IRREGULAR SAMPLING IN SHIFT-INVARIANT SPACES

by

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To the Electrical Engineering and Mathematics departments of Boğaziçi University,
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ABSTRACT

IRREGULAR SAMPLING IN SHIFT-INVARIANT SPACES

This thesis is an exposition of the concept of localization of frames in the problem of irregular sampling in shift-invariant spaces. The given definition of the localization of a frame will appear to be equivalent to an off-diagonal decay of the matrix corresponding to the frame operator. The proofs of some inverse-closedness theorems of certain classes of matrices having an off-diagonal decay will be given. These theorems imply the localization of the dual frame. Under these localization conditions, the Hilbert space theory can be extended to the family of associated Banach spaces. If the generator of a shift-invariant space satisfies necessary decay conditions, then it will be seen that its reproducing kernel frame will be a localized frame, and the theory will be applicable.

ÖZET

ÖTELEMELER ALTINDA DEĞİŞMEYEN UZAYLARDA DÜZENSİZ ÖRNEKLEME

Bu tezde çerçevelerin bölgeselliği kavramının, ötelemeler altında değişmeyen uzaylardaki düzensiz örnekleme problemine uygulanması işlenmektedir. Çerçeveler için tanımlanacak olan bölgesellik kavramının, çerçeve operatörüne karşılık gelen matrisin diyagonal dışı azalma özelliğine denk olduğu görülecektir. Belirli matris sınıflarında, diyagonal dışı azalma özelliğinin matris tersi alındığında korunmasıyla ilgili teoremlerin kanıtları verilecektir. Bu teoremler dual çerçevenin de bölgesel olmasını getirir. Bu koşullar altında Hilbert uzayları icin geçerli olan teori, ilgili Banach uzayları ailesi için geçerli olacak şekilde genişletilebilir. Eğer ötelemeler altında değişmeyen bir uzayın doğuray fonksiyonu belli azalma özelliklerine sahipse, çoğaltıcı çekirdek çerçevesinin bölgesel bir çerçeve olduğu görülecek ve söz edilen teori uygulanabilecektir.

TABLE OF CONTENTS

LIST OF SYMBOLS

$\square$	End of proof
$\subseteq$	Subset
$\subset$	Proper subset
$\equiv$	Equality that includes a definition
$\hat{f}$	The Fourier transform of f
$\operatorname{sinc}(x)$	The cardinal sine function, defined by $\operatorname{sinc}(x) = \sin(\pi x)/\pi x$
B_Ω	The space of functions whose Fourier transforms are supported on $[-\Omega, \Omega]$
$\|x\|_{l^p}$	l^p norm of the sequence x: $\left(\sum_k \|x_k\|^p\right)^{1/p}$
$\|x\|_{l^p_m}$	Weighted l^p norm of the sequence x: $\left(\sum_k (\|x_k\| m(k))^p\right)^{1/p}$
$\|f\|_{L^p}$	l^p norm of the function f: $\left(\int_{\mathbb{R}^d} \|f(x)\|^p\right)^{1/p} dx$
$\|f\|_{L^p_m}$	Weighted l^p norm of the function f: $\left(\int_{\mathbb{R}^d} (\|f(x)\| m(x))^p\right)^{1/p} dx$
$l^p(\mathbb{Z}^d)$	The vector space of sequences x over $\mathbb{Z}^d$ with $\|x\|_{l^p} < \infty$
$l^p_m(\mathbb{Z}^d)$	The vector space of sequences x over $\mathbb{Z}^d$ with $\|x\|_{l^p_m} < \infty$
$L^p(\mathbb{R}^d)$	The vector space of functions f over $\mathbb{R}^d$ with $\|f\|_{L^p} < \infty$
$L^p_m(\mathbb{R}^d)$	The vector space of functions f over $\mathbb{R}^d$ with $\|f\|_{L^p_m} < \infty$
$V^2(\varphi)$	The shift invariant space $\{\sum_k c_k \varphi(x-k) \colon (c_k) \in l^2(\mathbb{Z}^d)\}$
$V^p_m(\varphi)$	The shift invariant space $\{\sum_k c_k \varphi(x-k) \colon (c_k) \in l^p_m(\mathbb{Z}^d)\}$
$x_k * y_k$	Convolution of the sequences x_k and y_k: $\sum_l x_{k-l} y_l$
$\langle x \mid y \rangle$	An inner product that is linear in the second term, defined as $\sum_k \overline{x_k} y_k$
$\langle f, g \rangle$	The usual Hilbert inner product, given by $\int f(x) \overline{g(x)}\, dx$
A^*	Adjoint of the operator A
$a(s,t)$	The entry (s,t) of the matrix A
$Q_\alpha(T)$	The class of matrices B indexed by T that satisfy $\|b(s,t)\| \leq C(1 + \|s-t\|^\alpha)$ for some $C > 0$
$E_\gamma(T)$	The class of matrices B indexed by T that satisfy $\|b(s,t)\| \leq Ce^{\gamma' \|s-t\|}$ all $\gamma < \gamma$, for some $C > 0$ depending on γ'

$\Vert B\Vert$	Norm of B as an operator on l^2
$\Vert B\Vert_\alpha$	$\sup_{s,t\in T}(1+\Vert s-t\Vert_2)^\alpha \lvert b(s,t)\rvert$
$\Vert B\Vert_{l^p}$	$\max\left\{\sup_s \left[\sum_t \lvert a(s,t)\rvert^p\right]^{1/p},\ \ \sup_t \left[\sum_s \lvert a(s,t)\rvert^p\right]^{1/p}\right\}$
B_δ	A matrix derived from another matrix B, with its entry (s,t) defined to be $\Vert s-t\Vert_2^{d+\gamma}\, b(s,t)$
$\Vert T\Vert_{\mathrm{op}}$	Operator norm of the mapping T
$\asymp$	Norm equivalence: we write $\Vert\cdot\Vert_1 \asymp \Vert\cdot\Vert_2$ if there exist positive constants A and B such that $A\Vert\cdot\Vert_1 \le \Vert\cdot\Vert_2 \le B\Vert\cdot\Vert_1$.
$C_\mathcal{E}$	The coefficient operator corresponding to the frame $\mathcal{E} = (e_x)_{x\in\mathcal{X}}$, given by $f \mapsto (\langle f, e_x\rangle)_{x\in\mathcal{X}}$
$D_\mathcal{E}$	The synthesis operator corresponding to the frame $\mathcal{E} = (e_x)_{x\in\mathcal{X}}$ given by $(c_x)_{x\in\mathcal{X}} \mapsto \sum_{x\in\mathcal{X}} c_x e_x$
T_k	The translation operator defined by $T_k f(\,\cdot\,) = f(\,\cdot - k)$

1. INTRODUCTION

In his classic 1949 paper [1], Claude E. Shannon began with the following theorem: If the Fourier transform $\hat{f}$ of a function f in L^2 is supported on $[-\Omega, \Omega]$, then f can completely be reconstructed from its samples on the set $\{k\pi/\Omega : k \in \mathbb{Z}\}$ with the following formula [1]

$$f(x) = \sum_{k\in\mathbb{Z}} f(kT)\operatorname{sinc}(x/T - k), \tag{1.1}$$

where $T = \pi/\Omega$ and $\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$. The formula (1.1) is of theoretical importance, showing the possibility of representing continuous functions with a sequence of numbers, the idea that lead to the the *digital revolution.*

It can be shown that the space of L^2 functions whose Fourier transforms are supported on $[-\Omega, \Omega]$ can also be defined as

$$B_\Omega = \left\{ \sum_{k\in\mathbb{Z}} c_k \operatorname{sinc}(\Omega x/\pi - k) \colon c \in l^2(\mathbb{Z}) \right\}.$$

The functions in B_Ω have very good frequency localization: they all have compactly supported Fourier transforms. As a result of this, their time localizations are very poor. The function sinc decays with $1/x$ and this fact makes the formula (1.1) unsuitable for numerical implementations [2, 3]. Also, in practice, almost no signal is strictly bandlimited. This requires the signal to be filtered before the sampling process. This filtering operation corresponds to projecting the signal into the Hilbert space B_Ω [1, 4, 5]. However, depending on the structure of the signals, B_Ω might not be the optimal

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[1] Lebesque spaces are not function spaces, but their elements are equivalence classes of functions that agree almost everywhere. Therefore point evaluations $f(x)$ usually do not make sense. However, it can be shown that the functions whose Fourier transforms are compactly supported agree almost everywhere with a continuous function. We implicitly identify this subspace of L^2 with the space of continuous functions corresponding to it. We will study sampling only in continuous function spaces, and we will implicitly identify subspaces of Lebesque spaces with spaces of continuous functions when necessary.

space of approximation. So it would be advantageous to study sampling in a more flexible setting.

The focus of this thesis will be sampling theorems in the so called *shift invariant spaces*, which are roughly spaces of the form

$$V(\varphi) = \left\{ \sum_{k \in \mathbb{Z}^d} c_k \varphi(x-k) \right\}$$

where $c = (c_k)_{k \in \mathbb{Z}^d}$ comes from an appropriate sequence space. The classical sampling set up in B_Ω is a particular example of this, with $\varphi(x) = \operatorname{sinc}(x)$. We will study the shift invariant spaces that are subspaces of the weighted L^p spaces, which are denoted by L^p_m, and are defined by the norm $\|f\|_{L^p_m} \equiv \|mf\|_{L^p}$, where m is a positive weight function. The weight function controls the decay of the functions in the space. If $m(x) \to \infty$ as $|x| \to \infty$, then the functions in L^p_m will have a decaying property. If $m(x) \to 0$ as $|x| \to \infty$, then the functions in L^p_m may grow, up to some extent. The exponent p of the space L^p_m has also some effect on the decay of the function, but more than that p gives a control on the norm that we wish to use on the space, and it can be chosen in an optimal way according to the application [5].

Sampling on regular grids, i.e., sets of the form $\{kr : k \in \mathbb{Z}^d\}$ for some positive r, like in the classical sampling, is called *regular sampling*. Sampling on more arbitrary countable subsets of $\mathbb{R}^d$ is called *irregular sampling*. Irregular sampling may be unavoidable, or it might produce better results than regular sampling. Here are some cases when we encounter an irregular sampling problem [5, 6]:

- *Communication*: When some of the data of a uniformly sampled signal is lost, we end up with a problem of reconstruction of a function from irregular samples.
- *Astronomy*: Daylight periods and adverse nighttime weather conditions prevent regular data collection.
- *Medical imaging*: Computerized tomography and magnetic resonance imaging frequently use a nonuniform polar and spiral sampling sets.

Other application areas of irregular sampling include geophysics, spectroscopy, biomedical imaging [5]. In this thesis, we will not confine ourselves to regular sampling, instead we will follow a general approach that will treat both regular and irregular sampling together.

Frame theory turns out to play a key role in sampling theory. A countable set $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exists positive constants A and B such that

$$A \left\| f \right\|^2 \leq \sum_{x \in \mathcal{X}} \left| \langle f, e_x \rangle \right|^2 \leq B \left\| f \right\|^2$$

for all $f \in \mathcal{H}$. From the first inequality, it is seen that $\mathcal{E}$ is total in $\mathcal{H}$, and the second inequality brings a stability to the system. A frame is not necessarily a basis however, because the members of it are not necessarily linearly independent. Frames allow redundant representations of functions. The operator defined by

$$S \colon \mathcal{H} \to \mathcal{H}, \quad f \mapsto \sum_{x \in \mathcal{X}} \langle f, e_x \rangle \, e_x$$

is called the *frame operator*. The series converges unconditionally, and the frame operator is invertible. For every frame $(e_x)_{x \in \mathcal{X}}$ there exists a *dual frame* $(\tilde{e}_x)_{x \in \mathcal{X}}$ defined by

$$\tilde{e}_x = S^{-1} e_x,$$

and for every function $f \in \mathcal{H}$ we have

$$f = \sum_{x \in \mathcal{X}} \langle f, \tilde{e}_x \rangle \, e_x.$$

Although the above representation of a function f with respect to the frame is not unique, it is minimal with the respect to the l^2 norm of the coefficient sequence.

Now we can briefly describe the sampling scheme in general Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space of continuous functions over some space X. A countable subset $\mathcal{X} \subseteq X$ is called a *set of sampling* if there exists constants A and B such that

$$A \left\| f \right\|^2 \leq \sum_{x \in \mathcal{X}} \left| f(x) \right|^2 \leq B \left\| f \right\|^2 . \tag{1.2}$$

If this condition is satisfied, then it is easily seen from the first inequality that any function $f \in \mathcal{H}$ is uniquely determined by its values on the set $\mathcal{X}$. But (1.2) says more: the sampling operator $f \mapsto (f(x))_{x \in \mathcal{X}}$ is a linear, continuous, invertible mapping from, $\mathcal{H}$ to the sequence space $l^2(\mathcal{X})$, and its inverse is also contionuous. This allows stable sampling and reconstruction, i.e., a small variation in the sampled sequence causes a small variation in the reconstructed signal and vice versa. This is important, because there will always be additive noise, measurement and quantization errors on the samples [4, 5]. That is why we will only consider sets of sampling. Observe also that by (1.2), for a fixed point $x \in \mathcal{X}$ the mapping $f \mapsto f(x)$ is a bounded linear functional on $\mathcal{H}$. So, by the Riesz representation theorem, for every $x \in \mathcal{X}$ there exists a unique $K_x \in \mathcal{H}$ such that $f(x) = \langle f, K_x \rangle$. Putting this in (1.2), we get that $(K_x)_{x \in \mathcal{X}}$ is a frame for $\mathcal{H}$. Therefore we obtain the following reconstruction formula:

$$\begin{aligned} f &= \sum_{x \in \mathcal{X}} \langle f, K_x \rangle \tilde{K}_x \\ &= \sum_{x \in \mathcal{X}} f(x) \tilde{K}_x . \end{aligned}$$

In our setting, the space $\mathcal{H}$ will be the shift invariant space generated by a single function φ with specific decay conditions. We will obtain explicit formulas for $(K_x)_{x \in \mathcal{X}}$. The interpolating functions $\tilde{K}_x$ can be computed using the inverse of the frame operator S. That is why the inverse of the frame operator plays an important role. Then, our objective will be to extend this Hilbert space setting to the Banach spaces, generated by the same function, following Aldroubi and Gröchenig [5].

The general shift-invariant spaces are defined as follows:

$$V_m^p(\varphi) = \left\{ \sum_{k\in\mathbb{Z}^d} c_k\varphi(x-k) : c_k \in l_m^p(\mathbb{Z}^d) \right\},$$

where the sequence space $l_m^p(\mathbb{Z}^d)$ is the space defined by the norm

$$\|c\|_{l_m^p(\mathbb{Z}^d)} \equiv \left(\sum_{k\in\mathbb{Z}^d} |c_k m(k)|^p \right)^{1/p}.$$

Under some decay and stability conditions on φ, it will be guaranteed that the convergence will be unconditional, and the functions will agree almost everywhere with a continuous function. With the latter property, we can identify the space with a Banach space of continuous functions, hence the point evaluations will make sense. We can extend the definition of a set of sampling to V_m^p spaces as follows: a subset $\mathcal{X} \subseteq \mathbb{R}^d$ is called a *set of sampling* for V_m^p if there exists positive constants A and B such that

$$A\,\|f\|_{L_m^p} \leq \left(\sum_{x\in\mathcal{X}} |f(x)m(x)|^p \right)^{1/p} \leq B\,\|f\|_{L_m^p}. \tag{1.3}$$

Under some decay conditions, we will see that we still can apply the sampling formulas of the Hilbert space setting, and a remarkable result of Gröchenig follows that a set of sampling for $V^2(\varphi)$ is also a set of sampling for V_m^p for all $1 \leq p < \infty$ and all appropriate weight functions m, and the reconstruction formula of $V^2(\varphi)$ is valid for V_m^p [7]. This result depends on the concept of localization of frames, which is again introduced by Gröchenig in [7].

In this thesis, we will begin with a sampling set for $V^2(\varphi)$, and we will keep the problem of characterizing sampling sets out of the scope of this thesis. However we note that the Hilbert space theory is well understood, and sharp estimates of the form (1.2) exists for a wide class of generators [5, 7, 8].

The organization of the thesis will be as follows. In Chapter 2, we will introduce the mathematical preliminaries such as unconditional convergence, Riesz bases and Frames. In Chapter 3, we will prove theorems stating that some certain classes of infinite matrices having an off-diagonal decay are closed under matrix inversion. For the case of matrices having a polynomial decay, the proof of inverse-closedness theorem is difficult and we will give a proof which has been discovered by Jaffard and Journé [9]. These theorems play an important role in Chapter 4, where we will study the concept of localization of frames. We will see that a frame is localized with respect to a basis if and only if the matrix of the frame operator has an off-diagonal decay of the corresponding type. The inverse closedness theorems will imply that the dual frame has the same type of localization. For a Hilbert space $\mathcal{H}$, we will define a class of associated Banach spaces, and it will be shown that a localized frame for $\mathcal{H}$ will preserve its frame properties for the associated Banach spaces. In Chapter 5, we will study sampling in weighted shift invariant spaces, first in the $p = 2$, $m = 1$ case with a given set of sampling $\mathcal{X}$. Then we will extend the setting to the general Banach space case, and using the theory of Chapter 4, we will see that the same set $\mathcal{X}$ will also be a set of sampling, leading to a stable reconstruction formula.

2. MATHEMATICAL PRELIMINARIES

2.1. Unconditional Convergence of Series in Normed Vector Spaces

Let V be a normed vector space, and $(x_n)_{n\in\mathbb{N}}$ be a sequence of vectors in V. We say that the series $\sum_{n=1}^{\infty} x_n$ converges, if the sequence of partial sums $s_n = x_1 + \cdots + x_n$ converges to a vector $s \in V$ with respect to the metric induced by the norm. We denote this by $\sum_{n=1}^{\infty} x_n = s$. This definition depends on the order of the index set $\mathbb{N}$. We define a stronger version of convergence of series, which is independent of the order of the index set as follows:

Definition 2.1.1 *Let V be a normed vector space, $\mathcal{X}$ be an arbitrary index set, and let $\{v_t : t \in \mathcal{X}\}$ be a set of vectors in V. The series $\sum_{t\in\mathcal{X}} v_t$ is said to be* unconditionally convergent *to the vector $s \in V$, if for every $\epsilon > 0$, there exists a finite subset $\mathcal{F}$ of $\mathcal{X}$ such that*

$$\left\| \sum_{t\in\mathcal{F}'} v_t - s \right\| < \epsilon$$

for all finite subsets $\mathcal{F}'$ satisfying $\mathcal{F} \subseteq \mathcal{F}' \subseteq \mathcal{X}$; and this is denoted by $\sum_{t\in\mathcal{X}} v_t = s$.

The following proposition contains an alternative definition for unconditional convergence of countably many vectors:

Proposition 2.1.2 [10] *Let V be a normed vector space and $\{v_t\}_{x\in\mathcal{X}}$ be a countable set of vectors in V. The series $\sum_{t\in\mathcal{X}} v_t$ converges unconditionally to s, if and only if*

$$\sum_{n=1}^{\infty} v_{\tau(n)} = s$$

for any bijection τ from $\mathcal{X}$ to $\mathbb{N}$.

Proof. Assume that $\sum_{t\in\mathcal{X}} v_t$ converges unconditionally to s. Let τ be a permutation from $\mathcal{X}$ into $\mathbb{N}$. We want to show that $\sum_{n=1}^{\infty} v_{\tau(n)} = s$. Let $\epsilon > 0$ be given. We know that there exits a subset $\mathcal{F} \subseteq \mathcal{X}$ such that for all finite sets $\mathcal{F}'$ satisfying $\mathcal{F} \subseteq \mathcal{F}' \subseteq \mathcal{X}$ we have $\left\|\sum_{t\in\mathcal{F}'} v_t - s\right\| < \epsilon$. We choose an integer N such that $\mathcal{F}' \subseteq \{\tau(n)\}_{n=1}^{N}$, then for all $N' \geq N$ we have $\left\|\sum_{n=1}^{N'} v_{\tau(n)} - s\right\| < \epsilon$ as desired.

For the converse, assume that $\sum_{t\in\mathcal{X}} v_t$ is not unconditionally convergent. This means that for every $s \in V$, there exists an ϵ such that for all finite subsets $\mathcal{F}$ of $\mathcal{X}$ there exists a finite set $\mathcal{F}'$ with $\mathcal{F} \subseteq \mathcal{F}' \subseteq \mathcal{X}$ satisfying $\left\|s - \sum_{t\in\mathcal{F}'} v_t\right\| \geq \epsilon$. Using this, for a given $s \in V$ we will construct a bijection τ form $\mathbb{N}$ to $\mathcal{X}$ such that $\sum_{n=1}^{\infty} v_{\tau(n)}$ does not converge to s.

Let $(x_i)_{i\in\mathbb{N}}$ be an enumeration of $\mathcal{X}$. Let $\mathcal{F}_1$ be any subset of $\mathcal{X}$ containing $\{x_1\}$, and let $\mathcal{G}_1$ be a finite subset of $\mathcal{X}$ such that $\mathcal{F}_1 \subseteq \mathcal{G}_1 \subseteq \mathcal{X}$ and $\left\|s - \sum_{t\in\mathcal{G}_1} v_t\right\| \geq \epsilon$. For $k \in \mathbb{N}$, having defined $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k$ and $\mathcal{G}_1, \mathcal{G}_1, \cdots, \mathcal{F}_k$, we chose $\mathcal{F}_{k+1}$ to be any finite subset of $\mathcal{X}$ containing both $\mathcal{G}_k$ and $\{x_1, x_2, \cdots, x_{k+1}\}$; and $\mathcal{G}_{k+1}$ to be a finite subset of $\mathcal{X}$ with $\mathcal{F}_{k+1} \subseteq \mathcal{G}_{k+1} \subseteq \mathcal{X}$ and $\left\|s - \sum_{t\in\mathcal{G}_{k+1}} v_t\right\| \geq \epsilon$.

We have obtained a collection of finite sets $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{G}_3 \subseteq \cdots$ with $\bigcup_{k\in\mathbb{N}} \mathcal{G}_k = \mathcal{X}$. We can define τ to be a bijection from $\mathbb{N}$ to $\mathcal{X}$ such that for a given $k \in \mathbb{N}$, the image of the set $\{1, 2, \cdots, |\mathcal{G}_k|\}$ under τ is the set $\mathcal{G}_k$, where $|\mathcal{G}_k|$ denotes the number of elements in $\mathcal{G}_k$. With this τ it is clear that $\sum_{n=1}^{\infty} v_{\tau(n)}$ does not converge to s, as desired. □

2.2. Riesz Systems and Riesz Bases

Definition 2.2.1 *Let $\mathcal{H}$ be a Hilbert space. Then the set $\{y_t : t \in \mathcal{X}\}$ for some index set $\mathcal{X}$ is called a* Riesz system [2] *if there exists positive real numbers A and B such that*

$$A \sum_{t\in\mathcal{F}} |c_t|^2 \leq \left\|\sum_{t\in\mathcal{F}} c_t y_t\right\|^2 \leq B \sum_{t\in\mathcal{F}} |c_t|^2$$

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[2] The term “Riesz sequence” is used when the index set is $\mathbb{N}$.

for any finite subset $\mathcal{F}$ of $\mathcal{X}$.

Theorem 2.2.2 *Let $\mathcal{H}$ be a Hilbert space, and $\{y_t\colon t \in \mathcal{X}\}$ be countable a Riesz system in $\mathcal{H}$. Then for any sequence of scalars $(c_t)_{t\in\mathcal{X}}$ with $\sum_{t\in\mathcal{X}} |c_t|^2 < \infty$, the series $\sum_{t\in\mathcal{X}} c_t y_t$ converges unconditionally to some vector s in $\mathcal{H}$, and it satisfies*

$$A\sum_{t\in\mathcal{X}} |c_t|^2 \leq \left\|\sum_{t\in\mathcal{X}} c_t y_t\right\|^2 \leq B\sum_{t\in\mathcal{X}} |c_t|^2.$$

Proof. Assume $(y_t)_{t\in\mathcal{X}}$ is a square summable sequence. Let τ be a bijection from $\mathbb{N}$ to $\mathcal{X}$. Then that the series $\sum_{n=1}^{\infty} c_{\tau(n)} y_{\tau(n)}$ satisfies the Cauchy condition, since we have $\left\|\sum_{n=k}^{m} c_{\tau(n)} y_{\tau(n)}\right\|^2 \leq B\sum_{n=k}^{m} |c_{\tau(n)}|^2$ by the Riesz system assumption; hence the series converges to a vector $s \in \mathcal{H}$. Taking limits, we obtain the relation

$$A\sum_{t\in\mathcal{X}} |c_t|^2 \leq \left\|\sum_{n=1}^{\infty} c_{\tau(n)} y_{\tau(n)}\right\|^2 \leq B\sum_{t\in\mathcal{X}} |c_t|^2 \tag{2.1}$$

which is valid for any bijection τ. Now it only remains to show that the limit of $\sum_{n=1}^{\infty} c_{\tau(n)} y_{\tau(n)}$ is also independent of the bijection τ.

Let υ_1 and υ_2 be two arbitrary bijections from $\mathbb{N}$ to $\mathcal{X}$. We want to show that series $\sum_{n=1}^{\infty} c_{\upsilon_1(n)} y_{\upsilon_1(n)}$ and $\sum_{n=1}^{\infty} c_{\upsilon_2(n)} y_{\upsilon_2(n)}$ converge to the same limit. Let $\epsilon > 0$ be given. There exists an N_1 such that $\sum_{n=N_1+1}^{\infty} |c_{\upsilon_1(n)}|^2 < \epsilon$, and there exists an $N_2 > N_1$

such that $\{v_1(n)\}_{n=1}^{N_1} \subseteq \{v_2(n)\}_{n=1}^{N_2}$. We have

$$\begin{aligned}\left\| \sum_{n=1}^{\infty} c_{v_2(n)} y_{v_2(n)} - \sum_{n=1}^{\infty} c_{v_1(n)} y_{v_1(n)} \right\| &= \left\| \sum_{n=1}^{N_2} c_{v_2(n)} y_{v_2(n)} - \sum_{n=1}^{N_1} c_{v_1(n)} y_{v_1(n)} \right. \\ &\quad \left. + \sum_{n=N_2}^{\infty} c_{v_2(n)} y_{v_2(n)} - \sum_{n=N_1}^{\infty} c_{v_1(n)} y_{v_1(n)} \right\| \\ &\leq \left\| \sum_{n=1}^{N_2} c_{v_2(n)} y_{v_2(n)} - \sum_{n=1}^{N_1} c_{v_1(n)} y_{v_1(n)} \right\| \\ &\quad + \left\| \sum_{n=N_2}^{\infty} c_{v_2(n)} y_{v_2(n)} \right\| + \left\| \sum_{n=N_1}^{\infty} c_{v_1(n)} y_{v_1(n)} \right\| \\ &\leq 3\sqrt{B\epsilon},\end{aligned}$$

where the last inequality is a result (2.1) and the fact that all the coefficients in all of the three terms come from the set $\{c_{v_1(n)} : n > N\}$, the sum of the modulus squares of whose elements is known to be less than ϵ.

Since the norm of the difference can be made arbitrarily small, the two series converge to the same limit as desired. □

Remark 2.2.3 *We know that for series of complex numbers, unconditional convergence and absolute convergence are equivalent. This is not the case however for series in arbitrary normed vector spaces. We can give a counter example using the above theorem. Let* $\mathcal{H}$ *be an infinite dimensional Hilbert space, and let* $(x_n)_{n\in\mathbb{N}}$ *be orthonormal system in* $\mathcal{H}$*. Then the series* $\sum_{n=1}^{\infty} \frac{1}{n} x_n$ *is unconditionally convergent, since the coefficients are square summable and every orthonormal system is a Riesz system; while it is clearly not absolutely convergent. It can be shown however that absolute convergence implies unconditional convergence.*

Theorem 2.2.4 *Let* $\mathcal{H}$ *be a Hilbert space. Then the set* $\{y_t \colon t \in \mathcal{X}\}$ *be countable a Riesz system in* $\mathcal{H}$ *if and only if the Gram matrix* G *indexed by* $\mathcal{X} \times \mathcal{X}$ *whose entry* (s, t) *is given by* $g(s, t) = \langle y_s, y_t \rangle$ *is an invertible operator on* $l^2(\mathcal{X})$*.*

Definition 2.2.5 *Let $\mathcal{H}$ be a Hilbert space. Then the set Then the set $\{y_t \colon t \in \mathcal{X}\}$ is called a* Riesz basis *for $\mathcal{H}$ if it is a Riesz system, and for every $f \in \mathcal{H}$, there exists a sequence $(c_t) \in l^2(\mathcal{X})$ such that $f = \sum_{t\in\mathcal{X}} c_t y_t$.*

Theorem 2.2.6 *Let $\mathcal{H}$ be a Hilbert space and $\{y_t \colon t \in \mathcal{X}\}$ be a Riesz basis for $\mathcal{H}$. Then there exists another Riesz basis $\{\tilde{y}_t \colon t \in \mathcal{X}\}$ that satisfies*

$$\langle y_t, \tilde{y}_s \rangle = \begin{cases} 1 & \text{if} \quad s = t \\ 0 & \text{if} \quad s \neq t \end{cases},$$

which implies that each f in $\mathcal{H}$ can be represented as

$$f = \sum_{t\in\mathcal{X}} \langle f, \tilde{y}_t \rangle \, y_t = \sum_{t\in\mathcal{X}} \langle f, y_t \rangle \, \tilde{y}_t .$$

The proofs of Theorems 2.2.4 and 2.2.6 can be found in [11, Ch 1, Thm 9].

2.3. Frames

Definition 2.3.1 *Let $\mathcal{H}$ be a Hilbert space. A countable set $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ is called a* frame *if there exists positive constants A and B such that*

$$A \left\| f \right\|^2 \leq \sum_{x\in\mathcal{X}} \left| \langle f, e_x \rangle \right|^2 \leq B \left\| f \right\|^2$$

for all $f \in \mathcal{H}$.

Theorem 2.3.2 *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ be a frame for $\mathcal{H}$. Then*

(i) *For any $f \in \mathcal{H}$, the series $\sum_{x\in\mathcal{X}} \langle f, e_x \rangle \, e_x$ converges unconditionally.*

(ii) *The frame operator S, which defined by*

$$S\colon \mathcal{H} \to \mathcal{H}, \quad f \mapsto \sum_{x\in\mathcal{X}} \langle f, e_x\rangle\, e_x$$

is an invertible operator.

(iii) *The set $\{S^{-1}e_x : x \in \mathcal{X}\} \equiv \{\tilde{e}_x : x \in \mathcal{X}\}$ is also a frame for $\mathcal{H}$, which is called the dual frame of $\mathcal{E}$, and for any $f \in \mathcal{H}$ we have*

$$f = \sum_{x\in\mathcal{X}} \langle f, \tilde{e}_x\rangle\, e_x = \sum_{t\in\mathcal{X}} \langle f, e_x\rangle\, \tilde{e}_x,$$

where the series converge unconditionally.

The proof Theorem 2.3.2 can be found in any of [10, 11, 12, 13].

2.4. Weighted L^p and l^p Spaces

Let m be a positive weight function on $\mathbb{R}^d$. Let $p \in [1, \infty]$. We define the norm $\|\cdot\|_{L^p_m(\mathbb{R}^d)}$ as

$$\|f\|_{L^p_m(\mathbb{R}^d)} \equiv \|mf\|_{L^p(\mathbb{R}^d)}\,,$$

and we define the space $L^p_m(\mathbb{R}^d)$ as the space of functions that have a finite $\|\cdot\|_{L^p_m(\mathbb{R}^d)}$ norm. It can easily be verified that the L^p_m spaces are Banach spaces. Indeed, for $p \in [1, \infty)$ they coincide with the usual L^p spaces when we use a weighted Lebesque measure on $\mathbb{R}^d$, in which a subset $E \subseteq \mathbb{R}^d$ has the measure

$$\mu(E) = \int_E m(x)\, dx\,.$$

Analogously, we define the corresponding sequence space $l^p_m(\mathbb{Z}^d)$ with the norm

$\|\cdot\|_{l^p_m(\mathbb{Z}^d)}$ defined for a sequence $x = \{x_k\}_{k\in\mathbb{Z}^d}$ as

$$\|x\|_{l^p_m(\mathbb{Z}^d)} \equiv \left(\sum_{k\in\mathbb{Z}^d} (|x_k|m(k))^p\right)^{1/p}.$$

These spaces are also Banach spaces, and coincides with the l^p spaces over the set $\mathbb{Z}^d$ endowed with the weighted counting measure.

3. THEOREMS ABOUT INFINITE MATRICES THAT ARE WELL LOCALIZED AROUND THE DIAGONAL

In this chapter we are going to study the properties of matrices that are well-localized around the diagonal. Two classes of matrices will be examined: matrices that have a polynomial off-diagonal decay, and matrices that have an exponential off-diagonal decay. It will be shown that invertibility on l^2 implies that the inverse of the matrix has the same type of decay.

The case for the exponential decay has been a classical result in mathematical physics [7, 9]. Jaffard and Journé have proved the theorem for polynomial decay in [9], using analytical arguments. Recently, Gröchenig and Leinart have proved a stronger version of the theorem using Banach algebra techniques, which is applicable to a wider class of decay conditions [14]. In this chapter, our exposition will depend mostly on Jaffard's paper [9].

An infinite matrix is a function from $T \times T$ to $\mathbb{C}$, where T is a countable set. In order to be able to define the concept of being far away from the diagonal, a metric on T must be defined. For our purposes, it will be enough to study the case when T is a separated subset of $\mathbb{R}^d$ and the metric on it will be the Euclidean metric, $d(s,t) = \|s-t\|_2$. In the following section we will examine some properties of these kinds of sets.

3.1. Properties of the Separated Subsets of $\mathbb{R}^d$

Definition 3.1.1 *Let* (X, d) *be a metric space. A subset* T *of* X *is called a* separated set *if* $\inf\limits_{\substack{x,y\in T\\ x\neq y}} d(x,y) > 0$.

The following proposition shows that the separated subsets of $\mathbb{R}^d$ are very similar to subsets of $\mathbb{Z}^d$. It will enable us to adapt proofs from $\mathbb{Z}^d$ easily to the separated subsets of $\mathbb{R}^d$.

Proposition 3.1.2 *Let T be a separated subset of $\mathbb{R}^d$. Then there exists a one-to-one map τ from T into $\mathbb{Z}^d$, and there exists positive constants A and B, such that for all $x, y \in T$*

$$A \left\|\tau(x) - \tau(y)\right\|_2 \leq \left\|x - y\right\|_2 \leq B \left\|\tau(x) - \tau(y)\right\|_2 . \tag{3.1}$$

Proof. Since T is separated, there exists a $\sigma > 0$ such that $\|x - y\|_2 > \sigma$ for all $x, y \in T$. Let $r = \frac{\sigma}{\sqrt{d}}$. For $t \in \mathbb{Z}^d$ we define S_t to be the set $\prod_{i=1}^{d} [rt_i, r(t_i + 1))$, which is a half-open hypercube with side length r, whose corner with the minimum coordinates is rt. Clearly $\bigcup_{t \in \mathbb{Z}^d} S_t = \mathbb{R}^d$. Also for a given t, S_t can contain at most one point from T, because the diameter of the set S_t is equal to σ, whereas $\|x - y\|_2 > \sigma$ for all $x, y \in T$. So for a point $x \in T$ there exists a unique $t \in \mathbb{Z}^d$ such that $x \in S_t$; and we define $\tau(x) = t$. This map is one-to-one by its construction. Note that $r\,\tau(x) \in \mathbb{Z}^d$ is a corner of the hypercube to which x belongs, hence $\|r\,\tau(x) - x\|_2 < \sigma$ for all $x \in T$. Also we have $\|r\,\tau(x) - r\,\tau(y)\| > \sigma$ whenever $x \neq y$, because the points are the corners of distinct hyper cubes whose diagonal lengths are equal to σ. Now, using these facts, we can prove relation (3.1). The relation is trivial when $x = y$. When $x \neq y$, we have:

$$\begin{aligned}
\left\|\tau(x) - \tau(y)\right\|_2 &= \frac{1}{r}\left[\left\|r\,\tau(x) - x + x - y + y - r\,\tau(y)\right\|_2\right] \\
&\leq \frac{1}{r}\left[\left\|r\,\tau(x) - x\right\|_2 + \left\|x - y\right\|_2 + \left\|y - r\,\tau(y)\right\|_2\right] \\
&\leq \frac{1}{r}\left[\sigma + \left\|x - y\right\|_2 + \sigma\right] \\
&\leq \frac{1}{r}\left[\left\|x - y\right\|_2 + \left\|x - y\right\|_2 + \left\|x - y\right\|_2\right] \\
&= \frac{3}{r}\left\|x - y\right\|_2 \\
&\equiv \frac{1}{A}\left\|x - y\right\|_2 ,
\end{aligned}$$

and for the other direction:

$$\begin{aligned}
\|x-y\|_2 &= \|x - r\,\tau(x) + r\,\tau(x) - r\,\tau(y) + r\,\tau(y) - y\|_2 \\
&\leq \|x - r\,\tau(x)\|_2 + \|r\,\tau(x) - r\,\tau(y)\|_2 + \|r\,\tau(y) - y\|_2 \\
&\leq \sigma + r\ \|\tau(x) - \tau(y)\|_2 + \sigma \\
&\leq r\ \|\tau(x) - \tau(y)\|_2 + r\ \|\tau(x) - \tau(y)\|_2 + r\ \|\tau(x) - \tau(y)\|_2 \\
&= 3r\,\|\tau(x) - \tau(y)\|_2 \\
&\equiv B\,\|\tau(x) - \tau(y)\|_2\,,
\end{aligned}$$

as desired. $\square$

Now we give a lemma about convergence of two series:

Lemma 3.1.3 *Let d be a positive integer and $r > 0$. Then we have:*

(i) *The series $\sum_{s\in\mathbb{Z}^d} e^{-r\|s\|_2}$ converges,*

(ii) *For $\beta > d$, the series $\sum_{s\in\mathbb{Z}^d} (1 + r\,\|s\|_2)^{-\beta}$ converges.*

Proof. We will make use of the supremum norm $\|\cdot\|_\infty$ on $\mathbb{Z}^d$, which is defined as $\|x\|_\infty = \max_{1\leq i\leq d} |x_i|$. It can be seen that the number of points in the set $B(0,k) \equiv \{x \in \mathbb{Z}^d : \|x\|_\infty \leq k\}$ is equal to $(2k+1)^d = \mathcal{O}(k^d)$. Also the number of points in $S(0,k) \equiv \{x \in \mathbb{Z}^d : \|x\|_\infty = k\} = B(0,k)\backslash B(0,k-1)$ is equal to

$$\begin{aligned}
(2k+1)^d - (2k-1)^d &= \left[(2k)^d + (2k)^{d-1} + \cdots + 1\right] \\
&\quad - \left[(2k)^d - (2k)^{d-1} + \cdots \pm 1\right] \\
&= (2k)^{d-1} - (2k)^{d-2} + \cdots \mp 1
\end{aligned}$$

which is less than Ck^{d-1}, where C is a constant that depends on d. The Euclidean norms $\|\cdot\|_2$ of the points in $S(0,k)$ are at least equal to k, since the Euclidean norm is at least equal to the supremum norm for any point in $\mathbb{Z}^d$. So, for the first series we have

$$\begin{aligned}\sum_{s\in\mathbb{Z}^d} e^{-r\|s\|_2} &= \sum_{k=0}^{\infty} \sum_{\|s\|_\infty = k} \frac{1}{e^{r\|s\|_2}} \\ &\leq \sum_{k=0}^{\infty} C\frac{k^{d-1}}{e^{rk}} \\ &= \sum_{k=0}^{\infty} C\frac{k^{d-1}}{e^{\frac{rk}{2}}} e^{\frac{-rk}{2}} \\ &\leq \sum_{k=0}^{\infty} C' e^{\frac{-rk}{2}} < \infty\end{aligned}$$

where C' is an upper bound for $Ck^{d-1}e^{-\frac{\epsilon rk}{2}}$. Similarly, for the second series we have

$$\begin{aligned}\sum_{s\in\mathbb{Z}^d} (1 + r\left\|s\right\|_2)^{-\beta} &\leq \sum_{k=0}^{\infty} C\frac{k^{d-1}}{(1+rk)^{\beta}} \\ &\leq \sum_{k=0}^{\infty} \frac{C'}{(1+rk)^{\beta-d+1}} < \infty\end{aligned}$$

since we know that $\beta - d > 0$. $\square$

We will use the following property of separated subsets of $\mathbb{Z}^d$ frequently.

Proposition 3.1.4 *Let $T \subset \mathbb{R}^d$ be a separated set. Then*

(i) $\sup\limits_{t\in T} \sum\limits_{s\in T} e^{-\epsilon\|s-t\|_2} < \infty$, *and*

(ii) $\sup\limits_{t\in T} \sum\limits_{s\in T} (1 + \left\|s-t\right\|_2)^{-\beta} < \infty$

where $\epsilon > 0$ in the first series and $\beta > d$ in the second.

Proof. Let τ be a one-to-one function from T into $\mathbb{Z}^d$ satisfying (3.1), whose existence was proved in Proposition 3.1.2. For a fixed $t \in T$ we have

$$\begin{aligned}
\sum_{s\in T} e^{-\epsilon\|s-t\|_2} &\leq \sum_{s\in T} e^{-\frac{\epsilon}{A}\|\tau(s)-\tau(t)\|_2} \\
&\leq \sum_{x\in\mathbb{Z}^d} e^{-\frac{\epsilon}{A}\|x-\tau(t)\|_2} \\
&= \sum_{x\in\mathbb{Z}^d} e^{-\frac{\epsilon}{A}\|x\|_2} < \infty,
\end{aligned}$$

where the equality results from the fact that $\tau(t) \in \mathbb{Z}^d$, and any shift of $\mathbb{Z}^d$ by a point in $\mathbb{Z}^d$ is again $\mathbb{Z}^d$. Since this last sum is independent of t, it follows that the supremum over T is finite.

A very similar argument works to show that $\sup_{t\in T}\sum_{s\in T}(1+\|s-t\|_2)^{-\beta} < \infty$. $\square$

3.2. Definitions of the Classes Q_α and E_γ

We will denote the classes of polynomially decaying and exponentially decaying matrices by Q_α and E_γ respectively, where α and γ are related to the order of decay.

Definition 3.2.1 *Let $T \subset \mathbb{R}^d$ be a separated set. The class $E_\gamma(T)$ is defined to be the set of all matrices A indexed by $T \times T$ that satisfy*

$$\forall\, \gamma' < \gamma \;\; \exists\, C(\gamma') \;\text{ such that }\; |a(s,t)| \leq C(\gamma')\exp(-\gamma'\,\|s-t\|_2).$$

Remark 3.2.2 *Defining E_γ this way will make the class closed under matrix inversion. We shall see that an invertible matrix which has an off-diagonal decay of order γ has an inverse which has an off-diagonal decay of order γ' for all $\gamma' < \gamma$ but not necessarily of order γ. This is not the case for polynomial decay, i.e., an invertible matrix that has a polynomial decay of order α has an inverse with an off-diagonal decay of the same order α.*

Definition 3.2.3 *Let* $T \subset \mathbb{R}^d$ *be a separated set. Then for* $\alpha > d$*, the class* $Q_\alpha(T)$ *is defined to be the set of all matrices* A *indexed by* $T \times T$ *that satisfy*

$$|a(s,t)| \leq C(1 + \|s-t\|_2)^{-\alpha}$$

for some constant C *that depends on* α*.*

We sometimes will not write the T in the parentheses, and denote these spaces by E_γ and Q_α, when the context is not directly related to a specific T.

Proposition 3.2.4 *The matrices in* $E_\gamma(T)$ *and in* $Q_\alpha(T)$*are bounded operators on* $l^2(T)$*.*

Proof. This is a fact that follows from Schur's Lemma (Theorem A.2.1) and Proposition 3.1.4. For $A \in E_\gamma$ we have

$$\sup_{s \in T} \sum_{t \in T} |a(s,t)| \leq \sup_{s \in T} \sum_{t \in T} C \exp(-\gamma' \|s-t\|_2) < \infty.$$

Where $0 < \gamma' < \gamma$. In the same manner, we have $\sup\limits_{t \in T} \sum\limits_{s \in T} |a(s,t)| < \infty$. Similarly, for $A \in Q_\alpha$, using the second part of Proposition 3.1.4, one can show that $\sup\limits_{s \in T} \sum\limits_{t \in T} |a(s,t)|$ and $\sup\limits_{t \in T} \sum\limits_{s \in T} |a(s,t)|$ are both finite, and the claim follows from Schur's lemma.

3.3. The Classes Q_α and E_γ are Algebras

Theorem 3.3.1 *The class* $E_\gamma(T)$ *is an algebra for any separated set* $T \subset \mathbb{R}^d$*.*

Proof. Let A and B be in E_γ and we shall show that $C = AB$ is also in E_γ.

Let $\gamma' < \gamma$. We pick a γ'' such that $\gamma' < \gamma'' < \gamma$. We define $\epsilon = \gamma'' - \gamma'$. Then it follows that

$$\begin{aligned}
|c(s,u)| &\leq C \sum_{t \in T} |a(s,t)||b(t,u)| \\
&\leq C \sum_{t \in T} \exp(-\gamma'' \|s-t\|_2) \exp(-\gamma'' \|t-u\|_2) \\
&\leq C \sum_{t \in T} \exp(-\gamma'' \|s-t\|_2) \exp(-\gamma' \|t-u\|_2) \\
&= C \sum_{t \in T} \exp(-\gamma' \|s-t\|_2) \exp(-\epsilon \|s-t\|_2) \exp(-\gamma' \|t-u\|_2) \\
&= C \sum_{t \in T} \exp(-\gamma' [\|s-t\|_2 + \|t-u\|_2]) \exp(-\epsilon \|s-t\|_2) \\
&\leq C \sum_{t \in T} \exp(-\gamma' \|s-u\|_2) \exp(-\epsilon \|s-t\|_2) \\
&= C \left[\sum_{t \in T} \exp(-\epsilon \|s-t\|_2) \right] \exp(-\gamma' \|s-u\|_2) \qquad (3.2) \\
&\leq C'(\gamma') \exp(-\gamma' \|s-u\|_2),
\end{aligned}$$

where the part in brackets in (3.2) is finite by Proposition 3.1.4. Hence AB is in E_γ as desired. □

Now we will prove a similar result for Q_α. Before that, we define the norm $\|\cdot\|_\alpha$ on the set of all matrices indexed by $T \times T$ as follows:

$$\|B\|_\alpha \equiv \sup_{s,t} (1 + \|s-t\|_2)^\alpha |b(s,t)|. \qquad (3.3)$$

Note that $Q_\alpha(T)$ is the set of matrices that have a finite $\|\cdot\|_\alpha$ norm for $T \subset \mathbb{R}^d$ a separated set and for $\alpha > d$. Also note that $\|\cdot\|_\alpha$ is a weighted supremum norm, with the weight function $(1 + \|s-t\|_2)^\alpha$, and it is straight-forward to verify that it satisfies the norm axioms on $Q_\alpha(T)$.

Theorem 3.3.2 *Let T be a separated subset of $\mathbb{R}^d$. There exists a constant $C > 0$*

such that

$$\|AB\|_\alpha \leq C \|A\|_\alpha \|B\|_\alpha$$

for all $A, B \in Q_\alpha(T)$; and therefore $Q_\alpha(T)$ is closed under multiplication, hence, an algebra.

Proof. Let A and B be two matrices in Q_α. Let $D = AB$ be their product. Then we have

$$\begin{aligned}
|d(s,u)| &= \left|\sum_{t\in T} a(s,t)b(t,u)\right| \\
&\leq \sum_{t\in T} |a(s,t)||b(t,u)| \\
&\leq \sum_{t\in T} \|A\|_\alpha (1+\|s-t\|_2)^{-\alpha} \|B\|_\alpha (1+\|t-u\|_2)^{-\alpha} \\
&\leq \|A\|_\alpha \|B\|_\alpha \left[\sum_{\|s-t\|_2 \geq \frac{1}{2}\|s-u\|_2} (1+\|s-t\|_2)^{-\alpha}(1+\|t-u\|_2)^{-\alpha} \right. \\
&\qquad \left. + \sum_{\|t-u\|_2 \geq \frac{1}{2}\|s-u\|_2} (1+\|s-t\|_2)^{-\alpha}(1+\|t-u\|_2)^{-\alpha} \right].
\end{aligned}$$

The last inequality follows from the fact for any $t \in T$, at least one of the inequalities $\|s-t\|_2 \geq \frac{1}{2}\|s-u\|_2$ and $\|t-u\|_2 \geq \frac{1}{2}\|s-u\|_2$ is always true, and all the terms in the summations are positive. Let us examine the first term in the brackets:

$$\begin{aligned}
\|A\|_\alpha \|B\|_\alpha & \sum_{\|s-t\|_2 \geq \frac{1}{2}\|s-u\|_2} (1+\|s-t\|_2)^{-\alpha}(1+\|t-u\|_2)^{-\alpha} \\
&\leq \|A\|_\alpha \|B\|_\alpha \sum_{\|s-t\|_2 \geq \frac{1}{2}\|s-u\|_2} \left(1+\frac{1}{2}\|s-u\|_2\right)^{-\alpha} (1+\|t-u\|_2)^{-\alpha} \\
&\leq \|A\|_\alpha \|B\|_\alpha \left[2\sum_{t\in T}(1+\|t-u\|_2)^{-\alpha}\right](1+\|s-u\|_2)^{-\alpha} \qquad (3.4) \\
&\equiv \|A\|_\alpha \|B\|_\alpha 2S(1+\|s-u\|_2)^{-\alpha},
\end{aligned}$$

where $S = \sup_{s\in T} \sum_{t\in T} (1 + \|t-s\|_2)^{-\alpha}$, which we know to be finite by Proposition 3.1.4. A similar analysis can be done on the second term and we get that

$$|d(s,t)| \leq C \|A\|_\alpha \|B\|_\alpha (1 + \|s-u\|_2)^{-\alpha},$$

with $C = 4S$, which means

$$\|AB\|_\alpha \leq C \|A\|_\alpha \|B\|_\alpha . \quad \square$$

Corollary 3.3.3 *For $B \in Q_\alpha$, there exists an R_0 such that*

$$\|B^n\|_d \leq R_0^n.$$

Proof. By Theorem 3.3.2 we have $\|B^2\|_\alpha \leq C \|B\|_\alpha^2$ and continuing inductively, it follows that $\|B^n\|_\alpha \leq C^{n-1} \|B\|_\alpha^n$. Thus the result follows by taking

$$R_0 = \|B\|_\alpha \max\{1, C^2\}$$

and noting that $\|B^n\|_d \leq \|B^n\|_\alpha$.

3.4. E_γ is Closed under Matrix Inversion

In this section we shall show that if a matrix in $E_\gamma(T)$ is invertible as an operator on $l^2(T)$, then its inverse is also in $E_\gamma(T)$, i.e., it has the same decay property.

Theorem 3.4.1 *Let $T \subset \mathbb{R}^d$ be a separated set. If $A \in E_\gamma(T)$ and A is invertible on $l^2(T)$, then A^{-1} belongs to E_γ as well.*

Proof. We can assume that A is positive definite without loss of generality; for once we prove the theorem for positive definite matrices, then for any invertible matrix A

in E_γ we have

$$A^{-1} = A^*(AA^*)^{-1},$$

and since AA^* is always positive definite and E_γ is an algebra, it follows that $A^{-1} \in E_\gamma$. We can also assume that $\|A\| = 1$, since dividing by norm has no effect on the decay properties of a matrix.

Now, let $R = I - A$. Let $[m, M]$ be the smallest interval that contains the spectrum of A. Since A is positive definite, invertible and its norm is equal to 1, it follows that $[m, M]$ is a subset of $(0, 1]$. Hence

$$\begin{aligned} \|R\| &= \|I - A\| \\ &\leq \sup_{t \in [m,M]} |1 - t| \\ &< 1, \end{aligned}$$

so we can write the following Neumann Series:

$$A^{-1} = \sum_{n=0}^{n} R^n.$$

Now let $\gamma' < \gamma$ be given. Since E_γ is an algebra, R belongs to E_γ and we have $|r(s,t)| \leq C \exp(-\gamma' \|s - t\|_2)$. Let us pick a γ'' such that $\gamma' < \gamma'' < \gamma$, and define $\epsilon = \gamma'' - \gamma'$. Let us examine the nth term of the series:

$$\begin{aligned} &|r^n(s,t)| \leq \sum_{s_1} \cdots \sum_{s_{n-1}} |r(s, s_1)||r(s_1, s_2)| \cdots |r(s_{n-1}, t)| \\ &\leq \sum_{s_1} \cdots \sum_{s_{n-1}} C^n \exp(-\gamma'' \|s - s_1\|_2) \cdots \exp(-\gamma'' \|s_{n-2} - s_{n-1}\|_2) \exp(-\gamma'' \|s_{n-1} - t\|_2) \\ &\leq \sum_{s_1} \cdots \sum_{s_{n-1}} C^n \exp(-\gamma'' \|s - s_1\|_2) \cdots \exp(-\gamma'' \|s_{n-2} - s_{n-1}\|_2) \exp(-\gamma' \|s_{n-1} - t\|_2) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{s_1} \cdots \sum_{s_{n-1}} C^n \exp(-\gamma'' \|s - s_1\|_2) \cdots \\
&\qquad\qquad \cdots \exp(-\gamma'' \|s_{n-2} - s_{n-1}\|_2) \exp(-\gamma'' \|s_{n-1} - t\|_2) \exp(-\epsilon \|s_{n-1} - t\|_2) \\
&\leq \sum_{s_1} \cdots \sum_{s_{n-1}} C^n \exp(-\gamma'' \|s - s_1\|_2) \cdots \exp(-\gamma'' \|s_{n-2} - t\|_2) \exp(-\epsilon \|s_{n-1} - t\|_2) \\
&= \sum_{s_1} \cdots \sum_{s_{n-2}} C^n \exp(-\gamma'' \|s - s_1\|_2) \cdots \exp(-\gamma'' \|s_{n-2} - t\|_2) \sum_{s_{n-1}} \exp(-\epsilon \|s_{n-1} - t\|_2) \\
&\leq \sum_{s_1} \cdots \sum_{s_{n-2}} C^n K \exp(-\gamma'' \|s - s_1\|_2) \cdots \exp(-\gamma'' \|s_{n-2} - t\|_2) \\
&\qquad\qquad\qquad\qquad \vdots \\
&\leq C^n K^n \exp(-\gamma'' \|s - t\|_2) \\
&\leq C^n K^n \exp(-\gamma' \|s - t\|_2),
\end{aligned}$$

where $K = \sup_{s \in T} \sum_{t \in T} \exp(-\epsilon \|s - t\|_2)$. If $CK < 1$, then we have

$$\left| \sum_{n=1}^{n} r^n(s,t) \right| \leq \frac{1}{1 - CK} \exp(-\gamma' \|s - t\|_2),$$

for all n, and since convergence in the l^2 operator norm implies pointwise convergence, we get $A^{-1} \in E_\gamma$. Otherwise, if $CK \geq 1$, then we can consider $\frac{1}{2CK} R$ and use the fact that any scalar multiple of a matrix has the same type of decay. $\square$

3.5. Q_α is Closed under Matrix Inversion

In this section we will prove the analog of Theorem 3.4.1 for the case of polynomial decay, i.e., we will show that the class Q_α is closed under matrix inversion. Before giving the main theorem, we will prove several lemmas. The following notations will be used in this section:

$\|A\|$ is the norm of A as an operator on l^2.

$\|A\|_\alpha = \sup_{s,t} |a(s,t)|(1 + \|s - t\|_2)^\alpha$.

$$\|A\|_{l^p} = \max\left\{\sup_s \left[\sum_t |a(s,t)|^p\right]^{1/p},\ \ \sup_t \left[\sum_s |a(s,t)|^p\right]^{1/p}\right\}.$$

Proposition 3.5.1 *Let $p > 1$. Then there exists a constant C_p such that*

$$\|B\|_{l^p} \leq C_p \|B\|_d$$

for all B indexed by $\mathbb{Z}^d \times \mathbb{Z}^d$ with $\|B\|_d < \infty$.

Proof. We have

$$|b(s,t)| \leq \frac{\|B\|_d}{\|s-t\|_2^d},$$

hence

$$\begin{aligned}\left(\sum_{s\in\mathbb{Z}^d} |b(s,t)|^p\right)^{1/p} &\leq \|B\|_d \left(\sum_{s\in\mathbb{Z}^d} \frac{1}{\|s-t\|_2^{pd}}\right)^{1/p} \\ &= \|B\|_d \left(\sum_{s\in\mathbb{Z}^d} \frac{1}{\|s\|_2^{pd}}\right)^{1/p} \\ &\equiv \|B\|_d C_p,\end{aligned}$$

and a similar analysis can be performed on the columns, so the claim follows. □

Proposition 3.5.2 *Let $\beta > d$ and $\epsilon \leq \beta$. Then there exists a constant C depending on β and ϵ such that*

$$\sum_{u\in\mathbb{Z}^d} \frac{1}{(1+\|s-u\|_2)^\beta} \frac{1}{(1+\|u-t\|_2)^\epsilon} \leq \frac{C}{(1+\|s-t\|_2)^\epsilon}$$

Proof. We shall separate the sum into two parts, according to whether $\|s-u\|_2 \geq$

$\frac{1}{2}\left\|t-s\right\|_2$ or $\left\|u-t\right\|_2 \geq \frac{1}{2}\left\|t-s\right\|_2$. Let us examine the first term

$$\begin{aligned}
&\sum_{\|s-u\|_2\geq\frac{1}{2}\|t-s\|_2} \frac{1}{(1+\|s-u\|_2)^{\beta}}\,\frac{1}{(1+\|u-t\|_2)^{\epsilon}} \\
&\qquad\leq \sum_{\|s-u\|_2\geq\frac{1}{2}\|t-s\|_2} \frac{1}{(1+\|s-u\|_2)^{\beta-\epsilon}}\,\frac{1}{(1+\|s-u\|_2)^{\epsilon}}\,\frac{1}{(1+\|u-t\|_2)^{\epsilon}} \\
&\qquad\leq \sum_{\|s-u\|_2\geq\frac{1}{2}\|t-s\|_2} \frac{1}{(1+\|s-u\|_2)^{\beta-\epsilon}}\,\frac{1}{(1+\frac{1}{2}\|t-s\|_2)^{\epsilon}}\,\frac{1}{(1+\|u-t\|_2)^{\epsilon}} \\
&\qquad\leq \frac{1}{(1+\frac{1}{2}\|t-s\|_2)^{\epsilon}} \sum_{u\in\mathbb{Z}^d} \frac{1}{(1+\|s-u\|_2)^{\beta-\epsilon}}\,\frac{1}{(1+\|u-t\|_2)^{\epsilon}} \\
&\qquad= \frac{2}{(2+\|t-s\|_2)^{\epsilon}} \sum_{u\in\mathbb{Z}^d} \frac{1}{(1+\|s-u\|_2)^{\beta}} \left(\frac{1+\|s-u\|_2}{1+\|u-t\|_2}\right)^{\epsilon} \\
&\qquad\leq \frac{C}{(1+\|t-s\|_2)^{\epsilon}} \sum_{u\in\mathbb{Z}^d} \frac{1}{(1+\|u\|_2)^{\beta}} \\
&\qquad\equiv \frac{C'}{(1+\|t-s\|_2)^{\epsilon}},
\end{aligned}$$

where we have used the fact that the function $\left(\frac{1+\|s-u\|_2}{1+\|u-t\|_2}\right)^{\epsilon}$ approaches to 1 as $\|u\|_2$ goes to infinity, so it is bounded as a function of u.

In a similar manner, it can be seen that for the second term we have

$$\sum_{\|u-t\|_2\geq\frac{1}{2}\|t-s\|_2} \frac{1}{(1+\|s-u\|_2)^{\beta}}\,\frac{1}{(1+\|u-t\|_2)^{\epsilon}} \leq \frac{C''}{(1+\|t-s\|_2)^{\epsilon}}$$

hence the claim follows. $\square$

Lemma 3.5.3 *Let B be a matrix indexed by $\mathbb{Z}^d \times \mathbb{Z}^d$, with $\|B\|_d < \infty$ and $\|B\| < \infty$. Then for every $p \in (1,2]$ we have*

$$\|B\|_{l^p} \leq C(p)\,\|B\|_d^{2/p-1}\,\|B\|^{2-2/p}.$$

Proof. In the proof of this lemma, we shall use the supremum norm $\|\cdot\|_\infty$ on the index set $\mathbb{Z}^d$ and use the fact, which we have shown in the proof of Lemma 3.1.3, that the number of points in a ball of radius r is $\mathcal{O}(r^d)$ and in a sphere of radius r is $\mathcal{O}(r^{d-1})$ with respect to this norm.

For any finite set $\{a_k\}_{k\in S}$ of complex numbers, it follows from Hölder's inequality that:

$$\begin{aligned}\sum_{k\in S}|a_k|^p &= \sum_{k\in S} 1\cdot|a_k|^p \\ &\le \left[\sum_{k\in S} 1^{2/(2-p)}\right]^{(2-p)/2}\left[\sum_{k\in S}|a_k|^{p\,(2/p)}\right]^{p/2} \\ &= N^{(2-p)/2}\left[\sum_{k\in S}|a_k|^2\right]^{p/2},\end{aligned}$$

where N is the number of points in S. After taking the pth roots, we obtain

$$\left[\sum_{k\in S}|a_k|^p\right]^{1/p} \le N^{1/p-1/2}\left[\sum_{k=-N}^{N}|a_k|^2\right]^{1/2}. \tag{3.5}$$

Also for N a positive integer and s running over $\mathbb{Z}^d\backslash B(0,N)$, it can be shown using the standard integral estimate for p-series that

$$\begin{aligned}\sum_{\|s\|_\infty>N}\frac{1}{\|s\|_2^{p\,d}} &\le \sum_{k>N}\sum_{\|s\|_\infty=k}\frac{1}{k^{p\,d}} \\ &\le \sum_{k>N}\frac{Ck^{d-1}}{k^{p\,d}} \\ &\le \frac{C'}{N^{(p-1)d}}.\end{aligned} \tag{3.6}$$

Now we have

$$\begin{aligned}\left(\sum_{s\in\mathbb{Z}^d}|b(s,t)|^p\right)^{1/p} &\leq \left[\sum_{\|s-t\|_\infty\leq N}|b(s,t)|^p\right]^{1/p}+\left[\sum_{\|s-t\|_\infty>N}|b(s,t)|^p\right]^{1/p}\\ &\leq \left[\sum_{\|s-t\|_\infty\leq N}|b(s,t)|^p\right]^{1/p}+\left[\sum_{\|s-t\|_\infty>N}\frac{\|B\|_d^p}{N^{pd}}\right]^{1/p},\end{aligned}$$

where, in the last term we have used

$$\begin{aligned}\|B\|_d &\geq (1+\|s-t\|_2)^d|b(s,t)|\\ &\geq (\|s-t\|_2)^d|b(s,t)|\\ &> N^d|b(s,t)|\end{aligned}$$

whenever $\|s-t\|_2 > N$. Then, using the estimates (3.5) and (3.6) on the first and the second terms respectively, we get

$$\|B\|_{l^p} \leq C_1(p)N^{d(1/p-1/2)}\|B\|_{l^2} + C_2(p)N^{d(1/p-1)}\|B\|_d\,.$$

Taking an N such that $N^d \leq \left(\frac{\|B\|_d}{\|B\|_{l^2}}\right)^2 \leq (N+1)^d$ we obtain

$$\|B\|_{l^p} \leq C(p)\,\|B\|_d^{2/p-1}\,\|B\|_{l^2}^{2-2/p}\,.$$

Now observe that $\|B\|_{l^2}$ is the supremum of the norms of the images of the canonical basis elements of l^2 under B or B^*. Therefore we can infer that $\|B\|_{l^2} \leq \|B\|$ and therefore

$$\|B\|_{l^p} \leq C(p)\,\|B\|_d^{2/p-1}\,\|B\|^{2-2/p}\,. \qquad \square$$

Lemma 3.5.4 *Let $B \in Q_\alpha(\mathbb{Z}^d)$ with $d < \alpha < 2d$. Let δ be such that $0 \leq \delta < \alpha - d$,* *and let* B_δ *be the matrix defined by*

$$b_\delta(s,t) = \|s-t\|_2^{d+\delta}\, b(s,t).$$

Then there exists a $p \in (1,2)$*, such that for all matrices* M *and* N *with* $\|M\|_{l^p} < \infty$ *and* $\|N\|_{l^p} < \infty$ *the inequality*

$$|MB_\delta N(s,t)| \le C\, \|B_\delta\|_\alpha\, \|M\|_{l^p}\, \|N\|_{l^p}\,,$$

is satisfied, where C *depends on* α*,* d *and* δ*.*

Proof. First we will show that there exists a $p \in (1,2)$ such that the matrix B_δ is a bounded operator from l^p to $l^{p'}$. Here p' denotes the number satisfying $1/p + 1/p' = 1$. We have

$$|b_\delta(s,t)| \le \frac{\|B\|_\alpha}{\|s-t\|_2^{\alpha-d-\delta}} \equiv \|B\|_\alpha\, h(s-t),$$

and the sequence

$$h(t) \equiv \frac{1}{\|t\|_2^{\alpha-d-\delta}}$$

belongs to l^q whenever $q > \dfrac{d}{\alpha-d-\delta}$. We will use Young's inequality, which says that if

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \tag{3.7}$$

then $\|x_k * y_k\|_{l^r} \le \|x_k\|_{l^p}\, \|y_k\|_{l^q}$. Let p, q, r satisfy (3.7), with

$$q > \frac{d}{\alpha-d-\delta}. \tag{3.8}$$

Since we have

$$\begin{aligned}|B_\delta x(s)| &\leq \sum_{t\in\mathbb{Z}^d} |b_\delta(s,t)x(t)| \\ &\leq \sum_{t\in\mathbb{Z}^d} \frac{\|B\|_\alpha}{\|s-t\|_2^{\alpha-d-\delta}}|x(t)| \\ &= \|B\|_\alpha \, |h * x(s)|\end{aligned}$$

it follows that

$$\|B_\delta x\|_{l^r} \leq \|B\|_\alpha \|h\|_{l^q} \|x\|_{l^p} \equiv C \|B\|_\alpha \|x\|_{l^p}, \tag{3.9}$$

where $C = \|h\|_{l^q}$. We want to chose r such that $\frac{1}{p} + \frac{1}{r} = 1$, i.e, $r = p'$. Putting this in (3.7), we obtain the following relation between p and q:

$$q = \left(2 - \frac{2}{p}\right)^{-1}.$$

Note that q is a strictly decreasing function of p, and we have

$$q = \frac{d}{\alpha - d - \delta} \quad \text{when} \quad p = \frac{2d}{3d - \alpha + \gamma}.$$

So it is required that $p < \frac{2d}{3d - \alpha + \delta}$ in order to guarantee $q > \frac{d}{\alpha - d + \delta}$. Note that since $d < \alpha - \gamma \leq 2d$, we have

$$1 < \frac{2d}{3d - \alpha + \delta} \leq 2,$$

so we can choose p to be any number satisfying $1 < p < \frac{2d}{3d - \alpha + \delta}$.

In the following, we use the inner product $< x \mid y >$, which is given by:

$$< x \mid y >= \sum_{t \in T} x(t)^* y(t),$$

and M^* denotes the Hermitian of M given by

$$m^*(s,t) = m(t,s)^*.$$

Now, let (e_s) denote the sth canonical basis element of $l^2(T)$. We have

$$MB_\delta N(s,t) =< M^* e_s \mid B_\delta N e_t > .$$

Note that Ne_t is the column t of N, and since $\|N\|_{l^p} < \infty$ we have $Ne_t \in l^p$ with

$$\|Ne_t\|_{l^p} \leq \|N\|_{l^p} ,$$

hence $B_\delta N e_t \in l^{p'}$ and

$$\|B_\delta N e_t\|_{l^{p'}} \leq C \|B\|_\alpha \|N\|_{l^p} ,$$

by (3.9). Similarly, we have $\|M^* e_s\|_{l^p} \leq \|M\|_{l^p}$, and we can apply Hölder's inequality to conclude that

$$\begin{aligned} |MB_\delta N(s,t)| &= | < M^* e_s \mid B_\delta N e_t > | \\ &\leq C \|M\|_{l^p} \|B\|_\alpha \|N\|_{l^p} \end{aligned}$$

as desired. $\square$

Lemma 3.5.5 *Let $B \in Q_\alpha(\mathbb{Z}^d)$, and let $R > \|B\|$. Then there exists a constant C_R*

such that

$$\|B^n\|_d \leq C_R\, R^n$$

for all positive integers n.

Proof. The coefficient (s,t) of B^n, which we denote by $b^n(s,t)$, can be expanded as

$$b^n(s,t) = \sum_{u_1} \cdots \sum_{u_{n-1}} b(s,u_1) b(u_1,u_2) \cdots b(u_{n-1},t). \tag{3.10}$$

Using Jensen's inequality, and the fact that $x \mapsto \|x\|_2^d$ is a convex function, it follows that

$$\|a_1 + \cdots + a_n\|_2^d \leq n^{d-1} \left[\|a_1\|_2^d + \cdots + \|a_n\|_2^d \right]. \tag{3.11}$$

Now, using (3.10) and (3.11) we get

$$\begin{aligned}
\|s-t\|_2^d\, b_n(s,t) \leq\ & n^{d-1} \left| \sum_{u_1} \cdots \sum_{u_{n-1}} \|t-u_1\|_2^d\, b(s,u_1) b(u_1,u_2) \cdots b(u_{n-1},t) \right| \\
& + n^{d-1} \left| \sum_{u_1} \cdots \sum_{u_{n-1}} b(s,u_1)\, \|u_1-u_2\|_2^d\, b(u_1,u_2) \cdots b(u_{n-1},t) \right| \\
& \vdots \\
& + n^{d-1} \left| \sum_{u_1} \cdots \sum_{u_{n-1}} b(s,u_1) b(u_1,u_2) \cdots \|u_{n-1}-t\|_2^d\, b(u_{n-1},t) \right| \\
=\ & n^{d-1} \sum_{i=0}^{n-1} \left| B^i\ B_0\ B^{n-i-1}(s,t) \right|,
\end{aligned}$$

where B^0 is denotes the identity matrix, and B_0 is, in accordance with the notation of Lemma 3.5.4, defined to be the matrix whose entry (s,t) is given by

$$b_0(s,t) = \|s-t\|_2\, b(s,t).$$

Here we can apply Lemma 3.5.4 to obtain

$$\|B^n\|_d \leq Cn^{d-1} \sum_{i=0}^{n-1} \left\|B^i\right\|_{l^p} \|B\|_\alpha \left\|B^{n-i-1}\right\|_{l^p},$$

for some p in $(1,2)$, and C is a constant depending on α and d. Then we apply Lemma 3.5.3 to the above inequality to get

$$\|B^n\|_d \leq Cn^{d-1} \|B\|_\alpha \sum_{i=0}^{n-1} \left(\left\|B^i\right\|_d \left\|B^{n-i-1}\right\|_d\right)^{2/p-1} \|B\|^{(n-1)(2-2/p)}. \tag{3.12}$$

By Corollary 3.3.3, we have

$$\|B^n\|_d \leq R_0^n, \tag{3.13}$$

for some R_0. Using this estimate, we get

$$\begin{aligned}
\|B^n\|_d &\leq Cn^d R_0^{(n-1)(2/p-1)} \|B\|^{(n-1)(2-2/p)} \\
&= Cn^d \left[R_0^{(2/p-1)} \|B\|^{(2-2/p)}\right]^{n-1} \\
&\equiv Cn^d \left[f(R_0)\right]^{n-1},
\end{aligned}$$

where the function f is defined by

$$f(x) = x^{2/p-1} \|B\|^{2-2/p}.$$

Note that we have absorbed $\|B\|_\alpha$ in (3.12) into the constant C, so from now on C depends on B as well as on α and d. If we take $R_1 > f(R_0)$, then

$$\frac{Cn^d \left[f(R_0)\right]^{n-1}}{R_1^n}$$

becomes a bounded sequence of n, so there exists a C_1 such that

$$\|B^n\|_d \leq C_1 R_1^n,$$

for all n. Now by putting this back in (3.12) and continuing in the same way, we get that for any $R_2 > f(R_1)$, there exists C_2 such that $\|B^n\|_d \leq C_2 R_2^n$, for all n. Repeating the argument k times, we obtain that for any sequence $\{R_k\}_{k\in\mathbb{N}}$ that satisfies $R_{k+1} > f(R_k)$ and whose initial element satisfies (3.13), there exists a sequence of constants $\{C_k\}_{k\in\mathbb{N}}$ such that for any k

$$\|B^n\|_d \leq C_k R_k^n, \tag{3.14}$$

for all n.

We can assume $R_0 > \|B\|$, because if $R_0 \leq \|B\|$ the lemma directly follows from (3.13) with $C_R = 1$. Let $K > R_0$ and consider $f(x)$ over the interval $I = [\,\|B\|\,,\ K\,]$. Since we have $1 < p < 2$, $f(x)$ is of the form $x^a \|B\|^{1-a}$, with $0 < a < 1$. Using these facts, is easy to verify that f is a contraction over I with the unique fixed point $\|B\|$, and that $f(x) < x$ on $(\,\|B\|\,,\ K]$. Now let

$$g(x) = \frac{f(x) + x}{2}.$$

It can be seen that $g(x)$ is also a contraction over $[\,\|B\|\,,\ K]$ with the same fixed point $\|B\|$. Also, $g(x) > f(x)$ on $(\,\|B\|\,,\ K]$. Hence, starting with R_0, if we define $R_{k+1} = g(R_k)$, then by the Banach fixed point theorem we have

$$\lim_k R_k = \|B\|\,, \tag{3.15}$$

and for each k, there exists a C_k such that (3.14) is satisfied since $R_{k+1} > f(R_k)$. By (3.15), there exists a k_0 such that $\|B\| < R_{k_0} < R$ and the lemma follows by choosing $C_R = C_{k_0}$. $\square$

Lemma 3.5.6 *Let* $B \in Q_\alpha(\mathbb{Z}^d)$ *for some* $\alpha > d$ *and let* $R > \|B\|$*. Then there exists a* $\delta > 0$ *and a constant* C_R *such that*

$$\|B^n\|_{d+\delta} \le C_R R^n$$

for all n*.*

Proof. First note the that if $\alpha' > \alpha$ then $Q_{\alpha'} \subset Q_\alpha$. So if $B \in Q_\alpha(\mathbb{Z}^d)$ for some $\alpha \ge 2d$, then $B \in Q_\alpha(\mathbb{Z}^d)$ for all α satisfying $d < \alpha < 2d$. Therefore we can assume without loss of generality that $d < \alpha < 2d$.

Let us pick a $\delta > 0$ such that $d + \delta < \alpha$. Using Jensen's inequality with the fact that $x \mapsto \|x\|_2^{d+\delta}$ is a convex function, we get

$$\begin{aligned}\|s-t\|_2^{d+\delta} &\le n^{d+\delta-1}\left[\|s-u_1\|_2^{d+\delta} + \|u_1-u_2\|_2^{d+\delta} + \cdots + \|u_{n-1}-t\|_2^{d+\delta}\right]\\ &\le n^{2d}\left[\|s-u_1\|_2^{d+\delta} + \|u_1-u_2\|_2^{d+\delta} + \cdots + \|u_{n-1}-t\|_2^{d+\delta}\right].\end{aligned}$$

Then expand the term (s,t) of B^n as

$$b^n(s,t) = \sum_{u_1} \cdots \sum_{u_{n-1}} b(s,u_1)b(u_1,u_2)\cdots b(u_{n-1},t),$$

so it follows that

$$\begin{aligned}\|s-t\|_2^{d+\delta}\,|b^n(s,t)| \le n^{2d} &\sum_{u_1} \cdots \sum_{u_{n-1}} \|t-u_1\|_2^{d+\delta}\,|b(s,u_1)||b(u_1,u_2)\cdots b(u_{n-1},t)|\\ &+ n^{2d}\sum_{u_1} \cdots \sum_{u_{n-1}} |b(s,u_1)|\,\|u_1-u_2\|_2^{d+\delta}\,|b(u_1,u_2)\cdots b(u_{n-1},t)|\\ &\quad\vdots\\ &+ n^{2d}\sum_{u_1} \cdots \sum_{u_{n-1}} |b(s,u_1)b(u_1,u_2)\cdots b(u_{n-2},u_{n-1})|\\ &\qquad\cdot \|u_{n-1}-t\|_2^{d+\gamma}\,|b(u_{n-1},t)|\end{aligned}$$

$$= n^{2d} \sum_{i=0}^{n-1} \left| B^i B_\delta B^{n-i-1}(s,t) \right|, \tag{3.16}$$

where B^0 appearing in the last line is the identity matrix, and B_δ is the matrix whose entry (s,t) is defined to be

$$b_\delta(s,t) \equiv \|s-t\|_2^{d+\delta} |b(s,t)|,$$

as in Lemma 3.5.4. Then by the Lemma 3.5.4 applied to (3.16), there exists $p \in (1,2)$ and a constant C such that

$$\begin{aligned} \|s-t\|_2^{d+\delta} |b^n(s,t)| &\leq Cn^{2d} \sum_{i=0}^{n-1} \|B\|_{l^p}^i \|B\|_\alpha \|B\|_{l^p}^{n-i-1} \\ &= Cn^{2d+1} \|B\|_\alpha \|B\|_{l^p}^{n-1} \\ &\leq C' n^{2d+1} \|B\|_\alpha \|B\|_d^{n-1}, \end{aligned}$$

for all n; where the last inequality follows from Proposition 3.5.1. Then we pick a R' such that $\|B\| < R' < R$, and it follows from Lemma 3.5.5 that

$$\begin{aligned} \|s-t\|_2^{d+\delta} |b^n(s,t)| &\leq Cn^{2d+1} \|B\|_\alpha (R')^{n-1} \\ &\leq C' \|B\|_\alpha R^n, \end{aligned}$$

which means that $\|B\|_{d+\delta} \leq C' \|B\|_\alpha R^n$ as desired, hence the claim follows with $C_R = C' \|B\|_\alpha$. $\square$

Lemma 3.5.7 *Let $A \in Q_\alpha(\mathbb{Z}^d)$ be an invertible, positive definite operator on $l^2(\mathbb{Z}^d)$ with $\|A\| = 1$. Then there exists a $\delta > 0$ such that $A^{-1} \in Q_{d+\delta}$.*

Proof. Let $B = I - A$. Just as in the proof of Theorem 3.4.1, we have $\|B\| < 1$.

Hence we have $A^{-1} = \sum_{n=0}^{\infty} B^n$. We need to show that

$$\|A\|_{d+\delta} = \left\| \sum_{n=0}^{\infty} B^n \right\|_{d+\delta} < \infty.$$

But this claim is implied by Lemma 3.5.6 as follows: since we have $\|B\| < 1$, we can choose an R such that $\|B\| < R < 1$, and we have

$$\begin{aligned} \left\| \sum B^n \right\|_{d+\delta} &\leq \sum \|B^n\|_{d+\delta} \\ &\leq C_R R^n \\ &\leq C_R \frac{1}{1-R} < \infty \end{aligned}$$

as desired. □

After these preparations, we can now prove the main theorem of this section:

Theorem 3.5.8 *Let $T \subset \mathbb{R}^d$ be a separated set, and $\alpha > d$. Let $A \in Q_\alpha(T)$ be an invertible operator on $l^2(T)$. Then A^{-1} is also in $Q_\alpha(T)$.*

Proof. We will assume that A is a positive definite matrix with $\|A\| = 1$ and that $T = \mathbb{Z}^d$. It can be seen that the first assumption does not cause a loss of generality as in the proof of Theorem 3.4.1. Also, it can be shown that if the theorem holds for $T = \mathbb{Z}^d$, then it also holds when T is any separated subset of $\mathbb{R}^d$, using Proposition 3.1.2.

By Lemma 3.5.7 we have $A^{-1} \in Q_{d+\delta}$ for some $\delta > 0$. It follows that

$$|a^{-1}(s,t)| \leq \frac{C_1}{|s-t|^{d+\delta}}, \quad \text{and} \quad |s-t||a(s,t)| \leq \frac{C_2}{|s-t|^{\alpha-1}},$$

for some constants C_1 and C_2. Next consider the expression

$$g(s,t) \equiv \sum_{u\in\mathbb{Z}^d}\sum_{v\in\mathbb{Z}^d} a^{-1}(s,u)(u-v)a(u,v)a^{-1}(v,t).$$

We have

$$\begin{aligned}|g(s,t)| &\leq C\sum_{u\in\mathbb{Z}^d}\sum_{v\in\mathbb{Z}^d}\frac{1}{|s-u|^{d+\delta}}\frac{1}{|u-v|^{\alpha-1}}\frac{1}{|v-t|^{d+\delta}}\\ &\leq C'\frac{1}{1+|s-t|^{\alpha-1}}.\end{aligned}$$

by applying Proposition 3.5.2 twice.

Now we show that $g(s,t) = -(s-t)a^{-1}(s,t)$, which will imply the desired result:

$$\begin{aligned}g(s,t) &= \sum_{u\in\mathbb{Z}^d}\sum_{v\in\mathbb{Z}^d} a^{-1}(s,u)(u-v)a(u,v)a^{-1}(v,t)\\ &= \sum_{u\in\mathbb{Z}^d} a^{-1}(s,u)u\sum_{v\in\mathbb{Z}^d} a(u,v)a^{-1}(v,t) - \sum_{v\in\mathbb{Z}^d} va^{-1}(v,t)\sum_{u\in\mathbb{Z}^d} a^{-1}(s,u)a(u,v)\\ &= \sum_{u\in\mathbb{Z}^d} a^{-1}(s,u)u\delta(u,t) - \sum_{v\in\mathbb{Z}^d} va^{-1}(v,t)\delta(s,v)\\ &= -(s-t)a^{-1}(s,t)\end{aligned}$$

Therefore we have

$$|a^{-1}(s,t)| = \frac{g(s,t)}{|s-t|} \leq \frac{C'}{1+|s-t|^{\alpha}}$$

which finishes the proof of our theorem. $\square$

4. LOCALIZATION OF FRAMES

In this chapter, we will study the concept of localization of frames, introduced by Gröchenig in [7].

Given a Hilbert space $\mathcal{H}$, we will define an associated family of Banach spaces $\mathcal{H}_m^p$ following [7], and we will see that under the localization condition, a frame for $\mathcal{H}$ preserves its frame properties on this family Banach spaces.

The symbols $\mathcal{N}$ and $\mathcal{X}$ will always denote two separated subsets of $\mathbb{R}^d$ in this chapter. $\mathcal{N}$ will be used to index a Riesz basis, and $\mathcal{X}$ will be used to index a frame in a Hilbert space $\mathcal{H}$. The letter d will always be reserved to denote the dimension of the carrier space $\mathbb{R}^d$ containing the index sets we use. Unlike the previous chapter, we will donote the Euclidean norm on $\mathbb{R}^d$ with $|\cdot|$, instead of $\|\cdot\|_2$, since there will be no confusion with the supremum norm.

In this and the following chapter, our development will follow two parallel lines: sub-exponential and polynomial. Almost every definition and theorem will have one version of polynomial decay and one version of sub-exponential decay.

Let us begin with giving a definition of a localized frame:

Definition 4.0.9 *We say that the frame* $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ *is* polynomially localized *with respect to the Riesz basis* $\mathcal{G} = \{g_n\}$ *with decay* $s > 0$ *(or simply* s-localized*), if*

$$|\langle e_x, g_n \rangle| \leq C(1 + |x - n|)^{-s} \text{ and } |\langle e_x, \tilde{g}_n \rangle| \leq C(1 + |x - n|)^{-s} \tag{4.1}$$

for all $x \in \mathcal{X}$ *and* $n \in \mathcal{N}$.

In the same manner, $\mathcal{E}$ is called exponentially localized*, if for some $\alpha > 0$*

$$|\langle e_x, g_n\rangle| \leq Ce^{-\alpha|x-n|} \textit{ and } |\langle e_x, \tilde{g}_n\rangle| \leq Ce^{-\alpha|x-n|}. \tag{4.2}$$

Note that the localization of a frame depends on a particular Riesz basis given.

It might seem peculiar that this definition of localization of a frame also depends on the structure of the index sets $\mathcal{N}$ and $\mathcal{X}$. However, if $\mathcal{H}$ is a Hilbert space of functions on $\mathbb{R}^d$ and if the elements of the Riesz basis $\mathcal{G} = \{g_n : n \in \mathcal{N}\}$ are functions whose essential supports are centered around the point that they are indexed with, we can see that this localization of frames implies that the essential content of e_x is concentrated around x:

$$e_x = \sum_{n\in\mathcal{N}} \langle e_x, \tilde{g}_n\rangle\, g_n = \sum_{n\in\mathcal{N}} \langle e_x, g_n\rangle\, \tilde{g}_n.$$

Observe that as the n gets far away from x the coefficients get smaller.

4.1. Preliminaries

In this and the following chapter, we will assume the weight functions to be continuous, and to be of one of the following types: sub-exponential and s-moderate.

Definition 4.1.1 (i) *Let s be a positive real number. A nonnegative, continuous function $m : \mathbb{R}^d \to \mathbb{R}$ is called an* s-moderate weight function*, if there exists a constant $C > 0$ such that*

$$m(x+t) \leq C(1+|t|)^s m(x)$$

for all $x, t \in \mathbb{R}$.

(ii) *A nonnegative, continuous function* $m : \mathbb{R}^d \to \mathbb{R}$ *is called a* sub-exponential weight function *if there exists positive constants* C, γ *and* $\beta \in [0,1)$ *such that*

$$m(t+x) \leq Ce^{\gamma|t|^\beta} m(x)$$

for all $x, t \in \mathbb{R}$.

The following theorem will be used later in this chapter:

Theorem 4.1.2 [7, Lemma 2] *Let* $\mathcal{N}$ *and* $\mathcal{X}$ *be two separated subsets of* $\mathbb{R}^d$.

(i) *Let* A *be a matrix whose entry* (x,n) *is given by* $a(x,n) = (1+|x-n|)^{-s-d-\epsilon}$ *for some* $\epsilon > 0$. *Then* A *is a bounded operator from* $l^p_m(\mathcal{N})$ *to* $l^p_m(\mathcal{X})$ *for all* $p \in [1,\infty]$ *and for all* s*-moderate weights* m.

(ii) *Let* A *be a matrix whose entry* (x,n) *is given by* $a(x,n) = e^{-\alpha|x-n|}$. *Then* A *is a bounded operator from* $l^p_m(\mathcal{N})$ *to* $l^p_m(\mathcal{X})$ *for all* $p \in [1,\infty]$ *and for all sub-exponential weights* m.

Proof. For both parts (i) and (ii), we will prove the theorem for $p=1$ and for $p=\infty$. Then the general case will follow from the Riesz-Thorin interpolation theorem, which we state in the appendix, as Theorem A.3.1.

(i) Boundedness on l^1_m:

$$\begin{aligned}
\|Ac\|_{l^1_m(\mathcal{X})} &= \sum_{x\in\mathcal{X}} \left| \sum_{n\in\mathcal{N}} a(x,n)c_n \right| m(x) \\
&\leq \sum_{x\in\mathcal{X}} \sum_{n\in\mathcal{N}} (1+|x-n|)^{-s-d-\epsilon} |c_n| m(x) \\
&= \sum_{n\in\mathcal{N}} |c_n| m(n) \sum_{x\in\mathcal{X}} (1+|x-n|)^{-d-\epsilon} \left[(1+|x-n|)^{-s} m(x) m(n)^{-1} \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sup_{n\in\mathcal{N}}\sum_{x\in\mathcal{X}}(1+|x-n|)^{-d-\epsilon}\right)\left(\sup_{x\in\mathcal{X},n\in\mathcal{N}}(1+|x-n|)^{-s}m(x)m(n)^{-1}\right)\\
&\quad\times\sum_{n\in\mathcal{N}}|c_n|m(n) \qquad (4.3)\\
&\equiv C\left\|c\right\|_{l^1_m(\mathcal{N})}.
\end{aligned}
$$

The first supremum occurring in (4.3) is finite by Proposition 3.1.4. The second supremum is also finite because m is an s-moderate weight, therefore

$$m(x) = m(x-n+n) \leq C'(1+|x-n|)^s m(n).$$

Boundedness on l^∞_m:

$$
\begin{aligned}
\left\|Ac\right\|_{l^\infty_m(\mathcal{X})} &= \sup_{x\in\mathcal{X}}\left|\sum_{n\in\mathcal{N}}a(x,n)c_n\right|m(x)\\
&\leq \sup_{x\in\mathcal{X}}\sum_{n\in\mathcal{N}}(1+|x-n|)^{-s-d-\epsilon}|c_n|m(x)\\
&= \sup_{x\in\mathcal{X}}\sum_{n\in\mathcal{N}}(1+|x-n|)^{-d-\epsilon}(1+|x-n|)^{-s}|c_n|m(x)m(n)m(n)^{-1}\\
&\leq \left(\sup_{x\in\mathcal{X},n\in\mathcal{N}}(1+|x-n|)^{-s}m(x)m(n)^{-1}\right)\left(\sup_{x\in\mathcal{X}}\sum_{n\in\mathcal{N}}(1+|x-n|)^{-d-\epsilon}\right)\\
&\quad\times\left(\sup_{n\in\mathcal{N}}|c_n|m(n)\right)\\
&\leq C\left\|c\right\|_{l^\infty_m(\mathcal{N})}.
\end{aligned}
$$

(ii) Boundedness on l^1_m:

$$
\begin{aligned}
\left\|Ac\right\|_{l^1_m(\mathcal{X})} &= \sum_{x\in\mathcal{X}}\left|\sum_{n\in\mathcal{N}}a(x,n)c_n\right|m(x)\\
&\leq \sum_{x\in\mathcal{X}}\sum_{n\in\mathcal{N}}e^{-\alpha|x-n|}|c_n|m(x)\\
&= \sum_{n\in\mathcal{N}}|c_n|m(n)\sum_{x\in\mathcal{X}}e^{-\alpha|x-n|/2}e^{-\alpha|x-n|/2}m(x)m(n)^{-1}
\end{aligned}
$$

$$\begin{aligned}
&\le \left(\sup_{n\in\mathcal{N}} \sum_{x\in\mathcal{X}} e^{-\alpha|x-n|/2}\right)\left(\sup_{x\in\mathcal{X},n\in\mathcal{N}} e^{-\alpha|x-n|/2} m(x)m(n)^{-1}\right)\\
&\quad \times \sum_{n\in\mathcal{N}} |c_n| m(n)\\
&\equiv C\,\|c\|_{l^1_m(\mathcal{N})}.
\end{aligned}$$

The second supremum is finite, because m is a sub-exponential weight, and

$$\begin{aligned}
m(x-n+n) &\le Ce^{\gamma|x-n|^\beta} m(n)\\
&\le C' e^{\alpha|x-n|/2} m(n).
\end{aligned}$$

Boundedness on l^∞_m:

$$\begin{aligned}
\|Ac\|_{l^\infty_m(\mathcal{X})} &= \sup_{x\in\mathcal{X}} \left|\sum_{n\in\mathcal{N}} a(x,n)c_n\right| m(x)\\
&\le \sup_{x\in\mathcal{X}} \sum_{n\in\mathcal{N}} e^{-\alpha|x-n|} |c_n| m(x)\\
&\le \sup_{x\in\mathcal{X}} \sum_{n\in\mathcal{N}} e^{-\alpha|x-n|/2} e^{-\alpha|x-n|/2} |c_n| m(x)m(n)m(n)^{-1}\\
&\le \left(\sup_{n\in\mathcal{N}} |c_n| m(n)\right)\left(\sup_{x\in\mathcal{X}} \sum_{n\in\mathcal{N}} e^{-\alpha|x-n|/2} e^{-\alpha|x-n|/2} m(x)m(n)^{-1}\right)\\
&\le \|c\|_{l^\infty_m(\mathcal{N})}\left(\sup_{x\in\mathcal{X},n\in\mathcal{N}} e^{-\alpha|x-n|/2} m(x)m(n)^{-1}\right)\left(\sup_{x\in\mathcal{X}} \sum_{n\in\mathcal{N}} e^{-\alpha|x-n|/2}\right)\\
&\equiv C\,\|c\|_{l^\infty_m(\mathcal{N})}. \qquad \square
\end{aligned}$$

4.2. Banach Spaces Associated to a Hilbert Space

In this section we will define an abstract class of Banach spaces associated to a Riesz basis of a Hilbert space, following [7].

Definition 4.2.1 *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{G} = \{g_n : n \in \mathcal{N}\}$ be a Riesz basis of*

$\mathcal{H}$ with dual basis $\{\tilde{g}_n : n \in \mathcal{N}\}$. Let m be a weight function of $\mathbb{R}^d$ of polynomial or sub-exponential type, and let $p \in [1, \infty)$. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be the subspace defined as

$$\mathcal{H}_0 \equiv \left\{ \sum_{n \in \mathcal{N}} c_n g_n : (c_n) \in l^2(\mathcal{N}) \cap l^p_m(\mathcal{N}) \right\}.$$

We endow $f \in \mathcal{H}_0$ with the norm $\|f\|_{\mathcal{H}^p_m} = \|(c_n)_{n \in \mathcal{N}}\|_{l^p_m}$, where $f = \sum_{n \in \mathcal{N}} c_n g_n$. We define the Banach space $\mathcal{H}^p_m$ to be the completion of $\mathcal{H}_0$ with respect to this norm.

Remark 4.2.2

(i) *If $l^p_m(\mathcal{N}) \subseteq l^2(\mathcal{N})$, then $\mathcal{H}^p_m$ is a dense subspace of $\mathcal{H}$. The denseness results from the fact that l^p_m contains all sequences with only finite number of nonzero terms, which are dense in l^2. In general, however, the Banach space $\mathcal{H}^p_m$ is not necessarily a subset of $\mathcal{H}$, indeed it might not even be a function space even if $\mathcal{H}$ is a function space. On the other hand, in the next chapter, where we will use this theory, we will have additional conditions so that $\mathcal{H}^p_m$ will always be a function space.*

(ii) *Note that the definition of $\mathcal{H}^p_m$ depends on the given Riesz basis $\mathcal{G}$ for $\mathcal{H}$.*

(iii) *It can be seen that $\mathcal{G} = \{g_n : n \in \mathcal{N}\}$ is an unconditional basis for $\mathcal{H}^p_m$ with a norm equivalence.*

4.3. Frames and Frame Operators

Let $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ be a frame for $\mathcal{H}$ and $Sf \equiv \sum_{x \in \mathcal{X}} \langle f, e_x \rangle e_x$ be the corresponding frame operator. We recall that this series converges unconditionally, and the frame operator S is an invertible operator.

Let $\mathcal{G} = \{g_n : n \in \mathcal{N}\}$ be a Riesz basis for $\mathcal{H}$. We can expand any frame element e_x with respect to $\mathcal{G}$ as

$$e_x = \sum_{n \in \mathcal{N}} \langle e_x, \tilde{g}_n \rangle g_n = \sum_{n \in \mathcal{N}} \langle e_x, g_n \rangle \tilde{g}_n. \tag{4.4}$$

We will now try to find a expression for the matrix T corresponding to the frame operator S with respect to the Riesz basis $\mathcal{G}$:

$$\begin{aligned} Sf &= \sum_{x\in\mathcal{X}} \langle f, e_x\rangle\, e_x \\ &= \sum_{x\in\mathcal{X}} \sum_{n\in\mathcal{N}} f_n \langle g_n, e_x\rangle\, e_x \\ &= \sum_{x\in\mathcal{X}} \sum_{m\in\mathcal{N}} \sum_{n\in\mathcal{N}} f_n \langle g_n, e_x\rangle \langle e_x, \tilde{g}_m\rangle\, g_m \\ &= \sum_{m\in\mathcal{N}} \left(\sum_{n\in\mathcal{N}} \left(\sum_{x\in\mathcal{X}} \langle g_n, e_x\rangle \langle e_x, \tilde{g}_m\rangle \right) f_n \right) g_m. \end{aligned}$$

where we have used the fact that a Riesz basis is an unconditional basis in interchanging the orders of summations (see Theorem 2.2.2).

Inspecting the above equation, we can see that the matrix T whose entry (m, n) is given by

$$t(m,n) = \sum_{x\in\mathcal{X}} \langle g_n, e_x\rangle \langle e_x, \tilde{g}_m\rangle = \langle S g_n, \tilde{g}_m\rangle, \tag{4.5}$$

will be the infinite matrix corresponding to the frame operator with respect to the Riesz basis $\mathcal{G}$.

If we define Γ to be the mapping that maps $f \in \mathcal{H}$ to its coefficient sequence with respect to $\mathcal{G}$, i.e.,

$$\Gamma : \mathcal{H} \to l^2(\mathcal{N}), \qquad (\Gamma f)_n = \langle f, \tilde{g}_n\rangle. \tag{4.6}$$

we get $S = \Gamma^{-1}\, T\, \Gamma$, which we can see with the commutative diagram in Figure 4.1.

This observation extends to the associated Banach spaces $\mathcal{H}^p_m$, because Γ is an isometric isomorphism between $\mathcal{H}^p_m$ and $l^p_m(\mathcal{N})$. Therefore, we see that the S on $\mathcal{H}^p_m$ corresponds to the action of the matrix T on the sequence space $l^p_m(\mathcal{N})$.

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "S"] \arrow[d, "\Gamma"'] & \mathcal{H} \arrow[d, "\Gamma"] \\
l^2(\mathcal{N}) \arrow[r, "T"] & l^2(\mathcal{N})
\end{tikzcd}$$

Figure 4.1. Isomorphism between $\mathcal{H}$ and $l^2(N)$

For the investigation of T on $l^p_m(\mathcal{N})$ we will use Theorems 3.4.1 and 3.5.8 .

4.4. Localization of Frames

With the following lemma, we see that if $\mathcal{E}$ is a localized frame, then $\mathcal{E} \subseteq \mathcal{H}^p_m$ for all $p \in [1, \infty)$, and for all weight functions of the the type corresponding to the localization.

Lemma 4.4.1

(i) *Let $\mathcal{H}$ be a Hilbert space, with the Riesz basis $\mathcal{G}$, and let $\mathcal{E}$ be an $(s+d+\epsilon)$-localized frame with respect to $\mathcal{G}$ for some $\epsilon > 0$. Then $\mathcal{E} \subseteq \mathcal{H}^p_m$ for all s-moderate weights m and for all $p \in [1, \infty)$.*

(ii) *Let $\mathcal{H}$ be a Hilbert space, with the Riesz basis $\mathcal{G}$, and let $\mathcal{E}$ be an exponentially localized frame with respect to $\mathcal{G}$. Then $\mathcal{E} \subseteq \mathcal{H}^p_m$ for all sub-exponential weights m and for all $p \in [1, \infty)$.*

Proof. We can write $e_x \in \mathcal{E}$ with respect to $\mathcal{G}$ as $e_x = \sum_{n \in \mathcal{N}} \langle e_x, \tilde{g}_n \rangle g_n$. It will suffice to show that the coefficient sequence is in $l^p_m(\mathcal{N})$. For part (i), we have $| \langle e_x, \tilde{g}_n \rangle | \leq C(1 + |x - n|)^{-s-d-\epsilon}$ by the localization condition. Since the weight function m is s-moderate, it satisfies

$$\begin{aligned} m(n) &= m(x + n - x) \\ &\leq C m(x)(1 + |n - x|)^s \end{aligned}$$

and using the Lemma 3.1.3, we have $(\langle e_x, \tilde{g}_n\rangle\, m(n))_{n\in\mathcal{N}} \in l^p(\mathcal{N})$ for all $p \in [1,\infty)$.

The proof of (ii) is similar to part (i). $\square$

In the rest of the chapter, $\langle f, e_x\rangle$ denotes the linear functional on $\mathcal{H}_m^p$, whose restriction on $\mathcal{H} \cap \mathcal{H}_m^p$ is the usual inner product. It is well-defined, since the space $\mathcal{H} \cap \mathcal{H}_m^p$ is dense in $\mathcal{H}_m^p$.

With the following theorem, we extend the definition of frame operator from $\mathcal{H}$ to $\mathcal{H}_m^p$ for localized frames:

Theorem 4.4.2 [7] *Let p satisfy $1 \le p < \infty$, $s > 0$, and m be an s-moderate weight. Assume that $\mathcal{E}$ is a $(s+d+\epsilon)$-localized frame for some $\epsilon > 0$ with respect to a Riesz basis $\mathcal{G} = \{g_n \colon n \in \mathcal{N}\}$.*

- (i) *Then the coefficient operator defined by $C_{\mathcal{E}} f = (\langle f, e_x\rangle)_{x\in\mathcal{X}}$ is a bounded mapping from $\mathcal{H}_m^p$ to $l_m^p(\mathcal{X})$.*
- (ii) *The synthesis operator defined on finite sequences by $D_{\mathcal{E}} c = \sum_{x\in\mathcal{X}} c_x e_x$ extends to a bounded mapping from $l_m^p(\mathcal{X})$ to $\mathcal{H}_m^p$.*

If $\mathcal{E}$ is an exponentially localized frame, then these statements hold for weight functions that are sub-exponential.

Proof. We will only prove the theorem for $(s+d+\epsilon)$-localized frames. The proof for exponentially localized frames is very similar, except that the second part of Theorem 4.1.2 will be used instead of the first part.

(i) Let $f \in \mathcal{H}\cap\mathcal{H}_m^p$. Then $f = \sum_{n\in\mathcal{N}} f_n g_n$ for some $(f_n)_{n\in\mathcal{N}}$, and the localization condition implies that

$$\begin{aligned} |\langle f, e_x \rangle| &= \left| \sum_{n\in\mathcal{F}} f_n \langle g_n, e_x \rangle \right| \\ &\leq C \sum_{n\in\mathcal{F}} |f_n|(1+|x-n|)^{-s-d-\epsilon}. \end{aligned}$$

So by Theorem 4.1.2 we have

$$\|C_{\mathcal{E}} f\|_{l^p_m(\mathcal{X})} \leq C' \, \|(f_n)_{n\in\mathcal{N}}\| \, l^p_m(\mathcal{N}) = C' \, \|f\|_{\mathcal{H}^p_m}$$

Since $\mathcal{H} \cap \mathcal{H}^p_m$ is dense in $\mathcal{H}^p_m$, we can extend $C_{\mathcal{E}}$ to $\mathcal{H}^p_m$.

(ii) We need to show that the sequence with entries $(D_{\mathcal{E}} c)_n = \left\langle \sum_{x\in\mathcal{X}} c_x e_x, \tilde{g}_n \right\rangle$ is in $l^p_m(\mathcal{N})$ for any finite sequence c. With A as in Theorem 4.1.2 and using the localization condition, we obtain

$$\begin{aligned} |(D_{\mathcal{E}} c)_n| &\leq \sum_{x\in\mathcal{X}} |c_x| |\langle e_x, \tilde{g}_n \rangle| \\ &\leq C \sum_{x\in\mathcal{X}} |c_x|(1+|x-n|)^{-s-d-\epsilon} \\ &= C(A^*|c|)_n. \end{aligned}$$

Now, using Theorem 4.1.2 with $\mathcal{N}$ and $\mathcal{X}$ interchanged, we get the boundedness of $D_{\mathcal{E}}$:

$$\|D_{\mathcal{E}} c\|_{\mathcal{H}^p_m} = \|A^*|c|\|_{l^p_m(N)} \leq \|A^*\|_{\mathrm{op}} \, \|c\|_{l^p_m(\mathcal{X})} .$$

With this estimate, we can extend the synthesis operator $D_{\mathcal{E}}$ to the whole space l^p_m, since the set of finite sequences is dense in it. □

Now, using the above theorem, we will extend the definition of the frame operator to the associated Banach spaces $\mathcal{H}^p_m$:

Definition 4.4.3 *Let $\mathcal{H}$ be a Hilbert space with a Riesz basis $\mathcal{G} = (g_n)_{n\in\mathcal{N}}$. Let $\mathcal{E}$ be an $(s+d+\epsilon)$-localized frame or an exponentially localied frame with respect to $\mathcal{G}$. Then the frame operator S on $\mathcal{H}_m^p$ is defined to be the composition of the coefficient operator $C_\mathcal{E}$ and the synthesis operator $D_\mathcal{E}$, i.e., $S \equiv D_\mathcal{E} C_\mathcal{E}$, and*

$$S : \mathcal{H}_m^p \to \mathcal{H}_m^p, \quad f \mapsto \sum_{x\in\mathcal{X}} \langle f, e_x\rangle\, e_x. \tag{4.7}$$

Proposition 4.4.4 *The series in (4.7) converges unconditionally.*

Proof. We know that the sequence $(\langle f, e_x\rangle)_{x\in\mathcal{X}}$ is in l_m^p. Given $\epsilon > 0$, we choose a finite set $\mathcal{X}_0$ such that $\|(\langle f, e_x\rangle)_{x\notin\mathcal{X}_0}\|_{l_m^p(\mathcal{X})} \le \epsilon$. Let $\mathcal{X}_1$ be a set containing $\mathcal{X}_0$. Then by the boundedness of the operators $D_\mathcal{E}$ and $C_\mathcal{E}$ we get

$$\left\| Sf - \sum_{x\in\mathcal{X}_1} \langle f, e_x\rangle\, e_x \right\|_{\mathcal{H}_m^p} \le \|C_\mathcal{E}\|_{\text{op}} \, \|(\langle f, e_x\rangle\, e_x)_{x\notin\mathcal{X}_1}\| \le \|C_\mathcal{E}\|_{\text{op}}\, \epsilon,$$

which implies unconditional convergence. $\square$

Remark 4.4.5 *Note that the proof of Theorem 4.4.2 shows that the operator norms of $C_\mathcal{E}, D_\mathcal{E}$ and S can be bounded uniformly by a constant that depends only on s, but not on m nor p.*

Now we investigate how the localization of a frame affects the system matrix defined in (4.5).

Theorem 4.4.6 [7] (i) *Assume that $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ is polynomially localized with respect to the Riesz basis $\{g_n\}$ with decay $s > d$. Then*

$$|t(m,n)| \le C(1 + |m-n|)^{-s} \quad \textit{for } m, n \in \mathcal{N}. \tag{4.8}$$

(ii) *If $\mathcal{E}$ is exponentially localized, then for some $\alpha' > 0$*

$$|t(m,n)| \leq Ce^{-\alpha|m-n|} \quad \textit{for } m,n \in \mathcal{N}. \tag{4.9}$$

Proof. (i) By inserting the localization estimates in (4.1) into (4.5), we obtain

$$\begin{aligned} |t(m,n)| &\leq C \sum_{x\in\mathcal{X}} (1+|x-n|)^{-s}(1+|x-m|)^{-s} \\ &\leq C'(1+|m-n|)^{s}. \end{aligned}$$

by Proposition 3.5.2.

Part (ii) can be proved in a similar manner, using Theorem 3.3.1 in place of Proposition 3.5.2. □

4.5. Localization of the Dual Frame

With these concepts, we are now ready to prove the main theorem about the localization of frames and their dual frames. We recall that d is the dimension of the "carrier" space $\mathbb{R}^d$ and that all index sets $\mathcal{N}$ and $\mathcal{X}$ are separated subsets of $\mathbb{R}^d$. For polynomial decay the conditions depend on d.

Theorem 4.5.1 **[7]** *Assume that $\{e_x : x \in \mathcal{X}\}$ is a frame with polynomial decay $(s+d+\epsilon)$ with respect to the Riesz basis $\mathcal{G} = \{g_n \colon n \in \mathcal{N}\}$for some $\epsilon > 0$.*

(i) *Then the frame operator S is simultaneously invertible on all Banach spaces $\mathcal{H}^p_m$, where $1 \leq p < \infty$ and m is any s-moderate weight.*

(ii) *The dual frame $\{\tilde{e}_x = S^{-1}e_x : x \in \mathcal{X}\}$ is polynomially localized with the same decay $(s+d+\epsilon)$.*

(iii) *The frame expansion*

$$f = \sum_{x\in\mathcal{X}} \langle f, e_x\rangle\, \tilde{e}_x = \sum_{x\in\mathcal{X}} \langle f, \tilde{e}_x\rangle\, e_x \tag{4.10}$$

converges unconditionally in $\mathcal{H}^p_m$ *for* $1 \le p < \infty$.

(iv) *We have the norm equivalence*

$$\|f\|_{\mathcal{H}^p_m} \asymp \left(\sum_{x \in \mathcal{X}} |\langle f, e_x \rangle|^p \, m(x)^p \right)^{1/p} \asymp \left(\sum_{x \in \mathcal{X}} |\langle f, \tilde{e}_x \rangle|^p \, m(x)^p \right)^{1/p}. \tag{4.11}$$

Proof. (i) Consider the matrix T corresponding to the frame operator, defined in (4.5). By Theorem 4.4.6, T has the following off-diagonal decay:

$$t(m,n) = \mathcal{O}\left((1 + |m-n|)^{-s-d-\epsilon}\right) \tag{4.12}$$

Since the frame operator S is invertible, and T is the matrix representing S with respect to the Riesz basis $\mathcal{G}$, then T must also be invertible. So by the Theorem 3.5.8, with $\alpha = s + d + \epsilon > d$, we have that the inverse matrix satisfy

$$\left| t^{-1}(m,n) \right| \le C(1 + |m-n|)^{-s-d-\epsilon}. \tag{4.13}$$

By Theorem 4.1.2, T^{-1} extends to a bounded operator on all sequence spaces $l^p_m(\mathcal{N})$ simultaneously for all s-moderate weight functions m and all $p \in [1, \infty]$. By the isomorphism depicted in Figure 4.1, S is invertible on $\mathcal{H}^p_m$.

(ii) By the diagram in Figure 4.1 and (4.13) we have

$$\left| t^{-1}(m,n) \right| = \left| \left\langle S^{-1} g_n, \tilde{g}_m \right\rangle \right| \le C(1 + |m-n|)^{-s-d-\epsilon}.$$

If we expand $\tilde{e}_x$ with respect to the frame, we get:

$$\begin{aligned}\langle \tilde{e}_x, g_n\rangle &= \left\langle S^{-1}e_x, g_n\right\rangle \\ &= \left\langle e_x, S^{-1}g_n\right\rangle \\ &= \sum_{m\in\mathcal{N}} \langle e_x, g_m\rangle \left\langle \tilde{g}_m, S^{-1}g_n\right\rangle \\ &= \sum_{m\in\mathcal{N}} \langle e_x, g_m\rangle \, \overline{t^{-1}(m,n)}\end{aligned}$$

We now can use Proposition 3.5.2 to get the first localization estimate:

$$\begin{aligned}|\langle \tilde{e}_x, g_n\rangle| &\le C \sum_{m\in\mathcal{N}} (1+|x-m|)^{-s-d-\epsilon}(1+|m-n|)^{-s-d-\epsilon} \\ &\le C'(1+|x-n|)^{-s-d-\epsilon.}\end{aligned}$$

To get the second localization estimate, we repeat the steps:

$$\begin{aligned}\langle \tilde{e}_x, \tilde{g}_n\rangle &= \left\langle e_x, S^{-1}\tilde{g}_n\right\rangle \\ &= \sum_{m\in\mathcal{N}} \langle e_x, \tilde{g}_m\rangle \left\langle g_m, S^{-1}\tilde{g}_n\right\rangle \\ &= \sum_{m\in\mathcal{N}} \langle e_x, \tilde{g}_m\rangle \, t^{-1}(m,n)\end{aligned}$$

and the rest is of the proof exactly the same.

(iii) Since the series $\sum_{x\in\mathcal{X}} \langle f, e_x\rangle e_x$ converges unconditionally, the series

$$\begin{aligned}S^{-1}\left(\sum_{x\in\mathcal{X}} \langle f, e_x\rangle e_x\right) &= \sum_{x\in\mathcal{X}} \langle f, e_x\rangle S^{-1}e_x \\ &= \sum_{x\in\mathcal{X}} \langle f, e_x\rangle \tilde{e}_x\end{aligned}$$

also converges unconditionally.

(iv) Since $f = S^{-1}Sf = S^{-1}(D_{\mathcal{E}}D_{\mathcal{E}}f)$, we have

$$\|f\|_{\mathcal{H}^p_m} \leq \left\|S^{-1}\right\|_{\mathrm{op}} \|D_{\mathcal{E}}\|_{\mathrm{op}} \|C_{\mathcal{E}}f\|_{l^p_m(\mathcal{X})} \leq \left\|S^{-1}\right\|_{\mathrm{op}} \|D_{\mathcal{E}}\|_{\mathrm{op}} \|C_{\mathcal{E}}f\|_{l^p_m(\mathcal{X})}$$

and this is the first norm equivalence, since the middle term is equal to a constant times $\left(\sum_{x\in\mathcal{X}} |\langle f, e_x\rangle|^p \, m(x)^p\right)^{1/p}$. The second norm inequality can be obtained similarly, using the fact that $S^{-1}f = \sum_{x\in\mathcal{X}} \langle f, \tilde{e}_x\rangle \, \tilde{e}_x$. $\square$

We now state the version of Theorem 4.5.1 for exponentially localized frames. In this case the class of admissible weights can be extended to arbitrary sub-exponential weights.

Theorem 4.5.2 *Assume that $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ is an exponentially localized frame with respect to the Riesz basis $\{g_n\}$.*

- (i) *Then the frame operator S is invertible simultaneously on all Banach spaces $\mathcal{H}^p_m$ for all sub-exponential weight functions m.*
- (ii) *The dual frame $\{\tilde{e}_x : x \in \mathcal{X}\}$ is also exponentially localized.*

Furthermore, the frame expansion (4.10) converges unconditionally in $\mathcal{H}^p_m$ for $1 \leq p < \infty$ and all sub-exponential weights m, and the norm equivalence (4.11) holds.

Proof. The proof is similar to the proof of Theorem 4.5.1. We apply the part (ii) of Theorem 4.4.6, and see that Theorem 3.4.1 implies that $|t^{-1}(m,n)| \leq Ce^{-\alpha'|m-n|}$ for some $\alpha' < \alpha$. It follows that T^{-1} is bounded on all $l^p_m(\mathcal{N})$ for all $p \in [1,\infty)$, and all sub-exponential weights m by Theorem 4.1.2, part (ii). The rest is very similar. $\square$

5. SAMPLING THEOREMS IN SHIFT-INVARIANT SPACES

5.1. Stable Generators and Weighted Shift-Invariant Spaces

The weighted shift-invariant spaces are subspaces L^p_m that are spanned by the integer translates of a single function φ. See Chapter 2 for the definitions of L^p_m and l^p_m spaces. For the weight functions that we will use in this chapter, see Definition 4.1.1.

We will denote the translation operator by T_n, which is defined on $L^2(\mathbb{R}^d)$ by $T_n\varphi(\,\cdot\,) = \varphi(\,\cdot\, - n)$.

Definition 5.1.1 *A function* $\varphi \in L^2(\mathbb{R}^d)$ *is called a* stable generator *if* $\{T_k\varphi\}_{k\in\mathbb{Z}^d}$ *is a Riesz system in* $L^2(\mathbb{R}^d)$*, i.e., if there exists positive constants* A *and* B *such that*

$$A\sum_{n\in\mathcal{F}}|c_n|^2 \leq \left\|\sum_{n\in\mathcal{F}} c_n T_n\varphi\right\|^2 \leq B\sum_{n\in\mathcal{F}}|c_n|^2$$

for all finite subsets $\mathcal{F}$ *of* $\mathbb{Z}^d$*.*

Stable generators can also be characterized in the frequency domain with the following relation, which is usually easier to check. For the proof of it, one may refer to [15, Proposition 6.4.8] or [16, Lemma 1.8].

Proposition 5.1.2 *A function* $\varphi \in L^2(\mathbb{R}^d)$ *is a stable generator with bounds* A *and* B *if and only if its Fourier transfrom* $\hat{\varphi}$ *satisfies*

$$A \leq \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(\xi+k)|^2 \leq B.$$

In this chapter we will always require φ to be a continuous, stable generator satisfying one of the following conditions:

(i) When we are dealing with s-moderate weight functions, φ will satisfy

$$|\varphi(x)| \leq C(1+|x|)^{-d-s-\epsilon} \tag{5.1}$$

for some positive C and ϵ.

(ii) When we are dealing with sub-exponential weight functions, φ will satisfy

$$|\varphi(x)| \leq Ce^{-\alpha|x|} \tag{5.2}$$

for some positive constants C and α.

We will need the following lemma, which is a generalization of Young's convolution relation to weighted sequence spaces:

Lemma 5.1.3 **[5]** *Assume either that*

(i) $w(x) = (1+|x|)^s$ *for some* $s > 0$ *and* m *is an* s*-moderate weight*

or

(ii) $w(x) = e^{\gamma|x|^\beta}$ *for some* $\beta \in [0,1)$ *and* $\gamma > 0$ *and* m *is a sub-exponential weight.*

Let $d \in l^1_w$ *and* $c \in l^p_m$ *for some* $p \in [1,\infty]$. *Then we have*

$$\|c * d\|_{l^p_m} \leq C \|c\|_{l^p_m} \|d\|_{l^1_w}$$

for some constant C.

Proof. Note that we have $m(x+t) \leq C_1 w(x) m(t)$ for some C_1 and for all $x, t \in \mathbb{R}^d$. From this it follows that $m(k) = m(k-l+l) \leq C_1 m(k-l) w(l)$ for all $k, l \in \mathbb{Z}^d$. So we have

$$\begin{aligned}|(c*d)_k|m(k) &= \left|\sum_{l\in\mathbb{Z}^d} c_{k-l}d_l\right| m(k)\\ &\leq C_1\sum_{l\in\mathbb{Z}^d}|c_{k-l}|m(k-l)|d_l|w(l)\\ &= C_1(|c|m * |d|w)_k.\end{aligned}$$

By the assumptions, we have $(|c_k|m(k))_k \in l^p$ and $(|d_k|w(k))_k \in l^1$, so using the above relation and Young's inequality, we get that

$$\|c*d\|_{l^p_m} \leq C\,\|c\|_{l^p_m}\,\|d\|_{l^1_w}$$

for some C. $\square$

The following theorem shows that the weighted shift invariant spaces can be well defined.

Theorem 5.1.4 **[5]** *Let φ be continuous function in $L^2(\mathbb{R}^d)$. Assume either that*

(i) *m is an s-moderate weight function and φ satisfies condition (5.1),*

or

(ii) *m is a sub-exponential weight function, and φ satisfies condition (5.2).*

Let the sequence $c = (c_k)_{k\in\mathbb{Z}^d}$ be in $l^p_m(\mathbb{Z}^d)$. Then the series

$$\sum_{k\in\mathbb{Z}^d} c_k\varphi(\,\cdot\, - k)$$

is unconditionally convergent in L^p_m for all p satisfying $1 \leq p \leq \infty$. Moreover it is uniformly convergent on compact sets, which implies that the series also converges pointwise to a continuous function.

Proof. Let w be a as in Lemma 5.1.3, in accordance with types of m and φ. Let

$c = (c_k)_{k\in\mathbb{Z}^d}$ be in $l^p_m(\mathbb{Z}^d)$.

Let $f(x) = \sum_{k\in\mathbb{Z}^d} c_k\varphi(x-k)$, where the series converges pointwise because of the decay conditions on φ. We will show that it also converges in L^p_m.

Let (d_k) be the sequence defined as

$$d_k = \sup_{x\in k+[0,1]^d} |\varphi(x)|.$$

We can easily see that $d_k \in l^1_w$, as a result of the decay condition on φ. We also see that

$$\begin{aligned}
\|f\|^p_{L^p_m} &= \int_{\mathbb{R}^d} |f(x)m(x)|^p \, dx \\
&= \sum_{l\in\mathbb{Z}^d} \int_{l+[0,1]^d} |f(x)m(x)|^p \, dx \\
&= \sum_{l\in\mathbb{Z}^d} \int_{l+[0,1]^d} \left| \sum_{k\in\mathbb{Z}^d} c_k\varphi(x-k)m(x) \right|^p dx \\
&\leq C \sum_{l\in\mathbb{Z}^d} \left| \sum_{k\in\mathbb{Z}^d} c_k d_{l-k} m(l) \right|^p \\
&\leq C \sum_{l\in\mathbb{Z}^d} m(l)^p \left(\sum_{k\in\mathbb{Z}^d} |c_k| d_{l-k} \right)^p \\
&= C' \, \| \, |c| * d\|^p_{l^p_m} \, ,
\end{aligned}$$

and the same computation can easily be adapted for the case $p = \infty$. Using Lemma 5.1.3, we see that

$$\|f\|_{L^p_m} \leq C'' \, \|d\|_{l^1_w} \, \|c\|_{l^p_m} \, \equiv C_1 \, \|c\|_{l^p_m}$$

so we have that for all sequences (c_k), for all $p \in [1,\infty]$ and for all appropriate weight functions m,

$$\left\| \sum_{k\in\mathbb{Z}^d} c_k \varphi(\,\cdot\, - k) \right\|_{L^p_m} \leq C_1 \left\| c \right\|_{l^p_m} . \tag{5.3}$$

From the above relation, we can show unconditional convergence: Let ϵ be given. Let $\mathcal{F}$ be a finite subset of $\mathbb{Z}^d$ such that

$$\left(\sum_{k\in\mathbb{Z}^d/\mathcal{F}} |c_k|^p m(k)^p \right)^{1/p} < \epsilon / C_1.$$

Then for all sets $\mathcal{M}$ such that $\mathcal{F} \subseteq \mathcal{M} \subseteq \mathbb{Z}^d$ we have

$$\left\| \sum_{k\in\mathbb{Z}^d} c_k \varphi(\,\cdot\, - k) - \sum_{k\in\mathcal{M}} c_k \varphi(\,\cdot\, - k) \right\|_{L^p_m} < \epsilon,$$

from which the unconditional convergence in L^p_m follows.

Note that for sequence spaces we have $l^p_m \subseteq l^\infty_m$, and repeating the steps above for L^∞_m and l^∞_m, we get that $\sum_{k\in\mathbb{Z}^d} c_k \varphi(x-k) m(x)$ is uniformly convergent, and since m is continuous, it follows that $\sum_{k\in\mathbb{Z}^d} c_k \varphi(x-k)$ is uniformly convergent on compact subsets of $\mathbb{R}^d$, therefore the limit is continuous. $\square$

Now we can give the definition of weighted shift invariant spaces:

Definition 5.1.5 *Let φ and m be as in Theorem 5.1.4. We define the weighted shift invariant space $V^p_m(\varphi)$ as*

$$V^p_m(\varphi) \equiv \left\{ \sum_{k\in\mathbb{Z}^d} c_k T_k \varphi \colon (c_k) \in l^p_m(\mathbb{Z}^d) \right\}.$$

When m is the constant function 1*, we simply will denote the space by $V^p(\varphi)$.*

The set $\{T_k\varphi\}_{k\in\mathbb{Z}^d}$, which consists of the translates of a stable generator φ, forms a Riesz basis for the subspace $V^2(\varphi)$ of $L^2(\mathbb{R}^d)$. Therefore there exists a unique dual basis $\left\{\widetilde{T_k\varphi}\right\}_{k\in\mathbb{Z}^d}$ satisfying the biorthogonality relation $\left\langle\widetilde{T_k\varphi}, T_l\varphi\right\rangle = \delta_{kl}$. But from uniqueness, we can see that $\widetilde{T_k\varphi} = T_k\tilde{\varphi}$, where $\tilde{\varphi}$ is the dual element of $T_0\varphi = \varphi$.

Using the inverse closedness properties of matrices having an off-diagonal decay, which we studied in Chapter 3, we will show that $\tilde{\varphi}$ satisfies the same kind of decay condition as φ. Before that, we prove a simple lemma about infinite matrices:

Lemma 5.1.6 *Let B be an invertible infinite matrix indexed by $\mathbb{Z}^d \times \mathbb{Z}^d$ such that $b(s,t)$ is a function of $s-t$, that is, there exists a function $\gamma\colon \mathbb{Z}^d \to \mathbb{C}$ such that $b(s,t) = \gamma(s-t)$. Then B^{-1} is also a function of $s-t$.*

Proof. We want to show that there exists a function β such that $b^{-1}(s,t) = \beta(s-t)$. Let δ be defined as:

$$\delta(s,t) = \begin{cases} 1 & \text{if} \quad s = t \\ 0 & \text{if} \quad s \neq t \end{cases} \tag{5.4}$$

Let us define $\beta(u) = b^{-1}(u,0)$. We claim that $b^{-1}(s,t) = \beta(s-t)$ for all $s,t \in \mathbb{Z}^d$.

We have

$$\begin{aligned}
\sum_{u\in\mathbb{Z}^d} b(s,u)\beta(u-t) &= \sum_{u\in\mathbb{Z}^d} \gamma(s-u)\beta(u-t) \\
&= \sum_{u\in\mathbb{Z}^d} \gamma(s-u)b^{-1}(u-t,0) \\
&= \sum_{u\in\mathbb{Z}^d} \gamma(s-t-u)b^{-1}(u,0) \\
&= \sum_{u\in\mathbb{Z}^d} b(s-t,u)b^{-1}(u,0) \\
&= \delta(s-t,0) \\
&= \delta(s,t),
\end{aligned}$$

from which see that $\beta(s-t)$ is a right inverse, and from uniqueness it must be equal to $b^{-1}(s,t)$, as claimed. $\square$

Proposition 5.1.7 [7] *Let φ be a stable generator in $L^2(\mathbb{R}^d)$.*

(i) *If φ satisfies $|\varphi(x)| \leq C(1+|x|)^{-r}$ for some $r > d$ and a positive constant C, then the dual generator $\tilde{\varphi}$ satisfies*

$$|\tilde{\varphi}(x)| \leq C'(1+|x|)^{-r}$$

for some constant C'.

(ii) *If φ satisfies $|\varphi(x)| \leq Ce^{-\alpha|x|}$ for some $\alpha > 0$ and a constant C, then the dual generator $\tilde{\varphi}$ satisfies*

$$|\tilde{\varphi}(x)| \leq C''e^{-\alpha'|x|}$$

for some α' satisfying $0 < \alpha' < \alpha$ and a constant C''.

Proof.

(i) Let G be the Gram matrix of the Riesz basis $\{T_k\varphi\}_{k\in\mathbb{Z}^d}$, whose entry (k,l) is given by $g(k,l) = \langle T_k\varphi, T_l\varphi\rangle = \langle\varphi, T_{l-k}\varphi\rangle$, which is a function of $k-l$, and $g(k,l) = g^*(l,k)$. Since the translates of φ is a Riesz basis, this matrix is invertible as an operator on $l^2(\mathbb{Z}^d)$, and by the Lemma 5.1.6 its inverse G^{-1} is also a function of $k-l$. So we can write $g(k-l)$ and $g^{-1}(k-l)$, instead of $g(k,l)$ and $g^{-1}(k,l)$. We now will show

that G posesses a polynomial off-diagonal decay:

$$\begin{aligned}
|g(k-l)| &= |\langle \varphi, T_{k-l}\varphi\rangle| \\
&= \left| \int_{\mathbb{R}^d} \varphi(x)\overline{\varphi(x-k+l)}\, dx \right| \\
&\leq \int_{\mathbb{R}^d} |\varphi(x)|\, |\varphi(x-k+l)|\ dx \\
&\leq C \int_{\mathbb{R}^d} (1+|x|)^{-r}(1+|x+k-l|)^{-r}\, dx \\
&= C \sum_{t\in\mathbb{Z}^d} \int_{t+[0,1]^d} (1+|x|)^{-r}(1+|x+k-l|)^{-r}\, dx \\
&\leq C' \sum_{t\in\mathbb{Z}^d} \frac{1}{(1+|t|)^{-r}} \frac{1}{(1+|t+k-l|)^{-r}} \\
&= C' \sum_{t\in\mathbb{Z}^d} \frac{1}{(1+|k-t|)^{-r}} \frac{1}{(1+|t-l|)^{-r}} \\
&\leq C'' \frac{1}{(1+|k-l|)^{-r}},
\end{aligned}$$

where the last inequality results from Theorem 3.3.2. Then it follows by Theorem 3.5.8 that the inverse of G has the same type of polynomial decay, that is $|g^{-1}(k-l)| \leq C_1(1+|k-l|)^{-r}$ for some constant C_1.

Now, since $\tilde{\varphi} \in V^2(\varphi)$, we can expand it as

$$\tilde{\varphi} = \sum_{k\in\mathbb{Z}^d} b_k T_k \varphi$$

where $b = (b_k)_{k\in\mathbb{Z}^d}$ is a sequence in $l^2(\mathbb{Z}^d)$. Using the biorthogonality condition, we see that

$$\begin{aligned}\delta(0,l) &= \langle\tilde{\varphi}, T_l\varphi\rangle\\ &= \sum_{m\in\mathbb{Z}^d} b_m \langle T_m\varphi, T_l\varphi\rangle\\ &= \sum_{m\in\mathbb{Z}^d} b_m g(m-l)\\ &= \sum_{m\in\mathbb{Z}^d} g^*(l-m)b_m,\end{aligned}$$

and with a a matrix multiplication with the inverse of G^*, we see that the sequence b satisfies

$$\begin{aligned}|b_k| &= \left|\sum_{l\in\mathbb{Z}^d}\left(g^{-1}(k-l)\right)^*\delta(0,l)\right|\\ &= \left|\left(g^{-1}(k-0)\right)^*\right|\\ &= \left|g^{-1}(k)\right|\\ &\le C_1\frac{1}{(1+|k|)^r}.\end{aligned}$$

We know from Theorem 5.1.4 that the series $\sum_{k\in\mathbb{Z}^d} b_k\varphi(x-k)$ converges pointwise to $\tilde{\varphi}(x)$, therefore

$$\begin{aligned}|\tilde{\varphi}(x)| &= \left|\sum_{k\in\mathbb{Z}^d} b_k\varphi(x-k)\right|\\ &\le \sum_{k\in\mathbb{Z}^d}|b_k||\varphi(x-k)|\\ &\le C\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|k|)^r}\frac{1}{(1+|x-k|)^r}\\ &\le C'\frac{1}{(1+|x|)^r},\end{aligned}$$

where in the last inequality we have used Theorem 3.3.2 again.

(ii) The proof is very similar to part (i). ☐

In what follows, we will show that the weighted shift invariant spaces $V_m^p(\varphi)$ are closed subspaces of L_m^p with equivalent norms with the corresponding sequence spaces.

Theorem 5.1.8 [**5**] *Let φ be a continuous stable generator in $L^2(\mathbb{R}^d)$ satisfying one of the decay conditions (5.1) or (5.2), and let m be a weight function of the corresponding type, as in Theorem 5.1.4. Then for any $p \in [1, \infty)$ there exists constants m_p and M_p such that*

$$m_p \left\|c\right\|_{l_m^p} \le \left\| \sum_{k \in \mathbb{Z}^d} c_k \varphi(\,\cdot\, - k) \right\|_{L_m^p} \le M_p \left\|c\right\|_{l_m^p} \tag{5.5}$$

for all sequences $(c_k)_{k \in \mathbb{Z}^d}$. This relation also implies that $V_m^p(\varphi)$ is a closed subspace of L_m^p.

Proof. The right inequality was obtained in (5.3). We will only show the existence of the constant m_p and the left hand inequality.

To obtain the left hand inequality, we define the operator T to be the operator that maps a function $f \in V_m^p(\varphi)$ into its coefficient sequence, i.e.,

$$T \colon V_m^p(\varphi) \longrightarrow l_m^p, \quad \sum_{k \in \mathbb{Z}^d} c_k \varphi(\,\cdot\, - k) \mapsto (c_k)_{k \in \mathbb{Z}^d}.$$

It is enough to show that T is a bounded operator. We can see that if $f \in V_m^2(\varphi) \cap V_m^p(\varphi)$, then by the dual generator $\tilde{\varphi}$ we can define the action of T as

$$T \colon f \mapsto (\langle f, \tilde{\varphi}(\,\cdot\, - k)\rangle)_{k \in \mathbb{Z}^d}.$$

Let $V_0(\varphi)$ denote the set of finite linear combinations of the translates of φ. Clearly $V_0 \subseteq V_m^2(\varphi) \cap V_m^p(\varphi)$ and it is dense in $V_m^p(\varphi)$. Therefore, if the restriction of T on this set is bounded, then it is bounded on the whole $V_m^p(\varphi)$.

Let w be a weight function defined according to m as in Lemma 5.1.3. First we will show that for a fixed x, the sequence $\{f(x+j) : j \in \mathbb{Z}^d\}$ belongs to $l^p_m(\mathbb{Z}^d)$ for $f \in V_0(\varphi)$; and the sequence $\{\tilde{\varphi}(x-j) \colon j \in \mathbb{Z}^d\}$ belongs to $l^p_w(\mathbb{Z}^d)$. For the first claim, one can easily see that the sequence $\{\varphi(x+j) \colon j \in \mathbb{Z}^d\}$ belongs to $l^p_m(\mathbb{Z}^d)$ for any given x, and so does $\{f(x+j) : j \in \mathbb{Z}^d\}$, because it is a finite linear combination of this kind of sequences by the assumption that $f \in V_0(\varphi)$. The second claim is also not difficult to show, because $\tilde{\varphi}$ shares the same decay condition as φ by Proposition 5.1.7, so one can get it by plugging in the conditions (5.1) or (5.2) using the definition of w given in Lemma 5.1.3.

Now we will show the boundedness of T on $V_0(\varphi)$. Let $f \in V_0(\varphi)$ and $(c_k) = Tf$. Let $(d_k)_{k\in\mathbb{Z}^d}$ be the sequence defined as $d_k = \sup_{x\in[0,1]^d} |\tilde{\varphi}(x-k)|$. Again by Proposition 5.1.7 we have that $(d_k)_{k\in\mathbb{Z}^d}$ is in l^p_w. Now we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d} (|c_k| m(k))^p &= \sum_{k\in\mathbb{Z}^d} \left| \int_{\mathbb{R}^d} f(x)\overline{\tilde{\varphi}(x-k)}\, dx \right|^p m(k) \\
&= \sum_{k\in\mathbb{Z}^d} \left| \sum_{j\in\mathbb{Z}^d} \int_{j+[0,1]^d} f(x)\overline{\tilde{\varphi}(x-k)}\, dx \right|^p m(k) \\
&\le \int_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \left| f(x+j)\overline{\tilde{\varphi}(x+j-k)} m(k) \right|^p dx \\
&\le C^p \int_{[0,1]^d} \left(\sum_{k\in\mathbb{Z}^d} |f(x+k)m(k)|^p \right) \left(\sum_{k\in\mathbb{Z}^d} |\tilde{\varphi}(x-k)w(k)|^p \right) dx \\
&\le C' \|d\|^p_{l^1_w} \|f\|^p_{L^p_m} ,
\end{aligned}$$

where we have used Lemma 5.1.3. This estimate brings the boundedness of T on $V_0(\varphi$, hence on $V^p_m(\varphi)$.

Now we have obtained (5.5), which means that $V^p_m(\varphi)$ is homeomorphic to l^p_m with an equivalent metric. Since we know that l^p_m is complete, $V^p_m(\varphi)$ must also complete, and therefore closed. $\square$

5.2. Sampling in the Hilbert Space $V^2(\varphi)$

Let φ be a continuous, stable generator satisfying either (5.1) or (5.2). Then for every $f \in V^2(\varphi)$, there exists a sequence $c = (c_k)$ in $l^2(\mathbb{Z}^d)$ such that

$$f = \sum_{k \in \mathbb{Z}^d} c_k T_k \varphi$$

where the convergence is in L^2, and it is unconditional and uniform by Theorem 5.1.4.

Now let $\mathcal{X}$ be a set of sampling for $V^2(\varphi)$, that is, let $\mathcal{X}$ be a countable set satisfying

$$A \left\| f \right\|^2 \leq \sum_{x \in \mathcal{X}} |f(x)|^2 \leq B \left\| f \right\|^2 \tag{5.6}$$

for some positive constants A and B independent of $f \in V^2(\varphi)$. This also brings that the point evaluations $f \mapsto f(x)$ are bounded linear functionals on $V^2(\varphi)$, as a result of the second inequality. So the Riesz representation theorem asserts the existence of unique kernel functions $K_x \in V^2(\varphi)$ for each $x \in \mathcal{X}$ such that $f(x) = \langle f, K_x \rangle$. These kernel functions can be given explicitly with the formula

$$K_x = \sum_{k \in \mathbb{Z}^d} \overline{\varphi(x-k)} T_k \tilde{\varphi},$$

where the series converges in L^2, because the coefficient sequence $\{\overline{\varphi(x-k)}\}_{k \in \mathbb{Z}^d}$ belong to $l^2(\mathbb{Z}^d)$ as a result of the decay conditions imposed on φ. Let us now verify the above formula for K_x: Let f be a function in $V^2(\varphi)$. Then there exists a sequence $c \in l^2(\mathbb{Z}^d)$ such that $f = \sum_{k \in \mathbb{Z}^d} c_k T_k \varphi$. By the Theorem 5.1.4 we have pointwise convergence, so we have $f(x) = \sum_{k\in\mathbb{Z}^d} c_k\varphi(x-k)$. On the other hand we have

$$\begin{aligned}\langle f, K_x\rangle &= \left\langle \sum_{k\in\mathbb{Z}^d} c_k T_k\varphi, \sum_{l\in\mathbb{Z}^d} \overline{\varphi(x-l)} T_l\tilde{\varphi}\right\rangle \\ &= \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} c_k\varphi(x-l)\left\langle T_k\varphi, T_l\tilde{\varphi}\right\rangle \\ &= \sum_{k\in\mathbb{Z}^d} c_k\varphi(x-k) \\ &= f(x)\end{aligned}$$

as a result of the biorthogonality relation.

Replacing $\langle f, K_x\rangle$ with $f(x)$ in (5.6), we get

$$A\left\|f\right\|^2 \leq \sum_{x\in\mathcal{X}} |\left\langle f, K_x\right\rangle|^2 \leq B\left\|f\right\|^2,$$

which means that $\{K_x\}_{x\in\mathcal{X}}$ is a frame for $V^2(\varphi)$. We will refer to this frame as the *reproducing kernel frame*. Now we will see that this frame is also a localized frame with respect to the Riesz basis $\{T_k\varphi\}_{k\in\mathbb{Z}^d}$.

Proposition 5.2.1

(i) *Let φ be a continuous, stable generator which satisfies the decay condition (5.1). Let $\mathcal{X}$ be a set of sampling for $V^2(\varphi)$. Then the reproducing kernel frame $\{K_x\}_{x\in\mathcal{X}}$ is $(s+d+\epsilon)$-localized with respect to the Riesz basis $\{T_k\varphi\}_{k\in\mathbb{Z}^d}$.*

(ii) *Let φ be a continuous, stable generator which satisfies the decay condition (5.2). Let $\mathcal{X}$ be a set of sampling for $V^2(\varphi)$. Then the reproducing kernel frame $\{K_x\}_{x\in\mathcal{X}}$ is exponentially localized with respect to the Riesz basis $\{T_k\varphi\}_{k\in\mathbb{Z}^d}$.*

Proof.

(i) We have $T_k\varphi \in V^2(\varphi)$ for all k, so we can use the pointwise evaluation property

of the reproducing kernel frame:

$$|\langle K_x, T_k\varphi\rangle| = |\varphi(x-k)| \le C(1+|x-k|)^{-s-d-\epsilon}, \quad \text{and}$$

$$|\langle K_x, T_k\tilde{\varphi}\rangle| = |\tilde{\varphi}(x-k)| \le C'(1+|x-k|)^{-s-d-\epsilon},$$

as a result of the decay condition (5.1) on φ and also on $\tilde{\varphi}$ by Proposition 5.1.7.

(ii) Similar to part (i). ☐

5.3. Extending the Setting to Banach Spaces

Theorem 5.3.1 [**7**] *Assume that φ is a stable, continuous generator, and satisfies condition (5.1) for s-moderate weights, or (5.2) for sub-exponential weights. If $\mathcal{X}$ is a set of sampling for $V^2(\varphi)$ with the reproducing kernel frame K_x, then*

(i) *For any $f \in V_m^p(\varphi)$, $1 \le p < \infty$, we have*

$$A\,\|f\|_{L_m^p} \le \left(\sum_{x\in\mathcal{X}} |f(x)|^p m(x)^p\right)^{1/p} \le B\,\|f\|_{L_m^p}.$$

(ii) *Each $\tilde{K}_x$ satisfies the localization estimate (for polynomial localization)*

$$|\tilde{K}_x(t)| \le C(1+|t-x|)^{-s-d-\epsilon} \quad \text{for all } x\in\mathcal{X},\ t\in\mathbb{R}^d,$$

with a constant C independent of x; or the following (for exponential localization)

$$|\tilde{K}_x(t)| \le C' e^{-\alpha'|t-x|}.$$

(iii) *The reconstruction series*

$$f = \sum_{x\in\mathcal{X}} f(x)\tilde{K}_x$$

converges unconditionally in $V_m^p(\varphi)$ *for all* p *satisfying* $1 \le p < \infty$*.*

Proof. The proof depends on the theory that we studied in Chapter 4. We take $\mathcal{H} = V^2(\varphi)$. Then the spaces $\mathcal{H}_m^p$ coincide with $V_m^p(\varphi)$, as a result of Theorem 5.1.8. We have already proved that the reproducing kernel is a localized frame for $V^2(\varphi)$. So the parts (i) and (iii) follow directly from Theorem 4.5.1 or Theorem 4.5.2.

To prove part (ii), we use the fuction evaluation property of the reproducing kernels K_x:

$$\begin{aligned} |\tilde{K}_x(t)| &= |\langle K_x, K_t\rangle| \\ &= \left|\sum_{k\in\mathbb{Z}^d} \left\langle \tilde{K}_x, T_k\varphi \right\rangle \langle T_k\varphi, K_t\rangle \right|. \end{aligned}$$

We have $\langle T_k\varphi, K_t\rangle \le C(1+|k-t|)^{-s-d-\epsilon}$ since the reproducing kernel frame is localized. Also by Theorem 4.5.1, we have that the dual frame $\tilde{K}_x$ have the same type of localization as K_x. It follows that $\left\langle \tilde{K}_x, T_k\varphi \right\rangle \le C'(1+|x-k|)^{-s-d-\epsilon}$. Therefore, by Theorem 3.3.2, we have

$$\begin{aligned} |\tilde{K}_x(t)| &\le C \sum_{k\in\mathbb{Z}^d} (1+|x-k|)^{-s-d-\epsilon}(1+|k-t|)^{-s-d-\epsilon} \\ &\le C'(1+|x-t|)^{-s-d-\epsilon}. \end{aligned}$$

APPENDIX A: SOME BACKGROUND IN ANALYSIS

A.1. The Fourier Transform

This section is intended to fix the notation of the Fourier transform.

Definition A.1.1 *The Fourier transform of a function* $f \in L^1(\mathbb{R}^d)$ *is defined to be*

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx.$$

It can be shown that the restriction of the transform $f \mapsto \hat{f}$ *on* $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *is a bounded linear map, therefore it has a unique extension to* $L^2(\mathbb{R}^d)$*, which is by definition the Fourier transform on* $L^2(\mathbb{R}^d)$ *[17, 18].*

A.2. Schur's Condition for a Matrix to be a Bounded Operator on l^2

Theorem A.2.1 (Schur's Lemma) *Let* B *be a matrix defined on* $T \times T$ *for some countable set* T*. If there exists a positive number* C *such that*

$$\sum_{t \in T} |b(s,t)| < C \text{ for all } s \quad \text{and} \quad \sum_{s \in T} |b(s,t)| < C \text{ for all } t,$$

then B *is a bounded operator on* $l^2(T)$ *with its operator norm* $\|B\|$ *less than or equal to* C*.*

Proof. Let x be a sequence in $l^2(T)$. Then using the conditions on B and the Cauchy-

Schwartz inequality, we obtain

$$
\begin{aligned}
\|Bx\|_2^2 &= \sum_{s\in T}\left|\sum_{t\in T} b(s,t)x(t)\right|^2 \\
&\leq \sum_{s\in T}\left[\sum_{t\in T}\sqrt{|b(s,t)|}\left(\sqrt{|b(s,t)|}|x(t)|\right)\right]^2 \\
&\leq \sum_{s\in T}\left[\left(\sum_{t\in T}|b(s,t)|\right)\left(\sum_{t\in T}|b(s,t)||x(t)|^2\right)\right] \\
&\leq C\sum_{s\in T}\sum_{t\in T}|b(s,t)||x(t)|^2 \\
&\leq C\sum_{t\in T}|x(t)|^2\sum_{s\in T}|b(s,t)| \\
&\leq C^2\sum_{t\in T}|x(t)|^2.
\end{aligned}
$$

Therefore we have $\|Bx\|_2 \leq C\,\|x\|_2$ for all $x \in l^2(T)$, hence $\|B\| \leq C$.

A more general version can be found in [19, p. 30, problem 10].

A.3. Riesz-Thorin Interpolation Theorem

Theorem A.3.1 [15, Thm 3.2.6] *Let* (M,μ) , (N,ν) *be two measure spaces, and* (p_0,q_0), (p_0,q_0) *be two pairs of indices, where* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *with* $p_0 \neq p_1$, $q_0 \neq q_1$. *Let the operators* $A_0\colon L^{p_0}(M) \to L^{q_0}(N)$ *and* $A_1\colon L^{p_1}(M) \to L^{q_1}(N)$ *so that* $A_0 = A_1$ *on the common domain* $L^{p_0}(M) \cap L^{p_1}(M)$. *Let* $k_i = \|A_i\|_{p_i,q_i}$ *be the respective operator norms for* $i = 1,2$. *Let the pair* (p_t,q_t) *be defined by*

$$\frac{1}{p_t} = \frac{t}{p_1} + \frac{1-t}{p_0} \quad \text{and} \quad \frac{1}{q_t} = \frac{t}{q_1} + \frac{1-t}{q_0}.$$

Then there exists a linear operator $A_t\colon L^{p_t}(M) \to L^{q_t}(N)$ *that coincides with* A_i *on* $L^{p_0}(M) \cap L^{p_1}(M)$ *and whose operator norm satisfies*

$$\|A\|_{p_t,q_t} \leq k_0^{1-t}k_1^t.$$

A.4. Generalized Young's Convolution Inequality

Theorem A.4.1 [15, Thm 3.2.0.1] *Let* p, q, r *be real numbers with* $1 \leq p \leq \infty$, $1 \leq p \leq \infty$, $1/p + 1/q \geq 1$ *and*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Let $f \in L^p(\mathbb{R}^d, \mu)$ *and* $g \in L^q(\mathbb{R}^d, \mu)$ *for some measure* μ *defined on* $\mathbb{R}^d$*. Then the convolution* $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\, d\mu(y)$ *is defined* μ*-almost everywhere, and it satisfies*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

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