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DEPARTMENT OF MECHANICAL ENGINEERING

THESIS FOR M. S. DEGREE

" THE SHALLOW WATER ANALOGY IN GAS DYNAMICS "

INSTRUCTOR

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CONTENTS

Introduction ii
Theory of Shock Waves 1
Stagnation Properties 4
Shock Waves 7
Oblique Shock Waves 10
Fluid Motion Visualisation Methods Depending upon
Changes of Refractive Index...........................13
Schlieren Method......................................15
The Interferometer Method 17
The Shallow Water Analogy 21
The Shallow Water Analogy Tank 35
Plates 37
Graphs 42
Bibliography 49

INTRODUCTION

With the advance of supersonic flight the need for better and more powerful wind tunnels is greatly increased. When one considers how expensive the building, keeping and running of the simplest wind tunnel is, one may form an idea of the cost even a 2or3 Mach tunnel. This view has stimulated the scientists to look for a less expensive method of visualising supersonic flow.

One such method is the " Shallow Water Analogy " method. Analogies are oftenneeded in engineering when they give an aesier method of doing something - sometimes even impossible things, or as in our case, when they are more economical.

In this method we have a water flow of very small depth - about 5 to 10 mm. and the phenomena occuring in this flow are analogous to those occuring in gas flow. The tank in which the water is allowed to flow need not be very big of course, in fact the tank used in preparation of this thesis occupies a space nobigger than 40 cm. by 190 cm. by 110 cm. The power needed to run it is of course none but that of the normal city water supply, if this is not present, a simple water reservoir will do.

We shall now establish the analogy and give some results obtained from actual tests.

I. THEORY OF GAS FLOW AND SHOCK WAVES

The sonic velocity is the velocity at which a pressure disturbance travelsrelative to the gas.

From the fundamental theory of gas dynamics we know that the sonic velocity of a gas is given by: $a = \sqrt{\frac{dp}{d\rho}}$

Since for a polytropic gas $pv^k =$ const. where:

$K = \frac{c_p}{c_v}$ and $v = \frac{1}{g\rho}$ $\qquad \frac{p}{\rho^k} = \text{const.}$

Then: $\frac{dp}{p} - K\frac{d\rho}{\rho} = 0$ so $\frac{dp}{d\rho} = K\frac{p}{\rho}$

and from $\frac{p}{\rho} = g\bar{R}T$ $\qquad \frac{dp}{d\rho} = Kg\bar{R}T$

so: $\boxed{a = \sqrt{\frac{dp}{d\rho}} = \sqrt{Kg\bar{R}T}}$

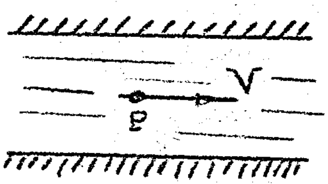


Fig. 1

Because at a given point P we have a given set of characteristic properties p, v, T we also have a fixed sonic velocity (a)

Then for that point,

$\boxed{\frac{V}{a} = M}$ is defined as the Mach number.

For $M < 1$ flow is subsonic

$M = 1$ flow is sonic

$M > 1$ flow is supersonic

Assume a non-flow medium with a disturbance at point O and the disturbance is an electric spark with a spark interval Δt of constant frequency. Since the propagation will be at sonic velocity at all points radially away from the disturbance center, the disturbances will propagate in the form of concentric spheres and the radial distance (wave length) will be $\lambda = a\Delta t$

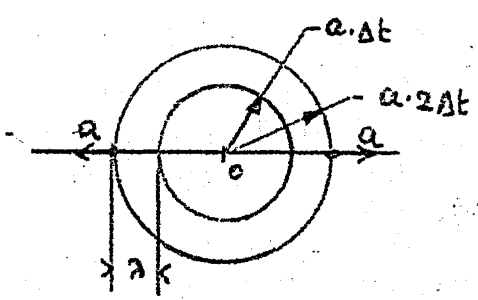


Fig. 2

Assuming now that the source of disturbance is in a flowing medium of velocity $V < a$ or $M < 1$

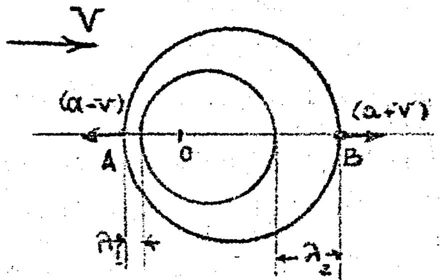


Fig. 3

Now the spheres of disturbance will no longer be concentric because the spheres will now propagate such that their geometric centers will move along with the fluid- downstream with a velocity V.

The upstream wave front will travel with a velocity $(a-V)$, the downstream wavefront will travel with a velocity $(a+V)$. And $\lambda_1 = (a-V)\Delta t$, $\lambda_2 = (a+V)\Delta t$

Now we shall assume that the flow is $V > a$ or $M > 1$ (supersonic). For this type of flow the spheres will no longer be within each other because the upstream wave front will have an absolute velocity of $(a-V)$,and since $V > a$ the wave front will have an absolute velocity in the downstream direction and so point A may never be upstream of O, and so when a new disturbance wave is produced the original sphere will have moved away from O.

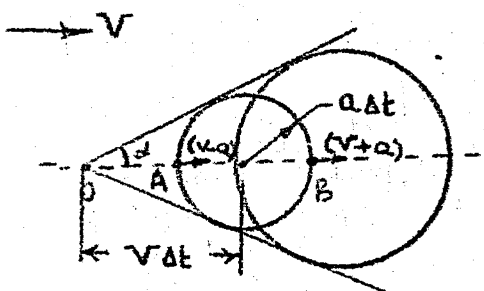


Fig. 4

Again the velocity at A will be $(V-a)$, the velocity at B $(V+a)$

Analyzing this further: $\sin\alpha = \frac{a \cdot t}{V \cdot t} = \frac{a}{V} = \frac{1}{M}$ $\qquad \alpha = \sin^{-1}\frac{1}{M}$

Therefore, the tangent does not depend on time and so will always be tangent. The cone formed by this tangent is called the Mach Cone, the volume contained is the 3 zone of action", the volume outside the cone is a place the disturbances shall never attain hence is called "the zone of silence" . For $M = 1$ (sonic flow) $\sin\alpha = 1$ and $\alpha = 90°$ the cone will become a plane, there will be an accumulation of pressure and we shall get a shock wave. Thus a shock wave is formed by an accumulation of disturbance such that there is am sudden change of pressureaccross it.This also tells us that a Mach Line is the weakest kind of shock wave since there is only one tangent sound wave at any given point.

We may add here that in a flowing medium the disturbance need not be an electric spark. Abody in the path of the flow would produce stagnatic pressure atleast at one point, and this rise of pressure would propagate at sonic velocity.

We shall now prove that shock waves can only occur in a decelerating flow.

Assuming a decelerating flow as shown below:

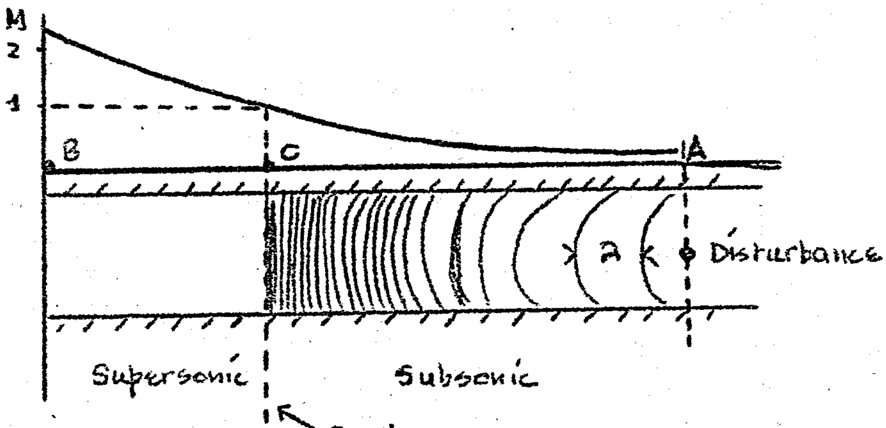


Fig. 5

A disturbance is propagating from A towards B. We know that the wavelength $\lambda = \Delta t(a-V)$ Then $\lambda = \Delta t \cdot a(1-M)$

At C we have $M = 1$ then $\lambda = 0$ and we get a shock wave.

Butt ifthe disturbance is at B then the wavelength will be

$\lambda = \Delta t(V+a) = \Delta t a(1+M)$ and so λ will never be zero.

Stagnation (total) Properties

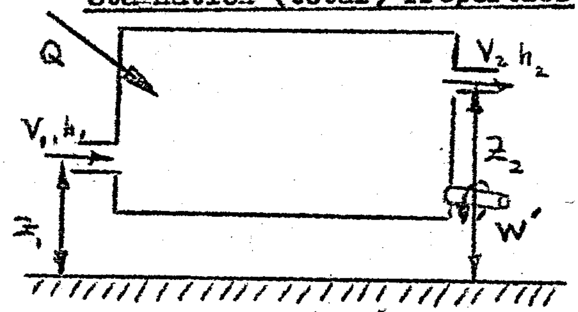


Fig. 6

The energy equation:

$$Q+h_1+\frac{V_1^2}{2gJ}+\frac{z_1}{J}=W'+h_2+\frac{V_2^2}{2gJ}+\frac{z_2}{J}$$

For no shaft work, adiabatic flow and when $z_2 - z_1$ is small, $\frac{1}{J}(z_2-z_1)$ is almost zero.

Then: $h_1+\frac{V_1^2}{2gJ}=h_2+\frac{V_2^2}{2gJ}=\text{const.}$

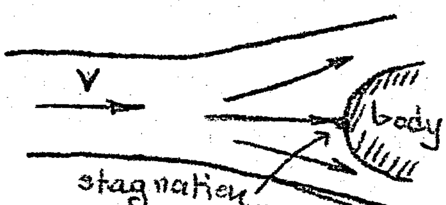


Fig. 7

If we introduce a body in the flow we have a point where $V = 0$ this is called stagnation point. Then

$$h_1+\frac{V_1^2}{2gJ}=h_2+0$$

h_2 is called the stagnation or total enthalpy and will be denoted by h_0

Differentiating $h+\frac{V^2}{2gJ}=\text{const}$:

$$dh+\frac{VdV}{gJ} \quad \text{since} \quad dh=c_p dT$$

$$dT+\frac{VdV}{gJc_p} \quad \text{then} \quad T+\frac{V^2}{2gJc_p}=\text{const.}$$

For the stagnation point: $T_1+\frac{V_1^2}{2gJc_p}=T_2=T_0$

Again T_0 is total or stagnation enthalpy.

At the same point we have then p_0, ρ_0, T_0, h_0 etc. Total properties.

Since this compression takes place in a very short time, it isadiabatic. So:

$$\left[\frac{p_0}{p}\right]^{\frac{k-1}{k}}=\frac{T_0}{T} \qquad \left[\frac{\rho_0}{\rho}\right]^{k}=\frac{p_0}{p}$$

and so on.....

Now let us look for $T=T(M)$, $p=p(M)$

$c_p - c_v = \frac{\bar{R}}{J}$ $\qquad 1-\frac{1}{K} = \frac{\bar{R}}{Jc_p}$ $\qquad Jc_p = \left(\frac{K}{K-1}\right)\bar{R}$

$TJc_p g = Tg\left(\frac{K}{K-1}\right)\bar{R}$ but $a^2 = gK\bar{R}T$

$Jc_p g = \frac{a^2}{T(K-1)}$ $\qquad T + \frac{V^2}{\frac{2a^2}{T(K-1)}} = T_o$

and $\frac{T_o}{T} = 1 + \frac{K-1}{2}M^2$ then $\frac{p_o}{p} = \left[1+\frac{K-1}{2}M^2\right]^{\frac{K-1}{K}}$

Derivation of Euler's Equation:

$dh + \frac{VdV}{gJ} = 0$ $\qquad h = u + \frac{pv}{J} \rightarrow dh = du + \frac{pdv}{J} + \frac{vdp}{J}$

but $dQ = du + \frac{pdv}{J} = 0$ therefore $dh = \frac{vdp}{J}$

$\frac{vdp}{J} + \frac{VdV}{gJ} = 0$ $\qquad \frac{dp}{\rho g} + \frac{VdV}{g} = 0 \rightarrow \boxed{\frac{dp}{\rho} + VdV = 0}$

for compressible or incompressible flow

Flow Systems

For most engineering applications we assume flow beginning from an infinite reservoir, hence flow velocity in reservoir is zero and conditions are stagnatic conditions.

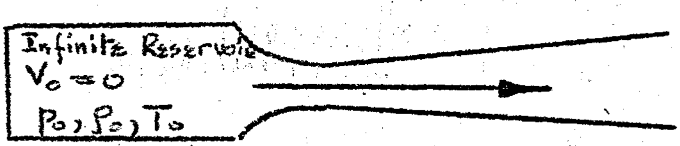


Fig.8

Weknow that for adiabatic flow :

$dh = \frac{vdp}{J} = \frac{dp}{g\rho J}$ or $c_p dT = \frac{dp}{g\rho J} \rightarrow \frac{dp}{\rho} = c_p g J dT$

also $\frac{dp}{\rho} + VdV = 0$ Euler's equation

then $c_p g J dT + VdV = 0$ also $c_p - c_v = \frac{\bar{R}}{J} \rightarrow c_p J = \left(\frac{K}{K-1}\right)\bar{R}$

so that $g\left(\frac{K}{K-1}\right)\bar{R}dT + VdV = 0$

then: $g\left(\frac{K}{K-1}\right)\bar{R}\int_{T_o}^{T} dT + \int_0^V VdV = 0$

Solving and simplifying: $\frac{a^2}{K-1} + \frac{V^2}{2} = \frac{a_o^2}{K-1}$

From the preceding equation we see that as V increases a decreases because $a_o = const$

When $a = V$, $(M = 1)$ $\quad a = a^*$ critical sonic velocity

$V = V^*$ critical flow velocity

at the same point we have p^*, ρ^*, T^*

At critical conditions: $\frac{a^{*2}}{K-1} + \frac{V^{*2}}{2} = \frac{a_o^2}{K-1}$ from $a^* = V^*$

$\sqrt{gK\bar{R}T^*} = \sqrt{\frac{2}{K-1}}\sqrt{gK\bar{R}T_o}$ then $V^* = a^* = \sqrt{\frac{2}{K-1}}\, a_o$

and $\frac{T^*}{T_o} = \frac{2}{K-1}$ since $\frac{p^*}{p_o} = \left[\frac{T^*}{T_o}\right]^{\frac{K}{K-1}}$

$\frac{p^*}{p_o} = \left[\frac{2}{K-1}\right]^{\frac{K}{K-1}}$ also $\frac{\rho^*}{\rho} = \left[\frac{2}{K-1}\right]^{\frac{1}{K-1}}$

again from $\frac{a^2}{K-1} + \frac{V^2}{2} = \frac{a_o^2}{K-1}$

$a = \sqrt{a_o^2 - \left(\frac{K-1}{2}\right)V^2}$ and $M = \frac{V}{\sqrt{a_o^2 - \left(\frac{K-1}{2}\right)V^2}}$

also: $M^* = \frac{V}{a^*} = \frac{V}{\sqrt{\frac{2}{K-1}}\, a_o}$

combining and simplifyng:

$$M^* = \sqrt{\frac{(K-1)M^2}{2 + (K-1)M^2}}$$

Thus for: $M = 1$, $M^* = 1$

and for : $M = \infty$, $M^* = \sqrt{\frac{K+1}{K-1}}$

Shock Waves

By the Kinetic Theory of Gas the thickness of a shock wave is given as 2 microns.

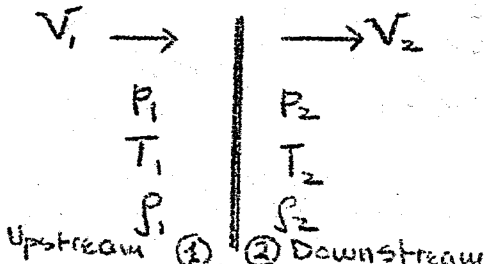


Fig.9

Therefore, the changes across a shock wave are adiabatic. We have a constant area, one dimensional flow but we also assume that the energy principle does not hold across a shock wave so that we have a change of states. Another assumption is that the total temperature does not change. As wild as these assumptions may seem, the derived equations based on these assumptions are in perfect conformity with obtained experimental data.

$\frac{dp}{\rho} + VdV = 0$ we cannot integrate this but,

$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$ since $A_1 = A_2$

$\rho V = \text{const.}$ $\qquad \frac{dp}{\rho V} + dV = 0$

$\frac{p_2 - p_1}{\rho V} + (V_2^2 - V_1^2) = 0$ and from $\rho = \frac{1}{vg}$

$$\boxed{(p_2 - p_1) + \left(\frac{V_2^2}{gv_2} - \frac{V_1^2}{gv_1}\right) = 0}$$

from
$$\left.\begin{aligned} p_1 v_1 &= \bar{R}T_1 \\ p_2 v_2 &= \bar{R}T_2 \end{aligned}\right\} \quad p_2 - p_1 = \frac{KV_1^2 p_1}{Kg\bar{R}T_1} - \frac{KV_2^2 p_2}{Kg\bar{R}T_2}$$

so
$$p_2 - p_1 = Kp_1M_1^2 - Kp_2M_2^2$$

and
$$\frac{p_2}{p_1} = \frac{1 + KM_1^2}{1 + KM_2^2}$$

But this tells us nothing since we do not know about the downstream Mach number.

We shall now try to relate the Mach number of the downstream side to the Mach number of the upstream side.

$$T_{o_1} = T_1\left[1+\left(\frac{K-1}{2}\right)M_1^2\right]$$

$$T_{o_2} = T_2\left[1+\left(\frac{K-1}{2}\right)M_2^2\right] \qquad \text{But } T_{o_1} = T_{o_2}$$

so $\dfrac{T_2}{T_1} = \dfrac{2+(K-1)M_1^2}{2+(K-1)M_2^2}$, $\dfrac{V}{v} = \text{const}$, $V = M\sqrt{gK\bar{R}T}$, $v = \dfrac{\bar{R}T}{P}$

combining: $\dfrac{M_2}{M_1} = \dfrac{p_1}{p_2}\cdot\dfrac{\sqrt{T_2}}{\sqrt{T_1}}$

Substituting: $\dfrac{M_2}{M_1} = \dfrac{1+KM_2^2}{1+KM_1^2}\cdot\left[\dfrac{2+(K-1)M_1^2}{2+(K-1)M_2^2}\right]^{1/2}$

Simplifying: $\boxed{M_2^2 = \dfrac{2+(K-1)M_1^2}{2KM_1^2-(K-1)}}$

We also know that: $M_1^{*2} = \dfrac{(K+1)M_1^2}{M_1^2(K-1)+2}$ and $M_2^{*2} = \dfrac{(K+1)M_2^2}{M_2^2(K-1)+2}$

so that $M_1^2 = \dfrac{2M_1^{*2}}{(K+1)-(K-1)M_1^{*2}}$ and $M_2^2 = \dfrac{2M_2^{*2}}{(K+1)-(K-1)M_2^{*2}}$

Substituting into the preceding formula and simplifying:

$$\boxed{M_1^* \cdot M_2^* = 1}$$ Rankine- Hugoniot equation

since $M_1^* = \dfrac{V_1}{a^*}$ and $M_2^* = \dfrac{V_2}{a^*}$

$$V_1 \cdot V_2 = a^*$$ Prandtl equation

Now we may come back and find $\dfrac{p_2}{p_1}$

$\dfrac{p_2}{p_1} = \dfrac{1+KM_1^2}{1+K\left[\dfrac{2+(K-1)M_1^2}{2KM_1^2-(K-1)}\right]}$ and $\boxed{\dfrac{p_2}{p_1} = \dfrac{2KM_1^2-(K-1)}{1+K}}$

by the same token: $\boxed{\dfrac{T_2}{T_1} = \dfrac{[2+(K-1)M_1^2][2KM_1^2-(K-1)]}{M_1^2(K+1)^2}}$

and from $\dfrac{\rho_2}{\rho_1} = \dfrac{p_2}{p_1}\cdot\dfrac{T_1}{T_2}$ $\boxed{\dfrac{\rho_2}{\rho_1} = \dfrac{(K+1)M_1^2}{2+(K-1)M_1^2}}$

We shall now that there is an energy loss across a shock wave, hence that it is irreversible.

Since the total pressure represents total energy, loss of total pressure represents loss of total energy.

$$\frac{p_{o1}}{p_1} = \left[1+\left(\frac{k-1}{2}\right)M_1^2\right]^{\frac{k}{k-1}} \qquad \frac{p_{o2}}{p_2} = \left[1+\left(\frac{k-1}{2}\right)M_2^2\right]^{\frac{k}{k-1}}$$

$$\frac{p_2}{p_1} = \frac{2kM_1^2-(k-1)}{(k+1)} \qquad M_2^2 = \frac{2+(k-1)M_1^2}{2kM_1^2-(k-1)}$$

Combining the above and simplifyng:

$$\frac{p_{o2}}{p_{o1}} = \frac{1}{(k+1)\left[2kM_1^2-(k-1)\right]^{\frac{1}{k-1}}}\left[\frac{(k+1)M_1^2}{2+(k-1)M_1^2}\right]^{\frac{k}{k-1}}$$

Since as $M \to \infty$, $\frac{p_{o2}}{p_{o1}} \to 0$ The increase of temperature is at the expense of total energy.

Oblique Shock Waves.

When the direction of a supersonic flow is suddenly changed, we get shock waves making an oblique angle with both flows. Also, when a supersonic flow crosses a shock wave in a non-perpendicular fashion it is deflected.

Now we shall analyse this phenomenon:

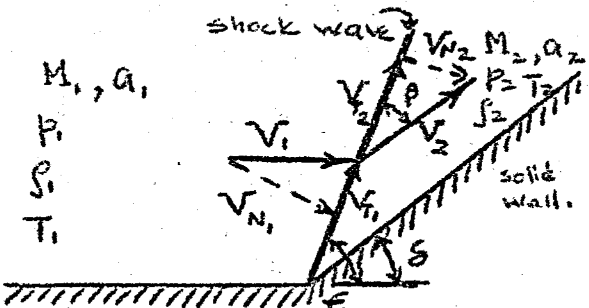


Fig. 10

δ = flow deflection angle

ϵ = shock angle

$\beta = \epsilon - \delta$

V_N = normal velocity component

V_T = tangential velocity component

$$V_{N_1} = V_1 \sin\epsilon \qquad V_{T_1} = V_1 \cos\epsilon$$

$$V_{N_2} = V_2 \sin\beta \qquad V_{T_2} = V_2 \cos\beta$$

We shall now take a very small stream tube and apply the continuity principle.

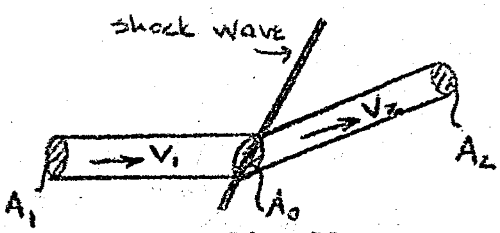


Fig. 11

$$A_1\rho_1V_1 = A_2\rho_2V_2$$

But $A_1 = A_0 \sin\epsilon$

$A_2 = A_0 \sin\beta$

$$A_0\rho_1V_1\sin\epsilon = A_0\rho_2V_2\sin\beta$$

So: $\boxed{V_{N_1}\rho_1 = V_{N_2}\rho_2}$

Taking a streamtube within the shock wave:

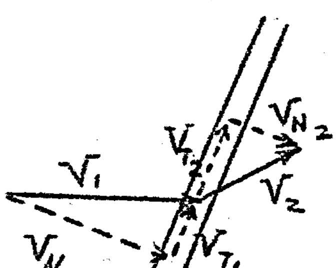


Fig. 12

From Euler's equation:

$$\frac{dp}{\rho} + VdV = 0$$

Since there is no change of pressure within and along this shock wave stream tube: $dp = 0$

hence: $VdV = 0$

So we get that : $\boxed{V_{T_1} = V_{T_2}}$

Also, applying the conservation of momentum principle:

$$A_0(p_1-p_2) = m\frac{dV_N}{dt} = m\frac{V_{N_2}-V_{N_1}}{t_2-t_1} \quad \text{where} \quad t_2-t_1 \cong dt$$

$$m = \rho_1 A_0 V_{N_1} dt \quad \text{or} \quad m = \rho_2 A_0 V_{N_2} dt$$

Substituting: $A_0(p_1-p_2) = m\frac{V_{N_2}}{dt} - m\frac{V_{N_1}}{dt} = \rho_2 A_0 V_{N_2} dt \frac{V_{N_2}}{dt} - \rho_1 A_0 V_{N_1} dt \frac{V_{N_1}}{dt}$

So: $$\boxed{p_1+\rho_1 V_{N_1}^2 = p_2+\rho_2 V_{N_2}^2}$$

Since $V_{T_1} = V_{T_2}$, $V_1 \cos\epsilon = V_2 \cos\beta$ and $\frac{V_2}{V_1} = \frac{\cos\epsilon}{\cos\beta}$

from: $\rho_2 V_{N_2} = \rho_1 V_{N_1}$

Combining we get: $\frac{\rho_2}{\rho_1} = \frac{V_{N_1}}{V_{N_2}} = \frac{\tan\epsilon}{\tan\beta}$

Then also: $c_p T_1 + \frac{V_1^2}{2gJ} = c_p T_2 + \frac{V_2^2}{2gJ}$

but $V_1^2 = V_{N_1}^2 + V_{T_1}^2$

$V_2^2 = V_{N_2}^2 + V_{T_2}^2$

Then: $$\boxed{c_p T_1 + \frac{V_{N_1}^2}{2gJ} = c_p T_2 + \frac{V_{N_2}^2}{2gJ}}$$

We observe that the fundamental relations for oblique shock waves are the same as those for normal shock waves with

V substituted by V_N

and M_1 substituted by $M \sin\epsilon$

so we shall change our equations accordingly.

Since: $V_1 > V_2$

$\frac{V_2}{V_1} < 1 \qquad \frac{\cos\epsilon}{\cos\beta} < 1$ therefore $\beta < \epsilon$

or, the flow is always deflected towards the shock wave.

Our new formulas are now:

1) $$\boxed{\frac{p_2}{p_1} = \frac{2k}{k+1}\left[M_1^2 \sin^2\epsilon - \frac{k-1}{2k}\right]}$$

2) $$\boxed{\frac{\rho_2}{\rho_1} = \frac{\tan\beta}{\tan\epsilon} = \frac{V_{N2}}{V_{N1}} = \frac{2}{k+1}\left[\frac{1}{M_1^2 \sin^2\epsilon} + \frac{k-1}{2}\right]}$$

3) $$\boxed{M_2^2 \sin^2\beta = \frac{2+(k-1)M_1^2 \sin^2\epsilon}{2kM_1^2 \sin^2\epsilon - (k-1)}}$$

Using eq. 3 and the fact that $\beta = \epsilon - \delta$ we get:

4) $$\boxed{\tan\delta = \frac{2(M_1^2 \sin^2\epsilon - 1)}{[(k+1)M_1^2 - 2(M_1^2 \sin^2\epsilon - 1)]\tan\epsilon}}$$

An interesting point is that $\tan\delta = 0$ for $M_1^2 \sin^2\epsilon = 1$ but we know that $\sin\epsilon = \frac{1}{M_1}$ is the equation of the Mach cone. So our assumption that Mach waves are the weakest waves is correct, since they are so weak that a flow may cross them without deflection.

Also $\tan\delta = 0$ for $\epsilon = \frac{\pi}{2}$ $\quad [\tan\epsilon = \infty \;, \; 1/\tan\epsilon = 0$

This corresponds of course to normal shock waves. Therefore between $\epsilon = \frac{\pi}{2}$ and $\epsilon = \sin^{-1}\frac{1}{M}$ we should have a maximum. We shall now look for it.

Taking $\frac{d(\tan\delta)}{d\epsilon} = 0$

we get: $$\boxed{\sin^2\epsilon_{max} = \frac{1}{kM_1^2}\left\{\frac{k+1}{4}M_1^2 - 1 + \left[(k+1)\left(1 + \frac{k-1}{2}M_1^2 + \frac{k+1}{16}M_1^4\right)\right]^{1/2}\right\}}$$

If δ is larger than δ_{max} corresponding to ϵ_{max}, we get shock dettachment.

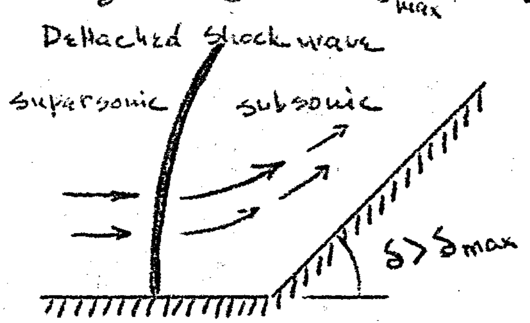


Fig. 13

In other words up to $\delta = \delta_{max}$ we shall have an attached shock wave making an angle ϵ, at $\delta = \delta_{max}$, shock wave will start to sepparate and when δ becomes larger than δ_{max} we get a dettached shock wave.

II- FLUID MOTION VISUALISATION METHODS DEPENDING UPON CHANGES OF REFRACTIVE INDEX

When the refractive index of an optical medium is close to unity, it can be related to the density with sufficient accuracy by the expression:

$$\frac{n-1}{\rho} = \text{const} \qquad \text{where } n = \text{refractive index}, \quad \rho = \text{density}$$

Therefore if the refractive index of the air(refractive index at standart conditions: 1.00029) can be measured as it flows around a body it may be possible to deduce the density and from a few assumptions get its Mach number etc.

The advantage of optical methods may be briefly listed as below:

a) Optical methods do not interfere with the subject being observed. Whereas the introduction of a body obstacle may be of great interference and will certainly distort the data at high Mach number flows.

b) By photographing the flow we may record all the necessary data in the minimum running time. This is particularly important at high Mach numbers since running the wind tunnel requires a great deal of power.

The disadvantages on the other hand are that all density changes in the path of the light are integrated and that they are restricted to two dimensional flows.

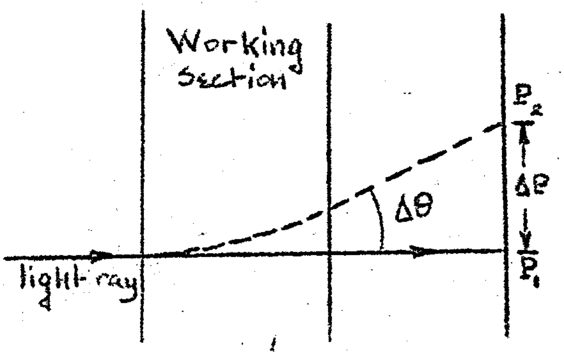


Fig. 14

A ray of light is sent through a working section and hits a screen. If the air in the working section is uniform throughout, the light will strike the screen at P_1

Now if an experimental model is placed in the working section and the density, hence the refractive index will no longer be uniform, the light will be refracted at an angle $\Delta\theta$ so that it strikes the screen at P_2. The time it takes to travel will now be changed by Δt . $\Delta\theta$ is measured by the Shlieren Method, Δt is measured by the Interferometer Method.

The general principle involved is as follows:

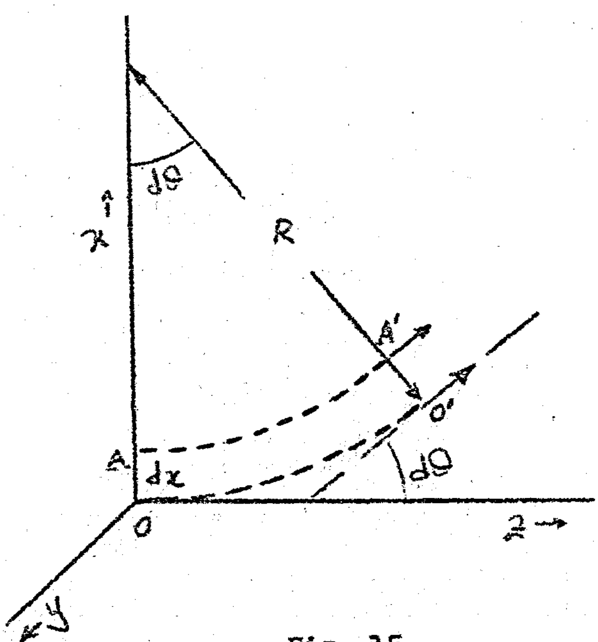


Fig. 15

Let Oz represent the path of the undisturbed light ray. A represents a point on an element of the wave front at O. But suppose that a disturbance is introduced in the path of the light with a constant gradiend of refractive index $\frac{\partial n}{\partial x}$ in the direction normal to Oz for an interval of dx.

n = refractive index at 0
n_A = refractive index at A
v = velocity of light ray at O
v_A = velocity of light ray at A

Then: $n_A = n + \frac{\partial n}{\partial x}dx$, by definition: $v_A n_A = v n$

So: $v_A = v \dfrac{n}{n + \frac{\partial n}{\partial x}dx}$

Taking R as the instatenous radius of curvature:

$$\frac{R}{v} = \frac{R-dx}{v_A} \quad \text{therefore:} \quad \frac{R}{v} = \frac{R-dx}{v\,\dfrac{n}{n+\frac{\partial n}{\partial x}dx}}$$

and:
$$R = \frac{(R-dx)(n+\frac{\partial n}{\partial x}dx)}{n}$$

or:
$$R = (R-dx)\left(1+\frac{1}{n}\frac{\partial n}{\partial x}dx\right)$$

$$R = R + R\frac{1}{n}\frac{\partial n}{\partial x}dx - dx - \frac{1}{n}\frac{\partial n}{\partial x}dx^2$$

neglecting second order differential terms:

$$\left(R\cdot\frac{1}{n}\frac{\partial n}{\partial x} - 1\right)dx = 0$$

Therefore:
$$\boxed{\frac{1}{R} = \frac{1}{n}\frac{\partial n}{\partial x}}$$

since $ds = R\,d\theta_x$, $d\theta_x = \frac{1}{R}ds$

So: $$\theta_x = \int \frac{1}{n}\frac{\partial n}{\partial x}ds$$

similarly: $$\theta_y = \int \frac{1}{n}\frac{\partial n}{\partial y}ds$$

If deflection is small:

$$\Delta\theta_x = \int \frac{1}{n}\frac{\partial n}{\partial x}dz$$

$$\Delta\theta_y = \int \frac{1}{n}\frac{\partial n}{\partial y}dz$$

The Schlieren Method.

In this method, some quantity is recorded from which it is possible to deduce the deflection $\Delta\theta$.

The basic Schlieren system is shown below:

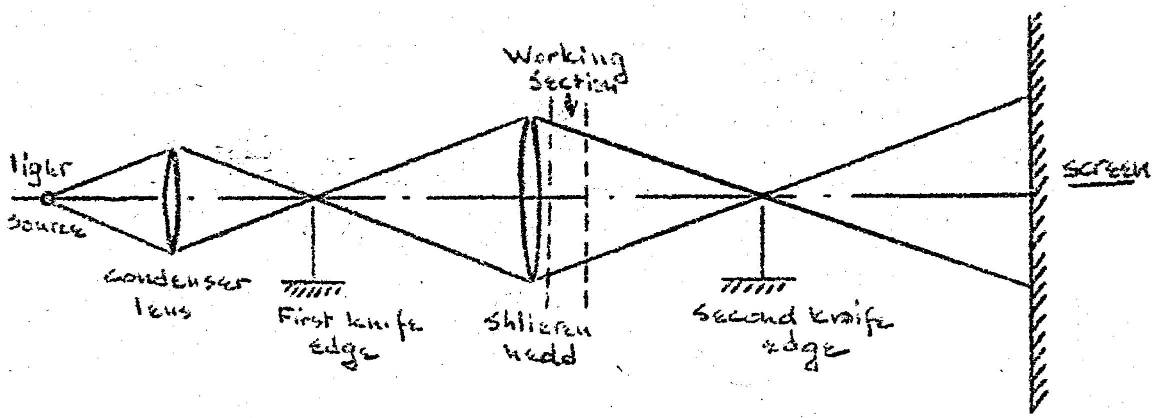


Fig. 16

Light from a source (preferably a line source instead of a point source) is focused by the condenser. At the focus a straight edge is placed such that by intercepting part of the light we get a sharply defined edge of light beam. This light then passes through the working section. At the new focus of this light we place a second knife edge such that it intercepts a part of the luminus flux and the intensity is thus diminished. The light is then recorded by a photographing Machine.

If there are no gradiends of refractive index within the working section, the amount of light reaching the film is fixed by the relative position of the two knife edges. If a gradiend normal to the plane of the knife edge exists however, the beam will be refracted up or down according to the gradiend- so that it either adds or sustracts from the light normally present on the screen. Thus, a working section involving a pattern of gases is reproduced in various tones on the film.

There are various methods in which the various elements of this method may be arranged.In general, because lenses of good quality are not available in sizes of more than a few inches in diameter, it is found more convenient to use concave paralabdlic mirrors.

The mostly widely used method is shown below:

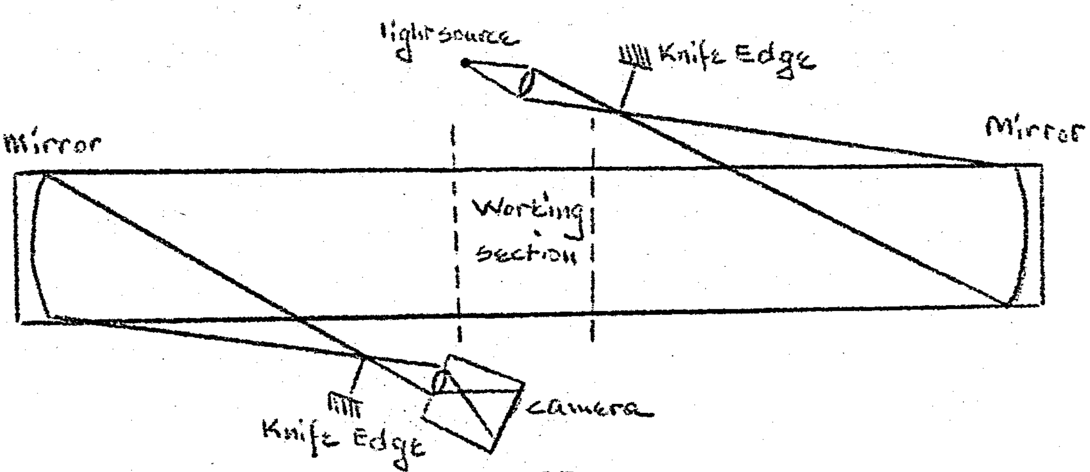


Fig. 17

This system has the advantage that parallel rays of light pass through the working section, thus producing an image of superior resolution. Further, because the light source and the camera are placed on opposite sides of the light path between the mirrors, coma is cancelled. Another advantage is that the working section is away from the mirrors; in fact since the light between the mirrors is parallel, they may be placed as far appart as desired - a characteristic very useful in wind tunnel testing or with explosion or high-temperature testing.

The knife edges are set up such that 50 per cent of the total illumination is cut off. <u>However,</u> care must be taken not to exceed theworking range, otherwise we shall get a black or white area wherethe changes z are not shown and so detail is lost.

<u>The Interferometer Method.</u>

Of all known optical methods this is the one most adaptable to quantitave research.

The basic principle of this method is as follows:

At any point where two or more trains of waves cross each other they are said to interfere. This means that the total effect will be determined by the algebraic addition of the individual effects.

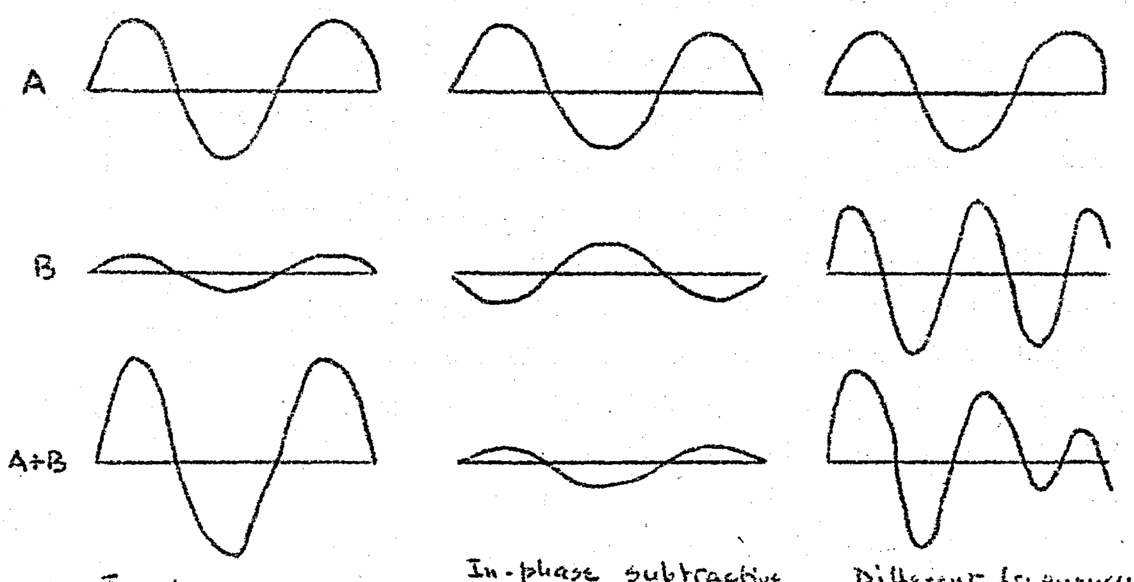


Fig. 18

Assume we have two points of light source on the line Ox whose lights are in phase (coherent sources).

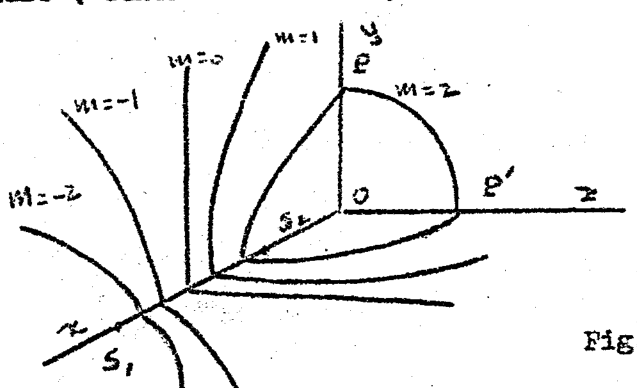


Fig. 19

There will be a point P such that $PS_1 - PS_2 = n\lambda$ or some integer multiple of the wavelengths. Then at P their effects will add and we shall have a fringe of light. There will also be a point P' of the same characteristic and in the y-z plane we shall have a circle formed by similar points. In the x-y and x-z planes such points will form a hyperbola (locus of points whose difference in distance to two given points is constant). As shown on the figure, we shall have many such points between S_1 and S_2.

If the distance between S_1 and S_2 is increased, we shall get many such lines, and they will become straighter and straighter. Also, a great number of bright and dark fringes will be formed on a plane parallel to the x axis and it is these that are recorded by interferometers.

In order to observe interference effects, it is necessary to have two sources emitting light waves which, from the start, are always in phase. This can never be accomplished with two sepparate sources since it is a fundamental property of atoms or molecules of a source that they are continually undergoing which produce frequent haphazard changes of phase in the light they emit. Hence it is necessary to use two rays ~~and to~~ split from up the light of a common source which are subsequently recombined. Moreover if the resulting fringes are to be sharply defined, it is necessary to use a monochromatic source. Under these conditions the fringe spacing depends upon the path difference of the two rays and on the wavelength of the light.

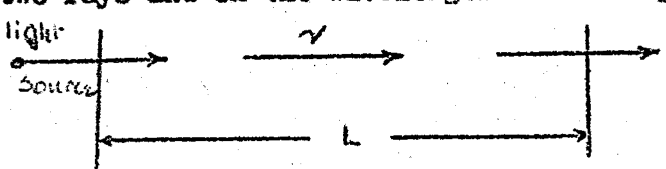


Fig. 20

By definition:

$$n = \frac{v_0}{v}$$

where: $v_0 =$ velocity of light in vacuum
$v =$ velocity of light in given medium

given two medii 1 and 2:

$$n_1 = \frac{v_0}{v_1} \rightarrow v_1 = \frac{v_0}{n_1} \qquad \text{also} \qquad v_1 = \frac{L}{t_1}$$

$$n_2 = \frac{v_0}{v_2} \rightarrow v_2 = \frac{v_0}{n_2} \qquad\qquad v_2 = \frac{L}{t_2}$$

where t is the time required to cross the medii. Then:

$$\left.\begin{array}{l}\frac{L}{t_1} = \frac{v_0}{n_1} \quad \rightarrow \quad L n_1 = v_0 t_1 \\ \frac{L}{t_2} = \frac{v_0}{n_2} \quad \rightarrow \quad L n_2 = v_0 t_2\end{array}\right\} \quad L(n_2 - n_1) = v_0 (t_2 - t_1)$$

$$\boxed{t_2 - t_1 = \frac{L(n_2 - n_1)}{v_0}}$$

Assuming the light to be travelling invacuum, the equivalent lengths are:

$$\left.\begin{array}{l}L_1 = v_0 t_1 \\ L_2 = v_0 t_2\end{array}\right\} \quad L_2 - L_1 = v_0 (t_2 - t_1)$$

so:

$$\frac{L_2 - L_1}{v_0} = \frac{L(n_2 - n_1)}{v_0}$$

and:

$$L_2 - L_1 = L(n_2 - n_1)$$

From the elementary theory of interference:

$$\frac{\Delta b}{b} = \frac{L(n_2 - n_1)}{\lambda_0}$$

where:

b=original fringe spacing

Δb=change of fringe spacing

λ_0=wavelength in vacuum

Since:

$$\frac{n_1 - 1}{\rho_1} = \frac{n - 1}{\rho} \qquad n_1 - 1 = \rho_1 \left(\frac{n-1}{\rho}\right)$$

$$\frac{n_2 - 1}{\rho_2} = \frac{n - 1}{\rho} \qquad n_2 - 1 = \rho_2 \left(\frac{n-1}{\rho}\right)$$

so:

$$n_2 - n_1 = (\rho_2 - \rho_1)\left(\frac{n-1}{\rho}\right)$$

Hence:

$$\boxed{\frac{\Delta b}{b} = \frac{L}{\lambda_0}(\rho_2 - \rho_1)\left(\frac{n-1}{\rho}\right)}$$

Which gives a quantitative method of determining the density change. It is important to note however that this method measures only the relative densities. Thus it is necessary to know the density at one fringe.

It may be easily seen that the quality of the optical instruments must be very high. To give an example: all optical flats must surfaced to within $\frac{1}{3}\lambda$ (including the tunnel windows). This comes to approximately 0.00015 mm.

The most widely used method is shown below:

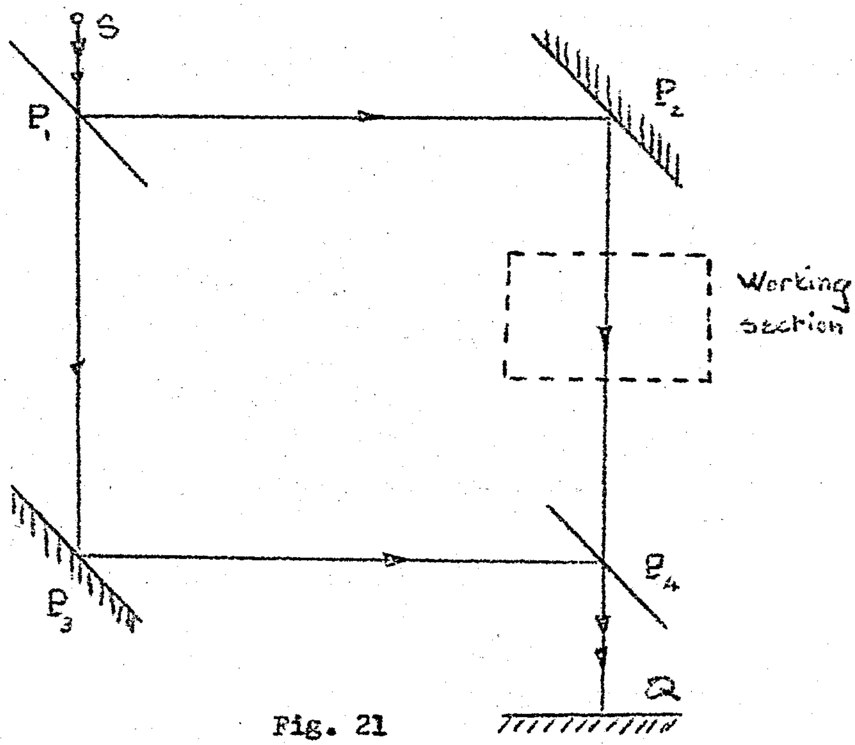


Fig. 21

Light from the source S is split up by the half-silvered plate P_1 into two beams. After reflection at the mirrors P_2 and P_3 the two rays are recombined at the second half-silvered plate P_4 and pass on tothe screen or photographic plate at Q.

The principal advantage of this system is that the two optical paths may be sepparated as far as necessary so that the working section may be placed in one without obstructing the other.

A major disadvantage of the interferometer is that the assembly cost is very great. Also, muchz care must be taken in adjusting the instruments.

THE SHALLOW WATER ANALOGY

Up to now we have seen the theory of compressible fluid flow and the means to visulize these flows.

Now we shall establish another method of visualisation: the Shallow Water method. In this method we shall use the fact that an open chanel liquid flow of very small depth has properties very similar to compressible fluid flow. By observing this analogous flow, then, it will be possible to get results similar to those obtained in wind tunnels.

The principle advantage of this method is that the need for very expensive wind tunnels is bypassed, instead of which very low cost equipment requiring practically no power to run is used.

In order to prove this similarity and to see what these similarities are we shall begin by a three dimensional analysis of a compressible fluid flow.

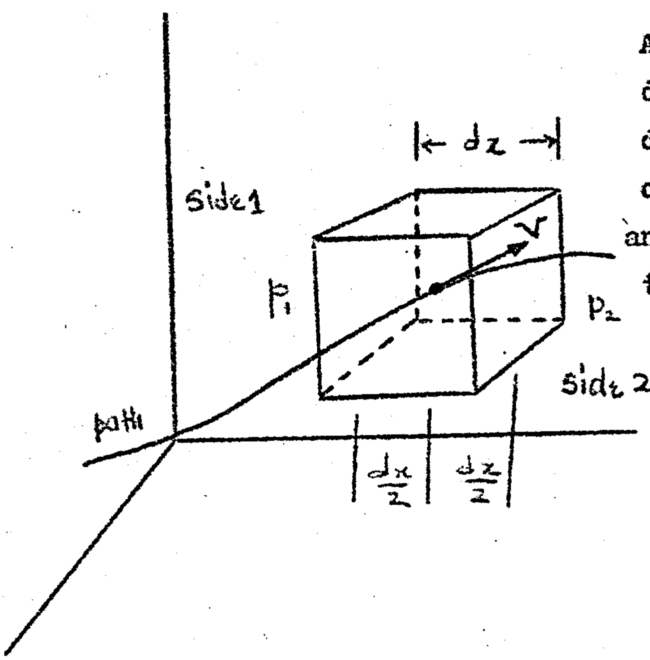


Fig. 22

Assume a flowing medium and a differential volume in this medium. Denoting the x component of V by u, the y component by v, and the z component by w, we have that in general:

$$u = u(x,y,z,t)$$
$$v = v(x,y,z,t)$$
$$w = w(x,y,z,t)$$

Applying Newton's principle to the volume:

$$\sum F_x = m a_x$$
$$\sum F_y = m a_y$$
$$\sum F_z = m a_z$$

But

$$\sum F_x = (p_1 - p_2)_x \, dy\, dz$$

where: $p_{2x} = p + \frac{\partial p}{\partial x}\cdot\frac{dx}{2}$, $p_{1x} = p - \frac{\partial p}{\partial x}\cdot\frac{dx}{2}$

substuting: $$\sum F_x = \left[\left(p - \frac{\partial p}{\partial x}\cdot\frac{dx}{2}\right) - \left(p + \frac{\partial p}{\partial x}\cdot\frac{dx}{2}\right)\right] dy\, dz$$

so: $$\sum F_x = -\frac{\partial p}{\partial x}\cdot dx\, dy\, dz$$

Similarly: $$\sum F_y = -\frac{\partial p}{\partial y}\cdot dx\, dy\, dz$$
$$\sum F_z = -\frac{\partial p}{\partial z}\cdot dx\, dy\, dz$$

Also: $$m = \rho\, dx\, dy\, dz$$

and $$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t} + \frac{\partial u}{\partial t}$$
$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot u + \frac{\partial u}{\partial y}\cdot v + \frac{\partial u}{\partial z}\cdot w + \frac{\partial u}{\partial t}$$

Similarly: $$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial x}\cdot u + \frac{\partial v}{\partial y}\cdot v + \frac{\partial v}{\partial z}\cdot w + \frac{\partial v}{\partial t}$$
$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot u + \frac{\partial w}{\partial y}\cdot v + \frac{\partial w}{\partial z}\cdot w + \frac{\partial w}{\partial t}$$

Therefore: $$\rho(dx\,dy\,dz)\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t}\right) = -\frac{\partial p}{\partial x}(dx\,dy\,dz)$$
$$\rho(dx\,dy\,dz)\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}\cdot v + \frac{\partial v}{\partial z}w + \frac{\partial v}{\partial t}\right) = -\frac{\partial p}{\partial y}(dx\,dy\,dz)$$
$$\rho(dx\,dy\,dz)\left(\frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w + \frac{\partial w}{\partial t}\right) = -\frac{\partial p}{\partial y}(dx\,dy\,dz)$$

Hence:

$$\rho\left(\frac{\partial u}{\partial x}u+\frac{\partial u}{\partial y}v+\frac{\partial u}{\partial z}w+\frac{\partial u}{\partial t}\right)=-\frac{\partial p}{\partial x}$$

$$\rho\left(\frac{\partial v}{\partial x}u+\frac{\partial v}{\partial y}v+\frac{\partial v}{\partial z}w+\frac{\partial v}{\partial t}\right)=-\frac{\partial p}{\partial y}$$

$$\rho\left(\frac{\partial w}{\partial x}u+\frac{\partial w}{\partial y}v+\frac{\partial w}{\partial z}w+\frac{\partial w}{\partial t}\right)=-\frac{\partial p}{\partial z}$$

Also, from the continuity equation:

$$\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho\vec{V})=0 \rightarrow \frac{\partial \rho}{\partial \rho}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

If Flow is two dimensional, $w=0$

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

And:

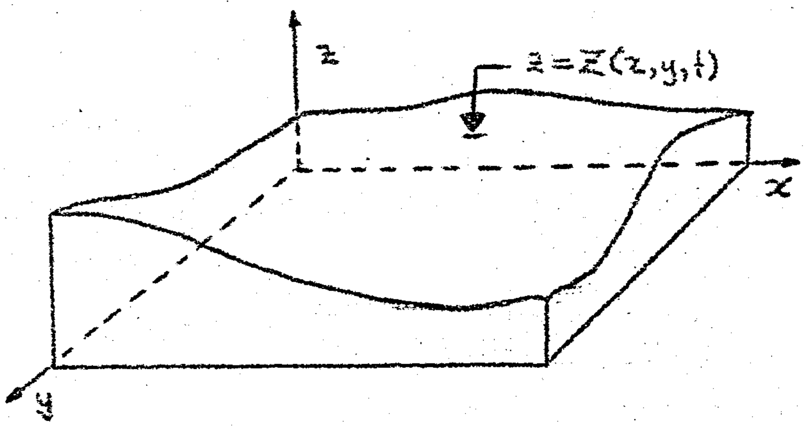
$$\rho\left(\frac{\partial u}{\partial x}u+\frac{\partial u}{\partial y}v+\frac{\partial u}{\partial t}\right)=-\frac{\partial p}{\partial x}$$

$$\rho\left(\frac{\partial v}{\partial x}u+\frac{\partial v}{\partial y}v+\frac{\partial v}{\partial t}\right)=-\frac{\partial p}{\partial y}$$

Giving us our equations for two dimensional compressible flow.

Now we shall make a similar study of an open channel incompressible fluid flow:

We place an (x, y, z) coordinate system in a space filled by water in such a way that the bottom surface is the plane and the top surface is given by a function $z = Z(x,y,t)$



Again:

$$u = u(x,y,z,t)$$
$$v = v(x,y,z,t)$$
$$w = w(x,y,z,t)$$

Fig. 23

From the continuity equation:

$$\operatorname{div}(\vec{V}) = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

Newton's Law: $F = m a$

in x direction:

$$F = -\frac{\partial p}{\partial x} \cdot dx\,dy\,dz$$

$$m = \rho(dx\,dy\,dz) \qquad a = \frac{du}{dt}$$

and:

$$-\frac{\partial p}{\partial x} dx\,dy\,dz = \rho \frac{du}{dt} dx\,dy\,dz$$

therefore:

$$-\frac{\partial p}{\partial x} = \rho \frac{du}{dt} \qquad (2a)$$

In y direction: $F = -\frac{\partial p}{\partial y}\, dx\,dy\,dz$

$$m = \rho(dx\,dy\,dz) \qquad a = \frac{dv}{dt}$$

therefore: $$-\frac{\partial p}{\partial y} = \rho \frac{dv}{dt} \qquad (2b)$$

In z direction: $F = \frac{\partial p}{\partial z}\, dx\,dy\,dz$

$$m = \rho(dx\,dy\,dz) \qquad a = \frac{dw}{dt} + g$$

therefore: $$-\frac{\partial p}{\partial z} = \rho\left(\frac{dw}{dt} + g\right) \qquad (2c)$$

At surface: $(z = Z)$

$$w_Z = \frac{dZ}{dt} \qquad \text{but} \quad Z = Z(x,y,t)$$

so: $$w_Z = \frac{\partial Z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial Z}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial Z}{\partial t}$$

and: $$w_Z = \frac{\partial Z}{\partial x} u + \frac{\partial Z}{\partial y} v + \frac{\partial Z}{\partial t} \qquad (3)$$

Boundary conditions:

$$p = 0 \quad \text{at} \quad z = Z$$

$$w = 0 \quad \text{at} \quad z = 0$$

Integrating the continuity equation (1) from $z = 0$ to Z

$$\int_{z=0}^{z=Z} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dz = 0$$

$$w_Z - w_0 + \int_0^Z \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = 0$$

but since: $w_0 = 0$ and from (3)

$$\frac{\partial Z}{\partial x} u + \frac{\partial Z}{\partial y} v + \frac{\partial Z}{\partial t} + \int_0^Z \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = 0$$

since: $$\frac{\partial Z}{\partial z} u = \int \left(\frac{\partial^2 Z}{\partial x \partial z} u + \frac{\partial Z}{\partial z} \frac{\partial u}{\partial z} \right) dz$$

and: $$\frac{\partial Z}{\partial y} v = \int \left(\frac{\partial^2 Z}{\partial y \partial z} v + \frac{\partial Z}{\partial y} \frac{\partial v}{\partial z} \right) dz$$

$$\frac{\partial Z}{\partial t} + \int^{Z} \left(\frac{\partial^2 Z}{\partial x \partial z} u + \frac{\partial Z}{\partial x} \frac{\partial u}{\partial z} + \frac{\partial^2 Z}{\partial y \partial z} v + \frac{\partial Z}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = 0$$

$$\frac{\partial Z}{\partial t} + \frac{\partial}{\partial x}\left[\int_0^{Z} \left(\frac{\partial Z}{\partial z} u + \frac{\partial Z}{\partial z} u + u \right) dz \right] + \frac{\partial}{\partial y}\left[\int_0^{Z} \left(\frac{\partial Z}{\partial z} v + \frac{\partial Z}{\partial z} v + v \right) dz \right] = 0$$

Then: $$\frac{\partial Z}{\partial t} + \frac{\partial}{\partial x}\left[\int_0^{Z} u \, dz \right] + \frac{\partial}{\partial y}\left[\int_0^{Z} v \, dz \right] = 0 \quad (4)$$

Now we shall make our basic assumption: that pressure variation along a vertical column is the same as in hydrostatics.

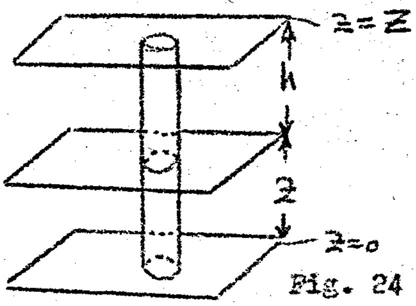


Fig. 24

$$p = \gamma h \qquad h = (Z - z)$$

$$\gamma = g\rho$$

$$p = g\rho (Z - z) \quad (5)$$

Then: $$\frac{\partial p}{\partial z} = g\rho \left(\frac{\partial Z}{\partial z} - \frac{\partial z}{\partial z} \right) = -g\rho \rightarrow -\frac{\partial p}{\partial z} = g\rho$$

From (2c): $$g\rho = \rho \left(\frac{dw}{dt} + g \right) = \rho \frac{dw}{dt} + g\rho$$

therefore $$\frac{dw}{dt} = 0$$

Since $w = 0$ at $z = 0$ then $w = 0$ for all z.

also $$\frac{\partial p}{\partial x} = g\rho \left(\frac{\partial Z}{\partial x} - \frac{\partial z}{\partial x} \right) = g\rho \frac{\partial Z}{\partial x} \quad (5a)$$

and $$\frac{\partial p}{\partial y} = g\rho \left(\frac{\partial Z}{\partial y} - \frac{\partial z}{\partial y} \right) = g\rho \frac{\partial Z}{\partial y} \quad (5b)$$

Therefore $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}$ are independent of z

hence from (2a) and (2b)

$$-g\rho \frac{\partial Z}{\partial x} = \rho \frac{du}{dt} \quad \text{and} \quad -g\rho \frac{\partial Z}{\partial y} = \rho \frac{dv}{dt}$$

Therefore $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$ are independent of z

$$\left.\begin{aligned} u &= \int_0^t \frac{du}{dt}\,dt = u(x,y,t) \\ v &= \int_0^t \frac{dv}{dt}\,dt = v(x,y,t) \end{aligned}\right\}$$

velocities are also independent of z

Since u and v are independent of z, and $w = 0$, flow is 2 dimensional

Coming back to equation (4) it becomes:

$$\frac{\partial Z}{\partial t} + \frac{\partial (Zu)}{\partial t} + \frac{\partial (Zv)}{\partial t} = 0 \qquad (6)$$

Also, combining equations (2a) and (5a)

$$\rho \frac{du}{dt} = -g\rho \frac{\partial Z}{\partial x}$$

but since $u = u(x,y,t)$

$$\rho \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial t} \right) = -g\rho \frac{\partial Z}{\partial x}$$

or:

$$\rho \left(\frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v + \frac{\partial u}{\partial t} \right) = -g\rho \frac{\partial Z}{\partial x} \qquad (6a)$$

by the same token - equations (2b) and (5b):

$$\rho \left(\frac{\partial v}{\partial x} u + \frac{\partial v}{\partial y} v + \frac{\partial v}{\partial t} \right) = -g\rho \frac{\partial Z}{\partial y} \qquad (6b)$$

In order to set up the analogy we shall now define equivalent densities and pressures such that:

$$\boxed{\bar{\rho} = \rho Z} \qquad (7)$$

and

$$\bar{p} = \int_0^Z p\,dz \qquad (8)$$

since $p = g\rho(Z - z)$

$$\boxed{\bar{p} = \tfrac{1}{2} g\rho Z^2} \qquad (8a)$$

Then: $\Xi = \frac{\bar{\rho}}{\rho} \rightarrow \frac{\partial \Xi}{\partial t} = \frac{\partial(\bar{\rho}/\rho)}{\partial t} = \frac{1}{\rho}\frac{\partial \bar{\rho}}{\partial t}$

$$\frac{\partial(\Xi u)}{\partial x} = \frac{\partial(\bar{\rho}/\rho\, u)}{\partial x} = \frac{1}{\rho}\frac{\partial(\bar{\rho}u)}{\partial x} \quad \text{and} \quad \frac{\partial(\Xi v)}{\partial y} = \frac{1}{\rho}\frac{\partial(\bar{\rho}v)}{\partial y}$$

and so eq. (6) becomes:

$$\frac{\partial \Xi}{\partial t} + \frac{\partial(\Xi u)}{\partial x} + \frac{\partial(\Xi v)}{\partial y} = \frac{1}{\rho}\frac{\partial \bar{\rho}}{\partial t} + \frac{1}{\rho}\frac{\partial(\bar{\rho}u)}{\partial x} + \frac{1}{\rho}\frac{\partial(\bar{\rho}v)}{\partial y}$$

Also: $\rho = \bar{\rho}/\Xi \; , \quad \frac{\partial \bar{p}}{\partial x} = g\rho\Xi\frac{\partial \Xi}{\partial x}$ and $\frac{\partial \bar{p}}{\partial y} = g\rho\Xi\frac{\partial \Xi}{\partial y}$

then eq. (6a) becomes: $\bar{\rho}\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial t}\right) = -g\rho\Xi\frac{\partial \Xi}{\partial x} = -\frac{\partial \bar{p}}{\partial x}$

and eq. (6b) becomes : $\bar{\rho}\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial t}\right) = -\frac{\partial \bar{p}}{\partial y}$

So now our three equations (6) , (6a) , (6b) have become:

$$\boxed{\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial(\bar{\rho}u)}{\partial x} + \frac{\partial(\bar{\rho}v)}{\partial y} = 0} \qquad (9)$$

$$\boxed{\bar{\rho}\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial t}\right) = -\frac{\partial \bar{p}}{\partial x}} \qquad (9a)$$

$$\boxed{\bar{\rho}\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial t}\right) = -\frac{\partial \bar{p}}{\partial y}} \qquad (9b)$$

These equations are exactly the same as those for two dimensional gas flow previously developed. The analogy is thus established.

In order to complete the analogy we shall now show that the gas represented by this analogl is a polytropic gas of polytropic gas constant equal to 2.

Combining equations (7) and (8a) :

$$\bar{p} = \frac{1}{2}g\rho\left(\frac{\bar{\rho}}{\rho}\right)^2 = g/2\rho \cdot \bar{\rho}^2$$

or:

$$\boxed{\frac{\bar{p}}{\bar{\rho}^2} = \text{const}} \qquad §10)$$

This is exactly the relation between pressure and density for a polytropic gas of $K = \frac{c_p}{c_v} = 2$ and from now on, we shall treat it as such.

Having established the analogy, we shall now look for analogous components.

The first thing that we shall look for is the equivalent of sonic velocity in a shallow water flow :

We know that for a gas flow the sonic velocity is given by $a = \sqrt{\frac{\partial p}{\partial \rho}}$ in general or $a = \sqrt{\frac{dp}{d\rho}}$ for a polytropic gas. So $a^2 = \frac{dp}{d\rho}$. For a water flow we shall use " equivalent sonic velocity " or " celerity " which will be designated by c and will of course have to be: $c^2 = \frac{d\bar{p}}{d\bar{\rho}}$

$$c^2 = \frac{d\bar{p}}{d\bar{\rho}} = \frac{d\bar{p}}{dZ} \cdot \frac{dZ}{d\bar{\rho}} \qquad \frac{d\bar{p}}{dZ} = g\rho Z \qquad \frac{dZ}{d\bar{\rho}} = \frac{1}{\rho}$$

$$c^2 = g\rho Z \cdot \frac{1}{\rho} = gZ \quad \text{or:} \quad \boxed{c = \sqrt{gZ}}$$

Checking the method by another:

$$\frac{\bar{p}}{\bar{\rho}^2} = \text{const} \qquad \frac{d\bar{p}}{\bar{p}} - 2\frac{d\bar{\rho}}{\bar{\rho}} = 0 \qquad \frac{d\bar{p}}{d\bar{\rho}} = 2\frac{\bar{p}}{\bar{\rho}}$$

$$c^2 = \frac{d\bar{p}}{d\bar{\rho}} = 2\,\frac{\frac{1}{2}g\rho Z^2}{\rho Z} = gZ \quad \text{or:} \qquad c = \sqrt{gZ}$$

We shall next show that the equivalent of Mach Number in shallow water flow is the Froude Number of the flow.

Proof:

$$F = \frac{V}{\sqrt{l\,\frac{\Delta\gamma}{\rho}}} \qquad \text{(by definition)}$$

where: V is the relative velocity of the flow
l is a typical linear dimension of the flow
$\Delta\gamma$ is the density difference of the two fluid media
ρ is the mass density of the fluid to be considered

Since $\gamma = g\rho \;, \; \Delta\gamma = g\Delta\rho$

When the two fluids in question are air and water $\Delta\rho \cong \rho$ the density of the water and:

$$F=\frac{V}{\sqrt{lg\rho/\rho}}=\frac{V}{\sqrt{lg}}$$

taking the typical linear dimension of the flow as its piezometric head (height of the water in the channel):

$$F=\frac{V}{\sqrt{gh}} \qquad \text{but since} \qquad c=\sqrt{gh}$$

$$F=\frac{V}{c} \qquad \text{which is analogous to} \qquad M=\frac{V}{a}$$

Now we shall look for the physical meaning of "equivalent sonic velocity" or " celerity ". Since our equations were derived for the surface of the water, the disturbance corresponding to pressure (sonic) disturbance will be a surface wave. We shall now check this.

Assuming a constant width open channel flow. We introduce a small disturbance travelling with a velocity c . To analyse it however, we shall assume that the disturbance is standing still while the water is moving with a velocity

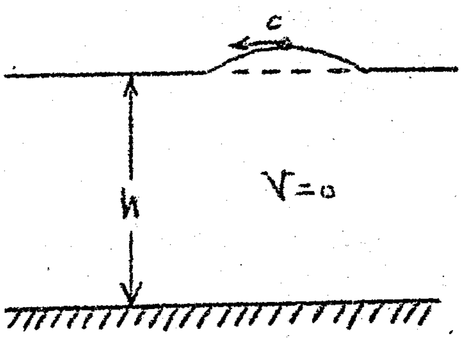


Fig. 25

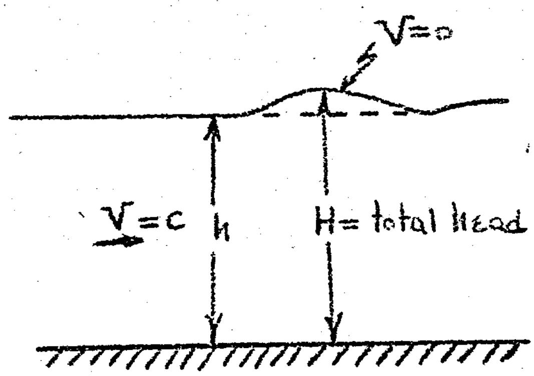


Fig. 26

We know that :

$$H=h+\frac{V^2}{2g}$$

Differentiating:

$$\frac{V\,dV}{g}+dh=0$$

From the continuity equation:

$\rho VA = const$ since $\rho = const$ and the width are constant,

$Vh = const.$ differentiating: $Vdh + hdV = 0$

so that: $dV = -\frac{V}{h} dh$

substituting: $dh - \frac{V\left(\frac{V}{h}dh\right)}{g} = 0$

Solving: $V^2 = gh$ or $V = c = \sqrt{gh}$

which is our " equivalent sonic velocity"

The celerity is the velocity at which a disturbance travels on the surface of the water. Here again the disturbance is in wave form. Everything said for gas flow may be said again in an equivalent form for shallow water flow eg:

For a non flow situation an intermittent disturbance will create a concentric waves propagating at velocity

For a flow of $V < c$ or $F = \frac{V}{c} < 1$ we shall again have eccentric rings whose upstream absolute velocity is $c - V$ and downstream absolute velocity is $c + V$

For a flow of $F > 1$, by the same analysis, we shall have rings all tangent to two lines drawn from the source of disturbance where the half-angle between shall be $\alpha = \frac{c}{V} = \frac{1}{F}$

At $V = c$ we shall again have an accumulation of waves at the disturbance - our equivalent shock waves.

As for a concrete example, what better example than the bow waves of a ship. Here again the disturbance is a body in the path of the flow such that at least at a point there is a stagnation causing a rise of head; and the head of course propagates at celeric velocity.

We have thus proved the similarity two dimensional gas flow and shallow water flow. To recapitulate briefly:

The analogy lies in taking the equivalent density for a water flow as:

$$\bar{\rho} = \rho Z$$

and the equivalent pressure as:

$$\bar{p} = \int_0^Z p \, dz$$

These two definitions reduce the flow equations of shallow water to exactly the same as those for two dimensional gas flow.

The polytropic gas constant of this gas was found to be $K = \frac{c_p}{c_v} = 2$

The elementary Kinetic Theory of gas gives that $1 < K \leq \frac{5}{3}$

Therefore there is no gas that corresponds exactly to this analog gas. This is not very important, however, because the qualitative analysis of this is not dependant on k and as for the quantitative analysis, the formulas derived in Gas Dynamics can still be calculated taking

The " sound wave " of the shallow water flow is in the form of a surface wave which moves at a relative velocity of $c = \sqrt{g h}$

The " shock wave " is represented by a stationary or moving wave front, and the sharper the ridge, the stronger the shock.

We shall now list our formulae for $K=2$

1) $$\boxed{F_2^2 = \frac{2+F_1^2}{4F_1^2-1}}$$

from the relation for p_{02}/p_{01}

2) $$\boxed{\frac{Z_{02}}{Z_{01}} = \frac{F_1^2}{2+F_1^2}\sqrt{\frac{3}{4F_1^2-1}}}$$

We get a total head drop

For Oblique Shock Waves:

3) $$\boxed{F_2^2 \sin^2\beta = \frac{2+F_1^2\sin^2\epsilon}{4F_1^2\sin^2\epsilon-1}}$$

4) $$\boxed{\tan\delta = \frac{2(F_1^2\sin^2\epsilon-1)}{[3F_1^2-2(F_1^2\sin^2\epsilon-1)]\tan\epsilon}}$$

5) $$\boxed{\sin^2\epsilon_{max} = \frac{1}{2F_1^2}\left\{\frac{3}{4}F_1^2-1+\left[3\left(1+\frac{1}{2}F_1^2+\frac{3}{16}F_1^4\right)\right]^{1/2}\right\}}$$

The mostx important formula that we shall need is one that will give us the Froude number of the flow by measurement of the flow depth. This is a necessity for experimental purposes. We shall have two such formulae: one for simple flow, and one for the presence of shock waves.

From the continuity equation: $h_1 b_1 V_1 = h_2 b_2 V_2$

from the energy equation: $\frac{V_1^2}{2g} + h_1 = \frac{V_2^2}{2g} + h_2$

since $V_1^2 = F_1^2 g h_1$

$$h_1^2 b_1^2 F_1^2 g h_1 = h_2^2 b_2^2 \left[2g(h_1 - h_2) + F_1^2 g h_1 \right]$$

$$F_1^2 (h_1^3 b_1^2 - h_2^2 b_2^2 h_1) = (2h_2^2 b_2^2 h_1 - 2h_2^3 b_2^2)$$

so:

$$F_1 = \sqrt{\frac{2h_2^2 b_2^2 (h_1 - h_2)}{h_1 (h_1^2 b_1^2 - h_2^2 b_2^2)}}$$

simplifying:

$$F_1^2 = \frac{2 \frac{h_2^3}{h_1^3} \left(\frac{h_1}{h_2} - 1 \right)}{\left(\frac{b_1^2}{b_2^2} - \frac{h_2^2}{h_1^2} \right)}$$

taking: $\frac{h_2}{h_1} = R \qquad \frac{b_1}{b_1} = B$

$$\boxed{F_1 = R \sqrt{\frac{2(1-R)}{(B^2 - R^2)}}}$$

In the presence of shock waves:

from:

$$\frac{\rho_2}{\rho_1} = \frac{(k-1) M_1^2}{2 + (k-1) M_1^2}$$

$$F_1 = \sqrt{\frac{2R}{3-R}}$$

THE SHALLOW WATER ANALOGY TANK

In order to make use of the foregoing knowledge in actual experimentation we shall make use of the " shallow water analogy tank". The tank- as shown in the plans in the back of this thesis- is made actually of two parts: 1- the secondary tank containing the adjustable overflow pipe. The function of this tank is to keep a constant hydraulic head. 2- the primary tank. This is the tank where the actual experimentation is carried on. As may be seen from the plans, the water comming from the secondary tank passes through a gauze which prevents any turbulence from being transmitted to the experimenting section. The bottom of the tank rises so that only a very shallow part of the water passes over the experimenting section. This section has a glass section so that light may be sent from the bottom and pictures may be takeng. Or, from the slot on the side, a striped ~~x~~ board may be put under the glass and pictures analogous to the interferometer pictures may be taken. On the glass we have a Laval nozzle with a wooden converging part and metal diverging part whose apex angle may be varried by using the nozzle regulators located at the side. We may study ~~x~~ constant area flow with the same apparatus by removing the nozzle and substituting parallel wooden blocks. We put wooden models that are weighed down by lead in the path of the flow and observe the flow formations. Direct pictures do not show the disturbances as well as interferometer pictures but the shape of the disturbances may be easily seen, wheras the interferometer pictures do not show the actual shape of the disturbances but instead show vividly the degree of disturbances.

In our experiment we tried to reproduce shock patterns at the entrance of a ramjet whose actual formation is shown below.

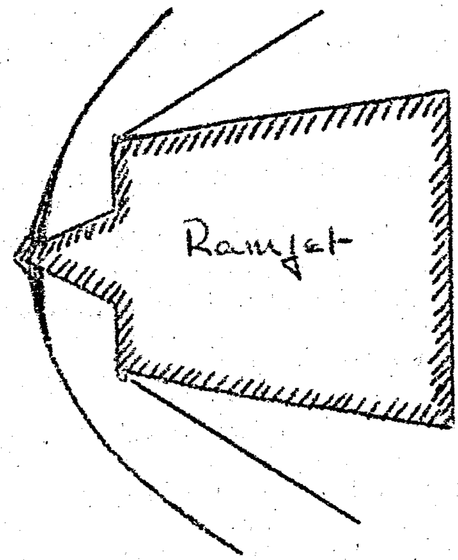


Fig. 27

As may be seen from the plates 11-12 the reproduction is not very exact due to the fact that k is 2 instead of 1.4 but the general shape of the two shock waves are strikingly similar. We must also take into accoumt that near the walls of the divergent nozzle, where there is boundary effect, the shock waves are ditorted, in fact, disappear since there is subsonic flow.

PLATES

We shall now use the photographs to test our formulae.

From Plate 1: $\delta = 6.5^\circ$ $\epsilon = 32.5^\circ$

a) From graph on page 44 : $F = 2.3$

b) From depthmeter readings: $h_1 = 0.11$, $h_2 = 0.13$

$$R = \frac{h_2}{h_1} = \frac{0.13}{0.11} = 1.18$$

From curve on page 47: $F_1 \sin\epsilon = 1.3$ and $F_1 = 2.42$

Changing the flow, Plate 2: $\delta = 21.5^\circ$ $\epsilon = 46.5^\circ$

a) From graph on page 44: $F = 3.15$

b) From depthmeter readings: $h_1 = 0.17$, $h_2 = 0.28$

$R = 1.65$, from curve on page 47:

$F_1 \sin\epsilon = 2.45$ and $F_1 = 2.94$

We get on the average 6% difference between Froude Numbers the above two methods. This is quite satisfactory if we take into account that the apparatus used in the test could be made much more sensitive than it was for our case.

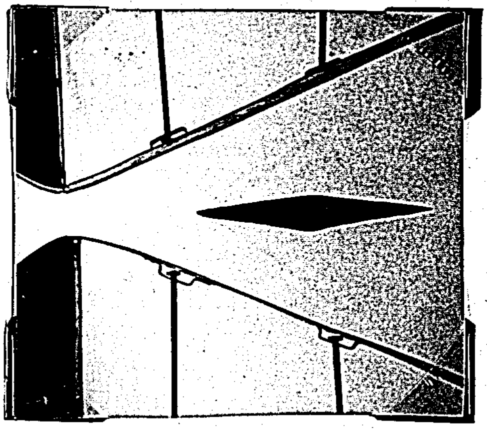


Plate 1

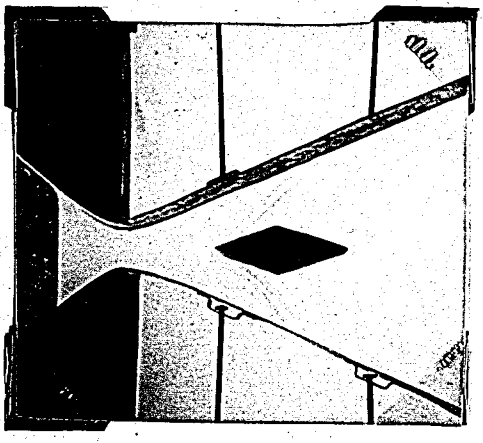


Plate 2

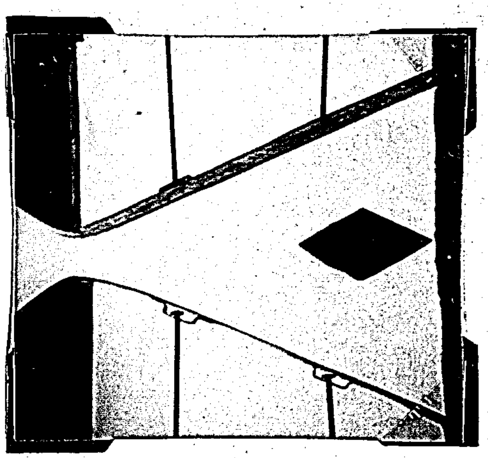


Plate 3

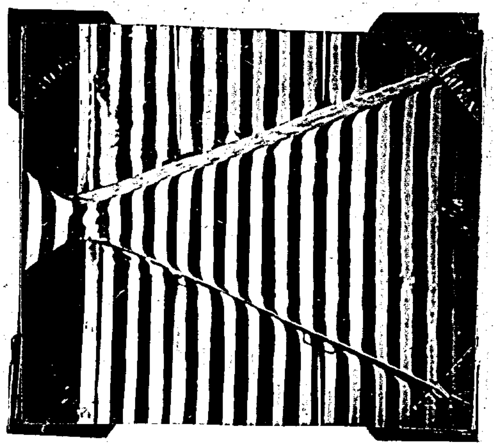


Plate 4

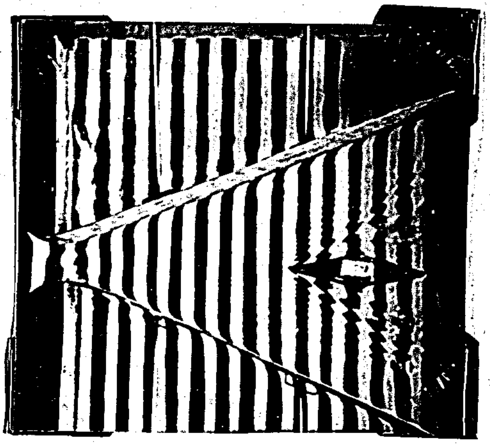


Plate 5

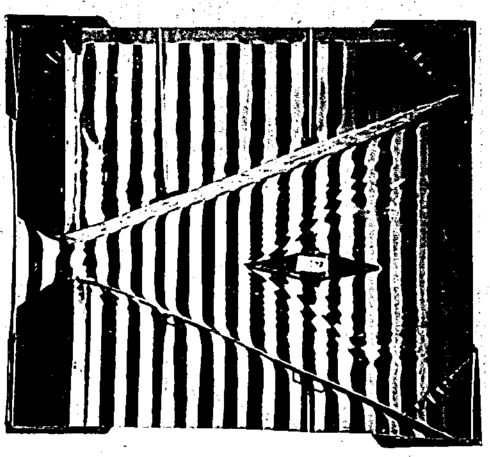


Plate 6

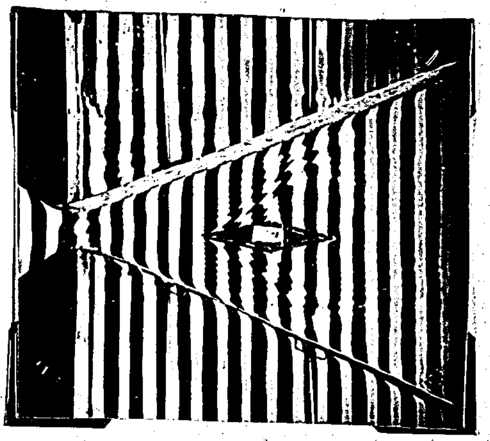


Plate 7

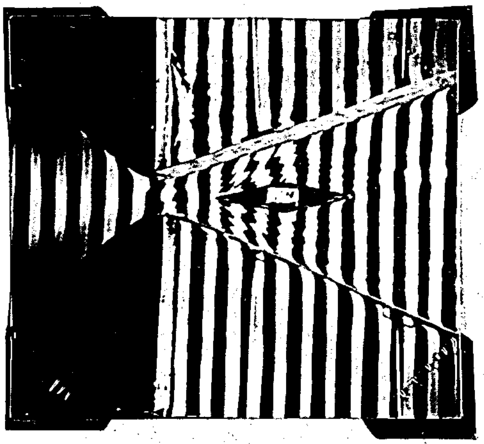


Plate 8

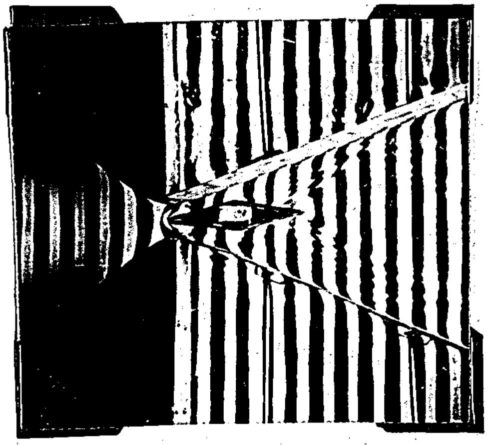


Plate 9

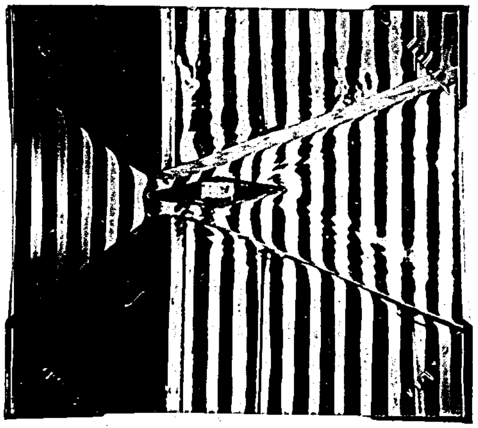


Plate 10

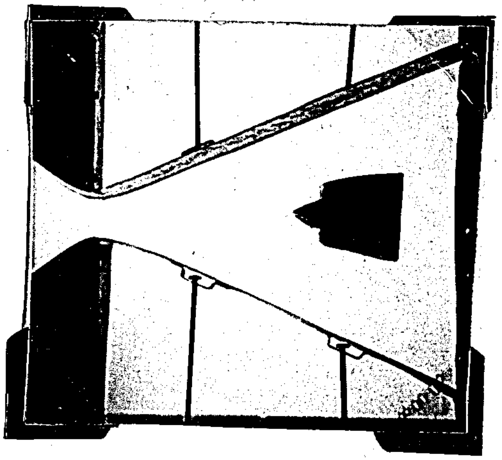


Plate 11

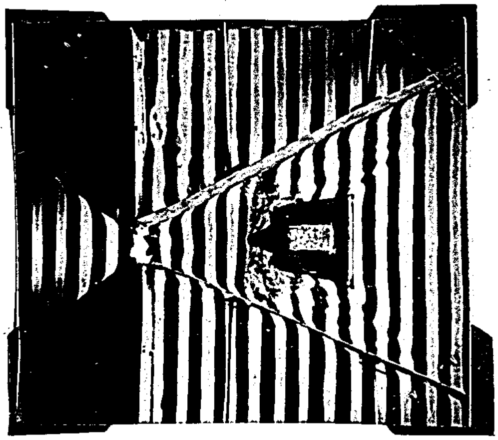


Plate 12

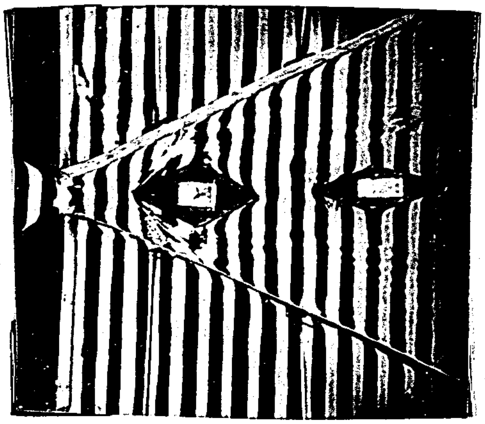


Plate 13

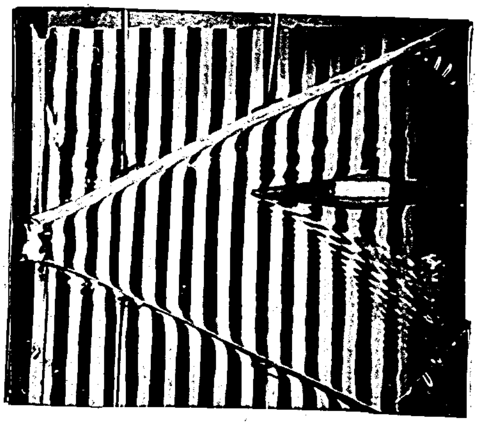


Plate 14

Plate 1- 3 : Oblique shock waves at the nose of 13°, 43° and 50° double wedges respectively.

Plate 4 : Undistorted Interferometer photograph (note distortion due to boundary layer effect at walls of nozzle)

Plate 5- 8 : Interferometer photographs of 30° dou ble wedge at various positions in the nozzle.

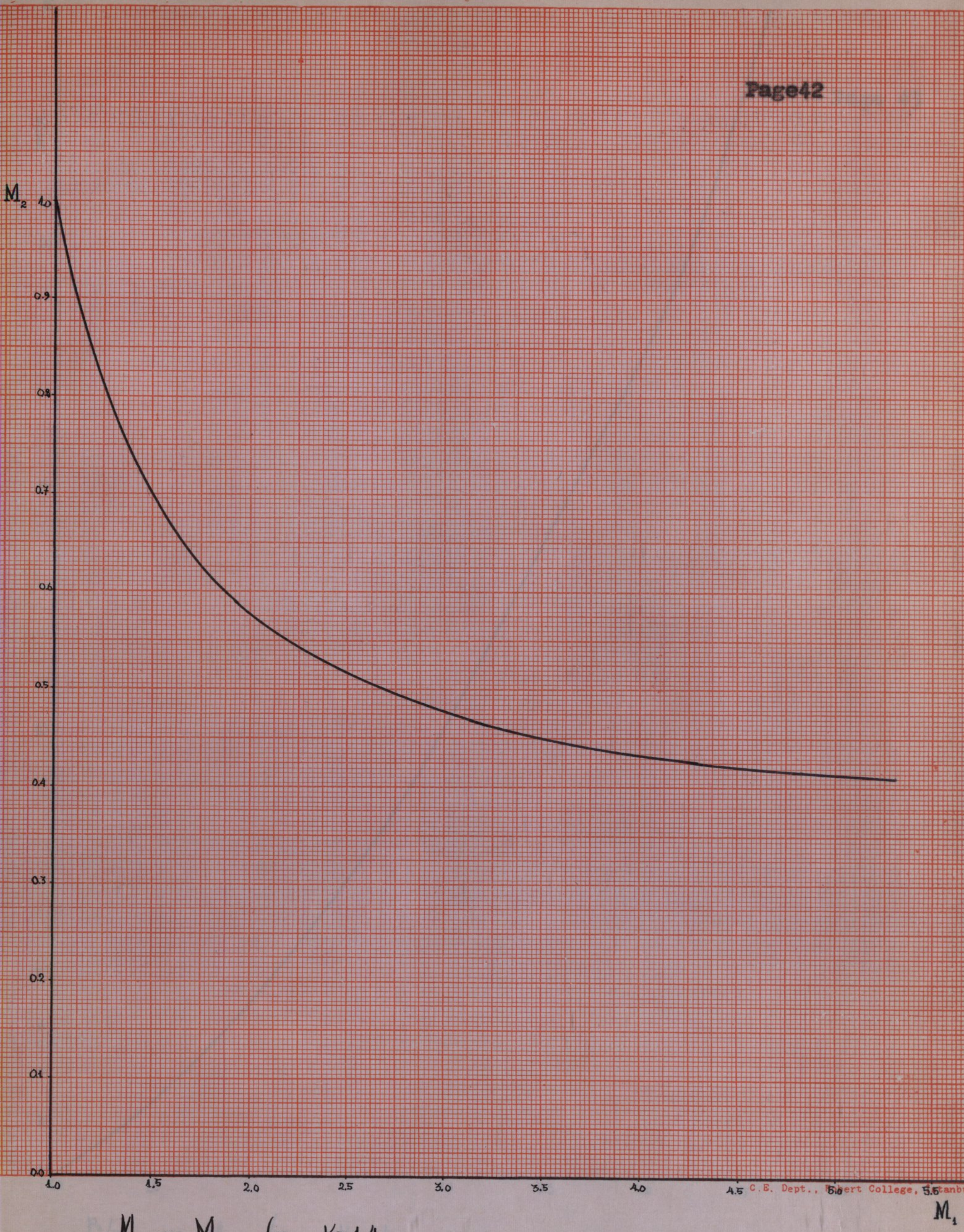
Plate 9-10 : Same as plates 5-8 but showing transonic disturbance at the throat ($F > 1$ for plate 9 and $F < 1$ for plate 10)

Plate 11-12 : Normal and Interferometer photographs of ram-jet model.

Plate 13: Showing the fact that flow is subsonic behind shock wave.

Plate 14: Showing subsonic flow near the walls of the nozzle due to eddys.

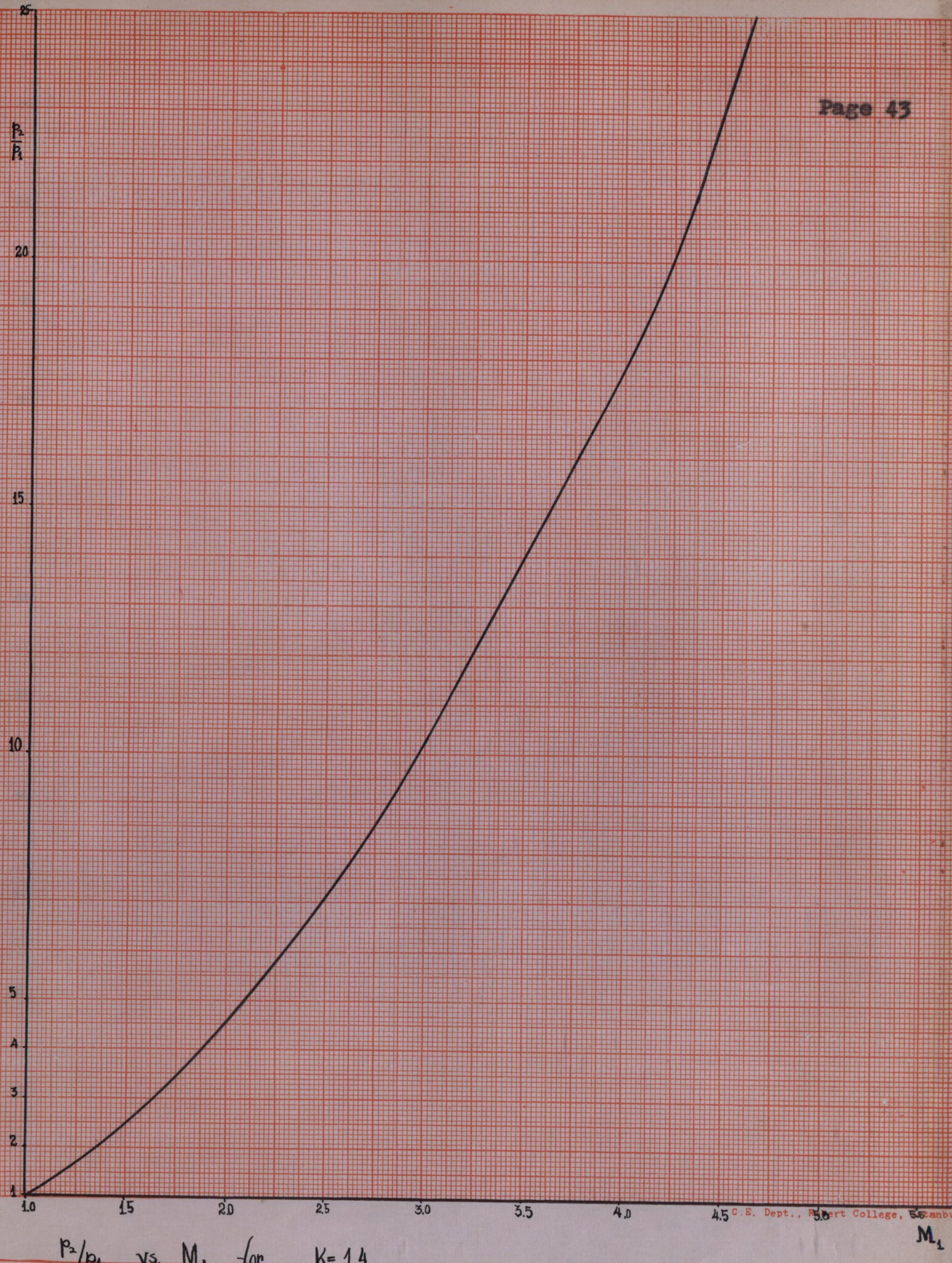
GRAPHS



M_2 vs. M_1 for $K = 1.4$

DATE

Hor. Vert. NAME Orhan Kural



p_2/p_1 vs. M_1 for $K = 1.4$

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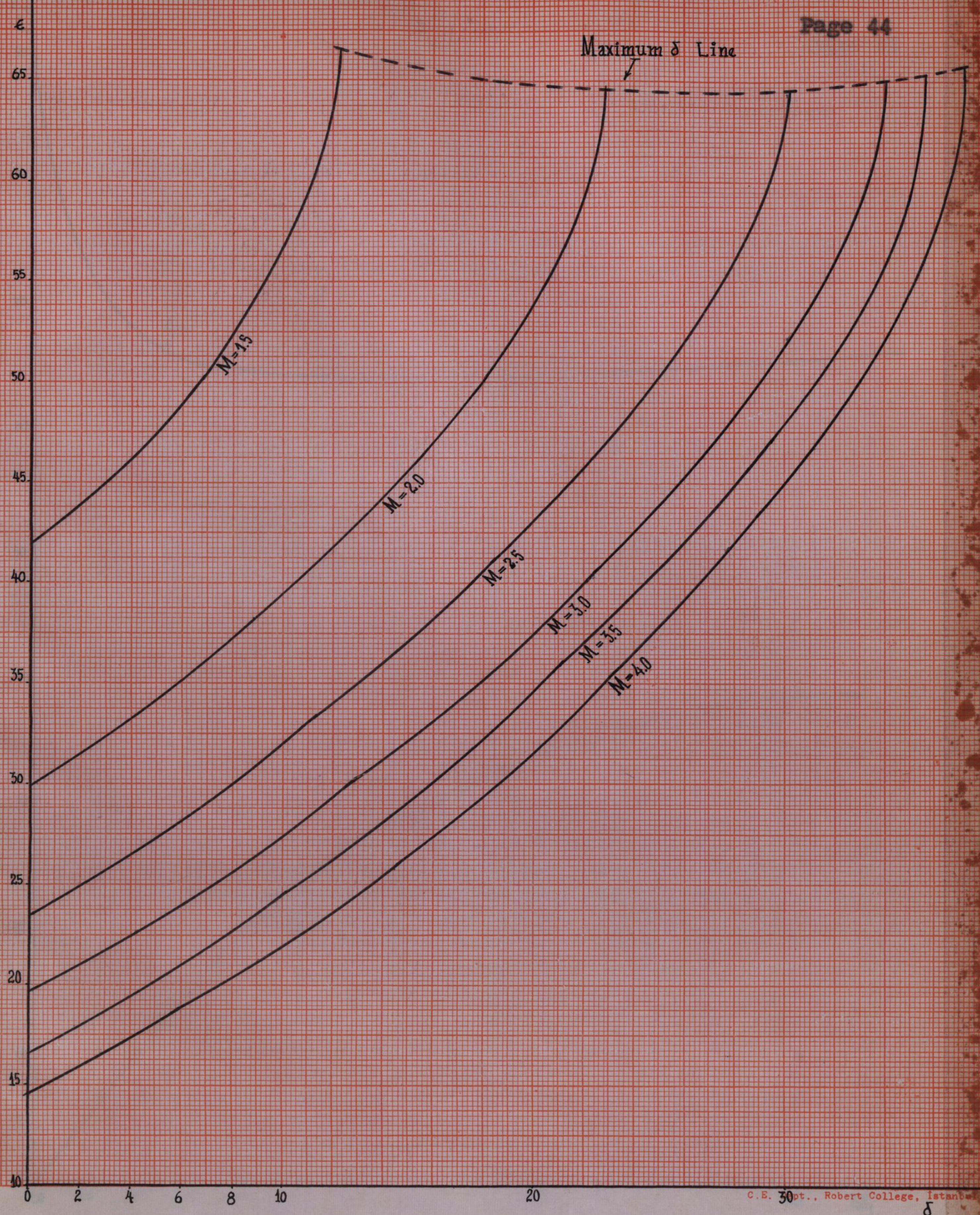

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°ε vs. °δ for K=1.4

DATE

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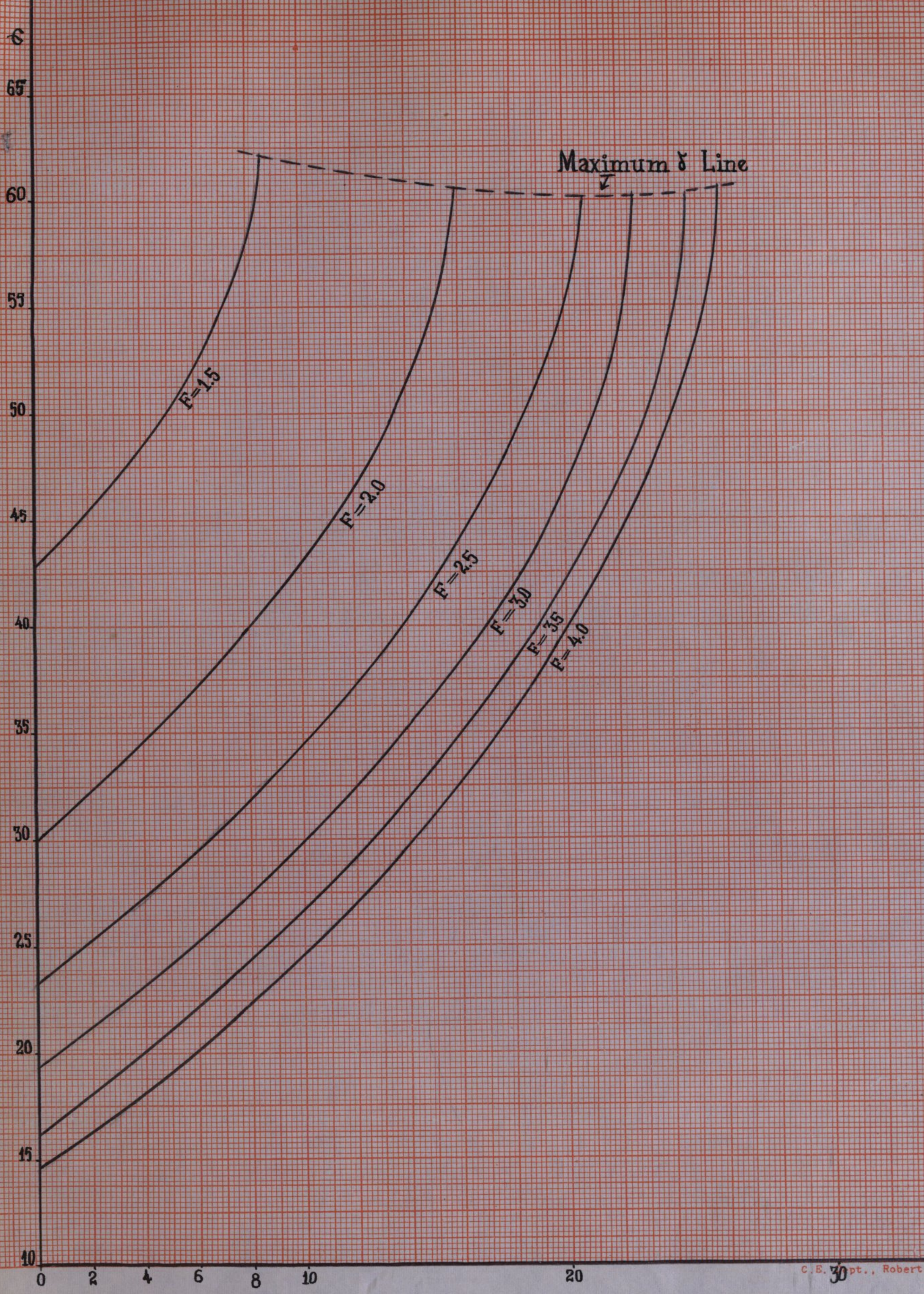

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°ε vs. °δ for K=2

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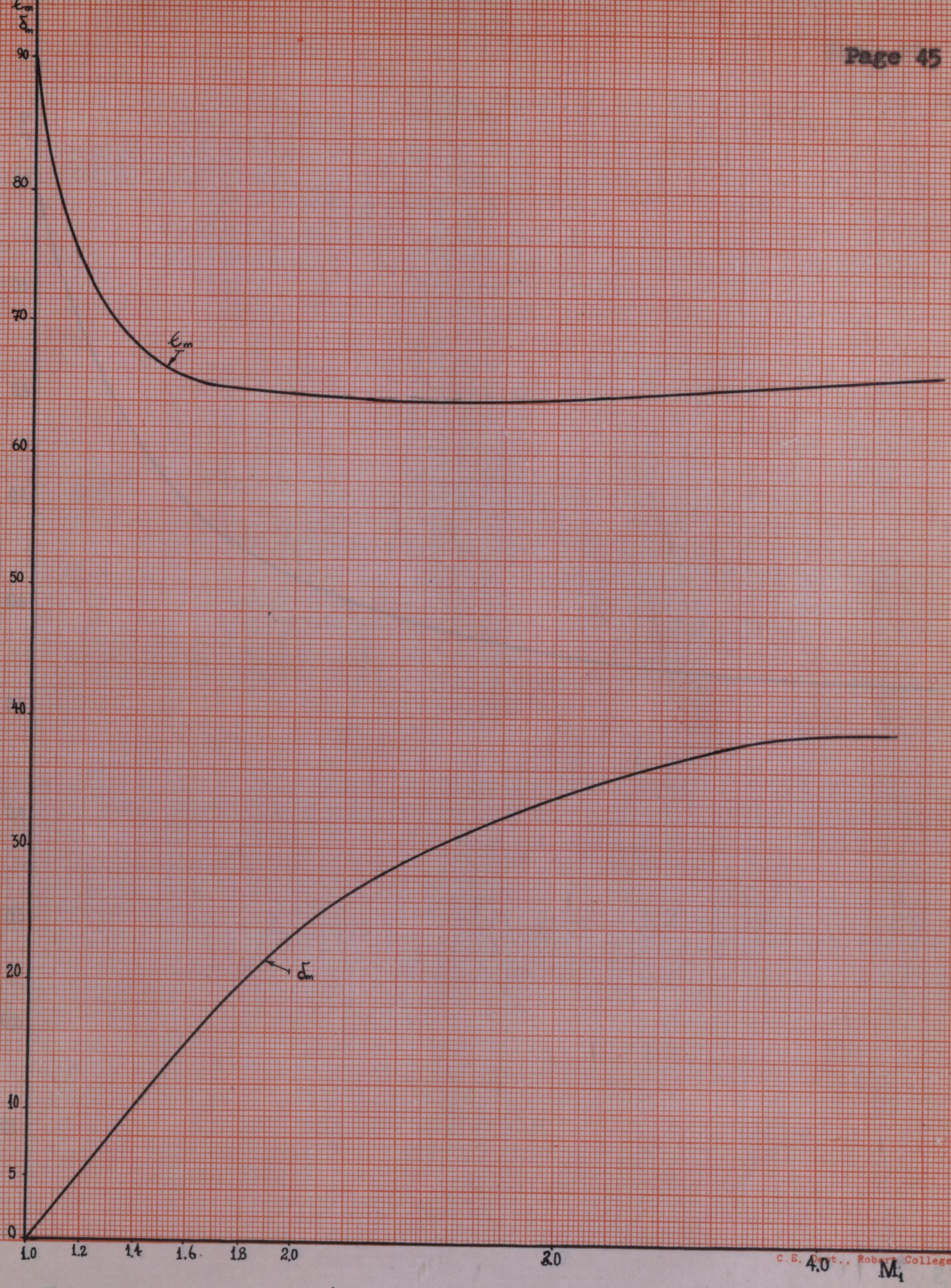
Orhan Kural



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$°\epsilon_{max}$, $°\delta_{max}$ vs. M_1 for $K = 1.4$

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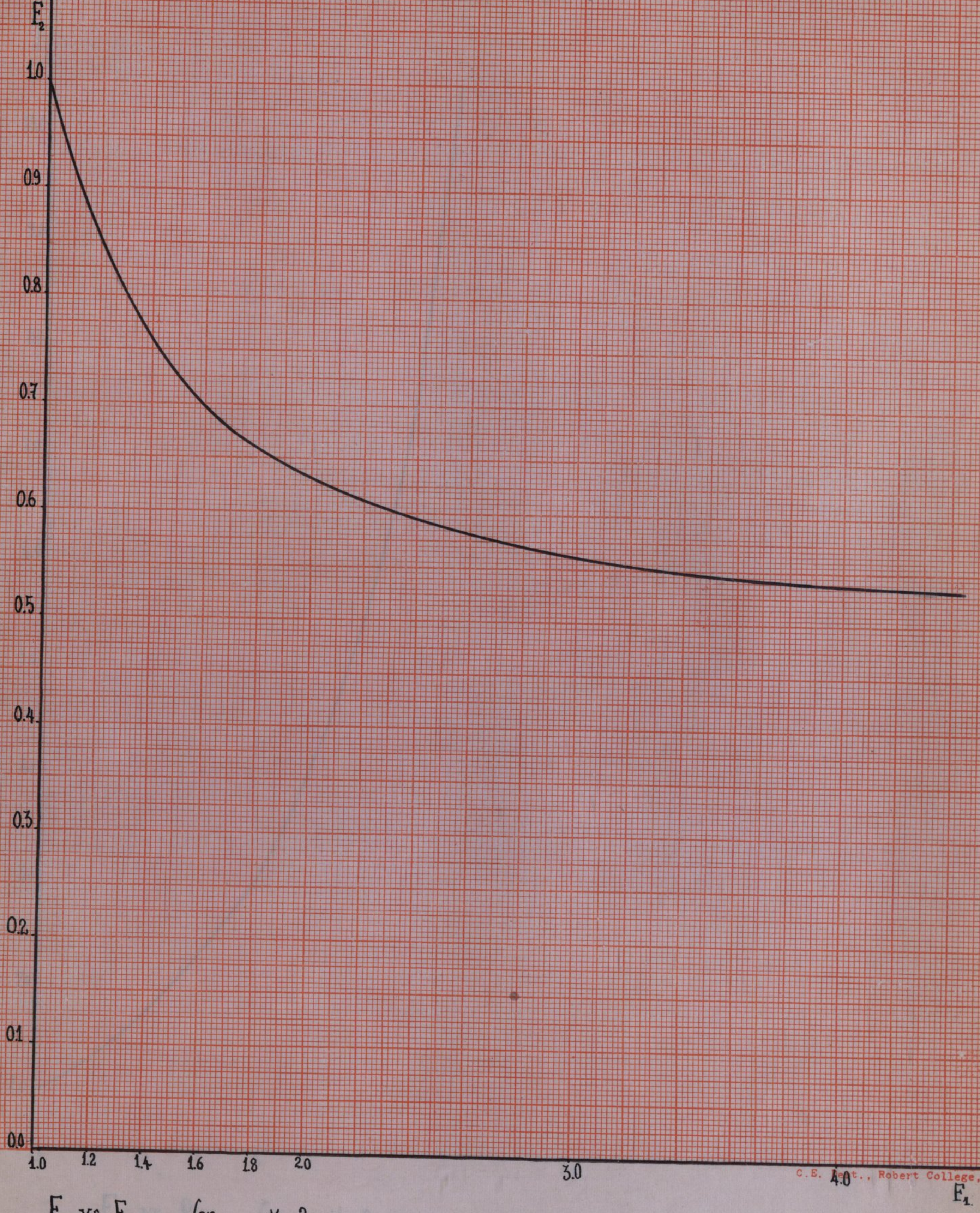

F_2 vs. F_1 for $K=2$

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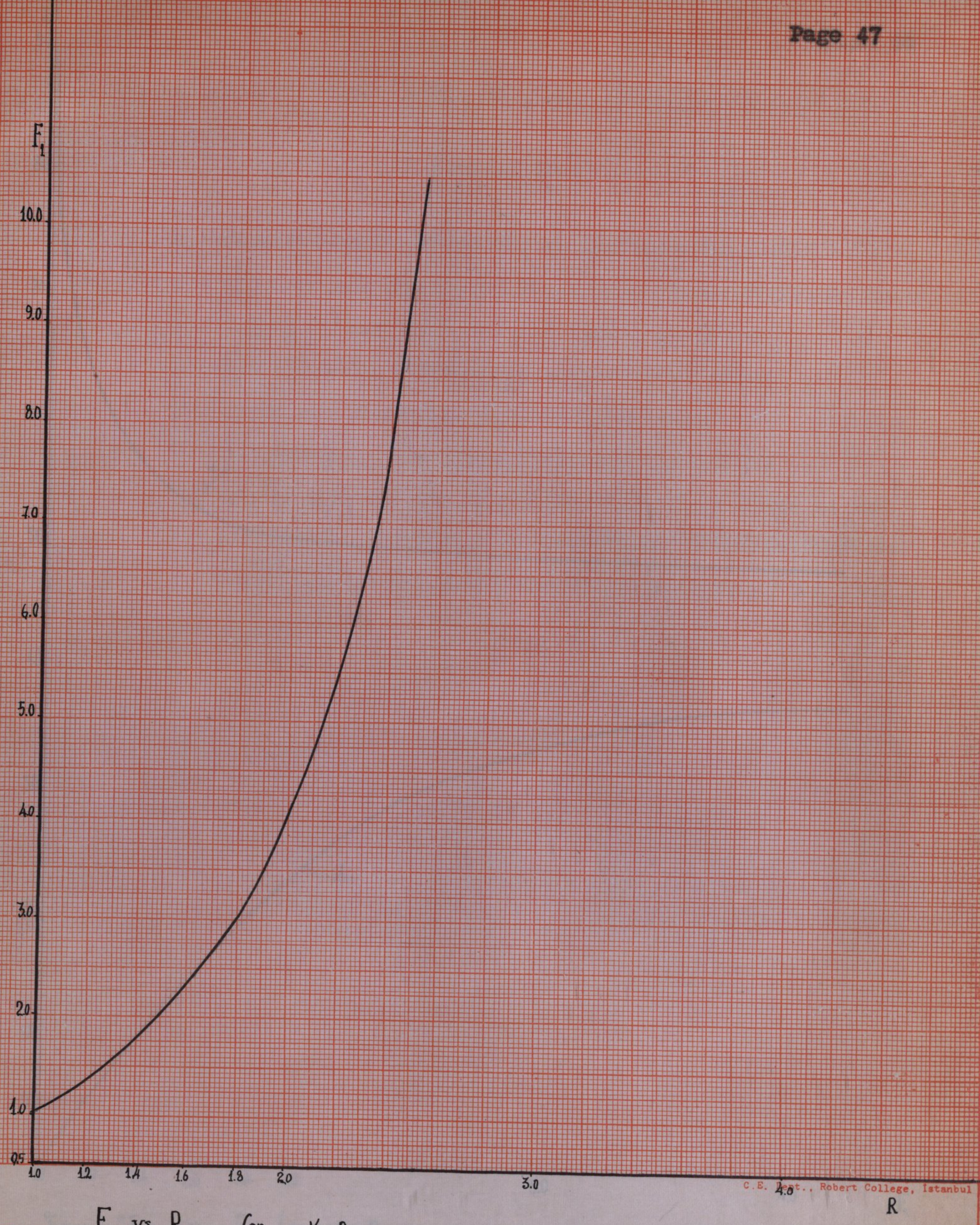

F_1 vs. R for K = 2

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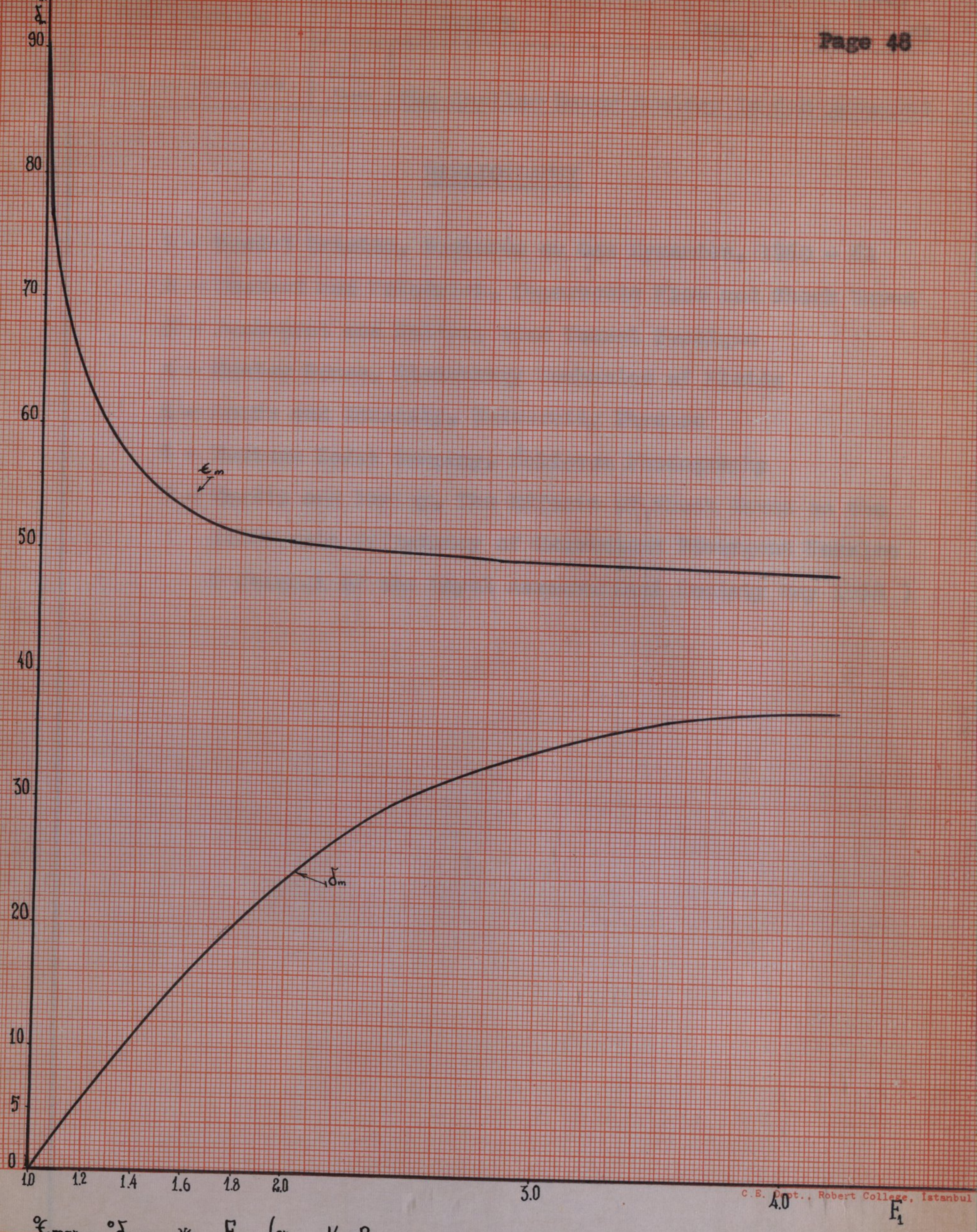

$°\epsilon_{max}$, $°\delta_{max}$ vs. F_1 for $K=2$

DATE

Hor. Vert. NAME Orhan Kural

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