phi\times 1)}\right).\end{aligned}$$

Note that $p_2\left(\Delta_\pi(G)\right) = p_2\left(\{(g,gN)\mid g\in G\}\right) = G/N$ and $p_1(H/N\times 1) = H/N$. Also we have the equalities $k_2\left(\Delta_\pi(G)\right) = \{gN\in G/N \mid (1,gN)\in\Delta_\pi(G)\} = N$ and $k_1\left(H/N\times 1\right) = \{gN\in G/N\mid (gN,1)\in H/N\times 1\} = H/N$. Therefore, $H_t = N$ and we have only one summand.

Moreover, we have

$$\begin{aligned}&\Delta_\pi(G)*{}^t(H/N\times 1)\\ &=\{(g,1)\in G\times 1\mid (g,hN)\in\Delta_\pi(G),(hN,1)\in{}^t(H/N\times 1)\text{ for some }hN\in G/N\}\\ &=H\times 1\end{aligned}$$

and

$$\begin{aligned}(1*(\phi\times 1))(h,1) &= 1(h,h'N)\cdot(\phi\times 1)(h'N,1)\\ &=\phi(h'N)\\ &=\phi(hN)\end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{Inf}_{G/N}^{G}[H/N,\phi]_{G/N} &\cong \left(\frac{G\times 1}{H\times 1,\bar{\phi}\times 1}\right)\\ &=[H,\bar{\phi}]_G,\end{aligned}$$

as required. □

Proposition 3.3. *Let $N\trianglelefteq G$ be a normal subgroup of G. Then, for any A-subelement*

$(H/N, hN)$ of G/N we have

$$\mathrm{Inf}_{G/N}^{G}(e_{H/N,hN}^{G/N}) = \sum_{(K,k)} e_{K,k}^{G}$$

where (K, k) runs over representatives of the G-classes of A-subelements of G such that (KN, k) is G-conjugate to (H, h).

Proof. At first we will demonstrate that for any A-subelement (K, k) of G and for any A-fibered G/N-set S, we have $s_{K,k}^{G}(\mathrm{Inf}_{G/N}^{G}(S)) = s_{KN/N,kN}^{G/N}(S)$. Since the inflation map is a ring homomorphism, it suffices to choose S transitive. Thus, observe that for any transitive A-fibered G/N-set $[V/N, \nu]_{G/N}$, we have

$$s_{K,k}^{G}(\mathrm{Inf}_{G/N}^{G}([V/N, \nu]_{G/N})) = s_{K,k}^{G}([V, \bar{\nu}]_{G}) = \sum_{gV} \bar{\nu}_g(k)$$

where gV runs over the fibers stabilized by K. But that fibers are also stabilized by KN/N. We have $\bar{\nu}_g(k) = \nu_g(kN)$ as well which implies

$$s_{K,k}^{G}(\mathrm{Inf}_{G/N}^{G}([V/N, \nu]_{G/N}) = s_{KN/N,kN}^{G/N}([V/N, \nu]_{G/N}).$$

Therefore, we obtain

$$\begin{aligned} s_{K,k}^{G}(\mathrm{Inf}_{G/N}^{G}(e_{H/N,hN}^{G/N}) &= s_{KN/N,kN}^{G/N}(e_{H/N,hN}^{G/N}) \\ &= \lfloor (KN/N, kN) =_{G/N} (H/N, hN) \rfloor \\ &= \lfloor (KN, k) =_{G} (H, h) \rfloor \end{aligned}$$

and the result follows. □

Lemma 3.4. *Let $N \trianglelefteq G$ be a normal subgroup of G. Then, for any transitive A-fibered*

G-set $[H,\phi]_G$, we have an isomorphism of A-fibered G/N-sets

$$\mathrm{Def}^G_{G/N}[H,\phi]_G \cong \begin{cases} [HN/N,\tilde{\phi}]_{G/N} & \text{if } H\cap N \leq \ker\phi \\ 0 & \text{otherwise} \end{cases}$$

where $\tilde{\phi}$ is the A-character of HN/N defined by $\tilde{\phi}(hN) = \phi(h)$.

Proof. By Proposition 2.2, we have

$$\begin{aligned} \mathrm{Def}^G_{G/N}[H,\phi]_G &= \left(\frac{G/N \times G}{{}_{\pi}\Delta(G), 1}\right) \otimes_{AG} \left(\frac{G \times 1}{H \times 1, \phi \times 1}\right) \\ &= \bigoplus_{\substack{t \in p_2({}_{\pi}\Delta(G))\backslash G/p_1(H\times 1) \\ 1|_{H_t} = {}^t\phi_1|_{H_t}}} \left(\frac{G/N \times 1}{{}_{\pi}\Delta(G) * {}^t(H \times 1), 1 * {}^t(\phi \times 1)}\right). \end{aligned}$$

Here $p_2({}_{\pi}\Delta(G)) = p_2(\{(gN,g) \mid g \in G\}) = G$ and $p_1(H\times 1) = H$. Thus, again, we have $t = 1$. Note that $k_2({}_{\pi}\Delta(G)) = \{g \in G \mid (N,g) \in {}_{\pi}\Delta(G)\} = N$ and $k_1(H\times 1) = \{g \in G \mid (g,1) \in H \times 1\} = H$. Therefore, the image is nonzero if and only if $\phi\mid_{H\cap N} = 1$.

Now, observe that

$$\begin{aligned} {}_{\pi}\Delta(G) * {}^t(H\times 1) &= \{(gN,1) \mid (gN,h) \in {}_{\pi}\Delta(G), (h,1) \in H \times 1 \text{ for some } h \in G\} \\ &= HN/N. \end{aligned}$$

Finally,

$$(1 * {}^t(\phi \times 1))(hN,1) = 1(hN,h)\cdot(\phi\times 1)(h,1) = \phi(h),$$

as required. ☐

Lemma 3.5. *Let $N \trianglelefteq G$ be a normal subgroup of G. Then, for any A-fibered G-set X* *and A-fibered G/N-set Y, we have the Frobenius relation*

$$Y \cdot \mathrm{Def}^G_{G/N}(X) = \mathrm{Def}^G_{G/N}\left(\mathrm{Inf}^G_{G/N}(Y) \cdot X\right). \tag{3.3}$$

Proof. By linearity, it is sufficient to take X and Y transitive. Let $Y = [H/N, \phi]_{G/N}$ and $X = [K, \psi]_G$ be transitive A-fibered G/N-set and G-set, respectively. If $N \cap K \leq \ker\psi$, then we have

$$\begin{aligned} Y \cdot \mathrm{Def}^G_{G/N}(X) &= [H/N, \phi]_{G/N} \cdot [KN/N, \tilde{\psi}]_{G/N} \\ &= \sum_{y \in (H/N)\backslash G/N/(KN/N)} \left[H/N \cap {}^{y}KN/N, \phi \cdot {}^{y}\tilde{\psi} \mid_{H/N \cap {}^{y}KN/N}\right]_{G/N}. \end{aligned}$$

Besides, we have

$$\begin{aligned} \mathrm{Def}^G_{G/N}(\mathrm{Inf}^G_{G/N}(Y) \cdot X) &= \mathrm{Def}^G_{G/N}\left([H, \bar{\phi}]_G \cdot [K, \psi]_G\right) \\ &= \mathrm{Def}^G_{G/N}\Big(\sum_{x \in H\backslash G/K} \left[H \cap {}^{x}K, \bar{\phi} \cdot {}^{x}\psi \mid_{H \cap {}^{x}K}\right]_G\Big) \\ &= \sum_{x \in H\backslash G/K} \left[(H \cap {}^{x}K)N/N, \phi \cdot {}^{x}\tilde{\psi} \mid_{(H \cap {}^{x}K)N/N}\right]_G. \end{aligned}$$

Note that the last equation holds since $N \leq \ker\phi$ implies that $N \cap H \cap {}^{x}K \leq \ker(\bar{\phi} \cdot {}^{x}\psi)$. Observe that, the map

$$f \colon [(H/N)\backslash G/N/(KN/N)] \to [H\backslash G/K]$$

sending gN to g is a bijection. Also note that

$$[H \cap {}^{x}K]N/N = [H \cap {}^{x}KN]/N = [H \cap {}^{x}(KN)]/N.$$

This, together with the above bijection proves the Frobenius relation for this case.

Now, assume $(N \cap K) \nleq \ker\psi$. Then $\mathrm{Def}^G_{G/N}(X) = 0$ implies $Y \cdot \mathrm{Def}^G_{G/N}(X) = 0$. Moreover, we have $(N \cap H \cap {}^{x}K) \nleq \ker(\phi \cdot {}^{x}\psi)$, since $N \leq \ker\phi$. Therefore, we have

$\mathrm{Def}^G_{G/N}(\mathrm{Inf}^G_{G/N}(Y) \cdot X) = 0.$ $\square$

Proposition 3.4. *Let $N \trianglelefteq G$ be a normal subgroup of G. Then, for any A-subelement (H, h) of G we have*

$$\mathrm{Def}^G_{G/N} e^G_{H,h} = m \cdot e^{G/N}_{HN/N,hN}$$

for some constant m.

Proof. Let S be an arbitrary A-fibered G/N-set. Then, using the Frobenius relation, we obtain

$$\begin{aligned} S \cdot \mathrm{Def}^G_{G/N} e^G_{H,h} &= \mathrm{Def}^G_{G/N}(\mathrm{Inf}^G_{G/N}(S) \cdot e^G_{H,h}) \\ &= \mathrm{Def}^G_{G/N}(s^G_{H,h}(\mathrm{Inf}^G_{G/N} S) \cdot e^G_{H,h}) \\ &= \mathrm{Def}^G_{G/N}(s^{G/N}_{HN/N,hN}(S) \cdot e^G_{H,h}) \\ &= s^{G/N}_{HN/N,hN}(S) \cdot \mathrm{Def}^G_{G/N}(e^G_{H,h}). \end{aligned}$$

However, $e^{G/N}_{HN/N,hN}$ is the unique element with the above property. Therefore, we conclude that

$$\mathrm{Def}^G_{G/N} e^G_{H,h} = m \cdot e^{G/N}_{HN/N,hN}$$

for some constant m. $\square$

4. MAIN RESULT

4.1. Minimal Groups

To find all composition factors of the fibered Burnside functor, we need to determine all possible minimal groups of a subfunctor F of kB^A.

Definition 4.1. *Let F be a subfunctor of kB^A. The group G is said to be a minimal group for F if $F(G)$ is non-zero and for any group H with $|H| < |G|$, we have $F(H) = 0$.*

Now, suppose G is a minimal group of a subfunctor F. We know, by Lemma 3.2, that $F(G)$ is an ideal of $B_k(G, A)$. Therefore, it is generated by a set of primitive idempotents $e^G_{H,h}$ of $B_k(G, A)$. Let X be a fibered (L, G)-biset for some group L. If X can be factored through a group K with $|K| < |G|$, then for any $e^G_{H,h} \in F(G)$ we should have $X \cdot e^G_{H,h} = 0$. This implies that to find the minimal groups, we need a deeper understanding of the effect of the fibered bisets that map to groups of smaller order on the idempotents.

If G is a minimal group, then $e^G_{H,h}$ is not contained in $F(G)$ for any proper subgroup $H < G$. Indeed, we have $\mathrm{Res}^G_H e^G_{H,h} = e^H_{H,h} \in F(H)$. Next consider the effect of the deflation map on the idempotents of the form $e^G_{G,g}$. Recall that if $N \trianglelefteq G$ is a normal subgroup of G, then we have

$$\mathrm{Def}^G_{G/N} e^G_{G,g} = m \cdot e^{G/N}_{G/N,gN} \tag{4.1}$$

for some constant m. Since G is minimal, for any non-trivial normal subgroup N of G the constant m should be zero. Following Bouc and Thevénaz, we will consider the elementary abelian p-groups and non-elementary abelian p-groups, separately. Let $\Phi(G)$ denote the Frattini subgroup of G.

Lemma 4.1. *For any p-group G and $g \in G$, we have*

$$\mathrm{Def}^G_{G/\Phi(G)} e^G_{G,g} = \frac{|O(G)|}{|N_G(G,g)|} \cdot |G/\Phi(G)| \cdot e^{G/\Phi(G)}_{G/\Phi(G),g\Phi(G)}.$$

Proof. Recall the idempotent formula

$$e^G_{G,g} = \frac{1}{|N_G(G,g)|} \sum_{(V,\nu)\in_G \mathcal{M}_G(A)} |V|\mu_G(V,\nu;G,g)[V,\nu]_G.$$

By Lemma 3.4, we have

$$\mathrm{Def}^G_{G/\Phi(G)} e^G_{G,g} = \frac{1}{|N_G(G,g)|} \sum_{\substack{(V,\nu)\in_G \mathcal{M}_G(A)\\ V\cap\Phi(G)\leq \ker\nu}} |V|\mu_G(V,\nu;G,g)[V\Phi(G)/\Phi(G),\bar{\nu}]_{G/\Phi(G)}.$$

The equality (4.1), then becomes

$$\begin{aligned}
&\frac{1}{|N_G(G,g)|} \sum_{\substack{(V,\nu)\in_G \mathcal{M}_G(A)\\ V\cap\Phi(G)\leq ker\nu}} |V|\mu_G(V,\nu;G,g)[V\Phi(G)/\Phi(G),\bar{\nu}]_{G/\Phi(G)}\\
&= \frac{m}{|N_{G/\Phi(G)}(G/\Phi(G),g\Phi(G))|} \sum_{(W,\omega)} |W|\mu_{G/\Phi(G)}(W,\omega;G/\Phi(G),g\Phi(G))[W,\omega]_{G/\Phi(G)}.
\end{aligned}$$

where (W,ω) runs over a set of representatives of the $G/\Phi(G)$-conjugacy classes of A-characters of $G/\Phi(G)$. Now, the coefficient of $[G/\Phi(G),1]_{G/\Phi(G)}$ in RHS is

$$\frac{m}{|N_{G/\Phi(G)}(G/\Phi(G),g\Phi(G))|}|G/\Phi(G)|\mu_{G/\Phi(G)}(G/\Phi(G),1;G/\Phi(G),g\Phi(G)).$$

We have

$$\begin{aligned}
&\mu_{G/\Phi(G)}\big(G/\Phi(G),1;G/\Phi(G),g\Phi(G)\big) =\\
&\sum_{(V,\nu)\in[G/\Phi(G),1]_{G/\Phi(G)}} \big|V\cap g\Phi(G)O\big(G/\Phi(G)\big)\big|\mu(V,G/\Phi(G))/|V| = \frac{1}{|G/\Phi(G)|}.
\end{aligned}$$

Thus, the coefficient is

$$\frac{m}{|N_{G/\Phi(G)}(G/\Phi(G), g\Phi(G))|} = \frac{m}{|G/\Phi(G)|}.$$

In LHS, the coefficient of $[G/\Phi(G), 1]_{G/\Phi(G)}$ is

$$\frac{1}{|N_G(G,g)|} \sum_{V\Phi(G)=G} |V|\mu_G(V, 1; G, g).$$

Since $\Phi(G)$ is the Frattini subgroup of G, the equality $V\Phi(G) = G$ implies that $V = G$. But, then

$$\begin{aligned}
\mu_G(G, 1; G, g) &= \sum_{(W,1)\in[G,1]_G} |W \cap gO(G))|\mu(W, G)/|W| \\
&= |gO(G))|/|G|
\end{aligned}$$

and then, the coefficient becomes

$$\frac{|gO(G)|}{|N_G(G,g)|}.$$

Since the coefficients in both sides are equal, we conclude that

$$m = \frac{|O(G)|}{|N_G(G,g)|} \cdot |G/\Phi(G)|.$$

□

Proposition 4.1. *If G is a minimal group of a subfunctor F of kB^A, then G is elementary abelian.*

Proof. Suppose G is non-elementary, then the Frattini subgroup $\Phi(G)$ of G is nontrivial. Also, observe that the coefficient $m = \frac{|O(G)|}{|N_G(G,g)|} \cdot |G/\Phi(G)|$ in the previous lemma is non-zero. Indeed, all the terms $|O(G)|$, $|G/\Phi(G)|$ and $|N_G(G, g)|$ are subgroups of a p-group G. Therefore, m is a power of p and the characteristic q of the field k does not divide

p, so m is non-zero. Since deflation of G to a non-trivial normal subgroup is non-zero, we conclude that G is not a minimal group.

□

Our next goal is to find which elementary abelian p-groups can be a minimal group of a subfunctor of kB^A. We need some preliminary results regarding the action of the deflation map on the primitive idempotents.

Lemma 4.2. *Let G be an elementary abelian group of order p^n and $H =< h >$ be a subgroup for some $h \in G$. Then, we have*

$$\mathrm{Def}^G_{G/H} e^G_{G,1} = \frac{1 - p^{n-1}}{p} \cdot e^{G/H}_{G/H,H}. \tag{4.2}$$

Proof. We have the idempotent formula

$$e^G_{G,1} = \frac{1}{|N_G(G,1)|} \sum_{(V,\nu)\in_G \mathcal{M}_G(A)} |V|\mu_G(V,\nu;G,1)[V,\nu]_G.$$

Here, we have

$$\mu_G(V,\nu;G,1) = \sum_{(V',\nu')\in[V,\nu]_G} \mu(V',\nu';G,1)$$

and

$$\mu(V',\nu';G,1) = (\nu')^{-1}(V' \cap 1 \cdot O(G))\mu(V',G)/|V'|.$$

Now, note that since G is abelian, $(V',\nu') \in [V,\nu]_G$ implies $V' = V$ and $\nu = \nu'$. Also, since G is elementary abelian, we have $O(G) = 1$. Hence, we obtain

$$\mu_G(V,\nu;G,1) = \mu(V,\nu;G,1) = \mu(V,G)/|V|.$$

Thus, the idempotent formula becomes

$$e_{G,1}^{G} = \frac{1}{|G|} \sum_{(V,\nu)\in\mathcal{M}_G(A)} \mu(V,G)[V,\nu]_G.$$

Observe that since G is elementary abelian, G/H is also elementary abelian. So, the arguments above hold for $e_{G/H,H}^{G/H}$, i.e. we have

$$e_{G/H,H}^{G/H} = \frac{1}{|G:H|} \sum_{(V,\nu)\in\mathcal{M}_{G/H}(A)} \mu(V,G/H)[V,\nu]_{G/H}.$$

Thus, we have the equality

$$\frac{1}{|G|} \sum_{(V,\nu)\in\mathcal{M}_G(A)} \mu(V,G)[VH/H,\bar{\nu}]_{G/H} = \frac{m}{|G:H|} \sum_{(W,\omega)\in\mathcal{M}_{G/H}(A)} \mu(V,G/H)[W,\omega]_{G/H}.$$

Note that the coefficient of $[G/H,1]_{G/H}$ in RHS is $\frac{m}{p^{n-1}}$. In LHS, it is $\frac{1}{|G|}\sum_V \mu(V,G)$ where V runs over the subgroups satisfying $VH = G$. But, this implies either that $V = G$ or that V is a complement of H in G. If V is a complement of H, then $|V| = p^{n-1}$. But, in this case, V is maximal subgroup of G. Thus, $\mu(V,G) = -1$. Note that there are p^{n-1} many complements of H. If $V = G$, then obviously we have $\mu(V,G) = 1$. Therefore, the coefficient in LHS becomes $\frac{1-p^{n-1}}{p^n}$. We conclude that $m = \frac{1-p^{n-1}}{p}$. □

Lemma 4.3. *Let G be an elementary abelian group of order p^n and $H =< h >$ be a subgroup for some $h \in G$. If g is an element of G such that $g \notin H$, we have*

$$\text{Def}_{G/H}^{G} e_{G,g}^{G} = \frac{1-p^{n-2}}{p} \cdot e_{G/H,gH}^{G/H}. \tag{4.3}$$

Proof. Recall the idempotent formula

$$e_{G,g}^{G} = \frac{1}{|N_G(G,g)|} \sum_{(V,\nu)\in_G\mathcal{M}_G(A)} |V|\mu_G(V,\nu;G,g)[V,\nu]_G.$$

Note that since G is elementary abelian, $[V,\nu]_G$ has only one element. Also, we have

$O(G) = 1$. Therefore the argument in the proof of the previous lemma applies and we get

$$e_{G,g}^G = \frac{1}{|G|} \sum_{(V,\nu) \in \mathcal{M}_G(A)} \nu^{-1}(g)\mu(V,G)[V,\nu]_G.$$

Similarly,

$$e_{G/H,gH}^{G/H} = \frac{1}{|G:H|} \sum_{(V,\nu) \in \mathcal{M}_G(A)} \nu^{-1}(gH)\mu(V,G/H)[V,\nu]_{G/H}.$$

Now, by applying the deflation map, we get the equality

$$\frac{1}{|G|} \sum_{(V,\nu)} \nu^{-1}(g)\mu(V,G)[VH/H,\bar{\nu}]_{G/H} = \frac{m}{|G:H|} \sum_{(W,\omega)} \omega^{-1}(gH)\mu(V,G/H)[W,\omega]_{G/H}.$$

where (V,ν) runs over characters of G such that $g \in V$ and similarly (W,ω) runs over characters of G/H such that $gH \in W$.

The coefficient of $[G/H,1]_{G/H}$ in RHS is $\frac{m}{p^{n-1}}$. In LHS, it is equal to the sum $\frac{1}{|G|}\sum_V \mu(V,G)$ where V runs over subgroups containing g and satisfying $VH = G$. But, again $VH = G$ implies $V = G$ or V is a complement of H in G. Observe that the number of complements containing g is p^{n-2}. Therefore, the coefficient becomes $\frac{1-p^{n-2}}{p^n}$. Thus, we conclude that $m = \frac{1-p^{n-2}}{p}$.

□

Lemma 4.4. *Let G be an elementary abelian group of order p^n and $H =< h >$ be a subgroup for some $h \in G$. If g is a non-trivial element of H, we then have*

$$\mathrm{Def}_{G/H}^G e_{G,g}^G = \frac{1}{p} \cdot e_{G/H,H}^{G/H}. \tag{4.4}$$

Proof. By similar arguments in the proof of the previous lemma, we have

$$\frac{1}{|G|}\sum_{(V,\nu)} \nu^{-1}(g)\mu(V,G)[VH/H,\bar{\nu}]_{G/H} = \frac{m}{|G:H|}\sum_{(W,\omega)} \omega^{-1}(gH)\mu(V,G/H)[W,\omega]_{G/H}.$$

where (V,ν) runs over characters of G such that $g \in V$ and similarly (W,ω) runs over characters of G/H such that $gH \in W$.

The coefficient of $[G/H,1]_{G/H}$ in RHS is $\frac{m}{p^{n-1}}$. In LHS, it is equal to the sum $\frac{1}{|G|}\sum_V \mu(V,G)$ where V runs over subgroups containing g and satisfying $VH = G$. Note that since $g \in H \cap V$ and $g \neq 1$, V cannot be complement of H and we have $V = G$. Thus, the coefficient becomes $\frac{1}{p^n}$ and we conclude that $m = \frac{1}{p}$.

□

We finally arrived at the main result of this chapter.

Proposition 4.2. *Let F be a subfunctor of kB^A and G be a minimal group of F. Then, G is elementary abelian of order p^r, where $p^{r-1} = 1 \bmod q$. Moreover, $F(G)$ is 1-dimensional generated by $e^G_{G,1}$.*

Proof. By Proposition 4.1, we know that G is elementary abelian. We also know that $F(G)$ is generated by the idempotents of the form $e^G_{G,g}$. Suppose the idempotent $e^G_{G,g}$ is contained in $F(G)$ for some $g \neq 1$. Then, $\mathrm{Def}^G_{G/<g>}(e^G_{G,g}) = \frac{1}{p}e^{G/<g>}_{G/<g>,<g>} \in F(G/<g>)$. But $F(G/<g>) = 0$, contradiction. Therefore, the idempotent $e^G_{G,1}$ generates $F(G)$. Moreover, we should have $p^{r-1} = 1 \bmod q$, by Lemma 4.2. □

4.2. Composition Factors of The Fibered Burnside Functor

In this chapter, we find the composition factors of the fibered Burnside functor kB^A. We need some preliminary results.

Lemma 4.5. *Let F be a fibered biset functor and $\mathcal{H}$ be a set of minimal groups of F. Then, the k-module $K_{\mathcal{H}}(G)$ defined by*

$$K_{\mathcal{H}}(G) = \bigcap_{\substack{{}_H X_G \\ H \in \mathcal{H}}} \ker\big(X : F(G) \to F(H)\big)$$

together with the induced actions of bisets is a subfunctor of F.

Proof. Let $x \in K_{\mathcal{H}}(G)$ be an arbitrary element and Y be an A-fibered (G, L)-biset. Then, for any $H \in \mathcal{H}$ and for any A-fibered (L, H)-biset X, we have

$$X \cdot (Y \cdot x) = (X \otimes_{AL} Y) \cdot x = 0$$

Therefore, by the definition of K_H, we get $Y \cdot x \in K_{\mathcal{H}}(L)$, as required. □

Lemma 4.6. *Let F be a subfunctor of kB^A and let the elementary abelian group E_r of rank r be the minimal group of F. Then, the idempotent $e^{E_r}_{E_r,1}$ generates F, as a fibered biset functor.*

Proof. Suppose, for a contradiction, that the idempotent $e^{E_r}_{E_r,1}$ does not generate F. Let K denote the subfunctor generated by $e^{E_r}_{E_r,1}$. Then, there exists a group G such that for some element $x \in F(G)$, we have $x \notin K(G)$. Since $F(G)$ is an ideal of $B_k(G, A)$, it is generated by a set I of primitive idempotents. Thus, in the primitive idempotent basis, we have $x = \sum_I x^G_{H,h} \cdot e^G_{H,h}$ for some $x^G_{H,h} \in k$. This implies that for some A-subelement (H, h) of G, the idempotent $e^G_{H,h} \notin K(G)$. Suppose G has minimal order with respect to $F(G) \neq K(G)$.

Suppose, for a contradiction, that $e^G_{H,h} \notin K(G)$ and $H \neq G$. Then, by the minimality of G, we have $\mathrm{Res}^G_H e^G_{H,h} \in K(H)$. So, for some A-fibered (H, G)-biset X, we have $\mathrm{Res}^G_H e^G_{H,h} = X \cdot e^{E_r}_{E_r,1}$. Thus, we have $\mathrm{Ind}^G_H \mathrm{Res}^G_H e^G_{H,h} = (\mathrm{Ind}^G_H X) e^{E_r}_{E_r,1}$. But note that

$$e^G_{H,h} \cdot \mathrm{Ind}^G_H \mathrm{Res}^G_H e^G_{H,h} = \alpha \cdot e^G_{H,h}$$

for some non-zero $\alpha \in k$. Thus, we have

$$e^G_{H,h} \cdot \big((\mathrm{Ind}^G_H X) e^{E_r}_{E_r,1}\big) = \alpha \cdot e^G_{H,h}$$

which implies that $e^G_{H,h} = \frac{1}{\alpha} \cdot (\widetilde{e^G_{H,h}} \cdot \mathrm{Ind}^G_H X) \cdot e^{E_r}_{E_r,1}$, contradiction. So we should have $H = G$.

Now, suppose, for a contradiction, that $e^G_{G,g} \notin K(G)$ and the Frattini subgroup $\Phi(G)$ is non-trivial. Then, again, by the minimality of G, we have $\mathrm{Def}^G_{G/\Phi(G)} e^G_{G,g} = X \cdot e^{E_r}_{E_r,1}$ for some A-fibered $(G/\Phi(G), G)$-biset X. Note that

$$e^G_{G,g} \cdot (\mathrm{Inf}^G_{G/\Phi(G)} \mathrm{Def}^G_{G/\Phi(G)} e^G_{G,g}) = \beta \cdot e^G_{G,g}$$

for some non-zero $\beta \in k$. Thus, we have

$$e^G_{G,g} \cdot (\mathrm{Inf}^G_{G/\Phi(G)} X \cdot e^{E_r}_{E_r,1}) = \beta \cdot e^G_{G,g}$$

which implies $e^G_{G,g} = \frac{1}{\beta} \cdot (\widetilde{e^G_{G,g}} \cdot \mathrm{Inf}^G_{G/\Phi(G)} X) \cdot e^{E_r}_{E_r,1}$, contradiction. So, we have $\Phi(G) = 1$ and G is elementary abelian.

Finally, suppose $e^G_{G,g} \notin K(G)$ and G is elementary abelian. If $g \neq 1$, then we have $\mathrm{Def}^G_{G/<g>} e^G_{G,g} = \frac{1}{p} e^{G/<g>}_{G/<g>,<g>}$ and by similar arguments, we obtain a contradiction. Therefore, suppose $g = 1$. Since E_r is a minimal group for F, we have $E_r \trianglelefteq G$. So, the idempotent $e^G_{G,1}$ is a summand of $\mathrm{Inf}^G_{E_r} e^{E_r}_{E_r,1}$. This is, again, a contradiction, since $\mathrm{Inf}^G_{E_r} e^{E_r}_{E_r,1} \in K(G)$. This proves the lemma.

□

Remark 4.1. By Lemma 4.6, the A-fibered Burnside functor of p-groups kB^A is uniserial. In other words, we have a composition series

$$kB^A = F_0 \supset F_1 \supset F_2 \supset F_3 \supset \cdots$$

of subfunctors such that F_{i+1} is the unique maximal subfunctor of F_i for each i.

Let $\mathcal{I} = \{0\} \cup \{r \in \mathbb{N} \mid p^{r-1} \equiv 1 \pmod{q}\}$ be the set of ranks of possible minimal groups for the subfunctors of kB^A. Enumerate the elements of $\mathcal{I} = \{r_i\}_{i=0}^{\infty}$ such that $i < j$ implies $r_i < r_j$.

Lemma 4.7. *Let F be a subfunctor of kB^A with minimal group E_{r_i}. If $L \subset F$ is the unique maximal subfunctor of F, then the minimal group of L is $E_{r_{i+1}}$.*

Proof. Suppose the minimal group of L is E_{r_j}. Since L is a proper subfunctor, by Lemma 4.6, we should have $i < j$. Let K denote the subfunctor of F generated by the idempotent $e^{E_{r_{i+1}}}_{E_{r_{i+1}},1}$. Then, K is a proper subfunctor of F. Indeed, every A-fibered $(E_{r_i}, E_{r_{i+1}})$-biset decomposes as in Theorem 2.1. However, the image of $e^{E_{r_{i+1}}}_{E_{r_{i+1}},1}$ under the restriction and the deflation maps are zero. Thus, we have $K(E_{r_i}) = 0$. Now, L being maximal guaranties that we have $K(E_{r_{i+1}}) \subseteq L(E_{r_{i+1}})$. Since $K(E_{r_{i+1}})$ is non-zero, we conclude that $j \leq i+1$ which implies $j = i+1$. □

Proposition 4.3. *Let K_0 denote kB^A. Define $\mathcal{H}_i$ as the minimal group for K_i, and recursively, $K_{i+1} = K_{\mathcal{H}_i}$. Then, K_{i+1} is the unique maximal subfunctor of K_i with the minimal group $E_{r_{i+1}}$.*

Proof. First of all, note that by Lemma 4.5, K_{i+1} is indeed a subfunctor. To see that it is maximal, let $F \subset K_i$ be a proper subfunctor. We need to show that for any group G, $F(G) \subseteq K_{i+1}(G)$ holds true. It suffices to show that $F(E_{r_i}) = 0$. Indeed, let $x \in F(G)$ be an arbitrary element. Then, for any A-fibered (E_{r_i}, G)-set ${}_{E_{r_i}}X_G$, we have ${}_{E_{r_i}}X_G \cdot x \in F(E_{r_i}) = 0$. It follows that $x \in K_{E_{r_i}}(G)$ by the definition of K_{i+1}. Now, note that we have $F(E_{r_i}) \subseteq K_i(E_{r_i}) \cong k \cdot e^{E_{r_i}}_{E_{r_i},1}$. Since F is proper and K_i is generated by $e^{E_{r_i}}_{E_{r_i},1}$, we conclude that $F(E_{r_i}) = 0$. This shows that K_{i+1} is the unique maximal subfunctor of K_i. The minimal group of K_{i+1} is E_{i+1} by Lemma 4.2. $\square$

We showed that for each $r_i \in \mathcal{I}$, K_i is a subfunctor of kB^A. Thus, we examine the set $\mathcal{I} = \{0\} \cup \{r \in \mathbb{N} \mid p^{r-1} \equiv 1 \pmod{q}\}$ more closely. If $q = 0$, then we have $\mathcal{I} = \{0, 1\}$. If $q \neq 0$, then $\mathcal{I}$ consists of all positive integers congruent to 1 modulo s where s is the order of p modulo q. Note that if q divides $p - 1$, then $s = 1$ and $\mathcal{I}$ consists of all positive integers.

It can be given as a remark that by this proposition for each i the functor K_i/K_{i+1} is a simple Burnside functor with minimal group E_{r_i}. We have the following theorem related to simple fibered biset functors. See appendix for the proof.

Theorem 4.1. *Let A be an abelian group satisfying Hypothesis A and k be a field. Then there is a bijective correspondence between*

(i) the set of isomorphism classes of simple A-fibered biset functors with an abelian minimal group and

(ii) the set of pairs (G, V) where G runs over all finite abelian groups, up to isomorphism, and for each G, V runs over the isomorphism classes of simple $k[G^* \rtimes \mathrm{Out}(G)]$-modules.

Here for each simple A-fibered biset functor S, the group G is minimal for S and $S(G) = V$. For each pair (G, V), we denote by $S_{G,V}$ the corresponding simple A-fibered biset functor.

Remark 4.2. This theorem still holds if A is a finite cyclic p-group and minimal groups are elementary abelian.

Proposition 4.4. *For each i, $K_i/K_{i+1}(E_{r_i})$ is the trivial $k[E^*_{r_i} \rtimes \mathrm{Out}(E_{r_i})]$-module.*

Proof. We have $K_i/K_{i+1}(E_{r_i}) = k \cdot e^{E_{r_i}}_{E_{r_i},1}$. Hence it suffices to check the effects of $\mathrm{Tw}^{\phi}_{E_{r_i},E_{r_i}}$ and $c^{\lambda}_{E_{r_i},E_{r_i}}$ on the idempotent $e^{E_{r_i}}_{E_{r_i},1}$ where $\phi \in E^*_{r_i}$ and $\lambda \in \mathrm{Out}(E_{r_i})$.

Observe that

$$\mathrm{Tw}^{\phi}_{E_{r_i},E_{r_i}} \cdot e^{E_{r_i}}_{E_{r_i},1} = \phi(1) \cdot e^{E_{r_i}}_{E_{r_i},1} = e^{E_{r_i}}_{E_{r_i},1} \tag{4.5}$$

and

$$c^{\lambda}_{E_{r_i},E_{r_i}} \cdot e^{E_{r_i}}_{E_{r_i},1} = e^{E_{r_i}}_{\lambda(E_{r_i}),\lambda(1)} = e^{E_{r_i}}_{E_{r_i},1} \tag{4.6}$$

□

We have proved the following theorem.

Theorem 4.2. *Let A be a cyclic p-group and k be a sufficiently large field of characteristic q with $q \neq p$. Then, A-fibered Burnside functor of p-groups kB^A has a unique filtration*

$$\begin{aligned} &kB^A = K_0 \supset K_1 \supset K_\infty = \{0\} && \textit{if } q = 0, \\ &kB^A = K_0 \supset K_1 \supset K_2 \supset K_3 \supset \cdots && \textit{if } q \neq 0 \textit{ and } q \mid p-1, \\ &kB^A = K_0 \supset K_1 \supset K_{1+s} \supset K_{1+2s} \supset \cdots && \textit{if } q \neq 0 \textit{ and } q \nmid p-1. \end{aligned}$$

Moreover, for each i, $K_i/K_{i+1}(E_{r_i})$ is the trivial $k[E^*_{r_i} \rtimes \mathrm{Out}(E_{r_i})]$-module and we have $K_i/K_{i+1} \cong S_{E_{r_i},1}$.

APPENDIX A: Fibered bisets for abelian groups

The whole work in this section is from unpublished notes of Coşkun.

In this section, we prove a decomposition theorem for fibred bisets for abelian groups under certain assumptions. This case is simpler then the general case given in Theorem 10.14 in [11] but still needs Hypothesis 10.1 in [11]. We recall this hypothesis below. Using this decomposition, we classify simple fibered biset functors with abelian minimal groups, which also turns out to be much simpler than the general case.

Hypothesis. For the rest of the paper, we assume that A satisfies the following hypothesis.

There is a (unique) set π of primes such that for every $n \in \mathbb{N}$, the n-torsion part of A is cyclic of order n_π, where n_π denotes the π-part of n.

Note that the cyclic groups μ_n of the previous section do not satisfy the above hypothesis. Thus the decomposition below cannot be applied to this case with all the generalities. However, if we further restrict to elementary abelian groups, as we did in the previous section, the decomposition will still hold although the hypothesis is not satisfied.

Now we have the following decomposition theorem. Let G, H be finite abelian groups and let A be an abelian group. Let also $(U, \phi) \in \mathcal{M}_{G\times H}(A)$. We write (P, K, η, Q, L) for the invariants $(p_1(U), k_1(U), \eta, p_2(U), k_2(U))$ where η is the canonical isomorphism between P/K and Q/L determined by U. We also write $\tilde{\phi} = \tilde{\phi}_1 \times \tilde{\phi}_2$ for an extension of ϕ to $P \times Q$ which exists since the group $P \times Q$ is abelian and A is divisible (by the hypothesis).

Theorem A.1. *Let G, H be finite abelian groups and let A be an abelian group satis-*

fying Hypothesis A. Let also $(U,\phi)\in\mathcal{M}_{G\times H}(A)$*. Then*

$$\Big(\frac{G\times H}{U,\phi}\Big)\cong \mathrm{Ind}_P^G\mathrm{Tw}_P^{\tilde\phi_1}\mathrm{Inf}_{P/K}^P\mathrm{c}_{P/K,Q/L}^{\eta}\mathrm{Def}_{Q/L}^Q\mathrm{Tw}_Q^{\tilde\phi_2}\mathrm{Res}_Q^H.$$

Proof. We evaluate the Mackey product on the right hand side. Without loss of generality, we may assume that $P=G$ and $Q=H$. Note that, in the above product, the stabilizer for $\mathrm{Tw}_P^{\tilde\phi_1}$ is $\Delta(P)$ and that for $\mathrm{Tw}_Q^{\tilde\phi_2}$ is $\Delta(Q)$. Also, it is clear from its definition that for any subgroups $V\le G\times P$ and $V'\le Q\times H$, we have $V*\Delta(P)=V$ and $\Delta(Q)*V'=V'$. Thus, the $*$-product of the stabilizers on the right hand side gives the subgroup U, by Bouc's theorem for ordinary bisets. Thus we only need to check that $\phi=(\tilde\phi_1*1)*(1*\tilde\phi_2)$. Let $(g,h)\in U$. Then we have

$$\begin{aligned}(\tilde\phi_1*1)*(1*\tilde\phi_2)(g,h) &= (\tilde\phi_1*1)(g,gK)\cdot(1*\tilde\phi_2)(hL,h)\\ &= \tilde\phi_1(g)\cdot\tilde\phi_2(h)\\ &= (\tilde\phi_1\times\tilde\phi_2)(g,h)\\ &= \phi(g,h)\end{aligned}$$

which completes the proof of the theorem. □

Next, we classify simple fibered biset functors with minimal non-zero evaluations at abelian groups. We need some notation. Let E_G denote the endomorphism ring of G in $\mathcal{C}$ and let $\bar E_G$ denote the subalgebra of E_G consisting of the A-fibered (G,G)-bisets which do not factor through a group of smaller order. The algebra $\bar E_G$ is crucial for the classification of the simple functors and its structure is described in [11, Section 8]. However, it is simpler when the group G is abelian. Indeed, suppose G is abelian and $\big(\frac{G\times G}{U,\phi}\big)$ be a transitive A-fibered (G,G)-biset which does not factor through a group of smaller order. Then by Theorem 2.1, we have

$$P=Q=G,\quad K=L=1,\quad U=\{(g,\lambda(g))\in G\times G|\lambda\in\mathrm{Out}(G)\}$$

and

$$\left(\frac{G\times G}{U,\phi}\right) = \mathrm{Tw}_G^{\tilde{\phi}_1} \otimes_{AG} \mathrm{c}_{G,G}^{\lambda} \otimes \mathrm{Tw}_G^{\tilde{\phi}_2}.$$

Now it follows easily from the Mackey product formula that

$$\mathrm{c}_{G,G}^{\lambda} \otimes_{AG} \mathrm{Tw}_G^{\phi} \cong \mathrm{Tw}_G^{\phi\circ\lambda} \otimes_{AG} \mathrm{c}_{G,G}^{\lambda}.$$

Therefore the algebra $\bar{E}_G$ is generated by all A-fibered (G,G)-bisets of the form $\mathrm{Tw}_G^{\phi}\otimes_{AG}$ $\mathrm{c}_{G,G}^{\lambda}$ where $\phi \in G^*$ and $\lambda \in \mathrm{Out}(G)$. For simplicity, we put

$$[\phi,\lambda]_G := \mathrm{Tw}_G^{\phi} \otimes_{AG} \mathrm{c}_{G,G}^{\lambda}.$$

To determine the algebra structure, let $\lambda,\mu \in \mathrm{Out}(G)$ and $\phi,\psi \in G^*$. Then it follows from the Mackey product formula that we have

$$[1,\lambda]_G \cdot [1,\mu]_G = [1,\lambda\cdot\mu]_G$$

$$[1,\lambda]_G \cdot [\phi,1]_G \cdot [1,\lambda^{-1}] = [\phi\circ\lambda,1]_G$$

and

$$[\phi,1]_G \cdot [\psi,1]_G = [\phi\circ\psi,1]_G.$$

In particular, we see that the algebra $\bar{E}_G$ is isomorphic with the group algebra $k[G^* \rtimes \mathrm{Out}(G)]$.

Simple functors. Let S be a simple fibered biset functor. Assume that $S(G) \neq 0$ for an abelian group G and $S(H) = 0$ for any group of order less than $|G|$, (so that G is a minimal group for S). It is well-known that, in this case, $S(G)$ is a simple

E_G-module.

Now since G is minimal for S, any A-fibered (G, G)-biset which factors through a group of smaller order annihilates $S(G)$. Therefore the evaluation $V = S(G)$ is actually a simple $\bar{E}_G$-module. Hence, by the previous paragraph, V is a simple $k[G^* \rtimes \mathrm{Out}(G)]$-module. Moreover, by [11, Proposition 9.4], any other minimal group for S should be isomorphic with the group G.

Indeed, by Theorem 9.2 in [11], any simple A-fibered biset functor is of the form $S_{G,K,\kappa,V}$ for some quadruple (G, K, κ, V). Here, under the hypothesis on A, G is a finite group, $K \leq Z(G) \cap G'$ is a subgroup of G, κ is a faithful character of K and V is a module for the group algebra $k\Gamma_{G,K,\kappa}$. (We refer to ?? for the definition of the group $\Gamma_{G,K,\kappa}$.) Therefore, if G is abelian, then there is only one choice for (K, κ), namely $(1, 1)$, the trivial group with the trivial character. Moreover, in this case, the definition of the group $\Gamma_{G,1,1}$ implies that $\Gamma_{G,1,1} \cong G^* \rtimes \mathrm{Out}(G)$.

On the other hand, let H be another minimal group for S. We claim that $H \cong G$. Indeed, by Theorem 9.2 in [11], we have $|H| = |G|$ and by Proposition 9.5 in [11], if $S(H)$ is non-zero then there is a section $H_1 \trianglelefteq H_2 \leq H$ of H and a subgroup L of $H^* = H_2/H_1$ such that $G \cong H^*/L$ and $L \cap (H^*)' = 1$. Now since $|G| = |H|$, we conclude that $G \cong H$.

As a result, we have proved the following theorem.

Theorem A.2. *Let A be an abelian group satisfying Hypothesis A and k be a field. Then there is a bijective correspondence between*

(i) the set of isomorphism classes of simple A-fibered biset functors with an abelian minimal group and

(ii) the set of pairs (G, V) where G runs over all finite abelian groups, up to isomorphism, and for each G, V runs over the isomorphism classes of simple $k[G^* \rtimes \mathrm{Out}(G)]$-modules.

The case of elementary abelian groups. Let p be a prime number and A be a finite cyclic p-group. Then the Hypothesis A is not satisfied. However, the previous results of this section holds for elementary abelian groups. Indeed, let G and H be elementary abelian p-groups and $(U, \phi) \in \mathcal{M}_{G\times H}(A)$. Let (P, K, η, L, Q) be the quintuple determined by U. Then we can extend the homomorphism ϕ to $P \times Q$ since $P \times Q$ is elementary abelian and A contains an element of order p. Therefore the decomposition of Theorem 2.1 holds for elementary abelian groups. Moreover the above classification of simple functors holds when we take A as a finite cyclic p-group and minimal groups as elementary abelian.

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