FREE VIBRATION ANALYSIS OF A NON-UNIFORM FREE-ENDED BEAM WITH WEIGHTED RESIDUAL METHODS

by

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ABSTRACT

FREE VIBRATION ANALYSIS OF A NON-UNIFORM FREE-ENDED BEAM WITH WEIGHTED RESIDUAL METHODS

In this study the transverse vibration of a non-uniform free-ended beam is analyzed by solving its governing differential equation based on Euler-Bernoulli Theory with weighted residual methods as approximate methods. First, several beam theories are reviewed and compared and among them Euler-Bernoulli theory is chosen for further study due to its basic structure. After studying the convergence of the approximate values obtained with Galerkin Method to the exact values of the natural frequencies for the first five modes of vibration of a uniform beam, several trials with combinations of different weighted residual methods and assumed basis functions are investigated to find approximate values for natural frequencies of a non-uniform elastic beam. Using Galerkin Method and power series as trial function, which resulted in best approximation, the variations of natural frequencies for the first six modes of vibration of the free-ended non-uniform beam with changing system parameters are investigated.

ÖZET

AĞIRLIKLI ARTIK METODLARI İLE SERBEST UÇLU DÜZGÜN OLMAYAN BİR KİRİŞİN SERBEST TİTREŞİM ANALİZİ

Bu çalışmada, iki ucu serbest, düzgün olmayan bir kirişin enine titreşimi, Euler-Bernoulli kiriş teorisine dayanan belirleyici diferansiyel denkleminin yaklaşık yöntem olarak ağırlıklı artık metodlarının uygulanarak çözülmesi ile analiz edilmiştir. Öncelikle, çeşitli kiriş teorileri gözden geçirilmiş ve karşılaştırılmış ve içlerinden çalışmanın devamı için temel yapısı sebebiyle Euler-Bernoulli kiriş teorisi seçilmiştir. Düzgün bir kirişin ilk beş titreşim modu için Galerkin Metodu ile bulunan dogal frekansların yaklaşık degerlerinin gerçek sonuçlara yakınsaması incelendikten sonra, düzgün olmayan elastik bir kirişin dogal frekanslarını yaklaşık olarak bulabilmek için farklı ağırlıklı artık metodları ve varsayılan yaklaşık fonksiyonların kombinasyonlarından oluşan denemeler yapılmıştır. En iyi yaklaşımla sonuçlanan Galerkin Metodu ve deneme fonksiyonu olarak kuvvet serisi kullanarak serbest uçlu düzgün olmayan elastik bir kirişin ilk altı titreşim modu için doğal frekanslarının değişen sistem parametrelerine göre değişimi incelenmiştir.

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a	shear correction factor
A_0	cross-sectional area of the mid-section of the beam
$A(x)$	cross-sectional area
b	coefficient determining the variation of the cross-sectional area
c	coefficient determining variation of the moment of inertia
c_b	longitudinal wave velocity in the Euler-Bernoulli beam
c_n	the coefficients of the shape function
c_s	velocity of the shear waves of the beam $c_s^{\ 2} = E_l / \rho$
c_t	velocity of the shear waves of the beam $c_t^{\ 2} = G / \rho$
C, C_1, C_2, C_3, C_4	coefficients determining the amplitude of the deflection
D_0	elastic sloping stiffness of the medium
$E(x)$	modulus of elasticity
E_l	longitudinal modulus of elasticity
E_r	error
$f(x,t)$	transverse external force
$f_b(x)$	function satisfying boundary conditions
$F_n(x)$	nth component of Fourier series
G	modulus of shear
$G_n(x)$	nth component of Legendre polynomial
$H_n(x)$	nth component of hyperbolic basis function
I_0	moment of inertia of the mid-section of the beam
$I(x)$	moment of inertia
k	consecutive roots of the general solution
k'	shear coefficient
k_0	wave number of Euler-Bernoulli beam
k_b	longitudinal wave number

k_t	shear wave number
k_w	wave number
L	length of the beam
$L\{.\}$	the functional for the differential equation
m	index number
$m(x)$	unit mass per unit length
M	order
$M(x,t)$	bending moment
n	index number
N	order
$P_N(x)$	approximating function of order N for a specific trial function
$T_n(x)$	n^{th} component of Chebishev Polynomials
$Q(x,t)$	the transverse shear force
$Q_N(x)$	the approximating function of order N
t	time
$u(x)$	the amplitude of the transverse deflection
x	spatial variable in the longitudinal direction
Δx_i	i^{th} subinterval
$y(x,t)$	bending deflection
α, β	limits of chosen interval in which the modified Chebishev Polynomial is defined
$\gamma(x)$	time dependent multiplier of bending deflection function
$\varepsilon_N(x,c)$	residual function of order N
θ	the phase angle
λ_i	consecutive root for the i th mode multiplied by the beam length
ν	the Poisson's ratio
ρ	density of the beam
$\phi_n(x)$	n^{th} component of the trial function (basis function)
$\Psi_n(x)$	n^{th} component of modified Chebishev Polynomial

φ	shear angle of the cross-section of the beam
ψ	the angle of rotation of the beam when shear is neglected
ω_i	i^{th} natural frequency
ω_n	natural frequency
ω_{exact}	exact value for natural frequency
ω_{appr}	approximate value for natural frequency

1. INTRODUCTION

1.1. Trial Solution Methods and their Applications to Vibration Problems

The mathematical procedure for vibrations of beams is by means of partial differential equations, in which the kinematic properties are functions of spatial variables and time. Beams are treated as continuous or distributed parameter systems of which mass and stiffness parameters are functions of spatial variable in general. Beams can be considered as consisting of infinite number of connected points. Therefore they have an infinite number of degrees of freedom. The position coordinate and time are independent variables of governing differential equations for vibration of beams. Boundary value problems for beams consist of fourth order partial differential equations with two boundary conditions for each end.

The first step of dynamic analysis of continuous systems is to evaluate its frequencies and mode shapes. For free vibration of undamped systems these are called natural frequencies and natural modes, which represent a characteristic property for the system. The natural frequencies together with modal vectors build up the natural modes of the system. The natural frequencies for a system are unique whereas only the shape of modal vectors is unique, their length is not.

For several beams having uniform properties along their lengths and having simple boundary conditions exact solutions for the displacements are obtained which depend on spatial variable and time. But in real life most systems do not have uniform properties. Determining the exact solution of fourth order differential equations of elastic beams having variable system parameters such as variable cross-section, mass or stiffness is mathematically complicated because the system parameters appear as coefficients depending on spatial variable in the differential equation. For those cases we must be content with approximate solutions (Meirovitch, 2001).

Among several approximation techniques which discretize the distributed parameter system by truncation, the trial solution methods are discussed here. We seek an approximate solution in terms of known functions which exactly satisfies the boundary conditions and approximately satisfies the governing differential equation.

The first step of any trial solution method is to construct an admissible solution in terms of known specific functions. After construction of a trial solution in form of a finite sum of series of known functions called as trial functions or basis or coordinate functions, which are forced to satisfy the boundary conditions, an optimization criterion is applied to find the best solution which will be as close as possible to the exact solution. At the end of the trial solution method the accuracy of approximation is estimated by examining the convergence of the solution.

Method of weighted residuals is one of the optimization criteria and it tries to find the best approximate solution by minimizing the expression for the residual error in the differential equation through weighted averages. The five criteria applied most commonly in weighted residual methods are collocation, subdomain, least-square, moment and the Galerkin methods. Of these methods Galerkin, subdomain and collocation methods will be discussed in later chapters.

The solution of a boundary value problem must satisfy the governing equation in the interior of the domain and the boundary conditions on the boundary of the domain. When constructing the trial functions for an approximate solution, the constraint that they must satisfy the boundary conditions exactly is imposed. These boundary conditions can be classified as essential and natural. For a beam, of which the governing differential equation is fourth order, the essential boundary conditions consist of equations containing the displacement function and its first derivative whereas the natural boundary conditions consist of its second and third derivatives. The satisfaction of all the boundary conditions is often problematic (Meirovitch, 2001).

1.2. Objective and Scope

The aim of this study is (1) to analyze the effect of trial functions and weighted residual methods on the convergence of approximate solutions for natural frequencies of transverse vibration of an elastic beam having variable system parameters, and (2) after determining the best approximation, to find the influence of changing these system parameters on the natural frequencies of the beam.

Firstly, different beam theories and constrained equations for boundary conditions for different boundary types are reviewed and compared. Among several beam theories Euler-Bernoulli beam model is chosen for further study which is the most commonly used because it is simple and provides reasonable engineering approximations for many problems (Han, 2001). Then trial solutions are constructed to find the best approximate solution for a beam having variable cross-section and moment of inertia with different trial functions such as power series, polynomials or trigonometric series. As optimization criteria among weighted residual methods, collocation, subdomain, and Galerkin methods are applied. The trial function and weighted residual method resulting in best approximation among others is used to examine the closeness of the approximate solution to the exact solution of a uniform beam. At the end of this study, using the same approximation, the effect of changing system parameters of an elastic beam on its natural frequencies is analyzed.

1.3. Literature Review

There are a vast number of studies about vibration of beams. Different theoretical beam models such as Euler-Bernoulli and Timoshenko are adopted to predict the natural frequencies of beams. Most of the research papers published recently dealt with Euler-Bernoulli beam model. Only a few researches have investigated the vibration of beams of different types based on Timoshenko beam model. The literature review within the scope of this study is bound with free vibrations of beams.

Maurizi *et al*. (1990) and Guiterrez *et al.* (1990) have studied free vibrations of Timoshenko beam having uniform and non-uniform cross-sections, respectively. Bruch and Mitchell (1987) and Abramovitch and Hamburger (1990) have investigated vibrations of a cantilever Timoshenko beam with a tip mass to determine the effect of rotary inertia and shear deformation which are neglected in classical Euler-Bernoulli beam.

Kang and Leissa (2004) have presented a three dimensional method of analysis based upon three-dimensional dynamic equations of elasticity for determining the free vibration frequencies and mode shapes of thick, tapered rods and beams having circular cross-section. The three dimensions are deflections in the radial, circumferential and axial directions. The displacement functions are assumed as algebraic polynomials and include functions which impose only the essential constraints. Ritz method is used to solve the eigenvalue problem. Fundamental frequencies are predicted using one term approximation.

There are a large number of studies for vibration of beams based on Euler-Bernoulli beam model. Hoffmann and Wertheimer (1999) have used a tapered cantilever beam with linear tapers as a function of beam stiffness to determine a relationship among the first-mode natural frequency, beam mass, and a mass distribution parameter.

Wu and Chiang (2004) have studied free vibrations of solid and hollow wedge beams based on Euler-Bernoulli beam theory. Solid and hollow wedge beams with square, rectangular and circular cross-sections with and without carrying any number of point masses are covered in this study. After determining the closed form solutions in terms of Bessel function for natural frequencies and normal mode shapes of an unloaded wedge beam, the partial differential equation for the case with point mass loads is transformed into matrix equation using the expansion theorem and finally the eigenvalue problem is solved.

Abrate (1994) has transformed the equation of motion for non-uniform rods and beams into the equation of motion for a uniform rod and beam in his study which is based on Euler-Bernoulli beam theory. He showed that if the ends are fixed the eigenvalues of non-uniform and uniform rods and beams are the same. To formulate the problem Rayleigh-Ritz approach is used and the displacements are replaced by a series of approximation polynomial functions that can satisfy the boundary conditions at one end and the other boundary condition is forced to satisfy the Lagrange multiplier method. Fundamental frequencies are predicted using one-term Rayleigh-Ritz approximation.

Alvarez *et al.* (1988) developed a simple algorithmic procedure using Rayleigh-Ritz approach to obtain an approximate solution for vibration of an elastically restrained non-uniform beam with translational and rotational springs and with a tip mass. Simple polynomials are used to approximate the mode shape functions.

Naguleswaran (1992a and 1992b) has investigated transverse vibrations of non-uniform Euler-Bernoulli beams. Analytical solutions for the mode shapes are derived by using method of Frobenius which is a power series solution to the problem of non-uniform beams, using different boundary conditions and truncation factors combinations. Other researchers who also applied power series expansion approach to the solution of frequencies are Kim and Kim (2000) and Zeng and Bert (2000) who have represented mode functions of a vibrating beam by Fourier and Taylor series respectively.

Kim (2000) derived the frequency equation of an elastic beam with generally restrained boundary conditions in matrix form using Fourier series in the Euler-Bernoulli theory. The boundary conditions comprise classical boundary conditions as well as non-classical boundary conditions such as restraints by rotational and translational springs. The numerical results are plotted in tables showing the effects of different restraints. To solve the free axial vibration of a tapered bar Zeng and Bert (2000) have used differential transformation method which is based on Taylor series expansion.

Naudi *et al.* (2000) have studied the free vibration behavior of a tapered beam with rectangular cross-section, linearly varying depth taper and non-linear end elastic restraints against rotation using finite element method. If the non-linear springs at the ends do not have identical stiffness then the solution is obtained using an iterative method. In this study numerical results for both identical and different springs at the ends are obtained and variation of frequency parameter with spring stiffness coefficients of both springs and depth ratio of the beam is presented in the form of tables.

Finally, Söylemez *et al*. (2005) have presented the eigensolutions of a Bernoulli-Euler Beam in which the approximate solutions are obtained by the use of Rayleigh-Ritz method based on the conservation of energy, and Galerkin method which is one of the weighted residual methods. In this study the vibration of a ship-hull is modeled as a beam of which thickness quadratically varies and ends are free. The solutions for frequencies are evaluated up to the fifth mode both for symmetric and antisymmetric shape functions. The approximate functions are chosen as power series and satisfy the homogenous boundary conditions. It is shown that if the same shape functions are used by two different methods the eigensolutions are identical. The Galerkin method is comparatively simple and easy to apply for the approximate solutions with a view to mathematical complexity and kinematic energy considerations of the Rayleigh-Ritz method (Söylemez *et al*., 2005).

2. REVIEW OF BEAM THEORIES

2.1. Equations of Transverse Vibration of Beams

The fundamental feature of all deformable systems is that their dynamic behavior is described by partial differential equations. To analyze the vibration of deformable systems the first thing to determine is the dynamic characteristics of the system such as eigenvalues and eigenfunctions.

In this chapter existing beam models are reviewed and the underlying assumptions used in the beam models are compared. There are several mathematical beam models which take into account different effects. These models are classified according to various theories of dynamic behaviors of beams. Beam models under consideration are transverse beam models which take into account displacement only in transverse direction of the beam and yield transverse displacement as a solution. The different beam theories which will be discussed are Euler-Bernoulli, Rayleigh, Shear, Timoshenko, Bress, Volterra and Ambartsumyan Theories, in which different governing equations arise because different forces are taken into account and different assumptions are made. As the slenderness ratio of the beam increases, the difference among Euler-Bernoulli, Shear, Rayleigh and Timoshenko theories magnifies (Han, 2001).

In considering the vibration of elastic beams it is assumed that the material of the beam is homogenous, isotropic and it follows Hook's law so that the material is linear elastic. The Poisson effect is also neglected. Elastic bodies are systems having infinite number of particles between which elastic forces are acting. Therefore there is infinite number of degrees of freedom and we can say that elastic bodies can have infinite number of natural modes of vibration theoretically.

Among various beam models Euler-Bernoulli beam model, which dates back to the eighteenth century, is the elementary beam model and it is the most commonly used one because of its simplicity and basic structure.

2.2. Euler-Bernoulli Beam Theory

The Euler-Bernoulli beam model includes the strain energy due to bending and kinetic energy due to lateral displacement. In this theory the inertia force due to transverse translation is taken into account and the inertia forces due to shear deflection and rotation are neglected. It is assumed that the rotation of the differential element is small compared to its translation and the angular distortion due to shear can be neglected. The cross-sections remain plane and orthogonal to the mid-plane of the beam after deformation.

To derive the equation of motion we use Newton's second law of motion. The equation of motion can also be obtained using Hamilton's variational principle (Meirovich, 2001). Considering the beam in Figure 2.1 and beam element in Figure 2.2 the equilibrium equation for the forces in the vertical direction of the beam in bending is given by:

$$\left(Q(x,t) + \frac{\partial Q(x,t)}{\partial x} dx \right) - Q(x,t) + f(x,t)dx = m(x)dx \frac{\partial^2 y(x,t)}{dt^2}, \quad 0 < x < L \qquad (2.1)$$

Furthermore the equilibrium equation for the moments for the beam element in bending assuming the mass moment of inertia of the element and the angular acceleration are negligible is given by:

$$\left(M(x,t) + \frac{\partial M(x,t)}{\partial x} dx \right) - M(x,t) + \left(Q(x,t) + \frac{\partial Q(x,t)}{\partial x} dx \right) dx + f(x,t)dx \frac{dx}{2} = 0, \quad 0 < x < L \qquad (2.2)$$

where *M(x,t)* denotes the bending moment, *Q(x,t)* is the shear force acting on the beam element, *f(x,t)* is the transverse force density, *m(x)* is the unit mass per unit length, and *I(x)* is the moment of inertia along the length of the beam. *E* is the modulus of elasticity.

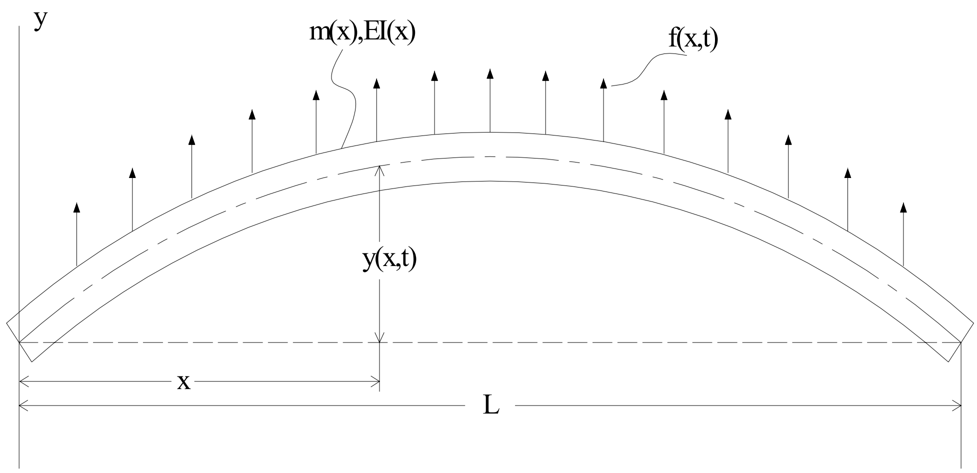

Figure 2.1. Beam in bending

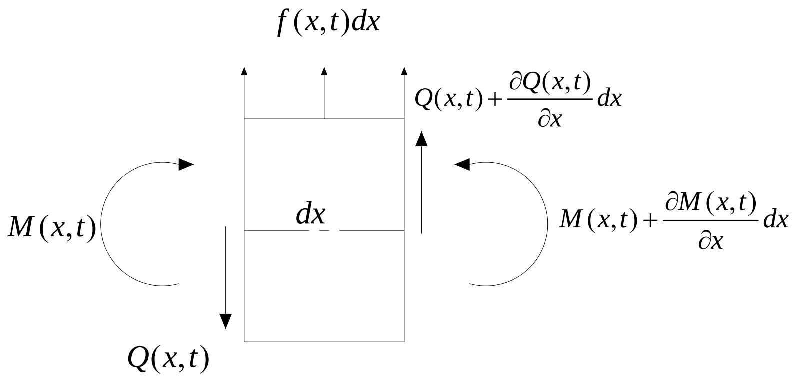

Figure 2.2. Free-body diagram for a beam element in bending

By ignoring the second-order terms in *dx* in Equation (2.2), inserting it in Equation (2.1), dividing Equation (2.1) by *dx* and canceling all appropriate terms we end up with the following equation:

$$-\frac{\partial^2 M(x,t)}{\partial x^2} + f(x,t) = m(x)\frac{\partial^2 y(x,t)}{dt^2}, \quad 0 < x < L \tag{2.3}$$

Equation (2.3) relates the bending moment *M (x,t)* and transverse force density *f(x,t)* to the bending deflection *y (x,t)*. We can write *M (x,t)* in terms of *y (x,t)* as follows:

$$M(x,t) = EI(x)\frac{\partial^2 y(x,t)}{\partial x^2} \tag{2.4}$$

By inserting Equation (2.4) in Equation (2.3) we can obtain a relationship between the displacement and transverse force density alone which is given by:

$$-\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right) + f(x,t) = m(x)\frac{\partial^2 y(x,t)}{\partial t^2}, \quad 0 < x < L \tag{2.5}$$

Equation (2.5) is a fourth order partial differential equation governing the bending vibrations of a beam. The equations of motion with boundary conditions form a boundary value problem which can be solved by the method of separation of variables. To complete the boundary value problem, we have to specify two boundary conditions for each end of the beam. These boundary conditions can be classified as essential and natural ones. For a beam governed by a fourth order differential equation, essential boundary conditions consist of equations containing the displacement function and its first derivative, whereas the natural boundary conditions consist of its second and third derivative.

For fixed ends the deflection and slope of the deflection curve must be zero so that the following boundary conditions both of which are essential must be satisfied:

$$y(x,t) = 0, \quad x = 0, L \tag{2.6a}$$

$$\frac{\partial y}{\partial x} = 0 \quad x = 0, L \tag{2.6b}$$

For pinned ends the displacements and the bending moments at the ends of the beam must be zero. The boundary conditions are given by:

$$y(x,t) = 0 \quad x = 0, L \tag{2.7a}$$

$$M(x,t) = EI(x)\frac{\partial^2 y(x,t)}{\partial x^2} = 0 \quad x = 0, L \tag{2.7b}$$

The first boundary condition is essential and the second is natural.

Finally for free ends the bending moment and shearing force at the ends of the beam must be zero. Both of the boundary conditions are natural. They are given by:

$$M(x,t) = EI(x)\frac{\partial^2 y(x,t)}{\partial x^2} = 0 \quad x = 0, L \tag{2.8a}$$

$$Q(x,t) = -\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2} \right) = 0 \quad x = 0, L \tag{2.8b}$$

The constrained equations satisfying these classical boundary conditions are homogenous. There are several other types of ends of beams for which various boundary conditions can be written. For beams with sliding ends, or ends restrained by translational or rotational springs, or beams resting on elastic foundations other boundary equations arise.

In the absence of external excitations the beam is vibrating freely and the term corresponding to the transverse force density *f(x,t)* in the governing partial differential equation is set to zero, so that Equation (2.5) reduces to:

$$m(x)\frac{\partial^2 y(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2} \right) = 0, \quad 0 < x < L \tag{2.9}$$

By taking the terms *m(x)* and *EI(x)* as constants for uniform beams Equation (2.9) is reduced to:

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \frac{m}{EI}\frac{\partial^2 y(x,t)}{\partial t^2} = 0, \quad 0 < x < L \tag{2.10}$$

The next step is to apply separation of variables to the term *y(x,t)* which is separable in the spatial variable *x* and time *t* because it is assumed that the beam is doing a synchronous motion during vibration which means that the shape or profile of the beam does not change only the amplitude of the profile changes with time during vibration. Application of separation of variables and boundary conditions will be discussed in detail in Chapter 3.

Equation (2.10) can be rewritten as:

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \frac{1}{D_0{}^4}\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{2.11a}$$

$$D_0{}^4 = \frac{EI}{m} \tag{2.11b}$$

If we assume that the displacement changes as $y = e^{ik_w x - i\omega_n t}$ where k_w is constant ave ($k_w = 2\pi / wavelength$) and ω_n is the natural frequency of vibration, then the dispersive relationship which establishes the relationship between k_w and ω_n may be given as follows (Karnovsky, 2004):

$$k_w{}^4 = \frac{\omega_n{}^2}{D_0{}^4} = k_0{}^4 \tag{2.12}$$

k_0 is the wave number for Euler-Bernoulli rod.

The degree of accuracy of a theory may be evaluated by its dispersive curve $k_w - \omega$ and its comparison with the exact dispersive curve. The corresponding curve to a dispersive equation is also referred to as propagation constant-frequency curve.

The elementary Euler-Bernoulli beam theory is valid if the ratio between the length of the beam and its depth is relatively large. (Meirovitch, 2001). This theory, however, tends to slightly overestimate the natural frequencies (Han, 2001).

2.3. Rayleigh Theory

This theory includes the effect of rotary inertia of the cross-section of the beam and provides an improvement to Euler-Bernoulli beam theory (Rayleigh, 1877). It is assumed that the cross-sections of the beam remain plane and orthogonal to the neutral axis (mid-plane) of the beam.

The governing differential equation of transverse vibration according to Rayleigh theory is given as follows:

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \frac{1}{D_0{}^4}\frac{\partial^2 y(x,t)}{\partial t^2} - \frac{1}{c_b{}^2}\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2} = 0 \tag{2.13a}$$

$$c_b{}^2 = \frac{E}{\rho} \tag{2.13b}$$

c_b is the longitudinal wave velocity in the Euler-Bernoulli rod. The third term of the left side of Equation (2.13b) represents the effect of rotary inertia. The dispersive equation is given as follows (Karnovsky, 2004):

$$2k_{w1/2}{}^2 = k_b{}^2 \pm \sqrt{k_b{}^2 + 4k_0{}^4} \tag{2.14a}$$

$$k_b{}^2 = \frac{\omega_n{}^2}{c_b{}^2} \tag{2.14b}$$

k_b is the longitudinal wave number and k_0 is the wave number for Euler-Bernoulli beam as mentioned before.

2.4. Euler-Bernoulli Modified Theory or Shear Model

In Euler-Bernoulli modified theory which is also called as shear model, the effect of shear distortion is taken into account but the effect of rotational inertia is neglected. In this case the cross-sections of the beam element remain planar but not orthogonal to the neutral axis because of the shear deformation (Figure 2.3).

The governing differential equation for transverse vibration of the beam is given by:

$$\frac{\partial^4 y(x,t)}{\partial x^4}+\frac{1}{{D_0}^4}\frac{\partial^2 y(x,t)}{\partial t^2}-\frac{1}{{c_t}^2}\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2}=0 \qquad (2.15a)$$

$${c_t}^2=\frac{G}{\rho} \qquad (2.15b)$$

G is the shear modulus which replaces the elasticity modulus E in the differential equation of Rayleigh Theory. c_t is the velocity of the shear waves in the thin rod. The dispersive relationship is given by:

$$2{k_{w1/2}}^2={k_t}^2 \pm \sqrt{{k_t}^2+4{k_0}^4} \qquad (2.16a)$$

$${k_t}^2=\frac{{\omega_n}^2}{{c_t}^2} \qquad (2.16b)$$

2.5. Bress and Volterra Theories

In both of these theories the effect of shear deformation, rotational inertia and their combined effects are added to the elementary Euler-Bernoulli beam theory. The differential equation of transverse vibration of beams in Bress Theory (Bress, 1859) is given by:

$$\frac{\partial^4 y(x,t)}{\partial x^4}+\frac{1}{D_0{}^4}\frac{\partial^2 y(x,t)}{\partial t^2}-\left(\frac{1}{c_b{}^2}+\frac{1}{c_t{}^2}\right)\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2}+\frac{1}{c_b{}^2 c_t{}^2}\frac{\partial^4 y(x,t)}{\partial t^4}=0 \qquad (2.17)$$

The differential equation of transverse vibration of beams in Volterra Theory (Volterra, 1955) is given by:

$$\frac{\partial^4 y(x,t)}{\partial x^4}+\frac{1-\nu^2}{D_0{}^4}\frac{\partial^2 y(x,t)}{\partial t^2}-\left(\frac{1}{c_s{}^2}+\frac{1}{c_t{}^2}\right)\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2}+\frac{1}{c_s{}^2 c_t{}^2}\frac{\partial^4 y(x,t)}{\partial t^4}=0 \qquad (2.18)$$

$$c_s{}^2 = E_1 / \rho \qquad (2.19a)$$

$$E_1 = E/(1-\nu^2) \qquad (2.19b)$$

E_1 is the longitudinal modulus of elasticity. The third and fourth terms in Equations (2.17) and (2.18) correspond to rotary inertia and shear deformation respectively. The last term represents their combined effect. Comparing Equations (2.17) and (2.18) it can be seen that the only difference between the two theories is that the modulus of elasticty according to Bress model is $(1-\nu^2)$ times greater than that given by Volterra model. This is because transverse compressive and tensile stresses are not allowed in Volterra Theory. In both of the theories the cross-sections are assumed to remain planar after deformation.

2.6. Ambartsumyan Theory

According to this theory the cross-section of the beam element does not remain planar after deformation so that the distortion of the cross-section is allowed (Ambartsumyan, 1956). This theory is a modified Volterra Theory, in which an additional constant is introduced, which depends on shear stress distribution along the cross-section and denoted by *a* (Figure 2.3).

The differential equation of transverse vibration is given by:

$$\frac{\partial^4 y(x,t)}{\partial x^4}+\frac{1-\nu^2}{D_0{}^4}\frac{\partial^2 y(x,t)}{\partial t^2}-\left(\frac{1}{c_s{}^2}+\frac{1}{ac_t^2}\right)\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2}+\frac{1}{ac_s{}^2 c_t^2}\frac{\partial^4 y(x,t)}{\partial t^4}=0 \qquad (2.20)$$

2.7. Timoshenko Theory

In elementary Euler-Bernoulli beam model the cross-sectional dimensions of the beam are assumed to be small compared to its length so that the rotary inertia and shear deflection are neglected. However the effect of cross-sectional dimensions on the frequency of vibration becomes important when studying vibrations of higher frequencies if the beam is subdivided into shorter portions. Therefore in Timoshenko Theory the effect of shearing force, rotary inertia and their combined effects are added to the Euler-Bernoulli beam model (Timoshenko, 1921).

Because during vibration the beam element does not only undergo translational motion but also rotates the effect of rotation is taken into account by modifying the corresponding terms. The moment exerted by inertia forces about the axis through the center of gravity of the beam element in Figure 2.2 is given by:

$$I\rho\frac{\partial^3 y}{\partial x \partial t^2}dx \qquad (2.21)$$

so that the first derivative of bending moment is given by:

$$\frac{\partial M(x,t)}{\partial x}=-\left(Q-I\rho\frac{\partial^3 y}{\partial x \partial t^2}dx\right) \qquad (2.22)$$

If we differentiate Equation (2.4) twice with respect to *x* we obtain:

$$\frac{\partial^2 M(x,t)}{\partial x^2}=EI\frac{\partial^4 y(x,t)}{\partial x^4} \qquad (2.23)$$

By inserting Equation (2.22) into Equation (2.23) we obtain:

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} = -m\frac{\partial^2 y(x,t)}{\partial t^2} + I\rho\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2} \tag{2.24}$$

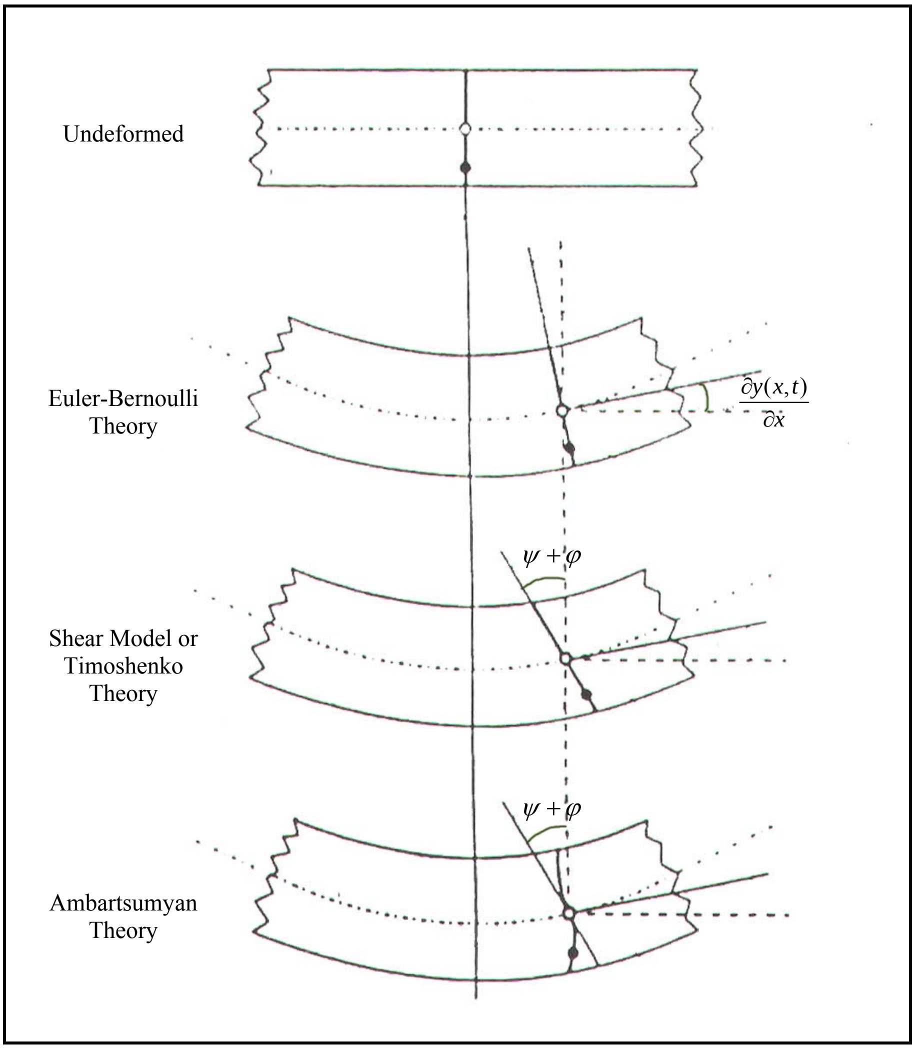

Figure 2.3. Deformation of a typical transverse normal line in various beam theories (Wang *et. al.* 2001)

Because the slope of the deflection curve depends on the rotation of the cross-section as well as on the shear, we will obtain more accurate results for vibration if we take in addition the deflection due to shear into account . If the angle of rotation of the beam when the shear force is neglected, is denoted by ψ and the shear angle of the cross-section of the beam is φ then the total angle is given by:

$$\frac{\partial y(x,t)}{\partial x} = \psi + \varphi \tag{2.25}$$

The bending moment and shear force are given by:

$$M(x,t) = EI(x)\frac{\partial \psi(x,t)}{\partial x} \tag{2.26}$$

$$Q = k'\varphi AG = k'\left(\frac{\partial y(x,t)}{\partial x} - \psi(x,t)\right)AG \tag{2.27}$$

In these equations k' represents a numerical factor which depends on the shape of the cross-section and also called as shear coefficient. For a rectangular cross-section it is taken as $k' = 2/3$ (Timoshenko, 1937). The final form of the governing differential equation of Timoshenko Theory for translational vibration of the beam is given by:

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y(x,t)}{\partial t^2} - \left(\rho I + \frac{EI\rho}{k'G}\right)\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2} + \rho I\frac{\rho}{k'G}\frac{\partial^4 y(x,t)}{\partial t^4} = 0 \tag{2.28}$$

In another formulation Equation (2.28) can be rewritten as:

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \frac{1}{D_0^{\,4}}\frac{\partial^2 y(x,t)}{\partial t^2} - \left(\frac{1}{c_b^{\,2}} + \frac{1}{k'c_t^{\,2}}\right)\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2} + \frac{1}{k'c_b^{\,2}c_t^{\,2}}\frac{\partial^4 y(x,t)}{\partial t^4} = 0 \tag{2.29}$$

The fundamental difference between Bress and Volterra Theories on one hand and the Timoshenko Theory on the other is that in Timoshenko Theory the correction factor is introduced in the initial equations whereas in other theories it appears as a result of shear and rotary effects. In Timoshenko Theory constant state of transverse shear strain and thus constant shear stress with respect to the thickness coordinate is included. Therefore the Timoshenko Beam Theory requires shear correction factor k' to compensate the error due to this constant shear stress assumption. The dispersive relationship is given as follows:

$$2k_{w1/2}^{\,2} = k_b^{\,2} + \frac{k_t^{\,2}}{k'} \pm \sqrt{\left(k_b^{\,2} - \frac{k_t^{\,2}}{k'}\right)^2 + 4k_0^{\,4}} \tag{2.30}$$

Timoshenko Model describes the vibration of short beams or high modes of a thin beam with high precision (Karnovsky, 2004).

There are several other theories about the dynamical behavior of beams. Those which are mentioned above are the most commonly encountered ones. Further details about the mentioned theories and other beam theories can be found in Karnovski (2004); Wang *et al*. (2001); Dökmeci (1972).

3. CONVERGENCE OF APPROXIMATE SOLUTIONS TO THE EXACT SOLUTIONS

3.1. Exact Solution of a Uniform Euler-Bernoulli Beam

Before weighted residual methods with several trial functions are explained in detail we want to compare in this chapter the results obtained with Galerkin method using power series as trial function with the exact solution of a uniform beam and investigate the convergence of approximate values to the exact solutions for natural frequencies. The choice of trial function and optimization criterion is very important for the best representation of the physical phenomena mathematically and affects the results significantly. When trying to find the eigen-solutions of the boundary value problem, choosing the approximating function is the crucial point. After determining the closeness of approximate values to the exact ones we will investigate in the next chapter other trials with several combinations of different weighted residual methods with different trial functions to find natural frequencies of a non-uniform beam with approximation.

The governing differential equation for free vibration of a uniform beam based on Euler-Bernoulli beam theory is given by:

$$\frac{d^4u(x)}{dx^4}-\left(\frac{\rho A_0\omega_n{}^2}{EI_0}\right)u(x)=0 \tag{3.1}$$

The general solution of Equation (4.1) can be written as:

$$u(x)=C_1\sin kx+C_2\cos kx+C_3\sinh kx+C_4\cosh kx \tag{3.2}$$

Where the coefficients C_1, C_2, C_3 and C_4 are to be determined for every particular case and k is given by:

$$k^4=\frac{\rho A_0\omega_n{}^2}{EI_0} \tag{3.3}$$

In considering particular cases of vibration it is useful to present the general solution (4.2) in the following form (Timoshenko,1937):

$$u(x) = C_1\left(\cos kx + \cosh kx\right) + C_2\left(\cos kx - \cosh kx\right) + C_3\left(\sin kx + \sinh kx\right) + C_4\left(\sin kx - \sinh kx\right) \tag{3.4}$$

For a beam with both ends free, the end conditions are:

$$\left(\frac{d^2u(x)}{dx^2}\right)_{x=0} = 0 \tag{3.5a}$$

$$\left(\frac{d^2u(x)}{dx^2}\right)_{x=L} = 0 \tag{3.5b}$$

$$\left(\frac{d^3u(x)}{dx^3}\right)_{x=0} = 0 \tag{3.5c}$$

$$\left(\frac{d^3u(x)}{dx^3}\right)_{x=L} = 0 \tag{3.5d}$$

In order to satisfy the conditions (3.5a) and (3.5c) we have to take in the general solution (3.4):

$$C_2 = C_4 = 0 \tag{3.6}$$

so that Equation (3.4) reduces to:

$$u(x) = C_1\left(\cos kx + \cosh kx\right) + C_3\left(\sin kx + \sinh kx\right) \tag{3.7}$$

To satisfy the conditions (3.5b) and (3.5d) the following relationships must hold:

$$C_1\left(-\cos kL+\cosh kL\right)+C_3\left(-\sin kL+\sinh kL\right)=0 \tag{3.8a}$$

$$C_1\left(\sin kL+\sinh kL\right)+C_3\left(-\cos kL+\cosh kL\right)=0 \tag{3.8b}$$

A solution for the coefficients C_1 and C_3 other than trivial solution can only be obtained in the case when the determinant of the Equations (3.8a) and (3.8b) is equal to zero. In this manner the following equation is obtained:

$$\left(-\cos kL+\cosh kL\right)^2-\left(\sinh^2 kL-\sin^2 kL\right)=0 \tag{3.9}$$

Using following properties:

$$\cosh^2 kL-\sinh^2 kL=1 \tag{3.10a}$$

$$\cos^2 kL+\sin^2 kL=1 \tag{3.10b}$$

Equation (3.9) is reduced to:

$$\cos kL\cosh kL=1 \tag{3.11}$$

The first six consecutive roots of the frequency Equation (3.11) are given in the table below (Timoshenko, 1937):

Table 3.1. First six roots of frequency equation of a free-free beam

k_0L	k_1L	k_2L	k_3L	k_4L	k_5L
0	4.730	7.853	10.996	14.137	17.279

We will use the following values for moment of inertia, modulus of elasticity, mass per unit length and length of the uniform beam which will also be used by solving the non-uniform beam in the following chapter for the purpose of comparison:

$$L = 100m; \;\; I_0 = 20m^4; \;\; E = 20x10^9 kgf / m^2; \;\; A_0 = 7m^2; \;\; \rho = 1000kg / m^3 \qquad (3.12)$$

If we substitute the consecutive roots of Equation (3.11) into Equations (3.8a) and (3.8b) we can calculate the ratios C_1 / C_3 for the corresponding modes of vibration and the shape of the deflection curve during vibration can be obtained from Equation (3.7). The mode shape function for the free-free beam will be as follows:

$$\{u(x)\}_{nth \;\; \text{mode}} = C\left\{r_n\left(\cos(k_n x) + \cosh(k_n x)\right) + \sin(k_n x) + \sinh(k_n x)\right\} \qquad (3.13a)$$

$$0 \leq x \leq L$$

where;

$$r_n = \left(C_1 / C_3\right)_n = \frac{\sin k_n L - \sinh k_n L}{\cosh k_n L - \cos k_n L} \qquad (3.13b)$$

The coefficient C specifies the amplitude of vibration and remains undetermined with the implication that only the mode shapes can be determined uniquely.

The first three natural modes are plotted in Figures 3.1 , 3.2 and 3.3 respectively.

The case where kL equals to zero corresponds to rigid body motion of the free-free beam. From Equation (3.3) we obtain for natural frequency:

$$\omega_n = k^2 \sqrt{\frac{EI_0}{\rho A_0}} \qquad (3.14)$$

If we let

$$\lambda_i = k_i L \qquad (3.15)$$

we can rewrite Equation (3.14) as follows:

$$\left(\omega_n\right)_i = \lambda_i^2 \sqrt{\frac{EI_0}{\rho A_0 L^4}} \qquad i = 1,2,3,\ldots n \tag{3.16}$$

If we insert the values given in Equation (3.12) and the roots given in Table (3.1) into Equation (3.16) we obtain the values for the natural frequencies of the uniform free-free beam which are given in the table below.

Table 3.2. The first six natural frequencies of the uniform free-free beam

ω_0	ω_1	ω_2	ω_3	ω_4	ω_5
0	16.91	46.86	91.89	151.89	226.91

3.2. Approximate Solution of the Uniform Euler-Bernoulli Beam

To solve the same problem with approximation we will use Galerkin method and power series as trial function therefore we substitute the term $u(x)$ representing the displacement in Equation (3.1) by the following approximating function which satisfies the boundary conditions:

$$P_N(x) = \left\{1 - \left(\frac{x}{L}\right)^2\right\}^4 \left(c_1 + c_2 x + c_3 x^2 + \ldots c_N x^{N-1}\right) \tag{3.17}$$

The residual equation for Galerkin method is given by:

$$\det \begin{bmatrix} \int_{-L}^{L} \varepsilon_1(x;c)\phi_1(x) & \cdots & \int_{-L}^{L} \varepsilon_1(x;c)\phi_N(x) \\ \vdots & \ddots & \vdots \\ \int_{-L}^{L} \varepsilon_N(x;c)\phi_1(x) & \cdots & \int_{-L}^{L} \varepsilon_N(x;c)\phi_N(x) \end{bmatrix} = 0 \tag{3.18}$$

where for the uniform beam the *Nth* residual is given by:

$$\varepsilon_N(x) = \frac{d^4 P_N(x)}{dx^4} - \left(\frac{\rho A_0 \omega_n^{\ 2}}{EI_0}\right) P_N(x) \tag{3.19a}$$

and

$$\phi_n = x^{n-1} \tag{3.19b}$$

The Index N indicates the number of terms taken in the power series and at the same time also the number of modes of vibration. The numerical values obtained for natural frequencies are given in the tables below.

Table 3.3. The first five natural frequencies of the uniform free-free beam obtained with Galerkin method and trial with power series using series length of one to eight

	Series Length							
Mode # N	1	2	3	4	5	6	7	8
1	24.48	24.48	20.79	20.79	19.33	19.33	18.58	18.58
2		61.62	61.62	55.25	55.25	52.24	52.24	50.61
3			112.79	112.79	105.29	105.29	100.80	100.80
4				177.25	177.25	170.35	170.35	164.46
5					255.35	255.35	250.39	250.39

Table 3.4. The first five natural frequencies of the uniform free-free beam obtained with Galerkin method and trial with power series using series length ten to thirty

	Series Length				
Mode # N	10	15	20	25	30
1	18.14	17.52	17.34	17.19	17.13
2	49.62	48.50	47.72	47.44	47.19
3	98.24	94.75	93.74	92.91	92.58
4	161.05	157.27	154.67	153.76	152.94
5	243.02	234.10	231.54	229.45	228.63

3.3. Convergence of the Approximate Solutions to the Exact Solutions

If we compare the approximate solutions for natural frequencies of the uniform Euler-Bernoulli beam obtained using Galerkin method which are given in Tables (3.3) and (3.4) with the exact values given in Table (3.2) we can easily see that the approximate values for each mode of vibration converge to the exact values by addition of successive terms to the trial series.

As the number of terms taken for approximating function increases, the rate of convergence decreases. When compared between series length of one and eight the convergence rate for the first mode of vibration is 31.75% whereas it is 5.90% between series length of ten and thirty. The error between the exact solutions and approximate solutions using series length of thirty is given in Table (3.5). It is calculated as follows:

$$\%\mathrm{E}_r = \left| \frac{\omega_{exact} - \omega_{appr}}{\omega_{exact}} x100 \right| \tag{3.20}$$

Table 3.5. Error for the first five modes with series length of thirty

Mode Number	1	2	3	4	5
Error in %	1.31	0.70	0.75	0.69	0.76

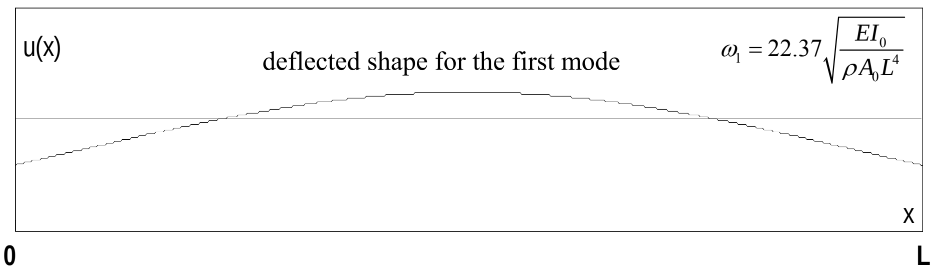

Figure 3.1. Deflected shape for the first natural mode of a free-free beam

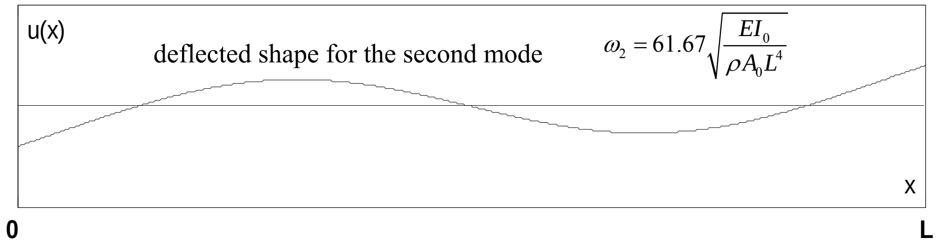

Figure 3.2. Deflected shape for the second natural mode of a free-free beam

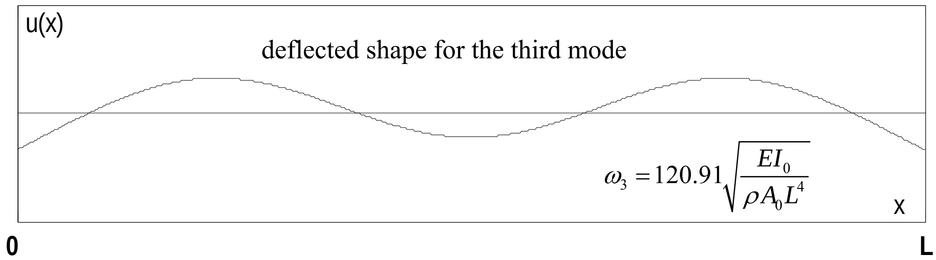

Figure 3.3. Deflected shape for the third natural mode of a free-free beam

The variation of the approximate solutions for the first five natural frequencies of the uniform free-free ended beam with increasing series length and convergence of them to the exact solution are plotted in the following figures.

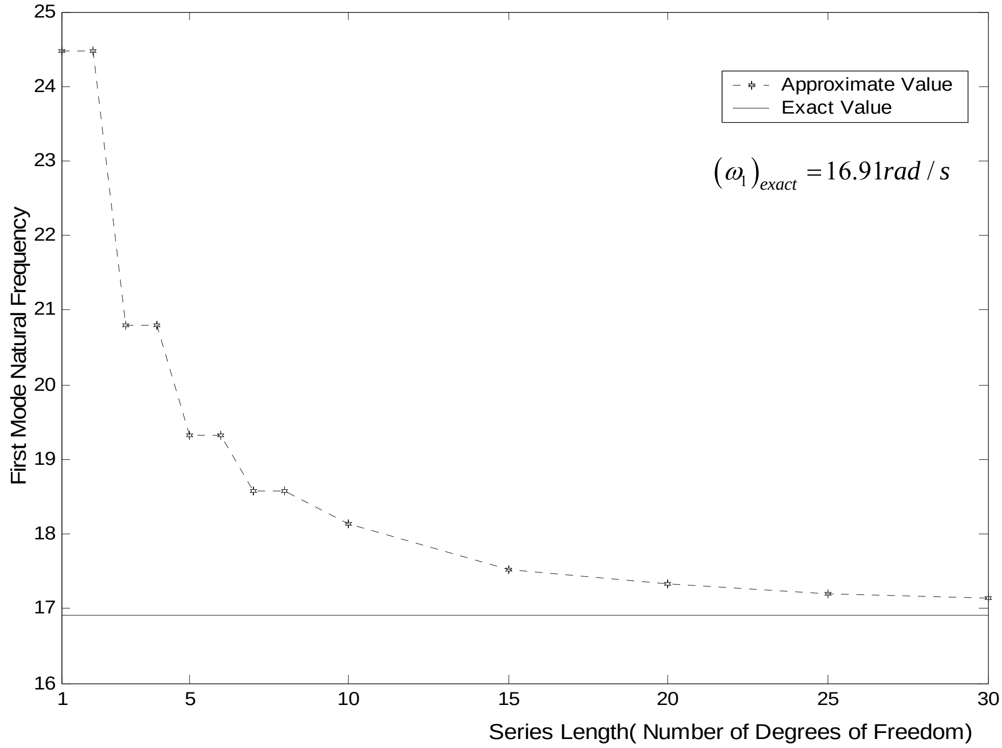

Figure 3.4. Exact and approximate values for the first mode natural frequency of the uniform free-free beam

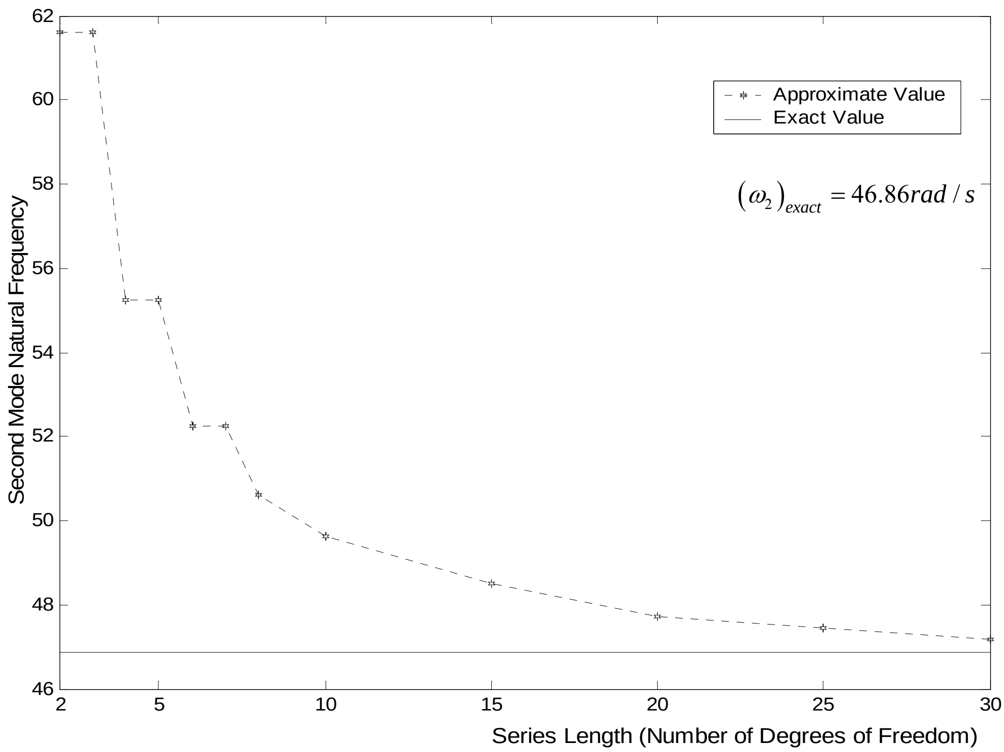

Figure 3.5. Exact and approximate values for the second mode natural frequency of the uniform free-free beam

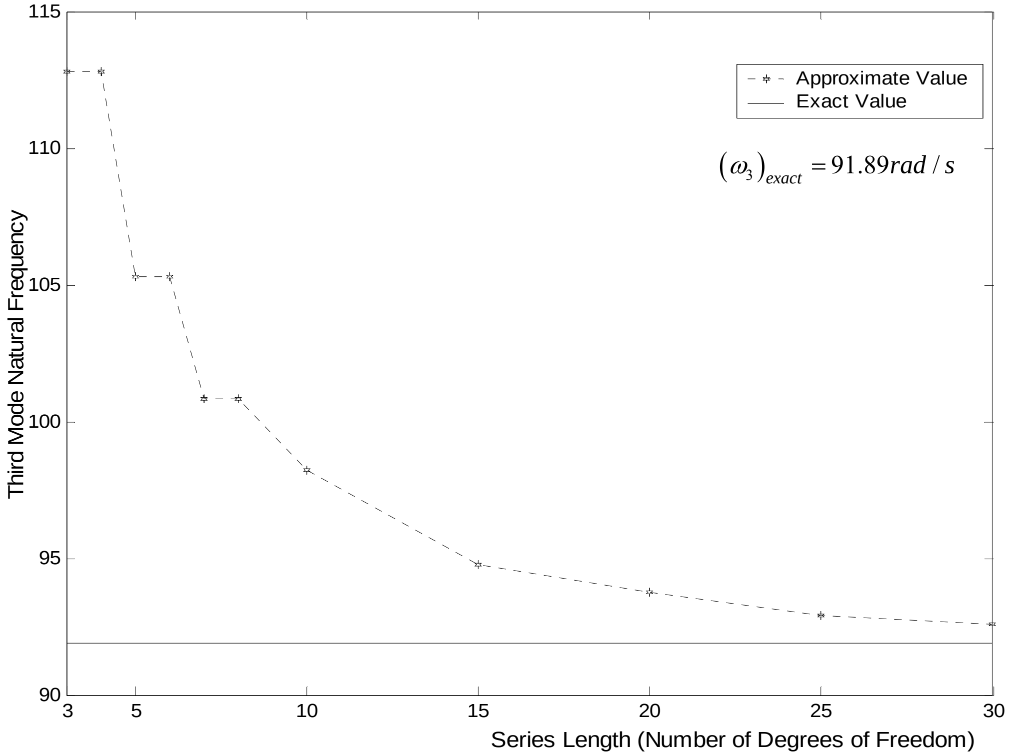

Figure 3.6. Exact and approximate values for the third mode natural frequency of the uniform free-free beam

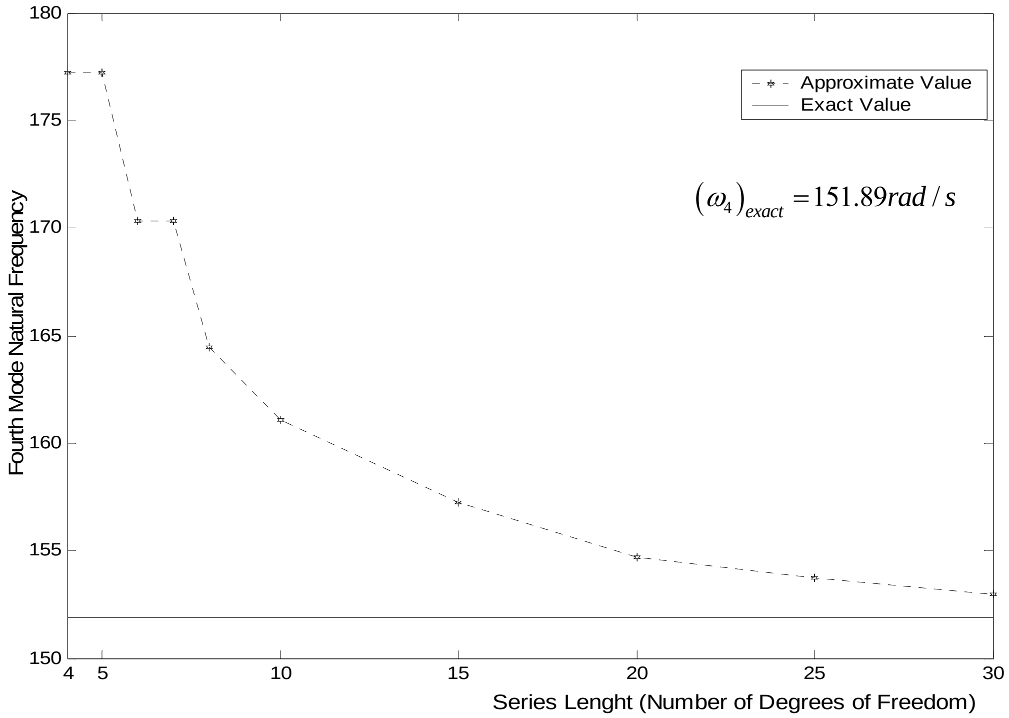

Figure 3.7. Exact and approximate values for the fourth mode natural frequency of the uniform free-free beam

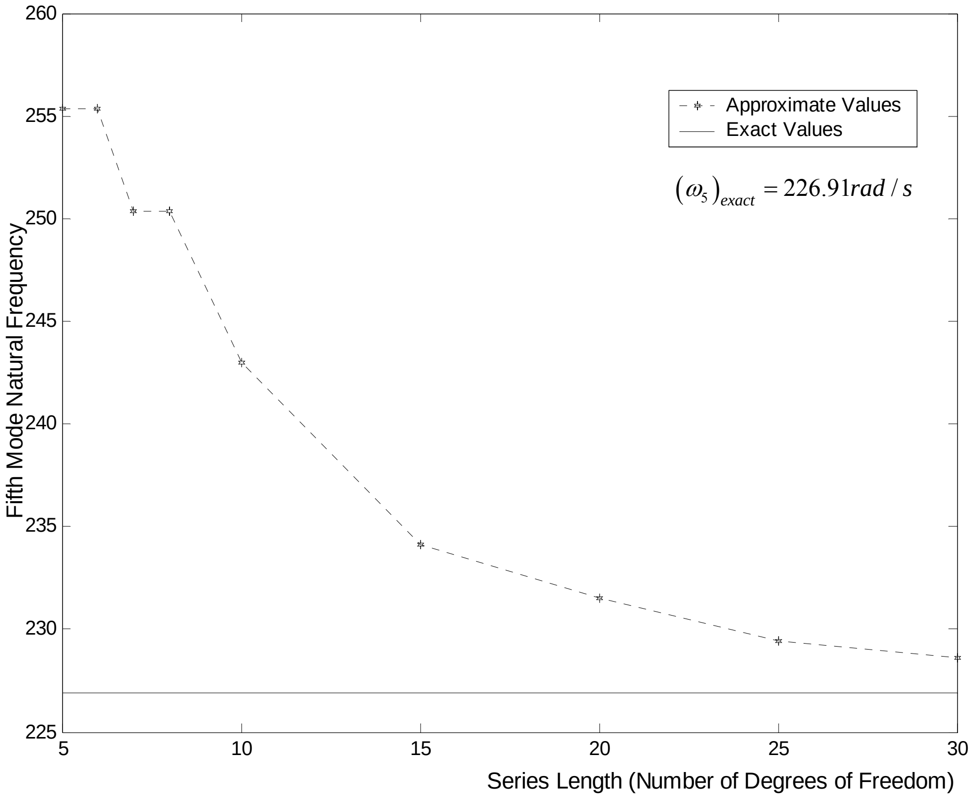

Figure 3.8. Exact and approximate values for the fifth mode natural frequency of the uniform free-free beam

4. WEIGHTED RESIDUAL METHODS AND TRIAL FUNCTIONS

4.1. Problem Definition and Mathematical Modeling

In this chapter we seek the approximate solution for the frequencies of a non-uniform Euler-Bernoulli beam of which ends are free. As mentioned before the Euler-Bernoulli beam theory considers the deflection of the beam due to bending only and neglects the shear deflection as well as rotary inertia. The Euler-Bernoulli beam model is chosen because it is most commonly used and has a basic and simple structure.

The beam under consideration has a span of *2L*, non-uniform cross-section and second moment of inertia. Both ends of the beam are free. The cross-sectional area *A(x)* and second moment of inertia *I(x)* are depicted in Figure 4.1 and in Figure 4.2 respectively and given as follows (Timoshenko, 1937):

$$A(x) = A_0(1 - cx^2) \tag{4.1a}$$

$$I(x) = I_0(1 - bx^2) \tag{4.1b}$$

A_0 and I_0 are the cross-sectional area and second moment of inertia at the mid-section of the beam where $x=0$ and c and b are constants. The identical example from Timoshenko (1937) which is also taken by Söylemez *et. al.* (2005) is used for the purpose of comparison.

The governing partial differential equation for free vibration of an Euler-Beam is given, as mentioned before, by the following differential equation:

$$m(x)\frac{\partial^2 y(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right) = 0, \quad 0 < x < L \tag{4.2}$$

If we replace the term *m(x)* representing the mass per unit length of the beam by $\rho A(x)$ the above equation takes the following form:

$$\rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right)=0 \tag{4.3}$$

In Equation (4.3) *y(x,t)* represents the transverse displacement, ρ the mass density of the beam material, *x* the cross-section abscissa and *t* time. *A(x)* and *I(x)* are two important terms which affect the magnitudes of the sectional mass $\rho A(x)$ and sectional stiffness $EI(x)$ in the beam equation and therefore the natural frequencies of the beam.

To solve Equation (4.3) we apply separation of variables to *y(x,t)* which is a function of *x* and *t*. By separation of variables we assume that *y(x,t)* is the product of two independent functions each of which depends only on *x* and *t*. We assume the solution in the following form:

$$y(x,t)=u(x)\gamma(t) \tag{4.4}$$

u(x) is the amplitude of the deflection and is only space dependent whereas $\gamma(t)$ represents the time dependent function. The deflection during vibration is assumed by Söylemez *et. al*. (2005) as follows:

$$y(x,t)=u(x)e^{i\omega_n t} \tag{4.5}$$

where ω_n represents the natural frequencies of the free-free beam. The assumption of Timoshenko (1937) for the deflection during vibration of beams of variable cross-section and free ends is given by:

$$y(x,t)=u(x)\cos\omega_n t \tag{4.6}$$

Equation (4.6) is a part of Equation (4.5) in which only the real part of the trigonometric expansion of $e^{i\omega_n t}$ is taken into account which can be written as:

$$e^{i\omega_n t} = \cos \omega_n t + i \sin \omega_n t \tag{4.7}$$

By inserting Equations (4.1a), (4.1b) and (4.5) into Equation (4.3) we obtain:

$$(1-bx^2)\frac{d^4u(x)}{dx^4} - 4bx\frac{d^3u(x)}{dx^3} - 2b\frac{d^2u(x)}{dx^2} - \left(\frac{\rho A_0 \omega_n{}^2}{EI_0}\right)(1-cx^2)u(x) = 0 \tag{4.8}$$

The above equation is homogenous because we want to determine the natural frequencies so that we assume there is no excitation. The solution function *u(x)* of this homogenous equation represents the amplitude of the displacement and defines the mode shapes of vibration. The crucial point in solving this equation is that the coefficients depend explicitly on the coordinate x and therefore finding the exact solution of *u(x)* becomes mathematically complicated.

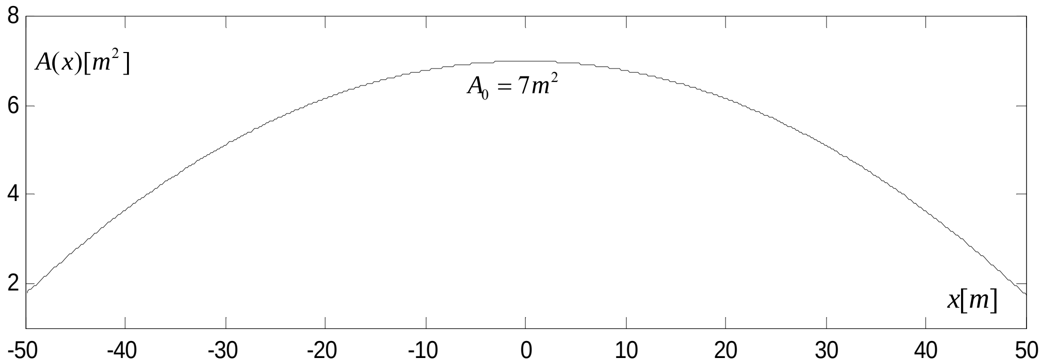

Figure 4.1.Variation of the cross-section of the beam along its length

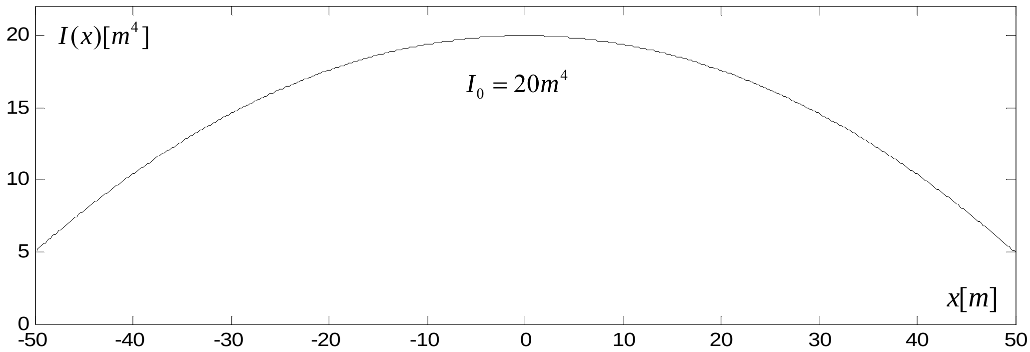

Figure 4.2. Variation of the second moment of inertia along the beam length

4.2. Approximate Solution of the Problem

We end up with a fourth order differential equation in which coefficients depend on the spatial variable x as given in Equation (4.8). To solve the governing differential equation describing the vibration of a free-free Euler-Bernoulli beam, weighted residual methods are used and several trial functions are considered to obtain the most efficient solution. The three principle operations of a trial solution procedure are (Burnett, 1988):

- Construction of a trial solution
- Application of an optimization criterion
- Estimation of the accuracy of the approximate solution

Application of weighted residual methods is the second step of trial solution procedure which is preceded by construction of a trial solution and it aims to implement an optimization criterion. Among several weighted residual methods Galerkin, subdomain and collocation methods are used to find the best solution for the problem which is as close as possible to the exact solution.

In order to solve the differential equation we think of an approximation function $Q(x)$ which can be represented in general as follows:

$$Q(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \tag{4.9}$$

where $\phi_n(x)$ is a given set of functions and c_n set of coefficients, which are regarded as the generalized coordinates of $Q_N(x)$ referred to the bases functions $\phi_n(x)$, of which each represent a degree of freedom of the vibration modes.

The main idea of approximation is to reduce the problem assuming a solution function with an infinite number of degrees of freedom into one of a finite number of degrees of freedom. Therefore we represent the displacement function of the beam $u(x)$ as a linear combination of given bases or trial functions where

$$Q_N(x) = \sum_{n=1}^{N} c_n \phi_n(x) \qquad n = 1, 2, ..., N \tag{4.10}$$

Equation (4.10) does not necessarily satisfy the boundary conditions itself. We have to modify the approximate solution by inserting the function $f_b(x)$ which satisfies the homogenous boundary conditions. For a free-free beam the bending moment and shear force at both ends of the beam are zero, therefore for bending moment

$$f_b''(L) = f_b''(-L) = 0 \tag{4.11}$$

and for shear force

$$f_b'''(L) = f_b'''(-L) = 0 \tag{4.12}$$

where $(')$ denotes derivative of the function with respect to spatial variable *x*. The equation satisfying both Equations (4.11) and (4.12) is given in the following form:

$$f_b(x) = \left[1 - \left(\frac{x}{L}\right)^2\right]^4 \tag{4.13}$$

The approximation function is modified as follows:

$$P_N(x) = f_b(x)Q_N(x) \tag{4.14}$$

We can choose for the function $Q_N(x)$ any form of trial functions such as power, trigonometric, or hyperbolic series to obtain the best approximation. To satisfy the homogenous boundary conditions we multiply it with the function $f_b(x)$. In this study the following series are chosen as trial functions which will be discussed later in detail:

- Power series
- Chebishev polynomials (Ordinary and modified)
- Legendre polynomials
- Fouries series
- Special trigonometric series

The second step of a trial solution method is to apply one of the optimization criteria which aims to determine the approximate solution as close as possible to the exact solution. Among several optimization criteria, weighted residuals methods seek to minimize an expression of error in the differential equation and not the unknown function. If we write the differential equation given in Equation (4.8) in a closed form as follows:

$$L\{u(x)\} = 0 \tag{4.15}$$

then the error function to be minimized is obtained by substitution of Equation (4.14) into Equation (3.15):

$$\varepsilon_N(x) \equiv L\left\{ f_b(x)\sum_{n=1}^{N} c_N \phi_N \right\} = 0 \tag{4.16}$$

At this point we are concerned with how close the residual is to zero, since this measures an error in satisfying the governing differential equation. Our primary interest is to minimize the error in the solution itself and minimizing the residual error tends to simultaneously minimize the solution error. This tendency is discussed in detail by Burnett

(1988). The methods we will apply to our problem using the residual given in Equation (3.16) are as follows:

- Galerkin Method
- Collocation Method
- Subdomain Method

For each criterion we will use several trial functions for the approximating function. Each different method of weighted residual criterion produces different sets of algebraic equations resulting in many different approximate solutions. Depending on the chosen trial functions and methods, the different solutions may be all close to each other and to the exact solution.

4.3. Galerkin Method

In this method, the weighted average of the residual $\varepsilon(x;c)$ over the entire domain should vanish. The weighting functions are the trial functions $\phi_n(x)$ associated with each parameter c_n. For an approximate solution with N parameters we obtain N residual equations as follows:

$$\int_{-L}^{L} \varepsilon(x;c)\phi_1(x)dx = 0 \tag{4.17a}$$

$$\int_{-L}^{L} \varepsilon(x;c)\phi_2(x)dx = 0 \tag{4.17b}$$

$$\vdots$$

$$\int_{-L}^{L} \varepsilon(x;c)\phi_N(x)dx = 0 \tag{4.17c}$$

We can write N independent equations for the chosen trial function with N parameters. Equation (4.17) which is the definition of the Galerkin method can be interpreted as making the residual orthogonal to members. In order to find the solution for natural frequencies ω_n we rewrite the above equation in matrix form. The residual function and weighting function have following dimensions in matrix form if an approximation function with N parameters is chosen:

$$\left[\varepsilon(x;c)\right]_{Nx1} \tag{4.18a}$$

$$\left[\phi_N(x)\right]_{1xN} \tag{4.18b}$$

The residual equation for Galerkin method is given by:

$$\det\begin{bmatrix} \int_{-L}^{L} \varepsilon_1(x;c)\phi_1(x)dx & \dots & \int_{-L}^{L} \varepsilon_1(x;c)\phi_N(x)dx \\ \vdots & \ddots & \vdots \\ \int_{-L}^{L} \varepsilon_N(x;c)\phi_1(x)dx & \cdots & \int_{-L}^{L} \varepsilon_N(x;c)\phi_N(x)dx \end{bmatrix} = 0 \tag{4.19}$$

By solving the above equation for natural frequencies we do not need to determine the coefficients c_n which are eliminated from the homogenous equation.

4.3.1. Galerkin Method with Power Series

To solve our problem we take first the trial function in the simple form of power series of which elements are composed of the basis functions and the generalized coordinates given by:

$$Q_N = c_1 + c_2 x + c_3 x^2 + \dots c_N x^{N-1} \tag{4.20}$$

where

$$\phi_n = x^{n-1} \tag{4.21}$$

Therefore the approximation function is given as follows:

$$P_N(x) = \left\{1 - \left(\frac{x}{L}\right)^2\right\}^4 \left(c_1 + c_2 x + c_3 x^2 + \dots c_N x^{N-1}\right) \tag{4.22}$$

We substitute the above equation for the term $u(x)$ in Equation (4.8) and solve for the natural frequencies by setting the determinant of the matrix given in Equation (4.19) equal to zero using MATLAB. The beam length and other constant terms in Equation (4.8) are taken as follows:

$$2L = 100m; \quad I_0 = 20m^4; \quad E = 20x10^9 kgf / m^2; \quad A_0 = 7m^2; \quad \rho = 1000kg / m^3$$

$$b = c = 0.0003 / m^2$$

In each approximation the number of elements is increased and convergence of the results is studied by the addition of every successive term to the trial series. Solution of the partial differential equation is sought for the best approximation function by increasing the number of terms of the polynomial function x^{n-1} from one to six. Results for the first six natural frequencies in rad./sec. are given in Table 4.1 below.

Table 4.1. Approximate values for the first natural frequencies obtained with Galerkin Method and trial with power series

	Series Length					
Mode # N	1	2	3	4	5	6
1	23.18	23.18	18.38	18.38	16.45	16.45
2		60.74	60.74	52.05	52.05	48.08
3			112.94	112.94	102.03	102.03
4				178.24	178.24	167.10
5					256.59	256.59
6						348.59

It can be seen from Table 4.1 that the number of calculated vibration modes is equal to the number of terms taken in the series. We also see immediately that convergence of the results is obtained by addition of every successive term to the polynomial. The results are consistent with those obtained by Söylemez *et. al*. (2005), who used both Galerkin and Rayleigh-Ritz methods.

4.3.2. Galerkin Method with Chebishev Polynomials

To solve the same problem we take this time as trial function Chebishev Polynomials that are used in many parts of numerical analysis and, more generally, in mathematics and physics. We have used both ordinary and modified Chebishev Polynomials for this study, of which properties are summarized briefly below.

Consider a function which is defined as:

$$T_n(x) = \cos(n\cos^{-1} x) \qquad -1 \le x \le 1 \tag{4.24}$$

which does not appear to be a polynomial but it is a polynomial of degree n. We can see it better if we simplify the above equation by introducing the terms:

$$\theta = \cos^{-1}(x) \tag{4.25}$$

$$x = \cos(\theta) \qquad 0 \le \theta \le \pi \tag{4.26}$$

Then,

$$T_n(x) = \cos(n\theta) \tag{4.27}$$

After using some trigonometric identities and addition formulas the triple recursion relationship of Chebishev Polynomials is obtained:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad n \ge 1 \tag{4.28}$$

Therefore we can write the first few Chebishev Polynomials as follows:

$$T_0(x) = 1 \tag{4.29}$$

$$T_1(x) = x \tag{4.30}$$

$$T_2(x) = 2x^2 - 1 \tag{4.31}$$

$$\vdots$$

After inserting Chebishev polynomials for trial function the approximation function becomes:

$$P_N(x) = \left\{1 - \left(\frac{x}{L}\right)^2\right\}^4 \sum_{n=1}^{N} c_n T_{n-1} \tag{4.32}$$

where

$$\phi_n = T_{n-1} \tag{4.33}$$

or in an expanded form:

$$P_N(x) = \left\{1 - \left(\frac{x}{L}\right)^2\right\}^4 \left(c_1 + c_2 x + c_3(2x^2 - 1) + \ldots + c_N T_{N-1}\right) \tag{4.34}$$

The modified Chebishev Polynomials are defined as follows:

$$\Psi_n(x) = T_{n-1}(x)\left(\frac{2x - \alpha - \beta}{\beta - \alpha}\right) \qquad \alpha \le x \le \beta \tag{4.35}$$

where the constants α and β for our problem are given by:

$$\alpha = -L = -50 \tag{4.36}$$

$$\beta = L = 50 \tag{4.37}$$

The approximation function for modified Chebishev Polynomials takes the following form:

$$P_N(x) = \left\{1 - \left(\frac{x}{L}\right)^2\right\}^4 \sum_{n=1}^{N} c_n \Psi_n \tag{4.38}$$

After inserting the approximation functions defined by both ordinary and modified Chebishev polynomials (Equations (4.34) and (4.38) respectively) into the residual term in Equation (4.16) and solving the matrix in Equation (4.19) we obtain in both cases the same results for natural frequencies which are given in Table (4.2). If we compare the Tables (4.1) and (4.2) we see easily that the approximate solutions for natural frequencies in both trials with ordinary and modified Chebishev Polynomials are exactly the same as the solutions with power series.

Table 4.2. Approximate values for the first natural frequencies obtained with Galerkin Method and trial with Chebishev Polynomials

	Series Length					
Mode # N	1	2	3	4	5	6
1	23.18	23.18	18.38	18.38	16.45	16.45
2		60.74	60.74	52.05	52.05	48.08
3			112.94	112.94	102.03	102.03
4				178.24	178.24	167.10
5					256.59	256.59
6						348.59

4.3.3. Galerkin Method with Legendre Polynomials

Let the approximation function be the Legendre Polynomial of degree n which is the solution of Legrendre's differential equation arising in numerous problems particularly in boundary value problems and denoted by $G_n(x)$. It can be described as follows:

$$G_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \tag{4.39}$$

We can write it in expanded form as:

$$G_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + - \ldots \tag{4.40}$$

where $M = n/2$ *or* $(n-1)/2$ whichever is an integer. The first few of Legendre Polynomials are given as follows:

$$G_0(x) = 1 \tag{4.41}$$

$$G_2(x) = \frac{1}{2}(3x^2 - 1) \tag{4.42}$$

$$G_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{4.43}$$

$$G_1(x) = x \tag{4.44}$$

$$G_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{4.45}$$

$$G_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \tag{4.46}$$

The approximation function used to solve the same problem using Legendre Polynomials can be written as:

$$P_N(x) = \left\{1 - \left(\frac{x}{L}\right)^2\right\}^4 \sum_{n=1}^{N} c_n G_{n-1} \tag{4.47}$$

In this case again the results for natural frequencies of the non-uniform beam are exactly the same as in the case with power series. The reason why we obtained the same approximate values for identical modes of vibrations although we have used different series for the trials such as power series, Chebishev Polynomials and Legendre Polynomials is that we can express all of these series in the form of powers of the variable x multiplied with some coefficients. These coefficients constitute the generalized coordinates referred to the bases functions. Deflection of the beam can be obtained by a superposition of successive normal modes $\phi_n(x)$ with the n^{th} mode participating with amplitude c_n. However the approximate values for frequencies of vibration modes do not depend on these coefficients which are eliminated by solving Equation (4.19) for the frequencies.

4.3.4. Galerkin Method with Fourier Series

When we study the normal modes of free vibration of a uniform beam the governing differential equation becomes by letting $b = c = 0$ in Equation (4.8) as follows:

$$\frac{d^4u(x)}{dx^4} - \left(\frac{\rho A_0 \omega_n^{\,2}}{EI_0}\right)u(x) = 0 \tag{4.48}$$

If we let $k^4 = \dfrac{\rho A_0 \omega_n^{\,2}}{EI_0}$ it can be easily verified that $\sin kx, \cos kx, \sinh kx, \cosh kx$ are particular solutions of the above equation. Therefore the general solution can be written as follows:

$$u(x) = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx \tag{4.49}$$

The coefficients C_1, C_2, C_3 and C_4 should be determined in every particular case according to the end conditions of the uniform beam.

When choosing suitable functions for trial series in our problem it would be a good idea to take the general solution of a uniform beam or some particular solutions of it to obtain a satisfactory approximation. Therefore we use Fourier Series which are series of cosine and sine functions and arise when representing general periodic functions for basis functions. They constitute an important tool in solving problems that involve ordinary and partial differential equations. We can rewrite the approximation function as follows:

$$P_N(x) = \left\{1 - \left(\frac{x}{L}\right)^2\right\}^4 \sum_{n=1}^{N} c_n F_n \tag{4.50}$$

where

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right) + \cos\left(\frac{n\pi x}{L}\right) \tag{4.51}$$

Results for the first six natural frequencies are given in Table (4.3). For the first five cases convergence of the results can be observed by increasing successively the series length. Some of the results however for the case with series length of six are complex numbers.

Table 4.3. Approximate values for the first natural frequencies obtained with Galerkin Method and trial with Fourier Series

Mode # N	1	2	3	4	5	6
1	65.38	50.66	43.30	38.71	35.52	33.83
2		203.79	171.75	153.73	141.71	39.68
3			412.89	365.11	336.32	18.17+37.01i
4				688.23	626.54	18.17-37.01i
5					1027.46	35.60+22.88i
6						35.60-22.88i

4.3.5. Galerkin Method with other Trigonometic Series

As the next trial we use the general solution of a uniform beam given in Equation (4.49) for the approximation function which is given by:

$$P_N(x) = \left\{1 - \left(\frac{x}{L}\right)^2\right\}^4 \sum_{n=1}^{N} c_n \left(\sin\frac{n\pi x}{L} + \cos\frac{n\pi x}{L} + \sinh\frac{n\pi x}{L} + \cosh\frac{n\pi x}{L} \right) \quad (4.52)$$

The solutions obtained with the approximation function given above and using method of Galerkin is presented in Table 4.4.

Table 4.4. Approximate values for the first six natural frequencies obtained with Galerkin Method and with approximation function given in Equation (4.52)

	Series Length					
Mode # N	1	2	3	4	5	6
1	39.00	37.64	35.75	35.59	30.98	29.19
2		115.49	116.14	87.13	83.03	6.90+31.29i
3			236.37	205.54	151.38	6.90-31.29i
4				434.17	405.37	23.93+24.65i
5					845.15	23.93-24.65i
6						32.29+10.13i

For a uniform free-free beam the moment and shear force at both ends must be zero so that the end conditions can be written as:

$$\left(u(x)''\right)_{x=\pm L} = \left(u(x)'''\right)_{x=\pm L} = 0 \quad (4.53)$$

The general solution given in Equation (3.49) for symmetrical modes of vibration should be taken in the form:

$$u(x) = C_1(\cos kx + \cosh kx) + C_2(\cos kx - \cosh kx) \quad (4.54)$$

Therefore for the last trial to solve the same problem with Galerkin Method we apply for basis function the trigonometric series which is given by:

$$H_n(x) = \cos\left(\frac{n\pi x}{L}\right) + \cosh\left(\frac{n\pi x}{L}\right) \quad (4.55)$$

The approximation function becomes:

$$P_N(x) = \left\{1 - \left(\frac{x}{L}\right)^2\right\}^4 \sum_{n=1}^{N} c_n \left(\cos\frac{n\pi x}{L} + \cosh\frac{n\pi x}{L}\right) \tag{4.56}$$

Comparing the results which are given in Table 4.5 with the numerical example given by Timoshenko, (1937) who solved the fundamental vibration of the same beam with Rayleigh-Ritz Method, it can be seen that the value obtained for the first mode is exactly the same. The values up to the trial with series length of five are close to the solutions obtained with power series.

Table 4.5. Approximate values for the first natural frequencies obtained with Galerkin Method and with approximation function given in Equation (4.56)

	Series Length					
Mode # N	1	2	3	4	5	6
1	21.63	20.72	20.67	19.63	19.68	11.55+24.68i
2		117.21	119.29	107.16	107.90	11.55-24.68i
3			331.23	337.53	293.02	23.50+17.16i
4				716.56	712.02	23.50-17.16i
5					1343.73	29.055+5.82i
6						29.055-5.82i

4.4. Collocation Method

In this method we force the residual at chosen points in the domain to be exactly zero. For a trial solution with *N* parameters we choose *N* points in the domain which are called as collocation points, where the residual is set to zero. These points may be located anywhere in the domain and on the boundary, not necessarily in any particular pattern. We therefore produce again a system of *N* residual equations for the same problem:

$$\varepsilon(x_1; c) = 0 \tag{4.57a}$$

$$\varepsilon(x_2;c) = 0 \tag{4.57b}$$

$$\vdots$$

$$\varepsilon(x_N;c) = 0 \tag{4.57c}$$

The residual equation for collocation method can be written as:

$$\det\begin{bmatrix} \varepsilon_1(x_1;c) & \dots & \varepsilon_1(x_N;c) \\ \vdots & \ddots & \vdots \\ \varepsilon_N(x_1;c) & \cdots & \varepsilon_N(x_N;c) \end{bmatrix} = 0 \tag{4.58}$$

We will insert power series, Chebishev Polynomials, Legendre polynomials, Fourier Series and the other trigonometric series we have used in the Galerkin Method in the residual equation of collocation method and solve for the natural frequencies of the same non-uniform beam of which properties are discussed before. The distinction in collocation method is that we obtain different numerical results for every different set of collocation points. The result is very dependent on where the collocation points are placed in the domain and on the boundary. It should be noted that the collocation points are the locations where the error in the residual is set to zero and not the error in the solution.

For trial functions with power series, Chebishev Polynomials and Legendre Polynomials we obtain again the same results as was in the case with Galerkin Method. All the numerical results with different trial functions and collocation points are given in tables which are represented below.

We first force the residual at the mid-section of the beam where $x = 0$ to be equal to zero and begin to spread the collocation points from midpoint to the boundaries over the beam span. At the same time we increase the series length. As can be seen from Tables 4.6 and 4.7 we have obtained convergence of the solutions for the trial with power series in this manner. We obtain reasonable results particularly in those cases where the collocation points are distributed along the mid-section of the beam.

Table 4.6. Approximate values for the first five natural frequencies obtained with collocation method and with power series (or Chebishev Polynomials or Legendre Polynomials) using series length of one to five

Series Length		1	1	2	3	4	5	5
x values of collocation points		0	5	0 ; 5	-5;0;5	-10;-5;5;10	-10;-5;0;5;10	-2;-1;0;1;2
Mode # N	1	37.77	36.05	37.77	28.49	27.58	24.05	24.49
	2			86.33	86.33	74.30	74.30	76.03
	3				141.52	141.68	142.95	145.98
	4					195.70	195.70	192.02
	5						235.08	227.21

Table 4.7. Approximate values for the first six natural frequencies obtained with collocation method and with power series (or Chebishev Polynomials or Legendre Polynomials) using series length of nine, eleven and thirteen

Series Length		9	11	11	11	13
x values of collocation points		-2;-1.5;-1;-0.5;0;0.5;1;1.5;2	-5;-4;-3;-2;-1;0;1;2;3;4;5	-25;-20;-15;-10;-5;0;5;10;15;20;25	-5; -2;-1.5;-1;-0.5;0;0.5;1;1.5;2;5	-25;-20;-15;-10;-5;-2;0;2;5;10;15;20;25
Mode # N	1	20.51	19.39	18.37	18.59	17.85
	2	62.32	58.70	55.49	56.23	53.91
	3	121.71	115.93	110.92	111.85	108.13
	4	204.46	198.54	188.96	190.56	183.86
	5	279.71	300.72	281.72	280.81	273.48
	6	322.44	341.50	369.53	397.86	414.63

For the trials with Fourier Series and other trigonometric series we again set at the collocation points the residual equal to zero in the same fashion as it was with power series. For the first trial we choose the midpoint of the beam as collocation point whereas for our last trial the collocation points are distributed along the beam length from -25meters to +25meters away from the midpoint. Most of the results however are this time complex numbers.

Table 4.8. Approximate values for the first five natural frequencies obtained with collocation method and with Fourier Series using series length of one to five

Series Length		1	1	2	3	4	5	5
x values of collocation points		0	5	0 ; 5	-5;0;5	-10;-5;5;10	-10;-5;0;5;10	-2;-1;0;1;2
Mode # N	1	81.98	86.97	167.46	37.67	31.03	22.64	20.13
	2			57.17i	177.59	143.69	112.00	104.49
	3				538.53	500.76	1544.22	1996.66
	4					683.50	419.52+201.85i	346.37+260.50i
	5						419.52-201.85i	346.37-260.50i

Table 4.9. Approximate values for the first six natural frequencies obtained with collocation method and with Fourier Series using series length of nine, eleven and thirteen

Series Length		9	11	11	11	13
x values of collocation points		-2;-1.5;-1;-0.5;0;0.5;1;1.5;2	-5;-4;-3;-2;-1;0;1;2;3;4;5	-25;-20;-15;-10;-5;0;5;10;15;20;25	-5; -2;-1.5;-1;-0.5;0;0.5;1;1.5;2;5	-25;-20;-15;-10;-5;-2;0;2;5;10;15;20;25
Mode # N	1	5.44	0.27+2.76i	11.28	0.17+2.69i	28.75
	2	0.98+4.57i	0.27-2.76i	0.11+10.23i	0.17-2.69i	0.46+3.43i
	3	0.98-4.57i	2.27+4.51i	0.11-10.23i	2.02+4.33i	0.46-3.43i
	4	4.08+6.55i	2.27-4.51i	4.28+12.60i	2.02-4.33i	1.50+5.30i
	5	4.08-6.55i	4.94+5.91i	4.28-12.60i	4.41+5.66i	1.50-5.30i
	6	7.97+7.67i	4.94-5.91i	9.18+13.80i	4.41-5.66i	3.73+7.51i

Table 4.10. Approximate values for the first five natural frequencies obtained with collocation method and with Equation (4.52) using series length of one to five

Series Length		1	1	2	3	4	5	5
x values of collocation points		0	5	0 ; 5	-5;0;5	-10;-5;5;10	-10;-5;0;5;10	-2;-1;0;1;2
Mode # N	1	47.96	54.85	69.09	30.13	7.77	27.77	34.07
	2			117.68i	101.43i	131.45	63.90	78.30
	3				1078.34i	325.66+544.94i	873.10i	3235.85i
	4					325.66-544.94i	1311.06	11645.34
	5						2150.36i	39022.07i

Table 4.11. Approximate values for the first six natural frequencies obtained with collocation method and with Equation (4.52) using series length of nine, eleven and thirteen

Series Length		9	11	11	11	13
x values of collocation points		-2;-1.5;-1;-0.5;0;0.5;1;1.5;2	-5;-4;-3;-2;-1;0;1;2;3;4;5	-25;-20;-15;-10;-5;0;5;10;15;20;25	-5; -2;-1.5;-1;-0.5;0;0.5;1;1.5;2;5	-25;-20;-15;-10;-5;-2;0;2;5;10;15;20;25
Mode # N	1	2.29+5.37i	0.94+2.11i	0.19+11.81i	0.46+2.52i	0.66+5.68i
	2	2.29-5.37i	0.94-2.11i	0.19-11.81i	0.46-2.52i	0.66-5.68i
	3	5.49+6.67i	2.84+3.59i	4.95+13.80i	2.36+4.18i	3.16+7.6i
	4	5.49-6.67i	2.84-3.59i	4.95-13.80i	2.36-4.18i	3.16-7.6i
	5	9.09+7.10i	5.33+4.81i	10.18+14.48i	4.88+5.52i	6.22+9.11i
	6	9.09-7.10i	5.33-4.81i	10.18-14.48i	4.88-5.52i	6.22-9.11i

Table 4.12. Approximate values for the first five natural frequencies obtained with collocation method and with Equation (4.56) using series length of one to five

Series Length		1	1	2	3	4	5	5
x values of collocation points		0	5	0 ; 5	-5;0;5	-10;-5;5;10	-10;-5;0;5;10	-2;-1;0;1;2
Mode # N	1	46.97	39.81	33.72	35.50	31.98	35.68	50.54
	2			1127.86i	744.36i	350.07	678.55i	181.33
	3					61119.16+61119.94i	3522.33i	1226743.47
	4					61119.16-61119.94i	3060.63+1645.25i	66.40i
	5						3060.63-1645.25i	2202.2i

Table 4.13. Approximate values for the first six natural frequencies obtained with collocation method and with Equation (4.56) using series length of nine, eleven and thirteen

Series Length		9	11	11	11	13
x values of collocation points		-2;-1.5;-1;-0.5;0;0.5;1;1.5;2	-5;-4;-3;-2;-1;0;1;2;3;4;5	-25;-15;-10;-5;0;5;10;15;20;25	-5; -2;-1.5;-1;-0.5;0;0.5;1;1.5;2;5	-25;-15;-10;-5;-2;0;2;5;10;15;20;25
Mode # N	1	2.18+5.92i	23.89	49.90	0.57+2.40i	0.52+2.31i
	2	2.18-5.92i	0.92+4.03i	0.28+4.85i	0.57-2.40i	0.52-2.31i
	3	5.65+7.34i	0.92-4.03i	0.28-4.85i	2.45+3.99i	2.23+3.93i
	4	5.65-7.34i	3.38+5.80i	3.00+6.89i	2.45-3.99i	2.23-3.93i
	5	9.55+7.81i	3.38-5.80i	3.00-6.89i	4.95+5.26i	4.52+5.35i
	6	9.55-7.81i	6.45+7.12i	6.45+8.36i	4.95-5.26i	4.52-5.35i

4.5. Subdomain Method

In this method we choose intervals in the domain and force the average of the residual in each interval to be zero. For a trial solution with *N* parameters we choose *N* intervals Δx_i so that we have system of *N* residual equations. The intervals Δx_i are called subdomains which may be chosen in any fashion, even overlapping or with gaps in between. If we recall the residual:

$$\varepsilon_N(x) = (1-bx^2)\frac{d^4P_N(x)}{dx^4} - 4bx\frac{d^3P_N(x)}{dx^3} - 2b\frac{d^2P_N(x)}{dx^2} - \left(\frac{\rho A_0\omega_n^{\ 2}}{EI_0}\right)(1-cx^2)P_N(x) = 0 \tag{4.59}$$

The *N* residual equation can be written as:

$$\frac{1}{\triangle x_1}\int_{\triangle x_1} \varepsilon(x;c)dx = 0 \tag{4.60a}$$

$$\frac{1}{\triangle x_2}\int_{\triangle x_2} \varepsilon(x;c)dx = 0 \tag{4.60b}$$

$$\vdots$$

$$\frac{1}{\triangle x_N}\int_{\triangle x_N} \varepsilon(x;c)dx = 0 \tag{4.60c}$$

Again in the subdomain method we obtain different numerical values for natural frequencies as in the collocation method if we choose different subintervals in which the residual is set to zero. The residual equation for method of subdomain is given by:

$$\det\begin{bmatrix} \int_{\triangle x_1} \varepsilon_1(x;c)dx & \dots & \int_{\triangle x_N} \varepsilon_1(x;c)dx \\ \vdots & \ddots & \vdots \\ \int_{\triangle x_1} \varepsilon_N(x;c)dx & \cdots & \int_{\triangle x_N} \varepsilon_N(x;c)dx \end{bmatrix} = 0 \tag{4.61}$$

For trial functions with power series, Chebishev Polynomials and Legendre Polynomials we obtain again the same results as was in the case with Galerkin Method. All the numerical results with different trial functions and several subdomain intervals are given in tables which are represented below.

We choose for the first subdomain the whole beam length and set the residual over the full length to be zero. For the next trial the beam length is divided into two equal parts and the residual is forced to be zero in both intervals. As the series length increases the subdomains are chosen in the vicinity of the mid-section of the free-free beam. We obtain better results as the sudomain intervals are concentrated across the mid-section.

We can conclude from the following tables that the location of subintervals affects the results for natural frequencies significantly and as the subdomains approach the region of the mid-section of the beam better results are obtained.

Table 4.14. Approximate values for the first six natural frequencies obtained with subdomain method and with power series (or Chebishev Polynomials or Legendre Polynomials) using series length of one to six

Series Length		1	2	3	4	5	6
x values of subdomain interval limits		(-50:50)	(-50:0) ;(0:50)	(-10:10);(-50:-10);(10:50)	(-5:0) ;(0:5);(-10:0);(0:10)	(-5:0) ;(0:5);(-10:0);(0:10);(-1:1)	(-5:0);(0:5);(-10:0);(0:10);(-15:-10);(10:15)
Mode # N	1	0.00	0.00	0.00	27.99	24.18	23.60
	2		50.08	35.25	74.89	74.9	65.83
	3			105.78	141.63	143.82	140.11
	4				194.42	194.41	242.80
	5					232.77	246.14+34.39i
	6						246.14-34.39i

Table 4.15. Approximate values for the first six natural frequencies obtained with subdomain method and with power series (or Chebishev Polynomials or Legendre Polynomials) using series length of ten and twelve

Series Length		10	12
x values of subdomain interval limits		(-5:0);(0:5);(-10:-5);(5:10);(-15:-10);(10:15);(-20:-15);(15:20);(-25:-20);(20:25)	(-5:0);(0:5);(-10:-5);(5:10);(-15:-10);(10:15);(-20:-15);(15:20);(-25:-20);(20:25);(-30:-25);(25;30)
Mode # N	1	19.26	17.88
	2	56.03	52.93
	3	115.60	109.45
	4	190.59	175.94
	5	365.70	301.46
	6	355.57	376.38+39.31i

Table 4.16. Approximate values for the first six natural frequencies obtained with subdomain method and with Fourier Series using series length of one to six

Series Length		1	2	3	4	5	6
x values of subdomain interval limits		(-50:50)	(-50:0) ;(0:50)	(-10:10);(-50:-10);(10:50)	(-5:0) ;(0:5);(-10:0);(0:10)	(-5:0) ;(0:5);(-10:0);(0:10);(-1:1)	(-5:0);(0:5);(-10:0);(0:10);(-15:-10);(10:15)
Mode # N	1	0.59i	143.88	117.97	28.25	21.11	18.04
	2		0.68i	340.11	135.85	107.23	39.47
	3			0.79i	569.88+128.27i	1729.13	17.60+32.78i
	4				569.88-128.27i	382.76+230.05i	17.60-32.78i
	5					382.76-230.05i	33.91+19.49i
	6						33.91-19.49i

Table 4.17. Approximate values for the first six natural frequencies obtained with subdomain method and with Fourier Series using series length of ten and twelve

Series Length		10	12
x values of subdomain interval limits		(-5:0);(0:5);(-10:-5);(5:10);(-15:-10);(10:15);(-20:-15);(15:20);(-25:-20);(20:25)	(-5:0);(0:5);(-10:-5);(5:10);(-15:-10);(10:15);(-20:-15);(15:20);(-25:-20);(20:25);(-30:-25);(25;30)
Mode # N	1	10.64	11.18
	2	4.22+15.83i	30.11
	3	4.22-15.83i	0.67+11.34i
	4	10.61+16.69i	0.67-11.34i
	5	10.61-16.69i	5.00+13.47i
	6	16.98+15.53i	5.00-13.47i

Table 4.18. Approximate values for the first six natural frequencies obtained with subdomain method and with Equation (4.52) using series length of one to six

Series Length		1	2	3	4	5	6
x values of subdomain interval limits		(-50:50)	(-50:0) ;(0:50)	(-10:10);(-50:-10);(10:50)	(-5:0) ;(0:5);(-10:0);(0:10)	(-5:0) ;(0:5);(-10:0);(0:10);(-1:1)	(-5:0);(0:5);(-10:0);(0:10);(-15:-10);(10:15)
Mode # N	1	6.30	40.01	55.85	130.61	31.16	34.67
	2		38.54i	84.19	12.36i	61.59	6.40+28.69i
	3			51.90i	318.32+699.08i	1814.81	6.40-28.69i
	4				318.32-699.08i	1015.69i	20.93+24.01i
	5					2809.06i	20.93-24.01i
	6						30.68+13.83i

Table 4.19. Approximate values for the first six natural frequencies obtained with subdomain method and with Equation (4.52) using series length of ten and twelve

Series Length		10	12
x values of subdomain interval limits		(-5:0);(0:5);(-10:-5);(5:10);(-15:-10);(10:15);(-20:-15);(15:20);(-25:-20);(20:25)	(-5:0);(0:5);(-10:-5);(5:10);(-15:-10);(10:15);(-20:-15);(15:20);(-25:-20);(20:25);(-30:-25);(25;30)
Mode # N	1	2.52+12.47i	27.08
	2	2.52-12.47i	1.51+7.60i
	3	7.50+13.82i	1.51-7.60i
	4	7.50-13.82i	4.66+9.39i
	5	12.72+13.71i	4.66-9.39i
	6	12.72-13.71i	8.29+10.52i

Table 4.20. Approximate values for the first six natural frequencies obtained with subdomain method and with Equation (4.56) using series length of one to six

Series Length		1	2	3	4	5	6
x values of subdomain interval limits		(-50:50)	(-50:0) ;(0:50)	(-10:10);(-50:-10);(10:50)	(-5:0) ;(0:5);(-10:0);(0:10)	(-5:0) ;(0:5);(-10:0);(0:10);(-1:1)	(-5:0);(0:5);(-10:0);(0:10);(-15:-10);(10:15)
Mode # N	1	1.44	0.66	3.80	31.06	38.34	18.78
	2		521.84i	59.75i	473.51	1199.13	6.22+12.04i
	3			5706.39i	7698.23	546.61i	6.22-12.04i
	4				26793.38i	601.20+1066.51i	12.92+10.14i
	5					601.20-1066.51i	12.92-10.14i
	6						17.37+5.55i

Table 4.21. Approximate values for the first six natural frequencies obtained with subdomain method and with Equation (4.56) using series length of seven, ten and twelve

Series Length		7	10	12
x values of subdomain interval limits		(-5:0);(0:5);(-10:0);(0:10);(-15:-10);(10:15);(-15:15)	(-5:0);(0:5);(-10:-5);(5:10);(-15:-10);(10:15);(-20:-15);(15:20);(-25:-20);(20:25)	(-5:0);(0:5);(-10:-5);(5:10);(-15:-10);(10:15);(-20:-15);(15:20);(-25:-20);(20:25);(-30:-25);(25;30)
Mode # N	1	0.97	0.98+2.04i	0
	2	2.01	0.98-2.04i	0.98
	3	3.34	2.88+3.37i	23.10
	4	5.02	2.88-3.37i	23.99
	5	10.52	5.41+4.32i	2.25+0.33i
	6	13.03	5.41-4.32i	2.25-0.33i

The variation of the first few natural frequencies, obtained with different methods and trial functions with increasing series length, are plotted in the following figures for the real valued solutions.

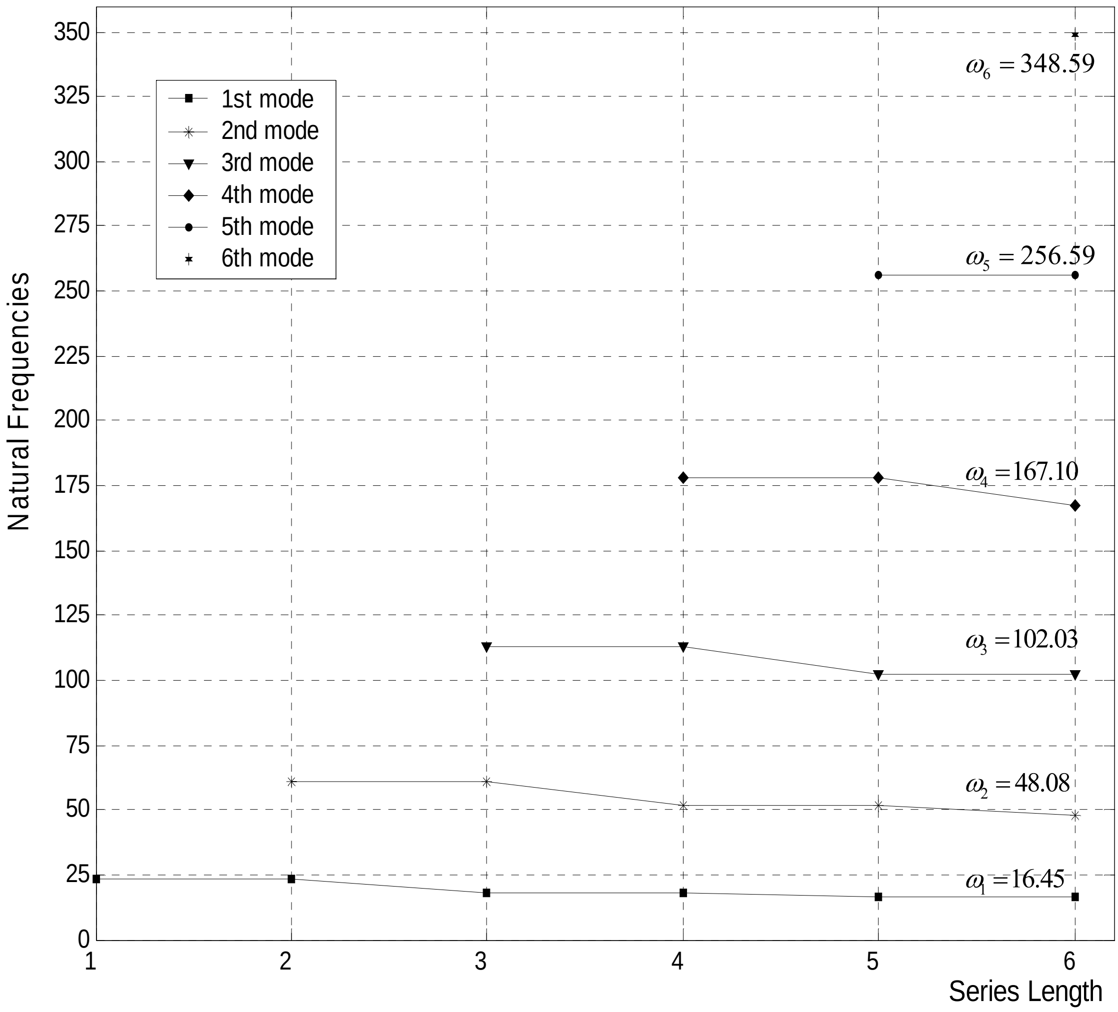

Figure 4.3. Variation of the first six natural frequencies in [rad./sec.] with increasing series length obtained with Galerkin Method using power series as trial function

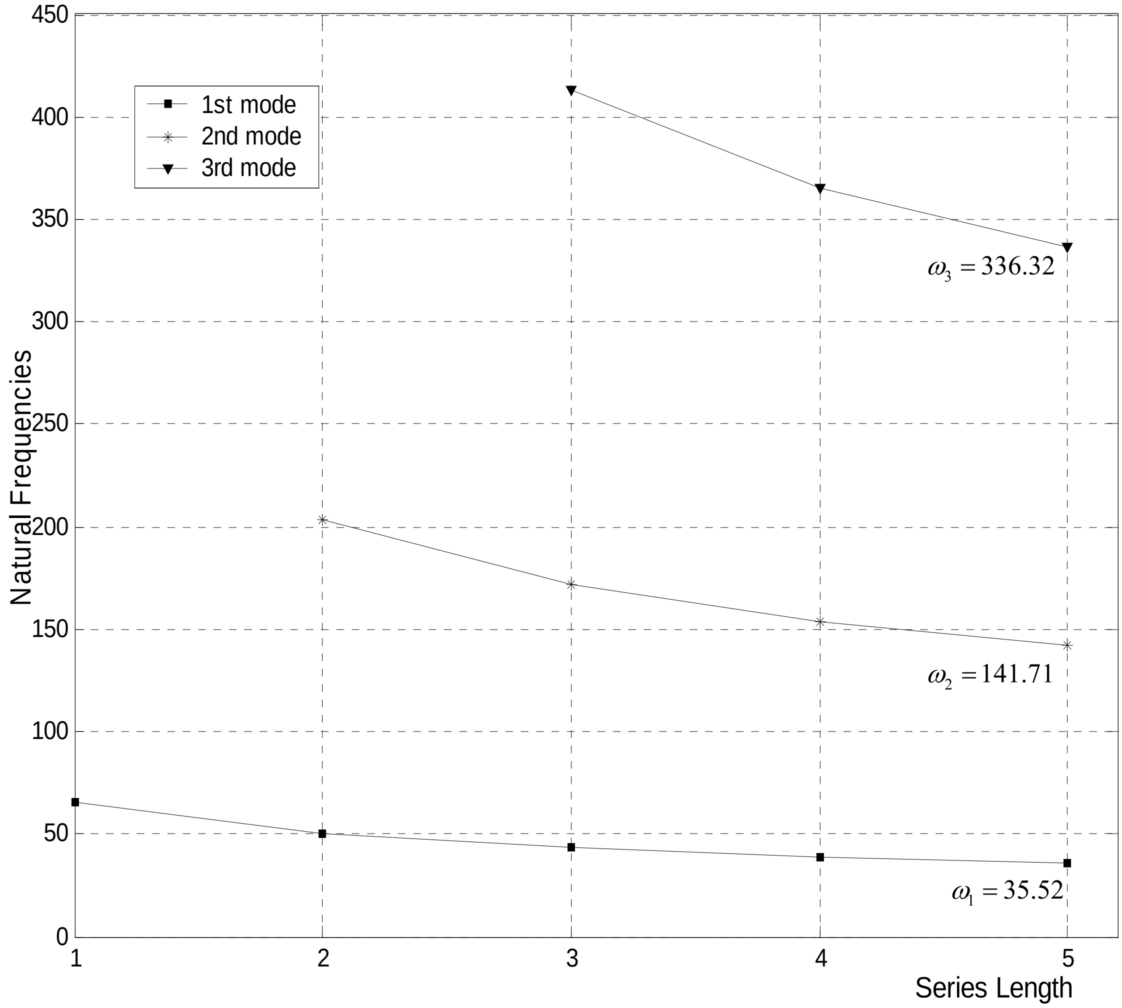

Figure 4.4. Variation of the first three natural frequencies in [rad./sec.] with increasing series length obtained with Galerkin Method using Fourier Series as trial function

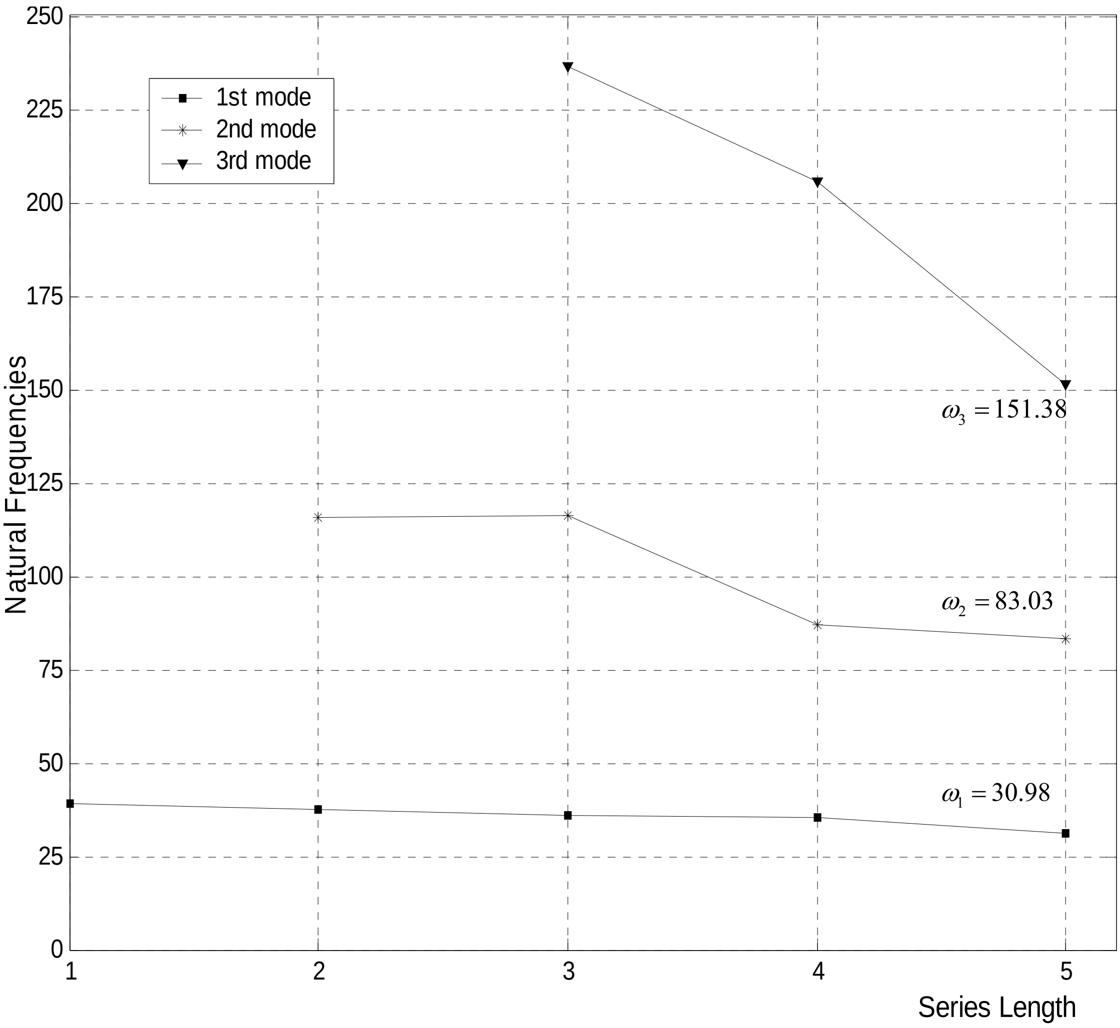

Figure 4.5. Variation of the first three natural frequencies in [rad./sec.] with increasing series length obtained with Galerkin Method using Equation (4.52) as trial function

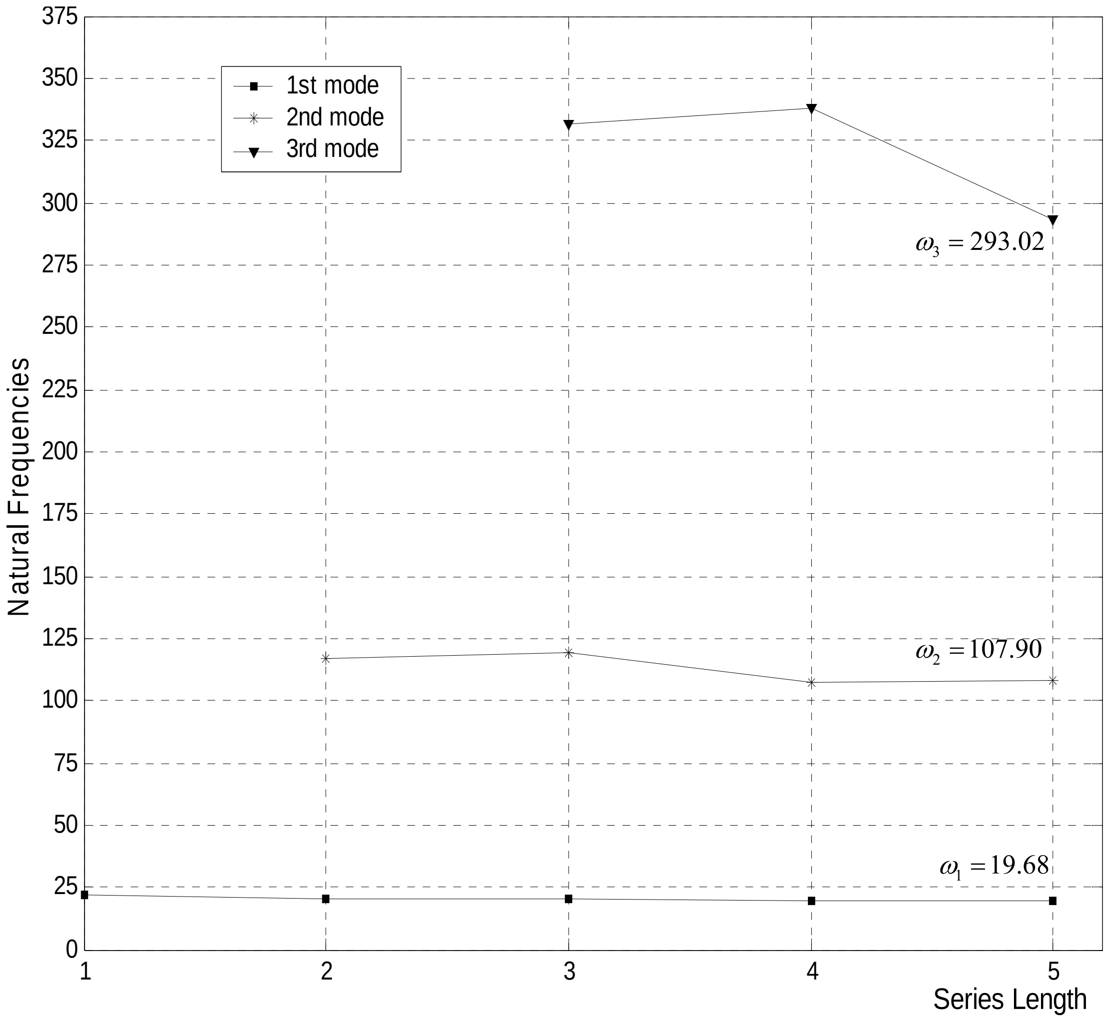

Figure 4.6. Variation of the first three natural frequencies in [rad./sec.] with increasing series length obtained with Galerkin Method using Equation (4.56) as trial function

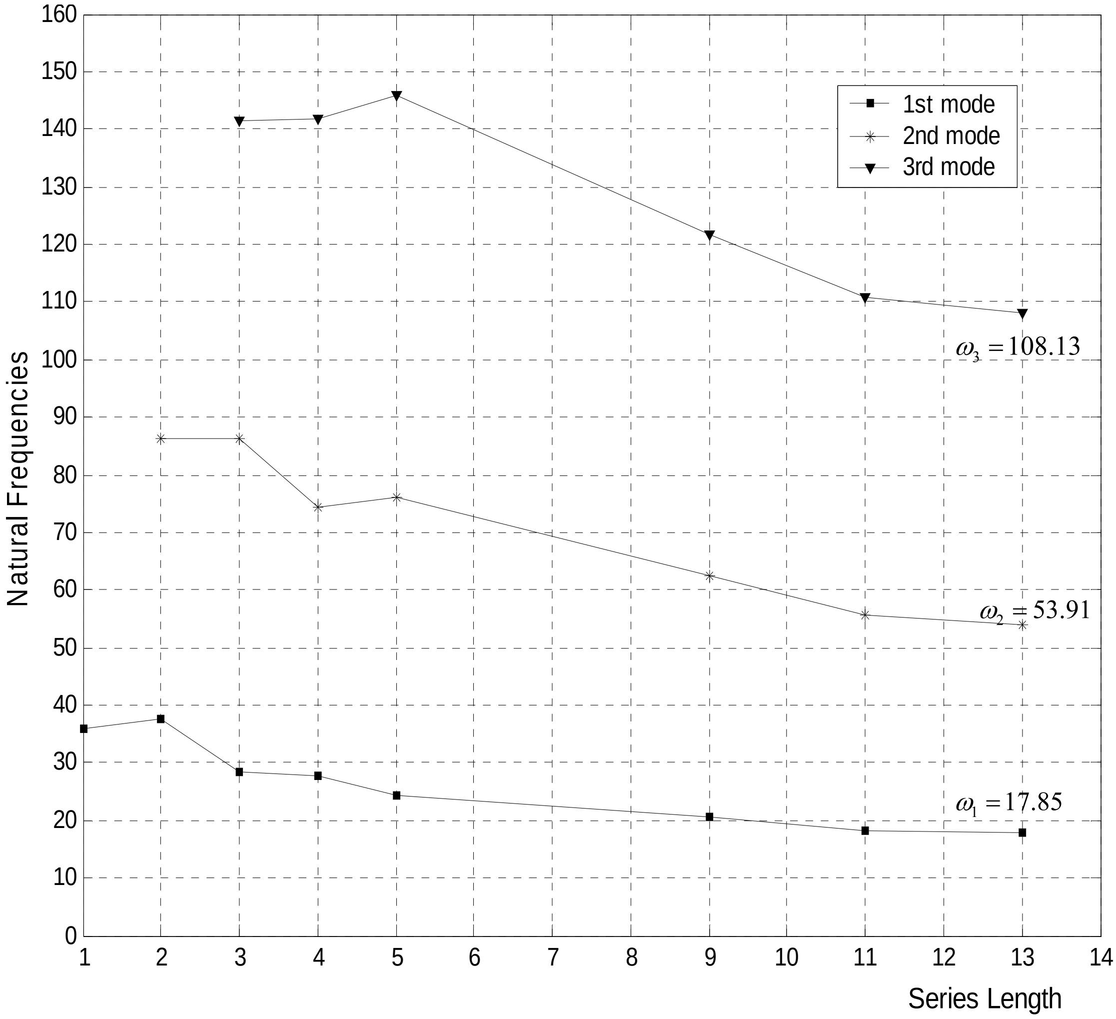

Figure 4.7. Variation of the first three natural frequencies in [rad./sec.] obtained with collocation method using power series as trial function

4.6. Comparison with Solutions Found with Finite Element Method

In this section we want to compare the results found with Galerkin Method using power series as trial function with the solutions for natural frequencies found with finite element method.

To model the beam in finite element method it is divided into twenty sub-elements each of which is five meters long. The cross-sectional area and moment of inertia of each beam element is given in the Table 4.22.

Table 4.22. Cross-sectional area and moment of inertia of beam elements in finite element method

Beam element	Area in m^2	Moment of inertia in m^4
1	1.75	5
2	2.75	7.85
3	3.64	10.4
4	4.43	12.65
5	5.11	14.6
6	5.69	16.25
7	6.16	17.6
8	6.53	18.65
9	6.79	19.4
10	7	20
11	7	20
12	6.79	19.4
13	6.53	18.65
14	6.16	17.6
15	5.69	16.25
16	5.11	14.6
17	4.43	12.65
18	3.64	10.4
19	2.75	7.85
20	1.75	5

The periods and natural frequencies for the first six modes of free vibration found by using computer program Sap 2000 is given in Table 4.23.

Table 4.23. The periods and frequencies of vibration of the non-uniform beam found with finite element method

Mode Number	Period in sec.	Frequency in rad./sec.
1	0.30	20.61
2	0.13	47.79
3	0.06	97.06
4	0.04	165.26
5	0.03	242.47
6	0.02	350.84

If we compare the results obtained using Galerkin Method with those obtained using finite element method we can easily see that they are very close. The difference between them is presented in Table 4.24 which is calculated as follows:

$$Difference \ \ in \ \ \% = \frac{Value_{Finite \ \ Element} - Value_{Galerkin \ \ Method}}{Value_{Finite \ \ Element}} x100 \qquad (4.62)$$

Table 4.24. Difference between the values obtained with Galerkin and finite element method

Finite Element Method	Galerkin Method	Difference in %
20.61	16.45	20.18
47.79	48.08	0.61
97.06	102.03	5.12
165.26	167.1	1.11
242.47	256.59	5.82
350.84	348.59	0.64

From the above table we see that the difference between the fundamental frequencies is bigger than those between the higher frequencies. The value for the fundamental frequency obtained with finite element method is consistent with the value obtained using Galerkin Method and Equation (4.56) as trial function and with the value obtained by Timoshenko, (1937) who used Rayleigh-Ritz Method.

5. VARIATION OF THE NATURAL FREQUENCIES WITH CHANGING SYSTEM PARAMETERS

5.1. The Effect of System Parameters on Natural Frequencies

The solutions for natural frequencies of beams of various configurations are of great importance to designers from many different fields. In this section we want to predict the influence of changing the system parameters on the natural frequencies of the non-uniform free ended beam. The partial differential equation for transverse vibration of the beam on the basis of Euler-Bernoulli Theory is converted to the following fourth order differential equation by the method of seperation of variables:

$$(1-bx^2)\frac{d^4u(x)}{dx^4}-4bx\frac{d^3u(x)}{dx^3}-2b\frac{d^2u(x)}{dx^2}-\left(\frac{\rho A_0\omega_n{}^2}{EI_0}\right)(1-cx^2)u(x)=0 \qquad (5.1)$$

where the cross-sectional area and moment of inertia is given, like in the third chapter, as follows:

$$A(x)=A_0(1-cx^2) \qquad (5.2a)$$

$$I(x)=I_0(1-bx^2) \qquad (5.2b)$$

As can be seen from Equation (5.1) the system parameters affecting the solution for natural frequencies of the free-free ended Euler-Bernoulli beam are the cross-sectional area A_0, moment of inertia I_0, modulus of elasticity E, the mass density ρ and the coefficients c and b which determine the variation in cross-sectional area and moment of inertia along the beam length.

In the preeceding chapter a covergence study is conducted for approximation of natural frequencies of a uniform beam of which exact solution is well known. We have compared the natural frequencies for transverse vibration of the uniform free-free beam obtained with Galerkin Method using power series expansion with the exact solutions and we have seen that the approximate solutions converge to the real values by addition of every successive term to the power series. Reasonable results which are close to the real values for each vibration modes are obtained where the total number of polynomial terms taken equals thirty and therefore they can be treated as acceptable. Thus we will use Galerkin method and power series as trial function to determine approximately the variation in frequencies with changing system parameters.

We can change the system parameters of the beam in any manner as desired and obtain the natural frequencies of vibration for the new configuration which is of interest in several situations in engineering design.

5.2. Influence of the Physical Properties of the Beam on its Natural Frequencies

Firstly we want to determine the effect of moment of inertia on the frequencies and change the value of it whereas keeping the other term constant as follows:

$$2L = 100m; \quad I_0 = 20m^4; \quad b = c = 0.0003/m^2; \quad A_0 = 7m^2; \quad \rho = 1000kg/m^3$$

The values for natural frequencies obtained with Galerkin Method using power series with number of terms taken as thirty is given in Table 5.1. We can clearly see from this table that as the modulus of elasticity of the beam increases the frequencies for each mode also increase.

Table 5.1. The natural frequencies for the first six modes of vibration in rad./sec.

ρ=1ton/m^3	Modulus of Elasticity in kgf/m^2					
Mode # N	0.5x10^9	1x10^9	5x10^9	10x10^9	15x10^9	20x10^9
1	2.14	3.03	6.78	9.59	11.74	13.56
2	6.57	9.28	20.76	29.36	35.96	41.52
3	13.55	19.16	42.84	60.59	74.21	85.69
4	22.95	32.45	72.57	102.62	125.69	145.13
5	34.81	49.23	110.09	155.69	190.68	220.18
6	49.03	69.34	155.06	219.29	268.57	310.12

The smallest value taken for modulus of elasticity which is $0.5x10^9$ kgf/m^2 is in the range of lower values of modulus of elasticty of concrete and the highest one which is $20x10^9$ kgf/m^2 correspondes to steel material.

As the next step we want to predict how the change in density influences the natural frequencies and therefore solve the vibration problem for different mass densities of beam. We begin with the smallest value of *0.6 ton/m*^{3} which corespondes to wood as beam material and end up with the highest value of *8 ton/m*^{3} which can be the density of steel.

Tables 5.2 and 5.3 are obtained by keeping the modulus of elasticity variable for two different values of density and by setting the following terms as constants:

$$2L = 100m; \quad I_0 = 20m^4; \quad b = c = 0.0003/m^2; \quad A_0 = 7m^2;$$

Table 5.2. The natural frequencies for the first six modes of vibration in rad./sec. with ρ=0.6 ton/m^3

ρ=0.6 ton/m^3	Modulus of Elasticity in kgf/m^2					
Mode # N	0.5x10^9	1x10^9	5x10^9	10x10^9	15x10^9	20x10^9
1	2.77	3.91	8.75	12.38	15.16	17.50
2	8.48	11.99	26.80	37.91	46.42	53.61
3	17.49	24.74	55.31	78.22	95.80	110.62
4	29.63	41.90	93.68	132.49	162.26	187.37
5	44.94	63.56	142.13	201.00	246.17	284.25
6	63.30	89.52	200.18	283.10	346.72	400.36

Table 5.3. The natural frequencies for the first six modes of vibration in rad./sec. with ρ=2 ton/m^3

ρ=2 ton/m^3	Modulus of Elasticity in kgf/m^2					
Mode # N	0.5x10^9	1x10^9	5x10^9	10x10^9	15x10^9	20x10^9
1	1.52	2.14	4.79	6.78	8.30	9.59
2	4.64	6.57	14.68	20.76	25.43	29.36
3	9.58	13.55	30.30	42.84	52.47	60.59
4	16.23	22.95	51.31	72.57	88.88	102.62
5	24.62	34.81	77.85	110.09	134.83	155.69
6	34.67	49.03	109.64	155.06	189.91	219.29

It can be seen from Tables 5.2, 5.3 and from Table 5.4, which is obtained by changing only the density and keeping other terms constant, that density increase results in frequency decrease. The results for the same density/modulus of elasticity ratio are the same.

Table 5.4. The natural frequencies for the first six modes of vibration in rad./sec. for different densities

$E=20x10^9 kgf/m^2$	Density in ton/m^3					
Mode # N	0.6	1	2	4	6	8
1	17.50	13.56	9.59	6.78	5.53	4.79
2	53.61	41.52	29.36	20.76	16.95	14.68
3	110.62	85.69	60.59	42.84	34.98	30.30
4	187.37	145.13	102.62	72.57	59.25	51.31
5	284.25	220.18	155.69	110.09	89.89	77.85
6	400.36	310.12	219.29	155.06	126.61	109.64

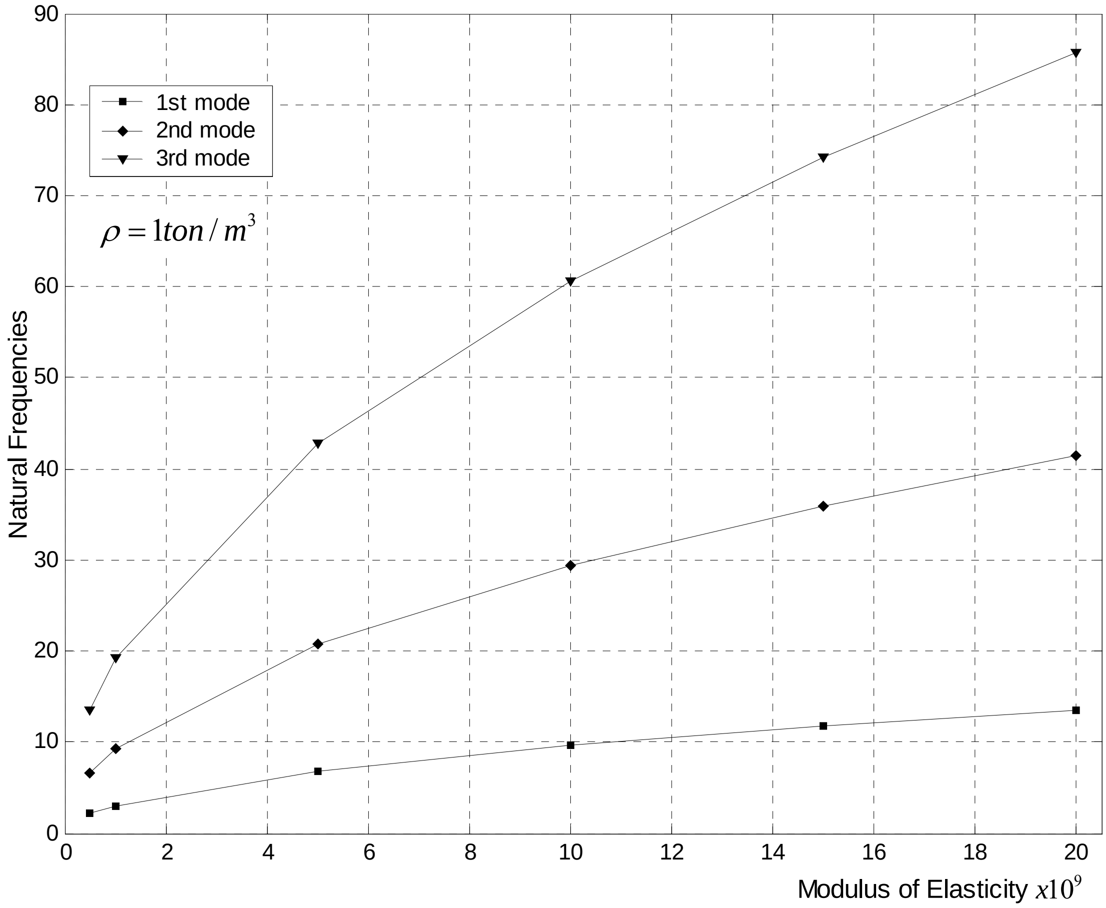

Figure 5.1. Variation of the first three natural frequencies in [rad./sec.] with modulus of elasticity in [kgf / m^2]

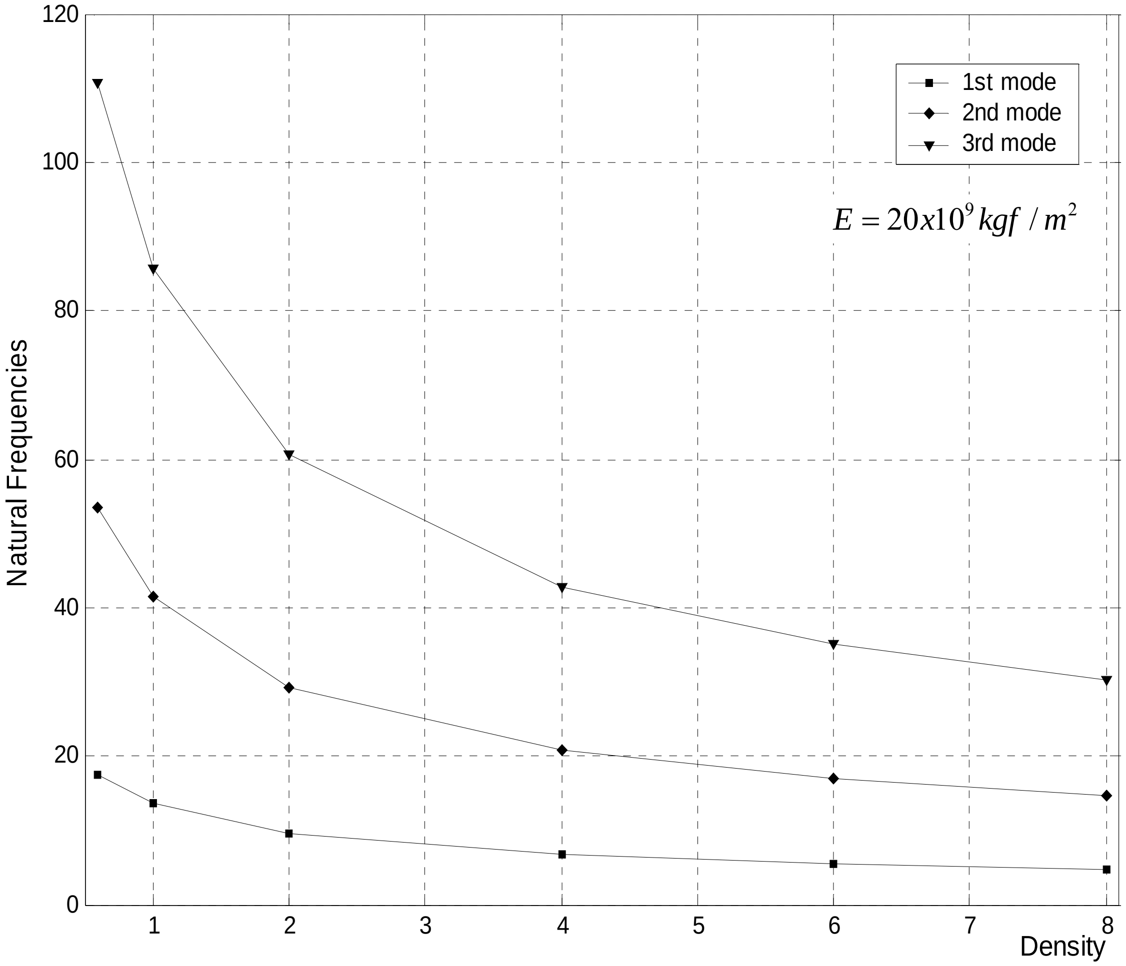

Figure 5.2. Variation of the first three natural frequencies in [*rad./sec.*] with mass density in [ton/m^3]

In Figure 5.3 we have plotted the fundamental frequency of the free ended beam by changing both its modulus of elasticity and density.

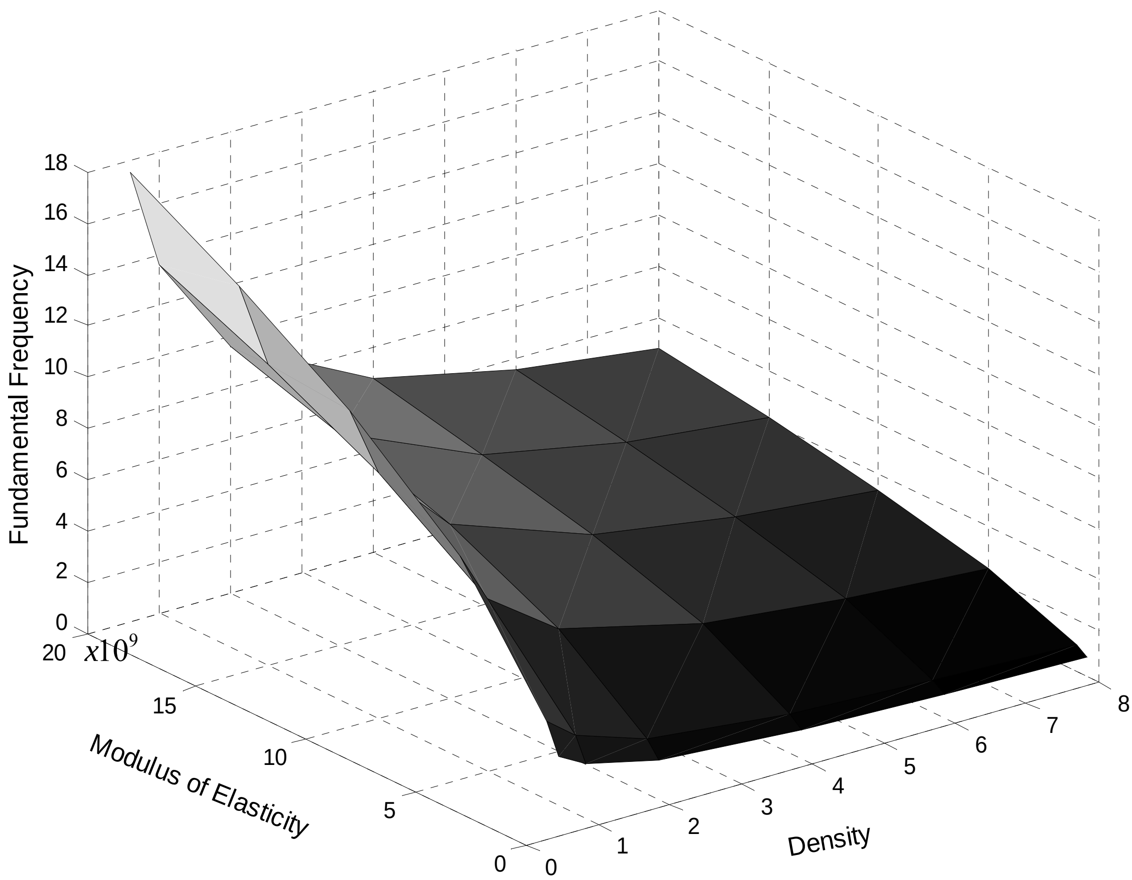

Figure 5.3. The variation of fundamental frequency in [*rad./sec*] with changing modulus of elasticy in [$kgf\ /\ m^2$] and density in [$ton\ /\ m^3$]

5.3. Influence of the Geometry of the Beam on its Natural Frequencies

This time we want to look at how the cross-sectional area $A(x)$ and moment of inertia $I(x)$ of the beam change its natural frequencies of vibration. These are two important terms in the equation of vibration of a non–uniform beam since both of them effect the magnitudes of the sectional mass ($\rho A(x)$) and sectional stiffness ($EI(x)$) of the non–uniform beam. The change of these parameters along the beam length is given in Equations (5.2a) and (5.2b).

We will change the central cross-section/ moment of inertia ratio as well as the coefficients c and b to observe the configurational influence. As can be deduced from Equation (5.1), it is the value of the $A_0\ /\ I_0$ ratio which is affecting the solution of vibration. The results for the natural frequencies obtained by changing this ratio are given in Table 5.5 where the following values are taken for other terms:

$$2L = 100m; \quad E = 20x10^9 kgf / m^2; \quad \rho = 1000kg / m^3 \quad b = c = 0.0003 / m^2$$

Table 5.5. The first six natural frequencies of vibration in [*rad./sec*]

	Ao/Io					
Mode # N	0.1	0.2	0.3	0.35	0.4	0.5
1	25.36	17.93	14.64	13.56	12.68	11.34
2	77.68	54.93	44.85	41.52	38.84	34.74
3	160.31	113.35	92.55	85.69	80.15	71.69
4	271.52	191.99	156.76	145.13	135.76	121.43
5	411.92	291.27	237.82	220.18	205.96	184.22
6	580.18	410.25	334.97	310.12	290.09	259.46

For the next step we look at the coefficients *b* and *c* which determine the variation of the cross-section and moment of inertia along the beam length. The cross-sectional areas corresponding to different coefficients *c* which are chosen as being equal to the coefficients *b* are depicted in the following figures.

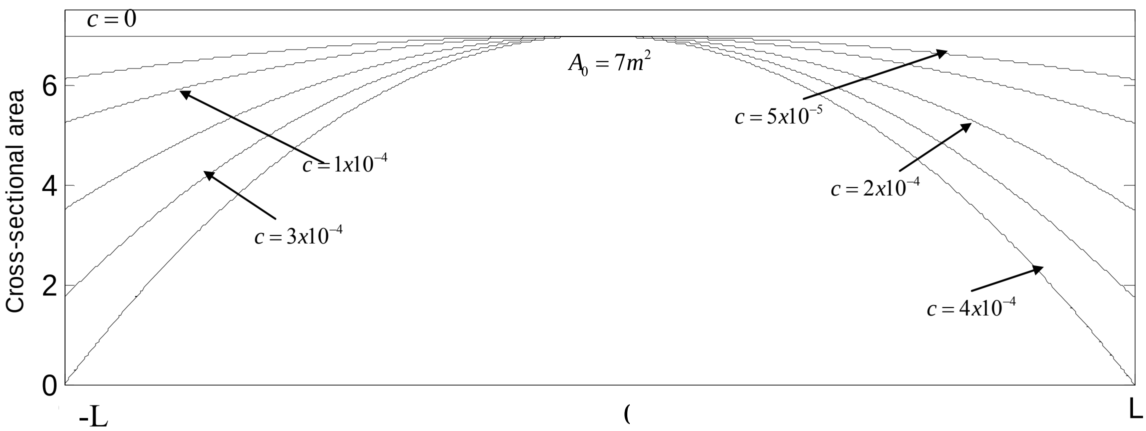

Figure 5.4. Variation of the cross section along the beam length for positive values of *c*

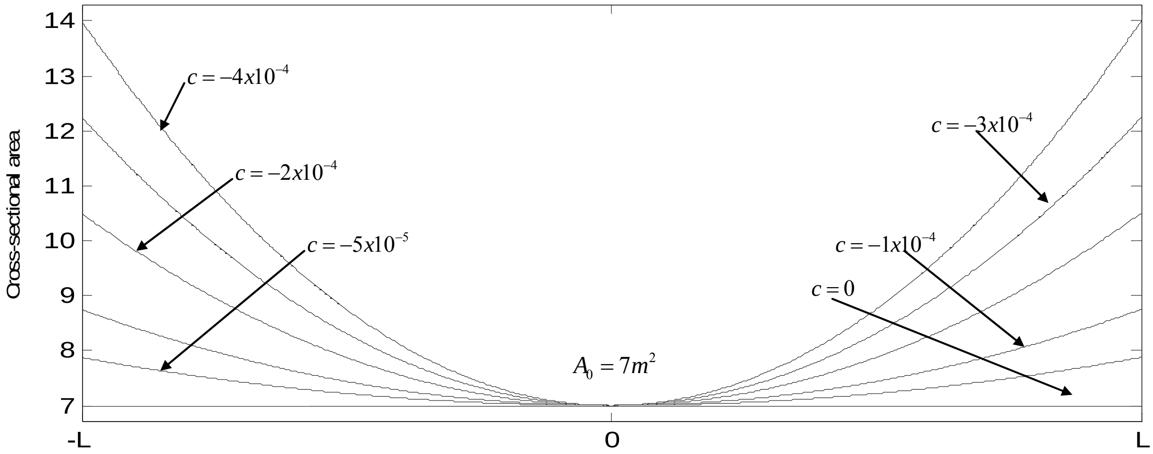

Figure 5.5. Variation of the cross section along the beam length for negative values of c

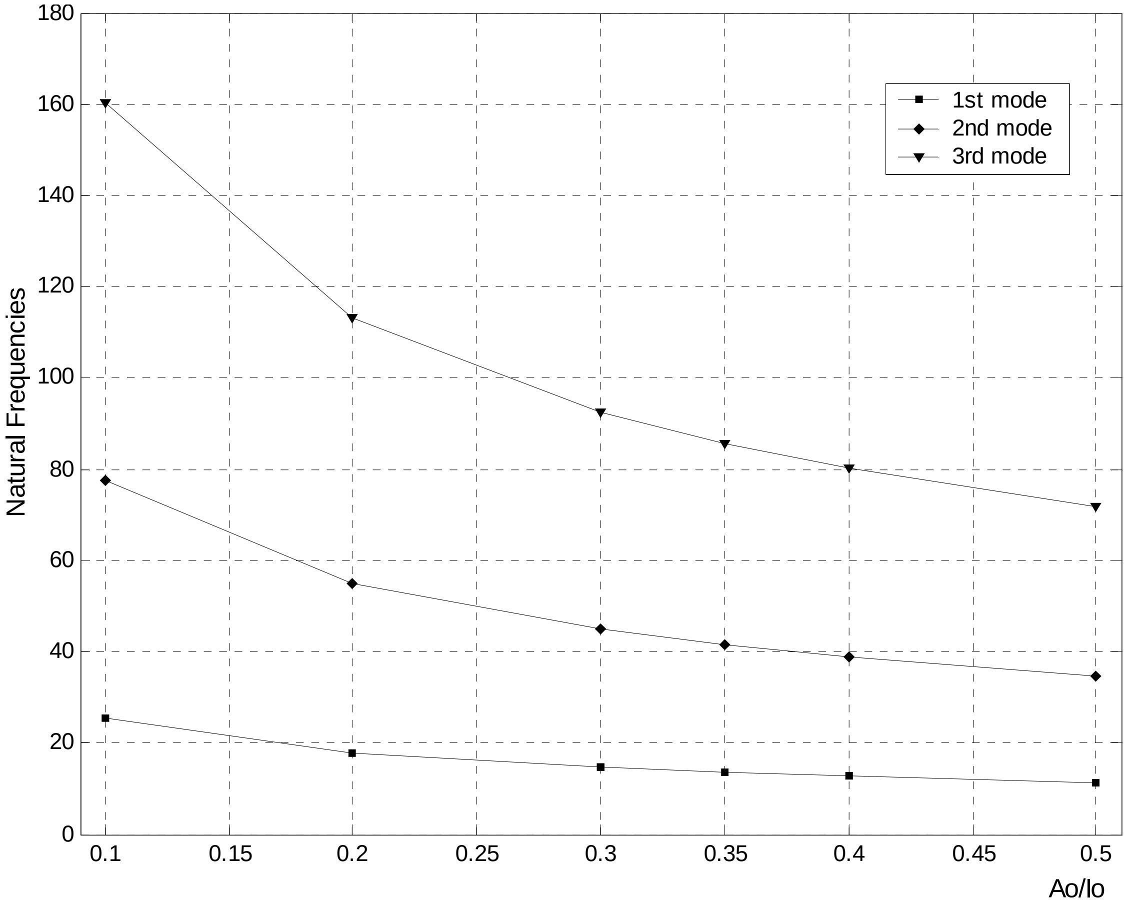

Figure 5.6. The variation of the first three frequencies in [*rad./sec.*]with varying A_0 / I_0 ratio

The natural frequencies for the first six modes of transverse vibration with different negative and positive values of *c* and *b* are given in Tables 5.6. and 5.7. The variation of the first three frequencies is plotted in Figure 5.7. It can be seen clearly from Figures 5.4., 5.5., and 5.7., that as the cross-sectional area and moment of inertia of the beam increase, the frequency of vibration also increases. By changing these coefficients the following values remain constant:

$$2L = 100m \quad \rho = 1000kg/m^3 \quad E = 20x10^9 kgf/m^2 \quad A_0 = 7m^2 \quad I_0 = 20m^4$$

Table 5.6. The natural frequencies in [rad./sec.] for positive *b* and *c*

	b=c					
Mode # N	0.0004	0.0003	0.0002	0.0001	0.00005	0
1	10.35	13.56	15.10	16.22	16.70	17.13
2	35.13	41.52	44.17	45.91	46.59	47.19
3	76.41	85.69	89.09	91.15	91.92	92.58
4	132.83	145.13	149.14	151.41	152.24	152.94
5	205.41	220.18	224.64	227.05	227.91	228.63
6	292.51	310.12	314.97	317.49	318.38	319.11

Table 5.7. The natural frequencies in [rad./sec.] for negative *b* and *c*

	b=c					
Mode # N	-0.0004	-0.0003	-0.0002	-0.0001	-0.00005	0
1	19.71	19.17	18.57	17.90	17.53	17.13
2	50.27	49.69	49.01	48.20	47.72	47.19
3	95.80	95.21	94.51	93.66	93.15	92.58
4	156.21	155.62	154.92	154.05	153.53	152.94
5	231.92	231.34	230.64	229.76	229.24	228.63
6	322.41	321.83	321.13	320.25	319.72	319.11

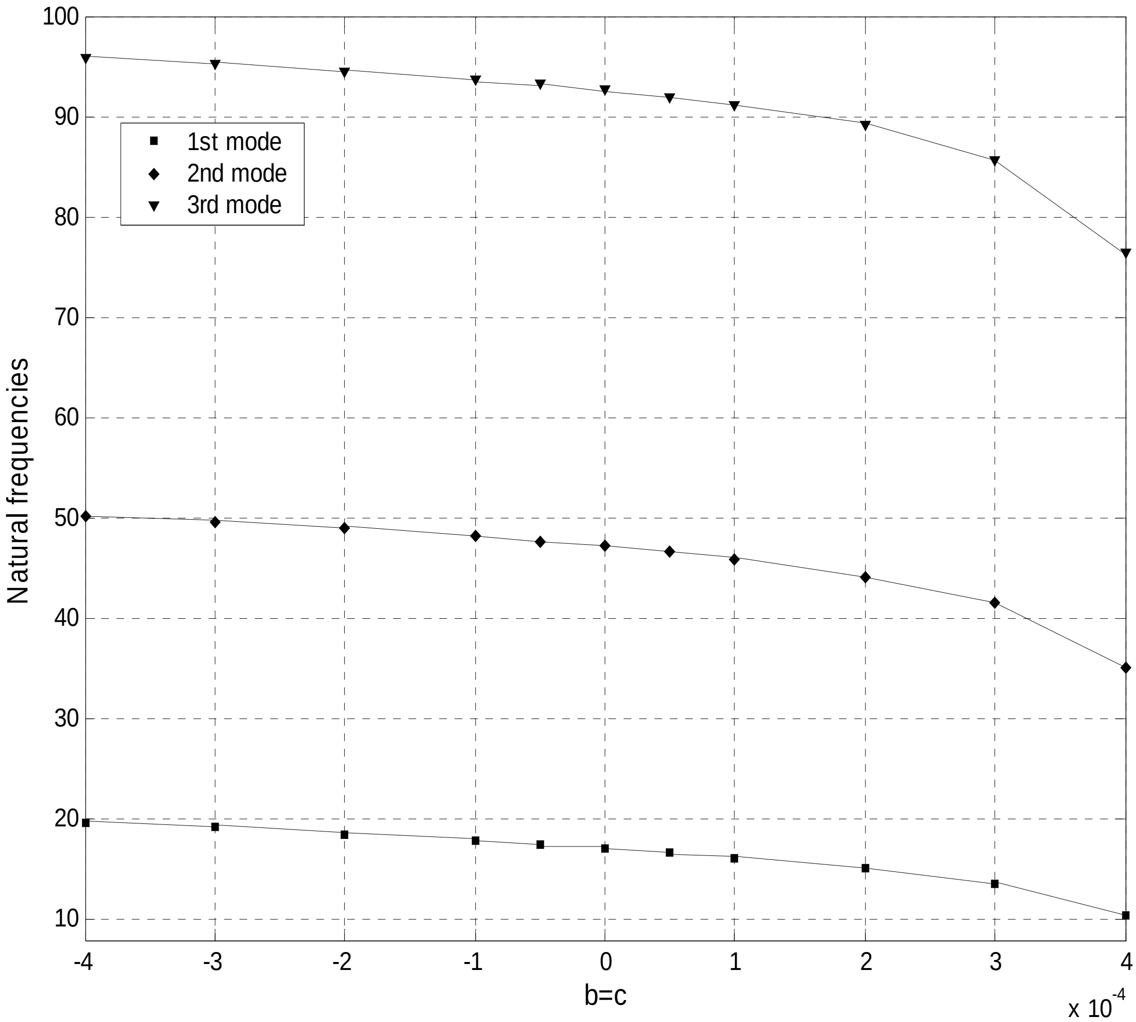

Figure 5.7. Natural frequencies of the first three modes of vibration for different cross-sections of the beam

6. CONCLUSIONS

In this study, the effect of weighted residual methods and trial functions on the convergence of approximate solutions for natural frequencies of transverse vibration of an elastic beam having variable system parameters is analyzed and after finding the best approximation, the variation of natural frequencies with changing system parameters is determined.

The resulting fourth order linear differential equation which is based on Euler-Bernoulli beam theory and governs the free vibration in transverse direction of the non-uniform beam cannot be solved analytically because the system parameters appear as coefficients depending on spatial variable. The closed form solution of that kind of equation is also not available in the literature. Therefore Galerkin, collocation and subdomain methods from weighted residual methods are applied to find an approximate solution. Weighted residual methods constitute the optimization step of the trial solution methods which discretize the distributed parameter system by truncation. Beams can be considered as systems having infinite number of degrees of freedom because they consist of infinite number of connected points. With trial solution methods we limit the number of degrees of freedom and find an approximate solution of the problem.

In Chapter 3 we have compared the exact solutions of a uniform free-ended beam for the first five natural frequencies with the approximate solutions found with Galerkin Method and power series. Although the approximate solutions did not approach to the exact values even for thirty degrees of freedom they appear to have a converging trend. The error for the fundamental frequency was 1.31% and for the fifth mode of vibration it was only 0.76%. Because there is a lack of similar study to the present one in the literature, we could not attain any further comparison.

In Chapter 4 we conducted several trials with weighted residual methods after constructing trial solutions in form of a finite sum of series of known functions called as trial functions or basis or coordinate functions which satisfy the boundary conditions and tried to find the best approximate solution by minimizing the expression for the residual error in the differential equation through weighted averages. We showed that Galerkin Method with power series as trial function is capable of finding the natural frequencies of the non-uniform free-ended beam easily without any need to find the coefficients of shape functions. subdomain and collocation methods were very sensitive to where the subdomain intervals and collocation points are located respectively. Putting these points correctly on the interior domain needs a strong theoretical background. Also the choice of the trial function for best approximation has crucial impact on the convergence of the solution. The values found for the first six natural frequencies using six degrees of freedom with Galerkin Method and power series were consistent with those obtained by Söylemez *et. al.*(2005) and the solution of fundamental frequency obtained using Galerkin Method with trial function which consists of *cos* and *cosh* series was consistent with the value obtained by Timoshenko (1937) who used Rayleigh-Ritz Method.

In Chapter 5 we tried to determine the natural frequencies of the free-ended beam for different system parameters. We have plotted the variation of the first few frequencies by changing the physical parameters such as mass density and modulus of elasticity and by changing the geometry of the beam such as the cross-sectional area and moment of inertia. In each case we have obtained reasonable results with Galerkin Method which was easy to construct and apply and results quickly.

The analysis in this study is limited to a definite geometry and free-ended boundary conditions and finding only the natural frequencies and not the mode shapes of vibration of the beam. In further studies the modes of beams with other geometry and boundary conditions may be investigated. It is well known that the eigenvalues which result from the set of homogenous equations defining the transverse displacement and provide the natural frequencies of vibration must be real numbers. The reason and further improvement of some complex values obtained with collocation and subdomain method may be issue of further studies.

APPENDIX A: MATLAB CODES

```
clear all
close all
digits(16);

% This code evaluates approximate frequencies of free-free beam

% with GALERKIN METHOD

% The trial function y is chosen as POWER SERIES

 n=6;     % n determines the lenght of the series

syms x a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 wn real;

E=20000000;
b=0.0003;
c=0.0003;
Ao=7;
Io=20;
rho=1;
L=50;

a=[a1;a2;a3;a4;a5;a6;a7;a8;a9;a10];

   for i=1:n;
      y(i)=a(i)*x.^(i-1);
   end

   Y=((1-(x/L)^2)^4).*y;

   Y4=diff(Y,'x',4);
   Y3=diff(Y,'x',3);
   Y2=diff(Y,'x',2);
   A=Ao*(1-c*x^2);
   I=Io*(1-b*x^2);

   Error=(1-b*x^2)*Y4-4*b*x*Y3-2*b*Y2-((rho*Ao*wn^2)/(E*Io))*(1-c*x^2)*Y;

   for j=1:n;
      Fi(j)=((1-(x/L)^2)^4)*x^(j-1);
   end

   for i=1:n;
      for j=1:n;
```

```
        edelE(i,j)=Error(i)*Fi(j);
      end
   end

   EdelE=int(edelE,x,-L,L);
   D=det(EdelE);
   d=vpa(D);

   S=solve(d,wn);
   s=vpa(S);     % s gives the frequencies
```

```
clear all
close all
digits(16);

% This code evaluates approximate frequencies of free-free beam

% with COLLOCATION METHOD

% The trial function y is chosen as POWER SERIES

 n=11;     % n determines the lenght of the series

syms x a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 a11 a12 a13 wn;

a=[a1;a2;a3;a4;a5;a6;a7;a8;a9;a10;a11;a12;a13];

E=20000000;
b=0.0003;
c=0.0003;
Ao=7;
Io=20;
rho=1;
L=50;

   for i=1:n;
      y(i)=a(i)*x.^(i-1);
   end

   Y=((1-(x/L)^2)^4).*y;

   Y4=diff(Y,'x',4);
   Y3=diff(Y,'x',3);
   Y2=diff(Y,'x',2);
```

```
    A=Ao*(1-c*x^2);
    I=Io*(1-b*x^2);

    Error=(1-b*x^2)*Y4-4*b*x*Y3-2*b*Y2-((rho*Ao*wn^2)/(E*Io))*(1-c*x^2)*Y;

    Err(1:n,1)=subs(Error,x,-25);
    Err(1:n,2)=subs(Error,x,-20);
    Err(1:n,3)=subs(Error,x,-15);
    Err(1:n,4)=subs(Error,x,-10);
    Err(1:n,5)=subs(Error,x,-5);
    Err(1:n,6)=subs(Error,x,0);
    Err(1:n,7)=subs(Error,x,5);
    Err(1:n,8)=subs(Error,x,10);
    Err(1:n,9)=subs(Error,x,15);
    Err(1:n,10)=subs(Error,x,20);
    Err(1:n,11)=subs(Error,x,25);

    D=det(Err);
    d=vpa(D);
    S=solve(d,wn);
    s=vpa(S);     % s gives the frequencies



clear all
close all
digits(16);

% This code evaluates approximate frequencies of free-free beam

% with SUBDOMAIN METHOD

% The trial function y is chosen as POWER SERIES

 n=6;     % n determines the lenght of the series

syms x a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 a11 a12 a13 wn;

a=[a1;a2;a3;a4;a5;a6;a7;a8;a9;a10;a11;a12;a13];

E=20000000;
b=0.0003;
c=0.0003;
Ao=7;
Io=20;
rho=1;
L=50;
```

```
   for i=1:n;
      y(i)=a(i)*x.^(i-1);
   end

   Y=((1-(x/L)^2)^4).*y;

   Y4=diff(Y,'x',4);
   Y3=diff(Y,'x',3);
   Y2=diff(Y,'x',2);
   A=Ao*(1-c*x^2);
   I=Io*(1-b*x^2);

   Error=(1-b*x^2)*Y4-4*b*x*Y3-2*b*Y2-((rho*Ao*wn^2)/(E*Io))*(1-c*x^2)*Y;

   Err(1:n,1)=int(Error,x,-5,0);
   Err(1:n,2)=int(Error,x,0,5);
   Err(1:n,3)=int(Error,x,-10,0);
   Err(1:n,4)=int(Error,x,0,10);
   Err(1:n,5)=int(Error,x,-15,-10);
   Err(1:n,6)=int(Error,x,10,15);
%      Err(1:n,7)=int(Error,x,-20,-15);
%      Err(1:n,8)=int(Error,x,15,20);
%      Err(1:n,9)=int(Error,x,-25,-20);
%      Err(1:n,10)=int(Error,x,20,25);
%      Err(1:n,11)=int(Error,x,-30,-25);
%      Err(1:n,12)=int(Error,x,25,30);


   D=det(vpa(Err));
   d=vpa(D);
   S=solve(d,wn);
   s=vpa(S);    % s gives the frequencies
```

```
clear all
close all
digits(16);

% This code evaluates approximate frequencies of free-free beam

% with SUBDOMAIN METHOD

% The trial function y is chosen as TRIGONOMETRIC SERIES
```

```
n=12;    % n determines the lenght of the series

E=20000000;
%  b=0.0003;
%  c=0.0003;
b=0;
c=0;

Ao=7;
Io=20;
rho=1;
L=50;
syms wn real;

h=0.1;
x=-50:h:50;

   for j=1:length(x);
      for i=1:n;
         Y(i,j)=((1-(x(j)./L).^2)^4).*(sin((i*pi*x(j))/L)+cos((i*pi*x(j))/L));
         %Y(i,j)=((1-(x(j)./L).^2)^4).*((sin((i*pi*x(j))/L)+cos((i*pi*x(j))/L))+...
            %(sinh((i*pi*x(j))/L)+cosh((i*pi*x(j))/L)));
         %Y(i,j)=((1-(x(j)./L).^2)^4).*(cos((i*pi*x(j))/L)+cosh((i*pi*x(j))/L));
         %Y(i,j)=((1-(x(j)./L).^2)^4).*(sin((i*pi*x(j))/L)+sinh((i*pi*x(j))/L));
      end
   end

z=-50:h:(50+4*h);

   for j=1:length(z);
      for i=1:n;
         y(i,j)=((1-(z(j)./L).^2)^4).*(sin((i*pi*z(j))/L)+cos((i*pi*z(j))/L));
         %y(i,j)=((1-(z(j)./L).^2)^4).*((sin((i*pi*z(j))/L)+cos((i*pi*z(j))/L))+...
               %(sinh((i*pi*z(j))/L)+cosh((i*pi*z(j))/L)));
         %y(i,j)=((1-(z(j)./L).^2)^4).*(cos((i*pi*z(j))/L)+cosh((i*pi*z(j))/L));
         % y(i,j)=((1-(z(j)./L).^2)^4).*(sin((i*pi*z(j))/L)+sinh((i*pi*z(j))/L));
      end
   end

% The derivatives of Y are evaluated using backward difference method.

  for i=1:n;
      for j=1:length(x);
         Y1(i,j)=(y(i,j+1)-y(i))./h;
         Y2(i,j)=(y(i,j+2)-2*y(i,j+1)+y(i,j))./(h^2);
         Y3(i,j)=(y(i,j+3)-3*y(i,j+2)+3*y(i,j+1)-y(i,j))/(h^3);
         Y4(i,j)=(y(i,j+4)-4*y(i,j+3)+6*y(i,j+2)-4*y(i,j+1)+y(i,j))./(h^4);
      end
   end
```

```
X=meshgrid(x,1:n);

   for i=1:length(x);
      A(i)=Ao.*(1-c.*x(i).^2);
      I(i)=Io.*(1-b.*x(i).^2);
   end

AA=meshgrid(A,1:n);
II=meshgrid(A,1:n);

   for i=1:n;
      for j=1:length(x);
         Error1(i,j)=II(i,j).*Y4(i,j)-4.*b.*X(i,j).*Y3(i,j)-(AA(i,j).*Y(i,j)).*...
               ((rho.*Ao.*wn.^2)./(E.*Io));
      end
   end

Error=vpa(simplify(Error1));


    for i=1:n;

        Err(i,1)=sum(Error(i,451:500)).*0.1;
        Err(i,2)=sum(Error(i,501:550)).*0.1;
        Err(i,3)=sum(Error(i,401:500)).*0.1;
        Err(i,4)=sum(Error(i,501:600)).*0.1;
        Err(i,5)=sum(Error(i,351:400)).*0.1;
        Err(i,6)=sum(Error(i,601:650)).*0.1;
        Err(i,7)=sum(Error(i,301:350)).*0.1;
        Err(i,8)=sum(Error(i,251:300)).*0.1;
        Err(i,9)=sum(Error(i,651:700)).*0.1;
        Err(i,10)=sum(Error(i,701:750)).*0.1;
        Err(i,11)=sum(Error(i,201:250)).*0.1;
        Err(i,12)=sum(Error(i,751:800)).*0.1;
   end


D=det(vpa((Err)));
d=vpa(D);

S=solve(d,wn);
s=vpa(S)    % S gives the frequencies
```

```
clear all
close all
digits(16);

% This code evaluates approximate frequencies of free-free beam

% with GALERKIN METHOD

% The trial function y is chosen as TRIGONOMETRIC SERIES

n=3;     % n determines the lenght of the series

E=20000000;
b=0.0003;
c=0.0003;
Ao=7;
Io=20;
rho=1;
L=50;
syms wn real;

h=0.1;
x=-50:h:50;

   for j=1:length(x);
      for i=1:n;
         Y(i,j)=((1-(x(j)./L).^2)^4).*(sin((i*pi*x(j))/L)+cos((i*pi*x(j))/L));
         %Y(i,j)=((1-(x(j)./L).^2)^4).*((sin((i*pi*x(j))/L)+cos((i*pi*x(j))/L))+...
         %      (sinh((i*pi*x(j))/L)+cosh((i*pi*x(j))/L)));
         %Y(i,j)=((1-(x(j)./L).^2)^4).*(cos((i*pi*x(j))/L)+cosh((i*pi*x(j))/L));
         %Y(i,j)=((1-(x(j)./L).^2)^4).*(sin((i*pi*x(j))/L)+sinh((i*pi*x(j))/L));
      end
   end

z=-50:h:(50+4*h);

   for j=1:length(z);
      for i=1:n;
         y(i,j)=((1-(z(j)./L).^2)^4).*(sin((i*pi*z(j))/L)+cos((i*pi*z(j))/L));
         %y(i,j)=((1-(z(j)./L).^2)^4).*((sin((i*pi*z(j))/L)+cos((i*pi*z(j))/L))+...
          %      (sinh((i*pi*z(j))/L)+cosh((i*pi*z(j))/L)));
         %y(i,j)=((1-(z(j)./L).^2)^4).*(cos((i*pi*z(j))/L)+cosh((i*pi*z(j))/L));
         % y(i,j)=((1-(z(j)./L).^2)^4).*(sin((i*pi*z(j))/L)+sinh((i*pi*z(j))/L));
      end
   end

% The derivatives of Y are evaluated using backward difference method.
```

```
  for i=1:n;
     for j=1:length(x);
        Y1(i,j)=(y(i,j+1)-y(i))./h;
        Y2(i,j)=(y(i,j+2)-2*y(i,j+1)+y(i,j))./(h^2);
        Y3(i,j)=(y(i,j+3)-3*y(i,j+2)+3*y(i,j+1)-y(i,j))/(h^3);
        Y4(i,j)=(y(i,j+4)-4*y(i,j+3)+6*y(i,j+2)-4*y(i,j+1)+y(i,j))./(h^4);
     end
  end

X=meshgrid(x,1:n);

  for i=1:length(x);
     A(i)=Ao.*(1-c.*x(i).^2);
     I(i)=Io.*(1-b.*x(i).^2);
  end

AA=meshgrid(A,1:n);
II=meshgrid(A,1:n);

  for i=1:n;
     for j=1:length(x);
        Error(i,j)=II(i,j).*Y4(i,j)-4.*b.*X(i,j).*Y3(i,j)-(AA(i,j).*Y(i,j)).*...
              ((rho.*Ao.*wn.^2)./(E.*Io));
     end
  end

T=Y';

  for i=1:n;
     Error(i,j)=0;
     T(j,i)=0;
  end

edelE=(Error*T).*h;
D=det(vpa(edelE));
S=solve(D,wn)      % S gives the frequencies




clear all
close all
digits(16);

% This code evaluates approximate frequencies of free-free beam
```

```
% with COLLOCATION METHOD

% The trial function y is chosen as TRIGONOMETRIC SERIES

n=2;    % n determines the lenght of the series

E=20000000;
b=0.0003;
c=0.0003;
Ao=7;
Io=20;
rho=1;
L=50;
syms wn real;

h=0.1;
x=-50:h:50;

  for j=1:length(x);
    for i=1:n;
      Y(i,j)=((1-(x(j)./L).^2)^4).*(sin((i*pi*x(j))/L)+cos((i*pi*x(j))/L));
      %Y(i,j)=((1-(x(j)./L).^2)^4).*((sin((i*pi*x(j))/L)+cos((i*pi*x(j))/L))+...
        %(sinh((i*pi*x(j))/L)+cosh((i*pi*x(j))/L)));
      %Y(i,j)=((1-(x(j)./L).^2)^4).*(cos((i*pi*x(j))/L)+cosh((i*pi*x(j))/L));
      %Y(i,j)=((1-(x(j)./L).^2)^4).*(sin((i*pi*x(j))/L)+sinh((i*pi*x(j))/L));
    end
  end

z=-50:h:(50+4*h);

  for j=1:length(z);
    for i=1:n;
      y(i,j)=((1-(z(j)./L).^2)^4).*(sin((i*pi*z(j))/L)+cos((i*pi*z(j))/L));
      %y(i,j)=((1-(z(j)./L).^2)^4).*((sin((i*pi*z(j))/L)+cos((i*pi*z(j))/L))+...
          %(sinh((i*pi*z(j))/L)+cosh((i*pi*z(j))/L)));
      %y(i,j)=((1-(z(j)./L).^2)^4).*(cos((i*pi*z(j))/L)+cosh((i*pi*z(j))/L));
      % y(i,j)=((1-(z(j)./L).^2)^4).*(sin((i*pi*z(j))/L)+sinh((i*pi*z(j))/L));
    end
  end

% The derivatives of Y are evaluated using backward difference method.

  for i=1:n;
    for j=1:length(x);
      Y1(i,j)=(y(i,j+1)-y(i))./h;
      Y2(i,j)=(y(i,j+2)-2*y(i,j+1)+y(i,j))./(h^2);
      Y3(i,j)=(y(i,j+3)-3*y(i,j+2)+3*y(i,j+1)-y(i,j))/(h^3);
      Y4(i,j)=(y(i,j+4)-4*y(i,j+3)+6*y(i,j+2)-4*y(i,j+1)+y(i,j))./(h^4);
    end
```

```
    end

X=meshgrid(x,1:n);

    for i=1:length(x);
        A(i)=Ao.*(1-c.*x(i).^2);
        I(i)=Io.*(1-b.*x(i).^2);
    end

AA=meshgrid(A,1:n);
II=meshgrid(A,1:n);

    for i=1:n;
        for j=1:length(x);
            Error(i,j)=II(i,j).*Y4(i,j)-4.*b.*X(i,j).*Y3(i,j)-(AA(i,j).*Y(i,j)).*...
                ((rho.*Ao.*wn.^2)./(E.*Io));
        end
    end


  Err(1:n,1)=Error(1:n,501);
  Err(1:n,2)=Error(1:n,551);
%   Err(1:n,3)=Error(1:n,501);
%   Err(1:n,4)=Error(1:n,511);
%   Err(1:n,5)=Error(1:n,521);
%   Err(1:n,6)=Error(1:n,481);
%   Err(1:n,7)=Error(1:n,501);
%   Err(1:n,8)=Error(1:n,521);
%   Err(1:n,9)=Error(1:n,551);
%   Err(1:n,10)=Error(1:n,601);
%   Err(1:n,11)=Error(1:n,651);
%   Err(1:n,12)=Error(1:n,701);
%   Err(1:n,13)=Error(1:n,751);
%   Err(1:n,12)=Error(1:n,701);
%   Err(1:n,13)=Error(1:n,751);
%   Err(1:n,13)=Error(1:n,501);
%   Err(1:n,13)=Error(1:n,501);



    Err1=vpa(simplify(Err));
    D=det(Err1);
    d=vpa(simplify(D));
    S=solve(d,wn);      % S gives the frequencies
```

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