THE EFFECT OF PRODUCT SUBSTITUTION ON NON-BOUNDARY INVENTORY LEVELS

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Dedicated to my parents...

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ABSTRACT

THE EFFECT OF PRODUCT SUBSTITUTION ON NON-BOUNDARY INVENTORY LEVELS

Competitive business world makes it difficult for companies to have high customer loyalty. As the number of manufacturer companies in markets gets higher, customers expect higher service level from these manufacturers. If a company wants to keep the customer satisfaction level high, it must present products to customers on the time and place customers desire. Stock out cases have become nightmares for companies especially in recent years. Inventory management and production control are two topics that careful attention must be paid in related decision making processes. In this thesis, studies are focused on this problem. A system with random yields and two different product classes is taken into consideration. A model is developed that simplifies the decision making processes related to production control and inventory management. In this model, product substitution method is applied at boundary and non-boundary states. Lost sales cost and substitution costs are valid in this model so that the difficulties manufacturers face in satisfying demands have been adapted to the model. Downward substitution is applied at boundary and non-boundary states both while upward substitution is applied only at boundary states. To simplify the decision making process, a methodology is presented that provides threshold values for production and substitution decisions. In addition, a simpler model with only downward substitution at boundary states is used to test our model. By running these two models under same parameter values, profit improvements that are obtained by applying the model developed in this research are observed.

ÖZET

SINIR HARİCİ ENVANTER SEVİYELERİNDE ÜRÜN İKAMESİNİN ETKİLERİ

Rekabetçi iş dünyası şirketler için müşteri sadakatine sahip olmayı zorlaştırmaktadır. Pazardaki üretici şirketlerin sayısı arttıkça, müşteriler de bu şirketlerden daha yüksek hizmet seviyesi beklerler. Bir şirket müşteri memnuniyet düzeyini yüksek tutmak istiyorsa, ürünlerini müşterilerin arzu ettikleri yer ve zamanda sunmak zorundadır. Stoksuz kalınan durumlar özellikle son yıllarda üreticilerin kabusu haline gelmiştir. Envanter yönetimi ve üretim kontrolü, ilgili karar alma süreçlerine ihtimam gösterilmesi gereken iki konudur. Bu tezde, çalışmalar bu soruna odaklanmıştır. Rassal ürün çıktısı ve iki farklı ürün sınıfının yer aldığı bir sistem dikkate alınmaktadır. Üretim kontrolü ve envanter yönetimi ile ilgili karar alma süreçlerini basitleştiren bir model geliştirilmektedir. Bu modelde, sınırdaki ve sınır haricindeki hallerde ürün ikamesi yöntemi uygulanmaktadır. Satış kaybı maliyeti ve ikame maliyeti bu modelde geçerlidir, böylece talepleri karşılamada üreticilerin karşılaştıkları zorluklar bu modele uyarlanmıştır. Aşağı yönlü ikame sınırdaki ve sınır haricindeki hallerde uygulanırken yukarı yönlü ikame yanlızca sınırdaki hallerde uygulanmaktadır. Karar alma sürecini basitleştirmek için, üretim ve ikame kararlarında eşik değerleri ortaya koyan bir metodoloji sunulmaktadır. İlaveten, modelimizi sınamak için sadece sınırdaki hallerde aşağı yönlü ikameye olanak tanıyan daha basit bir model kullanılmaktadır. Bu iki modeli aynı parametre değerleri altında çalıştırarak, bu araştırmada geliştirilen modeli uygulamanın sağlayacağı kâr atışları gözlemlenmektedir.

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LIST OF SYMBOLS

i	The class of a product, i=1, 2 where i=1 (lower quality), i=2 (higher quality)
x_i	On-hand inventory level of class i products
s_i	The price of class i products
h	Holding cost
p	The probability that the outcome of production is a class 1 product
Q	Threshold value for production decision
S	Threshold value for substitution decision in simpler model
$T1$	Threshold value for upward substitution decision in developed model
$T2$	Threshold value for upward substitution decision in developed model
c_i^L	Lost sales cost for class i products
$R_{i,k}(x_k^n)$	Amount of the revenue obtained when a demand for the product type i arrives and is satisfied by giving the customer the product type k, with inventory level of x_k at the arrival of the nth demand
$u_i(t)$	Control variable for substitution
$\dot{x}_i(t)$	Change in inventory level of class i products
t_i^d	Depletion time for class i products
t_i^r	Interrupt time for class i products
h_i	Holding cost for class i products
$x_i(t)$	Inventory level of class i products at time t
μ	Production rate of products
λ_i	Arrival rate of demands for class i products
δ	Ratio between sales prices of products (s_1/s_2)
α	Coefficient of c_i^L in additional cost for substitution
μ_i	Production rate for class i products

1. INTRODUCTION

Today’s rapidly changing world and the hard competition in the markets make today’s business world very challenging. As a result of the “Customer is always right” philosophy, companies aim to satisfy all of the expectations of customers, and to attain the highest possible customer satisfaction level. One of the most important criterions for customers in evaluating the companies is that whether their demands are satisfied by manufacturers on time and place customers want. Due to variations in demand rates or problems that occur in production processes, inventory level of some products drops to zero. In these no-stock cases, coming demands can’t be satisfied and turn into lost sales if backordering is not allowed. To decrease costs faced in these situations, various methods have been developed and applied by companies.

If a company wants to keep the customer satisfaction level high, it must always have products in stock to meet different demands of customers. When a company has many different types of products, it accumulates high storage and holding costs to reach high customer satisfaction target. Otherwise a customer that can not find the product he desires, he immediately goes to an other company for an equivalent product. This situation causes the first company not only lose a customer but also face an extra cost – lost sales cost – containing the cost of losing the customer and increasing the competitor’s customer number (market share). To overcome this problem, product substitution method is applied by companies to increase the customer satisfaction level while decreasing the storage and holding costs of different product types. This method is only applicable in companies offering substitutable products. By the application of this method, when no-stock state occurs, a substitution choice is offered to the customer usually also offering extra price advantages. Sometimes, substitution may occur when the stock level of a product type decreases a pre-determined level (but not necessarily zero).

Product substitution method is a very useful method especially for today’s big companies aiming to reach all classes of customers and providing a wide range of product variety. It can be applied by companies that produce substitutable products. That is, if the products of the company are substitutable, then when a customer can’t find the product he desires in the marketplace, the company may offer another product having similar properties to that customer. The properties of this product must be sufficient to meet the needs of the customer. By applying product substitution method, a company can decrease the number of lost sales cases. This will increase the customer satisfacton level and profit of the company.

Companies have to compete with their rivals to sell their products to customers. On the other hand, customers expect the products they buy to match their expectations as much as possible. Also globalization increases the product variety and the number of companies in local markets. When customers have the chance of making a choice between a high number of available products, customer loyalty becomes a much more difficult target to reach for companies. A company should always meet customer demands on time and should provide all product types and specifications to customers on the time and place customers expect. Of course this condition elevates the on hand inventory levels of companies.

Examples of product substitution by customers are numerous. Product substitution may be consumer driven or retailer – manufacturer driven. In consumer driven substitution, customer wants to substitute when he can not find the product that matches exactly his expectations. Consider a GAP clothing store that stocks 10 different styles of boxer short patterns: Given that consumers might substitute within a product line, what is the optimal stocking level of boxer shorts for the GAP store? The importance of accounting for substitution among different products is growing with the development of e-commerce. Some smart e-retailers are using product substitution to their advantage. A great example is JCrew.com. When making a purchase from the JCrew web site, customers can choose from a complete line of products, colors and sizes, even when some items are out of stock. However, if a customer attemps to place an order for an item that is not available, they will be notified that it is not available and offered a choice of several substitutes or postponed delivery. This is a retailer driven substitution. When the desired product isn’t present in marketplace, company offers products having similar properties to the customer. This way JCrew can track the selection process and use it to optimally stock merchandise. Product substitution is applicable in many sectors. Food sector is one of these sectors. For example, a retailer that sells biscuits produced by a company can apply product substitution by offering similar biscuit types when a customer can’t be satisfied by the product he aims to buy. Let these types of biscuits be Petit beurre, Sultani, Burçak and Kurabi. When a customer aiming to buy Sultani can’t find that product in the retailer, he can be offered to buy another type of biscuit of the same company, for example Burçak. Being offered some price advantages, the customer may accept the substitution offer if the offered product can meet his needs. Here, by applying product substitution method, the company prevents a lost sales case.

Substitution may be applied in two ways; one way (downward) substitution and two way (upward and downward) substitution. In downward substitution, you offer the customer a higher quality or more expensive product instead of the product customer aims to buy. Of course this offer is made at most for the price of the product the customer originally intended to buy. This may seem to be disadvantageous to the company but it avoids the company from losing the customer, and lets the company increase the customer satisfaction level and customer loyalty. In the two way substitution, higher and lower quality products may substitute each other. The substitution method is accepted to be a useful method, but it generates many difficult decisions that must be made carefully. Otherwise, this method may become a nightmare that results in lower profits, confusion and many other problems. A decision-maker should not lose so much time before decisions, so a procedure or a heuristic must be developed to follow during the processes.

Throughout this research, we determine our objective as to develop models that help decision makers, and that can be utilized to increase inventory holding efficiency, company profitability, customer satisfaction level, and to decrease the time spent making the necessary substitution and production decisions. In this research, we study the systems in which some useful methods such as the product substitution method are applied.

Many different aspects of production processes are examined from many different viewpoints in the past studies. Randomness of the yield is one of the related topics that has taken so much interest in recent years. In this research, a system including a production process with random yield is examined. The type of the end product can not be predetermined, it can either be one of two different product types. Different product types have different production probabilities, and the decision maker tries to make the right decisons according to these probabilities. Ramdomness of the yield increases the difficulty level of production planning and inventory control activities. Product classes can substitute each other. When offered, customer never rejects the substitution offer. Price advantages are also presented with the substitution offer in order to make the substitution offer advantageous for the customer.

The remainder of this thesis is organized as follows: We present and explain the results of our literature review studies in section 2. Section 3 contains the descriptions of our objectives in this research. Problem definition and model development phases are given in Section 4, while the results of the numerical analysis are explained in section 5. Section 6 is the sensitivity analysis section. Finally, in section 7, we summarize our results and discuss possible future researches. Some proofs are given in the appendix.

2. LITERATURE REVIEW

There are many studies in the literature about inventory related topics such as inventory holding, inventory holding costs, inventory allocation, product substitution, etc. In addition, we are also interested in the production systems and production control which is also a well studied topic in the literature. The problem in question in this study has two components. These components are inventory control and production control. In the following subsections, a brief review of related work in these topics will be given.

2.1. Studies on Inventory Management

Product substitution method is in the interest of many studies in the literature. In this study, we try to evaluate different substitution polices. Product substitution can be consumer driven or manufacturer driven. In the literature, one of the studies related to consumer driven cases is the study of Netessine and Rudi (2003) [1]. This paper examines a consumer-driven substitution problem and tries to find optimal inventory stocking policies for a given product line. The paper of Bassok *et al.* (1999) [2] studies a single period multi-product inventory model with stochastic demands, proportional revenues and costs, full downward substitution, and arbitrary starting inventory. They construct a model and show that the profit function is concave and sub-modular. Optimal policies are investigated by dividing the main problem into sub-problems and taking the partial derivatives of the profit function. They find some boundaries for the parameters of the problem and finally they show how parameters affect the profitability of the model.

Our interest is in manufacturer driven substitution rather than the consumer driven one. We consider a manufacturing system where the quality of the end product is uncertain and is graded into two quality levels after production. In many manufacturing environments, the quality of the end product is uncertain. Examples are in apparel manufacturing where firms will sell products with slight defects at large discounts, and electronics manufacturing where the clock speeds of chips produced by the same process is uncertain. In most of these environments, customers demanding the lower quality product may be offered a higher quality product. This act is called downward substitutability [3].

Another term related with random yield is the 'downgrading' term. Due to the uncertainty of the product yields, managers face difficulties in planning the amount of production for a certain period of time. This is also valid for systems carrying on batch production. Often, different customers need the same basic product, but some need tighter tolerances than others. One example is found in memory chips. Different customers need different sized chips. If there is a defect in a single quadrant of a large chip, it may not be necessary to scrap the entire unit. The remaining good quadrants can be separated and supplied as smaller chips. Downgrading is possible by separating the quadrants of a non defective large chip.

Some researchers such as Bitran and Dasu (1992) [4] consider a similar problem in electronics manufacturing where customers demanding slower chips can be satisfied by faster chips. Bitran and Dasu's study is one of the first papers in the limited literature on the optimal control of systems with random quality yields and substitution. In other studies, Bitran and Leong (1992) [5] and Bitran and Gilbert (1994) [6] assume that demand is deterministic and formulate the problem as a large scale stochastic program. Bitran and Gilbert study a system that consists of product classes with random yields and substitutable demands. These product classes differ according to their performances. They define the co-production term as follows: "A co-production process is one in which a family of several different products is produced simultaneously. Although these products perform the same basic function, they differ from one another according to one or more key performance parameters". The chip co-production problem is assumed to involve two stages in each period. In the first stage, the morning problem, a lot size is determined. In the second stage, the afternoon problem, the random yields have been observed, and downgrading decisions are made. Hsu and Bassok (1999) [7] analyze a single period problem where an order is placed once and random quantities of products of each class are received as a result.

Another important study in this literature is the study of Duenyas and Tsai (2000). In this paper, the authors try to find an answer to the question most manufacturers face; (i) how does one decide when to start/stop production? and (ii) when is it optimal to satisfy customers demanding the lower quality products by using the higher quality ones? They assume that the quality of the product is uncertain where it may be either high or low quality. High quality products can be used to substitute demands for low quality products.

They model this problem as an M/M/1 queuing system. They first investigate the two-products (quality levels) case. Then they extend their results to the multiple products environment. They also examine batch production and setup cost cases. They allow random production times and demands where they focus on characterizing the structure of the optimal policy and comparing developed heuristics to the optimal policy. Their formulation is for the infinite-horizon problem and they take queuing effects into account by focusing on a production system where production of each unit takes a random amount of time.

Another important topic in the related literature is the hybrid production systems with joint manufacturing and remanufacturing of items. Due to both economic incentives and legal pressure, more and more companies are going to be engaged in the product recovery business which refers to activities for regaining materials and value added in used products. A very important field of product recovery is remanufacturing which refers to those activities that bring used products or major modules back to such condition that remanufactured items are marketable again. This is widely performed for high-valued industrial products like copiers, computers, vehicle engines or medical equipment. In many industries, original equipment manufactures are active in the remanufacturing business themselves. Here two versions of the same product are the output of the hybrid system. Korugan and Gupta (2001) [8] consider a production system that satisfies the demand for a product with both remanufactured and newly manufactured items. In addition to its traditional manufacturing activities, the system accepts used product returns and brings them to ‘as good as new’ condition through disassembly, refurbishing, rework and upgrading activities to satisfy demand. In their study, substitution method is applied under different substitution rules and they try to find out which substitution rule works better than others.

Inderfurth (2002) [9] considers the coordination problem in hybrid production systems with remanufacturing. He focuses on a single-stage single-period problem with independent stochastic demands for both product types. The objective of the model formulated in this study is to maximize the expected profit from all decisions, i.e. from determining the manufacturing and the remanufacturing order quantity, given an arbitrary starting inventory of serviceable products. These analyses reveal how decision rules for hybrid manufacturing/remanufacturing systems in case of product substitutability can be developed which are applicable in practical decision making. Bayındır *et al.* (2002) [10] state the importance and possible gains of remanufacturing activities under one-way substitution in their report. They construct a steady state profit model under certain environmental assumptions on capacity requirements of operations, and revenue and cost schemes.

Another concept related to our work is inventory allocation. The allocation problem typically appears when a supplier maintains a common stock in order to satisfy different customers. The stocks, as well as the production capacity, are limited resources; therefore they must be rationed between the customers, possibly according to their relative economic importance for the supplier. Inventory allocation problem is one of the main interest areas of classical inventory theory. Although the literature in this domain is relatively rich because of the nature of the assumptions of uncapacitated production, constant lead times and identical clients, most of the research in this area does not directly address the issues of stock rationing [11].

Topkis (1968) [12] is one of the first researchers that focus on the inventory models that directly investigate the issues of stock rationing which fall into the single-location multiple- customer category. He considers how inventory should be allocated between demand classes within a single period of a periodic review model. The analysis is facilitated by breaking each review interval into a finite number of subperiods. At the end of each subperiod, the decision maker allocates inventory to demand that has been realized thus far. Within a single review interval, Topkis proves there exists optimal, nonnegative, rationing levels for each demand class which, under certain conditions, are decreasing in time. Besides Topkis's study, one of the other studies related with similar topics is the study of Nahmias and Demmy (1981) [13]. They are the first to analyze a rationing policy in a (Q,r) environment.

In the article of de Vericourt *et al.* (2002), the stock allocation problem is considered when there is a capacitated system that produces a single item for a demand with several classes of customers. Orders from different customers have different values for the supplier due to the backorder costs of these customers. They construct a model in which there is a supplier that produces the standard items and places them in a buffer in a make-to-stock manner. The dynamic decision problem of interest is to find an optimal stock allocation policy that minimizes average inventory holding and backorder costs.

In recent literature, one of the main studies about this topic is the study of Deshpande, *et al.* (2003) [14]. The authors analyze a method similar to the traditional (Q,r) control but also includes an inventory threshold level, K, which signals when to reserve stock for higher-priority customers. They assume a situation with two demand classes with high and low priorities. In the paper, four different control policies, priority cleaning, threshold cleaning, hybrid policy and optimal rationing policy are compared for the inventory management/rationing problem. Motivated by a study of military logistics, they consider an inventory replenishment policy supporting two demand classes, differing in delay and shortage penalty costs and demand arrival rates. Their rationing policy differs from Topkis's (1968) in three points: They make the decision of whether to fill or delay an order at the moment the order arrives while Topkis delays this decision until the end of each subperiod. Their rationing level is stationary, which is consistent with their continuous review environment, where there are no defined time intervals for revising decisions. And finally, their replenishment order cycles are based on the inventory position with respect to setup costs, lead times, and the possibility that multiple replenishment orders may be in the pipeline.

2.2. Studies on Production Control

A number of production-inventory problems for capacitated systems involving multiple customer classes have been studied in the context of the make-to-stock queue. The rationing problem in this setting resembles a closely related category of multi-item scheduling problems. In this latter category of problems, multiple classes of customers demand different products from the manufacturing system that has to schedule production by dynamically sharing capacity [15]. Veatch and Wein (1996) [16], and Pena-perez and Zipkin (1997) [17] propose heuristic solutions to this problem. Ha (1997a) [18] partially characterizes the structure of the optimal policy for two classes of customers.

Ha (1997a) investigates the switching curves that will simplify the decision making processes about when to produce and if production is warranted, which type of product to produce. He suggests that there are thresholds for each product such that it is optimal to satisfy the arriving demand from a customer from the on-hand stock if the stock level is above the threshold for that customer. Production priorities for backordered products can be determined by using these thresholds in a simple way. He mainly focuses on three different priority rules which he calls 'the $b\mu$ rule', 'the modified $b\mu$ rule' and 'the switching rule'. In our study, we use thresholds to determine the answers of two important questions, whether to produce or not and whether to substitute or not.

Ha (1997b) [19] formulates and studies the optimal rationing and production control of a multi-class system with lost sales cost. When the inventory is low, it is reasonable to ration inventory by rejecting demand from less valuable classes in an anticipation of future demand from the more valuable classes. He shows that the optimal production-control policy is a base-stock policy, and the optimal rationing policy is described by threshold levels corresponding to different demand classes. When the stock on hand is above the threshold level of a certain class of demand, it is satisfied from on hand stock, and otherwise it is lost. It is also mentioned in the paper that one example of the stock rationing problem is the common component of an assemble-to-order system. Component commonality is a strategy that uses one common component for two or more end products. With the recent trend of product proliferation and continuous shrinking of product life cycles, many manufacturing firms adopt a commonality strategy to simplify their engineering designs, achieve economies of scale in both R&D and production, and reduce the amount of safety stock due to risk pooling. The similarity between this study and ours is that the lost sales case is present in our study and thresholds are present and used to determine the answer of two important questions; whether to produce or not, and whether to substitute or not.

De Vericourt *et al.* (2000) [20] study the problem of dynamically allocating production capacity between two products to minimize backorder costs per unit time in a make-to-stock single machine system. They examine the optimal control problem by focusing on the optimal hedging point policies. The results obtained in this paper show that in the case where both products are backlogged, it is optimal to produce the most expensive item in terms of the backorder cost until its stock reaches a predetermined level before switching to save the less expensive product from backlog. It is also mentioned that this safety stock level does not depend on the level of backlogs of the less expensive product and can, in certain cases, be significantly large, depending on the cost and traffic parameters. In another study of de Vericourt *et al.* (2001), the improvements, which can be gained by effective allocation of production and inventory, in the cost performance of a system, a manufacturing facility, which produces a single type of product that is demanded by different types of classes of customers, are investigated. In the model, these customer classes differ in their demand rates, backorder costs and service levels. The authors analyze this model by using three different policies: The 'first-come-first-served' policy, the 'strict priority' policy and the 'multilevel rationing' policy.

3. OBJECTIVES

One of the most challenging tasks company managers face is to adapt the systems in companies to the rapidly changing conditions of the business world. The economic situations of countries, inflation rates, density rates of markets, globalization, and trends that shape the taste and preferences of customers are some of the main factors that cause the changes in the business environment. All of these factors affect the demand rates for products, production and inventory holding costs, prices of products and the profit margin obtained from sales, etc. Every change in these parameters makes it more difficult to obtain adaptation of the company to these conditions and to continue activities of the company effectively and successfully.

Every company has different working conditions. Some of them have high product variety, while some others have low product variety but execute mass production. Some of the companies have many production lines in which only one type of product can be produced in each, while some others have only one but very flexible production line that lets produce many different types. It is very difficult to develop models, heuristics, rules that are applicable to all of these systems effectively. That is, focusing on a specific system and studying on that is more reasonable.

In this research, a system with random yields is examined. There is only one production line and production can result with one of two different product classes. These product classes pass through the same production processes and differ in their quality levels. Class 2 products have higher quality levels than class 1 products. Each product class has its own demands and these demands occur with different arrival rates. In addition to different demand rates, sales prices of product classes are also different.

There are mainly two decisions to be examined in this research: One of them is related to the production control and the other one is related to the inventory management. We aim to simplify their decision making processes. Decision whether to produce or not is the first decision in question. The next decision studied is how to satisfy customer demands. These decisions have to be made by taking into consideration the current inventory levels of products. We determine the main objective of this study as to help the decision makers of the production facilities in their decision making processes by providing them a methodology in which after entering valid parameters for the current situation, the suggested decision is provided. More clearly, the mentioned methodology provides some threshold values based on the current inventory levels of the products that remove the uncertainty in minds about what decision should be preferred when. By following such a systematic, many benefits can be obtained such as time saving, more accurate decisions, less complexity and uncertainty, etc.

Production control is one of the main topics studied in this research. By constructing a model and applying this model to the system briefly described above, the right answer of the question when to produce is to be determined. Because of the randomness of the yield, the inventory levels of different product classes can’t be controlled independent of each other. Instead, the optimal total inventory level is taken into consideration. A model that gives a threshold value for the total inventory level is to be studied.

Inventory management is an important issue that must be handled by every manufacturer company very carefully. To offer customers higher service levels, a model should be developed that decreases the number of customers whose demands can’t be satisfied by the company. Otherwise, lost sales situations will cause money and market share losses. During the literature review studies, we realized that ‘two-way substitution case’ and ‘substitution at non-boundary states’ are two issues that had not taken enough interest and attention. Substitutions are usually assumed to be valid only at boundary states but not at non-boundary states. The insufficiency of the studies on these issues made us think that it would be better to build the basis of our project on them. We think that we can focus on these issues and develop a model and heuristic that will lessen the uncertainty and question marks in minds by removing the lack of information. In the model development phase of this research, product substitution method is taken into consideration in order to obtain higher service levels and manage uncontrollable increases in the inventory.

4. MODEL

4.1. Problem Definition

We examine the production control and inventory control problems in a system where substitution method is applied to increase the satisfaction levels of customers. We consider the same system as Duenyas and Tsai [3]. In this system, there is only one production line. Production in this line results with a random yield. The end product comes out in two different quality levels creating two product classes in the system. Class 2 products are higher quality products compared to class 1 products. These product classes have their own demands with different arrival rates. Downward substitution is considered when the inventory level of class 1 products is zero in Duenyas and Tsai. Produce, do not produce thresholds and product substitution is considered to control the inventory level. Two different cases with different substitution rules are examined in this research. This substitution policy is examined under different revenue levels with additional costs in this study. In addition, a model is developed that let downward substitution at boundary and non-boundary states both and upward substitution at boundary states.

We base our research on the models given in Duenyas and Tsai, and Korugan and Gupta [8]. In Duenyas and Tsai, a production and inventory control methodology is presented and examined while in Korugan and Gupta, different substitution policies for the finished goods inventory management are examined and compared for hybrid systems with manufacturing and remanufacturing activities. In this research, we combine the models in these works and introduce a new control methodology. For a better understanding of these studies and model, it will be better to give some brief information with explanations of the models in these two previous studies.

In the study of Duenyas and Tsai, a manufacturing system with downward substitutability is examined. Downward substitutability means that one can give the customer a product with higher quality than he wants for the price of the lower quality product. The production process has random yields in different quality levels. The quality level of the end product can't be predetermined; it can either be a low quality or a high quality product. Demands and production times are stochastic.

Simply, the research focuses on finding an answer to two questions:

- How does one decide when to start/stop production (i.e., what inventory levels are appropriate?).
- When is it optimal to satisfy customers demanding the lower quality products using the higher quality ones?

Duenyas and Tsai introduce a heuristic that considerably simplifies the implementation of the substitution method and the decision making process. They try to find the answers for the questions mentioned above. Here, they examine the one-way substitution case with substitution occurring only when the inventory level of class 1 products drops zero. In this research, we construct a model and form a heuristic that considers substitution at non-boundary states as well.

Korugan and Gupta consider a production system that satisfies the demand for a product with both remanufactured and newly manufactured items. This means that, besides its traditional manufacturing activities, the system accepts used product returns and restores them to working conditions through disassembly, refurbishing, rework and upgrading activities. Thus the output of the system is products of two different qualities but with the same capabilities. Here a substitution method is applied under different substitution rules to find out which substitution rule is better. They introduce a fluid approximation model for the substitution problem. Inspired by their efforts, we consider a fluid approximation in our model as well.

The notation we use is given below:

i: the class of a product, i=1, 2 where i=1 (lower quality), i=2 (higher quality).
x_i: on-hand inventory level of class i products.
s_i: the price of class i products.

h: holding cost.

μ: production rate of products.

p: the probability that the outcome of production is a class 1 product.

λ_i: arrival rate of demands for class *i* products.

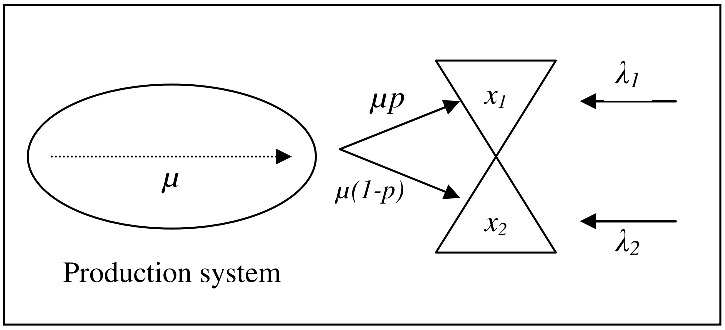

Figure 4.1. The material flow of the system

In the process, there is only one production line. This production line produces with the production rate μ and a random yield. The end product may be either a class 1 product or a class 2 product. The probability that the production process results with a production of a class 1 product is *p*, and the probability that the production process results with a class 2 product is 1-*p*. These two product classes have their own demands and these demands for different product classes are satisfied by the related product class if available.

The randomness in the yield makes the problem more complicated. When the yield of the production can be determined before production, the inventory levels of different product types can be planned and balanced easily. The production priority goes to the product type having the highest demand rate and/or the lowest inventory level at that time. In addition, when the inventory level of a product type falls under a pre-determined level, the system may start production process of that product. These possibilities facilitate the decision-making process in a production facility. Yet, with the randomness of the yield, controlling the production system and the inventory levels of products becomes a much more difficult task. The main reason for this difficulty is that, the type of the product that is going to be produced can't be determined before production. Consequently, to increase the inventory level of the right product type (product type having highest production priority), depending on the production probabilities, products of other types may be produced unnecessarily. This results in an unwanted increase in the inventory levels.

4.2. Introduction to Model

The main differences of our model from that of Duenyas and Tsai are the addition of extra parameters that allow changing the substitution rules and policies. In this part, we first try to explain these differences and additions. Later, we construct our own model, using MDPs (Markov Decision Processes) and mention our assumptions.

Our model suggests a two way substitution rule at the boundaries. That is lower quality products may be used to satisfy the demand of the higher quality ones if the higher quality products are out of stock at a certain cost and vice versa. In addition, downward substitution is allowed not only at the boundaries but also at non-boundary states whenever the determined conditions are met. It is assumed that the customer never rejects a substitution offer. When applying this method, the decisions in question are when to satisfy a demand for a product with the other one depending on the inventory levels in order to maximize the total profit. If a downward substitution policy is applied, only one decision is under consideration. But if a two-way substitution policy is applied and if both of the products can substitute each other, you have two decisions to be made.

For the decision about whether to produce or not to produce, at a given inventory position x(x_1, x_2), a threshold level Q is calculated. When the total inventory level of both classes of products is above the threshold level Q, we should not produce ($x_1+x_2 \geq Q$). On the other hand, when the total inventory level of two classes of products is below this threshold level Q, we should produce ($x_1+x_2<Q$). Decision epochs arise after each occurrence of events such as an arrival of a demand or completion of a production process. Figure 4.2 shows a flow chart of this decision process:

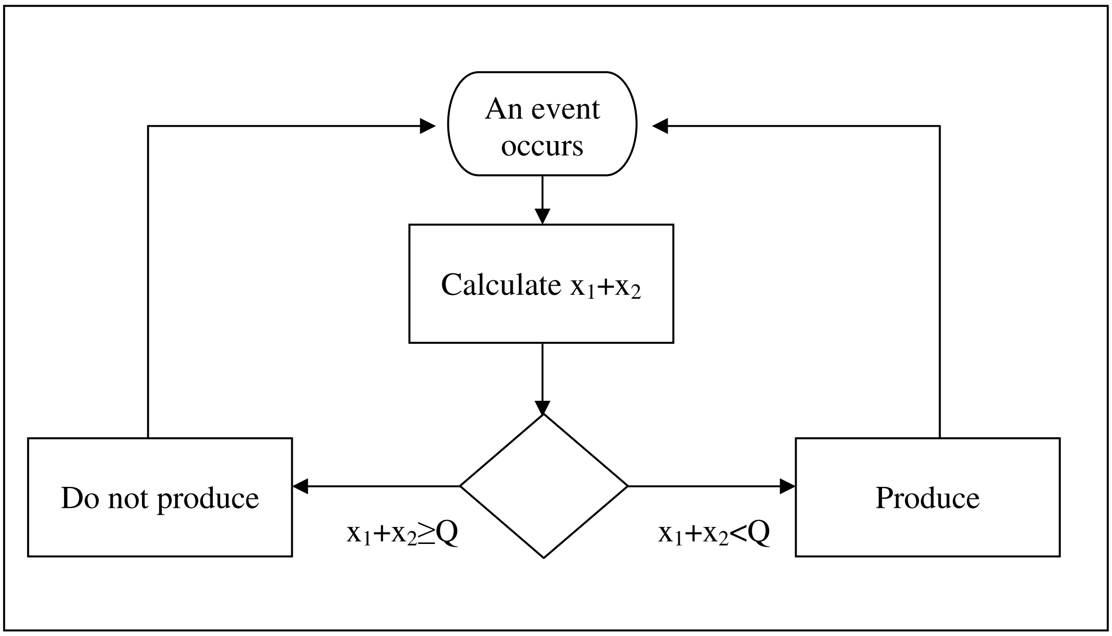

Figure 4.2. Flow chart for the decision processes whether to produce or not

After the production decision, the next phase is the substitution decision. Two threshold values T_1 and T_2 will be calculated for the substitution decisions at boundary states (when the inventory level of a product is zero). For example, if the inventory level of class 1 products is below the threshold value T_1 and the inventory level of class 2 products is zero, the system says 'do not substitute, demand is lost (lost sales case)'. If the inventory level of class 1 products is above the threshold value T_1 and the inventory level of class 2 products is zero, the system says 'substitute, satisfy the demand for class 2 product with class 1 product'. Similarly, if the inventory level of class 2 products is below the threshold value T_2 and the inventory level of class 1 products is zero, the system says 'do not substitute, the demand is lost (lost sales case)'. If the inventory level of class 2 products is above the threshold value T_2 and the inventory level of class 1 products is zero, the system says 'substitute and satisfy the demand for class 1 product with class 2 product'. These are the situations at boundary states. For the non-boundary states, only downward substitution is applied. Here, we have a trajectory and above this trajectory, substitution is made; below the trajectory, substitution is not made. Figure 4.3 and Figure 4.4 may be helpful in the understanding of these processes.

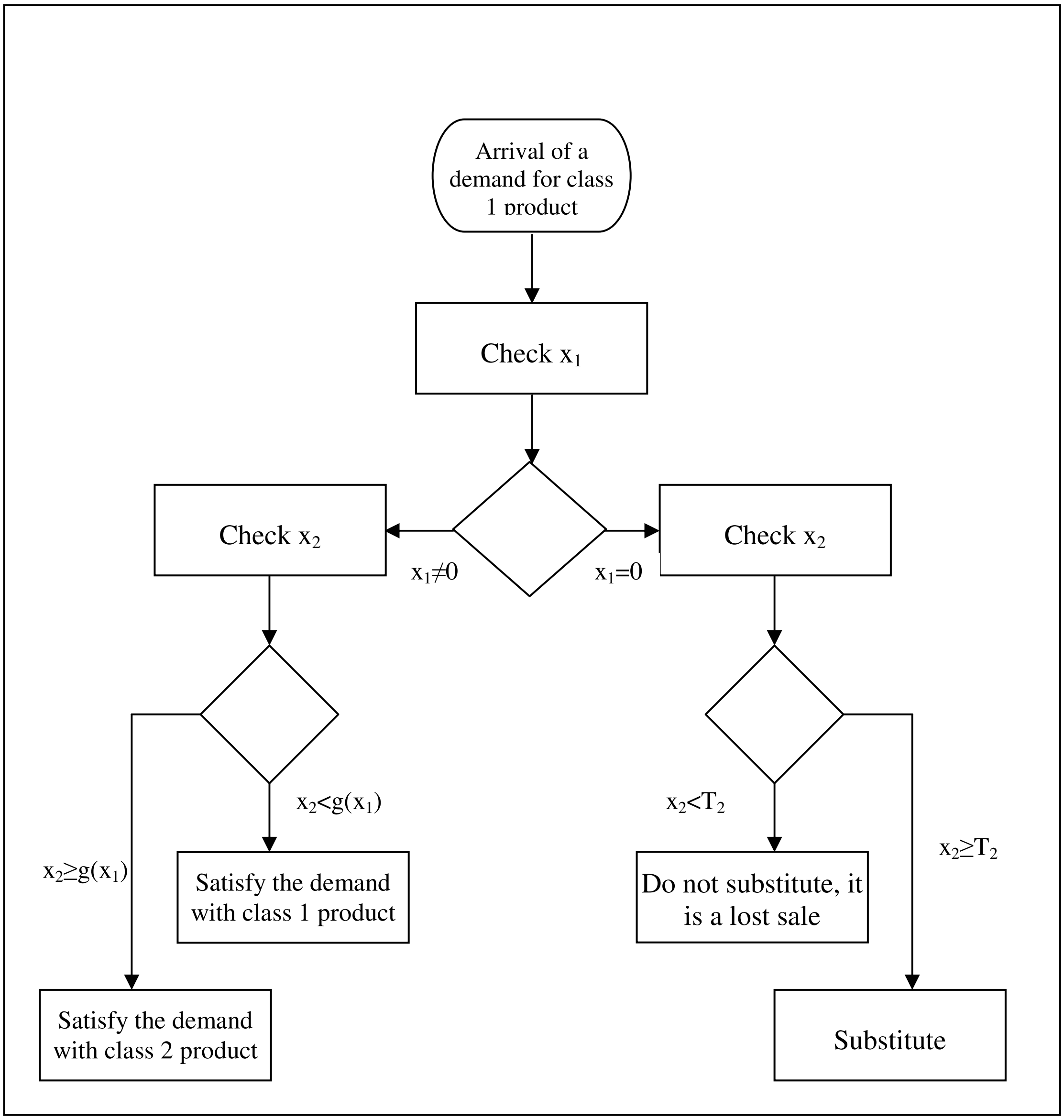

Figure 4.3. Flow chart for the process related to the decision how to satisfy demands for class 1 products

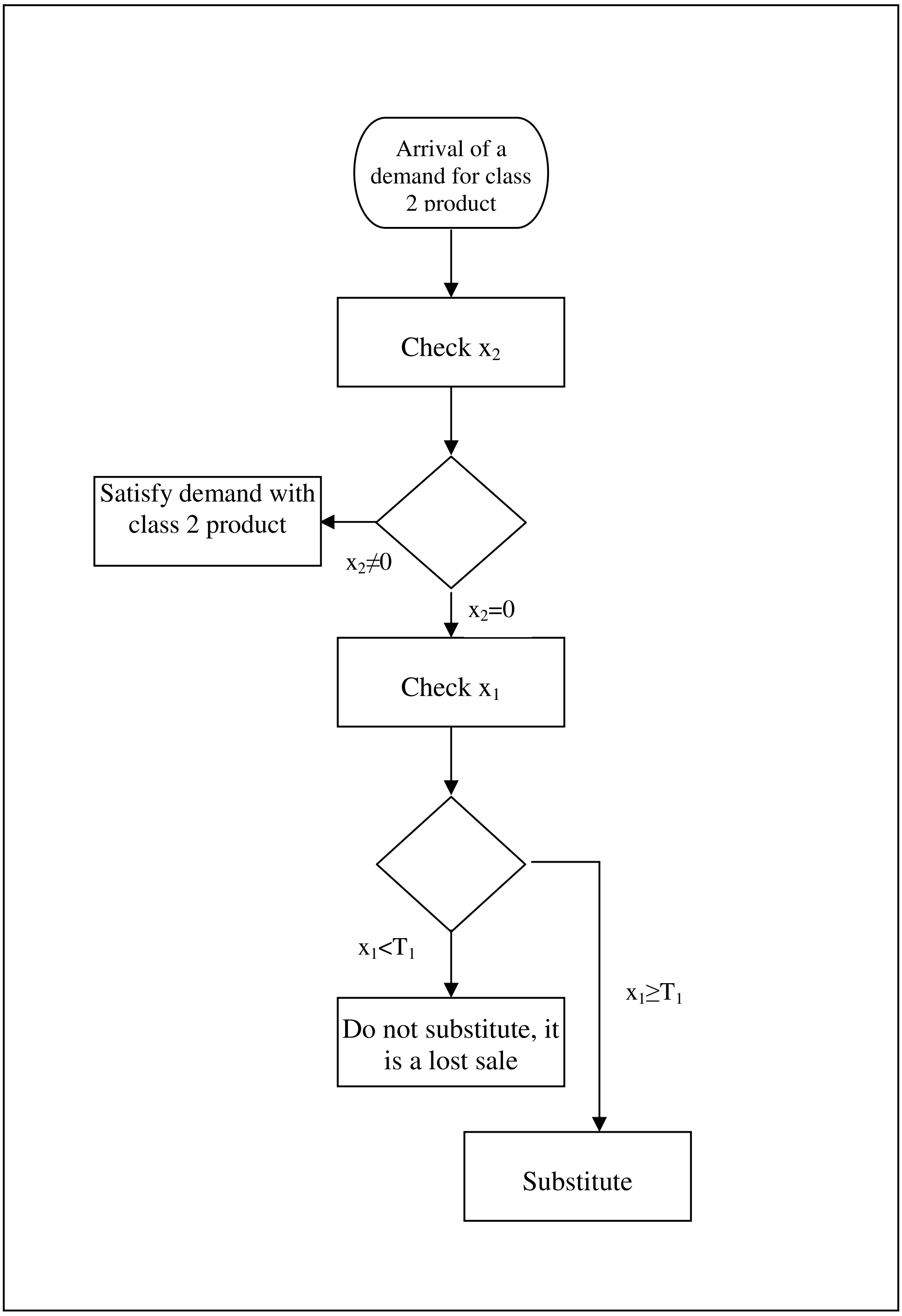

Figure 4.4. Flow chart for the process related to the decision how to satisfy demands for class 2 products

4.3. Mathematical Model

This section includes the information about our model and its development phase. We had briefly mentioned some of the abbreviations and symbols used throughout the thesis in the ‘problem definition’ part before. These symbols and abbreviations are not going to be repeated again in this part but when new ones are added to these symbols and abbreviations, they are going to be mentioned before the points they are used not to let any confusion appear. In the following lines, MDP (Markov Decision Process) models and value functions built for further necessary calculations are going to be presented. For the simplicity of understanding, necessary explanations about equations and models are given besides the mathematical models, and the assumptions made in the preparation of these models are declared.

Model development part is followed by the ‘numerical results’ section. ‘Numerical results’ section consists of mainly two parts. In the first part, only downward substitution is considered and the model is built according to this assumption. The revenues and costs are different and simpler in this part than the ones in the following part. The main idea of this difference is that one of our objectives is to develop and improve the model in Duenyas and Tsai’s paper and obtain a better and more practical model. As a result, firstly the assumptions and cost functions used in this previous study are used at that point, to make the comparison between these two models easier. Latter, besides downward substitution, two-way substitution is applied with additional costs related with substitution and lost sales cases. To establish a good and strong coordination level between two sections, the differences that occur in the models used in these two parts are mentioned in the explanation and presentation phases of the mathematical models in the following lines.

Firstly, the first case mentioned above will be taken into consideration. In this case, only downward substitution is applied at boundary states and non-boundary states both. The amounts of revenues obtained after the sales in this model are assumed to be as mentioned below. In this display, the $R_{i,k}(x_k^n)$ term represents the amount of the revenue obtained when a demand for the product type i arrives and is satisfied by giving the customer the product type k, with inventory level of x_k at the arrival of the nth demand.

$$R_{i,k}(x_k^n) = \begin{cases} s_1 & i,k = \{1\}, x_1^n > 0. \\ s_1 & i = \{1\}, k = \{2\}, (x_1^n = 0, x_2^n > T_2) or (x_1^n > 0, x_2^n > g(x_1)). \\ 0 & i = \{1\}, x_1^n = 0, x_2^n < T_2. \\ s_2 & i,k = \{2\}, x_2^n > 0. \\ 0 & i = \{2\}, x_2^n = 0. \end{cases} \tag{4.1}$$

As seen above, revenue of s_1 is obtained when we satisfy a demand for class 1 products. At this point, the type of the product that is given to the customer does not affect the revenue obtained; we can satisfy the demand by a class 1 product or a class 2 product and obtain the revenue of s_1. On the other hand, there is no substitution option when satisfying a demand for class 2 product. We may satisfy the demand for class 2 product by a class 2 product and obtain a revenue of s_2. If this demand can't be satisfied, there is no chance for substitution and this demand becomes a lost sales case. When a demand can't be satisfied, that demand is lost but no lost sales cost is applied in this case. The assumption $s_2 \geq s_1$ is almost always valid since a class 2 product provides higher quality level than a class 1 product and consequently is sold for a higher price.

We can write our value function with the assumptions mentioned above for the first case in which only downwards substitution is allowed as follows:

$$\begin{aligned} &\frac{g}{\mu+\lambda_1+\lambda_2} + \upsilon(x_1,x_2) = \\ &\frac{1}{\mu+\lambda_1+\lambda_2}\Bigg\{ -h(x_1+x_2) + \mu \max\begin{pmatrix} p\upsilon(x_1+1,x_2) + (1-p)\upsilon(x_1,x_2+1) \\ \upsilon(x_1,x_2) \end{pmatrix} \\ &\qquad + \lambda_2\Big(\big(\upsilon(x_1,x_2-1) + R_{2,2}^{\Pi}(x_2^n)\big) I_{x2>0} + \big(\upsilon(x_1,0)\big) I_{x2=0}\Big) \\ &\qquad + \lambda_1 \max\begin{pmatrix} \big(\upsilon(x_1-1,x_2) + R_{1,1}^{\Pi}(x_1^n)\big) I_{x1>0} + \big(\upsilon(0,x_2)\big) I_{x1=0} \\ \big((\upsilon(x_1,x_2-1) + R_{1,2}^{\Pi}(x_2^n)\big) I_{x2>0} + \big(\upsilon(x_1,0)\big) I_{x2=0} \end{pmatrix} \Bigg\} \end{aligned} \tag{4.2}$$

In this value function, class 2 products can be used to satisfy demands for class 1 products while, on the other hand, class 1 products can't be used to satisfy demands for class 2 products. This downward substitution option is valid not only at boundary states, but also at non-boundary states. Here, $I_.$ is the indicator function. We assume that customer never rejects the substitution offer. Production results with a class 1 product with probability p_1 (p_1=p), and results with a class 2 product with probability p_2 (p_2=1-p). For the calculation of average profit, g function and right hand side of the equation is divided by "$\mu+\lambda_1+\lambda_2$" to obtain the unit occurrence rate of events. The set of possible events contains the production of products or arrivals of demands for each type of products. We can either produce or keep the station idle according to the maximization function that is multiplied by the μ term in the equation above. The production decision will increase the inventory level of one of the products according to the random yield of the production and takes us to state $v(x_1+1,x_2)$ with probability p, and to state $v(x_1,x_2+1)$ with probability '1-p'. When a demand for class 2 product arrives, it can only be satisfied by class 2 product or this demand may turn into a lost sale. The same conditions are also valid for the arrival of a demand for class 1 product with the addition of the substitution option. When a demand for class 1 product arrives, this demand can be satisfied by a class 2 product, it can be satisfied by a class 1 product or this demand may turn into a lost sale. If a demand is satisfied by a class 1 product, it takes us to state $v(x_1-1,x_2)$, and if a demand is satisfied by a class 2 product, it takes us to state $v(x_1,x_2-1)$. We will examine these situations more detailed in the following parts.

At this point, the second case in which two way substitution is allowed with new revenue and cost values can be introduced. The revenues given in (4.1) change with the addition of new costs and become:

$$R_{i,k}(x_k^n)=\begin{cases} s_1 & i,k=\{1\},x_1^n>0. \\ s_1-(s_2-s_1) & i=\{1\},k=\{2\},(x_1^n=0,x_2^n>T_2) \\ & or(x_1^n>0,x_2^n>g(x_1). \\ -c_1^L & i=\{1\},x_1^n=0,x_2^n<T_2. \\ s_2 & i,k=\{2\},x_2^n>0. \\ s_1-\alpha c_2^L-(s_2-s_1) & i=\{2\},k=\{1\},x_2^n=0,x_1^n>T_1. \\ -c_2^L & i=\{2\},x_2^n=0,x_1^n<T_1. \end{cases} \tag{4.3}$$

The amounts of revenues seen above in (4.3) are lower than the amounts of revenues that exist in (4.1). There are two different costs added in the latter case (in (4.3)). These costs are lost sales costs (c_2^L *and* c_1^L) and substitution costs (($s_2 - s_1$) and ($\alpha c_2^L + (s_2 - s_1)$)).

Lost sales costs are costs faced when a demand can't be satisfied and the opportunity to make a sale is lost. The lost sales cost amounts are assumed to be 1.2 times the sales prices of products. That is;

$$\begin{aligned} c_1^L &= 1.2 s_1 \\ c_2^L &= 1.2 s_2 \end{aligned} \tag{4.4}$$

The number 1.2 is used in this formula because the costs faced by the company in lost sales cases are higher than the price of the product. That is, the company not only loses a sale, but also loses its customers. At least, the opinions of customers about the company are affected negatively. As a result, possible future sales to these customers may also be lost.

The other cost term is the substitution cost. Substitution cost comes out because of the need for an extra advantage presented to the customer and offered together with the substitution offer. Otherwise, the customer will leave the company and prefer a product of another company when he can't find the product he desires available at the market place. The amount of lost opportunity cost is "$s_2 - s_1$". That is, when we satisfy a demand for class 1 product by a class 2 product, we lose the opportunity to earn an amount of s_2 units by selling this product to a customer demanding class 2 product. The amount lost in this case is "$s_2 - s_1$". On the other hand, when we satisfy a demand for class 2 product by a class 1 product, we again lose the opportunity to earn an amount of s_2 units by selling a class 2 product to this customer. The term α is used for determining the extra advantage amount offered to the customer to accept the substitution offer of the company when we want to satisfy a demand for class 2 product by a class 1 product. The amount of this cost term is considered to be a ratio of the lost sales cost amounts of products. The addition of this new cost term is necessary because the main objective in this study is to obtain a practical model that is very useful and applicable under real life conditions. In today's competitive market, companies have to offer extra advantages to customers to keep them and convince them to buy their products. Especially, if the customer is offered a product with lower quality level than the product he desires, this extra advantage amount offered to the customer may become really high. The value of α is determined as 0.2 and this assumption is valid throughout the studies.

Substitution cost and lost sales cost are two cost terms seen in the revenue scheme. Normally, for the substitution decision to be profitable, the condition that the revenue amount obtained by satisfying the demand by substitution must be higher than the lost sales cost should hold. Let us examine these conditions for both upward and downward substitution. The revenues obtained are "$s_1-(s_2-s_1)$" for downward substitution and "$s_1-(s_2-s_1)-\alpha c_2^L$" for upward substitution. These revenues should be compared with lost sales costs to examine the profitability of the substitution decision. For downward substitution, the condition for substitution decision to be profitable is "$s_1-(s_2-s_1)>-c_1^L$". Further evaluating this condition, also by taking the "$c_1^L=1.2s_1$" equation into consideration, we obtain the final condition for downward substitution decision to be profitable as "$3.2s_1>s_2$". On the other hand, for upward substitution, the condition in question becomes "$s_1-(s_2-s_1)-\alpha c_2^L>-c_2^L$". By taking the equations "$c_2^L=1.2s_2,\ \mathit{and}\ \ \alpha=0.2$" into consideration, further calculations are made and the final condition "$50s_1>s_2$" is obtained. Finally, it can be said that, if the condition "$3.2s_1>s_2$" holds, downward substitution decision is profitable, and if the condition "$50s_1>s_2$" holds, upward substitution decision is profitable.

The value function presented in (4.2) changes with the addition of upward substitution option in addition to the downward substitution option in second case. The revenues are seen as the same ones with the revenues in (4.2) but the amounts of these revenue terms are different in two cases. The new value function becomes:

$$
\begin{aligned}
&\frac{g}{\mu+\lambda_1+\lambda_2}+\upsilon(x_1,x_2)= \\
&\frac{1}{\mu+\lambda_1+\lambda_2}\left\{-h(x_1+x_2)+\mu\max\begin{pmatrix} p\upsilon(x_1+1,x_2)+(1-p)\upsilon(x_1,x_2+1) \\ \upsilon(x_1,x_2)\end{pmatrix}\right. \\
&\qquad +\lambda_2\max\begin{pmatrix}\big(\upsilon(x_1,x_2-1)+R^{\Pi}_{2,2}(x_2^n)\big)I_{x2>0}+\big(\upsilon(x_1,0)-c_2^L\big)I_{x2=0} \\ \big((\upsilon(x_1-1,0)+R^{\Pi}_{2,1}(x_1^n)\big)I_{x1>0}+\big(\upsilon(0,0)-c_2^L\big)I_{x1=0}\end{pmatrix} \\
&\qquad \left.+\lambda_1\max\begin{pmatrix}\big(\upsilon(x_1-1,x_2)+R^{\Pi}_{1,1}(x_1^n)\big)I_{x1>0}+\big(\upsilon(0,x_2)-c_1^L\big)I_{x1=0} \\ \big((\upsilon(x_1,x_2-1)+R^{\Pi}_{1,2}(x_2^n)\big)I_{x2>0}+\big(\upsilon(x_1,0)-c_1^L\big)I_{x2=0}\end{pmatrix}\right\}
\end{aligned}
\tag{4.5}
$$

There are three decisions to be made in the more general case – the second case mentioned above:

- When to start and stop production?
- When to satisfy class 2 product demands with class 1 products?
- When to satisfy class 1 product demands with class 2 products?

For the first case with only downward substitution allowed, the second decision is not applicable and is not valid. There are two decisions to be made in this first case.

We transform these decision making processes into a simpler methodology in the following way:

- Whenever the total inventory level of class 1 and class 2 products is below the pre-determined level Q (if $x_1+x_2<Q$), produce; otherwise (if $x_1+x_2\geq Q$) do not produce.

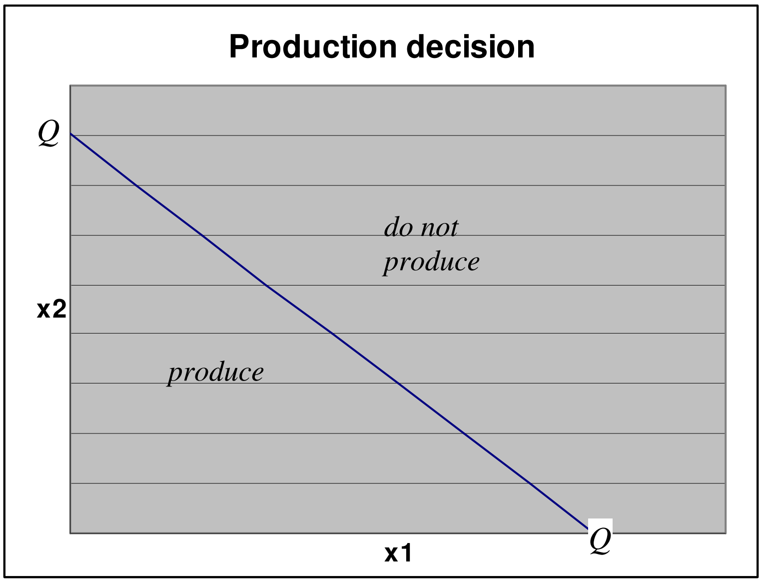

Figure 4.5. Production decision areas at state space

- Whenever the inventory level of class 2 items is zero (at boundary states) and the inventory level of class 1 items is higher than a pre-determined level T_1 (x_2=0, $x_1 \geq T_1$), satisfy class 2 product demands with class 1 products; otherwise do not substitute and do not satisfy class 2 demands (lost sales). (This option is not valid again for the first case with only downward substitution allowed).

- Whenever the inventory level of class 1 items is zero (at boundary states) and the inventory level of class 2 items is higher than a pre-determined level T_2 (x_1=0, $x_2 \geq T_2$), and at non-boundary states whenever the inventory level of class 2 products is higher than a trajectory $g(x_1)$ ($x_1 \neq 0$, $x_2 \geq g(x_1)$), satisfy class 1 product demands with class 2 products; otherwise do not substitute.

Applying this methodology simplifies the decision making process for decision-makers. The effectiveness of the methodology is dependent on the parameter values. In this study, we also look at the impact of these values using the MDP models.

For the decision whether to produce or not, the same MDP models as given in (4.2) and (4.5) are used. Using this MDP models, the Q value (the threshold value) is determined by making necessary calculations and choosing the first value of (x_1+x_2) for which $\upsilon(x_1, x_2) > (p\upsilon(x_1+1, x_2) + (1-p)\upsilon(x_1, x_2+1))$. After determining the Q threshold value, one can easily decide whether to produce or not according to the on hand inventory level at that time. Of course, any change in the parameter values will affect (and probably change) the value of the *Q* term, so we should make the calculations and run the model from the beginning after every parameter change.

For the decision whether to satisfy class 2 product demands by class 1 products or not, a new MDP is constructed. In this MDP only the inventory level of class 1 products is mentioned because this decision comes out only when the on hand inventory level of class 2 products is zero (x_2=0). If the inventory level of class 2 products is higher than zero, no substitution is necessary and demands for class 2 products are satisfied by class 2 products. So, we write our MDP in the following way;

$$
\begin{aligned}
&\frac{g}{p\mu+\lambda_1+\lambda_2}+\upsilon(x_1)= \\
&\frac{1}{p\mu+\lambda_1+\lambda_2}\left\{-hx_1+p\mu\max\begin{pmatrix}\upsilon(x_1+1)\\ \upsilon(x_1)\end{pmatrix}\right. \\
&\qquad +\lambda_1\left((\upsilon(x_1-1)+R^{\Pi}_{1,1}(x_1^n))I_{x1>0}+(\upsilon(0)-c_1^L)I_{x1=0}\right) \\
&\qquad \left.+\lambda_2\max\begin{pmatrix}(\upsilon(x_1-1)+R^{\Pi}_{2,1}(x_1^n))I_{x1>0}+(\upsilon(0)-c_2^L)I_{x1=0}\\ \upsilon(x_1)-c_2^L\end{pmatrix}\right\} \\
&\qquad\qquad x_1=0,1,2,\ldots
\end{aligned}
\tag{4.6}
$$

The total inventory level is equal to the inventory level of class 1 products, so when we produce a product, the total inventory level becomes 'x_1+1'. We decide between producing or not according to the comparison of the value functions $\upsilon(x_1)$ *and* $\upsilon(x_1+1)$. When a demand arrives for class 1 product, it can be satisfied by class 1 product if available at stock or lose the sale. On the other hand, related with the arrival of a demand for class 2 products, there is one more option, we may choose not to satisfy the demand even though there are some class 1 products available at the stock. The revenue obtained from satisfying a demand for class 2 products by a class 1 product is $s_1-\alpha c_2^L-(s_2-s_1)$. This substitution option is valid when we are applying downwards and upwards substitution at the same time. That is, this option is not taken into consideration in the first part of the numerical results.

In the MDP above, by making necessary calculations, the T_1 threshold value can be found by choosing the first value of x_1 that satisfies "$\left[\left(\upsilon(x_1-1)+R_{2,1}^{\Pi}(x_1^n)\right)I_{x1>0}+\left(\upsilon(0)-c_2^L\right)I_{x1=0}\right]>\left(\upsilon(x_1)-c_2^L\right)$". The first value that satisfies this equation determines the value of T_1; and for the values higher than T_1, class 2 product demands are satisfied by class 1 products, while for the values lower than T_1, demands for class 2 products can't be satisfied and these demands are lost sales for the company.

And for the decision whether to satisfy class 1 product demands by class 2 products or not, another MDP is built to determine the T_2 threshold value that helps us in making the substitution decision in boundaries. In non-boundary states, our decisions are made according to a trajectory determined by utilizing a fluid approximation model. Firstly the MDP model that helps us in determining the value of T_2 is presented below, and later we will continue with the fluid approximation model.

$$\begin{aligned}
&\frac{g}{(1-p)\mu+\lambda_1+\lambda_2}+\upsilon(x_2)= \\
&\frac{1}{(1-p)\mu+\lambda_1+\lambda_2}\left\{-hx_2+(1-p)\mu\max\begin{pmatrix}\upsilon(x_2+1)\\ \upsilon(x_2)\end{pmatrix}\right. \\
&\qquad +\lambda_2\left((\upsilon(x_2-1)+R_{2,2}^{\Pi}(x_2^n))I_{x2>0}+(\upsilon(0)-c_2^L)I_{x2=0}\right) \\
&\qquad \left.+\lambda_1\max\begin{pmatrix}\left(\upsilon(x_2-1)+R_{1,2}^{\Pi}(x_2^n)\right)I_{x2>0}+\left(\upsilon(0)-c_1^L\right)I_{x2=0}\\ \upsilon(x_2)-c_1^L\end{pmatrix}\right\}
\end{aligned} \tag{4.7}$$

This MDP is very similar to the previous one. The main difference is the replacement of x_1 parameter with x_2 parameter. For normalization, the equation (of the MDP) is divided by "$(1\text{-}p)\mu+\lambda_1+\lambda_2$". Since this MDP only considers the states when the inventory level of class 1 products is zero ($x_1=0$), the production rate of class 1 products does not exist in this denominator term. In addition, the revenue obtained from satisfying the demands for class 1 products by class 2 products is $R_{1,2}^{\Pi}(x_2^n)$. In the second case with additional cost terms, possible profit amount of "$s_2\text{-}s_1$" that we would gain if we could sell this product (class 2 product) to a customer demanding class 2 product is added to the model as an additional cost for the company. The result of the comparison between the value functions

$\upsilon(x_2)$ *and* $\upsilon(x_2+1)$ helps the decision-maker during the decision process about whether to produce or not. The completion of a production process takes us to state $\upsilon(x_2+1)$ from the state $\upsilon(x_2)$ with a rate of $(1-p)\mu$. Demands for class 2 products may be satisfied by class 2 products or they are lost sales. Similarly, demands for class 1 products may be satisfied by class 2 products or they are lost sales, too. Lost sales cost is present in this MDP, but for the first case with simpler cost terms and revenue amounts, the lost sales cost term does not exist and no cost is faced when a demand can't be satisfied. Besides, the revenues are different in two cases, but the presentation of the value function is not affected from this situation since the differences are present and included under the cost titles $R_{i,k}^{\Pi}(x_k^n)$.

Now, we can look at the condition of satisfying class 1 product demands by class 2 products at non-boundary states. Korugan and Gupta [8] generate an approximation for a downward substitution rule at non-boundary states using a fluid model. Using the same approach, we generate a fluid model for the problem of interest to us as:

$$\min \int_0^T \left[h\left(x(t)\right)x(t) - s_2\lambda_1\left((1-u_2(t))\delta + u_2(t)(2\delta-1)\right) - s_2\lambda_2 \right] dt \tag{4.8}$$

where $u_2(t)$ is the control variable for substitution, and

$$x(t)>0,\ x(0)=x,$$
$$0\leq u(t)\leq 1,$$
$$\delta = \frac{s_1}{s_2}.$$

In equation (4.8), when $u_2(t)$ is 1 (when the substitution control is on and indicating that demands for class 1 products are satisfied by class 2 products) the revenue obtaining rate depending on the demand rates become:

$$\begin{aligned}
s_2\lambda_1\left((1-u_2(t))\delta+u_2(t)(2\delta-1)\right)-s_2\lambda_2 & \\
&= s_2\lambda_1(2\delta-1)-s_2\lambda_2 \\
&= 2\delta s_2\lambda_1-s_2\lambda_1-s_2\lambda_2 \\
&= 2s_1\lambda_1-s_2\lambda_1-s_2\lambda_2 \\
&= \lambda_1(s_1-(s_2-s_1))-s_2\lambda_2
\end{aligned} \tag{4.9}$$

As seen above when the substitution control is on, demands for class 1 products are satisfied by class 2 products if available and a revenue amount of '$s_1-(s_2-s_1)$' is obtained. Demands for class 2 products are satisfied from class 2 products inventory if available and a revenue amount of s_2 is obtained.

On the other hand, when $u_2(t)$ is 0 (when the substitution control is off and indicates that demands for class 1 products are satisfied by class 1 products) the revenue obtaining rate depending on the demand rates become:

$$\begin{aligned}
s_2\lambda_1\left((1-u_2(t))\delta+u_2(t)(2\delta-1)\right)-s_2\lambda_2 & \\
&= s_2\lambda_1\delta-s_2\lambda_2 \\
&= s_1\lambda_1-s_2\lambda_2
\end{aligned} \tag{4.10}$$

In the equation above, no substitution is applied since the substitution control is off. As a result, when we satisfy a demand for class 1 products, we give the customer a class 1 product if available and obtain a revenue of s_1. Besides, we can only satisfy a demand for class 2 product by a class 2 product and obtain a revenue of s_2 from this sale.

After the introduction of the fluid model, we can write the probable changes in inventory levels of product classes in the following way;

$$\dot{x}_1(t)=p\mu-\lambda_1+\lambda_1u_2(t) \tag{4.11}$$

$$\dot{x}_2(t)=(1-p)\mu-\lambda_2-\lambda_1u_2(t) \tag{4.12}$$

$\dot{x}_i(t)$ =change in inventory of product class *i*.

The first terms in the right hand side of these equations are the production rates of products; $p\mu$ for class 1 products and $(1-p)\mu$ for class 2 products. The second terms are the demand rates for products. The last terms are the changes in inventory levels due to the substitution at non-boundary states. We add the number of demands satisfied by substitution to the inventory level of class 1 product and subtract this amount from the inventory level of class 2 products.

The Hamiltonian function for the equations above is given below;

$$\begin{aligned} H = & -(h_1 x_1 + h_2 x_2) + s_2(\lambda_2 + \delta\lambda_1) \\ & + y_1(t)(\mu_1 - \lambda_1) + y_2(t)(\mu_2 - \lambda_2) \\ & + u_2(t) s_2 \lambda_1 (\delta - 1 + y_1(t) - y_2(t)). \end{aligned} \tag{4.13}$$

In this function and in some of the following functions until the end of this part, μ_1 and μ_2 terms will be used for the ease of usage and writing ($\mu_1 = \mu p$, $\mu_2 = \mu(1-p)$). Besides these, h_1 and h_2 terms will be used in these equations but they are assumed to be equal for both types of products ($h_1 = h_2 = h$) in our models and studies. The aim for the usage of these different holding cost terms is to make the model easier to apply for possible future studies and models that will apply different holding costs to different product classes and types.

From this Hamiltonian function, we get the continuity equation as;

$$k_2(t) = s_2 \lambda_1 \left(\delta - 1 + y_1(t) - y_2(t)\right) \tag{4.14}$$

It is assumed that decision changes occur either as interrupts or depletions. Interrupts are decision changes that take place in the interior of the state space while depletions are forced decision changes where the product i is depleted, i.e. when $u_2(t)=1$, all the demand is satisfied by class 2 products until $x_2(t)=0$ where $0 \le t \le T$ and $u_2(t)=0$. The decision change instances are shown as t_i^d for depletions and t_i^r for interrupts, where $0 \le t_j^d$, $t_j^r \le T$.

If at time $t=t_2^r=0$ an interrupt takes place and $u_2(t)=1$, since decision changes are optimal control points for H, the continuity equation $\partial H/\partial u=0$. So, $k_2(t_2^r)=0$ and therefore,

$$y_2(0)=y_1(0)-(1-\delta) \tag{4.15}$$

Similarly, for the depletion of class 2 products;

$$y_2(t_2^d)=y_1(t_2^d)-(1-\delta) \tag{4.16}$$

Besides, since for $t\le t_2^d$, $\dot{y}_2=h_2$, we obtain,

$$\int_0^{t_2^d}\dot{y}_2dt=h_2(t-t_2^d)+c_2 \tag{4.17}$$
$$t\le t_2^d$$

Using these equations, when we subtract equality in (4.15) from the equality in (4.16), we get,

$$\begin{aligned}h_2t_2^d&=y_1(t_2^d)-y_1(0)\\&=\int_0^{t_2^d}\dot{y}_1dt\\&=h_1t_2^d+c_1\end{aligned} \tag{4.18}$$

Again, by evaluating these equations further, we reach;

$$y_1(t)=h_1(t-t_1^d)+(h_2-h_1)t_2^d \tag{4.19}$$

$$y_2(t)=h_2(t-t_2^d)+(h_1-h_2)t_1^d \tag{4.20}$$

When an interrupt occurs, the system starts replenishing both types of demand with product 2 until another decision change occurs. We consider the possibility of the control $u_2(t)$ becomes 0 before the class 2 products are depleted. Let t_1^r ($>t_2^r$) be the time when an interrupt occurs and the system changes to the no substitution decision. As a result, the depletion times of both products are:

$$x_1(t_1^d) = x_1(0) + \mu_1 t_1^i + (\mu_1 - \lambda_1)(t_1^d - t_1^i) \tag{4.21}$$

$$x_2(t_2^d) = x_2(0) + (\mu_2 - \lambda_2 - \lambda_1)t_1^i + (\mu_2 - \lambda_2)(t_2^d - t_1^i) \tag{4.22}$$

Let us give some explanations about these equations. In the first of these two equations above, '$x_1(0)$' term is the starting inventory level for class 1 product. '$\mu_1 t_1^i$' term shows that until the first decision change, production of class 1 products continues while the demands for class 1 products are satisfied by class 2 products. So, until the time of first decision change, only the production of class 1 products is observed. '$(\mu_1 - \lambda_1)(t_1^d - t_1^i)$' term tells us that after the first decision change, the demands for class 1 products are satisfied by class 1 products. So, the change in inventory level of class 1 products materialize with a rate of production rate-demand rate. For the second equation, similar explanations are valid. '$x_2(0)$' term is the starting inventory level for class 2 products. '$(\mu_2 - \lambda_2 - \lambda_1)t_1^i$' term tells us that until the first decision change demands for bothtypes of products are met by class 2 products. '$(\mu_2 - \lambda_2)(t_2^d - t_1^i)$' term shows that after this decision change, class 2 products in inventory are only used to satisfy demands for class 2 products.

After building the model and necessary equations as mentioned above, our studies continue by equalizing the last two equations to zero, because they are the inventory levels of the product classes at depletion times. When they are equalized to zero and necessary calculations are made for further inferences, we get;

$$t_1^d = -\frac{x_1(0) + \lambda_1 t_1^i}{(\mu_1 - \lambda_1)} \tag{4.23}$$

and

$$t_2^d = -\frac{x_2(0) - \lambda_1 t_1^i}{(\mu_2 - \lambda_2)} \tag{4.24}$$

The detailed steps in calculations of these equations are present in the Appendix (Appendix A and B) section at the end of this research.

Since at time t_1^i an interrupt occurs, we obtain;

$$y_2(t_1^i) = y_1(t_1^i) - (1-\delta) \tag{4.25}$$

By evaluating this equation further, we get;

$$t_1^i = \frac{b_1 x_1(0) - \beta x_2(0) - (1-\delta)}{\theta} \tag{4.26}$$

where,

$$b_1 = \frac{2h_1 - h_2}{\mu_1 - \lambda_1}, \tag{4.27}$$

$$\beta = \frac{2h_2 - h_1}{\mu_2 - \lambda_2}, \tag{4.28}$$

and

$$\theta = \left[\frac{\lambda_1(-2h_2 + h_1)}{(\mu_2 - \lambda_2)} + \frac{\lambda_1(-2h_1 + h_2)}{(\mu_1 - \lambda_1)} \right] \tag{4.29}$$

The detailed steps in calculations of these equations are present in the Appendix (Appendix C) section at the end of this report.

Finally, when we calculate the continuity equation for the initial interrupt at $t_2^i = 0$, we obtain the equation

$$x_2(0) = \frac{b_1}{\beta} x_1(0) - \frac{(1-\delta)}{\beta} \tag{4.30}$$

Since the substitution of demands for class 1 products and satisfaction of these demands by class 2 products starts at the point T_2 when the inventory level of class 1 products is zero, we build the line that determines the substitution decision at non-boundary states as follows:

$$x_2(0) = T_2 + \left|\frac{b_1}{\beta}\right| x_1(0) - \frac{(1-\delta)}{\beta} \tag{4.31}$$

where all states lying above this function are instructed to substitute the class 1 product demand with class 2 products while the states below it supply this demand with class 1 products. The absolute value of the term $\frac{b_1}{\beta}$ is used since the slope of the line that will be used for substitution decision should be positive in a way that reflects the decrease in the need for substitution as the inventory level of class 1 products increases.

5. NUMERICAL RESULTS

5.1. Downward Substitution at Boundary and Non-boundary States

In this part, three different application cases of downward substitution are examined. Different substitution policies and rules are applied in these different cases. In the first case, the same assumptions as in Duenyas and Tsai [3] are valid. Under these assumptions, substitution is possible only at boundary states and only downward. In the second case, in addition to the first case, substitution is possible also at non-boundary states. Only downward substitution is allowed again. The substitution decisions at non-boundary states are made according to the iterations and calculations using value functions and MDP models given before. For each state in state space, these calculations and comparison between the cases in which substitution is applied or not applied are made. In the third case, downward substitution is possible at boundary and non-boundary states. At boundary states, decisions are made using the same heuristic with the first two cases. But at non-boundary states, the line equation that is obtained using the fluid approximation model is used to make decisions related to the substitution.

The revenue function given in (4.1) is used throughout the analysis in this part. In this function, lost sales costs and substitution costs are not valid. Firstly, the case in which the substitution is only possible at boundary states (when x_1=0) will be examined. The parameters combination of the example set used for the analysis in this part are as follows:

Table 5.1. Parameters for the example set used for numerical calculations

Example number	**h**	**μ**	**p**	**λ_1**	**λ_2**	**s_1**	**s_2**
1	5	0,5	0,4	0,2	0,2	500	1000
2	1	0,5	0,4	0,2	0,2	500	1000
3	10	0,5	0,4	0,2	0,2	500	1000
4	5	0,3	0,4	0,2	0,2	500	1000
5	5	0,7	0,4	0,2	0,2	500	1000
6	5	0,5	0,4	0,1	0,35	500	1000
7	5	0,5	0,4	0,3	0,2	500	1000
8	5	0,5	0,4	0,2	0,1	500	1000
9	5	0,5	0,4	0,2	0,3	500	1000
10	5	0,5	0,4	0,2	0,2	200	1000
11	5	0,5	0,4	0,2	0,2	800	1000
12	5	0,5	0,4	0,2	0,2	500	500
13	5	0,5	0,4	0,2	0,2	500	1500
14	5	0,5	0,2	0,2	0,2	500	1000
15	5	0,5	0,6	0,2	0,2	500	1000
16	5	0,5	0	0,2	0,2	500	1000
17	5	0,5	0,1	0,2	0,2	500	1000
18	5	0,5	0,9	0,2	0,2	500	1000
19	5	0,5	1	0,2	0,2	500	1000
20	5	0,5	0,4	1	0,1	500	1000
21	5	0,5	0,4	0,1	1	500	1000
22	5	0,5	0,4	0,2	0,2	500	5000

In this table;

h: Holding cost.
μ: production rate.
p: probability that production results with class 1 product.
λ_1: Demand rate for class 1 products.
λ_2: Demand rate for class 2 products.
s_1: Price for class 1 products.
s_2: Price for class 2 products.

The table above shows the parameter values used in the 22 examples which are used to test our heuristic and to perceive the improvements achieved by the application of the heuristic. Example one is used as the basic case. From example two to 15, mostly one (or sometimes two) parameter is either increased or decreased in each example. Examples 16 to 22 cover highly unbalanced cases where the problem parameters for one class are significantly different than problem parameters for the other class. In Examples 16 to 19, *p* is set to either zero or one so that producing only one of the classes is possible. When *p*=0, the production system only produces class 2 items. However, in the heuristics in question the system still has the option of selling class 2 products to class 1 customers at the lower price, s_1. When *p*=1, the production system only produces class 1 items.

In the first case to be examined, substitution is only made available at boundary states. Besides, only downwards substitution is allowed so that substitution is only possible at one side of the boundary. Class 2 products may substitute demands for class 1 products when the inventory level of class 1 products drops to zero. The substitution decision is made according to the *S* threshold value, if the inventory level of class 2 products is below the value *S*, no substitution is made and incoming demands for class 1 products are lost. Otherwise, if the inventory level of class 2 products is equal to or higher than the threshold value *S*, substitution is made and the demands for class 1 products are satisfied by class 2 products.

The results of the numerical calculations obtained using the rules and assumptions mentioned above are listed in the table below. These values are the profit values of the model of the first case under the revenue amounts mentioned in (4.1).

Table 5.2. Results of numerical calculations for the first case

Example number	**Q**	**S**	**Heuristic**	**Optimal**	**Percentage difference (%)**
1	7	3	275,63	280,25	1,65
2	13	6	314,25	316,63	0,75
3	5	2	244,00	250,06	2,42
4	16	6	297,50	297,63	0,04
5	5	1	228,88	237,25	3,53
6	7	8	176,25	181,75	3,03
7	11	4	289,00	290,50	0,52
8	5	1	213,75	214,63	0,41
9	12	7	306,63	307,75	0,37
10	7	4	214,75	217,38	1,21
11	8	2	342,25	345,00	0,80
12	6	1	176,63	178,25	0,91
13	8	4	380,75	386,13	1,39
14	7	1	288,63	289,50	0,30
15	7	6	213,50	215,56	0,96
16	7	1	291,00	291,25	0,09
17	7	1	290,13	290,63	0,17
18	4	16	113,94	115,13	1,03
19	3	21	93,19	93,19	0,00
20	72	3	178,13	178,13	0,00
21	7	71	105,56	110,44	4,41
22	11	8	1138,50	1148,50	0,87
Average			280,58		

In this table, under the ‘Heuristic’ column, there are profit values calculated using the heuristic that is mentioned above. This heuristic works by the calculation of Q and S values. Q value is the maximum total inventory level that can be hold at one time. On the other hand, S value is the threshold value that determines whether the demand should be satisfied by substitution or not. The column with the title ‘optimal’ includes the optimal profit values that can be achieved by using the same assumptions. In the optimal case, every state is considered separately and the optimal decision is made for that state. But substitution is allowed in the same region, which is only in one side of the boundaries in this case. The average of the profit values for the 22 examples in the example set is 280.58. By applying the heuristic developed for the substitution decision at non-boundary states, we aim to obtain an increase in these profit values in the following two cases.

After the numerical results for the first case, the next case to be examined is the second case in which substitution is allowed also at non-boundary states. Only downward substitution is allowed again. Substitution decisions at non-boundary states are made through the calculations and comparisons using the MDP and value functions. At boundary states, the same heuristic that is used in the first case is used, but for the non-boundary states, there are no certain rules and threshold values. For every state, the situations in which substitution is made and not made are compared using the value functions and the optimal decision is made regardless of the decisions at other states. The revenues are accepted as the same ones assumed in the first case (that are shown in (4.1)).

The numerical results obtained using the rules and assumptions mentioned above for the second case are as follows:

Table 5.3. Results of numerical calculations for the second case

Example number	**Q**	**S**	**Our model (Downward subs. Only, MDP)**
1	8	3	278,13
2	15	6	315,50
3	6	2	248,00
4	18	6	297,50
5	6	1	233,25
6	4	8	178,06
7	12	4	289,13
8	5	1	213,75
9	13	7	307,25
10	7	4	214,75
11	9	2	343,50
12	7	1	177,75
13	9	4	383,38
14	8	1	288,75
15	6	6	213,50
16	7	1	291,00
17	7	1	290,13
18	3	16	114,44
19	3	21	93,19
20	63	3	178,13
21	4	71	110,13
22	12	8	1142,00
Average			281,87

The values of the variables Q and S and the profit values that are calculated for the second case are listed in table above. The values for the S variable in the first two cases are the same because the same heuristic is used in both of the cases to determine whether to substitute or not at boundary states. Q values are slightly different in two cases due to the addition of the substitution option at non-boundary states in second case. The profit values obtained in this case are equal to or higher than the profit values calculated in the first case. The table below shows the profit values found in the first case and the second case together and shows the percentage differences in profits of the two cases.

Table 5.4. Comparison of results for first and second cases

Example number	**Second Case (Downward subs. Only, MDP)**	**First Case**	**Percentage difference (%)**
1	278,13	275,63	0,91
2	315,50	314,25	0,40
3	248,00	244,00	1,64
4	297,50	297,50	0,00
5	233,25	228,88	1,91
6	178,06	176,25	1,03
7	289,13	289,00	0,04
8	213,75	213,75	0,00
9	307,25	306,63	0,20
10	214,75	214,75	0,00
11	343,50	342,25	0,37
12	177,75	176,63	0,64
13	383,38	380,75	0,69
14	288,75	288,63	0,04
15	213,50	213,50	0,00
16	291,00	291,00	0,00
17	290,13	290,13	0,00
18	114,44	113,94	0,44
19	93,19	93,19	0,00
20	178,13	178,13	0,00
21	110,13	105,56	4,32
22	1142,00	1138,50	0,31
Average	281,87	280,58	0,46

Profit values of the second case in these 22 examples have increased approximately 0,46 per cent when compared with the profit values of the first case in the 22 examples.

The average profit value for these 22 examples is 281.87 for the second case while it is 280.58 for the first case. This improvement in profit values is achieved by the availability of the substitution option at non-boundary states. The improvement may seem so small but higher improvements will be achieved in the second part of the numerical results section in the following pages by the addition of lost sales cost, substitution cost and two way substitution option.

The results for the first two cases have been examined. In the third case, similar assumptions with the assumptions of the second case are made. Downward substitution is allowed at boundary and non-boundary states both. The same revenues with the ones used in the first two cases are applied. The difference comes out with the application method of the substitution policy at non-boundary states. In the second case, as mentioned above, there was not any strict rule or threshold value to be used in the decision process related to the substitution at non-boundary states. This situation makes it difficult to make good decisions and judgments related to substitution. Decision making process becomes a very time-consuming and confusing process. In this case, we are applying an easy to implement rule to overcome this problem. We will use the line equation obtained in the previous mathematical model section by using the fluid approximation model. At non-boundary states, we will use this line equation and decide whether to substitute or not according to the position of the current state. If the current state is above this line, substitution is made and the demand for class 1 product is satisfied by a class 2 product. Otherwise, substitution is not made. The equation of the line that is used in this case is slightly different than the equation in (4.31) due to the differences in revenues. In Appendix, calculation steps for this new equation are present (Appendix D). In this case, there are no costs related to lost sales case or substitution. The equation that will be used at this point becomes:

$$x_2(0) = T2 + \left|\frac{b_1}{\beta}\right| x_1(0) \tag{5.1}$$

After the explanations of the general assumptions and rules that are valid in this case, the numerical results for this case can be examined. The profit values obtained by applying the rules for this case mentioned above are as follows:

Table 5.5. Results of numerical calculations for the third case

Example number	**Q**	**S**	**Third Case (Downward subs. Only, fluid)**
1	8	3	278,13
2	15	6	315,50
3	6	2	248,00
4	15	6	296,85
5	6	1	230,25
6	4	8	178,06
7	11	4	289,00
8	5	1	213,75
9	12	7	306,75
10	7	4	214,75
11	9	2	343,50
12	7	1	177,69
13	9	4	383,38
14	7	1	288,25
15	6	6	213,13
16	7	1	291,00
17	7	1	289,73
18	3	16	114,44
19	3	21	93,19
20	62	3	178,13
21	4	71	110,13
22	12	8	1142,00
Average			281,62

In the table above, threshold values together with the profit values are shown. *S* values do not differ in three cases since the same heuristic is followed at boundary states. *Q* values may show some differences between three cases due to the different rules applied at non-boundary states. The table below shows the profit values for the first and third cases in one table and list the percentage differences between the profit values of these two cases.

Table 5.6. Comparison of results for first and third cases

Example number	**Third Case (Downward subs. Only, fluid)**	**First Case**	**Percentage difference (%)**
1	278,13	275,63	0,91
2	315,50	314,25	0,40
3	248,00	244,00	1,64
4	296,85	297,50	-0,22
5	230,25	228,88	0,60
6	178,06	176,25	1,03
7	289,00	289,00	0,00
8	213,75	213,75	0,00
9	306,75	306,63	0,04
10	214,75	214,75	0,00
11	343,50	342,25	0,37
12	177,69	176,63	0,60
13	383,38	380,75	0,69
14	288,25	288,63	-0,13
15	213,13	213,50	-0,18
16	291,00	291,00	0,00
17	289,73	290,13	-0,14
18	114,44	113,94	0,44
19	93,19	93,19	0,00
20	178,13	178,13	0,00
21	110,13	105,56	4,32
22	1142,00	1138,50	0,31
Average	281,62	280,58	0,37

The table above shows the percentage differences of the profit values for first and third cases in 22 examples applied. The usage of our substitution rule at non-boundary states enabled us to obtain 0.37 per cent improvement in average when compared with the first case. The profit values of second case were higher than the profit values of the third case. This is an expected result because in second case optimal decision is searched at each of the non-boundary states. Instead in the third case, we applied an easier to implement substitution rule that took us to better profit values than the ones in the first case. While obtaining such an improvement in profit values, no difficulty or confusion is faced. The substitution decision is made very easily after determining the line equation according to the parameters of the current state.

The next part of the numerical results section includes the addition of the lost sales cost, substitution cost and two-way substitution availability at boundary states. These new revenue-cost combinations and substitution policies change the profit values in large amounts. The detailed expressions and explanations are made in the next part of the numerical results section in the following pages.

5.2. Addition of Upward Substitution Option and New Costs

Only downward substitution was available in the previous part. But upward substitution may be useful especially when there are costs related to the lost sales cases. In this second part of the numerical results section, upward substitution is made available in addition to the downward substitution. Besides, new cost terms are added to the model to make the assumptions and the model itself nearer to real life's conditions. These cost terms are lost sales costs and substitution costs. After the addition of these cost terms to the model, the revenue scheme takes the form shown in (4.3). Substitution becomes more important especially after the addition of lost sales costs to the model. A large amount of cost is faced when a demand can't be satisfied and turned back. So, the availability of upward substitution option may affect the profit of the company positively and considerably when the higher monetary value of the class 2 products is taken into consideration.

In the following phases of the analysis, the case in which upward and downward substitution is available at boundary states and only downward substitution is available at non-boundary states will be compared with the case in which only downwards substitution is available only at boundary states. In the first case, substitution decisions at boundary states are made using the same heuristic that is used in the second case (and also used in three cases in the first part of the numerical analysis section). Downward substitution decisions at non-boundary states are made according to the line equation that is formed using the fluid approximation model (explained in mathematical model part before) shown in (4.31). The implementation of this policy is simple; if the current state is above the line, substitution is made, otherwise substitution is not made. This simple method is very easy to implement and at the same time increases the profit values and works effectively.

The same example set (containing 22 examples) that was also used in previous part of the numerical results section will be used in this analysis, too. The tables below show the results of the numerical calculations for the first case. Firstly, the values of the decision variables for the expanded case in which upwards and downwards substitution is available will be given. And in the next table, profit values calculated using the substitution rules and assumptions mentioned above are listed. Besides, in the same table, optimal values for these example set under same substitution policies are given.

Table 5.7. Parameters calculated for the model developed

Example number	T_1	T_2	**Q**	**Slope of $g(x_1)$**
1	1	5	11	0.1/0=∞
2	1	9	19	0.1/0=∞
3	1	4	8	0.1/0=∞
4	1	13	28	0.25
5	1	3	7	2.75
6	1	18	13	0.5
7	1	6	16	1
8	1	2	6	0.2/0=∞
9	1	12	19	0/0
10	1	7	6	0.1/0=∞
11	1	4	12	0.1/0=∞
12	1	1	8	0.1/0=∞
13	1	7	11	0.1/0=∞
14	1	3	9	2
15	1	10	10	0
16	1	2	8	1.5
17	1	2	9	1.67
18	1	49	9	0.6
19	1	65	8	0.67
20	1	4	99	0.25
21	1	n/a	n/a	7
22	1	10	9	0.1/0=∞

Using these parameter values for T_1, T_2 and Q terms, profit values are calculated. T_1 value mentioned above is the threshold value for substituting a class 2 demand by class 1 products. T_2 value is the threshold value for satisfying a class 1 demand by a class 2 product. Above these threshold values, substitution is made; otherwise, no substitution is made. Q value is the maximum total inventory level. In addition to these threshold levels, slope of $g(x_1)$ function is also shown in Table 5.7. Slope of $g(x_1)$ function have positive values in some experiments while in some others it has values 0 or ∞. When the slope of this function is 0, $g(x_1)$ function behaves like a horizontal line passing through the point $(0,T_2)$. On the other hand, when the slope is equal to ∞, there is no substitution opportunity at non-boundary states because of the sudden and huge increase in the slope of the function.

The profit values calculated for the first case of this part using the T_1, T_2 and Q values are shown in the table below:

Table 5.8. Profit values calculated for the model developed

Example number	**Profit Values for Our Heuristic**	**Optimal Profit Values**	**Percentage difference (%)**
1	245,88	248,00	0,86
2	289,13	291,75	0,90
3	208,38	210,75	1,13
4	182,81	183,25	0,24
5	212,50	215,88	1,56
6	263,75	263,75	0,00
7	217,88	221,25	1,53
8	157,75	158,13	0,24
9	285,38	286,75	0,48
10	172,44	182,50	5,51
11	323,13	327,50	1,34
12	173,31	173,38	0,04
13	329,13	332,50	1,02
14	212,75	213,25	0,23
15	223,88	226,00	0,94
16	172,81	172,88	0,04
17	192,75	193,13	0,19
18	104,06	104,06	0,00
19	61,34	61,34	0,00
20	-112,38	-112,38	0,00
21	n/a	n/a	n/a
22	1038,00	1068,00	2,81
Average	235,94		

The profit values above gives an average value of 235.94 for the example set consisting of 22 examples mentioned before. In the 21st example, the program software used for calculations couldn't give the results due to the incapacity of the program limits. The percentage differences between the profit values of our heuristic and optimal profit values have taken the highest value in tenth example with the value of 5.51. And in some of the examples, the heuristic developed behaves as good as the optimal one. This values show that the heuristic works effectively while it simplifies the decision processes related to the production and substitution.

Let us shortly look at the changes in the $g(x_1)$ function and produce – do not produce regions in state space that are observed in some of these experiments. Experiments four, five, 16 and 19 are going to be examined. In experiments four and five, production rates of products are different, that is 0.3 in experiment four and 0.7 in experiment five. On the other hand, in experiments 16 and 19, probabilities that the production process results with a class 1 product are different, that is 0 in experiment 16 and 1 in experiment 19.

For experiment four, the $g(x_1)$ function and current produce – do not produce regions can be given as follows:

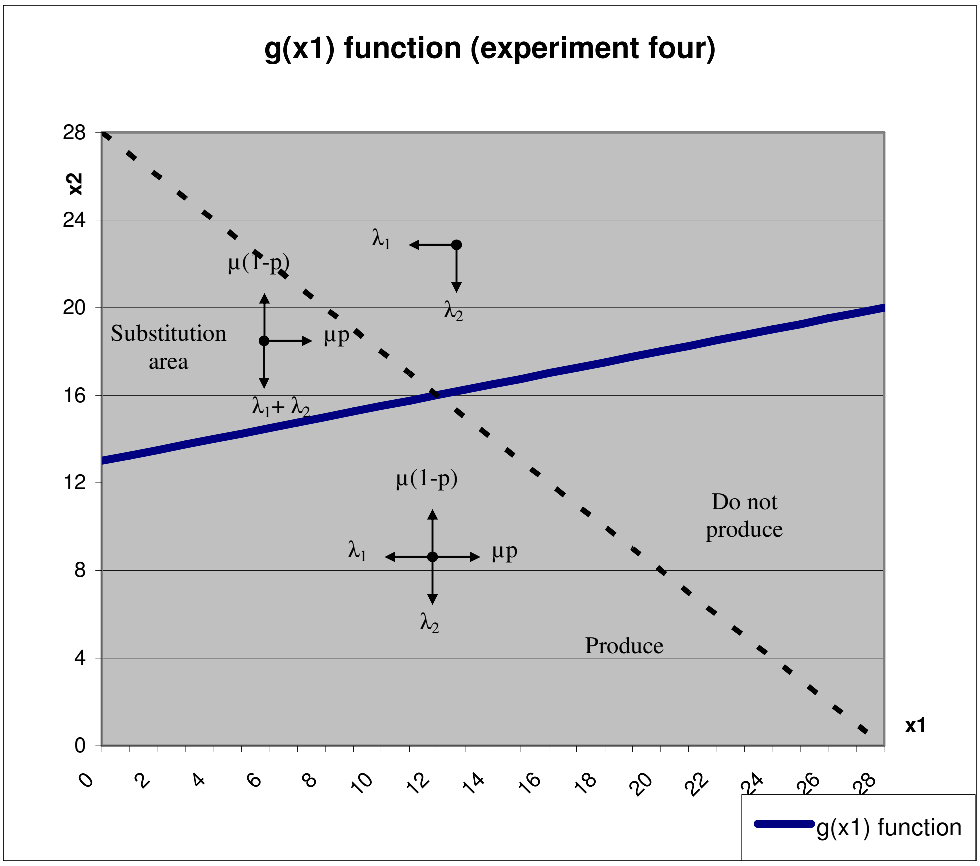

Figure 5.1. The $g(x_1)$ function and produce – do not produce regions in experiment four

In figure 5.1, the line starts from T_2 threshold value (13) and has a slope of 0.25. The dots show the produce - do not produce regions for the experiment four, that is Q value is equal to 28 in this experiment. Downward substitution area at non-boundary states is bounded by the triangle in the up-left side of the state space formed by the $g(x_1)$ function line, x_2 line and the dots come out due to the Q threshold value. The possible changes in inventory levels in different regions of the space are shown in the figure by arrows and inventory changing rates next to these arrows. The profit value obtained in experiment four is 182.81.

For experiment five, the $g(x_1)$ function and current produce – do not produce regions can be given as follows:

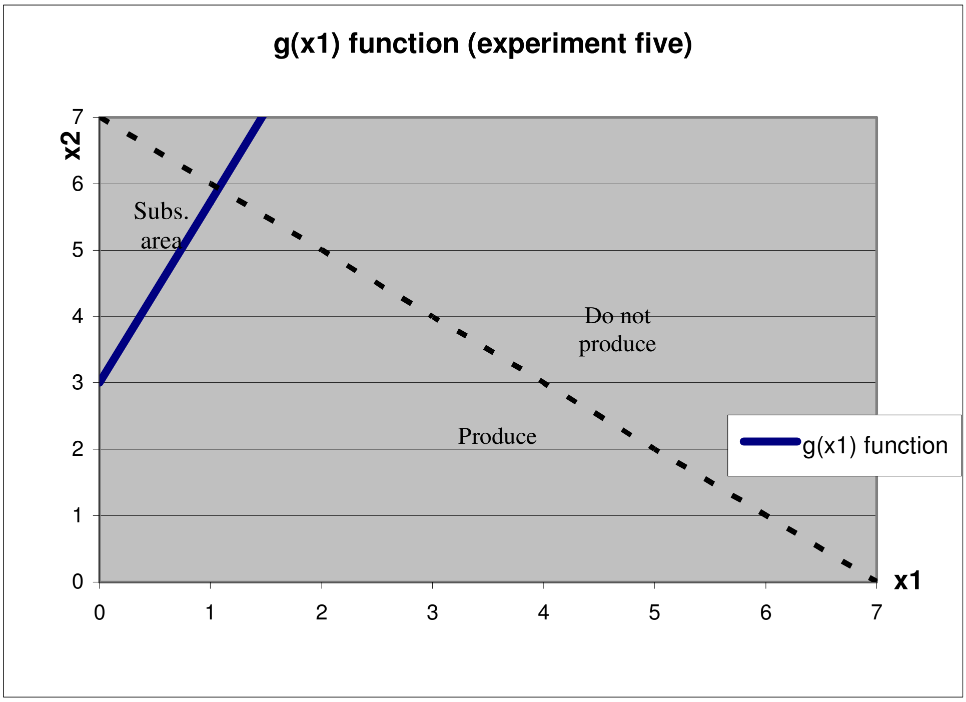

Figure 5.2. The $g(x_1)$ function and produce – do not produce regions in experiment five

In figure 5.2, the line starts from T_2 threshold value (3) and has a slope of 2.75. The dots show the produce - do not produce regions for the experiment five, that is Q value is equal to 7 in this example. Downward substitution area at non-boundary states is bounded by the triangle in the up-left side of the state space formed by the $g(x_1)$ function line, x_2 line and the dots come out due to the Q threshold value. Lower Q threshold value makes the state space in use narrower, and also substitution area at non-boundary states is narrower. This is the expected result of higher production rate. An increase in the production rate decreases the necessary total inventory level (Q). This level is 7 in experiment five while it is 28 in experiment four. Downward substitution area also gets narrower as the state space for the experiment gets narrower. Higher production rate decreases the need for substitution at non-boundary states because the probability of stock out cases decrease and the system tends to make less substitution at non-boundary states to face lower substitution costs. These changes in threshold values and substitution area increases the profit of the system and the profit value of 212.50 is obtained in experiment five while the profit value for experiment four is 182.81.

For experiment 16, the $g(x_1)$ function and current produce – do not produce regions can be given as follows:

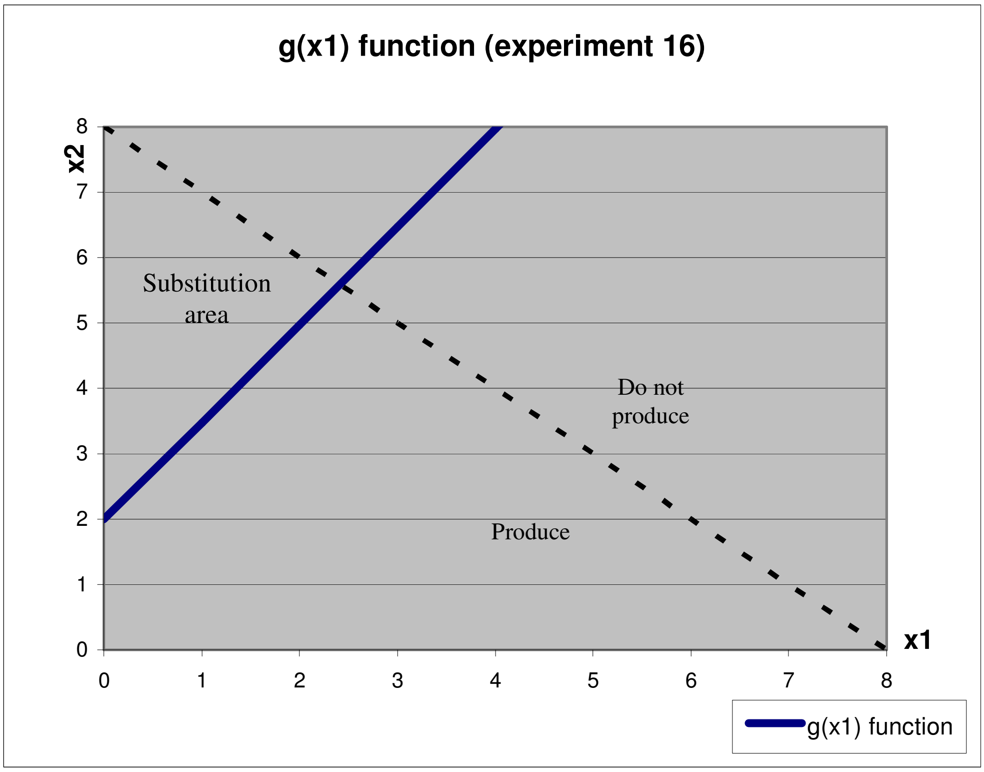

Figure 5.3. The $g(x_1)$ function and produce – do not produce regions in experiment 16

In figure 5.3, the line starts from T_2 threshold value (2) and has a slope of 1.5. The dots show the produce - do not produce regions for the experiment 16, that is Q value is equal to 8 in this example. Downward substitution area at non-boundary states is bounded by the triangle in the up-left side of the state space formed by the $g(x_1)$ function line, x_2 line and the dots come out due to the Q threshold value. Profit values obtain in experiment 16 is 172.81.

For experiment 19, the $g(x_1)$ function and current produce – do not produce regions can be given as follows:

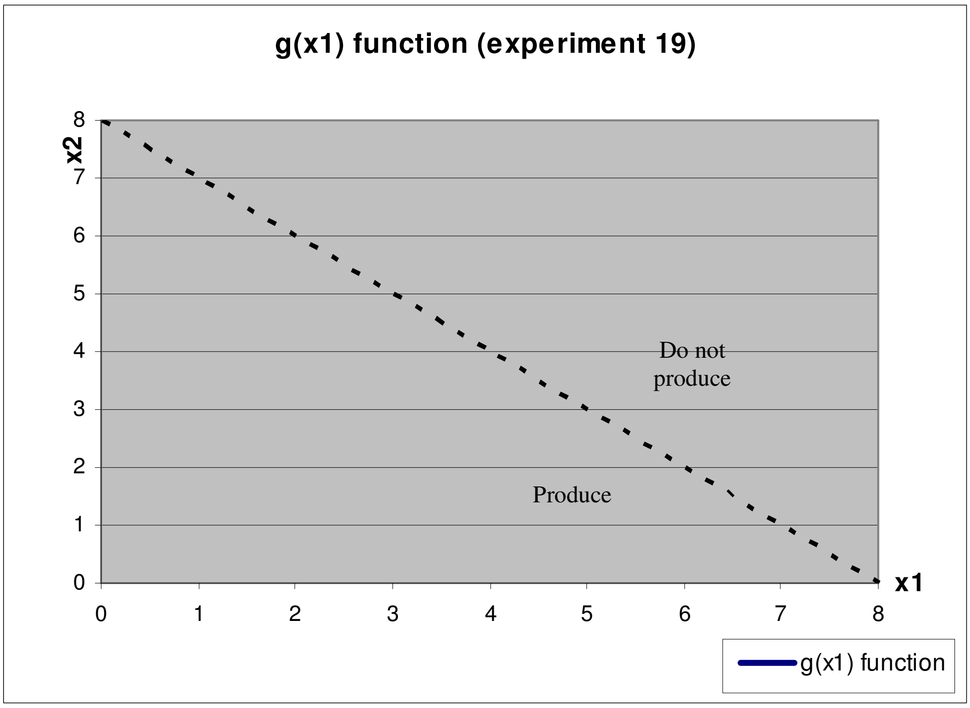

Figure 5.4. The $g(x_1)$ function and produce – do not produce regions in experiment 19

In figure 5.4, the line starts from T_2 threshold value (65) and has a slope of 0.66. The dots show the produce - do not produce regions for the experiment 19, that is Q value is equal to 8 in this example. There is no downward substitution area at non-boundary states since the T_2 threshold value is greater than Q, and this implies that no downward substitution is going to be made at non-boundary states. This is the result of $p = 1$ equality that means no class 2 products can be produced in the production line. All products produced in this line are class 1 products so there is no need to substitute class 1 demands. Demands for class 2 products can only be satisfied by upward substitution since there is no production of class 2 products. This situation increases the substitution costs faced. Only class 1 products are produced so that the revenues obtained from sales are "s_1" if a class 1 product demand is satisfied and "$s_1 - (s_2 - s_1) - \alpha c_2^L$" if a class 2 product demand is satisfied by substitution. These revenues are lower than the revenues that would be valid if these demands were satisfied by class 2 products. As a result of these, a profit value of 61.34 is obtained in experiment 19 while the profit value for experiment 16 is 172.81. This result shows that our model is robust under extreme p values.

In the second case of this analysis, the same assumptions that were made in the first case of the first part of the numerical results section are made. Substitution is possible only downward and only at boundaries. The only difference comes out with the revenues and costs used in this latter case. In the first case of the first part, the revenue scheme shown in (4.1) was used, while in this case, the revenue scheme shown in (4.3) is used. In other words, lost sales costs and substitution costs are added to the model. The results of the numerical calculations for this second case are as follows:

Table 5.9. Parameters calculated for the basic case with additional costs

Example number	**Q**	**S**	**Profit Values**
1	11	5	240,75
2	19	9	288,75
3	9	4	198,63
4	30	13	179,88
5	8	2	205,88
6	6	20	-78,69
7	16	6	217,50
8	6	2	155,88
9	19	13	255,88
10	7	6	164,00
11	12	4	317,00
12	9	1	163,13
13	12	7	325,75
14	9	2	212,50
15	8	10	109,81
16	8	2	172,75
17	9	2	193,00
18	4	49	-127,44
19	4	65	-178,25
20	99	4	-112,38
21	5	225	-530,50
22	9	10	1019,00
average			154,22

As seen in the table above, the average profit values of 22 examples for the second case is 154.22. This value is quite small when compared to the profit values obtained in the first case. The profit value of the 21st example couldn't have been calculated in the first case so it may be better to compare these values after taking out the profit value of the same example for the second case. Under these circumstances, the average profit value obtained from the 21 examples of our example set becomes 186.82 which is higher than the value 154.22 but is still quite lower than the average profit value calculated in the first case, 235.94.

It may be appropriate to see the profit values obtained in two different cases together. The table below shows these values together to make the comparisons between two cases easier.

Table 5.10. Comparison of profit values of two cases

Example number	**Profit Values for the Second Case**	**Profit Values for Our Heuristic (First Case)**
1	240,75	245,88
2	288,75	289,13
3	198,63	208,38
4	179,88	182,81
5	205,88	212,50
6	-78,69	263,75
7	217,50	217,88
8	155,88	157,75
9	255,88	285,38
10	164,00	172,44
11	317,00	323,13
12	163,13	173,31
13	325,75	329,13
14	212,50	212,75
15	109,81	223,88
16	172,75	172,81
17	193,00	192,75
18	-127,44	104,06
19	-178,25	61,34
20	-112,38	-112,38
21	-530,50	???
22	1019,00	1038,00
average	154,22	235,94

The values calculated and mentioned up to now show that the application of downward and upward substitution at boundary states and downward substitution at non-boundary states using the rules and heuristics explained in previous parts enables the companies and decision-makers to obtain higher profit values than the profit values that are obtained when only downward substitution is allowed only at boundary states. The improvements achieved by the usage of these heuristics increase with the addition of lost sales costs and substitution costs. The results of some of the examples indicate losses instead of profits especially in the second case. These are the normal results of the addition of lost sales costs, and the application of the model mentioned in the first case of this part enables us to increase our profits or at least decrease the losses faced considerably.

6. SENSITIVITY ANALYSIS

In this section, we examine the effects of different parameters on the profitability and the efficiency of the system. This analysis will help us to understand the system better. The cases in which higher improvement rates can be obtained by the application of substitution rules developed so far are determined. These results will show us when the application of the heuristics developed is more crucial and when it is less effective. Also, some strategies can be developed that can be applied when some changes in parameter values occur that have negative effects on profitability.

The sensitivity analysis studies start with the analysis of the effects of changes in the ratio between sales prices of two product classes (the ratio s_1 / s_2). After that, the ratio between the holding cost of products and the sales prices of products ($h / s_1 \ \ or \ \ h / s_2$) is taken into consideration. The effects of changes in this ratio on the effectiveness of the model developed are tried to be found out in this part. After these, it is investigated that how the changes in the ratio between the total demand rates of two product classes and the total supply (production) rate ($(\lambda_1 + \lambda_2) / \mu$) affect the effectiveness of the model developed. The coefficient used in calculating the lost sales cost amounts for product classes, and changes in p values are the last two factors analyzed in this section.

6.1. Analysis of the Effects of Changes in the Ratio s_1/s_2

Sales prices of products with different specifications and different quality levels may show great differences in some cases. Usually, these sales prices are determined in market conditions without the control of the supplier. As the difference between the sales prices of product classes increases, some of the product classes start to possess higher importance and priority level compared to the others.

In our studies, we have two product classes with sales prices s_1 and s_2. The ratio between these two sales prices may have strong effects on the effectiveness of the model developed. Especially, with the addition of lost sales costs to the model, the importance of substitution availability at non-boundary states and upward substitution option will probably increase considerably.

During the analysis of the effects of changes in ratio s_1 / s_2, the following experiment set is used:

Table 6.1. Parameter values of the experiments used for the analysis of the effects of changes in the ratio s_1 / s_2

Experiment number	h	μ	p	λ_1	λ_2	s_1	s_2
1	5	0,5	0,4	0,2	0,2	240	1200
2	5	0,5	0,4	0,2	0,2	480	2400
3	5	0,5	0,4	0,2	0,2	960	4800
4	5	0,5	0,4	0,2	0,2	240	600
5	5	0,5	0,4	0,2	0,2	480	1200
6	5	0,5	0,4	0,2	0,2	960	2400
7	5	0,5	0,4	0,2	0,2	240	400
8	5	0,5	0,4	0,2	0,2	480	800
9	5	0,5	0,4	0,2	0,2	960	1600
10	5	0,5	0,4	0,2	0,2	240	300
11	5	0,5	0,4	0,2	0,2	480	600
12	5	0,5	0,4	0,2	0,2	960	1200
13	5	0,5	0,4	0,2	0,2	240	240
14	5	0,5	0,4	0,2	0,2	480	480
15	5	0,5	0,4	0,2	0,2	960	960

In the 15 experiments shown in Table 6.1, the ratio between the sales prices of class 1 products and class 2 products take 5 different values between 0.2 and 1. The values of this ratio increases in steps of 0.2 units and for every value of the ratio, 3 different experiments are present.

In Table 6.2, the values of the ratio s_1 / s_2 are shown besides the sales prices of the products.

Table 6.2. Values of the ratio s_1 / s_2 in 15 experiments

Experiment number	**s_1**	**s_2**	**s_1/s_2**
1	240	1200	0,2
2	480	2400	0,2
3	960	4800	0,2
4	240	600	0,4
5	480	1200	0,4
6	960	2400	0,4
7	240	400	0,6
8	480	800	0,6
9	960	1600	0,6
10	240	300	0,8
11	480	600	0,8
12	960	1200	0,8
13	240	240	1
14	480	480	1
15	960	960	1

As seen above, the ratio increases by 0.2 in each three experiments. It starts with a value of 0.2 and in the last three experiments, reaches to the value of 1. The sales price of class 1 products takes three different values; 240, 480 and 960. The values of the sales prices of class 2 products change in accordance with the values of s_1 and the ratio between two sales prices.

In the numerical calculations phase using the experiment set shown above, the same order with the second part of the numerical results section will be followed. That is, firstly, the extended case in which downward substitution is available at boundary and non-boundary states both, and upward substitution is available at boundary states only will be examined. The values for the parameters T_1, T_2 and Q will be calculated. Using these parameter values, profit values for the model will be exposed. After the calculations for this case, the case in which only downward substitution is available at boundary states will be taken into consideration. The profit values for this case will be calculated using the same experiment set. After these numerical calculations, the findings obtained from these results will be analyzed. The effectiveness of the model will be examined under different ratio levels of the sales prices by comparing the profit values of two cases.

In table 6.3, values for parameters T_1, T_2, Q and profit values obtained by running the experiments mentioned above for the first case are shown:

Table 6.3. Threshold and profit values calculated for the first case of the analysis of the effects of changes in the ratio s_1 / s_2

Experiment number	T_1	T_2	Q	Our Heuristic
1	1	7	6	212,00
2	1	8	7	469,25
3	1	10	9	1019,00
4	1	5	8	116,50
5	1	6	11	273,63
6	1	8	14	597,25
7	1	3	8	87,81
8	1	5	10	206,63
9	1	6	13	460,75
10	1	2	7	75,19
11	1	3	9	178,63
12	1	4	12	397,75
13	1	1	6	69,69
14	1	1	8	165,25
15	1	1	10	365,75
Average				313,00

The values in Table 6.3 show that upward substitution is useful during the sales prices of the product with higher quality is higher than the low quality one. As the ratio between two sales prices approach to 1, in other words, as the difference between two sales prices decreases, *T2* values decrease, too, indicating that substitution becomes profitable in the lower inventory levels of class 2 products. For the total inventory level, it can be said that as the sales prices of products increase, the need for holding more inventory increase. Table 6.3 also shows the profit values for the first case. The average value of the profit values for the first case in these experiments are 313.

For the second case, in which only downward substitution is available only at boundary states, the same experiment set is applied. The values obtained in these experiments are shown in Table 6.4:

Table 6.4. Threshold and profit values calculated for the second case of the analysis of the effects of changes in the ratio s_1 / s_2

Experiment number	**Q**	**S**	**Heuristic**
1	6	7	198,56
2	7	8	450,25
3	9	10	1000,00
4	9	5	112,31
5	11	6	268,63
6	14	8	593,25
7	8	3	81,72
8	11	5	202,25
9	14	6	457,25
10	7	2	67,56
11	10	3	172,06
12	13	4	391,88
13	7	1	61,13
14	9	1	155,13
15	12	1	355,25
Average			304,48

In the table above, when compared to the first case, lower profit values are present. This is the result of the additional substitution options in the first case. The average profit value of 15 experiments for the second case is 304.48. But, the more important point here is the effectiveness of the model developed in this study. To see this, let us gather the profit values of two cases in a table:

Table 6.5. Comparison of profit values of two cases for the analysis of the effects of changes in the ratio s_1 / s_2

Experiment no	First case	Second case	Improvement	s_1 values	s_2 values	s_1/s_2
1	212,00	198,56	0,0634	240	1200	0,2
4	116,50	112,31	0,0359		600	0,4
7	87,81	81,72	0,0694		400	0,6
10	75,19	67,56	0,1014		300	0,8
13	69,69	61,13	0,1229		240	1
2	469,25	450,25	0,0405	480	2400	0,2
5	273,63	268,63	0,0183		1200	0,4
8	206,63	202,25	0,0212		800	0,6
11	178,63	172,06	0,0367		600	0,8
14	165,25	155,13	0,0613		480	1
3	1019,00	1000,00	0,0186	960	4800	0,2
6	597,25	593,25	0,0067		2400	0,4
9	460,75	457,25	0,0076		1600	0,6
12	397,75	391,88	0,0148		1200	0,8
15	365,75	355,25	0,0287		960	1

In Table 6.5, the profit values for two cases are present. Besides, the ratios between the differences in profits and the profit values for the first case are shown in the table. The order of the experiments used during the calculations is changed so that it is easier to understand the effect of the changes in the s_1 / s_2 ratio. Before making comments on these values, it will be better to see these results on a figure as follows:

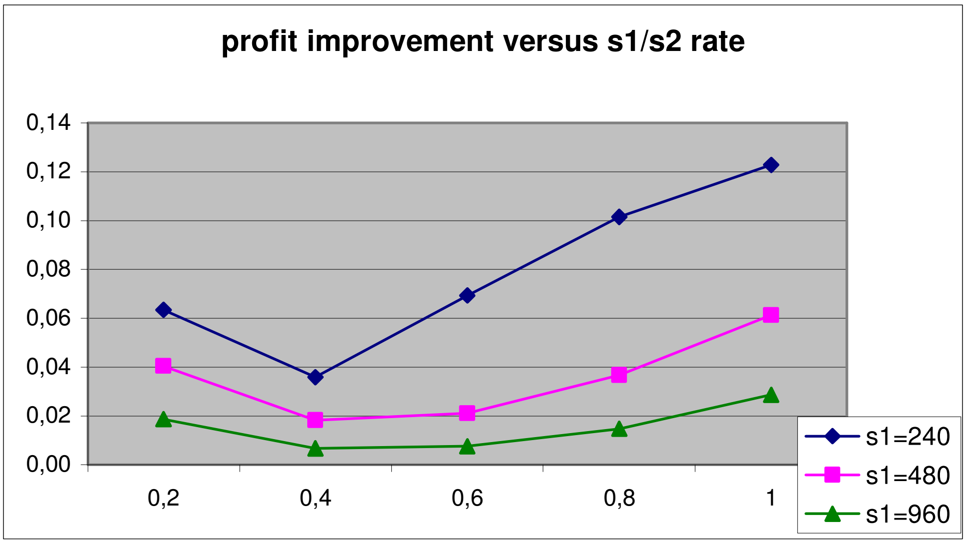

Figure 6.1. Effectiveness of the model developed versus different values of s_1 / s_2 ratio

Figure 6.1 shows the effects of the changes in s_1 / s_2 ratio on the effectiveness of the model clearly. The improvement obtained in the first case decreases as we go to the ratio value of 0.4 from 0.2. But after this point, the improvement rate starts to increase and this increase in the effectiveness of the model continues during the path from the value 0.4 to1. It can be said that as the difference between two sales prices decrease, the effectiveness of our model increases. In our 15 experiments, the effectiveness of our model took its lowest values at point 0.4. This value is a minimum point according to the results obtained from our 15 experiments.

Another inference that can be obtained from the results above is that the effectiveness of the model developed in this study is higher when the products have lower sales prices. Three lines in the figure represent different levels of sales prices. The line at the top of three represents the situation in which lower sales prices are valid for the products.

6.2. Analysis of the Effects of Changes in the Ratio *h/s*

The values of the holding costs of products are usually determined according to the volumes that the products occupy in the warehouses. Sometimes, some other factors are also added to the volume factor such as the extra costs related to the storage of the products like maintaining the optimum temperature level or moisture level for the storage of products in warehouses or the losses in the monetary value of the product that arise due to time passing. During this study, we made the assumption that the value of the holding cost term was the same for both of the product classes. These two product classes are passing through same production processes and they have same volumes and production costs, they necessitate similar storage conditions.

As all other cost terms, an increase in the value of holding cost term decreases the profits of the company. The main question in this analysis is that how the probable changes in the ratio between holding cost and sales prices affect the effectiveness of the model developed. In other words, we do not investigate the effects of an increase in holding cost on the profit of the company; we investigate the effects of changes in the ratio h / s on the improvements achieved by applying the model developed throughout this study. The term

h / s is used since during the numerical calculations in this part, the ratio between the sales prices are not altered so that an increase in h / s_1 ratio also indicates an increase in the ratio h / s_2 with the same rate. For simplicity, during the experiments, the ratio h / s_1 is used as the ratio in question.

A new experiment set is formed for this analysis. The experiments and parameter combinations for each experiment are shown in Table 6.6:

Table 6.6. Parameter values of the experiments used for the analysis of the effects of changes in the ratio h / s_1

Experiment number	**h**	**μ**	**p**	**λ_1**	**λ_2**	**s_1**	**s_2**
1	2	0,5	0,4	0,2	0,2	500	1000
2	3	0,5	0,4	0,2	0,2	750	1500
3	4	0,5	0,4	0,2	0,2	1000	2000
4	4	0,5	0,4	0,2	0,2	500	1000
5	6	0,5	0,4	0,2	0,2	750	1500
6	8	0,5	0,4	0,2	0,2	1000	2000
7	6	0,5	0,4	0,2	0,2	500	1000
8	9	0,5	0,4	0,2	0,2	750	1500
9	12	0,5	0,4	0,2	0,2	1000	2000
10	8	0,5	0,4	0,2	0,2	500	1000
11	12	0,5	0,4	0,2	0,2	750	1500
12	16	0,5	0,4	0,2	0,2	1000	2000
13	10	0,5	0,4	0,2	0,2	500	1000
14	15	0,5	0,4	0,2	0,2	750	1500
15	20	0,5	0,4	0,2	0,2	1000	2000

15 experiments are present in the experiment set. In these experiments, production rate (μ), production probability of class 1 products (p) and demand rates (λ_1 and λ_2) are held constant while sales prices of products are changed in each experiment to one of the three value pairs. Holding cost term takes values between 2 and 20. And these parameter combinations lead us to the following h / s_1 ratios:

Table 6.7. Values of the ratio h / s_1 in 15 experiments

Experiment number	**h**	s_1	s_2	**h/s₁**
1	2	500	1000	0,004
2	3	750	1500	0,004
3	4	1000	2000	0,004
4	4	500	1000	0,008
5	6	750	1500	0,008
6	8	1000	2000	0,008
7	6	500	1000	0,012
8	9	750	1500	0,012
9	12	1000	2000	0,012
10	8	500	1000	0,016
11	12	750	1500	0,016
12	16	1000	2000	0,016
13	10	500	1000	0,02
14	15	750	1500	0,02
15	20	1000	2000	0,02

In the experiment set mentioned above, three different value pairs are used for sales prices of products. These value pairs are: $s_1 = 500,\ s_2 = 1000;\ s_1 = 750,\ s_2 = 1500;\ s_1 = 1000,\ s_2 = 2000$. The ratio h / s_1 is used as the ratio in question and appropriate holding cost values are used to obtain ratio levels from 0.004 to 0.02. The steps taken while going on this path is 0.004. That is in every three experiments, the value of the h / s_1 ratio increases by 0.004. The value of s_2 term is twice of the value of s_1 term in all of the 15 experiments.

In the numerical calculations phase using the experiment set shown above, the same order with the first part of the sensitivity analysis section will be followed. That is, firstly, the extended case in which downward substitution is available at boundary and non-boundary states both, and upward substitution is available at boundary states only will be examined. The values for the parameters T_1, T_2 and Q will be calculated. Using these parameter values, profit values for the model will be exposed. After the calculations for this case, secondly, the case in which only downward substitution is available at boundary states will be taken into consideration. The profit values for this case will be calculated using the same experiment set. After these numerical calculations, the findings obtained from these results will be mentioned. The effectiveness of the model will be examined under different ratio levels between holding cost values and sales prices of class 1 products by comparing the profit values of two cases.

In Table 6.8, values for parameters T_1, T_2 and Q obtained by running the experiments mentioned above for the first case are shown:

Table 6.8. Threshold and profit values calculated for the first case of the analysis of the effects of changes in the ratio h / s_1

Experiment number	**T_1**	**T_2**	**Q**	**Our Heuristic**
1	1	7	15	275,88
2	1	7	15	413,75
3	1	7	15	551,75
4	1	6	12	253,88
5	1	6	12	380,75
6	1	6	12	507,75
7	1	5	10	236,75
8	1	5	10	355,00
9	1	5	10	473,50
10	1	4	9	222,88
11	1	4	9	334,25
12	1	4	9	445,75
13	1	4	8	208,38
14	1	4	8	312,50
15	1	4	8	416,75
Average				359,30

The upward substitution is profitable again at all of the boundary states when the inventory level of class 2 products is zero. On the other hand, as the ratio h / s_1 increases, in other words, as the value of holding cost increases, the tendency towards downward substitution at boundary states increases. This increase in the tendency towards downward substitution decreases the values of *T2* parameter as the h / s_1 ratio increases. Holdings cost term is closely related to the total inventory level of products. Higher unit holding costs result with lower total inventory levels. The table above also shows the profit values for the first case. The average value of the profit values for the first case in these experiments is 359.3.

For the second case, in which only downward substitution is available only at boundary states, the same experiment set is applied. The values obtained in these experiments are shown in Table 6.9:

Table 6.9. Threshold and profit values calculated for the second case of the analysis of the effects of changes in the ratio h/s_1

Experiment number	**Q**	**S**	**Heuristic**
1	15	7	274,38
2	15	7	411,63
3	15	7	548,75
4	12	6	250,13
5	12	6	375,25
6	12	6	500,25
7	10	5	230,00
8	10	5	345,00
9	10	5	460,00
10	9	4	213,25
11	9	4	320,00
12	9	4	426,75
13	9	4	198,56
14	9	4	297,88
15	9	4	397,13
Average			349,93

In Table 6.9, when compared to the first case, lower profit values are present. This is the result of the additional substitution options in the first case. The average value of the profit values for the second case in these experiments is 349.93. But, the more important point here is the effectiveness of the model developed in this study. To see this, let us gather the profit values of two cases in a table:

Table 6.10. Comparison of profit values of two cases for the analysis of the effects of changes in the ratio h / s_1

Experiment number	First case	Second case	Improvement	h	s_1 - s_2	h/s_1
1	275,88	274,38	0,0054	2	500 - 1000	0.004
4	253,88	250,13	0,0148	4		0.008
7	236,75	230,00	0,0285	6		0.012
10	222,88	213,25	0,0432	8		0.016
13	208,38	198,56	0,0471	10		0.02
2	413,75	411,63	0,0051	3	750 - 1500	0.004
5	380,75	375,25	0,0144	6		0.008
8	355,00	345,00	0,0282	9		0.012
11	334,25	320,00	0,0426	12		0.016
14	312,50	297,88	0,0468	15		0.02
3	551,75	548,75	0,0054	4	1000 - 2000	0.004
6	507,75	500,25	0,0148	8		0.008
9	473,50	460,00	0,0285	12		0.012
12	445,75	426,75	0,0426	16		0.016
15	416,75	397,13	0,0471	20		0.02

In Table 6.10, the profit values for two cases are present. Besides, the ratios between the differences in profits and the profit values for the first case are shown in the table. The order of the experiments used during the calculations is changed so that it is easier to understand the effect of the changes in the h / s_1 ratio. Before making comments on these values, it will be better to see these results on a figure as follows:

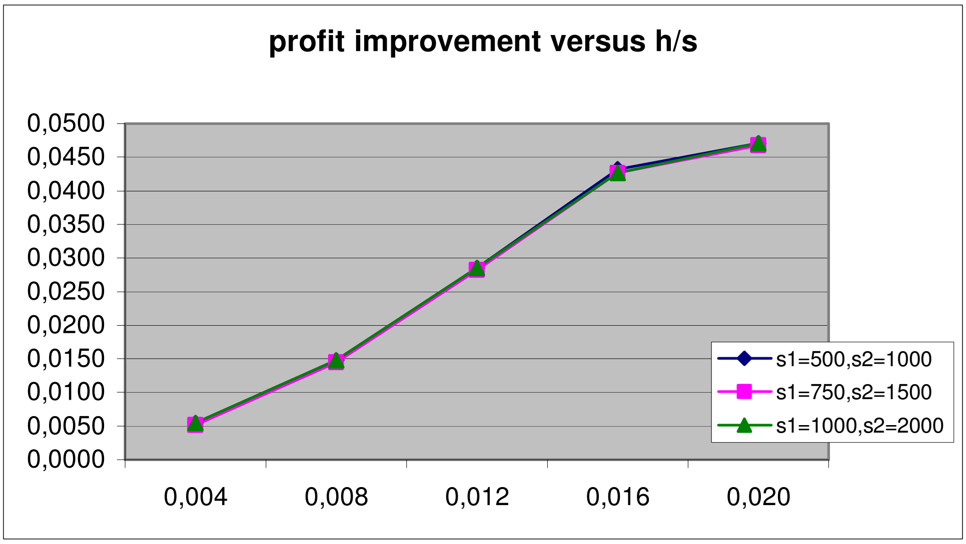

Figure 6.2. Effectiveness of the model developed versus different values of h / s_1 ratio

Figure 6.2 shows the effects of the changes in h/s_1 ratio on the effectiveness of the model clearly. As the h/s_1 ratio increases, the efficiency of our model increases continuously. The improvement achieved by the application of the model is approximately 0.005 per cent for situations where the h/s_1 ratio is 0.004 while the improvement achieved increases to 5 per cent when the h/s_1 ratio reaches to the value of 0.02. After these findings, it can be said that the necessity for the application of our model increases as the h/s_1 ratio increases.

6.3. Analysis of the Effects of Changes in the Ratio $(\lambda_1+\lambda_2)/\mu$

Companies concentrate all their efforts on satisfying the demands for their products by utilizing their production capacity as effectively as possible. Sometimes, higher demand rates are faced so that the company can't satisfy all demands and face lost sales cases and costs related to this situation. On the other hand, if the production capacity of the company is higher than the demand rates for products, the company faces higher inventory levels and higher holding costs related to the higher inventory levels. Besides, as the value of the utilization ratio decreases, the production facilities will have to be kept idle so that productivity of production process will decrease.

All companies aim to make as many sales as possible. To accomplish this, one may claim that the production capacity of the company should be increased so that all demands coming for the products can be satisfied without any problems. But in real life, this is not so easy. The demand rates for products are not so stable. As a result of this, there may be strong variations in demand rates. Usually, the average of this changing demands or the expected value of them are used to determine the most appropriate production capacity. This situation causes insufficiency of capacity sometimes.

In this part of the sensitivity analysis section, the effects of changes in the ratio between demand rates and production rate are examined. A new experiment set is formed for this analysis. The experiments and parameter combinations for each experiment are shown in the table below:

Table 6.11. Parameter values of the experiments used for the analysis of the effects of changes in the ratio $(\lambda_1 + \lambda_2)/\mu$

Experiment number	**h**	**μ**	**p**	**λ_1**	**λ_2**	**s_1**	**s_2**
1	5	0,5	0,4	0,1	0,25	500	1000
2	5	0,5	0,4	0,25	0,1	500	1000
3	5	0,5	0,4	0,15	0,2	500	1000
4	5	0,5	0,4	0,2	0,15	500	1000
5	5	0,5	0,4	0,15	0,3	500	1000
6	5	0,5	0,4	0,3	0,15	500	1000
7	5	0,5	0,4	0,2	0,25	500	1000
8	5	0,5	0,4	0,25	0,2	500	1000
9	5	0,5	0,4	0,2	0,35	500	1000
10	5	0,5	0,4	0,35	0,2	500	1000
11	5	0,5	0,4	0,25	0,3	500	1000
12	5	0,5	0,4	0,3	0,25	500	1000
13	5	0,5	0,4	0,25	0,4	500	1000
14	5	0,5	0,4	0,4	0,25	500	1000
15	5	0,5	0,4	0,3	0,35	500	1000
16	5	0,5	0,4	0,35	0,3	500	1000

The experiment set shown in Table 6.11 is composed of 16 experiments. In these experiments the values of holding cost terms, production rates of products, the production probabilities of class 1 products and sales prices of products are held constant. To obtain different values of ratio $(\lambda_1 + \lambda_2)/\mu$, demand rates of products are altered in each of the experiments. Now, let us see the values of the ratio $(\lambda_1 + \lambda_2)/\mu$ in these experiments:

Table 6.12. Values of the ratio $(\lambda_1 + \lambda_2)/\mu$ in 16 experiments

Experiment number	μ	λ1	λ2	demand/supply
1	0,5	0,1	0,25	0,7
2	0,5	0,25	0,1	0,7
3	0,5	0,15	0,2	0,7
4	0,5	0,2	0,15	0,7
5	0,5	0,15	0,3	0,9
6	0,5	0,3	0,15	0,9
7	0,5	0,2	0,25	0,9
8	0,5	0,25	0,2	0,9
9	0,5	0,2	0,35	1,1
10	0,5	0,35	0,2	1,1
11	0,5	0,25	0,3	1,1
12	0,5	0,3	0,25	1,1
13	0,5	0,25	0,4	1,3
14	0,5	0,4	0,25	1,3
15	0,5	0,3	0,35	1,3
16	0,5	0,35	0,3	1,3

The values for the ratio between demand rates for products and production rates of products are shown above. These values start from 0.7 and reach the value of 1.3. The increase in these values occurs in steps of 0.2 units. For each of these values, 4 different experiments are applied. The production rates of products are held constant, while the values of demand rates differ in each experiment. Sometimes, demand rates for class 1 products are higher while in some of them demands for class 2 products are higher.

In the numerical calculations phase using the experiment set shown above, the same order with the first and second parts of the sensitivity analysis section will be followed. That is, firstly, the extended case in which downward substitution is available at boundary and non-boundary states both, and upward substitution is available at boundary states only will be examined. The values for the parameters T_1, T_2 and Q will be calculated. Using these parameter values, profit values for the model will be exposed. After the calculations for this case, secondly, the case in which only downward substitution is available at boundary states will be taken into consideration. The profit values for this case will be calculated using the same experiment set. After these numerical calculations, the findings obtained from these results will be mentioned. The effectiveness of the model will be examined under different ratio levels between total demand rates of products and production rates of products by comparing the profit values of two cases.

In Table 6.13, values for parameters T_1, T_2 and Q obtained by running the experiments mentioned above for the first case are shown:

Table 6.13. Threshold and profit values calculated for the first case of the analysis of the effects of changes in the ratio $(\lambda_1 + \lambda_2)/\mu$

Experiment number	**T_1**	**T_2**	**Q**	**Our Heuristic**
1	1	6	7	245,13
2	1	3	8	151,50
3	1	4	9	246,38
4	1	3	8	207,13
5	1	12	14	282,25
6	1	4	12	187,75
7	1	8	14	272,88
8	1	6	13	232,75
9	1	21	28	272,63
10	1	6	20	198,50
11	1	13	22	265,13
12	1	9	22	235,63
13	1	35	55	200,75
14	1	9	34	178,63
15	1	21	43	213,50
16	1	13	35	204,50
Average				224,69

The values of T_1 parameters do not change under different values of $(\lambda_1 + \lambda_2)/\mu$ ratio. On the other hand, as the values of this ratio increase, the values of *T2* and *Q* parameters increase, too. This is the expected result of the higher demand rates so that the system tends to hold higher inventory amounts in stocks not to lose any demands. Table 6.13 also shows the profit values for the first case. The average value of the profit values for the first case in these experiments is 224.69.

For the second case, in which only downward substitution is available only at boundary states, the same experiment set is applied. The values obtained in these experiments are shown in Table 6.14:

Table 6.14. Threshold and profit values calculated for the second case of the analysis of the effects of changes in the ratio $(\lambda_1 + \lambda_2)/\mu$

Experiment number	**Q**	**S**	**Heuristic**
1	6	6	41,52
2	8	3	151,25
3	10	4	221,00
4	8	3	203,88
5	10	12	147,88
6	12	4	187,56
7	15	8	262,00
8	13	6	231,75
9	20	21	214,63
10	20	6	198,38
11	26	13	262,50
12	22	9	235,13
13	48	35	187,00
14	34	9	178,50
15	47	21	209,50
16	39	13	204,00
Average			196,03

In Table 6.14, when compared to the first case, lower profit values are present. This is the result of the additional substitution options in the first case. But, the more important point here is the effectiveness of the model developed in this study. To see this, let us gather the profit values of two cases in a table:

Table 6.15. Comparison of profit values of two cases for the analysis of the effects of changes in the ratio $(\lambda_1 + \lambda_2)/\mu$

Experiment number	**First case**	**Second case**	**Improvement**	**μ**	**λ_1**	**λ_2**	**λ_1-λ_2**	**$(\lambda_1+\lambda_2)/\mu$**
1	245,13	41,52	0,8306	0,5	0,1	0,25	-0,15	0,7
5	282,25	147,88	0,4761	0,5	0,15	0,3		0,9
9	272,63	214,63	0,2127	0,5	0,2	0,35		1,1
13	200,75	187,00	0,0685	0,5	0,25	0,4		1,3
2	151,50	151,25	0,0017	0,5	0,25	0,1	0,15	0,7
6	187,75	187,56	0,0010	0,5	0,3	0,15		0,9
10	198,50	198,38	0,0006	0,5	0,35	0,2		1,1
14	178,63	178,50	0,0007	0,5	0,4	0,25		1,3
3	246,38	221,00	0,1030	0,5	0,15	0,2	-0,05	0,7
7	272,88	262,00	0,0399	0,5	0,2	0,25		0,9
11	265,13	262,50	0,0099	0,5	0,25	0,3		1,1
15	213,50	209,50	0,0187	0,5	0,3	0,35		1,3
4	207,13	203,88	0,0157	0,5	0,2	0,15	0,05	0,7
8	232,75	231,75	0,0043	0,5	0,25	0,2		0,9
12	235,63	235,13	0,0021	0,5	0,3	0,25		1,1
16	204,50	204,00	0,0024	0,5	0,35	0,3		1,3

In Table 6.15, the profit values for two cases are present. Besides, the ratios between the differences in profits and the profit values for the first case are shown in the table. The order of the experiments followed during the calculations is changed so that it is easier to understand the effect of the changes in the $(\lambda_1 + \lambda_2)/\mu$ ratio. Before making comments on these values, it will be better to see these results on a figure as follows:

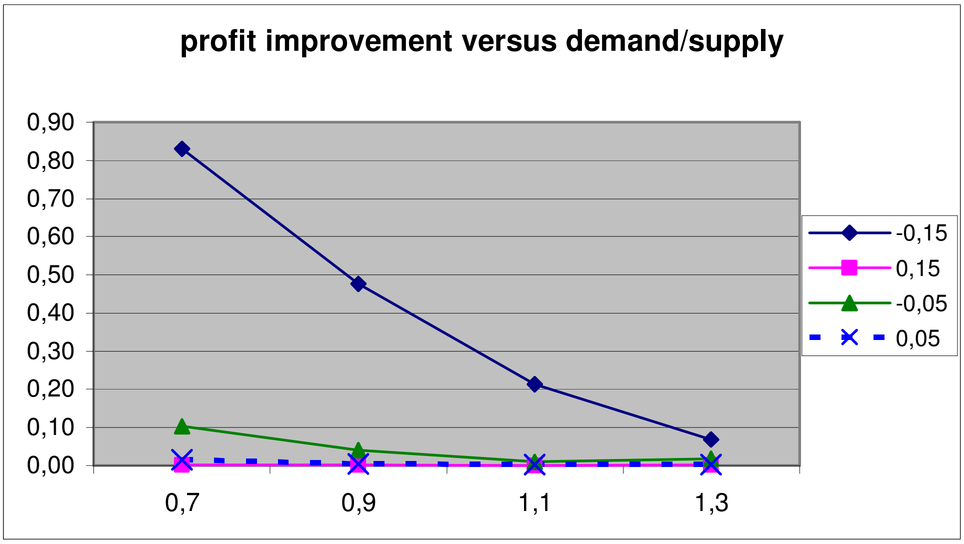

Figure 6.3. Effectiveness of the model developed versus different values of $(\lambda_1 + \lambda_2) / \mu$ ratio

Figure 6.3 shows that in three different levels of the difference between demands rates, the model developed do not change the profit value considerably. But for the difference level of –0.15, as the demand/supply ratio increases, the effectives of our model decreases. For smaller values of this ratio, the model and heuristics developed in this study work really effectively add so much value is added to the productivity of the company.

6.4. Analysis of the Effects of Changes in the Lost Sales Cost Coefficient

Lost sales cost is one of the most important costs manufacturers face. The company loses the opportunity for making a sale in these cases. In addition, the company loses its customers unless their demands are satisfied. These customers prefer other companies if they can’t find the desired products available on time. This factor increases the amount of cost manufacturers face. That is, the amount of cost faced in lost sales cases is usually higher than the cost of the product.

In this research, factors mentioned above are taken into consideration and up to this point the amount of lost sales costs are determined by multiplying sales prices of products by 1.2. The fact that the amount of costs faced in these situations are usually greater than the sales prices are products guided us in determining the value of lost sales cost coefficient.

In this part of the study, effects of changes in lost sales cost coefficients on the effectiveness of the model are examined. The current value of this coefficient is 1.2 and during this analysis, it is appointed different values between 0.8 and 1.6. A new experiment set is formed for this analysis. The experiments and parameter combinations for each experiment are shown in Table 6.16:

Table 6.16. Parameter values of the experiments used for the analysis of the effects of changes in lost sales cost coefficient

Experiment number	**h**	**μ**	**p**	**λ_1**	**λ_2**	**s_1**	**s_2**
1	8	0,5	0,4	0,2	0,2	750	1500
2	8	0,5	0,4	0,2	0,2	1000	2000
3	8	0,5	0,4	0,2	0,2	1250	2500
4	8	0,5	0,4	0,2	0,2	750	1500
5	8	0,5	0,4	0,2	0,2	1000	2000
6	8	0,5	0,4	0,2	0,2	1250	2500
7	8	0,5	0,4	0,2	0,2	750	1500
8	8	0,5	0,4	0,2	0,2	1000	2000
9	8	0,5	0,4	0,2	0,2	1250	2500
10	8	0,5	0,4	0,2	0,2	750	1500
11	8	0,5	0,4	0,2	0,2	1000	2000
12	8	0,5	0,4	0,2	0,2	1250	2500
13	8	0,5	0,4	0,2	0,2	750	1500
14	8	0,5	0,4	0,2	0,2	1000	2000
15	8	0,5	0,4	0,2	0,2	1250	2500

In Table 6.16, there are 15 experiments. The parameter values except sales prices of product are same in these experiments. Sales prices of products take three different values. These value pairs are $s_1 = 750,\ s_2 = 1500;\ s_1 = 1000,\ s_2 = 2000;\ s_1 = 1250,\ s_2 = 2500$. Amounts of lost sales costs faced are determined as a proportion of sales prices. In the first three experiments, the lost sales cost coefficient is 0.8 ($c_1^L = 0.8s_1,\ c_2^L = 0.8s_2$). The value of this coefficient is increased by 0.2 in each three experiments and reaches to 1.6 in the last three experiments.

Table 6.17 shows the sales prices of products, the values of lost sales cost coefficients and amounts of lost sales costs in the experiments:

Table 6.17. Values of the lost sales cost coefficients in 15 experiments

Experiment number	s_1	s_2	c_i^L coef.	c_1^L	c_2^L
1	750	1500	0,8	600	1200
2	1000	2000	0,8	800	1600
3	1250	2500	0,8	1000	2000
4	750	1500	1	750	1500
5	1000	2000	1	1000	2000
6	1250	2500	1	1250	2500
7	750	1500	1,2	900	1800
8	1000	2000	1,2	1200	2400
9	1250	2500	1,2	1500	3000
10	750	1500	1,4	1050	2100
11	1000	2000	1,4	1400	2800
12	1250	2500	1,4	1750	3500
13	750	1500	1,6	1200	2400
14	1000	2000	1,6	1600	3200
15	1250	2500	1,6	2000	4000

As seen above in each three experiment from the beginning, the lost sales cost coefficient is increased by 0.2 units. The actual amounts of lost sales costs are presented in the last two columns. These values are proportional to the sales prices of products.

In the numerical calculations phase using the experiment set shown above, the same order with the first and second parts of the sensitivity analysis section will be followed. That is, firstly, the extended case in which downward substitution is available at boundary and non-boundary states both, and upward substitution is available at boundary states only will be examined. The values for the parameters T_1, T_2 and Q will be calculated. Using these parameter values, profit values for the model will be exposed. After the calculations for this case, secondly, the case in which only downward substitution is available at boundary states will be taken into consideration. The profit values for this case will be calculated using the same experiment set. After these numerical calculations, the findings obtained from these results will be mentioned. The effectiveness of the model will be examined under different values of lost sales cost coefficient by comparing the profit values of two cases.

In Table 6.18, values for parameters T_1, T_2 and Q obtained by running the experiments mentioned above for the first case are shown:

Table 6.18. Threshold and profit values calculated for the first case of the analysis of the effects of changes in lost sales cost coefficient

Experiment number	**T_1**	**T_2**	**Q**	**Our Heuristic**
1	2	5	10	369,25
2	3	6	11	513,00
3	3	6	12	662,00
4	1	5	10	366,25
5	1	6	12	510,50
6	1	6	12	658,50
7	1	5	11	364,25
8	1	6	12	507,75
9	1	6	13	656,50
10	1	5	11	362,00
11	1	6	12	504,75
12	1	6	13	653,50
13	1	5	11	359,75
14	1	6	13	503,00
15	1	6	13	650,50
Average				509,43

The values of T_1 parameter start with values of two or three in the first three experiments in which the lost sales cost coefficient is 0.8. As the value of coefficient increases, values of T_1 decrease to 1. No change is observed in T_2 values, while total inventory level slightly increases as the value of coefficient increases. This is a precaution of the model to prevent the company from high lost sales costs. Table 6.18 also shows the profit values for the first case. The average value of the profit values for the first case in these experiments is 509.43.

For the second case, in which only downward substitution is available only at boundary states, the same experiment set is applied. The values obtained in these experiments are shown in Table 6.19:

Table 6.19. Threshold and profit values calculated for the second case of the analysis of the effects of changes in lost sales cost coefficient

Experiment number	**Q**	**S**	**Heuristic**
1	10	5	362,50
2	11	6	507,00
3	12	6	656,00
4	10	5	358,00
5	12	6	504,50
6	13	6	653,50
7	11	5	356,25
8	12	6	500,25
9	13	6	649,50
10	11	5	352,75
11	13	6	498,50
12	13	6	645,50
13	12	5	351,25
14	13	6	495,00
15	14	6	644,00
Average			502,30

In Table 6.19, when compared to the first case, lower profit values are present. This is the result of the additional substitution options in the first case. But, the more important point here is the effectiveness of the model developed in this study. To see this, let us gather the profit values of two cases in a table:

Table 6.20. Comparison of profit values of two cases for the analysis of the effects of changes in lost sales cost coefficient

Experiment number	**First case**	**Second case**	**Improvement**	c_i^L **coef.**	s_1 **-** s_2
1	369,25	362,50	0,0183	0,8	750-1500
4	366,25	358,00	0,0225	1	
7	364,25	356,25	0,0220	1,2	
10	362,00	352,75	0,0256	1,4	
13	359,75	351,25	0,0236	1,6	
2	513,00	507,00	0,0117	0,8	1000-2000
5	510,50	504,50	0,0118	1	
8	507,75	500,25	0,0148	1,2	
11	504,75	498,50	0,0124	1,4	
14	503,00	495,00	0,0159	1,6	
3	662,00	656,00	0,0091	0,8	1250-2500
6	658,50	653,50	0,0076	1	
9	656,50	649,50	0,0107	1,2	
12	653,50	645,50	0,0122	1,4	
15	650,50	644,00	0,0100	1,6	

In Table 6.20, the profit values for two cases are present. Besides, the ratios between the differences in profits and the profit values for the first case are shown in the table. The order of the experiments followed during the calculations is changed so that it is easier to understand the effect of the changes in the value of lost sales cost coefficients. Before making comments on these values, it will be better to see these results on a figure as follows:

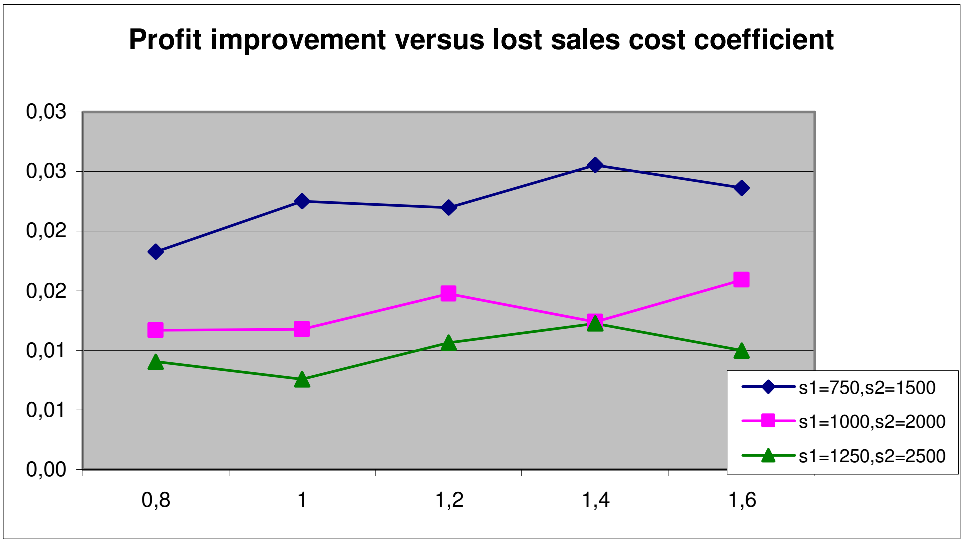

Figure 6.4. Effectiveness of the model developed versus different values of lost sales cost coefficient

Figure 6.4 shows that as the value of lost sales cost coefficient increases, effectiveness of our model slightly increases, too. For each of three different value pairs of sales prices, it can be said that the profit improvement obtained by the application of the model is higher for the coefficient value of 1.6 than for the coefficient value of 0.8. Even though the increase in the effectiveness is not so high, the availability of upward substitution option at boundaries and downward substitution at non-boundaries makes an increasing effect on the profit values.

6.5. Analysis of the Effects of Changes in *p* parameter values

The probability that production process results with a class 1 product (*p*) forms one of the parameters that may have significant effects on the effectiveness of our model. In this part, the effects of *p* parameter values on the effectiveness of our model will be analyzed. A new experiment set is formed for this analysis. The experiments and parameter combinations for each experiment are shown in Table 6.21:

Table 6.21. Parameter values of the experiments used for the analysis of the effects of changes in *p* parameter values

Experiment number	h	μ	p	λ_1	λ_2	s_1	s_2
1	5	0,5	0,2	0,2	0,2	600	1200
2	5	0,5	0,2	0,3	0,2	600	1200
3	5	0,5	0,2	0,2	0,3	600	1200
4	5	0,5	0,4	0,2	0,2	600	1200
5	5	0,5	0,4	0,3	0,2	600	1200
6	5	0,5	0,4	0,2	0,3	600	1200
7	5	0,5	0,6	0,2	0,2	600	1200
8	5	0,5	0,6	0,3	0,2	600	1200
9	5	0,5	0,6	0,2	0,3	600	1200
10	5	0,5	0,8	0,2	0,2	600	1200
11	5	0,5	0,8	0,3	0,2	600	1200
12	5	0,5	0,8	0,2	0,3	600	1200
13	5	0,5	1	0,2	0,2	600	1200
14	5	0,5	1	0,3	0,2	600	1200
15	5	0,5	1	0,2	0,3	600	1200

The experiment set shown in Table 6.21 is composed of 15 experiments. In these experiments the values of holding cost terms, production rates of products and sales prices of products are held constant. To be able to analyze the effects of *p* parameter, the values of *p* parameter and demand rates of products are altered throughout these experiments in order to obtain different combinations of probability values and demand rates of products. As seen in the table, *p* parameter takes values between 0.2 and 1 in these experiments. In the first three, the value of *p* is 0.2, and this value increases by 0.2 units in each three experiments. In the last three experiments, the value of *p* parameter reaches to 1.

In the numerical calculations phase using the experiment set shown above, the same order with the previous parts of the sensitivity analysis section will be followed. That is, firstly, the extended case in which downward substitution is available at boundary and non-boundary states both, and upward substitution is available at boundary states only will be examined. The values for the parameters T_1, T_2 and Q will be calculated. Using these parameter values, profit values for the model will be exposed. After the calculations for this case, the case in which only downward substitution is available at boundary states, will be taken into consideration. The profit values for this case will be calculated using the same experiment set. After these numerical calculations, the findings obtained from these results will be mentioned. The effectiveness of the model will be examined under different values of *p* parameter by comparing the profit values of two cases.

In Table 6.22, values obtained in the first case for parameters T_1, T_2 and Q obtained for experiments in Table 6.21 are shown:

Table 6.22. Threshold and profit values calculated for the first case of the analysis of the effects of changes in *p* parameter values

Experiment number	**T_1**	**T_2**	**Q**	**Our Heuristic**
1	1	3	10	263,00
2	1	4	15	223,00
3	1	7	17	330,88
4	1	6	12	302,75
5	1	6	17	271,25
6	1	14	21	353,50
7	1	11	11	278,38
8	1	12	20	297,88
9	1	40	19	263,75
10	1	40	10	183,13
11	1	40	17	211,38
12	1	78	17	151,75
13	1	77	9	80,44
14	1	77	16	99,81
15	1	116	16	39,06
Average				223,33

The values of T_1 parameters do not change under different values of *p* parameter. On the other hand, as *p* increases, the values of T_2 increase too. The *Q* values are not affected from changes in *p* values considerably. It can be said that the system tends to make less downward substitution as *p* values increase. Higher production frequency of class 1 products decreases the need for substitution of class 1 product demands because as *p* values increase, the probability of class 1 products being out of stock decreases while the probability of class 2 products being out of stock increases. In this case, the system tends to satisfy class 2 product demands by class 2 products instead of class 1 product demands. Thus, the system can balance and coordinate its production and inventory management activities more effectively, and can prevent itself to face higher lost sales costs. Table 6.22 also shows the profit values for the first case. The average value of the profit values for the first case in these experiments is 223.33. These profit values behave like a concave function as *p* values increase. For example in experiments two, five, eight, 11 and 14 the profit values 223, 271.25, 297.88, 211.38 and 99.81 are obtained successively under *p* values 0.2, 0.4, 0.6, 0.8 and 1. Profit values increase as *p* values increase up to the level of "$\lambda_1/(\lambda_1+\lambda_2)$", and for higher values of *p*, profit values decrease as *p* values increase. The system becomes more profitable when the production probabilities of product classes are closer to demand rates ratios of product classes. Under these circumstances, the system is more balanced, that is production process can satisfy demands more effectively.

For the second case, in which only downward substitution is available only at boundary states, the same experiment set is applied. The values obtained in these experiments are shown in Table 6.23:

Table 6.23. Threshold and profit values calculated for the second case of the analysis of the effects of changes in *p* parameter values

Experiment number	**Q**	**S**	**Heuristic**
1	10	3	263,00
2	15	4	222,75
3	17	7	330,75
4	12	6	298,13
5	17	6	271,13
6	21	14	321,75
7	8	11	138,88
8	20	12	276,38
9	8	40	-5,18
10	5	40	-73,31
11	9	40	41,63
12	5	78	-208,50
13	4	77	-210,00
14	6	77	-139,13
15	4	116	-333,00
Average			79,68

In Table 6.23, when compared with the first case, lower profit values are present. In some of the experiments negative profit values (losses) are obtained. The reason for these decreases in profit values is the additional substitution option in the first case. Without these additional substitution options, the system faces higher lost sales costs and obtain lower profit values. The profit values for two cases are examined but the more important point here is the effectiveness of the model developed in this study. To see this, let us gather the profit values of two cases in Table 6.24:

Table 6.24. Comparison of profit values of two cases for the analysis of the effects of changes in *p* parameter values

Experiment number	First case	Second case	Improvement	p	λ_1 - λ_2
1	263,00	263,00	0,0000	0,2	0,2 - 0,2
4	302,75	298,13	0,0153	0,4	
7	278,38	138,88	0,5011	0,6	
10	183,13	-73,31	1,4003	0,8	
13	80,44	-210,00	3,6107	1	
2	223,00	222,75	0,0011	0,2	0,3 - 0,2
5	271,25	271,13	0,0005	0,4	
8	297,88	276,38	0,0722	0,6	
11	211,38	41,63	0,8031	0,8	
14	99,81	-139,13	2,3939	1	
3	330,88	330,75	0,0004	0,2	0,2 - 0,3
6	353,50	321,75	0,0898	0,4	
9	263,75	-5,18	1,0197	0,6	
12	151,75	-208,50	2,3740	0,8	
15	39,06	-333,00	9,5248	1	

In Table 6.24, the profit values for two cases are present. Besides, the ratios between the differences in profits and the profit values for the first case are shown in the table. The order of the experiments followed during the calculations is changed so that it is easier to understand the effect of the changes in the *p* parameter value. Before making comments on these values, it will be better to see these results on a figure as follows:

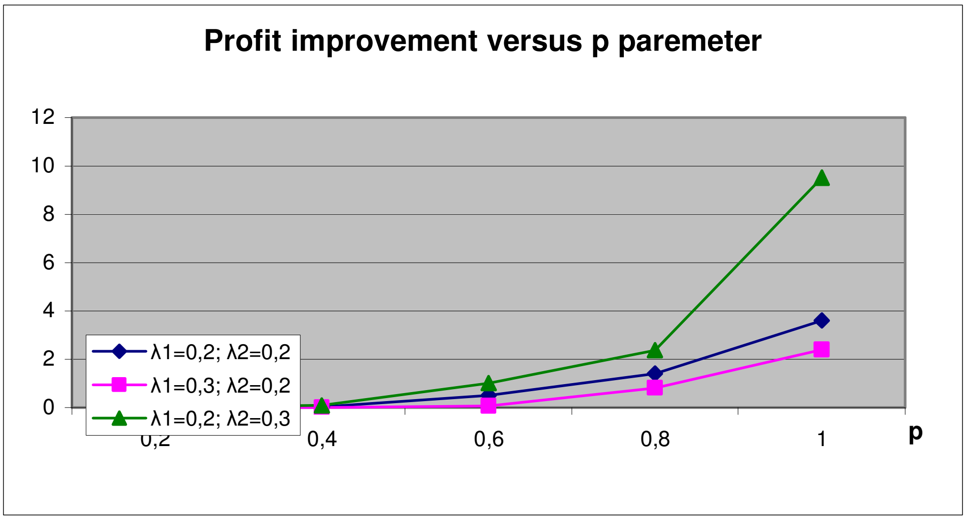

Figure 6.5. Effectiveness of the model developed versus different values of *p* parameter

Figure 6.5 shows that the effectiveness of our model increases as the values of *p* parameter increases. Three different cases with different demand rates confirm these findings. Especially the addition of upward substitution option makes our model work more effectively than the simpler one under higher *p* values. As *p* values get higher, the need for satisfying class 2 product demands by class 1 products increases due to the lower production frequency of class 2 products. In the first case, upward substitution option is valid and the decreases in producton rate of class 2 products can be compensated by substitution option. In the second case, upward substitution is not allowed and as *p* values increase, the number of lost sales cases for class 2 product demands increases. The figure 6.5 supports this idea; in this figure, the effectiveness of the model for the case with higher arrival rate of class 2 product demands is higher than the other two cases. Lost sales costs are higher in this case due to higher demand rates for class 2 product demands. Finally, it can be said that, higher *p* values increases the effectiveness of the model, and for higher level of *p* values, the effectiveness of our model is higher in cases with higher value of λ_2.

7. CONCLUSIONS

In this research, we constructed a model and developed a heuristic that simplifies the decision making processes in systems with different quality levels of products, product substitution and random production yields. Making decisions related to production control and inventory management is more difficult in systems with random production yields. The type of the end product cannot be predetermined. Thus in order to increase the inventory level of a specific product class, many products of other classes may be produced unnecessarily. On the other hand, the inventory levels of these products can not be managed independent of each other. That is, when making decisions related to production control, only the total inventory level of these products is taken into consideration.

A system with random yields and substitutable products is examined in this research and we constructed a model that allows product substitution. At boundary states, this model provides threshold values for substitution. At non-boundary states, using a fluid model, a line equation is determined that divides the non-boundary state space into two parts. At states above this line, substitution is made, and below this line, substitution is not made. Downward substitution is valid at boundary and non-boundary states both while upward substitution is valid only at boundary states.

After this model development phase, we continued our studies with the numerical analysis of our model. We tried to understand how our model behaved against the previous models in the literature. A predetermined example set is used in these calculations in which different cases are examined in each example. Numerical results section consists of two parts. In the first part, only downward substitution is allowed and three different cases are examined. In the first case, downward substitution is only valid at boundary states. In the second and third cases in addition to boundary states, downward substitution is also valid at non-boundary states. Lost sales cost and substitution cost is not valid in this first part. When we compared these three cases, we realized that an improvement in profits is achieved about 0.46 per cent in second case and 0.37 per cent in third case compared to the first case.

In the second part, we added lost sales cost and substitution costs to the model. Upward substitution is valid at boundary states in addition to downward substitution at boundary and non-boundary states. The line equation that is used to determine substitution decision at non-boundary states is examined under different values of production rate and production probabilities of product classes. We compared our model to the case in which only downward substitution is valid at boundary states using new revenue scheme for both cases. The improvements achieved in the second are higher than the ones obtained in the first part. This is the result of the increase in the effectiveness of our model obtained by addition of the upward substitution option, and lost sales cost and substitution costs. The profit improvement obtained in the second part of numerical results section is more than 26 per cent.

Finally we made a sensitivity analysis to understand better the effects of the model parameters on the effectiveness of the model. We found out the conditions in which the application of our model enables the user to obtain higher profit improvements. We tested our model under different values of “s_1 / s_2” ratio, “h / s” ratio, “total demand rate/production rate” ratio and lost sales cost coefficient. The results of these analyses showed us that as the difference between the sales prices of two product classes (s_1 / s_2) decreases, the effectiveness of our model increases. In addition, as the ratio between holding cost and sales prices of products (h / s) increases, the effectiveness of our model also increases. The effectiveness of our model decreases as the ratio “$(\lambda_1 + \lambda_2) / \mu$” increases. In other words, when the total demand rate is smaller than the production rate, our model works more effectively since in this case inventories are positive. Finally, increases in the lost sales cost coefficient and *p* parameter values also increase the effectiveness of our model, but this increase is not so high for lost sales cost coefficient. Profit values increase as *p* values increase up to the level of "$\lambda_1 / (\lambda_1 + \lambda_2)$", and for higher values of *p*, profit values decrease as *p* values increase. The effectiveness of the model gets higher as *p* values increase to higher values than "$\lambda_1 / (\lambda_1 + \lambda_2)$".

Our studies and numerical analysis show us that our model is working well. The results obtained from our model are better than the results obtained from the previous models in the literature. In addition, our model and heuristic are providing very close results to the optimal profit values.

There are a number of future research ideas that are left to investigate such as the multiple products case (a system with more than two product / quality classes), a case with backorders where unmet demand is backordered and satisfied when profitable and the two way substitution case considered at non-boundary states. Some other extensions of the model and system examined in this research such as the different holding costs and batch production may also be studied in the future.

Finally, looking at the results obtained throughout this research we can say that we reached our objective that we had determined at the beginning of our studies. We developed a practical, applicable and useful model that makes it very easy to make the decisions related to production and inventory management. The application of this model will enable the users to increase their profits and decrease the difficulty level faced during the decision making processes.

APPENDIX A: PROOF FOR THE CALCULATION STEPS OF THE DEPLETION TIME FOR CLASS 1 PRODUCTS

We start with the equation below:

$$x_1(t_1^d) = x_1(0) + \mu_1 t_1^i + (\mu_1 - \lambda_1)(t_1^d - t_1^i) \tag{A.1}$$

By equalizing the left hand side zero due to the fact that the inventory level of the products at depletion times is zero, we get:

$$0 = x_1(0) + \mu_1 t_1^i + (\mu_1 - \lambda_1)(t_1^d - t_1^i) \tag{A.2}$$

When we take out the interior terms from the last parantesis;

$$0 = x_1(0) + \mu_1 t_1^i + (\mu_1 - \lambda_1)t_1^d - (\mu_1 - \lambda_1)t_1^i \tag{A.3}$$

By passing the term that includes t_1^d to the left hand side of the equation, we get;

$$(\mu_1 - \lambda_1)t_1^d = x_1(0) + \mu_1 t_1^i - (\mu_1 - \lambda_1)t_1^i \tag{A.4}$$

This time, we take out the terms from the parantesis in the right hand side of the equation;

$$(\mu_1 - \lambda_1)t_1^d = x_1(0) + \mu_1 t_1^i - \mu_1 t_1^i + \lambda_1 t_1^i \tag{A.5}$$

Further calculations take us to;

$$(\mu_1 - \lambda_1)t_1^d = x_1(0) + \lambda_1 t_1^i \tag{A.6}$$

And finally, passing the terms in parantesis from left hand side to the right hand side of the equaiton, we reach;

$$t_1^d = \frac{x_1(0) + \lambda_1 t_1^i}{(\mu_1 - \lambda_1)} \tag{A.7}$$

APPENDIX B: PROOF FOR THE CALCULATION STEPS OF THE DEPLETION TIME FOR CLASS 2 PRODUCTS

We start with the equation below:

$$x_2(t_2^d) = x_2(0) + (\mu_2 - \lambda_2 - \lambda_1)t_1^i + (\mu_2 - \lambda_2)(t_2^d - t_1^i) \tag{B.1}$$

By equalizing the left hand side zero again due to the fact that the inventory level of the products at depletion times is zero, we get:

$$0 = x_2(0) + (\mu_2 - \lambda_2 - \lambda_1)t_1^i + (\mu_2 - \lambda_2)(t_2^d - t_1^i) \tag{B.2}$$

When we collect the terms in the right hand side of the equation by the multiplication coefficients t_1^i and t_2^d, we obtain;

$$0 = x_2(0) + t_1^i(\mu_2 - \lambda_2 - \lambda_1 - \mu_2 + \lambda_2) + t_2^d(\mu_2 - \lambda_2) \tag{B.3}$$

By making the necessary eliminations and passing the term that includes t_2^d to the left hand side of the equation, we get;

$$-t_2^d(\mu_2 - \lambda_2) = x_2(0) + t_1^i(-\lambda_1) \tag{B.4}$$

And finally, passing the terms in parantesis from left hand side to the right hand side of the equaiton, we reach;

$$t_2^d = -\frac{x_2(0) - t_1^i\lambda_1}{(\mu_2 - \lambda_2)} \tag{B.5}$$

APPENDIX C: PROOF FOR THE CALCULATION STEPS OF THE LINE EQUATION USED FOR SUBSTITUTION DECISION AT NON-BOUNDARY STATES

From the equations given in the model section of the report, we have;

$$y_2(0) = y_1(0) - (1-\delta) \tag{C.1}$$

$$y_1(t) = h_1(t - t_1^d) + (h_2 - h_1)t_2^d \tag{C.2}$$

$$y_2(t) = h_2(t - t_2^d) + (h_1 - h_2)t_1^d \tag{C.3}$$

When we put the values of y terms that are mentioned in second and third equations to their places in the first equation, we get;

$$h_2(0 - t_2^d) + (h_1 - h_2)t_1^d = h_1(0 - t_1^d) + (h_2 - h_1)t_2^d - (1-\delta) \tag{C.4}$$

When we open the parantesis;

$$-h_2t_2^d + h_1t_1^d - h_2t_1^d = -h_1t_1^d + h_2t_2^d - h_1t_2^d - (1-\delta) \tag{C.5}$$

By continuing further calculations, we obtain;

$$t_2^d(-h_2 - h_2 + h_1) = t_1^d(-h_1 - h_1 + h_2) - (1-\delta) \tag{C.6}$$

$$t_2^d(-2h_2 + h_1) = t_1^d(-2h_1 + h_2) - (1-\delta) \tag{C.7}$$

Then, we put the values of t_2^d and t_1^d to their places in the equation above, we get;

$$\frac{-x_2(0) + \lambda_1 t_1^i}{(\mu_2 - \lambda_2)}(-2h_2 + h_1) = \frac{-x_1(0) - \lambda_1 t_1^i}{(\mu_1 - \lambda_1)}(-2h_1 + h_2) - (1-\delta) \tag{C.8}$$

These calculations take us finally to;

$$\lambda_1 t_1^i \left[\frac{(-2h_2 + h_1)}{(\mu_2 - \lambda_2)} + \frac{(-2h_1 + h_2)}{(\mu_1 - \lambda_1)} \right] = \frac{x_1(0)(2h_1 - h_2)}{(\mu_1 - \lambda_1)} - \frac{x_2(0)(2h_2 - h_1)}{(\mu_2 - \lambda_2)} - (1 - \delta) \qquad \text{(C.9)}$$

From this last equation, we obtain the values of coefficient terms as follows:

$$b_1 = \frac{2h_1 - h_2}{\mu_1 - \lambda_1} \qquad \text{(C.10)}$$

$$\beta = \frac{2h_2 - h_1}{\mu_2 - \lambda_2} \qquad \text{(C.11)}$$

$$\theta = \left[\frac{\lambda_1(-2h_2 + h_1)}{(\mu_2 - \lambda_2)} + \frac{\lambda_1(-2h_1 + h_2)}{(\mu_1 - \lambda_1)} \right] \qquad \text{(C.12)}$$

and we can say that

$$t_1^i = \frac{b_1 x_1(0) - \beta x_2(0) - (1 - \delta)}{\theta} \qquad \text{(C.13)}$$

Finally, when we calculate the continuity equation for the initial interrupt at $t_2^i = 0$, we obtain the equation,

$$x_2(0) = \frac{b_1}{\beta} x_1(0) - (1 - \delta) \qquad \text{(C.14)}$$

APPENDIX D: PROOF FOR THE DIFFERENCES IN LINE EQUATION THAT OCCUR WHEN DIFFERENT REVENUES ARE APPLIED

The fluid approximation model in (4.9) takes shape as follows under the revenue scheme in (4.2):

$$\min \int_0^T \left[h\left(x(t)\right) x(t) - s_2 \lambda_1 \delta - s_2 \lambda_2 \right] dt \tag{D.1}$$

Inventory levels of the product classes change with the same rates as mentioned in ‘mathematical model’ part. Under these conditions, the Hamiltonian function becomes:

$$\begin{aligned} H = & -(h_1 x_1 + h_2 x_2) + s_2(\lambda_2 + \delta \lambda_1) \\ & + y_1(t)(\mu_1 - \lambda_1) + y_2(t)(\mu_2 - \lambda_2) \\ & + u_2(t) s_2 \lambda_1 (y_1(t) - y_2(t)). \end{aligned} \tag{D.2}$$

From this Hamiltonian function, we get the continuity equation as;

$$k_2(t) = s_2 \lambda_1 \left(y_1(t) - y_2(t) \right) \tag{D.3}$$

Since the decision changes occur at optimal control points, the continuity equation seen above is equal to zero at $t=0$ if an interrupt takes place at $t_2^r=0$. Therefore;

$$y_2(0) = y_1(0) \tag{D.4}$$

Similarly, for the depletion of class 2 products;

$$y_2(t_2^d) = y_1(t_2^d) \tag{D.5}$$

The equations (4.14) to (4.21) are valid in this part, too. Continuing from that point; since at time t_1^i an interrupt occurs, we obtain;

$$y_2(t_1^i) = y_1(t_1^i) \tag{D.6}$$

By evaluating this equation further, we get;

$$t_1^i = \frac{b_1 x_1(0) - \beta x_2(0)}{\theta} \tag{D.7}$$

where;

$$b_1 = \frac{2h_1 - h_2}{\mu_1 - \lambda_1}, \tag{D.8}$$

$$\beta = \frac{2h_2 - h_1}{\mu_2 - \lambda_2}, \tag{D.9}$$

and

$$\theta = \left[\frac{\lambda_1(-2h_2 + h_1)}{(\mu_2 - \lambda_2)} + \frac{\lambda_1(-2h_1 + h_2)}{(\mu_1 - \lambda_1)} \right] \tag{D.10}$$

Finally, when we calculate the continuity equation for the initial interrupt at $t_2^i = 0$, we obtain the equation,

$$x_2(0) = \frac{b_1}{\beta} x_1(0) \tag{D.11}$$

Since the substitution of demands for class 1 products and satisfaction of these demands by class 2 products starts at the point T2 when the inventory level of class 1 products is zero, we build the line that determines the substitution decision at non-boundary states as follows:

$$x_2(0) = T2 + \left| \frac{b_1}{\beta} \right| x_1(0) \tag{D.12}$$

REFERENCES

1. Netessine, S. and N. Rudi, 2003, “Centralized and competitive inventory models with demand substitution”, *Operations Research*, Vol. 51, No. 2, pp. 329-335.

2. Bassok, Y., R. Anupindi.and R. Akella, 1999, “Single-period multiproduct inventory models with substitution”, *Operations Research*, Vol. 47, No. 4, pp. 632-642.

3. Duenyas, I. and C. Tsai, 2000, “Control of a manufacturing system with random product yield and downward substitutability”, *IIE Transactions*, Vol. 32, pp. 785-795.

4. Bitran, G. R. and S. Dasu, 1992, “Ordering policies in an environment of stochastic yields and substitutable demands”, *Operations Research*, Vol. 40, pp. 999-1017.

5. Bitran, G. R. and T. Leong, 1992, “Deterministic approximations to co-production problems with service constraints and random yields”, *Management Science*, Vol. 38, pp. 724-742.

6. Bitran, G. R. and S. M. Gilbert, 1994, “Co-production processes with random yields in the semiconductor industry”, *Operations Research*, Vol. 42, pp. 476-491.

7. Hsu, A. and Y. Basok, 1999, “Random yield and random demand in a production system with downward substitution”, *Operations Research*, Vol. 47, pp. 227-290.

8. Korugan, A. and S. M. Gupta, 2001, *Product Substitution in Hybrid Systems: Rules and Policies*, Northeastern University, Laboratory for Responsible Manufacturing, Boston LRM 2001-003.

9. Inderfurth, K., 2002, “Optimal policies in hybrid manufacturing/remanufacturing systems with product substitution”, *International Journal of Production Economics*, Vol. 90, No. 3, pp. 325-343.

10. Bayındır, Z. P., N. Erkip and R. Güllü, 2002, *Assessing the benefits of remanufacturing option under one-way substitution*, Middle East Technical University Industrial Engineering Department, Technical Report No: 02-02.

11. De Vericourt, F., F. Karaesmen and Y. Dallery, 2002, “Optimal stock allocation for a capacitated supply system”, *Management Science*, Vol. 48, No. 11, pp. 1486-1501.

12. Topkis, D. M., 1968, “Optimal ordering and rationing policies in a non-stationary dynamic inventory model with n demand classes”, *Management Science*, Vol. 15, No. 3, pp. 160-176.

13. Nahmias, S. and W. S. Demmy, 1981, “Operating characteristics of an inventory system with rationing”, *Management Science*, Vol. 27, No. 11, pp. 1236-1245.

14. Deshpande, V., M. A. Cohen and K. Donohue, 2003, “A threshold inventory rationing policy for service-differentiated demand classes”, *Management Science*, Vol. 49, no. 6, pp. 683-703.

15. De Vericourt, F., F. Karaesmen, and Y. Dallery, 2001, “Assessing the benefits of different stock-allocation policies for a make-to-stock production system”, *Manufacturing& Service Operations Management*, Vol. 3, No. 2, pp. 105-121.

16. Veatch, M. and L. M. Wein, 1996, “Scheduling a make-to-stock queue: Index policies and hedging points”, *Operation Research*, Vol. 44, No. 4, pp. 634-647.

17. Pena-Perez, A. and P. Zipkin, 1997, “Dynamic scheduling rules for a multiproduct make-to-stock queue”, *Operations Research*, Vol. 45, No. 6, pp. 919-930.

18. Ha, A., 1997a, “Optimal dynamic scheduling policy for a make-to stock production system”, *Operations Research*, Vol. 45, No. 1, pp. 42-53.

19. Ha, A., 1997b, “Inventory rationing in a make-to-stock production system with several demand classes and lost sales”, *Management Science*, Vol. 43, No. 8, pp. 1093-1103.

20. De Vericourt, F., F. Karaesmen and Y. Dallery, 2000, “Dynamic scheduling in a make-to-stock system: A partial characterization of optimal policies”, *Operations Research*, Vol. 48, No. 5, pp. 811-819.