# COSMOLOGY AND SPONTANEOUS SYMMETRY BREAKING

by

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# Abstract

We investigate two problems in this work. In the first, the basic motivation is to incorporate spontaneous symmetry breaking and gravitation into the context of a scalar-tensor theory. Non-zero value of the minimum of the potential provides the present-day value of the gravitational constant. We present sample solutions for different potentials, however, our main emphasis is on the globally conformal invariant case. The solutions are approximate in the sense that they are characterized by the scale of the vacuum expectation value of the scalar field. In the Einstein frame, an equation of state is derived. According to this equation of state, cosmology is dominated by the scalar field both in the early and the present universe and the scalar field supplies the critical density of the universe today. The mass of the scalar field, depending on its vacuum expectation value, is determined by the present-day values of the Planck mass and the Hubble constant. For Electroweak Theory Higgs field this leads to a long-range coupling stronger than gravity, between the scalar field and matter. The second problem contains systematic means for finding inhomogeneous solutions. In the second problem, we apply the long-wavelength approximation to the low-energy effective string action in the context of the Hamilton-Jacobi theory. The Hamilton-Jacobi equation for the effective string action is explicitly invariant under scale factor duality. We present the leading-order, general solution of the Hamilton-Jacobi equation. The Hamilton-Jacobi approach yields a solution consistent with the Lagrange formalism. The momentum constraints take an elegant, simple form. Furthermore, this general solution reduces to the quasi-isotropic one, if the evolution of the gravitational radiation is neglected. The duality transformation for the general solution is written as a coordinate transformation in an abstract space of fields.

# Özet


Bu çalışmada iki problem incelendi. İlkinde temel amaç simetri bozulması ve gravitasyonun bir skalar-tensor teorisi çerçevesinde birleştirilmesi olup, Newton'un gravitasyon sabitinin bugünkü değeri skalar alanın vakum beklenen değeriyle belirlenmektedir. Değişik potansiyeller için örnek çözümler sunulmakla birlikte, özellikle global konformal invaryans üzerinde duruldu. Bu durum için skalar alanın vakum beklenen değeriyle karakterize edilen yeni, yaklaşık çözümler sunuldu. Einstein çerçevesinde bir hal denklemi çıkarıldı, buna göre Evren'in başlangıcında ve bugün, kozmolojinin skalar alanca belirlendiği ve Evren'in kritik kütlesinin bu skalar alan tarafından sağlandığı gösterildi. Ayrıca skalar alanın kütlesi Planck sabiti ve Hubble sabiti'nin bugünkü değerleri cinsinden hesaplandı. Bu, örneğin Elektrozayıf etkileşme'nin Higgs alanı için gravitasyondan daha kuvvetli ve uzun mesafeli bir etkileşmeye neden olmaktadır. İkinci problemde ise uzun-dalgaboyu yaklaşımı, Hamilton-Jacobi denklemi çerçevesinde, düşük enerji etkin sicim eylemine uygulandı. Hamilton-Jacobi denklemi'nin evrenin ölçek çarpan dualitesi altında değişmez olduğu gösterildi. Sunulan çözüm birinci mertebe olup Lagrange denklemleriyle tutarlıdır. Momentum denklemleri ise daha basit bir şekilde yazıldı. Eğer gravitasyon alanı ihmal edilirse bu çözüm yarı-isotropik çözüme inmektedir. Ayrıca dualite dönüşümü soyut bir uzayda koordinat dönüşümü olarak yazıldı.

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# List of Symbols

$G$: Gravitational constant.

$G_F$: Fermi coupling constant.

$e$: Electric charge of the electron.

$m_e$: Mass of the electron.

$m_p$: Mass of the proton.

$c$: Speed of light.

$\hbar$: Planck's constant.

$M_p$: Planck mass.

$l_p$: Planck length.

$t$: Age of the Universe (Section 2. 1. 2).

$H$: Hubble parameter.

$N$: Number of nucleons in the Universe (Section 2. 1. 2).

$R$: Radius of the Universe (Section 2. 1. 2).

$M_u$: Total mass of the Universe.

$S$: Action.

$\phi$: Scalar field (Section 2.2).

$g_{\mu\nu}$: Metric.

$R_{\mu\nu}$: Ricci tensor.

$R$: Scalar curvature.

$V(\phi)$: Potential for the scalar field.

$a$: Scale factor in Brans-Dicke frame.

$k$: Curvature parameter.

$b$: Scale factor in Einstaein frame.

$\Omega$: Conformal factor.

$\Lambda$: Cosmological constant.

$\lambda$: Scaled cosmological constant.

$N$: Lapse function (Section 6).

$N_i$: Shift vector (Section 6).

$\gamma_{ij}$: 3-metric.

$\pi^{\phi}$: Momentum of the dilaton.

$\pi^{ij}$: Momenta of the gravitational field.

$Tr$: Trace.

# 1. INTRODUCTION

We investigate two problems in this thesis. The first problem aims to incorporate gravitation and spontaneous symmetry breaking. In the second problem we formulate the long-wavelength approximation for low energy effective string action.

Among the fundamental interactions, gravitation describes the large scale structure of the Universe. The Electromagnetic, Weak and Strong forces describe the world of particles. Compared to gravity these forces are effectively short ranged and much stronger. In the standard model of Strong and Electroweak interactions, particles are fundamentally assumed to be massless. They acquire mass through the spontaneous symmetry breaking mechanism which has a central role in the Standard theory of Electroweak interactions. One speaks of spontaneous symmetry breaking when a system possesses a symmetry that is not displayed by the ground state. According to General Relativity, spacetime curvature is determined by the energy and matter distribution. The curvature and the energy density have the dimensions of *(energy)*$^2$ and *(energy)*$^4$ respectively. The dimensional factor relating the curvature and the energy density is provided by the gravitational constant $G \doteq 6.7 \times 10^{-39} (GeV)^{-2}$. It also determines the strength of the gravitational interaction. In scalar-tensor theories of gravitation, the gravitational constant is determined by a scalar field. Therefore the spatially and temporally constant vacuum expectation value of the scalar field plays the role of a modern aether which fills all of space for all times.

If we compare the standard theory of fundamental particles and gravitation, mass, which is produced by the spontaneous symmetry breaking mechanism, determines the curvature of spacetime. In other words, on one hand we have spontaneous symmetry breaking for explaining how fundamental particles acquire mass, on the other hand mass distribution determines the curvature of spacetime. Therefore, we are lead to expect a relation between spontaneous symmetry breaking and spacetime curvature. This is the basic motivation of this thesis. More specifically, if we compare the Standard model of Electroweak interactions and gravitation, we expect a relation which is similar to the one in Electroweak theory [1], [2].

$$G_F \sim \frac{1}{\phi_o^2} \tag{1.1}$$

where $G_F$ is the Fermi coupling and $\phi_o$ is the vacuum expectation value of a scalar field. Therefore we replace the gravitational constant $G$ by a scalar field. We also include a potential for the scalar field. In certain cases this model posseses conformal invariance. However choice of a certain value for the scalar field in accordance with the gravitational constant causes breaking of this symmetry. We investigate the cosmological consequences of spontaneous breaking of conformal symmetry in the context of a scalar-tensor theory. However, contrary to the Weak interaction, gravity is long ranged and the graviton is massless. A nonminimal coupling of gravitation with the scalar field leads to complications as we shall see.

Scalar-tensor theories have a long history. They date back to theories of Jordan [3, 4] and Brans-Dicke [5, 6]. In the Brans-Dicke theory, which is mainly motivated by Mach's principle and Large Numbers Hypothesis [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], the gravitational constant $G$ is replaced by a scalar field. Mach's principle can be briefly stated as 'energy-matter there governs inertia here'. The hypothesis that large dimensionless numbers should be proportional to the cosmic time $t$ raised to some simple power is known as the Large Numbers Hypothesis (LNH). The appearance of large numbers when comparing the properties of particles and those of the universe has attracted the attention of physicists. One such large number is the ratio of the electric to the gravitational force between an electron and a proton in a hydrogen atom, which is about $2\times 10^{39}$. Dirac's Large Numbers Hypothesis and Mach's principle have motivated many ideas for incorporating gravitation with the other interactions.

The idea of deriving $G$ through a spontaneous symmetry-breaking mechanism has been studied in the context of scalar-tensor theories. Ideas underlying spontaneous symmetry breaking in gravitation can be traced back to Zel'dovich [24] and Sakharov [25]. Zel'dovich suggested that the gravitational interactions of virtual particles in vacuo (vacuum fluctuations) may endow empty space with an effective energy density and pressure which can be interpreted as a cosmological constant. This led Sakharov to propose that vacuum fluctuations actively contribute to the properties of the gravitational field in vacuo. That is, he suggested that there should exist a relationship between the quantum nature of the vacuum and the curvature of space. He postulated that the lagrangian density for these vacuum fluctuations may be written as a series expansion in the curvature and the momenta of the virtual particles. This line of work is followed by Adler [26, 27, 28] who has proposed that a $\phi^2 R$ term would be induced by a composite field $\phi$, and that gravity will arise as a result of spacetime acquiring curvature under quantum corrections.

Similar models have also been studied by Smolin [29] and Zee [30, 31, 32]. We are going to explain the relation of our work to that of Zee and Smolin in the next section. These models and variants of it (with or without a potential) have also been studied in the context of inflationary cosmology. They are generically called induced gravity or extended inflationary models, depending on the type of inflationary behaviour, [33, 34, 35]. They are improvements upon the original old inflationary scenario [36, 37, 38, 39, 40].

In this thesis our aim is to investigate a model which incorporates gravitation and spontaneous symmetry breaking in the context of a scalar-tensor theory. In the second section, after giving a brief account of Mach's Principle and the Large Numbers

Hypothesis, we present the model and the field equations. The third section contains sample solutions which are homogeneous and isotropic. However our emphasis will be on the globally conformal invariant case. In particular, we will investigate this model for a flat universe and a quartic self-coupling . We also give a qualitative account of a non-flat universe for this potential. In the fourth section we derive an equation of state for the Einstein frame variables, which is determined by an interplay between the kinetic and the potential terms of the scalar field. If the scalar field at present is oscillating about the minimum of its potential, then its energy density may provide the critical density of the universe [41], [42]. We show that in this model the critical density of our universe is provided by the scalar field. We also derive the mass of the scalar field for this potential classically. The mass of the scalar field, depending on its vacuum expectation value, is determined by the present-day values of the Planck mass and the Hubble constant. We also discuss the scale of the scalar field $\phi$ and the related parameters. This enables us to study the cosmological implications of this model. Although this result is subject to quantum corrections, it gives an estimate of the validity of the classical results of the model. Then in the fifth section we start to investigate the second problem. We present a brief summary of hamilton formalism and the canonical transformations for low energy effective string action in the sixth section. In the seventh section we show that the Hamilton-Jacobi equation is invariant under scale factor duality. In section eight and nine we present the solution and we write the momentum constraint in a simpler form. We investigate the duality property of the solution and we write it as a coordinate transformation in an abstract space of fields in the tenth section. In the last section we present a summary of the problems and the results. The appendix contain the conventions, necessary constants of Nature and other equations used in the text.

# 2. THE GENERAL FRAMEWORK

As it was explained very briefly in the previous section the scalar field is an important ingredient of the models describing fundamental interactions of Nature. We are going to investigate the cosmological consequences of a nonminimal coupling between gravitation and the scalar field. We first present a brief account of Mach's principle and Dirac's Large Numbers Hypothesis since they are the underlying motivation for scalar-tensor theories historically. Meanwhile the tests of Mach's principle may also provide cosmological constraints on our model.

## 2.1. Mach's Principle and The Large Numbers Hypothesis

The Large Numbers Hypothesis and Mach's principle have inspired many ideas for incorporating gravitation with the other interactions. In this section, we present a brief account of Mach's Principle and the Large Numbers Hypothesis.

### 2.1.1. Mach's Principle

Mach's principle can be shortly stated as 'matter-energy there governs inertia here'. In the special theory of relativity a specific set of reference frames, the inertial frames, is singled out. If one inertial frame is known, then all the others are obtained from the given frame by carrying out Lorentz transformations. But in a particular region of space we have no principle for determining the inertial frames a-priori. We have to carry out dynamical experiments in that region to determine them.

According to the principle of relativity, we know that velocities are not absolute. We can not ask about the velocity of a particle in an otherwise empty space, we can measure only relative velocities. This circumstance is reflected in the mathematical structure as the equivalence of inertial frames. However the same is not true for acceleration. The acceleration of a test particle is the same in all inertial frames. Therefore we can speak of the acceleration of a test particle even in the absence of other matter. Mach interpreted this as a lack of correspondence between the mathematical machinery and the physics it is designed to describe. The Newton argument in favour of the notion of absolute acceleration was the rotating bucket experiment: whether the water in a bucket is rotating or not can be determined by observing whether the surface of the water is concave or flat, without the use of external reference points. However later it was argued that the water surface is flat when the water is at rest relative to the stars, so that the experiment determines rotation relative to the fixed stars, and therefore rotation is after all a relative rather than not an a bsolute concept. Mach developed this idea further, by putting forward the view that the inertial forces on a body are brought about by the acceleration of the body relative to the distant matter of the universe, and that distant matter is somehow responsible for the inertial properties of local phenomena. According to this view, the local inertial frames are determined as those which are not accelerated relative to some averaged motion of distant matter. In other words, the distribution and motion of all the matter in the universe determines which reference frames are the inertial ones.

In the General Theory of Relativity Mach's principle is incorporated partially. For example consider a completely empty, flat spacetime ($R_{\alpha\beta\mu\nu} = 0$), the inertial frames ($g_{\mu\nu} = \eta_{\mu\nu}$) can be defined even in the complete absence of matter. Another less trivial example is provided by Gödel's solution, which is a cosmological solution without the usual restriction that the geodesics of matter are orthogonal to the 3-spaces, and it describes a universe that is homogeneous but not isotropic. The local inertial frames in this solution are not those that are unaccelerated relative to the matter as a whole. If Newton's rotating bucket experiment were carried out in a Gödel universe, the water surface would be concave when the water was at rest relative to the 'fixed stars', and when the water surface was flat there would be a relative angular velocity of the water relative to the stars. Thus there exist solutions of the equations of General Relativity that violate Mach's principle. In order to incorporate this principle, Brans and Dicke proposed to replace the gravitational constant by a scalar field which accounts for the effect of the distant matter. However this causes non-Einsteinian effects such as the violation of the equivalence principle. Therefore classical tests of General Relativity provide constraints on these type of models. Meanwhile inclusion of a potential for the scalar field also changes the non-Einsteinian effects. Nevertheless, the known constraints are not as precise as to discriminate between different models.

### 2.1.2. The Large Numbers Hypothesis

A general belief among physicists is that dimensionless numbers, like the fine structure constant, derived from the combinations of fundemantal physical constants will eventually emerge from some as yet nonexistent theory. Some of these dimensionless numbers resulting from combining atomic and cosmological constants are, however, exceedingly large to be expected to emerge from fundamental theories that usually involve numbers like 2 or $\pi$. The appearance of large numbers when comparing the properties of particles and those of the universe has attracted the attention of physicists. The hypothesis that large dimensionless numbers should be proportional to the cosmic time $t$ raised to some simple power is known as the Large Numbers Hypothesis.

One such large number is the ratio of the electrostatic to the gravitational force between an electron and a proton (say in a hydrogen atom),

$$\frac{electric\, force}{gravitational\, force} = \frac{e^2}{Gm_pm_e} \sim 2 \times 10^{39} \quad . \tag{2.1}$$

Many years ago Dirac proposed an explanation for such large numbers. Dirac noticed that if the present age of the universe is expressed in a natural unit which is the time taken for light to travel the classical electron radius,

$$\frac{e^2}{m_ec^3} \sim 10^{-23}\, s, \tag{2.2}$$

then the age of the universe is

$$t_o = \frac{H_o^{-1}}{e^2/m_ec^3} \sim 10^{40} \quad , \tag{2.3}$$

where $H_o$ is the Hubble constant today. The suprising similarity of the two numbers above prompted Dirac to propose that the above ratio is not in fact constant but is proportional to the age of the Universe $t_o$, i.e.,

$$\frac{e^2}{Gm_em_p} \sim t_o. \tag{2.4}$$

Dirac assumed that the atomic constants such as $e$, $m_e$, $m_p$ and $\hbar$ are truly constants. In order for the above relation to be valid $G$ must be inversely proportional to $t$

$$\frac{\dot{G}}{G} = -\frac{1}{t_o} \sim 8.33 \times 10^{-11} \qquad yr^{-1} \quad . \tag{2.5}$$

According to Dirac, the number $e^2/Gm_em_p$ is large today simply because the Universe is now old.

Besides the relations above we can introduce three other relations from which many interesting large numbers follow. If we calculate the ratio of the visible radius of the

Universe, defined as $c/H_o$, to the classical electron radius, $e^2/m_ec^2$, we obtain,

$$\frac{R_o}{e^2/m_ec^2} = \frac{cH_o^{-1}}{e^2/m_ec^2} \sim 10^{40} \sim t_o. \tag{2.6}$$

According to the LNH we must have, at any time $t$,

$$\frac{m_ec^3}{e^2H} \sim t \quad . \tag{2.7}$$

Next if the estimated mass of all the matter in the Universe ($\rho \sim 10^{-31}\ \ grcm^{-3}$) is divided by the proton mass $m_p$, we derive an estimate of the total number of nucleons in the Universe at the present time. This number $N_o$ turns out to be $\sim 10^{78}$ which is remarkably close to the square of the number above. According to LNH, at any time $t$ the total number of nucleons $N$ in the universe must then have been proportional to $t^2$

$$N \sim t^2 \quad . \tag{2.8}$$

This is perhaps the most important consequence of Dirac's LNH, since it implies that the matter in the Universe increases with time.

Finally, if we take the ratio of the classical electron radius $r_e$, to the present Planck length $l_p$ defined as

$$l_p = (\hbar G/c^3)^{1/2} \sim 10^{-33}\, cm, \tag{2.9}$$

we obtain

$$\frac{e^2}{(m_e^2\hbar cG_o)^{1/2}} \sim 10^{20} \sim t_o^{1/2} \quad , \tag{2.10}$$

and so quite generally, at any time $t$,

$$\frac{e^2}{(m_e^2\hbar cG)^{1/2}} \sim t^{1/2} \quad . \tag{2.11}$$

Multiplying equations (2.7), (2.11) and using (2.8) to eliminate $t$ we obtain

$$R = c/H \sim N^{3/4}(\hbar G/c^3)^{1/2}, \tag{2.12}$$

which expresses the radius of the Universe in terms of the total number of nucleons and other fundamental physical parameters. Combining (2.7) and (2.8) we get

$$m_e c^2 = \frac{e^2 N^{1/2}}{c/H} = \frac{e^2 N^{1/2}}{R} \quad . \tag{2.13}$$

This relation has been arrived at by Narlikar, who assumed a departure from strict charge neutrality in the Universe, i.e., he assumed that charge balance is a statistical effect. Hence the above equation would be interpreted as saying that the electrostatic energy of charge fluctuations in the Universe is equal to the rest mass energy of the electron. If we now eliminate the radius of the Universe between the equations (2.12) and (2.13), we obtain a relation for the mass of the Universe $M_u$

$$M_u = m_p N \sim (e^2/m_e c^2)^4 (c^3/\hbar G)^2 m_p \quad . \tag{2.14}$$

Finally if we combine equations (2.4) and (2.7) we obtain the relation

$$GM_u H/c^3 = 1 \quad , \tag{2.15}$$

which is usually regarded as expressing Mach's principle

$$mc^2 = mGM_u/R \quad . \tag{2.16}$$

Since Dirac didn't want the LNH to contradict Einstein's General Theory of Relativity, he introduced two metrics, the atomic metric and the Einstein or the mechanical metric. The atomic metric is measured by atomic instruments and therefore results from experimental measurements. On the other hand, the Einstein field equations are written in the Einstein metric with constant $G$. As Dirac has stressed, Einstein units should be used when one deals with mechanical laws like the motion of stars, planets, etc., whereas the atomic units are to be employed when one deals with atomic or nuclear phenomena. Alternatively, Einstein units can be characterized by saying that

$$G_E \sim constant \quad , \qquad M_E \sim constant \quad , \tag{2.17}$$

whereas in atomic units

$$e,\, \hbar,\, m \sim constant. \tag{2.18}$$

How do $G$ and $M$ vary in atomic units? How do $e$, $\hbar$ and $m$ vary in Einstein units? In Einstein units

$$G_E \sim constant \quad , \qquad \left(\frac{e^2}{m^2}\right) \sim t \quad . \tag{2.19}$$

In the atomic units

$$e,\, \hbar,\, m \sim constant \quad , \qquad G \sim t^{-1} \quad . \tag{2.20}$$

From the definition of total mass of an object

$$M \sim mN \quad . \tag{2.21}$$

Therefore

$$m_E \sim t^{-2} \quad , \tag{2.22}$$

i.e., the mass of every elementary particle decreases like $(cosmic\ time)^{-2}$, when measured in Einstein units. From the equation above we obtain

$$e_E^2 \sim t^{-3} \quad , \qquad \hbar_E \sim t^{-3} \quad , \tag{2.23}$$

if $e^2/\hbar c$ is constant, as it appears to be.

## 2.2. The Action and the Field Equations

A comparison of the theories which describe the universe with those which describe the particles naturally leads to a connection between spacetime curvature and spontaneous symmetry breaking. In order to incorporate spontaneous symmetry breaking, we are going to investigate a model which is defined by the action

$$S = \int \left[ -\frac{1}{6}\kappa\phi^2 R + \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \mathcal{L}(\phi, g^{\mu\nu}, \cdots) \right] \sqrt{-g} d^4x \quad . \tag{2.24}$$

Here, in order to incorporate spontaneous symmetry breaking, depending on dimensional arguments, we put $\kappa\phi^2/6$ in place of $1/16\pi G$ in the Einstein-Hilbert action. The kinetic and the potential terms for the modulus, $\phi$, of the scalar field are written explicitly. We study only the case of real scalar field. $\mathcal{L}$ denotes the rest of the lagrangian for other matter fields. It is a scalar built out of matter fields in the usual way, but with fermion masses replaced by $g\phi$ where g are appropriate coupling constants. Gauge field couplings are left in their customary form without mass terms. They acquire mass via the minimal coupling of $\phi$ to them [43]. However matter fields are excluded in this work in order to keep things simple. For a quartic potential, we can choose it as the Standard model lagrangian [44]. In this form the action does not contain any dimensional parameters, and it is globally conformal invariant. Although we restrict ourselves to the classical aspects, we should notice that such theories are apparently renormalizable by power counting [44, 45]. We should also state that in accordance with Sakharov and Adler we regard the $\phi^2 R$ term as the entire gravitational action, rather than as just an additional contribution to the gravitational action. In accordance with Dirac's terminology we call the metric in action (2.24) the atomic metric. It is possible to write it with an Einstein metric by performing a conformal transformation. However, the atomic metric is the physical frame since it is the one measured in experiments.

Furthermore, a cosmological solution with a negative Ricci scalar can induce spontaneous breaking of conformal symmetry. The action above can be cast in to Brans-Dicke form with an additional potential, by a simple redefinition of the scalar field: $\varphi = \frac{1}{6}\kappa\phi^2$. For Brans-Dicke theory radar echo delay experiments provide an upper bound on $\omega$, and therefore on $\kappa = 3/4\omega \leq 1.5 \times 10^{-3}$ [46], where $\omega$ is the Brans-Dicke coupling constant. However if the potential has a minimum with a non-zero value, which is proportional to the Planck mass, then this constraint is invalid. Basically because the scalar field is massive in this case, this reduces the range of non-Einsteinian gravitational effects. This is a point of departure from the usual Brans-Dicke theory. We will derive sample solutions for different potentials. However our emphasis will be on the globally conformal invariant case.

The action above yields the field equations

$$\kappa\phi^2 G^{\alpha\beta} - \kappa C^{\alpha\beta} - 3\left(g^{\alpha\mu}g^{\beta\nu} - \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}\right)\phi_{;\mu}\phi_{;\nu} - 3g^{\alpha\beta}V(\phi) = 0 \quad , \tag{2.25}$$

for the metric and

$$3\left(g^{\mu\nu}\phi_{;\nu}\right)_{;\mu}+\kappa\phi R+3\frac{\partial V}{\partial\phi}=0 \quad , \tag{2.26}$$

for the scalar field. The extra terms arising from the nonminimal coupling of the scalar field and the gravitation are included in $C^{\alpha\beta}$:

$$C^{\alpha\beta}=\left(g^{\alpha\mu}g^{\beta\nu}-g^{\alpha\beta}g^{\mu\nu}\right)\left(\phi^2\right)_{;\mu;\nu} \quad . \tag{2.27}$$

Using the Bianchi identity and

$$V^{\lambda}_{\ ;\mu;\nu}-V^{\lambda}_{\ ;\nu;\mu}=-V^{\sigma}R^{\lambda}_{\ \sigma\mu\nu} \quad , \tag{2.28}$$

we find that the covariant derivative of the left-hand side of equation (2.25) is given by

$$g^{\alpha\beta}\phi_{;\beta}\left[3\left(g^{\mu\nu}\phi_{;\nu}\right)_{;\mu}+\kappa\phi R+3\frac{\partial V}{\partial\phi}\right] \quad , \tag{2.29}$$

which is identically zero by equation (2.26). This ensures the conservation of energy-momentum. Trace of equation (2.25) yields

$$\kappa\phi^2 R-3\kappa g^{\mu\nu}\left(\phi^2\right)_{;\mu;\nu}-3g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}+12V(\phi)=0 \quad . \tag{2.30}$$

Combining this with (2.26) we find that

$$\left(\kappa+\frac{1}{2}\right)\left[g^{\mu\nu}\left(\phi^2\right)_{;\nu}\right]_{;\mu}=4V-\phi\frac{\partial V}{\partial\phi} \quad . \tag{2.31}$$

A homogeneous and isotropic cosmological solution can be obtained by using the Robertson-Walker metric

$$ds^2=g^{\mu\nu}dx^{\mu}dx^{\nu}=N^2(t)\,dt^2-a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right] \quad . \tag{2.32}$$

where $k$ is the curvature parameter, with $k = -1, 0, +1$ corresponding to open, flat or closed universes respectively. Here $N$ and $a$ are only functions of time $t$. Due to reparametrization invariance, different choices of $N(t)$ correspond to choosing different time coordinates. For example $N(t) = 1$ corresponds to cosmological time.

Action (2.24) with the metric (2.32) and $\phi(t)$ reduces to a new action containing three dynamical variables, $N(t)$, $a(t)$ and $\phi(t)$. Equations of motion derived from the reduced action are equivalent to the equations (2.25) and (2.26) when metric (2.32) and the scalar field $\phi(t)$ are substituted. After eliminating the boundary terms (total divergences) we find that the reduced action is

$$S[a,\phi,N] = I \int \left[ -2\kappa a^2 \dot{a}\phi\dot{\phi}\frac{1}{N} - \kappa a\dot{a}^2\phi^2\frac{1}{N} + \kappa k a\phi^2 N + \frac{1}{2}a^3\dot{\phi}^2\frac{1}{N} - Na^3V(\phi)\right] dt \quad , \tag{2.33}$$

where $I$ contains the spatial integrals. We are going to use the reduced action for conformal transformation to Einstein Frame. The field equations are

$$\kappa\phi^2\left(a\dot{a}^2 + kaN^2\right)\frac{1}{N^2} + 2\kappa a^2\dot{a}\phi\dot{\phi}\frac{1}{N^2} - \frac{1}{2}\frac{a^3\dot{\phi}^2}{N^2} - a^3V = 0 \quad , \tag{2.34}$$

$$\kappa\phi^2\left(2a\ddot{a} + \dot{a}^2 + kN^2\right)\frac{1}{N} + 4\kappa\frac{a\dot{a}\phi\dot{\phi}}{N} + 2\kappa\frac{a^2\phi\ddot{\phi}}{N} + \left(2\kappa + \frac{3}{2}\right)\frac{a^2\dot{\phi}^2}{N} - 2\kappa\frac{a\dot{a}\phi^2\dot{N}}{N^2}$$

$$-2\kappa\frac{a^2\phi\dot{\phi}\dot{N}}{N^2} - 3a^2V = 0 \quad , \tag{2.35}$$

$$2\kappa a\phi\left(a\ddot{a} + \dot{a}^2 + kN^2\right)\frac{1}{N} - \frac{a^3\ddot{\phi}}{N} - 3\frac{a^2\dot{a}\dot{\phi}}{N} + \frac{a^3\dot{\phi}\dot{N}}{N^2} - 2\kappa\frac{a^2\dot{a}\phi\dot{N}}{N^2} - a^3\frac{\partial V}{\partial\phi} = 0 \quad . \tag{2.36}$$

If $\phi$ attains a constant value which is the minimum of the potential, in accordance with the present-day value of the gravitational constant $G$, then these equations reduce to those of standard cosmology. Therefore in this case, this theory is indistinguishable from the standard cosmology except for extreme cases.

We will use conformal transformations for solving the field equations whenever we need. Under the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu} \quad , \tag{2.37}$$

the curvature scalar changes as

$$R = \Omega^{-2} \left[ \tilde{R} - 6\tilde{g}^{\mu\nu} \left(\ln \Omega\right)_{;\mu;\nu} - 6\tilde{g}^{\mu\nu} \left(\ln \Omega\right)_{;\mu} \left(\ln \Omega\right)_{;\nu} \right] \quad , \quad g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu} \quad . \tag{2.38}$$

Then the action becomes

$$S = \int \left[ -\frac{1}{6}\kappa\phi^2\Omega^2\tilde{R} + \kappa\phi^2\tilde{g}^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega + \frac{1}{2}\Omega^2\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \Omega^4 V\left(\phi\right) \right] \sqrt{-\tilde{g}} d^4x$$

$$+ \int \kappa\phi^2\Omega^2\tilde{g}^{\mu\nu} \left(\ln \Omega\right)_{;\mu;\nu} \sqrt{-\tilde{g}} d^4x \quad . \tag{2.39}$$

Choosing the conformal factor appropriately we can solve the field equations for different potentials.

The action (2.24) has also been studied by Smolin [29] and Zee [30, 31, 32]. Zee proposed that the scalar field $\phi$ in (2.24) is the modulus of the Grand Unified Theory Higgs field. In this respect our point of view is close to that of Zee. In Smolin's model, the action does not contain any dimensional parameters. Moreover additional structures (extra internal symmetry) have been used for local scale invariance. However we choose this symmetry to be a global symmetry. For a quartic potential the lagrangian is globally conformal invariant under $\phi \to s\phi$, $g_{\mu\nu} \to s^{-2} g_{\mu\nu}$. If we include any other field in the matter lagrangian, $\mathcal{L}(\phi, g^{\mu\nu}, ...)$, then it transforms as $\Psi \to s^d\Psi$, where $d$ is the dimension of the field. The field equations for a quartic potential are also invariant under the conformal transformation in which

$$\Omega = \tilde{\phi}^{\frac{q}{1+q}} \quad , \tag{2.40}$$

where $\tilde{\phi}$ is defined as

$$\tilde{\phi} = \phi^{1+q} \quad . \tag{2.41}$$

Under this transformation, after eliminating the boundary term, we get the action as

$$S = A \int \left[ -\frac{1}{6} K \tilde{\phi}^2 \tilde{R} + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - V(\tilde{\phi}) \right] \sqrt{-\tilde{g}} d^4x \quad , \tag{2.42}$$

where $A$ and $K$ are

$$A = \frac{1 - 2\kappa q(2+q)}{(1+q)^2} \quad , \qquad K = \frac{\kappa}{A} \quad , \tag{2.43}$$

and the potential is

$$V(\tilde{\phi}) = \frac{1}{A} \tilde{\phi}^{\frac{4q}{1+q}} V(\phi) \quad . \tag{2.44}$$

The action has the same form except for the constant $A$. We see that for a quartic potential, $\lambda\phi^4$, the field equations are invariant if we define the new coefficient as $\Lambda = \frac{\lambda}{A}$. For $q = 1$, a constant potential reduces to that of a massive field. This transformation satisfies

$$\phi^2 g_{\mu\nu} = \tilde{\phi}^2 \tilde{g}_{\mu\nu} \quad . \tag{2.45}$$

In this thesis our aim is to present sample solutions of the field equations as well as to investigate cosmological consequences of this model. Our emphasis will be on globally conformal invariant case, especially on a quartic potential. In doing this we employ new variables which are defined by a conformal transformation of the metric and which measure the spacetime intervals in terms of the time varying Planck length. The solutions are characterized by a dimensionless parameter which is determined by the scale of the vacuum expectation value of the scalar field. We especially present solutions for which the parameter $\kappa$ is very large. The opposite case in which $\kappa = 3/4\omega \to 0$ corresponds to General Relativistic limit [7].

In terms of the new variables, we are going to discuss our model in a more general setting and present an equation of state which is determined by an interplay between the kinetic and the potential terms of the scalar field. If the scalar field at present is oscillating about the minimum of its potential, then its energy density can provide the critical density of the universe [41], [42]. These models may provide an explanation for dark matter. In the framework of our model, we show that the scalar field supplies the critical density of the universe. The phrase dark matter means matter whose existence has been inferred only through its gravitational effects. The strongest evidence for dark matter is spiral galaxies. The circular velocity of hydrogen clouds surrounding the galaxies typically imply a total mass of about ten times the visible mass [47]. The mass of the scalar field, depending on its vacuum expectation value, is determined by the present-day values of the Planck mass and the Hubble constant. We will also discuss the scale of the scalar field $\phi$ and of the parameters $\kappa$ and $\lambda$ and their physical implications.

We should also note that if the fields other than the gravitation and the dilaton is set to zero then the low-energy effective string action can be written as a special case of the Brans-Dicke action with $\omega = -1$ [48, 49, 50, 51, 52, 53, 54, 55]. Therefore,the action (2.24) contains the low-energy effective string action as a special case. Similar to that which has been done in the low energy effective string action, it is possible to introduce scale factor duality and Pre-Big Bang scenarios in this context. This has been done by Lidsey et al., [56, 57].

# 3. SOLUTIONS

In this section we present sample solutions for various potentials. First we study a flat universe in which the potential for the scalar field is zero or that of a massive field. Then we focus on our main concern which is the globally conformal invariant case. In the first case the potential is zero and the curvature parameter is different from zero. In the second case the universe is flat but the potential for the scalar field is quartic. Then we present a qualitative analysis of a non-flat universe with a quartic potential.

## 3.1. Sample Solutions

In this subsection we present sample solutions of the model. The field equations yield, for $N = 1$,

$$\begin{aligned} \left[a^3\left(\phi^2\right)^{\cdot}\right]^{\cdot} &- 2\frac{(\nu^2-1)}{\nu^2}a^3\left[4V - \phi\frac{\partial V}{\partial \phi}\right] = 0 \quad , \\ \left[\left(a^3\right)^{\cdot}\phi^2\right]^{\cdot} &+ 6ka\phi^2 - 3\frac{(\nu^2-1)}{\nu^2}a^3\left[\left(6\nu^2-4\right)V + \phi\frac{\partial V}{\partial \phi}\right] = 0 \quad , \end{aligned} \tag{3.1}$$

where $\nu^2$ is

$$\nu^2 = 1 + \frac{1}{2\kappa} \quad . \tag{3.2}$$

If the potential is zero or that of a massive scalar field, for $k = 0$, these equations are homogeneous in $\phi^2$ and $a^3$.

### 3.1.1. Flat Universe without a Potential

We find, using the above equations, that for $k = 0$ and $V(\phi) = 0$ the solution is

$$\begin{aligned} a^3(t) &= At^C \quad , \\ \phi^2(t) &= Bt^{1-C} \quad , \end{aligned} \tag{3.3}$$

where $A$ and $B$ are arbitrary constants and $C$ satisfies the equation

$$\left(9\nu^2 - 1\right) C^2 - 6\left(3\nu^2 - 1\right) C + 9\left(\nu^2 - 1\right) = 0 \quad . \tag{3.4}$$

Here it is interesting to look at the limiting cases in which $\kappa \to \infty$ and $\kappa \to 0$. For $\kappa \to \infty$, $\nu^2 \to 1$ and this yields $C = 0$ or $C = \frac{3}{2}$. For $\kappa \to 0$, $\nu^2 \to \infty$, and $C = 1$. This solution corresponds to equation of state $p = \rho$, in standard cosmology, (appendices equation 12.9).

### 3.1.2. Flat Universe with a Massive Scalar Field

For $V = m^2\phi^2$, the action can be written as a Brans-Dicke theory with a cosmological constant. For this potential the above equations are homogeneous.

$$\begin{aligned} \left[a^3\left(\phi^2\right)^{\cdot}\right]^{\cdot} &- 4\frac{(\nu^2-1)}{\nu^2}m^2a^3\phi^2 = 0 \quad , \\ \left[\left(a^3\right)^{\cdot}\phi^2\right]^{\cdot} &- 6\frac{(\nu^2-1)\left(3\nu^2-1\right)}{\nu^2}m^2a^3\phi^2 = 0 \quad . \end{aligned} \tag{3.5}$$

We can simply write these equations as

$$(y\dot{z})^{\cdot} - cyz = 0 \qquad , \qquad (\dot{y}z)^{\cdot} - c'yz = 0 \quad , \tag{3.6}$$

where $y = a^3$, $z = \phi^2$ and the coefficients $c$ and $c'$ are

$$c = 4\frac{(\nu^2-1)}{\nu^2}m^2 \quad , \quad c' = 6\frac{(\nu^2-1)\,(3\nu^2-1)}{\nu^2}m^2 \quad . \tag{3.7}$$

If we use $u$ and $v$ which are defined as

$$u^2 = \frac{y^{\frac{c'-c}{c'}}}{z^{\frac{c'-c}{c}}} \quad , \quad v^{2\frac{c'-c}{c'+c}} = y^{\frac{c'-c}{c'}} \, . \, z^{\frac{c'-c}{c}} \quad , \tag{3.8}$$

this yields the solution as

$$\begin{aligned} a^3(t) &= AF^{\gamma}(t)G^{\beta}(t) \quad , \\ \phi^2(t) &= BF^{1-\gamma}(t)G^{-\beta}(t) \quad . \end{aligned} \tag{3.9}$$

Here $F$ and $G$ are

$$F(t) = \sinh(\omega t) \quad , \quad G(t) = \tanh\left(\frac{\omega t}{2}\right) \quad , \tag{3.10}$$

$A$ and $B$ are arbitrary constants and the parameters $\omega$, $\gamma$ and $\beta$ are

$$\begin{aligned} \omega^2 = 2\frac{(\nu^2-1)(9\nu^2-1)}{\nu^2}m^2 \quad , \quad \gamma &= 3\frac{3\nu^2-1}{9\nu^2-1} \quad , \\ \beta^2 &= 36\frac{\nu^2}{(9\nu^2-1)^2} \quad . \end{aligned} \tag{3.11}$$

However we do not go into details of the physical properties of this solution since this has been examined before in [58].

## 3.2. Globally Conformal Invariant Case

In this subsection we present globally conformal invariant solutions. First we present the general method for the solution. Then we investigate three cases. The first case contains a curved universe without a potential. This case can be written as a pure Brans-Dicke theory. The second case contains a flat universe with a quartic potential for the scalar field. As for the third case, we investigate a curved universe with a quartic potential. The rest of this study is devoted to the solution and the properties of this model.

Taking the action (2.24) at face value and denoting the present (approximately constant) value of $\phi$ by $\phi_o$, the transformation

$$\tilde{g}_{\mu\nu} = \frac{\phi^2}{\phi_o^2} g_{\mu\nu} \quad , \tag{3.12}$$

yields

$$S = \int \left[ -\frac{1}{6}\kappa\phi_o^2\tilde{R} + \frac{2\kappa+1}{2}\phi_o^2\tilde{g}^{\mu\nu}\partial_\mu \ln\phi\, \partial_\nu \ln\phi - \lambda\phi_o^4 \right] \sqrt{-\tilde{g}}\, d^4x \quad . \tag{3.13}$$

Here $1/16\pi G = \kappa\phi_o^2/6$. In accordance with Dirac's terminology, we call the metric in (2.24) the atomic metric and the metric in (3.13) the Einstein metric. The potential $\lambda\phi^4$ in the original action (2.24), transforms into a cosmological constant in terms of the Einstein variables.

To exhibit the integrability of the field equations it is most convenient to express the action (2.24) in terms of the Einstein metric defined by (3.12). This corresponds to defining

$$\begin{aligned} b &= \frac{\phi}{\phi_o} a \quad , \\ n &= \frac{\phi}{\phi_o} N \quad , \end{aligned} \tag{3.14}$$

where $b$ is the size of the universe measured in terms of the time varying Planck length. The reduced action containing new variables is

$$S = I \int \phi_o^2 \left[ (\kappa + \frac{1}{2}) \frac{b^3 \dot{\phi}^2}{n\phi^2} - \kappa \frac{b\dot{b}^2}{n} + \kappa k n b - \phi_o^2 n b^3 \frac{1}{\phi^4} V(\phi) \right] dt \quad . \tag{3.15}$$

The equations of motion written in terms of the new variables are

$$-\left(\kappa + \frac{1}{2}\right) \frac{b^3 \dot{\phi}^2}{n^2 \phi^2} + \kappa \frac{b\dot{b}^2}{n^2} + \kappa k b - \phi_o^2 b^3 \frac{1}{\phi^4} V(\phi) = 0 \quad , \tag{3.16}$$

$$3\left(\kappa + \frac{1}{2}\right) \frac{b^2 \dot{\phi}^2}{n\phi^2} + \kappa \frac{\dot{b}^2}{n} + 2\kappa \frac{b\ddot{b}}{n} - 2\kappa \frac{b\dot{b}\dot{n}}{n^2} + \kappa k n - 3\phi_o^2 n b^2 \frac{1}{\phi^4} V(\phi) = 0 \quad , \tag{3.17}$$

$$\begin{aligned} -2\left(\kappa + \frac{1}{2}\right) \left( 3\frac{b^2 \dot{b} \dot{\phi}}{n\phi^2} + \frac{b^3 \ddot{\phi}}{n\phi^2} - \frac{b^3 \dot{\phi} \dot{n}}{n^2 \phi^2} - \frac{b^3 \dot{\phi}^2}{n\phi^3} \right) - \phi_o^2 n b^3 \frac{\partial}{\partial \phi} \frac{1}{\phi^4} V(\phi) \\ = -2 \frac{\left(\kappa + \frac{1}{2}\right)}{\phi} \left( \frac{b^3 \dot{\phi}}{n\phi} \right)^{\cdot} - \phi_o^2 n b^3 \frac{\partial}{\partial \phi} \frac{1}{\phi^4} V(\phi) = 0 \quad . \end{aligned} \tag{3.18}$$

For a vanishing or a quartic potential, the second term in (3.18) is identicaly zero. Then it readily integrates to

$$\frac{b^3\dot{\phi}}{n\phi} = c \quad . \tag{3.19}$$

If we replace this in the equation (3.16), we find $b$ in terms of $\phi$,

$$\phi^2 \left(\frac{db}{d\phi}\right)^2 = \nu^2 b^2 - \frac{k}{c^2} b^6 + \frac{\alpha^2}{c^2} b^8 \quad , \tag{3.20}$$

where $\alpha^2$ is

$$\alpha^2 = \frac{\lambda}{\kappa}\phi_o^2 \quad , \tag{3.21}$$

and $\nu$ was defined in equation (3.2). Then we can find their time dependence explicitly using the equation (3.19) again. At this point, we should remark the relation between the equations (3.16) and (3.17). The equation (3.19) appears as the integrability condition. In other words, equation (3.16) together with (3.19) implies equation (3.17). The equation (3.20) yields elliptic integrals containing both the first and the third kind, [59]. Therefore, we first present a sample solution for a non-flat universe without a potential and then we will continue with the conformally invariant case with a quartic potential. Meanwhile for a flat universe without a potential, we find the solution (3.3) again.

### 3.2.1. Curved Universe without a Potential

Here, we present sample solutions for no potential but $k \neq 0$, in the limiting case $\nu = -1$, which corresponds to $\kappa \to \infty$. The solution for $k = 1$ and $V(\phi) = 0$ is

$$\phi^4 = \phi_o^4 \frac{t^2}{2c - t^2} \quad ,$$

$$b^4 = t^2\left(2c - t^2\right) \quad , \tag{3.22}$$

$$a^4 = \left(2c - t^2\right)^2 \quad .$$

The solution for $k = -1$ and $V(\phi) = 0$ is

$$\phi^4 = \phi_o^4 \frac{t^2}{t^2 - 2c} \quad ,$$

$$b^4 = t^2\left(t^2 - 2c\right) \quad , \tag{3.23}$$

$$a^4 = \left(t^2 - 2c\right)^2 \quad .$$

We should point that the time parameter $t$ is the cosmological time, that is $N(t) = 1$.

### 3.2.2. Flat Universe with a Quartic Potential

Hereafter we are going to investigate this case and its properties, since it is the most interesting one among the other solutions presented before. This case is globally conformal invariant. Furthermore the field equations are invariant under the local conformal transformation defined by (2.40) and (2.41). A negative scalar curvature can induce a spontaneous breaking of the global conformal invariance. If we look at the equation (3.19), we see that for a universe expanding infinitely ($b \to \infty$), the scalar field $\phi$ converges to a constant value which we denote by $\phi_o$. We will assume that the present epoch corresponds essentially to this limiting value of $\phi$. Thus at the present time $\kappa\phi^2/6 \cong \kappa\phi_o^2/6 = 1/16\pi G_o$ where $G_o$ is the present value of the Newtonian gravitational constant, and $b_o \cong a_o$. Note that both $\phi^2 R$ and $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ terms in the action (2.24) have contributed to the $b^3\dot{\phi}^2/n\phi^2$ term in (3.15). The relative amount of their contribution depends on the value of $\kappa$ which inturn is determined by the scale of $\phi_o$. For example if $\phi$ is identified with the modulus of the Higgs field in Standard model then $\phi_o \approx 10^2$ GeV, thus $\kappa \sim 10^{34}$, and the contribution of the kinetic term of the Higgs field can thus be neglected. Therefore for large scale cosmology the $\phi^2 R$ term generates an effective kinetic term for the Higgs field. As $\phi_o$ increases, the relative contribution of the other term increases. As $\phi^2 \to M_p$ their contribution becomes comparable.

If we integrate equation (3.20), for $k = 0$, we find

$$\begin{aligned} b^3 &= 2\frac{c\nu}{\alpha}\frac{\varphi^{3\nu}}{1-\varphi^{6\nu}} \quad , \qquad \varphi = \frac{\phi}{\phi_o} \quad , \\ a^3 &= 2\frac{c\nu}{\alpha}\frac{\varphi^{3(\nu-1)}}{1-\varphi^{6\nu}} \quad . \end{aligned} \tag{3.24}$$

This explicitly shows that $\phi \to \phi_o$ as $a \to \infty$. We find, using equation (3.19) with the cosmological time $N = 1$, that

$$\dot{\varphi} = \frac{\alpha}{2\nu}\varphi^2\left(\varphi^{-3\nu} - \varphi^{3\nu}\right) \quad . \tag{3.25}$$

The following simple, explicit solution can be obtained by setting $\nu = 1/3$,

$$\begin{aligned} \phi^2 &= \frac{\phi_o^2}{1+e^{-3\alpha t}} \quad , \\ b^3 &= \frac{1}{4}a_1^3 e^{3\alpha t}(1+e^{-3\alpha t})^{1/2} \quad , \\ a^3 &= \frac{1}{4}a_1^3 e^{3\alpha t}(1+e^{-3\alpha t})^2 \quad , \qquad a_1^3 = \frac{8c}{3\alpha} \quad . \end{aligned} \tag{3.26}$$

where $a_1$ is the value of $a$ at $t = 0$, which corresponds to the time when $a$ is minimum. This can be taken to be the Big Bang instant. The universe, beginning with a size of $a_1$, expands to infinity. Meanwhile $\phi^2$ doubles its initial value. Equivalently, the gravitational constant G reduces to half of its initial value. In fact, the solution shows that the universe at infinite past comes from a contracting phase, where everything is massless and develops to the present phase where $\phi = \phi_o$, [48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. However this solution is not realistic since it requires a negative $\kappa$ which corresponds to changing the sign of the Einstein-Hilbert action.

For a more realistic solution we choose $\nu = 1$ which corresponds to $\kappa >> 1$, and obtain

$$
\begin{aligned}
-3\alpha(t - t_i) &= \ln \frac{(\varphi^2 - 1)^2}{\varphi^4 + \varphi^2 + 1} + \sqrt{3} \arctan \frac{\sqrt{3}}{2\varphi^2 + 1} \quad , \\
b^3 &= a_1^3 \frac{\varphi^3}{1 - \varphi^6} \quad , \qquad\qquad (3.27) \\
a^3 &= a_1^3 \frac{1}{1 - \varphi^6} \quad , \qquad\qquad a_1^3 = \frac{2c}{\alpha} \quad .
\end{aligned}
$$

Here we have two different solutions which correspond to different choices for the sign of $c$. In solution I, which corresponds to $c > 0$, $\phi$ is an increasing function of time. It starts from 0, and $\phi \to \phi_o$ as $t \to \infty$ and the universe starts from a finite region. In solution II, $c < 0$ and $\phi$ starts at infinity. It decreases in time and approaches $\phi_o$ and the universe starts with a Big Bang. Using the integration constant $t_i$ one can adjust the time variable so that $t = 0$ corresponds to the minimum value of $a$. Thus $t_i = \sqrt{3}\pi/9\alpha$ for solution I and $t_i = 0$ for solution II. The time dependence of $\varphi = \phi/\phi_o$ and $a/a_1$ for these two solutions is shown in Fig. 3.1.

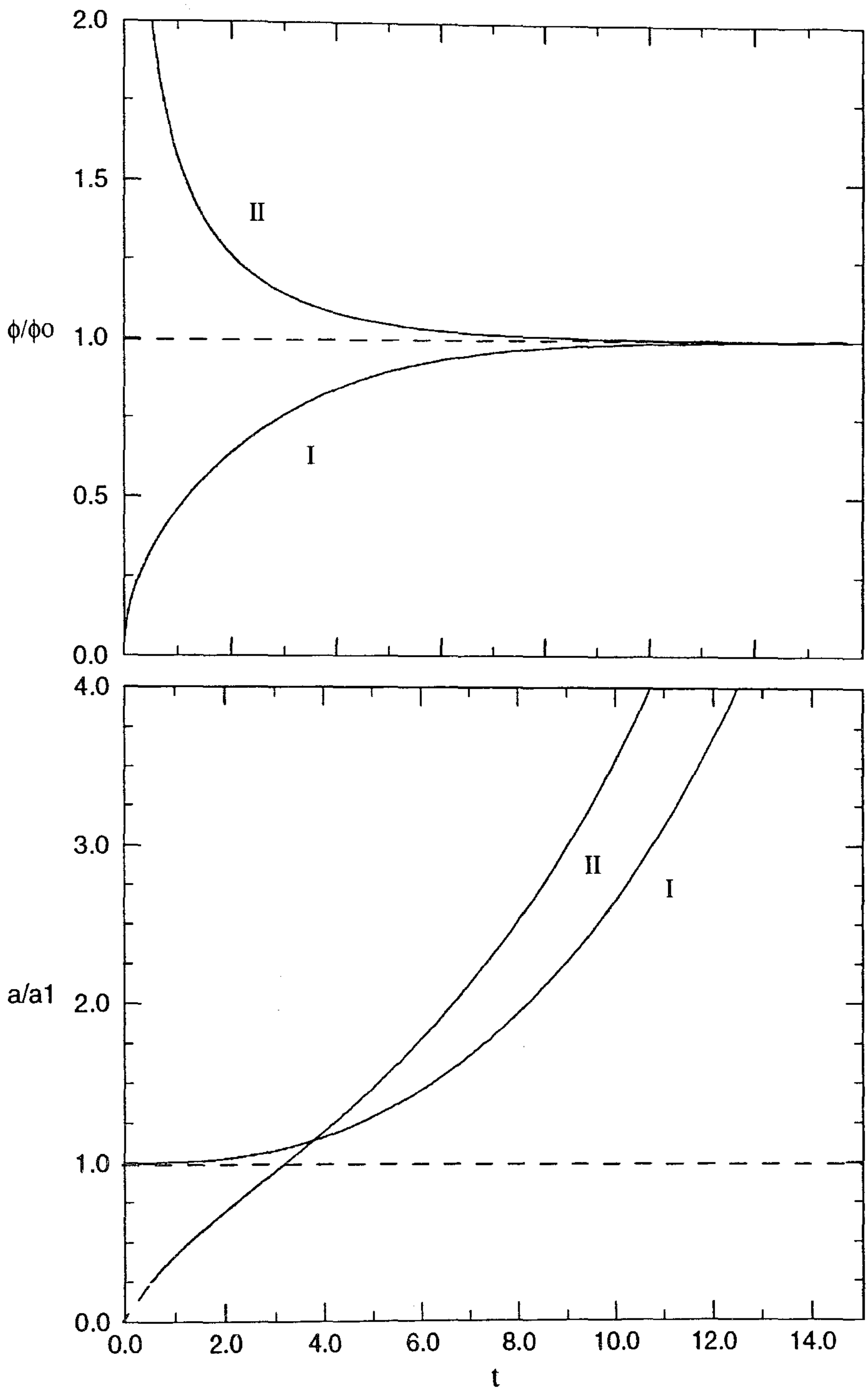


Figure 3.1: The upper and the lower figures show the time dependence of $\varphi = \phi/\phi_o$ and $a/a_1$ respectively.

### 3.2.3. Curved Universe with a Quartic Potential

From equations (3.15) and (3.19) the $k \neq 0$ case can be investigated by studying the analogue hamiltonian, (3.16),

$$\dot{b}^2 + V(b) = -k \quad , \tag{3.28}$$

where dot denotes derivative with respect to the conformal time variable corresponding to $n = 1$, and the potential is

$$V(b) = -\alpha^2 b^2 - \nu^2 c^2 \frac{1}{b^4} \quad , \tag{3.29}$$

[60]. The potential has an extremum at $b^6 = 2\nu^2 c^2/\alpha^2$ which is given by

$$V_{max} = -\frac{3}{2^{2/3}}(\nu^2 c^2 \alpha^4)^{1/3} \quad . \tag{3.30}$$

Its behaviour can be seen in Fig. 3.2. For small $b$ it is similar to that of the Brans-Dicke Theory. However, due to inclusion of the $\lambda\phi^4$ potential, the situation changes dramatically for large $b$. A flat or open universe expands to infinity. A closed universe may also expand to infinity unless $V_{max} > -1$, in which case it is confined by the potential barrier. Then, it may tunnel through the barrier quantum mechanically towards the region where it can expand freely [61]. For $V_{max} = 0$, the theory either reduces to General Relativity with a cosmological constant ($c = 0$) or to the Brans-Dicke Theory ($\alpha = 0$) or $\kappa$ changes sign ($\nu = 0$).

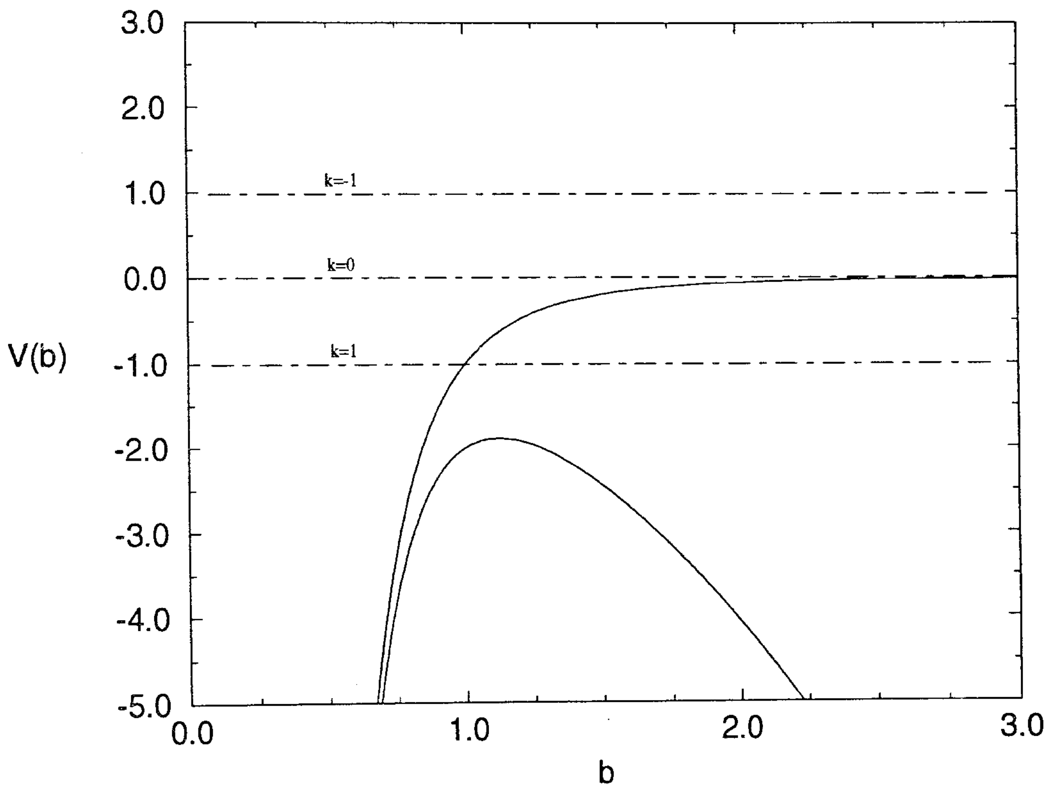


Figure 3.2: The lower solid line shows the behaviour of the potential $V(b)$. The alternately dotted and dashed lines are the $k = -1$, $k = 0$ and $k = +1$ lines respectively and the upper solid line represents $\alpha = 0$, which corresponds to the Brans-Dicke Theory.

# 4. THE GLOBALLY CONFORMAL INVARIANT MODEL: CONSEQUENCES AND IMPLICATIONS

In this section we discuss the properties of the globally conformal invariant model with a quartic potential. By doing this we can compare the model with observations. Although we presented specific solutions corresponding to $\nu = 1/3$ and $\nu = 1$, the analysis hereafter is valid for arbitrary $\nu$.

## 4.1. Equation of State

We discuss the thermodynamics in the Einstein frame. We start from the Einstein form (3.13) of the action. Using (3.15), (3.19), equations (3.16) and (3.17) reduce to

$$\begin{aligned} \frac{8\pi G_o}{3}\rho &\equiv \left(\frac{\dot{b}}{b}\right)^2 + \frac{k}{b^2} = \frac{\nu^2 c^2}{b^6} + \frac{\Lambda}{3} \quad , \\ \frac{8\pi G_o}{3}p &\equiv -\frac{1}{3}\left[2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2 + \frac{k}{b^2}\right] = \frac{\nu^2 c^2}{b^6} - \frac{\Lambda}{3} \quad , \qquad \Lambda = 3\alpha^2 \quad . \end{aligned} \tag{4.1}$$

In order to derive a relation between $b$ and temperature $T$ we choose the temperature and the volume as independent variables and we assume that the energy density $\rho$ and the pressure $p$ in the universe are uniquely determined by the temperature. Then using the thermodynamic relation

$$TdS = dE + pdV \quad , \tag{4.2}$$

and integrating the identity $\partial^2 S/\partial T\partial V = \partial^2 S/\partial V\partial T$, with the help of equations (4.1), we find

$$b = b_1 \left(\frac{T}{T_1}\right)^{-1/3} \quad . \tag{4.3}$$

Then the energy density and the pressure are given by

$$\begin{aligned} \frac{8\pi G_o}{3}\rho &= \frac{\nu^2 c^2}{b_1^6 T_1^2} T^2 + \frac{\Lambda}{3} \quad , \\ \frac{8\pi G_o}{3} p &= \frac{\nu^2 c^2}{b_1^6 T_1^2} T^2 - \frac{\Lambda}{3} \quad . \end{aligned} \tag{4.4}$$

In the equation of state, for Big Bang $T \to \infty$, $p \to \rho$, and when the universe cools so that $T \to 0$, $p \to -\rho$. These correspond to the two extreme cases for pressure. That is, they satisfy the dominant energy condition $-\rho \leq p \leq \rho$, [62]. The $T^2$ term in (4.4) arises from the $\dot{\phi}^2$ term in (3.15) and the $\Lambda$ term is directly related to the $\phi^4$ term in (2.24). Therefore, the initial and the final behaviour of the universe is determined by the scalar field. Because this model reduces to General Relativity as $\phi \to \phi_o$ and $b_o \cong a_o$, we end up with $T \to 0$ and $p \to -\rho$, today. Here we should remark that, in this frame, the expansion is adiabatic.

## 4.2. Critical Density

We can calculate, using (3.24) and (3.25), the Hubble constant and the temporal variation of the Newtonian gravitational constant for any $\nu$

$$-\frac{\nu - 1}{2}\frac{\dot{G}}{G} = H - \alpha\varphi^{3\nu+1} = \alpha\frac{\nu - 1}{2\nu}\frac{1 - \varphi^{6\nu}}{\varphi^{3\nu-1}} \quad . \tag{4.5}$$

Experimentally [46, 63, 64]

$$\frac{\dot{G}}{G} \leq 10^{-43} GeV \quad , \tag{4.6}$$

and a measurement of $\dot{G}/G$, depending on the value of $\nu$, will indicate how close $\varphi = \phi/\phi_o$ is to 1. In this model the Planck mass $M_p$ changes with time as

$$M_p^2 \equiv G^{-1} = \frac{8}{3}\pi\kappa\phi_o^2\varphi^2 \quad . \tag{4.7}$$

In the present epoch $a_o(t) \cong b_o(t)$ and we have the relation

$$H_o^2 = \frac{\Lambda_o}{3} \quad , \tag{4.8}$$

which holds provided $b(t)$ is big enough. This yields

$$\begin{aligned} \lambda_o &= \frac{\Lambda_o}{3H_o^2} = 1 \quad , \\ \rho_c &= \frac{3H_o^2}{8\pi G_o} = \frac{\Lambda_o}{8\pi G_o} \quad , \end{aligned} \tag{4.9}$$

where $\lambda_o$ and $\rho_c$ are the scaled cosmological constant and the critical density for Standard cosmology respectively. From (4.1) it then follows that $\rho_o = \rho_c$ and therefore $\Omega = 1$. Thus in the framework of our model the vacuum expectation value of the scalar field supplies the critical energy density. When the action is expressed in terms of the Einstein metric (3.13) this energy density plays the role of the cosmological constant for all times, whereas in terms of the atomic metric, (2.24), it is the $\lambda\phi^4$ term in the scalar potential.

## 4.3. The Mass of the Scalar Field and the Scale of the Parameters

For the mass of the scalar field, expanding $\phi$ around $\phi_o$, which we choose as the ground state of the scalar field, we find

$$m^2 = \left(-6\nu^2 + 6\nu + 16\right) \lambda {\phi_o}^2 \quad . \tag{4.10}$$

Eliminating the coupling constants between the equations (4.5), (4.7) and (4.10), we obtain a relationship between the present-day values of $m$, $\phi$, $M_p$ and $H$

$$m^2 \phi_o^2 = \frac{3\left(-6\nu^2 + 6\nu + 16\right)}{8\pi} M_p^2 H_o^2 \quad . \tag{4.11}$$

The factor $-6\nu^2 + 6\nu + 16$ is positive for $\kappa \geq 3(13 - \sqrt{105})/64$. This provides an upper bound for the ground state of the scalar field: $\phi_o \leq \sqrt{(13 + \sqrt{105})/8\pi}\, M_p$. We have no preference for the sign of $\nu$. It may be negative as well as positive. This bound is valid for $\nu$ positive. If we choose the negative one, then we should replace $-(+)$ with $+(-)$ above. We should remark that this result is classical and it is subject to quantum fluctuations.

Choosing the vacuum expectation value, $\phi_o$, of the scalar field appropriately, one can tune the dimensionless parameters $\kappa$ and $\lambda$ according to the present day values of the Planck mass and the Hubble constant. First let us consider the case where $\phi_o$ is very small compared to the Planck mass. Specifically we could identify the field $\phi$ with the modulus of the Electroweak Higgs field and then this would imply $\phi_o \approx 10^2\ GeV$. Then from equation (4.11), the Higgs mass is

$$m_H \approx 10^{-25}\ GeV \quad . \tag{4.12}$$

It was claimed by Dreitlein [65] that coupling of the Higgs field with matter gives rise to a force of range $10^{16}$ cm, and if the proton and electron in an atom is coupled with equal but opposite strength with the scalar field of mass less than $2.4 \times 10^{-27}$ $Mev$, then no major contradictions result. However, on the contrary it was shown by Veltman, [66], that this does cause contradictions. On one hand the coupling of the electron to the scalar field $\phi$ is given by

$$-\frac{gm_e}{2M}\phi(\bar{\psi}\psi) \quad , \; g^2 = 8M^2G_F \quad , \tag{4.13}$$

$$G_F = 1.02 \times 10^{-5} m_p^{-2}/\sqrt{2} \quad ,$$

where $m_e$, $m_p$ and $M$ are the electron, proton and intermediate vector-meson masses respectively. On the other hand the coupling of the electron to the gravitational field is given by

$$\sqrt{G} h_{\mu\nu} \bar{\psi} \gamma^{\mu} \partial_{\nu} \psi \quad . \tag{4.14}$$

If we consider a nonrelativistic electron in the ground state of a hydrogen atom, for the time components we can replace the derivatives with $im$ and the relative magnitude of the coupling constants is

$$\frac{\frac{gm_e}{2M}}{m_e\sqrt{G}} \sim 10^{16} \quad . \tag{4.15}$$

Moreover for the excited states the kinetic energies are of the order of, say 1 $eV$ and this gives rise to an additional term of the order of $v^2/c^2 \sim 10^{-6}$ in favour of the gravitational coupling. Further, because of the difference between the masses of protons and electrons we have a factor of 2000 in favor of gravity. As a result the Higgs coupling remains at least 7 orders of magnitude stronger than the gravitational coupling. Therefore, the Higgs field with such a small mass would have a long range coupling with matter stronger than gravity, which is not observed.

We can also try to identify the scalar field with the modulus of the Grand Unified Theory Higgs field. The energy scale of Grand Unified Theories is between $10^{15}$ GeV and $10^{19}$ GeV. In this case equation (4.11) yields even lower values of the order of $10^{-38}\ GeV$ for the mass of the scalar field. At this point we should mention the axion which is introduced for the elimination of CP violation in Strong interactions. It was expected to have a very small mass, about $10^{-8}\ eV$ in some models, [67, 68, 69, 70, 71]. However, from our analysis we expect a scalar particle with a mass less than those expected for the axion. Our analysis is only classical and any comment on this requires more detailed calculations.

As for the cosmological constant, equation (4.8) yields $\Lambda_o = 3\,H_o^2 = 3\frac{\lambda}{\kappa}\phi_o^2 \sim 10^{-83}\,(GeV)^2$. Hence, in the two cases above, the comparison yields extraordinarily large and small values for the dimensionless parameters $\kappa$ and $\lambda$ respectively. Therefore, we have to fine tune these constants in accordance with the observations. However we are unable to explain why such different orders of magnitudes are present in the model.

We want to point to inhomogeneous models of this type. In the present model the Newtonian gravitational constant is determined by the vacuum expectation value of a homogeneous scalar field. Therefore, a natural generalization of the present work would be to study inhomogeneous models. This can be done systematically using the long-wavelength expansion. This method has been applied to low energy effective string cosmology by Veneziano [72] and Saygili, [73]. For Brans-Dicke theory a quasi-isotropic solution has been found by Soda et al., [74]. But a more general solution is expected to yield more precise results and clues for the formation of large scale structure of the Universe. This will be the subject of the rest of this thesis.

# 5. INHOMOGENEOUS PRE-BIG BANG COSMOLOGY

Hereafter we apply the long-wavelength approximation to the low-energy effective string action in the context of the Hamilton-Jacobi theory.

The correct theory of quantum gravity is generally believed to be string theory. The low energy effective action of string theory has the form of a Brans-Dicke (BD) action with parameter $\omega_{BD} = -1$. Duality symmetries play an important role in string theory. In the Pre-Big Bang scenario [48, 49, 50], scale factor duality associates inflationary solutions to non-inflationary ones [51, 52, 53, 54, 55, 75, 76, 77].

Long-wavelength gravity (gradient expansion) has proved to be a fruitful method for studying (slightly) inhomogeneous fields. It is a significant improvement over those of homogeneous minisuperspace. The main idea of this scheme is that when the scale of spatial variations of the fields are larger than the Hubble radius, the equations can be solved neglecting the second-order spatial gradients. It was first introduced by Lifschitz and Khalatnikov. Later, Tomita developed this approximation as the anti-Newtonian scheme. For general relativity, it was formulated either directly in terms of Lagrange equations [78, 79], or in the framework of the Hamilton-Jacobi equation [80, 81, 82]. It was also studied for the BD theory in the Hamilton-Jacobi framework [74].

Recently, an inhomogeneous version of Pre-Big Bang cosmology has been investigated using the Lagrange equations [72, 83, 84]. In this scenario, the universe, with very perturbative (i.e. weak coupling and very small curvature) but otherwise arbitrary initial conditions, is followed towards the Big Bang singularity in the future. Quasi-homogeneous regions, which exhibit Pre-Big Bang behaviour, eventually fill almost all of space, and within these regions the Universe appears homogeneous, flat, and isotropic [72, 83].

First we solve the long-wavelength problem for low-energy string cosmology in the framework of the Hamilton-Jacobi (HJ) equation. We also show that the HJ equation is invariant under scale factor duality (SFD) transformation. We work directly in the physical string frame. The HJ approach has importance for quantum cosmology since it is the lowest-order equation in the WKB approximation of the Wheeler-De Witt equation. Secondly, following Salopek [80], we write the momentum constraints in a simple, elegant form.

In section 6, we write the action in the Hamiltonian form and we derive the equations for the fields and for their conjugate momenta. The action also gives rise to the Hamiltonian and the momentum constraints. For completeness, we also give a brief summary of the canonical transformations via the generating functional technique. In section 7, we write the Hamilton-Jacobi equation, and we show that it is invariant under SFD transformation. The solutions represent a universe evolving towards a Big Bang singularity in the future. Section 8 includes the most general solution near a singularity. In this section, the generating functional is taken as a function of both the dilaton and the metric. This dependence is chosen in such a way that it will effectively decompose the gravitational momentum tensor into a trace contribution and a traceless part. Furthermore, with the help of this choice, the Hamilton-Jacobi equation reduces to that of massless scalar fields. We write the momentum constraint in a simple form. The momentum constraints state that the generating functional is invariant under spatial coordinate transformations. We write them in terms of the new canonical variables. Then, in section 9, we present a quasi-isotropic solution of Pre-Big Bang cosmology. The general solution reduces to the quasi-isotropic one, if the evolution of the gravitational radiation is neglected. In section 10, we briefly discuss duality transformation for the general solution. We represent the evolution of the universe in a space of fields where the duality transformation can be written as the transformation of an angle in a suitable plane.

# 6. HAMILTON FORMALISM AND THE CANONICAL TRANSFORMATIONS

The low-energy effective string action, in the string frame, is

$$\Gamma = \int e^{-\phi}(R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)\sqrt{-g}d^4x \quad , \tag{6.1}$$

where $R$ is the scalar curvature, $\phi$ is the dilaton. We set the antisymmetric tensor field $B_{\mu\nu}$ to zero. Here we remark that the dilaton and the scalar field used in the previous problem are different scalar fields. The HJ equation for this action can be obtained using the ADM formalism in which the space-time is foliated by space-like hypersurfaces. In the ADM formalism the metric is parametrized as

$$ds^2 = -N^2(t)dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad , \tag{6.2}$$

where $N$ and $N^i$ are the lapse and shift functions, respectively, and $\gamma_{ij}$ is the 3-metric. We work in the synchronous gauge $N = 1$, $N^i = 0$.

The action written in Hamiltonian form becomes

$$\Gamma = \int (\pi^{ij}\dot{\gamma}_{ij} + \pi^\phi\dot{\phi} - N\mathcal{H} - N^i\mathcal{H}_i)d^4x \quad , \tag{6.3}$$

where $N$ and $N^i$ act as Lagrange multipliers. Their variation give rise to the Hamiltonian constraint

$$\begin{aligned} \mathcal{H} &= \gamma^{-1/2}e^{\phi}\left[\pi^{ij}\pi^{kl}\gamma_{ik}\gamma_{jl} + \frac{1}{2}(\pi^\phi)^2 + \pi\pi^\phi\right] \\ &\quad -\gamma^{1/2}e^{-\phi}R - \gamma^{1/2}e^{-\phi}\gamma_{ij}\partial_i\phi\partial_j\phi + 2\gamma^{1/2}\triangle e^{-\phi} = 0 \quad , \end{aligned} \tag{6.4}$$

and to the momentum constraints

$$\mathcal{H}_i = -2(\gamma_{ik}\pi^{kj})_{,j} + \pi^{kl}\gamma_{kl,i} + \pi^\phi\phi_{,i} = 0 \quad . \tag{6.5}$$

Variation with respect to the canonical variables yield the evolution equations

$$-2K_{ij} = \frac{1}{N}\left(\dot{\gamma}_{ij} - N_{i;j} - N_{j;i}\right) = \gamma^{-1/2}e^\phi\left(2\pi^{kl}\gamma_{ik}\gamma_{jl} + \gamma_{ij}\pi^\phi\right) \quad , \tag{6.6}$$

$$\frac{1}{N}\left(\dot{\phi} - N^i\phi_{,i}\right) = \gamma^{-1/2}e^\phi\left(\pi^\phi + \pi\right) \quad , \tag{6.7}$$

$$\frac{1}{N}\left[\dot{\pi}^i{}_j - \left(N^m\pi^i{}_j\right)_{,m} + N^i{}_{,m}\pi^m_j - N^m_{,j}\pi^i{}_m\right] \tag{6.8}$$
$$= \quad \frac{1}{2}\gamma^{-1/2}e^\phi\delta^i_j\left[\pi^{mn}\pi^{kl}\gamma_{mk}\gamma_{nl} + \frac{1}{2}\left(\pi^\phi\right)^2 + \pi\pi^\phi\right] \quad ,$$

$$\frac{1}{N}\left[\dot{\pi}^\phi - (N^i\pi^\phi)_{,i}\right] = -\gamma^{-1/2}e^\phi\left[\pi^{ij}\pi^{kl}\gamma_{ik}\gamma_{jl} + \frac{1}{2}\left(\pi^\phi\right)^2 + \pi\pi^\phi\right] \quad . \tag{6.9}$$

Here $K_{ij}$ is the extrinsic curvature, which is the relevant object in Lagrange formalism.

The basic idea of canonical transformations is well known [80]. New fields are defined, which we denote by tilde, so that Hamilton's equations are preserved. This implies that the new action has the same form as the original, except that it may have a total time derivative added to it:

$$\tilde{\Gamma} = \int(\tilde{\pi}^{ij}\dot{\tilde{\gamma}}_{ij} + \tilde{\pi}^\phi\dot{\tilde{\phi}} - N\tilde{\mathcal{H}} - N^i\tilde{\mathcal{H}}_i)d^4x + \int\dot{\mathcal{S}}dt \quad , \tag{6.10}$$

where $\mathcal{S}$ is a functional that depends on the old and the new field variables. It is assumed that $\mathcal{S}$ does not depend on time explicitly. Applying the chain rule

$$\dot{\mathcal{S}} = \int\left[\frac{\delta\mathcal{S}}{\delta\phi(x)}\dot{\phi}(t,x) + \frac{\delta\mathcal{S}}{\delta\tilde{\phi}(x)}\dot{\tilde{\phi}}(t,x) + \frac{\delta\mathcal{S}}{\delta\gamma_{ij}(x)}\dot{\gamma}_{ij}(t,x) + \frac{\delta\mathcal{S}}{\delta\tilde{\gamma}_{ij}(x)}\dot{\tilde{\gamma}}_{ij}(t,x)\right]d^3x \quad , \tag{6.11}$$

and comparing eq. (6.11) with (6.10), the canonical transformation linking the various variables is derived

$$\mathcal{H}(x) = \tilde{\mathcal{H}}(x) \quad , \quad \mathcal{H}_i(x) = \tilde{\mathcal{H}}_i(x) \tag{6.12}$$

$$\begin{aligned} \pi^{\phi}(x) &= \frac{\delta \mathcal{S}}{\delta \phi(x)} \quad , \quad \pi^{ij}(x) = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \quad , \\ \tilde{\pi}^{\phi}(x) &= -\frac{\delta \mathcal{S}}{\delta \tilde{\phi}(x)} \quad , \quad \tilde{\pi}^{ij}(x) = -\frac{\delta \mathcal{S}}{\delta \tilde{\gamma}_{ij}(x)} \quad . \end{aligned} \tag{6.13}$$

The new variables, denoted by a tilde, will be chosen such that the new Hamiltonian density strongly vanishes. Therefore they are constant in time.

# 7. THE HAMILTON-JACOBI EQUATION

The HJ equation is given by the Hamiltonian constraint (6.4) after expressing the momenta through eq. (6.13). In the long-wavelength limit, we neglect the last three terms since they involve two spatial derivatives. We thus obtain the HJ equation in the lowest order term

$$\gamma^{-1/2}e^{\phi}\left[\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\frac{\delta\mathcal{S}}{\delta\gamma_{kl}}\gamma_{ik}\gamma_{jl}+\frac{1}{2}\left(\frac{\delta\mathcal{S}}{\delta\phi}\right)^{2}+\gamma_{ij}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\frac{\delta\mathcal{S}}{\delta\phi}\right]=0 \quad . \tag{7.1}$$

The HJ equation is interesting because of its intimate relation to quantum gravity. The Wheeler-De Witt equation and the momentum constraint for the effective string action are given by

$$\mathcal{H}\Psi=0 \quad , \qquad \mathcal{H}_i\Psi=0 \quad , \tag{7.2}$$

where the canonical commutation relations

$$\left[\gamma_{ij}(x),\pi^{kl}(x')\right]=\frac{i}{2}(\delta^k_i\delta^l_j+\delta^k_j\delta^l_i)\delta(x-x') \quad , \tag{7.3}$$
$$\left[\phi(x),\pi^{\phi}(x')\right]=i\delta(x-x') \quad ,$$

are used. If we consider the WKB approximation, we get the HJ equation at the lowest order.

The HJ equation takes a simpler form by using, instead of $\phi$, the shifted dilaton $\Phi=\phi-\ln\gamma^{1/2}$. We then find

$$e^{\Phi}\left[Tr\left(\gamma\frac{\delta\mathcal{S}}{\delta\gamma}\gamma\frac{\delta\mathcal{S}}{\delta\gamma}\right)-\frac{1}{4}\left(\frac{\delta\mathcal{S}}{\delta\Phi}\right)^{2}\right]=0 \quad . \tag{7.4}$$

Equation (7.4) is invariant under SFD transformation, which is defined as

$$[\gamma] \longrightarrow [\gamma]^{-1} \quad , \quad \Phi \longrightarrow \Phi \quad . \tag{7.5}$$

Notice that, although the HJ equation is SFD-invariant, the momentum constraints are not. We can write the momentum constraints, via the eqs. (6.5) and (6.13):

$$\mathcal{H}_i = -2\left(\gamma_{ik}\frac{\delta\mathcal{S}}{\delta\gamma_{kj}}\right)_{,j} + \frac{\delta\mathcal{S}}{\delta\gamma_{kl}}\gamma_{kl,i} + \frac{\delta\mathcal{S}}{\delta\phi}\phi_{,i} = 0 \quad . \tag{7.6}$$

They state that the generating functional is invariant under spatial diffeomorphisms.

# 8. GENERAL SOLUTION NEAR A SINGULARITY

## 8.1. Ansatz for the Generating Functional

In this section, we investigate the full classical long-wavelength problem of the low energy string cosmology. The generating functional $\mathcal{S}$ is assumed to be a function of both the scalar field and the gravitational field. Adapting an ansatz used by Salopek [80], the dependence on the gravitational field is chosen in a specific way, without losing the generality of the solution.

We choose the ansatz given below for the lowest-order generating functional:

$$\mathcal{S}^o = -2\int e^{-\frac{3}{2}(\phi-\tilde{\phi})} H(\phi, h_{ij}; \tilde{\phi}, \tilde{h}_{ij})\sqrt{\gamma} d^3x \quad , \tag{8.1}$$

where $h_{ij} = \gamma^{-1/3}\gamma_{ij}$ and $\tilde{h}_{ij} = \tilde{\gamma}^{-1/3}\tilde{\gamma}_{ij}$ are the unimodular conformal 3-metric. Note that we introduced the $e^{-\frac{3}{2}\tilde{\phi}}$ factor only for convenience, as will become clear in the following. Since

$$\frac{\partial H}{\partial \gamma_{ij}} = \gamma^{-1/3}\left[\frac{\partial H}{\partial h_{ij}} - \frac{1}{3}\frac{\partial H}{\partial h_{kl}} h_{kl} h^{ij}\right] \quad , \tag{8.2}$$

we can write the HJ equation in terms of the conformal metric $h_{ij}$:

$$H^2 = \frac{8}{3}\frac{\partial H}{\partial h_{ij}}\frac{\partial H}{\partial h_{kl}}\left(h_{ik}h_{jl} - \frac{1}{3}h_{ij}h_{kl}\right) + \frac{4}{3}\left(\frac{\partial H}{\partial \phi}\right)^2 \quad . \tag{8.3}$$

This is referred to as the separated HJ equation.

Inserting (8.1) in (6.13) we find the new and the old momenta

$$\pi^{ij} = -2\gamma^{1/2} e^{-\frac{3}{2}(\phi-\tilde{\phi})}\left[\frac{1}{2}\gamma^{ij} H + \gamma^{-1/3}\left(\frac{\partial H}{\partial h_{ij}} - \frac{1}{3}\frac{\partial H}{\partial h_{kl}} h_{kl} h_{ij}\right)\right] \quad , \tag{8.4}$$

$$\tilde{\pi}^{ij} = 2\gamma^{1/2}\tilde{\gamma}^{-1/3}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\left[\frac{\partial H}{\partial \tilde{h}_{ij}} - \frac{1}{3}\frac{\partial H}{\partial \tilde{h}_{kl}}\tilde{h}_{kl}\tilde{h}_{ij}\right] \quad , \tag{8.5}$$

$$\pi^{\phi} = -2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\left[-\frac{3}{2}H + \frac{\partial H}{\partial \phi}\right] \quad , \tag{8.6}$$

$$\tilde{\pi}^{\phi} = 2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\left[\frac{3}{2}H + \frac{\partial H}{\partial \tilde{\phi}}\right] \quad . \tag{8.7}$$

The trace of the gravitational momentum is proportional to $H$:

$$\pi = \pi^i_i = -3\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})}H \quad , \tag{8.8}$$

and this is proportional to the integrand in eq. (8.1).

The specific choice for the dependence of the function $H$ on the metric, through the combination $h_{ij} = \gamma^{-1/3}\gamma_{ij}$, effectively decomposes the gravitational momentum tensor into a trace contribution and a traceless part, which describes the evolution of gravitational radiation. Similarly, the new gravitational momentum tensor is traceless, $\tilde{\pi}^{ij}\tilde{\gamma}_{ij} = 0$.

Furthermore, we attempt the following solution to the separated HJ equation (8.3):

$$H(\phi, h_{ij}; \tilde{\phi}, \tilde{h}_{ij}) \equiv H(\phi, \tilde{\phi}, z) \quad , \tag{8.9}$$

where $z$ is defined as [80]

$$z^2 = \frac{1}{2}Tr\left[\ln\left([h][\tilde{h}]^{-1}\right)\ln\left([h][\tilde{h}]^{-1}\right)\right] \quad . \tag{8.10}$$

Here $[h]$ and $[\tilde{h}]^{-1}$ are matrices with components $h_{ij}$ and $\tilde{h}^{ij}$, respectively. This variable may be thought of as the "distance", in field space, between the old conformal metric $h_{ij}$ and the new one $\tilde{h}_{ij}$ [80]. We do not lose any information, because six constants of integration have been introduced through $\tilde{h}_{ij}$, which are sufficient to describe the dynamics of the gravitational field.

In terms of this variable, the separated Hamilton-Jacobi equation reduces to that of massless scalar fields $z$ and $\phi$,

$$H^2 = \frac{4}{3}\left[\left(\frac{\partial H}{\partial z}\right)^2 + \left(\frac{\partial H}{\partial \phi}\right)^2\right] \quad , \tag{8.11}$$

where we used

$$\frac{\partial H}{\partial h_{ij}} = \frac{\partial H}{\partial z}\frac{\partial z}{\partial h_{ij}} = \frac{1}{2}\frac{\partial H}{\partial z} z^{-1}\left[[h]^{-1}\ln\left([h][\tilde{h}]^{-1}\right)\right]^{ij} \quad . \tag{8.12}$$

Similarly

$$\frac{\partial H}{\partial \tilde{h}_{ij}} = \frac{\partial H}{\partial z}\frac{\partial z}{\partial \tilde{h}_{ij}} = -\frac{1}{2}\frac{\partial H}{\partial z} z^{-1}\left[[h]^{-1}\ln\left([h][\tilde{h}]^{-1}\right)[h][\tilde{h}]^{-1}\right]^{ij} \quad . \tag{8.13}$$

Equations (8.12) and (8.13) yield

$$h_{ij}\frac{\partial H}{\partial h_{ij}} = 0 \quad , \qquad \tilde{h}_{ij}\frac{\partial H}{\partial \tilde{h}_{ij}} = 0 \quad . \tag{8.14}$$

Therefore, the equations for the momenta become

$$\pi^{ij} = -2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\left[\frac{1}{2}\gamma^{ij}H + \gamma^{-1/3}\frac{\partial H}{\partial h_{ij}}\right] \quad , \tag{8.15}$$

$$\tilde{\pi}^{ij} = 2\gamma^{1/2}\tilde{\gamma}^{-1/3}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\frac{\partial H}{\partial \tilde{h}_{ij}} \quad , \tag{8.16}$$

$$\pi^{\phi} = -2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\left[-\frac{3}{2}H + \frac{\partial H}{\partial \phi}\right] \quad , \tag{8.17}$$

$$\tilde{\pi}^{\phi} = 2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\left[\frac{3}{2}H + \frac{\partial H}{\partial \tilde{\phi}}\right] \quad . \tag{8.18}$$

Note that the new gravitational momentum is related to the old one through the reciprocity relation

$$\pi^{ij}\gamma_{jl} = \frac{1}{3}\pi\delta^i_{\ l} + \tilde{\pi}^{ij}\tilde{\gamma}_{jl} \quad . \tag{8.19}$$

We will also need the equations for $z$ and its momentum. They can be written in a manner similar to those of the dilaton:

$$\dot{z} = \gamma^{-1/2}e^{\phi}(\pi^z + \pi) \quad , \tag{8.20}$$

$$\pi^z = -2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\left[-\frac{3}{2}H + \frac{\partial H}{\partial z}\right] \quad . \tag{8.21}$$

It will become clear below that these equations are consistent and provide the correct evolution for $z$.

## 8.2. Momentum Constraints

The momentum constraints admit a simple expression through the solution (8.9) [80]. The gravitational momentum tensor can be decomposed into a trace and a traceless part, which we denote by an overbar

$$\pi^{ij} = \frac{1}{3}\pi\gamma^{ij} + \bar{\pi}^{ij} \quad . \tag{8.22}$$

The momentum constraints become

$$\mathcal{H}_i = -\frac{2}{3}\pi_{,i} - 2(\bar{\pi}^{jl}\gamma_{li})_{,j} + \pi^{kl}\gamma_{kl,i} + \pi^{\phi}\phi_{,i} = 0 \quad . \tag{8.23}$$

Using eq. (8.8), the generating functional can be written in terms of the trace of the gravitational momentum

$$\mathcal{S} = \frac{2}{3}\int \pi\left(\phi(x), h_{ij}(x); \tilde{\phi}(x), \tilde{h}_{ij}(x)\right) d^3x \quad . \tag{8.24}$$

Therefore the new and the old canonical variables can be expressed as partial derivatives of $\pi$. The spatial derivative of $\pi$ can be written as

$$\pi_{,i} = \frac{3}{2}\pi^{kl}\gamma_{kl,i} - \frac{3}{2}\tilde{\pi}^{kl}\tilde{\gamma}_{kl,i} + \frac{3}{2}\pi^{\phi}\phi_{,i} - \frac{3}{2}\tilde{\pi}^{\phi}\tilde{\phi}_{,i} \quad . \tag{8.25}$$

If we insert this into the eq. (8.23), using the reciprocity relation, we can write the momentum constraints in terms of the new variables

$$\tilde{\mathcal{H}}_i = -2(\tilde{\gamma}_{ik}\tilde{\pi}^{kj})_{,j} + \tilde{\pi}^{kl}\tilde{\gamma}_{kl,i} + \tilde{\pi}^{\phi}\tilde{\phi}_{,i} = 0 \quad . \tag{8.26}$$

Here one effectively performs a Legendre transformation between the new and the old variables.

The evolution equations for the new variables are given by the new action, which was written in eq. (6.10):

$$\tilde{\Gamma} = \int(\tilde{\pi}^{ij}\dot{\tilde{\gamma}}_{ij} + \tilde{\pi}^{\phi}\dot{\tilde{\phi}} - N^i\tilde{H}_i)d^4x \quad . \tag{8.27}$$

One can easily see that, if the shift function $N_i$ vanishes, then the new canonical variables are independent of time, but they can depend on space coordinates. They are restricted by the momentum constraints (8.26). We can write the momentum constraints in terms of $\tilde{h}_{ij}$,

$$\tilde{\mathcal{H}}_i = -2\left(\tilde{\gamma}^{1/3}\tilde{\pi}^{jk}\tilde{h}_{ki}\right)_{,j} + \tilde{\gamma}^{1/3}\tilde{\pi}^{kl}\tilde{h}_{kl,i} + \tilde{\pi}^{\phi}\tilde{\phi}_{,i} = 0 \quad . \tag{8.28}$$

Since the theory does not depend on the parametrization of the spatial coordinates, the momentum constraints may be written in terms of a covariant derivative with respect to $\tilde{h}_{ij}$,

$$\tilde{\mathcal{H}}_i = -2(\tilde{\gamma}^{1/3}\tilde{\pi}^j_i)_{;j} + \tilde{\pi}^\phi \tilde{\phi}_{,i} = 0 \quad . \tag{8.29}$$

## 8.3. Solution

The solution of eq. (8.11) is given by

$$H = -\frac{2}{3t_o e^{\tilde{\phi}}} \exp\left\{\frac{\sqrt{3}}{2}\left[\left(\phi - \tilde{\phi}\right)^2 + (z - \tilde{z})^2\right]^{1/2}\right\} \quad , \tag{8.30}$$

where a tilde refers to the initial value of the corresponding variable. The initial value of $H$ is chosen in order to have a Pre-Big Bang behaviour and $t_o$ is an arbitrary constant [72, 83]. Its meaning will become apparent below. Here one should note the rotational symmetry of the solution.

Using the evolution equations for $\phi$ and $z$ (6.7), (8.20), and the equations for their conjugate momentum (8.17), (8.21), we find

$$\dot{\phi} = -2e^{\phi} e^{-\frac{3}{2}(\phi - \tilde{\phi})} \frac{\partial H}{\partial \phi} \quad , \tag{8.31}$$

$$\dot{z} = -2e^{\phi} e^{-\frac{3}{2}(\phi - \tilde{\phi})} \frac{\partial H}{\partial z} \quad . \tag{8.32}$$

If we use the new variables $x$ and $y$ defined as

$$x = \frac{\sqrt{3}}{2}(\phi - \tilde{\phi}) \quad , \quad y = \frac{\sqrt{3}}{2}(z - \tilde{z}) \quad , \quad r^2 = x^2 + y^2 \quad , \tag{8.33}$$

then we find

$$\dot{x} = \frac{1}{t_o} \frac{x}{\sqrt{x^2+y^2}} e^{-\frac{1}{\sqrt{3}}x + \sqrt{x^2+y^2}} \quad , \tag{8.34}$$

$$\dot{y} = \frac{1}{t_o} \frac{y}{\sqrt{x^2+y^2}} e^{-\frac{1}{\sqrt{3}}x + \sqrt{x^2+y^2}} \quad . \tag{8.35}$$

Because of rotational symmetry, it is natural to use the polar coordinates in the $(\phi, z)$-plane. One then finds that the angular coordinate is constant in time; it depends only on the spatial coordinates. The radial coordinate is given by

$$r = \frac{-1}{1 - \frac{1}{\sqrt{3}}\cos\varphi} \ln\left(1 - \frac{t}{\tilde{t}}\right) \quad , \quad \tilde{t} = \frac{t_o}{1 - \frac{1}{\sqrt{3}}\cos\varphi} \quad . \tag{8.36}$$

We find, using $x = r\cos\varphi$, $y = r\sin\varphi$ and eq. (8.33),

$$\phi = \tilde{\phi} + \beta \ln\left(1 - \frac{t}{\tilde{t}}\right) \quad , \quad \beta = \frac{-\frac{2}{\sqrt{3}}\cos\varphi}{1 - \frac{1}{\sqrt{3}}\cos\varphi} \quad , \tag{8.37}$$

$$z = \frac{-\frac{2}{\sqrt{3}}\sin\varphi}{1 - \frac{1}{\sqrt{3}}\cos\varphi} \ln\left(1 - \frac{t}{\tilde{t}}\right) \quad . \tag{8.38}$$

Here, notice that $\tilde{z} = 0$ by definition. In the $(\varphi, z)$-plane, circles concentric with the origin correspond to constant $H$ surfaces. The evolution of the fields $\phi$ and $z$ at a fixed spatial point are given by the rays originating from the origin. These remain orthogonal to the uniform $H$ surfaces every time. One can see, using eqs. (8.15) and (8.17), that the momenta for the gravitational field and the dilaton are constant in time. But they can have spatial dependence:

$$\pi^i_j = \lambda^i_j(x) \quad . \tag{8.39}$$

The evolution of the unimodular conformal metric $h_{ij}$ is given by

$$\dot{h}_{ij} = -4e^{\phi}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\frac{\partial H}{\partial h_{kl}}h_{ki}h_{lj} \quad . \tag{8.40}$$

Here eqs. (6.6), (8.15) and (8.17) are used. At this point, by a direct application of the chain rule, one can check that eqs. (8.20) and (8.21) for $\dot{z}$ and $\pi^z$ are consistent so that they lead to the correct expression for the evolution of $z$. We can find the evolution of the unimodular conformal metric using eq. (8.40) and the solution (8.38) for $z$

$$\left[\ln\left([h][\tilde{h}]^{-1}\right)\right] = z\,[p(x)] \quad . \tag{8.41}$$

Here the matrix $[p(x)]$ satisfies

$$Tr\,([p][p]) = 2 \quad , \qquad Tr\,([p]) = 0 \quad . \tag{8.42}$$

Hereafter, in order to avoid repetition, we will write the results without going into the details of algebraic manipulations. Using eq. (8.15), we find

$$\frac{1}{\lambda}[\lambda] = \frac{1}{3}\left(I + \frac{\sqrt{3}}{2}\sin\varphi[p]\right) \quad . \tag{8.43}$$

At this stage, eliminating $[p]$ in favour of $[\lambda]$, we define the matrix $[\alpha]$:

$$[\alpha(x)] = I - \frac{2}{1-\frac{1}{\sqrt{3}}\cos\varphi}\frac{1}{\lambda}[\lambda] \quad . \tag{8.44}$$

Then the trace of $[\alpha]$ is

$$\alpha = \frac{1-\sqrt{3}\cos\varphi}{1-\frac{1}{\sqrt{3}}\cos\varphi} \quad . \tag{8.45}$$

The matrix $[\alpha]$ can be simplified further as below. Equation (8.41) yields

$$[h(t,x)] = \exp\left\{2\left([\alpha]-\frac{1}{3}\alpha I\right)\ln\left(1-\frac{t}{\bar{t}}\right)\right\}\left[\tilde{h}(x)\right] \quad . \tag{8.46}$$

If we replace eqs. (8.15) and (8.17) in eq. (6.6), we find that the determinant of the metric evolves according to

$$\gamma = \left(\frac{\lambda \tilde{t} e^{\tilde{\phi}}}{3-\alpha}\right)^2 \left(1 - \frac{t}{\tilde{t}}\right)^{2\alpha} \quad . \tag{8.47}$$

Then we find the evolution of the metric $\gamma_{ij} = \gamma^{1/3} h_{ij}$:

$$[\gamma(t,x)] = \exp\left\{2[\alpha(x)] \ln\left(1 - \frac{t}{\tilde{t}}\right)\right\} [\tilde{\gamma}(x)] \quad , \tag{8.48}$$

where

$$[\tilde{\gamma}(x)] = \left(\frac{\lambda \tilde{t} e^{\tilde{\phi}}}{3-\alpha}\right)^{2/3} \left[\tilde{h}(x)\right] \quad . \tag{8.49}$$

For the following discussion of duality, it is possible to introduce local coordinates in which the matrix $[\alpha]$ and the metric are diagonal. One can also write, arranging eqs. (8.37) and (8.38), the scalar field $\phi$ and $z$ in terms of trace $\alpha$. Here, it is illuminating to find the extrinsic curvature. We obtain, using eq. (6.6),

$$[K] = \frac{[\alpha]}{\tilde{t} - t} \quad , \tag{8.50}$$

where $[K]$ is the matrix representation of the extrinsic curvature, with entries $K^i{}_j$. This yields the expansion rate of the universe as

$$K = -\frac{1}{2}\frac{\dot{\gamma}}{\gamma} = \frac{\alpha}{\tilde{t} - t} \quad . \tag{8.51}$$

Meanwhile we can write $H$ as

$$H = -\frac{3-\alpha}{3\tilde{t} e^{\tilde{\phi}}} \left(1 - \frac{t}{\tilde{t}}\right)^{-\frac{3-\alpha}{2}} \quad . \tag{8.52}$$

$K$ is proportional to $e^{-\frac{\phi}{2}} H$. Equation (8.45) yields $-\sqrt{3} \leq \alpha \leq \sqrt{3}$. The condition for quasi-homogeneous regions to undergo superinflation, $\alpha < 0$, corresponds to the region

$\cos\varphi > 1/\sqrt{3}$ in the $(\phi, z)$-plane [72, 83]. The maximal rate of expansion is reached for $\alpha = -\sqrt{3}$. This corresponds to the quasi-isotropic case $\varphi = 0$, as explained in the next chapter. $[\alpha]$ and the matrix of gravitational momentum $[\lambda]$ are related as

$$\frac{1}{\lambda}[\lambda] = \frac{1}{3-\alpha}\left(I - [\alpha]\right) \quad . \tag{8.53}$$

The relations (8.42) and (8.43) yield

$$\frac{1}{\lambda^2}\lambda^i_j\lambda^j_i = \frac{1}{6}\left(3 - \cos^2\varphi\right) \quad , \tag{8.54}$$

which gives the condition

$$\frac{1}{3} \leq \frac{1}{\lambda^2}\lambda^i_j\lambda^j_i \leq \frac{1}{2} \quad . \tag{8.55}$$

Here, the lower limit corresponds to the quasi-isotropic case. Therefore we have a Kasner-like solution:

$$Tr\left([\alpha][\alpha]\right) = 1 \quad , \qquad \beta = -1 + \alpha \quad . \tag{8.56}$$

Momenta $\pi^\phi$ and $\pi^z$ are related to the angle $\varphi$ and to the trace of the gravitational momentum $\lambda$ as follows

$$\pi^\phi = \left(-1 + \frac{1}{\sqrt{3}}\cos\varphi\right)\lambda \quad , \qquad \pi^z = \left(-1 + \frac{1}{\sqrt{3}}\sin\varphi\right)\lambda \quad , \tag{8.57}$$

and they satisfy

$$\left(\pi^\phi + \lambda\right)^2 + \left(\pi^z + \lambda\right)^2 = \frac{1}{3}\lambda^2 \quad . \tag{8.58}$$

The initial momenta for $z$, $\phi$, and the gravitational field are

$$\tilde{\pi}^{\phi} = \pi^{\phi} \quad , \qquad \tilde{\pi}^{z} = \pi^{z} \quad , \tag{8.59}$$

$$\tilde{\pi}^{i}_{\ j} = \lambda^{i}_{\ j} - \frac{1}{3}\delta^{i}_{\ j}\lambda \quad . \tag{8.60}$$

# 9. QUASI-ISOTROPIC SOLUTION

In this section we consider a quasi-isotropic space. This is a special case of the general solution, as is explained below. If we use the quasi-isotropic ansatz

$$\mathcal{S}^o = -2\int e^{-\frac{3}{2}\phi} H(\phi)\sqrt{\gamma}d^3x \quad , \tag{9.1}$$

the Hamilton-Jacobi equation reduces to

$$H^2 = \frac{4}{3}\left(\frac{\partial H}{\partial \phi}\right)^2 \quad . \tag{9.2}$$

The momentum constraints (diffeomorphism invariance)

$$H_{,i} = \frac{\partial H}{\partial \phi}\phi_{,i} \tag{9.3}$$

are automatically satisfied by this ansatz. Meanwhile the equations of motion for the fields are

$$\dot{\phi} = -2e^{-\frac{1}{2}\phi}\frac{\partial H}{\partial \phi} \quad , \qquad \dot{\gamma}_{ij} = e^{-\frac{1}{2}\phi}\left[H - 2\frac{\partial H}{\partial \phi}\right]\gamma_{ij} \quad . \tag{9.4}$$

They immediately yield a quasi-isotropic solution $\gamma_{ij} = a^2(\phi)h_{ij}(x)$. Here

$$a^2 = \exp\left\{-\frac{1-\sqrt{3}}{2}\int \frac{H}{\partial H/\partial \phi}d\phi\right\} \quad , \tag{9.5}$$

and $h_{ij}$ is the seed metric. If we solve the equations and the constraints explicitly, we find, for the scalar field and the gravitational field:

$$\phi = \frac{2}{1-\sqrt{3}}\ln\left(1 - \frac{t}{\tilde{t}}\right) \quad , \tag{9.6}$$

$$\tag{9.7}$$

$$[\gamma] = \left(1 - \frac{t}{\bar{t}}\right)^{-\frac{2}{\sqrt{3}}} [h(x)] \quad .$$

Here $t_o$ is rescaled by a factor of $(1 - 1/\sqrt{3})^{-1}$. We obtain, using eq. (6.13) and the solution, that the momentum of the gravitational field and the scalar field are constant which are independent of space-time coordinates. Furthermore the traceless part of the gravitational momentum is zero and the evolution of the gravitational radiation is neglected. $H$ is given by

$$H = \frac{2}{\sqrt{3}(1-\sqrt{3})} \frac{1}{\bar{t}} \left(1 - \frac{t}{\bar{t}}\right)^{\frac{\sqrt{3}}{1-\sqrt{3}}} \quad . \tag{9.8}$$

Meanwhile we obtain, for the extrinsic curvature,

$$[K] = -\frac{1}{\sqrt{3}} \frac{1}{\bar{t} - t} I \quad . \tag{9.9}$$

This yields, for the expansion rate of the quasi-isotropic universe,

$$K = -\frac{\sqrt{3}}{\bar{t} - t} \quad . \tag{9.10}$$

We expect that the general solution contains the quasi-isotropic one as a special case. If, in the general solution, we consider the case $\varphi = 0$, then the traceless part of $[\alpha]$ and the momentum $[\lambda]$ disappear. They contain only trace parts, which are independent of space coordinates. The unimodular conformal metric $[h]$ becomes independent of time. One should notice that we introduced the factor $e^{-\frac{3}{2}\bar{\phi}}$ in the generating function by hand, only for convenience. As a result the metric $[\gamma]$ and the scalar field $\phi$ reduce to those of the quasi-isotropic space. Similarly, the extrinsic curvature reduces to the corresponding quasi-isotropic one. This correspondence can be checked explicitly, by putting $\phi = 0$ in the general solution.

# 10. DUALITY

In this section we briefly discuss the duality property of the general solution. It is apparent, from the form of the solution, that the transformation $\alpha_a \longrightarrow \alpha'_a = -\alpha_a$ (for all $a$) generates a dual solution. This yields $\alpha \longrightarrow \alpha = -\alpha$ for the trace of the matrix $[\alpha]$. We can perform this as a transformation of the angular coordinate $\varphi$ in the $(\phi, z)$-plane:

$$\cos(\pi - \varphi') = \frac{\cos\varphi - \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}\cos\varphi} \quad . \tag{10.1}$$

One can easily check that this transformation is equivalent to $\alpha \longrightarrow \alpha'$, using eq. (8.45) explicitly. The dual solution is of the same form. This transformation is well defined since $-1 < \cos\varphi' < 1$ and $-\sqrt{3} < \alpha' < \sqrt{3}$. The transformation of the radius can be found by using eq. (8.36). Using $x = r\cos\varphi$ and $y = r\sin\varphi$ we find the known result

$$\phi \longrightarrow \phi' = \phi - \ln\gamma \quad . \tag{10.2}$$

However, $z$ does not experience any change except for an additional constant contribution (remember $\tilde{z} = 0$ by definition). This can be seen more easily if it is written in terms of $\alpha$.

We can decompose the gravitational momentum tensor into a trace contribution and a traceless part as

$$\frac{1}{\lambda}[\lambda] = \frac{1}{3}I + [q] \quad . \tag{10.3}$$

Then we find that the traceless part $[q]$ transforms as

$$[q'] = -\frac{\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{1 - 6|q|^2}}{1 + \frac{\sqrt{3}}{2}\sqrt{1 - 6|q|^2}}[q] \quad , \tag{10.4}$$

under the duality transformation. Here $|q|^2 = Tr([q][q])$. We have to impose the momentum constraints simultaneously.

Meanwhile, eq. (10.1) yields that the dual of the quasi-isotropic solution is again a quasi-isotropic one. However, it is not contained in the superinflationary section of the plane.

# 11. CONCLUSION

In the first part of this work, we studied the cosmological consequences of a scalar-tensor theory which is expected to incorporate spontaneous symmetry breaking with gravitation. In this model, the Newtonian gravitational constant is induced by the vacuum expectation value of the scalar field. We presented homogeneous and isotropic solutions. First, we presented sample solutions for various potentials. Nevertheless, our main emphasis is on the globally conformal invariant case with a quartic potential, in accordance with our main motivation. The solutions we presented are approximate in the sense that, they are characterized by the dimensionless parameter $\nu$, which, in turn depends on the scale of the ground state of the scalar field. The model contains two types of solutions. In the first, the scalar field $\phi$ increases in time and in the second it decreases in time. These correspond to decreasing and increasing Newtonian gravitational constant, $G$, respectively. The universe may begin from a finite region or with a Big Bang, depending on the behaviour of the scalar field.

This model reduces to the Brans-Dicke Theory as the conformal size of the universe $b \to 0$. But as $b \to \infty$ the potential term $\lambda\phi^4$ dominates. In these two limits the equation of state satisfies $p = \rho$ and $p = -\rho$ respectively, and the behaviour of the universe in both limits is determined by the scalar field. We also showed that $\rho \to \rho_c$ and thus $p \to -\rho_c$ as $b \to \infty$, and in the present epoch the vacuum expectation value of the scalar field supplies the critical energy density. The present model may provide a natural explanation for the dark matter in the Universe.

We also presented an analysis of the cosmological consequences of the model for arbitrary $\nu$. This enables us to test the solutions and the model itself. We can also comment on the scalar field. For example identification of the scalar field with Standard model Higgs field leads to a very small mass and a long range coupling which is stronger than gravity, with matter. A comparison with Grand Unified Theories leads to smaller values for the mass. However our analysis is classical and further comments need more elaborate calculations. Nevertheless, cosmological parameters lead to different orders of magnitude for the dimensionless parameters of the model. We are unable to explain why different orders of magnitude are present in the action. Such extremely large and small values are inconsistent with the current theoretical notion of naturalness. If an explanation for their presence can be found, this type of theory will be attractive since it has the advantages of relating all physical constants with dimensions of mass to the same source and explaning why a closed universe can become as large as ours.

In the second problem we applied the long-wavelength approximation to the low-energy effective string action in the context of the Hamilton-Jacobi theory. The Hamilton-Jacobi equation for the effective string action in four dimensions is invariant under SFD transformation. However, the momentum constraints are not invariant under this transformation. Long-wavelength gravity (gradient expansion) has proved to be a fruitful method for studying slightly inhomogeneous cosmology. It is a significant improvement over those of homogeneous minisuperspace.

We presented a leading-order solution of the HJ equation. This is the most general solution near a singularity. We solved the HJ equation, including the evolution of the gravitational radiation. In order to do this, we performed a transformation to new canonical variables where the Hamiltonian density vanishes strongly. Therefore, the new variables are constant in time, if the shift function vanishes. However, they depend on the spatial coordinates. In the separated Hamilton-Jacobi equation, the gravitational degrees of freedom can be reduced to that of a single massless scalar field. However, the gravitational field is fundamentally different from massless scalar fields. For example, it carries spin angular momentum, and the momentum constraints restrict the longitudinal modes of the gravitational momentum tensor. Then we presented the quasi-isotropic solution. The general solution contains this one as a special case. In this case, gravitational radiation is neglected. The Hamilton-Jacobi approach yields a result consistent with that derived by using the Lagrange equations [72, 83].

In the Hamilton-Jacobi approach, we can simply represent physically important cases in the $(\phi, z)$-plane. Constant $H$ surfaces are circles concentric with the origin. The evolution of the fields $\phi$ and $z$, at a fixed spatial point, are given by trajectories coming out of the origin. These trajectories remain orthogonal to uniform $H$ surfaces every time. The region $\cos\varphi > 1/\sqrt{3}$ corresponds to superinflationary solutions. Meanwhile, we obtain a quasi-isotropic universe on the $\phi$-axis. The Big Bang instant corresponds to a point at infinity. The momentum constraints admit a simple expression in terms of the new canonical variables. This form is useful for general discussions. We also performed the duality transformation as a transformation of angle in $(\phi, z)$-plane, $\varphi \longrightarrow \varphi'$. We have to impose the momentum constraints simultaneously. However, this transformation and its relation to the momentum constraints need further clarification.

# 12. APPENDIX

## 12.1. Units, Dimensions and Constants

We use the units $\hbar = c = 1$. Therefore, we write the necessary constants in terms of energy [47].

Gravitational constant: $G_o = 6.7 \times 10^{-39}\ (GeV)^{-2}$

Planck mass: $M_p = \sqrt{\frac{1}{G_o}} = 1.2 \times 10^{19}\ GeV$

Hubble constant: $H_o = 2.1\, h_o \times 10^{-42}\ GeV \quad , \qquad 0.5 < h_o < 0.85$

Age of the Universe: $t_u \cong 0.8 \times 10^{42}\ (Gev)^{-1}$

Scaled cosmological constant: $\quad -1 < \frac{\Lambda}{3H_o^2} < 2$

Critical density: $\rho_c = 7.5\, h_o^2 \times 10^{-47}\ (GeV)^4$

Density parameter of the Universe: $\Omega_o \equiv \frac{\rho_o}{\rho_c},\ 0.1 < \Omega_o < 2$

Rate of change of the Gravitational constant: $\frac{\dot{G}}{G} \leq 1.9 \times 10^{-43}\ GeV$

Electroweak Theory: $\phi_o = 246\ GeV \quad , \qquad m_H > 50\ GeV$

Grand Unified Theories: $\phi_o \geq 10^{15}\ GeV$

## 12.2. The Robertson-Walker Metric

We use the signature $(+,-,-,-)$ for the metric. The convention for the Ricci tensor is

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\alpha} \quad . \tag{12.1}$$

The Robertson-Walker metric is given by

$$ds^2 = g^{\mu\nu}dx^\mu dx^\nu = N^2\,(t)\,dt^2 - a^2\,(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right] \quad . \tag{12.2}$$

The determinant of this metric is

$$g = -a^6 N^2 \frac{r^4}{1-kr^2}\sin^2\theta \quad . \tag{12.3}$$

The Einstein tensor and its components are

$$G^\mu_{\ \nu} = R^\mu_{\ \nu} - \frac{1}{2}\delta^\mu_{\ \nu}R \quad , \tag{12.4}$$

$$G^0_{\ 0} = \frac{3}{a^2 N^3}\left(\dot{a}^2 N + kN^3\right) = 3\left[\left(\frac{\dot{a}}{a}\right)^2 \frac{1}{N^2} + \frac{k}{a^2}\right] \quad , \tag{12.5}$$

$$\begin{aligned} G^1_{\ 1} = G^2_{\ 2} = G^3_{\ 3} &= \frac{1}{a^2 N^3}\left(2a\ddot{a}N + \dot{a}^2 N + kN^3 - 2a\dot{a}\dot{N}\right) \\ &= \frac{1}{N^2}\left[2\left(\frac{\dot{a}}{a}\right)^{\cdot} + 3\frac{\dot{a}}{a}\right] + \frac{k}{a^2} - 2\frac{\dot{a}}{a}\frac{\dot{N}}{N^2} \quad , \end{aligned} \tag{12.6}$$

The Friedmann equations are

$$\frac{8\pi G_o}{3}\rho = \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{\Lambda}{3} \quad , \tag{12.7}$$

$$\frac{8\pi G_o}{3}p = -\frac{1}{3}\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] + \frac{\Lambda}{3} \quad . \tag{12.8}$$

If we use $a = At^\alpha$ and $p = \beta\rho$, for a flat universe , $k = 0$, without a cosmological constant, $\Lambda = 0$, Friedmann equations yield

$$\alpha = \frac{2}{3}\frac{1}{1+\beta} \quad . \tag{12.9}$$

## 12.3. Brans-Dicke Action and the Field Equations

Brans-Dicke action is given by

$$S_{BD} = \int \left[ -\phi R + \omega \frac{\phi_{,\alpha}\phi^{,\alpha}}{\phi} \right] \sqrt{-g} d^4x \quad . \tag{12.10}$$

Brans-Dicke equations are

$$G^{\alpha\beta} = \frac{\omega}{\phi^2} \left( g^{\alpha\mu} g^{\beta\nu} - \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \right) \phi_{;\mu}\phi_{;\nu} + \frac{1}{\phi} \left( g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu} \right) \phi_{;\mu;\nu} \quad , \tag{12.11}$$

$$2\frac{\omega}{\phi} \left( g^{\mu\nu} \phi_{;\nu} \right)_{;\mu} - \frac{\omega}{\phi^2} g^{\mu\nu} \phi_{;\mu}\phi_{;\nu} + R = 0 \quad . \tag{12.12}$$

Combining the two equations we get

$$\left( g^{\mu\nu} \phi_{;\nu} \right)_{;\mu} = 0 \quad . \tag{12.13}$$

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