HEAT KERNEL METHODS IN MANY BODY PROBLEMS

by

Fatih Erman

B.S., Physics, Ege University, 1998

M.S., Physics, Izmir Institute of Technology, 2003

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Graduate Program in Physics

Boğaziçi University

2010

ACKNOWLEDGEMENTS

First, I would like to express my deep and sincere gratitude to my supervisor Prof. O. Teoman Turgut for suggesting me the subject, his foresight and guidance throughout my thesis studies. He always encouraged me when I faced with a problem and he helped me with his wide knowledge and intuition. I have also always received his friendly and kind behavior from the first time when I come to Boğaziçi University.

I would also like to thank my collaborator Barış Altunkaynak for useful discussions that we have made when I started to learn the subject. We have started together to work on this subject and some results about the scattering cross section calculations of the second chapter in my thesis is based on his master thesis.

I am indebted to my thesis committee, Professors Nihat Sadık Değer, Ersan Demiralp, Ali Mostafazadeh and Tonguç Rador for their corrections.

Furthermore, I owe a huge debt of gratitude to my friend Haydar Uncu for the insightful discussions. Without his endless support, I would not have a chance to complete this thesis in Boğaziçi University. I also wish to express my warm and sincere thanks to my friend Cenk Akyüz for his help while I was writing my thesis with LaTeX. In addition, I gratefully acknowledge my friends Elif Demiryas, Hakan Erkol, Erol Ertan and Aylin Yıldız for their helps and assistance during writing my thesis.

I owe my deepest gratitude to the librarians Oya Özdoğan and Mustafa Cevizbaş for their untiring helps using the library.

Needles to say, I would also like to extend my deepest gratitude to my family for their everlasting support.

ABSTRACT

HEAT KERNEL METHODS IN MANY BODY PROBLEMS

The main goal of this thesis is to construct a non-perturbative renormalization for several models in quantum mechanics and field theory by means of the heat kernel in two or three dimensional Riemannian manifolds and study their spectral properties with several approximation and heat kernel techniques. All models investigated here are formulated in terms of a finite well defined operator, including all the information about the interaction, and it is called the principal operator. As a first model, a particle interacting with finitely many Dirac delta potentials in two and three dimensional manifolds is considered and the problem is renormalized with two different methods. The relation between the self-adjoint extensions and the renormalization approach to the same problem is emphasized via a kind of Krein's formula that we obtain. We then give a comparison theorem between the bound state energy of N and $N+1$ point interactions. The estimate of the bound state energies in the tunnelling regime is calculated by applying the perturbation theory. Moreover, the pointwise upper bounds on the wave function corresponding to the bound states are obtained and the existence of the Hamiltonian operator from the resolvent is established. Using Geršgorin theorem, the ground state energy is proven to be bounded from below for compact and Cartan-Hadamard manifolds. In addition to these, non-degeneracy and uniqueness of the ground state is found as a simple consequence of the Perron-Frobenius theorem. The renormalization group equations are also derived and the β function is exactly calculated. In the second model, the renormalization of the non-relativistic Lee model in two and three dimensional manifolds is constructed and its ground state energy is proven to be bounded from below for compact and Cartan-Hadamard manifolds. Then, a kind of mean field approximation is applied to the model in two and three dimensions separately. Finally, the construction of the renormalization of the non-relativistic limit of $\lambda\phi^4$ model in two dimensional manifolds is given.

ÖZET

ÇOK PARÇACIKLI PROBLEMLERDE ISI ÇEKİRDEĞİ YÖNTEMLERİ

Bu tezin ana amacı iki ve üç boyutlu Riemann katmanlı uzaylarında ısı çekirdeği aracılığıyla kuantum mekaniği ve alan teorisindeki çeşitli modellerin pertürbasyon dışı bir renormalizasyonunu kurmak, ve bazı yaklaşıklık ve ısı çekirdeği teknikleri kullanarak ele alınan modellerin spektral özelliklerini çalışmaktır. Burada incelenen tüm modeller, etkileşim hakkındaki bütün bilgiyi içeren sonlu ve iyi tanımlı bir operator cinsinden formüle edilmiştir ve bu operator esas operator olarak adlandırılır. İlk model olarak iki ve üç boyutlu katmanlı uzaylarda sonlu sayıda Dirac delta potensiyeliyle etkileşen bir parçacık ele alınmıştır ve problem iki farklı yöntemle renormalize edilmiştir. Kendi-eşlenik genişleme yaklaşımı ile renormalizasyon yöntemi arasındaki ilişki buldugumuz bir çeşit Krein formülü vasıtasıyla vurgulanmıştır. Daha sonra, N ve $N+1$ tane Dirac delta potensiyeli içeren sistemlerin bağlanma enerjisini karşılaştıran bir teorem verilmiştir. Tünelleme rejimindeki bağlanma enerjisi pertürbasyon teorisi kullanarak hesaplanmıştır. Ayrıca, bağlı hallere karşılık gelen dalga fonksiyonlarının üst limitleri elde edilmiş ve çözen yardımıyla Hamilton operatorünün varlığı gösterilmiştir. Geršgorin teoremi kullanılarak, taban durum enerjisinin tıkız ve Cartan-Hadamard katmanlı uzayları için aşağıdan sınırlı oldugu ispat edilmiştir. Bunlara ek olarak, Perron-Frobenius teoreminin basit bir sonucu olarak taban halin yozlaşmasının olmadığı ve tek olduğu gösterilmiştir. Renormalizasyon grup denklemleri ayrıca türetilmiş ve β fonksiyonu tam olarak hesaplanmıştır. İkinci modelde, iki ve üç boyutlu katmanlı uzaylarda göreceli olmayan Lee modelinin renormalizasyonu oluşturulmuş ve taban durum enerjisinin tıkız ve Cartan-Hadamard katmanlı uzayları için aşağıdan sınırlı olduğu ispat edilmiştir. Daha sonra, bir ceşit ortalama alan yaklaşımı iki ve üç boyutlu durumlarda ayrı olarak modelimize uygulanmıştır. Son olarak, iki boyutlu katmanlı uzaylarda $\lambda\phi^4$ modelinin göreceli olmayan limitinin renormalizasyonu verilmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

$C^2(\mathcal{M})$	The space of all differentiable functions whose second derivatives are continuous on manifold
$d(x,y)$	Geodesic distance between x and y, where $x, y \in \mathcal{M}$
$\mathrm{d}_g^D x$	D dimensional Riemannian volume element with the metric structure g
D	Dimension of space
$\mathcal{H}$	Hilbert space
$K_0(x)$	Modified Bessel function of the third kind
$K_t(x,y;g)$	Heat kernel defined on Riemannian manifold $(\mathcal{M}, g)$
$L^2(\mathcal{M})$	Space of square integrable functions on manifold
$(\mathcal{M}, g)$	Riemannian manifold with the metric structure g
n	Number of bosons
N	Number of point interactions
$\Re(z)$	Real part of the complex number z
$\mathbb{R}^D$	D dimensional Euclidean space
$V(\mathcal{M})$	Volume of the Riemannian manifold $\mathcal{M}$
$\delta_g^D(x,a)$	D dimensional Dirac delta function on $(\mathcal{M}, g)$ at the position $a \in \mathcal{M}$
ϵ	Cut-off parameter in time
λ	Coupling constant
Λ	Cut-off parameter in momentum
$\Phi(E)$	Principal operator
∇_g^2	Laplace-Beltrami operator with the metric structure g

1. INTRODUCTION

There are particularly simple and instructive examples of regularization and renormalization in standard quantum mechanics. One such example is the two and three dimensional Dirac delta potentials (which are also called zero range, contact or point interactions, or Fermi pseudopotentials in the literature). The studies of Dirac delta interactions in quantum mechanics date back to the work of Kronig and Penney [1] who introduced the periodic point interactions describing the non-relativistic electrons moving in a one dimensional fixed crystal lattice and it is one of the few periodic potential Hamiltonians completely solvable in quantum mechanics. The studies of the point interaction in higher dimensions started with the study of Bethe and Peierls [2]. Although it was Thomas [3] who pointed out that the problem of point interactions in three dimensions could not be physically acceptable due to the ultra-violet divergences, Thorn [4] realized that we did not have to abandon these interactions and physical results could be obtained after the so-called regularization and renormalization procedures. The detailed historical development of point interactions has been given extensively in the monograph [5]. There are large amount of works on the renormalization of point interactions in the literature from several point of views: Regularization schemes can be performed either in coordinate space [6, 7, 8, 9] or in momentum space [4, 10, 11, 12, 13, 14, 15, 16]. The path integral approach of the two dimensional case has been also investigated in [17]. The two dimensional delta potentials are additionally interesting since it is an explicit example of the dimensional transmutation and quantum mechanical symmetry breaking or anomaly. This can be seen as follows: Consider the two dimensional Dirac delta potential in natural units ($\hbar = 2m = 1$ in which all dimensions can be expressed in terms of lengths),

$$-\nabla^2\psi(\mathbf{x}) - \lambda\delta^2(\mathbf{x})\psi(\mathbf{x}) = E\psi(\mathbf{x}) \tag{1.1}$$

then, the standard dimensional analysis for estimating the energy of the system do not work since there is no intrinsic scale for the problem (the coupling constant λ is dimensionless). Without any quantity with the dimension of length, we can not determine the bound state energy (for attractive case) and it will give an infinite result which can be seen from the variational principle: Let $E[\psi] = \int_{\mathbb{R}^2} \mathrm{d}^2 x \; |\nabla \psi(\mathbf{x})|^2 - \lambda |\psi(\mathbf{0})|^2$ be the expectation value of the Hamiltonian with the normalized wave function $\psi(\mathbf{x})$. If we define a new normalized wave functions $\psi_\alpha(\mathbf{x}) = \alpha \psi(\alpha \mathbf{x})$, we can find another energy functional $E[\psi_\alpha]$ associated with the scaled wave function $\psi_\alpha(\mathbf{x})$. Then, one can easily show that $E[\psi_\alpha] = \alpha^2 E[\psi]$. This tells us that if we have a negative energy with a square integrable wave function in the problem, we must also admit the scaled energy solutions. Since α^2 is positive and arbitrary, there is no reason to keep the energy bounded from below: $E \to -\infty$. These puzzles can be resolved by means of the regularization and renormalization procedures. What renormalization does is to introduce a scale into this problem by hand, thus breaking the scale invariance of the classical Hamiltonian. This phenomenon is known as the anomaly in quantum mechanics or quantum mechanical symmetry breaking [11, 18] and is an example of a dimensional transmutation [4, 10, 19, 20, 21], which occurs when a dimensionless quantity (the coupling constant) is traded in for a quantity with a dimensional parameter (the experimentally measured bound state energy). The concept of renormalization is associated with a function, called β function, whose zeroes correspond to the fixed points of the theory. However, in most situations in quantum field theory, this function can only be calculated perturbatively. It would therefore be interesting to find simple examples where the function can be calculated in a nonperturbative way and its zeroes can be determined explicitly. Point interactions in quantum mechanics again provides a nonperturbative solution to this problem and it has been discussed in [6, 16, 17] and the β function has been calculated exactly, in which the theory has been found as asymptotically free in two dimensional Euclidean spaces. A rather simple problem constitutes an analytical example for several field theoretical concepts such as regularization, renormalization, dimensional transmutation, quantum anomaly, exact nonperturbative solutions to renormalization group equations, etc. Therefore, working on this elementary example may help us to understand the several ideas in the renormalization in a more elementary context rather than field theory.

There is another treatment of the two and three dimensional point interactions, which is first developed by Berezin and Faddeev [22] for three dimensions. In this approach, the Hamiltonian of the system is rigorously written as a self-adjoint operator derived by Krein's theory of self-adjoint extensions of symmetric operators [23] and a detailed exposition of this subject has been discussed in the monograph [5]. There are two different approaches to the definition of self-adjoint operators describing point interactions: the first one is based on the von Neumann formula, the second one uses a relation called Krein formula, which we will shortly discuss in 4.4. The result of self-adjoint extension method is identical to that of the renormalization method if a certain relation between the parameter in the extension and the renormalized (or bare) coupling constant is satisfied [11]. Let us note that the two dimensional Dirac delta potential and the equivalent self-adjoint extension has been discussed in the context of point particle dynamics in (2+1) dimensional gravity [24] and in Chern-Simons gauge theory (Aharonov-Bohm/Ehrenberg-Siday interaction)[25]. In this thesis, we approach the point interactions from the renormalization point of view without going into the details of self-adjoint extensions.

A new nonperturbative renormalization method has been proposed by S.G. Rajeev [26] that can be applied to some simple problems in quantum mechanics and some field theories with the hope for applying these ideas to more realistic situations, like QCD. The basic idea in [26] is based on finding a well defined finite expression of the resolvent operator of the Hamiltonian in terms of new operator, called principal operator $\Phi(E)$. It is free of divergences and can be determined explicitly (usually as a matrix or an integral operator) and is quite simple in almost all cases. There are no subtleties in the definition of its domain as in the self-adjoint extension method. The main advantage of this method is that, the renormalization is performed without actually solving the dynamics in contrast to the usual perturbative treatment of quantum field theory. Once we have a finite expression of the principal operator, the spectral information can be obtained exactly or by approximation methods. The complete information about the system is hidden in the resolvent and the interaction is described by a term in the formula for $\Phi(E)$. The eigenvalues of energy are given by the solutions to $\Phi(E)A = 0$ and the scattering amplitude is determined by the inverse of $\Phi(E)$. No matter how complicated it is, the standard approximation methods can be applied, such as variational principles or perturbation theory to the principal operator, since now we have a finite formulation of the problem. What seems more remarkable is that the systems need not be exactly solvable in order to do renormalization. The method has wide applicability, such as quantum mechanical model with singular potentials, some many body problems and non-relativistic field theories. We shall give the details when we shall consider our problems in this context without reviewing.

Our main goal in this thesis is to extend the novel approach of Rajeev, which is developed for the renormalization of some simple models in quantum mechanics and some non-relativistic field theories in flat spaces, onto the Riemannian manifolds. However, there is no direct way to do it and there is no trivial way of removing the singular part of the dynamics, so we need a new mathematical tool which allows us to perform the renormalization of the problem on manifolds. It is well known that the essential tool is the heat kernel [27], which plays very important role in mathematical physics [28], geometric analysis [29, 30], quantum field theory [31]. For example, it has been successfully applied to the analysis of the structure of ultra-violet divergences and the calculation of the quantum effective action in the presence of background fields [31] (such as gravity background). Therefore, it is natural to ask whether the same tool can also be applied to our problem. Following the original ideas developed in [26], we try to extend it to the several other models onto the Riemannian manifolds with the help of heat kernel techniques:

(i) Two and three dimensional point interactions in quantum mechanics,

(ii) Two and three dimensional non-relativistic Lee model (a short review of the subject in flat spaces will be given in the relevant chapter of the thesis),

(iii) Non-relativistic limit of $\lambda\phi^4$ model in (2+1) dimensions.

With these attempts, we hope that the nature of renormalization on general curved spaces can be understood better.

We shall now describe the content and the organization of this thesis:

In Chapter 2, we first summarize the one dimensional Dirac delta potential with emphasizing the two equivalent way of solving the problem. Then we show that the same method developed for one dimensional Dirac delta potential problem do not work in higher dimensions. The new idea or methods which are needed are regularization and renormalization. Without going into details of the renormalization of the point interactions in the current literature, we just review the idea of S. G. Rajeev on the nonperturbative renormalization of the point interactions in quantum mechanics by means of a new operator, called principal matrix [26]. This operator will be very useful when we try to extend it to Riemannian manifolds and many body systems. Most of results given in this chapter can be found in [26, 32] and it would be useful for the reader to read the details through the paper [26]. Despite the simplicity of the model, the findings of this chapter constitutes one of the essential part of this thesis.

In Chapter 3 we give some basic definitions and relevant theorems (for our own calculations) about the heat kernel defined on Riemannian manifolds without proving them. All proofs can be found in the mathematics literature [30, 33, 34]. Section 3.1 contains the definition of the Laplace operator in Riemannian manifolds. The spectral theorem is explained in Section 3.2 and the definition of the heat kernel on Riemannian manifolds is presented in the subsequent section. Afterwards, some crucial properties of the heat kernel are briefly summarized and the scaling property of the heat kernel under the scaling of the metric is derived. In Section 3.5, we have compiled the short-time asymptotics of the heat kernel for different cases. At the end of the chapter, we give a brief exposition of the upper and lower bounds on the heat kernel for two topologically different class of manifolds, namely compact and Cartan-Hadamard manifolds. Some bounds are directly taken form the mathematics literature, whereas the others are the simplified versions of existing results.

Chapter 4 is devoted to our studies [35, 36] of the point interactions on two and three dimensional Riemannian manifolds by generalizing the method, developed by S. G. Rajeev [26], for the flat spaces. In Section 4.1, we indicate how the quantization on manifolds could be performed and state our point of view. In Section 4.2 we present a heuristic renormalization of the model in which a non-relativistic point particle living in two or three dimensional Riemannian manifolds interacts with finitely many Dirac delta potentials located on the manifold. This preliminary work is established in analogy with the same model in flat spaces. A rigorous construction of the model is given in Section 4.3 by means of the heat kernel. The resolvent of the Hamiltonian of the system is expressed in terms of the heat kernel and the divergence in two and three dimensions can be removed using the substraction of the short-time asymptotic of the heat kernel (even in the case where we have not an explicit expression for it). The resolvent is then finite and well-defined expression given in terms of the heat kernel. Then, we finally express the wave function for the bound states in terms of the well-defined finite expression including the heat kernel. Since the resolvent includes all the information about the spectrum, we give the equation in order to determine the bound state energies. At the end of this section, we briefly mention that our problem can also be considered as a dimensional transmutation. In Section 4.4, we briefly sketch the self-adjoint extension form Von Neumann and Krein's point of view and show heuristically that our problem can also be considered as a self-adjoint extension. Section 4.5 is intended to carry our investigation of the point interactions in Riemannian manifolds to the many body problems. We alternatively construct the same problem but for n bosons living in two and three dimensional Riemannian manifolds interacting with N external Dirac delta interactions. This construction is motivated by the work [26] in which the many body version of this problem is renormalized non-perturbatively in flat spaces. This can be accomplished by extending our original Fock space by defining new creation and annihilation operators. After establishing the finite formulation of the problem, we show that the result is consistent with the previous section for one particle boson sector. In section 4.6, we find the wave function for the bound states in a more elegant way and show that the ground state energy of $N+1$ center case is smaller than the N center case using Cauchy interlacing theorem. Since the system can not be exactly soluble, the bound state energies in the tunnelling regime is calculated using a version of perturbation theory in Section 4.7. Section 4.8 presents pointwise bounds on the wave function using the upper bounds of the heat kernel given in the previous chapter and it is shown that the result is consistent with the classical result found in the standard quantum mechanics. This Section ends with another supporting idea that the problem can be heuristically considered as a self-adjoint extension of a formal free hamiltonian by calculating the expectation value of the free Hamiltonian between the bound state energy that we have found. After that we are concerned with the proof of the existence of the self-adjoint densely defined closed Hamiltonian is explicitly given. Section 4.10 establishes the lower bound of the ground state energy by using Geršgorin theorem. The proof is given for compact and Cartan-Hadamard manifolds separately, including some explicit examples: $\mathbb{S}^2$, $\mathbb{H}^2$ and $\mathbb{H}^3$. Then, in section 4.11, non-degeneracy and positivity of the ground state is proven with the help of Perron-Frobenius theorem. A simple proof of the theorem is also included. Finally, we proceed with the study of the renormalization group equations and the β function is calculated exactly.

In Chapter 5, we examine the many body or field theoretical models by following the same approach developed in the previous chapter. Section 5.1 is devoted to our studies on the non-relativistic Lee model on Riemannian manifolds. Before the construction of the model, we introduce the original relativistic Lee model and discuss the non-relativistic Lee model in $\mathbb{R}^3$, and we give the analogy between the Lee model and two level atom - field systems. In Subsection 5.1.4, we establish the renormalization of the non-relativistic Lee model on two and three dimensional Riemannian manifolds in Fock space formalism and is mainly based on our work [37]. Subsection 5.1.5 provides a detailed analysis on the proof that the ground state energy is bounded from below after the renormalization in two and three dimensions. We proceed with a kind of mean field approximation for large number of bosons in two and three dimensional manifolds separately. Section 5.2 deals with the construction of the non-relativistic $\lambda\phi^4$ model in two dimensional Riemannian manifolds based on the similar approach developed in 4.5.

Conclusion is given in Chapter 6 with some comments, future directions and open problems.

2. POINT INTERACTIONS IN EUCLIDEAN SPACES

2.1. A Single Point Interaction in One Dimensional Euclidean Space

The time-independent Schrödinger equation of a particle interacting with a Dirac delta potential at the origin in one dimension is

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda\delta(x)\right]\psi(x) = E\psi(x) \ , \tag{2.1}$$

where m is the mass of the particle and λ, the so-called coupling constant, is the free positive parameter which determines the strength of the Dirac delta function, . The location of the interaction can generally be any point in the real line but we can always shift the coordinate system such that the location of the point interaction is at the origin for simplicity. One can then directly solve the bound state energy and the scattering problem by going to the momentum space since the calculations there are relatively simpler. For the bound state problem, we write down the equation (2.1) in momentum space and parametrize the bound state energy $E = -\nu^2$, so that we get the integral equation for the momentum space wave function

$$\left(\frac{p^2}{2m} + \nu^2\right)\tilde{\psi}(p) = \lambda \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi\hbar}\tilde{\psi}(q) \equiv C \ , \tag{2.2}$$

where $\psi(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi\hbar} e^{ipx/\hbar}\tilde{\psi}(p)$. Then the above equation implies that

$$\tilde{\psi}(p) = \frac{C}{\frac{p^2}{2m} + \nu^2} \ , \tag{2.3}$$

where C can now be determined from the normalization condition. Substituting the solution (2.3) into the definition of C, we arrive at a consistency condition:

$$\lambda \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi\hbar} \frac{1}{\frac{q^2}{2m} + \nu^2} = 1 \ . \tag{2.4}$$

After evaluating the integral in (2.4), we easily find that there exists one bound state and its energy is given by $E = -\nu^2 = -\frac{m\lambda^2}{2\hbar^2}$. From the normalization condition in momentum space, we find $C = \sqrt{m\lambda^3/\hbar^2}$ and then taking the inverse Fourier transform of the wave function $\tilde{\psi}(p)$, we obtain

$$\psi(x) = \sqrt{\frac{m\lambda}{\hbar^2}} e^{-m\lambda|x|/\hbar^2} \ . \tag{2.5}$$

For the scattering problem, Lippmann-Schwinger equation in momentum space gives the scattering solution

$$\tilde{\psi}(p) = (2\pi\hbar)\delta(p - p_0) + \frac{\lambda}{\frac{p^2}{2m} - \frac{p_0^2}{2m} - i\epsilon}\psi(0) \ , \tag{2.6}$$

and then it is easy to find the reflection and transmission coefficients by returning to the coordinate space wave function: $R = 1/(1 + \frac{2p_0^2}{m\lambda^2})$ with $T = 1 - R$. Since Dirac delta functions are generalized functions defined as the limit of a sequence of ordinary functions [38], one may naturally ask: "Would we get the same result for the above problem if we had started with a chosen sequence of Dirac delta function and took the limit after solving the problem?" Answer is not so obvious due to the non-commutativity of integration and limiting procedures in general. Let us take the simplest possible family of the sequence of Dirac delta functions: the well potential. We may consider the well potential whose width is $2a$ centered at the origin and whose depth is $-V_0$ and start to solve bound state and scattering problem for that. The solution is a standard elementary textbook question which can be found in any elementary quantum mechanics textbook [39]. Interesting point is that if we take the zero range limit $a \longrightarrow 0$ in such a way that the area $2aV_0$ is kept constant (getting deeper and narrower wells), we obtain exactly the same result for the bound state energy and scattering data (R, T) if we had started directly with the Dirac delta potential (see problem 2.31 in [39]). Therefore, it makes no difference whether we had directly started with Dirac delta potential or its finite range sequences in one dimension.

2.2. A Single Point Interaction in Two Dimensional Euclidean Space

We may wonder whether the conclusion drawn at the end of the previous section is still true in higher dimensions? To find the answer, one may first mimic the direct method described above. Therefore, following the same steps, one obtains the same integral equation (2.2) written in two dimensions and formally the same solution to the momentum space wave function given in (2.3), whereas the constant C is now defined as

$$C = \lambda \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 q}{(2\pi\hbar)^2} \tilde{\psi}(\mathbf{q}) \; . \tag{2.7}$$

The problem immediately appears when we substitute the solution $\tilde{\psi}(\mathbf{p})$ into the above equation. The consistency equation becomes:

$$\lambda \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{1}{\frac{\mathbf{p}^2}{2m} + \nu^2} = 1 \; . \tag{2.8}$$

If we express the integral in polar coordinates, the integrand goes as $1/p$ for large values of p so that the integral is logarithmically divergent for large values of p, which puts us into trouble. Note that the problem is still divergent when we go to the higher dimensions [1] . Since the integral is divergent for high values of momenta, we call this ultra-violet divergence, which is coming from the terminology of quantum field theory. Indeed, this problem occurs also for Aharonov-Bohm potential [7], the one dimensional inverse square potential [41], and even more general class of potentials in quantum mechanics, called singular potentials [42]. At this stage, one may argue that there is no point interactions whatsoever, since all known interactions in nature has finite range so that the infinity that appear in the problem must not be a fundamental problem. On the other hand, the zero range potentials can well be used as a physical model when de Broglie wavelength of the particle is large compared to the range of the potential. Therefore, we should not dismiss the infinite result and say that the problem is physically artificial to work with. Furthermore, since we have the complete

[1] One can possibly extend the Dirac delta potentials to higher dimensions without worrying about the infinities by preserving its one dimensional structure, i.e., radially symmetric potentials [40].

solution of the same problem in one dimension, we must at least understand the puzzle of divergence that we have encountered in two or higher dimensions, as well.

In Section 2.1 we have solved problem in two ways: first directly or secondly by choosing a sequence of the Dirac delta function and taking the zero range limit in an appropriate way at the end. It seems that the direct solution, that is, trying to use the zero range potential formally in the Schrödinger equation does not work in two and higher dimensions at all. We may then consider another treatment of solving the problem which takes into account the zero range of the potential. It is natural to assume that there is a bound for the validity of the potential in short distances, or equivalently, the momentum of the particle must be bounded. One can interpret this bound as the point beyond which the theory is no longer valid. Therefore we must first solve the finite range interaction before taking the zero range limit. When we tried to solve directly the Dirac delta potential in two dimensions, we must have exceeded the domain of validity. Since we are going to solve our problem in momentum space, the finite range condition restricts the momentum of the particle, that is, the momentum cannot be arbitrarily high and it must have an upper bound Λ. This upper bound Λ is called a cut-off. We will restrict all the upper bounds of momentum integrals to a large but finite value Λ in our equations. The procedure of doing this is called cut-off regularization. Therefore we get the following regularized integral equation

$$\left(\frac{\mathbf{p}^2}{2m} + \nu^2\right) \tilde{\psi}_\Lambda(\mathbf{p}) = \lambda \Theta_\Lambda(\mathbf{p}) \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 q}{(2\pi\hbar)^2} \Theta_\Lambda(\mathbf{q}) \tilde{\psi}_\Lambda(\mathbf{q}) \;, \tag{2.9}$$

where $\Theta_\Lambda(\mathbf{q})$ is the step function which cuts off the upper bound of all momentum integrals. This is an eigenvalue problem of an operator with the following symmetric integral kernel:

$$(2\pi\hbar)^2 \frac{\mathbf{q}^2}{2m} \delta^2(\mathbf{q}-\mathbf{p}) - \lambda \Theta_\Lambda(\mathbf{p}) \Theta_\Lambda(\mathbf{q}) \;. \tag{2.10}$$

Notice that we must put the second step function in front of the integral to get the symmetric integral kernel (2.10) since it preserves the Hermitian property of the corresponding operator. Incidentally, this choice of regularization is not the only possible way. For example, the upper bound of the momentum does not have to be sharp. There are other regularization methods as well, such as dimensional regularization (DR) and Pauli-Villars regularization, widely used in quantum field theory. We will not bother on the type the regularization method since they will give the same result for $D = 2, 3$ [13]. With our cut-off regularization scheme, we obtain

$$\lambda \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{\Theta_\Lambda(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} + \nu^2} = 1 \; . \tag{2.11}$$

One can now solve the bound state energy E as a function of Λ

$$E = -\nu^2 = -\frac{\Lambda^2}{2m} \left(\frac{1}{e^{2\pi\hbar^2/m\lambda} - 1} \right) \; . \tag{2.12}$$

If λ is kept constant as cut-off Λ goes to infinity, the bound state energy (2.12) still apparently diverges. We will then be stuck with the infinity again. In analogy with the quantum field theory, we must now follow a second procedure after we have applied the regularization. We shall take the limit in such a way that the bound state energy (and other observables) is finite and fixed to the experimentally measured value. This requires that the coupling constant (or other parameters if they exist) must vary as a function of the cut-off Λ. The determination of the dependence of the coupling constant on the cut-off so as to get a finite limit is called renormalization. One way to determine this dependence is to interpret the equation (2.11) differently and consider it as a determining equation for λ. That is, we define the cut-off dependent coupling constant (or sometimes called bare coupling constant) as

$$\frac{1}{\lambda(\Lambda)} = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{\Theta_\Lambda(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} + \mu^2} = \frac{m}{2\pi\hbar^2} \ln \left(\frac{\frac{\Lambda^2}{2m} + \mu^2}{\mu^2} \right) \; , \tag{2.13}$$

where we have a new scale $-\mu^2$ and it is the experimentally measured bound state energy of the particle. One can check that plugging (2.13) into (2.12) in the limit $\Lambda \to \infty$, the bound state energy becomes $E = -\mu^2$ and momentum space wave function for the bound state becomes

$$\tilde{\psi}(\mathbf{p}) = \mu \sqrt{\frac{2\pi\hbar^2}{m}} \frac{1}{\frac{\mathbf{p}^2}{2m} + \mu^2} \ , \tag{2.14}$$

or in coordinate space

$$\psi(\mathbf{x}) = \mu \sqrt{\frac{2m}{\pi\hbar^2}} K_0 \left(\frac{\sqrt{2m}}{\hbar} \mu |\mathbf{x}| \right) \ , \tag{2.15}$$

where K_0 is the modified Bessel function of the third kind or MacDonald's function [43]. Although this function is singular at the origin, it is square integrable, i.e, $\psi(\mathbf{x}) \in L^2(\mathbb{R}^2)$.

Now, the next question that we would like to ask is how can we be sure that the above procedure is sufficient to yield well-defined finite predictive results with the other experimentally measured quantities, like scattering cross section or phase shift. If there is still divergence we must make additional regularization and renormalization procedures depending on the parameters in the system. In this system we are in a good situation, only coupling constant renormalization is enough to ensure us to get finite results for all observables. Before explicitly showing this, let us recall some basic ideas and tools in scattering theory.

In scattering problems, we usually encounter operators of the form $(A - E)^{-1}$ in the Lippmann-Schwinger equations, where E is in general a complex number and A is a linear operator in a Hilbert space $\mathcal{H}$. If A is any linear operator in $\mathcal{H}$, the resolvent $R(E)$ of A is the operator-valued function

$$R(E) = (A - EI)^{-1} \ , \tag{2.16}$$

defined in all complex plane at which the inverse exists. Here, I is the identity operator. Then, the resolvent operator of the Hamiltonian H or simply the resolvent of H is defined as $R(E) = (H - EI)^{-1}$. (Some authors prefer to define the resolvent of H

by $(EI - H)^{-1}$). The S-matrix, which is of great importance in solving the scattering problems, is related to the resolvent $R(E)$ or the Green's operator in physics literature. It is convenient to express the resolvent operator in terms of a new operator $T(E)$ [44] as,

$$R(E) = R_0(E) - R_0(E)T(E)R_0(E) \ , \tag{2.17}$$

where $R_0(E) = (H_0 - EI)^{-1}$. Sometimes, we shall omit writing the identity next to E for simplicity.

Therefore, the Lippmann-Schwinger equation for the matrix elements of the regularized operator $T^\Lambda(E)$ for our system is

$$\begin{aligned} \langle \mathbf{p}'|T^\Lambda(E)|\mathbf{p}\rangle &= \langle \mathbf{p}'|V^\Lambda|\mathbf{p}\rangle + \langle \mathbf{p}'|V^\Lambda \frac{1}{H_0 - EI} T^\Lambda(E)|\mathbf{p}\rangle \\ &= -\lambda(\Lambda) - \lambda(\Lambda) \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 q}{(2\pi\hbar)^2} \frac{\Theta_\Lambda(\mathbf{q})}{\frac{\mathbf{q}^2}{2m} - E} \langle \mathbf{q}|T^\Lambda(E)|\mathbf{p}\rangle \ . \end{aligned} \tag{2.18}$$

Since $\langle \mathbf{p}'|T^\Lambda(E)|\mathbf{p}\rangle$ is independent of $\mathbf{p}'$, we can take that outside of the integral and get

$$\langle \mathbf{p}'|T^\Lambda(E)|\mathbf{p}\rangle = \left(I^\Lambda(E) - \frac{1}{\lambda(\Lambda)} \right)^{-1} \ , \tag{2.19}$$

where

$$I^\Lambda(E) = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{\Theta_\Lambda(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} - E_r - i\epsilon} \ , \tag{2.20}$$

and $E = E_r + i\epsilon$, $E_r = \Re(E) > 0$. If we take the limit $\Lambda \to \infty$, the integral above is divergent. Nevertheless, the choice of the bare coupling constant given in (2.13) leads to the following finite expression

$$\langle \mathbf{p}'|T(E)|\mathbf{p}\rangle = \frac{2\pi\hbar^2}{m} \left(\frac{1}{-\ln(E_r/\mu^2) + i\pi} \right) \ , \tag{2.21}$$

after taking the limits $\Lambda \to \infty$ and $\epsilon \to 0^+$. The finiteness of the matrix element of the operator T allows us to get finite physical results since scattering cross section can be completely determined in terms of the matrix element of the operator T [10]

$$\sigma = \frac{64\hbar}{\pi\sqrt{2mE_r}} \left(\frac{1}{\ln^2(E_r/\mu^2) + \pi^2} \right) . \tag{2.22}$$

It is important to notice that this expression clearly violates the naive scale invariance of the Hamiltonian due to the logarithmic term, as noticed by [11]. Hence two dimensional Dirac delta potential is an example of an anomaly in quantum mechanics. As we have seen that the choice of the coupling constant (2.13) automatically makes the other observables finite and we do not need to do the extra renormalization procedures for the other parameters (such as mass in the model). Of course this is not true in general. One may also think the following way, which we usually make it in quantum field theory: We can start with the scattering problem and choose the bare coupling constant in such a way that the experimentally measured quantity, the cross section or phase shift, is kept fixed. After that, one can also prove that the bound state energy is finite with that choice of the coupling constant. Therefore, we have a completely well-defined and finite formulation of our problem since all the physical observables are finite. One crucial point which is inherently different from the standard renormalization procedure in quantum field theory should now be emphasized. Here, we renormalize the problem exactly without applying the perturbation theory and the results agree with the perturbation theory approach [45].

Although we introduced the concept of the resolvent above, we did not use its full power for investigating the spectrum of our system. The resolvent operators play very important role in functional analysis and are very efficient in the sense that we can examine the bound states and scattering states of the system together. Once we have found the resolvent of a given operator, all the information about its spectrum can be found in principle. Therefore, the renormalization method for our model could be better described in this more general formalism. In order to find the resolvent of the full Hamiltonian of our system, we must try to find the solution to the inhomogenous version of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x}) - \lambda\delta^2(\mathbf{x})\psi(\mathbf{x}) - E\psi(\mathbf{x}) = \rho(\mathbf{x}) \ , \tag{2.23}$$

where $E \in \mathbb{C}$, $\mathbf{x} \in \mathbb{R}^2$, and $\rho(\mathbf{x})$ is a sufficiently smooth function. One point about the notation here must be emphasized to avoid confusion. We shall use the same letter ψ for the solution of (2.23) with the one for the wave function of the system.

The first step is now to regularize the equation (2.23) in momentum space, so that we find the function

$$\tilde{\psi}^{\Lambda}(\mathbf{p}) = \frac{\tilde{\rho}^{\Lambda}(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} - E} + \frac{\Theta_{\Lambda}(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} - E} C(\Lambda) \ , \tag{2.24}$$

where

$$C(\Lambda) = \lambda(\Lambda) \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 q}{(2\pi\hbar)^2} \Theta_{\Lambda}(\mathbf{q}) \tilde{\psi}^{\Lambda}(\mathbf{q}) \ . \tag{2.25}$$

Substituting (2.24) into (2.25), we get

$$C(\Lambda) \left[\frac{1}{\lambda(\Lambda)} - \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{\Theta_{\Lambda}(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} - E} \right] = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{\Theta_{\Lambda}(\mathbf{p}) \tilde{\rho}^{\Lambda}(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} - E} \ . \tag{2.26}$$

Considering $\lambda(\Lambda)$ in terms of the experimentally measured ground state energy $-\mu^2$, given in (2.13), and taking the limit $\Lambda \to \infty$, we obtain

$$\tilde{\psi}(\mathbf{p}) = \frac{\tilde{\rho}(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} - E} + \frac{2\pi\hbar^2}{m} \frac{1}{\ln(-E/\mu^2)} \frac{1}{\frac{\mathbf{p}^2}{2m} - E} \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{\tilde{\rho}(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} - E} \ . \tag{2.27}$$

From this, we can easily read the well-defined finite expression of the resolvent kernel in momentum space

$$R(\mathbf{p}, \mathbf{q}|E) = \frac{(2\pi\hbar)^2 \delta^2(\mathbf{p} - \mathbf{q})}{\frac{\mathbf{p}^2}{2m} - E} + \frac{2\pi\hbar^2}{m} \frac{1}{\ln(-E/\mu^2)} \frac{1}{\frac{\mathbf{p}^2}{2m} - E} \frac{1}{\frac{\mathbf{q}^2}{2m} - E} \ , \tag{2.28}$$

where $R(\mathbf{p},\mathbf{q}|E)$ is defined by $\tilde{\psi}(\mathbf{p}) = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 q}{(2\pi\hbar)^2} R(\mathbf{p},\mathbf{q}|E)\tilde{\rho}(\mathbf{q})$. The last equation can be put into the following closed form

$$R(\mathbf{p},\mathbf{q}|E) = R_0(\mathbf{p},\mathbf{q}|E) + \int_{\mathbb{R}^4} \frac{\mathrm{d}^2 k}{(2\pi\hbar)^2} \frac{\mathrm{d}^2 k'}{(2\pi\hbar)^2} R_0(\mathbf{p},\mathbf{k}|E)\Phi^{-1}(E)R_0(\mathbf{k}',\mathbf{q}'|E) \ , \quad (2.29)$$

where R_0 is the free resolvent and the function

$$\Phi(E) = \frac{m}{(2\pi\hbar^2)} \ln(-E/\mu^2) \quad (2.30)$$

is defined for later possible generalizations, and it is called the principal function, which was first introduced in a field theory context by S. G. Rajeev in [26]. Note that

$$\Phi^{-1}(E) = -\langle \mathbf{p}'|T(E)|\mathbf{p}\rangle \ . \quad (2.31)$$

Let us look at what the above equation (2.28) says in a little more detail. We can find the whole spectrum of the problem by working out the resolvent due to following definition [46]:

The set of all complex numbers E, at which the resolvent $R(E)$ of H is a bounded operator, defined on a dense set [2] in a Hilbert space $\mathcal{H}$ is called the resolvent set of H. Its complement is called the spectrum $\sigma(H)$ of H. The spectrum $\sigma(H)$ is the disjoint union of the point spectrum (or discrete spectrum) $\sigma_p(H)$, continuous spectrum $\sigma_c(H)$, and the residual spectrum $\sigma_r(H)$. The point spectrum $\sigma_p(H)$ is the set such that $R(E)$ does not exist. The continuous spectrum $\sigma_c(H)$ is the set such that $R(E)$ exists but unbounded. The residual spectrum $\sigma_r(H)$ is the set such that $R(E)$ exists but it is not defined on a dense subset of $\mathcal{H}$. The self-adjoint operators have no residual spectrum [46].

Therefore, since the bound state in the point spectrum corresponds to the simple pole of resolvent kernel on the negative real axis, the simple pole of the logarithm in

[2] A set of vectors (for example the domain of an operator) is called dense if for every vector ψ there is a sequence of vectors ψ_n in the set such that $\psi_n \to \psi$.

our resolvent kernel (2.28) at $E = -\mu^2$ is the bound state energy, as we have already suggested. We also note that the free Green's functions $R_0(\mathbf{p}, \mathbf{q}|E)$ will not contain any poles on the negative real axis, so all the poles on the negative real axis will come from the poles of $\Phi^{-1}(E)$. Since the eigenvalues are isolated (just one simple pole), we can find the projection operator to the subspace corresponding to the bound state by the following contour integral [46]:

$$\begin{aligned} \langle \mathbf{p}|\mathbb{P}|\mathbf{q}\rangle = \tilde{\psi}(\mathbf{p})\tilde{\psi}^*(\mathbf{q}) &= -\frac{1}{2\pi i}\oint_\gamma \mathrm{d}E\ R(\mathbf{p}, \mathbf{q}|E) \\ &= -\mathrm{Res}_{E=-\mu^2} R(\mathbf{p}, \mathbf{q}|E) \\ &= \frac{2\pi\hbar^2\mu^2}{m}\frac{1}{\frac{\mathbf{p}^2}{2m}+\mu^2}\frac{1}{\frac{\mathbf{q}^2}{2m}+\mu^2} , \end{aligned} \tag{2.32}$$

where γ is a small contour enclosing the isolated eigenvalue $E = -\mu^2$. From the equation above, we can easily read off the ground state wave function by calculating the residue of the resolvent kernel, and obtain exactly the same result as (2.14). The resolvent kernel also gives information about the continuous spectrum, and it has a branch cut along the positive real axis. The discontinuity of the function $\frac{2\pi\hbar^2}{m}\frac{1}{\ln(-E/\mu^2)}$ across the branch cut is $\frac{(2\pi\hbar)^2}{m}\frac{1}{\ln^2(E_r/\mu^2)+\pi^2}$ and the scattering cross section can be calculated in terms of this discontinuity and gives exactly (2.22).

2.3. Finitely Many Point Interactions on Plane

The simplest possible extension of the previous problem is to consider several point interactions on the plane. In this case, time-independent Schrödinger equation reads

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \sum_{i=1}^{N}\lambda_i\ \delta^2(\mathbf{x}-\mathbf{a}_i)\right]\psi(\mathbf{x}) = E\psi(\mathbf{x}) , \tag{2.33}$$

where N is the number of delta potentials and $\mathbf{a}_i$ is the location of the i-th delta potential. We assume that $\mathbf{a}_i \neq \mathbf{a}_j$ for $i \neq j$. We can do exactly the same analysis for finitely many Dirac delta interactions. Let us start with the bound state problem. For simplicity, we parametrize the energy $E = -\nu^2$. The Schrödinger equation in momentum space is then

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} e^{\frac{i\mathbf{p}.\mathbf{x}}{\hbar}} \left(\frac{\mathbf{p}^2}{2m}\tilde{\psi}(\mathbf{p}) + \nu^2 \tilde{\psi}(\mathbf{p}) - \sum_{i=1}^{N} \lambda_i A_i e^{-\frac{i\mathbf{p}.\mathbf{a}_i}{\hbar}} \right) = 0 \ , \tag{2.34}$$

where $A_i = \psi(\mathbf{a}_i)$ and we have used the fact that $\delta^2(\mathbf{x}-\mathbf{a}_i)\psi(\mathbf{x}) = \delta^2(\mathbf{x}-\mathbf{a}_i)\psi(\mathbf{a}_i)$. Since the set of functions $e^{\frac{i\mathbf{p}.\mathbf{x}}{\hbar}}$ form a complete orthonormal system, we find

$$\tilde{\psi}(\mathbf{p}) = \sum_{i=1}^{N} \lambda_i A_i \frac{e^{-\frac{i\mathbf{p}.\mathbf{a}_i}{\hbar}}}{\frac{\mathbf{p}^2}{2m} + \nu^2} \ , \tag{2.35}$$

and wave function in coordinate space

$$\psi(\mathbf{x}) = \sum_{i=1}^{N} \lambda_i A_i \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{e^{\frac{i\mathbf{p}.(\mathbf{x}-\mathbf{a}_i)}{\hbar}}}{\frac{\mathbf{p}^2}{2m} + \nu^2} \ . \tag{2.36}$$

For consistency condition on $A_i = \psi(\mathbf{x} = \mathbf{a}_i)$, we must have

$$A_i = \psi(\mathbf{a}_i) = \sum_{j=1}^{N} \lambda_j A_j \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{e^{\frac{i\mathbf{p}.(\mathbf{a}_i-\mathbf{a}_j)}{\hbar}}}{\frac{\mathbf{p}^2}{2m} + \nu^2} \ . \tag{2.37}$$

Extracting the $j = i$ term from the the right hand side, we obtain

$$\left(\lambda_i^{-1} - \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{1}{\frac{\mathbf{p}^2}{2m} + \nu^2} \right) A_i - \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\lambda_j}{\lambda_i} A_j \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{1}{\frac{\mathbf{p}^2}{2m} + \nu^2} = 0 \ . \tag{2.38}$$

This equation is the analog of the equation (2.11) written for one Dirac delta potential and it can be written as a matrix equation $\Phi_{ij}(E = -\nu^2)A_j = 0$, where

$$\Phi_{ij}(E) = \begin{cases} \lambda_i^{-1} - \displaystyle\int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{1}{\frac{\mathbf{p}^2}{2m} + \nu^2} & \text{if } i = j \\ \\ -\displaystyle\frac{\lambda_j}{\lambda_i} \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{e^{\frac{i\mathbf{p}.(\mathbf{a}_i-\mathbf{a}_j)}{\hbar}}}{\frac{\mathbf{p}^2}{2m} + \nu^2} & \text{if } i \neq j \ . \end{cases} \tag{2.39}$$

This $N \times N$ matrix is called principal matrix or principal operator, which is the generalization of the principal function defined for the one delta potential problem. When we discuss the generalization of this model to the many body and field theoretical models, we will see that this can also be a differential or integral operator in general. Similar to the one delta potential case, we have divergent results, that is the diagonal part of the principal operator is ultra-violet divergent for large momenta, which we have expected. The well-known method to remove the divergence is to put a cut-off Λ to the integral's upper limit and consider the equation as a determining equation of bound state energy for a given coupling constant λ as we have done for one delta potential. If this regularization is performed, we realize that as the cut-off goes to infinity, ground state energy becomes divergent. In order to get a physically acceptable result, one assumes that the coupling constant depends on this cut-off and performs the limit $\Lambda \to \infty$ in such a way that bound state energy remains finite. These infinities should be removed properly since all the physical observables are measured experimentally as finite quantities. The cut-off regularized principal matrix is then

$$\Phi^{\Lambda}_{ij}(E) = \begin{cases} \lambda_i^{-1}(\Lambda) - \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{\Theta_\Lambda(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} + \nu^2} & \text{if } i = j \\ \\ -\frac{\lambda_j(\Lambda)}{\lambda_i(\Lambda)} \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{e^{\frac{i\mathbf{p}.(\mathbf{a}_i - \mathbf{a}_j)}{\hbar}}}{\frac{\mathbf{p}^2}{2m} + \nu^2} \Theta_\Lambda(\mathbf{p}) & \text{if } i \neq j \ . \end{cases} \tag{2.40}$$

We again consider λ_i as a function of Λ in such a way that all observables are finite. A possible choice for $\lambda_i(\Lambda)$ in order to get a finite answer is

$$\frac{1}{\lambda_i(\Lambda)} = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{\Theta_\Lambda(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} + \mu_i^2} \ , \tag{2.41}$$

where $-\mu_i^2$ is the experimentally measured bound state energy of the i-th Dirac delta center, so that we obtain the diagonal part of (2.40) by taking the limit $\Lambda \to \infty$

$$\lim_{\Lambda \to \infty} \left[(2\pi) \int_0^{\Lambda} \frac{\mathrm{d}p \, p}{(2\pi\hbar)^2} \left(\frac{1}{\frac{p^2}{2m} + \mu_i^2} - \frac{1}{\frac{p^2}{2m} + \nu^2} \right) \right] = \frac{m}{\pi\hbar^2} \ln(\nu/\mu_i) \ , \tag{2.42}$$

and the off diagonal term of (2.40) at the same limit

$$\lim_{\Lambda\to\infty} \frac{\lambda_j(\Lambda)}{\lambda_i(\Lambda)} \int_0^{\Lambda} \frac{\mathrm{d}p\; p}{(2\pi\hbar)^2} \frac{1}{\frac{p^2}{2m}+\nu^2} \left[\int_0^{2\pi} \mathrm{d}\theta\; e^{\frac{ip|\mathbf{a}_i-\mathbf{a}_j|\cos\theta}{\hbar}}\right]$$
$$= \frac{1}{2\pi\hbar^2} \int_0^{\infty} \mathrm{d}p\; p\; J_0\left(\frac{p|a_i-a_j|}{\hbar}\right) \frac{1}{\frac{p^2}{2m}+\nu^2} = \frac{m}{\pi\hbar^2} K_0\left(\frac{\nu\sqrt{2m}|a_i-a_j|}{\hbar}\right) , \tag{2.43}$$

where we have used the integral representation of Bessel function of the first kind J_0 [38]

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\; e^{ix\cos\theta} \tag{2.44}$$

and the useful relation between the integral of J_0 and K_0 in [43] for $a, b > 0$:

$$\int_0^{\infty} \mathrm{d}x\; \frac{x J_0(bx)}{(x^2+a^2)} = K_0(ab) . \tag{2.45}$$

Also, we have used the fact that $\lim_{\Lambda\to\infty} \frac{\lambda_j(\Lambda)}{\lambda_i(\Lambda)} = 1$, which can be easily proved from (2.41). Hence, the well-defined and finite principal matrix comes out to be

$$\Phi_{ij}(E=-\nu^2) = \frac{m}{\pi\hbar^2} \begin{cases} \ln(\nu/\mu_i) & \text{if } i=j \\ \\ -K_0\left(\nu/\mu_{d_{ij}}\right) & \text{if } i\neq j . \end{cases} \tag{2.46}$$

where we have defined $\mu_{d_{ij}}^2 = \frac{\hbar^2}{2m|\mathbf{a}_i-\mathbf{a}_j|^2}$ for abbreviation. We can also show that the other regularization methods (Pauli-Villars and dimensional regularization) give the same result. Therefore, the matrix equation, after the renormalization has been performed, becomes

$$\Phi_{ij}(E)A_j = 0 , \tag{2.47}$$

where the principal matrix is given by (2.46), that is, it is well-defined and finite. It has a non trivial solution if only if $\det\Phi(E) = 0$. This is not a standard form of the eigenvalue equation $H\psi = E\psi$ in quantum mechanics because the principal matrix is a complicated function of E in which one usually can not find the exact solution of the problem for N arbitrarily given distribution of Dirac delta centers. The equation (2.47) is called a non-linear eigenvalue problem. Nevertheless, the special case $N = 2$ allows us to analyze the problem analytically. This problem can be considered as a very elementary model for ionized diatomic molecule (for example: H_2^+) [47, 48] so we digress a little on this case now. Then, the condition $\det \Phi(E) = 0$ gives a transcendental equation

$$\ln(\nu/\mu_1)\ln(\nu/\mu_2) = K_0^2\left(\nu/\mu_d\right) \ , \tag{2.48}$$

where $\mu_d^2 = \frac{\hbar^2}{2m|\mathbf{a}_1-\mathbf{a}_2|^2}$. In order to find the solution, let us write the equation above in terms of dimensionless variables: $\ln(\alpha_1 x)\ln(\alpha_2 x) = K_0^2(x)$, where $x = \frac{\nu}{\mu_d}$ and $\alpha_1 = \frac{\mu_d}{\mu_1}$, $\alpha_2 = \frac{\mu_d}{\mu_2}$. Without loss of generality, we can choose $\alpha_1 > \alpha_2$. Although the exact analytical roots are unfortunately difficult to find, we can at least tell how many bound states we may have and investigate the asymptotic cases. The function $\ln(\alpha_1 x)\ln(\alpha_2 x)$ is positive decreasing function when $0 < x < \frac{1}{\alpha_1}$ and positive increasing one when $x > \frac{1}{\alpha_2}$, whereas it is negative between $x = \frac{1}{\alpha_1}$ and $x = \frac{1}{\alpha_2}$ with two zeros at $x = \frac{1}{\alpha_1}$ and $x = \frac{1}{\alpha_2}$. It is also easy to see that this function has a local minimum at $x = \frac{1}{\sqrt{\alpha_1\alpha_2}}$ ($\frac{1}{\alpha_1} < \frac{1}{\sqrt{\alpha_1\alpha_2}} < \frac{1}{\alpha_2}$). No matter how α_1 and α_2 is chosen, we expect there is at least one

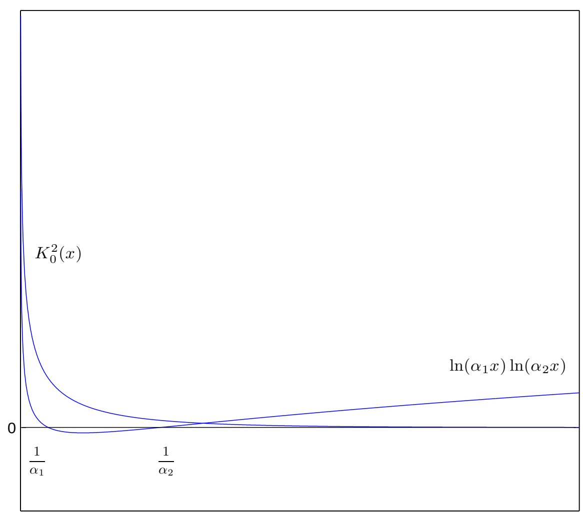

Figure 2.1. Graphics of $K_0^2(x)$ and $\ln(\alpha_1 x)\ln(\alpha_2 x)$

solution because the function $\ln(\alpha_1 x)\ln(\alpha_2 x)$ eventually goes to zero and intersect the monotonically positive decreasing function $K_0^2(x)$. The asymptotic expansion of $K_0(x)$ [43] as $x \to \infty$ is given as $K_0(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} (1 + O(1/x))$. Hence, we have

$$K_0^2(x) \sim \frac{\pi}{2x} e^{-2x} (1 + O(1/x)) \ , \tag{2.49}$$

where we have used the fact that if $f(x) = \frac{e^{-x}}{x^{1/2}} \sum_{n=0}^{N} \left(\frac{a_n}{x^n} + O(x^{-(N+1)}) \right)$ as $x \to \infty$, then the square of the function possess an asymptotic power series $f^2(x) = \frac{e^{-2x}}{x} \sum_{n=0}^{N} (\frac{c_n}{x^n} + O(x^{-(N+1)}))$, where $c_n = a_0 a_n + a_1 a_{n-1} + \cdots + a_n a_0$.

From its graph, we see that, we may have a second root if we impose the condition that $\ln(\alpha_1 x)\ln(\alpha_2 x)$ is able to catch the function $K_0^2(x)$ near $x = 0$. Therefore it is necessary to impose $\ln(\alpha_1 x)\ln(\alpha_2 x) > K_0^2(x)$ for $x \ll \frac{1}{\alpha_1}$ in order to get a second bound state. Since $K_0(x)$ has the following power series expansion for small x [43]:

$$K_0(x) \simeq -\ln\left(\frac{x}{2}\right) + \gamma \ , \tag{2.50}$$

where γ is the Euler's gamma constant, we have the following condition on the distance between two Dirac delta centers in order to have a second bound state

$$d > \frac{2\hbar e^{\gamma}}{\sqrt{2m\mu_1\mu_2}} \ . \tag{2.51}$$

This tells us that we have at least one and at most two bound states for two delta centers and we have exactly two bound states if the distance between the delta centers is larger than the above critical value. We must also check that the derivative of $\ln(\alpha_1 x)\ln(\alpha_2 x)$ must be greater than the derivative of the function $K_0^2(x)$ for small values of x. We can easily see that if the lower bound of d is chosen as in (2.51), it is automatically satisfied. It is not easy to tell what condition must be met for any given number of delta centers N since the root of the equation $\det \Phi(-\nu^2) = 0$ is a complicated nonlinear function of ν.

Let us again formulate the problem in the context of the resolvent of the system.

We follow the same steps for one Dirac delta center, and find the function

$$\tilde{\psi}(\mathbf{p}) = \frac{\tilde{\rho}(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} - E} + \sum_{i=1}^{N} \lambda_i A_i \frac{e^{-\frac{\mathbf{p}.\mathbf{a}_i}{\hbar}}}{\frac{\mathbf{p}^2}{2m} - E} \, , \tag{2.52}$$

where again $A_i = \psi(\mathbf{a}_i)$ formally (recall that $\psi(\mathbf{a}_i)$ is actually divergent). If we insert the above equation into the definition of A_i, the divergent terms will again appear. So we apply exactly the same regularization and renormalization procedures that we have done for one Dirac delta center with the same choice of the bare coupling constant (2.41) and obtain the well-defined and finite principal matrix defined on the complex energy plane (see [32] for more details):

$$\Phi_{ij}(E) = \begin{cases} \frac{m}{2\pi\hbar^2} \ln(-E/\mu_i^2) & \text{if } i = j \\ \\ -\frac{m}{\pi\hbar^2} K_0\left(\sqrt{-E/\mu_{d_{ij}}}\right) & \text{if } i \neq j \, , \end{cases} \tag{2.53}$$

Hence, the resolvent kernel in momentum space becomes

$$R(\mathbf{p},\mathbf{q}|E) = \frac{(2\pi\hbar)^2 \delta^2(\mathbf{p}-\mathbf{q})}{\frac{\mathbf{p}^2}{2m} - E} + \sum_{i,j=1}^{N} \frac{e^{-\frac{\mathbf{p}.\mathbf{a}_i}{\hbar}}}{\frac{\mathbf{p}^2}{2m} - E} \, \Phi_{ij}^{-1}(E) \, \frac{e^{\frac{\mathbf{q}.\mathbf{a}_j}{\hbar}}}{\frac{\mathbf{q}^2}{2m} - E} \, . \tag{2.54}$$

or in coordinate space

$$\begin{aligned} R(\mathbf{x},\mathbf{y}|E) = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \, \frac{e^{i\mathbf{p}.(\mathbf{x}-\mathbf{y})}}{\frac{\mathbf{p}^2}{2m} - E} + \sum_{i,j=1}^{N} \left[\int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \, \frac{e^{i\mathbf{p}.(\mathbf{x}-\mathbf{a}_i)}}{\frac{\mathbf{p}^2}{2m} - E} \right] \\ \times \Phi_{ij}^{-1}(E) \left[\int_{\mathbb{R}^2} \frac{\mathrm{d}^2 q}{(2\pi\hbar)^2} \, \frac{e^{i\mathbf{q}.(\mathbf{a}_j-\mathbf{y})}}{\frac{\mathbf{q}^2}{2m} - E} \right] . \end{aligned} \tag{2.55}$$

Note that, we have a matrix operator Φ_{ij} in the resolvent kernel instead of a function as in the case of the single center case (2.29). Here, Φ_{ij}^{-1} denotes for the ij-th element of the inverse of the principal matrix and the explicit form of principal operator $\Phi_{ij}(E)$ must be considered as the analytical continuation of the principal matrix $\Phi_{ij}(-\nu^2)$ onto the largest set in the entire complex energy plane E.

The equation (2.54) includes all the information about the spectrum of our problem. Since $\Phi_{ij}^{-1}(E) = \frac{C_{ij}(E)}{\det \Phi(E)}$, the poles of the matrix $\Phi_{ij}^{-1}(E)$ are due to zeros of $\det \Phi(E)$. There is no singularity coming from the cofactor C_{ij} of the matrix $\Phi_{ij}(E)$ for $\Re(E) < 0$. Therefore, the energy eigenvalues of the system for the bound states are just the solution of the equation $\det \Phi(E) = 0$ or we can equivalently state that the bound states correspond to the solution of the eigenvalue equation (2.47). From the explicit expression of the principal operator (2.53), we also have an important property of the principal matrices

$$\Phi_{ij}^{\dagger}(E) = \Phi_{ij}(E^*) \tag{2.56}$$

for all $E \in \mathbb{C}$. This says that the value of the principal matrix at the point E^* is the same as its adjoint at the point E.

As for the scattering states, we must look at the branch cut of inverse principal matrix Φ^{-1}. The branch cut structure of Φ^{-1} coincides with that of the Φ because the cofactor C_{ij} and $\det \Phi_{ij}(E) = \sum_{i_1, i_2 \ldots i_N}^{N} \varepsilon_{i_1 i_2 \ldots i_N} \Phi_{1 i_1} \Phi_{2 i_2} \ldots \Phi_{1 i_N}$ consist of the matrix elements $\ln(-E/\mu_i^2)$ and $K_0\left(\sqrt{-E/\mu_{d_{ij}}}\right)$, which have a common branch cut along the positive real axis. Discontinuity of $\Phi_{ij}^{-1}(E)$ across the branch cut is related to the scattering matrix but it is difficult to find this inverse matrix explicitly. We can also find the scattering amplitude from the relation between T matrix and principal matrix

$$\langle \mathbf{p} | T(E) | \mathbf{q} \rangle = - \sum_{i,j=1}^{N} e^{-\frac{\mathbf{p} . \mathbf{a}_i}{\hbar}} \, \Phi_{ij}^{-1}(E) \, e^{-\frac{\mathbf{q} . \mathbf{a}_j}{\hbar}} \,, \tag{2.57}$$

which can be seen from (2.17) and (2.54). The scattering amplitude and cross section can also be explicitly calculated for two Dirac delta centers with equal strengths and given explicitly in [32]. We can similarly extend our problem to the three dimensions except the divergence is linear in this case and the principal operator is obtained [32]

$$\Phi_{ij}(-\nu^2) = \frac{\nu}{4\pi(\frac{\hbar^2}{2m})^{3/2}} \begin{cases} 1 - \dfrac{\mu_i}{\nu} & \text{if } i = j \\ \\ -\dfrac{e^{-\nu/\mu_{d_{ij}}}}{\nu/\mu_{d_{ij}}} & \text{if } i \neq j \; . \end{cases} \tag{2.58}$$

Before completing the discussion of the point interactions in flat spaces, let us emphasize that we have non-perturbatively renormalized the point interactions in two or three dimensional Euclidean spaces by writing the resolvent of the Hamiltonian in terms of a finite principal matrix, which is first introduced by S. G. Rajeev [26] in the field theory context. After the renormalization, the spectrum can completely be determined by the principal matrix (or operator in general as we will see in later chapters), that is, one can find the energy eigenvalues by a non-linear eigenvalue problem $\Phi(E)A = 0$. As we will mention later on that the point interactions can be rigorously defined in the context of self-adjoint extension theory. Therefore, one advantage of using the principal matrix is that the interactions are completely described by an explicit formula of it instead of dealing with the domain issues of the operators. Moreover, we do not have to find the spectrum of the problem in order to do renormalization. The finite formulation of the problem is given once the finite well defined principal matrix has been found.

3. THE HEAT KERNEL ON RIEMANNIAN MANIFOLDS

3.1. Some Basic Concepts in Riemannian Geometry

Let $\mathcal{M}$ be a smooth manifold with dimension D (smoothly glued coordinate patches, where each patch is an open region of $\mathbb{R}^D$). A mapping $\xi : C^\infty(\mathcal{M}) \to \mathbb{R}$ is called an $\mathbb{R}$-differentiation at a point $x_p \in \mathcal{M}$ if

(i) ξ is linear

(ii) ξ satisfies the Leibniz rule: $\xi(fg) = \xi(f)g(x_p) + \xi(g)f(x_p)$ for all $f, g \in C^\infty(\mathcal{M})$ [30, 49].

The set of all $\mathbb{R}$-differentiations at a point x_p is denoted by $T_p(\mathcal{M})$ and it is easy to show that it is a linear vector space of dimension D. This space is called the tangent space of $\mathcal{M}$ at x_p and its elements are called tangent vectors at x_p. Let $x^1, \dots, x^D$ be local coordinates in a chart U in a neighborhood of x_p. One can show that all the partial derivatives $\frac{\partial}{\partial x^i}$ evaluated at x_p are $\mathbb{R}$-differentiations at x_p and they are linearly independent(where $i = 1, \dots, D$). Any tangent vector $\xi \in T_p(\mathcal{M})$ can be represented in the form $\xi = \xi^i \frac{\partial}{\partial x^i}$, where $\xi^i = \xi(x^i)$. Here we are using the Einstein summation convention, that is, all repeated indexes are summed over unless otherwise stated. Then, for any smooth function f on $\mathcal{M}$, the directional derivative of f along the direction ξ is given by $\xi(f) = \xi^i \frac{\partial f}{\partial x^i}$. Also, a vector field on a smooth manifold $\mathcal{M}$ is a family $\{X(x)\}_{x\in\mathcal{M}}$ of tangent vectors such that $X(x) \in T_x(\mathcal{M})$ for any $x \in \mathcal{M}$. In local coordinates, we have $X(x) = X^i(x)\frac{\partial}{\partial x^i}$. A Riemannian metric on $\mathcal{M}$ is a family $\{g(x)\}_{x\in\mathcal{M}}$ such that $g(x)$ is a symmetric, positive definite, bilinear form on the tangent space $T_x\mathcal{M}$. Then, one can define an inner product $\langle , \rangle$ in any tangent space $T_x\mathcal{M}$ by $\langle \xi, \eta \rangle = g_{ij}(x)\xi^i\eta^j$ in local coordinates, where $\xi, \eta \in T_x(\mathcal{M})$. Then, a Riemannian manifold is defined as a couple $(\mathcal{M}, g)$. One can naturally define a dual vector space of the tangent space $T_p(\mathcal{M})$ and this dual is called cotangent space, denoted by $T_p^*(\mathcal{M})$. If $x^1, \dots, x^D$ are local coordinates in a neighborhood of x_p, then $\{\frac{\partial}{\partial x^i}\}_{i=1}^D$ is a basis in the tangent space, whereas the corresponding dual basis in the cotangent space is $\{\mathrm{d}x^i\}_{i=1}^D$ since $\langle \mathrm{d}x^i, \frac{\partial}{\partial x^j}\rangle = \delta^i_j$. Let $(\mathcal{M}, g)$ be a Riemannian manifold and f be a smooth function on it. The gradient of the function f is defined by $\langle \nabla f, \xi\rangle = \xi(f)$ for all $\xi \in T_p(\mathcal{M})$. In local coordinates, we have

$$(\nabla f)^i = g^{ij}\frac{\partial f}{\partial x^j} \; . \tag{3.1}$$

The volume form of any oriented Riemannian manifold is given by

$$\mathrm{dvol} = \mathrm{d}^D_g x = \sqrt{\det g}\; \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D = \sqrt{\det g}\; \mathrm{d}^D x \; , \tag{3.2}$$

where $g = (g_{ij})$ and $\wedge$ stands for the wedge product or exterior product [50]. Then, the volume of the manifold $\mathcal{M}$ is defined to be

$$V(\mathcal{M}) = \int_{\mathcal{M}} \mathrm{d}^D_g x \; . \tag{3.3}$$

(It may be infinite). For any C^∞ vector field $X(x)$ on a Riemannian manifold, there exists a unique smooth function on it, denoted by $\nabla \cdot X$ and called divergence, such that the following inequality holds

$$\int_{\mathcal{M}} \mathrm{d}^D_g x \; Y(\nabla \cdot X) = -\int_{\mathcal{M}} \mathrm{d}^D_g x \; \langle X, \nabla Y\rangle \; , \tag{3.4}$$

for any $Y \in C_0^\infty(\mathcal{M})$. Here $C_0^\infty(\mathcal{M})$ is the subspace of $C^\infty(\mathcal{M})$ which consists of functions whose support (support of f is the closure of the set $\{x \in \mathcal{M} : f(x) \neq 0\}$) is compact. This condition allows us to ignore the boundary terms when we make integration by parts. From this definition, the divergence in local coordinates can be found as

$$\nabla \cdot X = \frac{1}{\det g}\partial_i\left(\sqrt{\det g}\; X^i\right) \; , \tag{3.5}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ for simplicity. The Laplace operator or the Laplacian (called also the Laplace-Beltrami operator) on any Riemannian manifold acting on any smooth function f can be defined as

$$\nabla^2 f = \nabla \cdot (\nabla f) \,, \tag{3.6}$$

or in local coordinates

$$\nabla^2 = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{D} \partial_i \left(g^{ij} \sqrt{\det g} \partial_j \right) \ . \tag{3.7}$$

From this point on, we shall usually denote it as ∇_g^2 to specify which metric structure on Riemannian manifold it is associated with. When necessary, we use the notation $\nabla_{g,x}^2$, $\nabla_{g,y}^2$, etc. to indicate on which variable the Laplacian acts.

3.2. The Spectral Theorem

Having defined the Laplacian on Riemannian manifolds, we now consider the closed eigenvalue problem on $\mathcal{M}$

$$-\nabla_g^2 u(x) = \sigma u(x) \,, \tag{3.8}$$

that is, we are trying to find all real numbers σ for which a nontrivial solution $u \in C^2(\mathcal{M})$ exists. Without going into the technical details, we at first consider the compact manifolds and then assume that we can naturally extend these ideas into the appropriate noncompact manifolds that we are interested in. The compactness is a topological concept and a compact manifold is a manifold that is compact as a topological space. Intuitively, a compact space is a space in which any infinite sequence of points must eventually accumulate at some point within the space, or even more roughly, a compact space does not go off to infinity and it does not have holes cut out of it nor parts of its boundary removed [51]. D dimensional sphere $\mathbb{S}^D$ and torus $\mathbb{T}^D$ are well known examples to the compact manifolds, whereas the D dimensional Euclidean space $\mathbb{R}^D$, Hyperbolic spaces $\mathbb{H}^D$, the open unit disc, and the closed disc with the center removed are the examples to the noncompact manifolds. In some sense, the compact manifolds are nice due to the fact that they can be covered by finitely many coordinate charts and any continuous real valued function is bounded on a compact manifold. Therefore, let us start with the compact manifolds.

Let $(\mathcal{M}, g)$ be a compact connected Riemannian manifold, then there exists a complete orthonormal system of C^∞ eigenfunctions $\{f_l\}_{l=0}^\infty$ in $L^2(\mathcal{M})$ and the spectrum $\sigma(\mathcal{M}, g) = \{\sigma_l\} = \{0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \dots\}$, with σ_l tending to infinity as $l \to \infty$ and each eigenvalue has finite multiplicity: $-\nabla_g^2 f_l(x; g) = \sigma_l f_l(x; g)$. Some eigenvalues are repeated according to their multiplicity. The multiplicity of the first eigenvalue $\sigma_0 = 0$ is one and corresponding eigenfunction is constant and given by $f_0(x; g) = 1/\sqrt{V(\mathcal{M})}$. This theorem is also called spectral theorem (or also Hodge theorem for functions [34, 49]) and is still valid for Neumann, Dirichlet and mixed eigenvalue problems except for $\sigma_0 > 0$, provided that the appropriate boundary condition is imposed. The operator $-\nabla_g^2$ is formally self-adjoint or symmetric with respect to the inner product $L^2(\mathcal{M})$ defined by

$$(\psi_1, \psi_2)_{L^2} = \int_{\mathcal{M}} \mathrm{d}_g^D x \; \psi_1^*(x) \psi_2(x) \; . \tag{3.9}$$

Then, we can prove that $-\nabla_g^2$ is a non-negative operator:

$$\begin{aligned} (-\nabla_g^2 \psi, \psi)_{L^2} &= \int_{\mathcal{M}} \mathrm{d}_g^D x \; \left(-\nabla_g^2 \psi(x)\right)^* \psi(x) = \int_{\mathcal{M}} \mathrm{d}_g^D x \; \left(-\nabla \cdot \nabla \psi(x)\right)^* \psi(x) \\ &= \int_{\mathcal{M}} \mathrm{d}_g^D x \; |\nabla \psi(x)|^2 \geq 0 \; . \end{aligned} \tag{3.10}$$

We can think of the eigenfunctions $f_l(x; g)$ as real-valued functions without loss of generality since the operator $-\nabla_g^2$ is symmetric and real [52]. The spectral theorem provides us with all the tools of Fourier analysis, so that we can expand any function $\psi(x) \in L^2(\mathcal{M})$ in terms of the complete orthonormal eigenfunctions $f_l(x; g)$

$$\psi(x) = \sum_{l=0}^{\infty} \left(\psi(x), f_l(x; g)\right)_{L^2} \; f_l(x; g) = \sum_{l=0}^{\infty} C_l \, f_l(x; g) \; , \tag{3.11}$$

where C_l's are the expansion coefficients. If there is a degeneracy, the additional indices must be taken into account in the sum. The orthogonality and the completeness of the eigenfunctions on compact manifolds means that

$$\begin{aligned} \int_{\mathcal{M}} \mathrm{d}_g^D x \; f_k(x;g) f_l(x;g) &= \delta_{kl} \;, \\ \sum_{l=0}^{\infty} f_l(x;g) f_l(a_i;g) &= \delta_g^D(x,a_i) \;. \end{aligned} \tag{3.12}$$

D - dimensional Dirac delta function $\delta_g^D(x,a_i)$ at point $a_i \in \mathcal{M}$ can be defined as

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \; \delta_g^D(x,a_i) f(x) = f(a_i) \;, \tag{3.13}$$

in terms of a well-behaved test functions $f(x)$ defined on manifold. The normalization is also given by

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \; \delta_g^D(x,a_i) = 1 \;. \tag{3.14}$$

In fact, the Hodge theorem or spectral theorem given above is similar to the basic well-known theorem of Sturm-Liouville theory [53]. It is given as a remark in [49] and stated explicitly as lemma [29] for an even more general case (self-adjoint pseudodifferential operators [3]). In our case, Lemma 1.6.3 in [29] is simplified for any second order self-adjoint elliptic operator P acting on functions and can be given as a generalization of the spectral theorem for the Laplace operator. The most general linear second order partial differential equations in D variables $(x^1, \ldots, x^D)$ can be written as [53]

$$Pu = A_{ij}(x^k) \frac{\partial^2 u}{\partial x^i x^j} + B_j(x^k) \frac{\partial u}{\partial x^j} + C(x^k) u = 0 \;, \tag{3.15}$$

[3]The idea of introducing pseudodifferential operators is to think of a differential operator acting on a function as the inverse Fourier transform of a polynomial in the Fourier variable times the Fourier transform of the function. This integral representation leads to a generalization of differential operators, which correspond to functions other than polynomials in the Fourier variable, as long as the integral converges [29].

where A_{ij} are assumed to be symmetric and each elements of the coefficients A_{ij}, B_j and C are smooth functions of x. Since A_{ij} can be diagonalized to $A_i(x^k)$, we have simply

$$A_i(x^k)\frac{\partial^2 u}{\partial x^{i\,2}} + F\left(x^k, u, \frac{\partial u}{\partial x^k}\right) = 0 \ , \tag{3.16}$$

where the last term includes all the other terms except the second derivatives. We say that P is an elliptic operator at x_0^k if all the coefficients $A_i(x_0^k)$ are nonzero and have the same sign. If P is elliptic at every point of its domain, it is said to be elliptic type. As an example, the Laplacian is an elliptic operator.

Let $(\mathcal{M}, g)$ be a compact connected Riemannian manifold, then there exists a complete orthonormal system of C^∞ eigenfunctions $\{f_l\}_{l=0}^\infty$ in $L^2(\mathcal{M})$ of a self-adjoint elliptic differential operator P,

$$P f_l(x; g) = \sigma_l f_l(x; g) \ , \tag{3.17}$$

and its spectrum is given by $\sigma(\mathcal{M}, g) = \{\sigma_l\} = \{|\sigma_0| \leq |\sigma_1| \leq |\sigma_2| \leq \ldots\}$, with $|\sigma_l|$ tending to infinity as $l \to \infty$ and each eigenvalue has finite multiplicity. Furthermore, the eigenvalues of a second order of elliptic differential operator obey the Weyl's asymptotic formula [34].

$$\sigma_l^{D/2} \sim \frac{2^{D-1}\pi^{D/2} D\Gamma(D/2)}{V(\mathcal{M})}\, l \ , \tag{3.18}$$

as $l \to \infty$. Moreover, one may extend heuristically the problem onto some noncompact manifolds in such a way that spectral theorem still applies the relations such as completeness and orthogonality relations are defined by an appropriate generalization of the measures to the continuous distributions in the sense of [54]. Since the spectrum does not have to be discrete for noncompact manifolds, we may have in general

$$\psi(x) = \int \mathrm{d}\mu(l)\ \psi(l)\ f_l(x; g) \ ,$$

$$\begin{aligned} \int_{\mathcal{M}} \mathrm{d}_g^D x \; f_k(x;g) f_l(x;g) &= \delta_{kl} \; , \\ \int \mathrm{d}\mu(l) \; f_l(x;g) f_l(a_i;g) &= \delta_g^D(x, a_i) \; , \end{aligned} \tag{3.19}$$

where $\mathrm{d}\mu(l)$ is the spectral measure and it includes continuous spectrum as well as point spectrum. Noncompact manifolds with infinite volume do not admit discrete spectrum so that the measures do not contain the discrete part in this case. This generalization should be taken with a grain of salt and one must not forget that we may not have a rigorous proof of the spectral theorem for some special noncompact manifolds. From now on, we assume that we are dealing with manifolds which do not have such pathologies, that is, the spectral theorem is applicable.

3.3. Definition of the Heat Kernel

Having defined the Laplacian in Riemannian manifolds, in the Section 3.1, one may wonder whether the associated Cauchy problem of the heat equation is well posed. Luckily, we can naturally extend the problem and the definition of the heat kernel into Riemannian manifolds [34] by analogy with the definition of it in $\mathbb{R}^D$ [55]. The "heat" operator L on $\mathcal{M}$ is defined to be the parabolic differential operator

$$L = -\frac{\hbar^2}{2m} \nabla_g^2 + \hbar \frac{\partial}{\partial t} \; , \tag{3.20}$$

and "heat" equation is given by $Lu = 0$. In order to have a dimensionally consistent formulation with our works that we shall describe later on, we have attached the factors $\hbar$ and m in a dimensionally proper way into the dimensionless heat operator, given in standard mathematics literature. The initial value problem (or the so-called Cauchy problem) is

$$\begin{aligned} \frac{\hbar^2}{2m} \nabla_g^2 u &= \hbar \frac{\partial u}{\partial t} \; , \\ u(x, t=0) &= f(x) \; . \end{aligned} \tag{3.21}$$

Considering the Riemannian manifold $\mathcal{M}$ as a homogenous and isotropic medium and $u = u(x,t)$ as the "temperature" of the point $x \in \mathcal{M}$ at time t, then the equation above determines the conduction of "heat" through the medium with the assumption that there is no source or sink of heat.

There are several ways to introduce the heat kernel. Our viewpoint is the following: A fundamental solution of the "heat" equation on $\mathcal{M}$ or the heat kernel of $\mathcal{M}$ [34] is a continuous function $K_t(x,y;g)$ defined on $\mathbb{R}^+ \times \mathcal{M} \times \mathcal{M}$, which is C^2 with respect to the variable x and y, C^1 with respect to the variable t, and which satisfies

$$\begin{aligned} L_x K_t(x,y;g) = \left[-\frac{\hbar^2}{2m}\nabla^2_{g,x} + \hbar\frac{\partial}{\partial t}\right] K_t(x,y;g) &= 0 \ , \\ \lim_{t\to 0^+} K_t(.,y;g) &= \delta_g(.,y) \ , \end{aligned} \tag{3.22}$$

where the limit must be taken in the sense of distributions, that is, for all bounded continuous functions f on $\mathcal{M}$, we have

$$\lim_{t\to 0^+} \int_{\mathcal{M}} \mathrm{d}^D_g x \ K_t(x,y;g) f(x) = f(y) \qquad \text{for every} \ \ y \in \mathcal{M} \ . \tag{3.23}$$

If the initial temperature distribution is $u(x,t=0) = f(x)$, then the solution to Cauchy problem (3.21) is given by

$$u(x,t) = \int_{\mathcal{M}} \mathrm{d}^D_g y \ K_t(x,y;g) f(y) \ . \tag{3.24}$$

Heat kernel can be considered as an integral kernel of the one parameter family of bounded operators $e^{\frac{t}{\hbar}\left(\frac{\hbar^2}{2m}\nabla^2_g\right)}$ for $t > 0$. This is the formal solution to heat equation and the solution to Cauchy problem is given by $u = e^{\frac{t}{\hbar}\left(\frac{\hbar^2}{2m}\nabla^2_g\right)} f$. We can sometimes write the heat kernel in Dirac's bra-ket notation as

$$K_t(x,y;g) = \langle x | e^{\frac{\hbar t}{2m}\nabla^2_g} | y \rangle \ . \tag{3.25}$$

Here one point must again be emphasized. Although it is not obvious, one can properly define an exponential of an unbounded operator ∇_g^2 thanks to the spectral theorem (c.f. chapter IX of Kato's book [56] and [30]). The heat kernel for $\mathcal{M} = \mathbb{R}^D$ can be explicitly computable by various techniques and is given by the classical formula

$$K_t(x, y; g_{\mathbb{R}^D}) = \frac{1}{(4\pi\hbar t/2m)^{D/2}} \exp\left(-\frac{m|\mathbf{x}-\mathbf{y}|^2}{2\hbar t} \right). \tag{3.26}$$

One way to find (3.26) is to transform the original initial value problem to Fourier space and then solve the resulting problem in the transformed space (which converts the problem into either algebraic equations or differential equations with fewer variables), and finally take inverse transform [55]. This can also be a useful idea in calculating the heat kernel for hyperbolic spaces $\mathbb{H}^D$ by the analog of Fourier transform, namely Mehler transform [34]. The explicit formulas [57, 58, 59] for the heat kernel of $\mathbb{H}^2$ and $\mathbb{H}^3$ are

$$\begin{aligned} K_t(x, y; g_{\mathbb{H}^2}) &= \frac{\sqrt{2}}{(4\pi \left[\frac{\hbar}{2mR^2}\right] t)^{3/2}} \frac{e^{-\frac{\hbar}{2mR^2}\frac{t}{4}}}{R^2} \int_{\frac{d(x,y)}{R}}^{\infty} \mathrm{d}r \, \frac{r e^{-\frac{r^2}{4}\frac{2mR^2}{\hbar}\frac{1}{t}}}{\sqrt{\cosh r - \cosh \frac{d(x,y)}{R}}} \\ K_t(x, y; g_{\mathbb{H}^3}) &= \frac{1}{R^3} \frac{\frac{d(x,y)}{R}}{\left(4\pi \left[\frac{\hbar}{2mR^2}\right] t\right)^{3/2} \sinh \frac{d(x,y)}{R}} \exp\left(-\frac{\hbar t}{2mR^2} - \frac{m d(x,y)^2}{2\hbar t} \right), \end{aligned} \tag{3.27}$$

where $d(x, y)$ is the geodesic distance [4] between any two points x and y on $\mathbb{H}^2$ or $\mathbb{H}^3$ and R is the scaling parameter. The geodesic distance $d(x, y)$ is just $|\mathbf{x} - \mathbf{y}|$ for $\mathbb{R}^D$ and will be explicitly given for hyperbolic spaces by the formula (4.227). Although the notion of heat kernel can be defined on any Riemannian manifold, the explicit formulae exist only for some special class of manifolds, which have enough symmetries, such as hyperbolic spaces $\mathbb{H}^D$ [57, 58, 59] or some noncompact symmetric spaces [60].

When we do not have an exact expression of the heat kernel on an arbitrary manifold $\mathcal{M}$, any knowledge about it, such as its short and/or long time behavior, and its upper and/or lower bound can help in solving the various problems about the spectrum of the Laplacian and the local and/or global geometrical properties of the manifold in question. If we compare the explicit form of the heat kernels (3.26) and

[4]The geodesic distance $d(x, y)$ is defined as the infimum of the lengths of all admissible curves from x to y, where $x, y \in \mathcal{M}$.

(3.27), we notice that there are certain similarities between them. First of all, they depend only on x and y via the geodesic distance $d(x,y)$ and time t. The gaussian factor $\exp\left(-\frac{md(x,y)^2}{2\hbar t}\right)$ appears for $\mathbb{R}^D$ and $\mathbb{H}^D$ and it may reflect the structure of the heat equation rather than the geometry and we will see that this factor will always appear when we try to estimate the heat kernel on several classes of manifolds.

3.4. Some Properties of the Heat Kernel

If $\mathcal{M}$ is a compact manifold, then as a consequence of the spectral theorem, the heat kernel can be expanded $K_t(x,y;g) = \sum_{l\geq 0} a_l(x,t)\ f_l(y;g)$ for fixed x and t. Thus, $a_l(x,t) = \int_{\mathcal{M}} \mathrm{d}_g^D y\ K_t(x,y;g) f_l(y;g)$. Then, we have

$$\begin{aligned} \hbar\partial_t a_l(x,t) &= \hbar\partial_t \int_{\mathcal{M}} \mathrm{d}_g^D y\ K_t(x,y;g) f_l(y;g) \\ &= \frac{\hbar^2}{2m}\int_{\mathcal{M}} \mathrm{d}_g^D y\ \nabla_{g,y}^2 K_t(x,y;g) f_l(y;g) \\ &= \frac{\hbar^2}{2m}\int_{\mathcal{M}} \mathrm{d}_g^D y\ K_t(x,y;g) \nabla_{g,y}^2 f_l(y;g) \\ &= -\frac{\hbar^2\sigma_l}{2m}\int_{\mathcal{M}} \mathrm{d}_g^D y\ K_t(x,y;g) f_l(y;g) \\ &= -\frac{\hbar^2\sigma_l}{2m} a_l(x,t)\ . \end{aligned} \tag{3.28}$$

This leads to the solution $a_l(x,t) = k_l(x) e^{-\frac{\hbar t}{2m}\sigma_l}$. We can express any function $f \in L^2(\mathcal{M})$ as $f(x) = \sum_{l\geq 0} c_l f_l(x;g)$. Then,

$$\begin{aligned} f(x) &= \lim_{t\to 0^+} \int_{\mathcal{M}} \mathrm{d}_g^D y K_t(x,y;g) f(y) \\ &= \lim_{t\to 0^+} \int_{\mathcal{M}} \mathrm{d}_g^D y \sum_{l=0}^{\infty} e^{-\frac{\hbar t}{2m}\sigma_l} k_l(x) f_l(y;g) \sum_{m=0}^{\infty} c_m f_m(y;g) \\ &= \lim_{t\to 0^+} \sum_{l=0}^{\infty} e^{-\frac{\hbar t}{2m}\sigma_l} k_l(x) c_l = \sum_{l=0}^{\infty} c_l k_l(x)\ , \end{aligned} \tag{3.29}$$

from which we conclude that $k_l(x) = f_l(x; g)$. Thus, we obtain the following expansion of the heat kernel

$$K_t(x, y; g) = \sum_{l=0}^{\infty} e^{-\frac{\hbar t}{2m}\sigma_l} f_l(x; g) f_l(y; g) \ , \tag{3.30}$$

which converges uniformly on $\mathcal{M} \times \mathcal{M}$ (cf. Chapter 3 in [49] for the proof of uniform convergence). This tells us that if we can calculate exactly eigenfunctions and eigenvalues of a given Laplacian, we can find the heat kernel as an infinite sum. One can easily find an important result for the large time behavior of the heat kernel for compact manifolds

$$K_t(x, y; g) = \frac{1}{V(\mathcal{M})} + \sum_{l=1}^{\infty} e^{-\frac{\hbar t}{2m}\sigma_l} f_l(x; g) f_l(y; g) \ , \tag{3.31}$$

so that $K_t(x, y; g) = \frac{1}{V(\mathcal{M})} + O(e^{-\frac{\hbar t}{2m}\sigma_1})$ or

$$K_t(x, y; g) \to \frac{1}{V(\mathcal{M})} \ , \tag{3.32}$$

as $t \to \infty$. The analog of the eigenfunction expansion of the heat kernel on noncompact manifolds can be heuristically given as

$$K_t(x, y; g) = \int \mathrm{d}\mu(l) e^{-\frac{\hbar t}{2m}\sigma(l)} f_l(x; g) f_l(y; g) \ . \tag{3.33}$$

Let us now state the important basic properties of the heat kernel without going into details:

(i) Heat Equation: It satisfies heat equation since it is a fundamental solution to it by definition.

$$\hbar \frac{\partial K_t(x, y; g)}{\partial t} - \frac{\hbar^2}{2m} \nabla^2_{g,x} K_t(x, y; g) = 0 \ . \tag{3.34}$$

(ii) Initial condition: It solves the Cauchy problem, that is,

$$\lim_{t\to 0^+} K_t(.,y;g) = \delta_g(.,y) \ , \tag{3.35}$$

in the sense of distributions.

(iii) Semi-group Property: It is the consequence of the operator identity $e^{\frac{\hbar(t_1+t_2)}{2m}\nabla_g^2} = e^{\frac{\hbar t_1}{2m}\nabla_g^2}\, e^{\frac{\hbar t_2}{2m}\nabla_g^2}$, that is,

$$\int_{\mathcal{M}} \mathrm{d}_g^D z \; K_{t_1}(x,z;g) K_{t_2}(z,y;g) = K_{t_1+t_2}(x,y;g) \ , \tag{3.36}$$

for all $x, y \in \mathcal{M}$ and $t_1, t_2 > 0$.

(iv) Symmetry Property: Although it is not so obvious from the definition of the heat kernel, it can be directly seen from the eigenfunction expansion of the heat kernel (3.30) that we have

$$K_t(x,y;g) = K_t(y,x;g) \quad \text{for all } x, y \in \mathcal{M} \ \text{ and } \ t \geq 0 \ . \tag{3.37}$$

(v) Positivity Property: The properties of the heat kernel above follows mainly from spectral properties of the Laplacian. However, there are other properties which can not be derived only from the spectral properties of the Laplacian. One such property is the positivity

$$K_t(x,y;g) > 0 \quad \text{for all } x, y \in \mathcal{M} \ \text{ and } \ t > 0 \ , \tag{3.38}$$

which is not at all obvious from the eigenfunction expansion since they are signed except the first eigenfunction. The proof of this property is based on maximum/minimum principle [34, 61].

(vi) Another nontrivial property is

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \; K_t(x,y;g) \leq 1 \; . \tag{3.39}$$

If the inequality is saturated by 1, then the manifold is said to be stochastically complete, that is,

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \; K_t(x,y;g) = 1 \; . \tag{3.40}$$

This means that in terms of the probabilistic interpretation of the heat kernel, the total probability of the particle being found in $\mathcal{M}$ is one, that is, it reflects the conservation of probability. This property of the heat kernel is related to the asymptotic completeness of the manifold [5] . A stochastically complete manifold does not have to be geodesically complete [63]. The well-known Hopf-Rinow theorem [62] provides a simple criterion to determine whether a Riemannian manifold is geodesically complete or not. It states that a connected Riemannian manifold is geodesically complete if and only if it is complete as a metric space. On a geodesically complete manifold, the Laplacian is essentially self-adjoint in $L^2(\mathcal{M})$ [64]. The natural question one may then ask is this: do we have a test for stochastic completeness. The answer is affirmative and given by the following theorem. Let $\mathcal{M}$ be a geodesically complete manifold. Assume that, for some point $x \in \mathcal{M}$,

$$\int^{\infty} \mathrm{d}r \; \frac{r}{\ln V(x,r)} = \infty \; , \tag{3.41}$$

where $V(x,r) = \mu(B(x,r))$ is the volume of the geodesic ball $B(x,r)$ on $\mathcal{M}$ of radius r centered at x [6] . Then $\mathcal{M}$ is stochastically complete. The condition (3.41) holds if $V(x,r) \leq c_1 e^{c_2 r^2}$, where c_1, c_2 are appropriate dimensional constants. When the manifold $\mathcal{M}$ is a geodesically complete Riemannian manifold with Ricci curvature

[5] A Riemannian manifold is geodesically complete if every maximal geodesic is defined for all t [62]. Intuitively, the geodesics may be continued indefinitely and the manifold does not have any holes or boundary.

[6] $B(x,r) = \{y \in \mathcal{M} : d(x,y) < r\}$, where $d(x,y)$ is the geodesic distance between x and y. μ is the Riemannian volume or measure.

$Ric(\mathcal{M})$ [7] bounded from below, then from the Bishop-Gromov volume comparison theorem we have $V(x,r) \leq c_3 e^{c_4 r}$, so that $\mathcal{M}$ is stochastically complete [61, 65]. In particular, for $\mathbb{R}^D$, $\mathbb{H}^D$ and for compact manifolds stochastic completeness is always satisfied [66]. A manifold $\mathcal{M}$ is called a Cartan-Hadamard manifold [61] if $\mathcal{M}$ is a geodesically complete, simply connected, non-compact Riemannian manifold with non-positive sectional curvature everywhere. The D-dimensional flat $\mathbb{R}^D$ and hyperbolic spaces $\mathbb{H}^D$ are the best known examples of Cartan-Hadamard manifolds. A Cartan-Hadamard manifold is stochastically complete provided its sectional curvature (c.f. [62] for the definition) is bounded below [63].

In what follows, assuming the stochastic completeness one can derive how the heat kernel transforms under the scaling transformation of the metric $g_{ij} = \alpha^{-2}\hat{g}_{ij}$, where $\alpha \in \mathbb{R}$. From the definition of the Laplacian (3.7) and the Riemannian volume element (3.2), we have

$$\begin{aligned} \nabla^2_{\alpha^{-2}\hat{g}} &= \alpha^2 \nabla^2_{\hat{g}} \ , \\ \mathrm{d}^D_{\alpha^{-2}\hat{g}} x &= \alpha^{-D} \mathrm{d}^D_{\hat{g}} x \ , \end{aligned} \tag{3.42}$$

and then the eigenvalue problem becomes

$$\nabla^2_g f_l(x;g) = \nabla^2_{\alpha^{-2}\hat{g}} f_l(x;\alpha^{-2}\hat{g}) = \alpha^2 \nabla^2_{\hat{g}} f_l(x;\alpha^{-2}\hat{g}) = \sigma_l f_l(x;\alpha^{-2}\hat{g}) \ . \tag{3.43}$$

Hence, this tells us that the spectrum of the new scaled Laplacian $\nabla^2_{\hat{g}}$ is $\{\alpha^{-2}\sigma_l\}$. Scaling property of the eigenfunctions is obtained from their orthogonality or completeness condition and we get

$$f_l(x;\hat{g}) = \alpha^{-D/2} f_l(x;\alpha^{-2}\hat{g}) = \alpha^{-D/2} f_l(x;g) \ , \tag{3.44}$$

that is, $\{\alpha^{-D/2} f_l(x;g)\}$ form a orthonormal basis of $L^2(\mathcal{M};\alpha^2 g)$ if $f_l(x;g)$ is an orthonormal basis of $L^2(\mathcal{M};g)$. Also, from the normalization of Dirac-Delta functions

[7] The components of Ric are denoted by R_{ij} and defined in terms of the Riemann curvature tensor: $R_{ij} := g^{km} R_{kijm}$.

on manifolds (3.14) and the transformation rule of the Riemannian volume element in (3.42), we easily find

$$\delta_g(x) = \delta_{\alpha^{-2}\hat{g}}(x) = \alpha^D \delta_{\hat{g}}(x) \ . \tag{3.45}$$

The heat kernel satisfies heat equation under scaling transformation of the metric, that is,

$$\hbar \frac{\partial K_t(x,y;\alpha^{-2}\hat{g})}{\partial t} = \frac{\hbar^2}{2m} \nabla^2_{\alpha^{-2}\hat{g}} K_t(x,y;\alpha^{-2}\hat{g}) = \alpha^2 \frac{\hbar^2}{2m} \nabla^2_{\hat{g}} K_t(x,y;\alpha^{-2}\hat{g}) \ . \tag{3.46}$$

If we also scale the time $t = s/\alpha^2$, then we find

$$\hbar \frac{\partial K_{s/\alpha^2}(x,y;\alpha^{-2}\hat{g})}{\partial s} - \frac{\hbar^2}{2m} \nabla^2_{\hat{g}} K_{s/\alpha^2}(x,y;\alpha^{-2}\hat{g}) = 0 \ . \tag{3.47}$$

Therefore $K_{s/\alpha^2}(x,y;\alpha^{-2}\hat{g})$ satisfies the same form of heat equation with the time parameter s and the metric $\hat{g}$. In other words, this solution is proportional to $K_s(x,y;\hat{g})$ but up to an overall factor which could depend on the scaling parameter α due to the linearity of heat equation:

$$K_{s/\alpha^2}(x,y;\alpha^{-2}\hat{g}) = N_\alpha K_s(x,y;\hat{g}) \ . \tag{3.48}$$

Scaling back to original time variable we obtain $K_t(x,y;\alpha^{-2}\hat{g}) = N_\alpha K_{\alpha^2 t}(x,y;\hat{g})$. The overall constant N_α can be determined from the stochastic completeness condition of the heat kernel (3.40). Hence we get

$$K_{\alpha^2 t}(x,y;\hat{g}) = \alpha^{-D} K_t(x,y;\alpha^{-2}\hat{g}) \ , \tag{3.49}$$

equivalently

$$K_t(x,y;g) = \alpha^D K_{\alpha^2 t}(x,y;\alpha^2 g) \ . \tag{3.50}$$

Another useful and important fact about the heat kernel is its relation with the resolvent. It was first realized by Fock [67] that it is useful to express the resolvent kernel or the Green's function as an integral over a time variable (so-called "proper time") of the heat kernel and later on Schwinger [68] showed that this idea made many issues related to the renormalization and gauge invariance more transparent. In order to see the relation between the resolvent kernel and heat kernel, we need the following operator identity, which can be proven rigorously in semigroup theory [56]

$$\left(-\frac{\hbar^2}{2m}\nabla_g^2 - E\right)^{-1} = \frac{1}{\hbar}\int_0^\infty \mathrm{d}t\; e^{-\frac{t}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla_g^2 - E\right)}, \tag{3.51}$$

for $\Re(E) < 0$. This shows that the resolvent of the free Hamiltonian $-\frac{\hbar^2}{2m}\nabla_g^2$ is nothing but the Laplace transform of the semigroup $e^{\frac{\hbar t}{2m}\nabla_g^2}$. In fact, the above operator identity is not restricted to the Laplacian but valid for any linear self-adjoint operator [56]. The result should be continued analytically to its largest set in the entire complex plane E. As a result we find the integral representation of the free resolvent kernel in terms of the heat kernel

$$R_0(x, y|E) = \langle x| \left(-\frac{\hbar^2}{2m}\nabla_g^2 - E\right)^{-1} |y\rangle = \frac{1}{\hbar}\int_0^\infty \mathrm{d}t\; e^{\frac{Et}{\hbar}} K_t(x, y; g). \tag{3.52}$$

3.5. Short-Time Asymptotics of the Heat Kernel

Apart from these facts, there is a well-known short-time asymptotic of the diagonal heat kernel given for any manifolds [29, 49]

$$K_t(x, x; g) \sim \frac{1}{(4\pi\hbar t/2m)^{D/2}} \sum_{k=0}^{\infty} u_k(x, x)(\hbar t/2m)^{k/2}\,, \tag{3.53}$$

for every $x \in \mathcal{M}$ as $t \to 0^+$. Here the functions $u_k(x, x)$ are scalar polynomials in curvature tensor of the manifold and its covariant derivatives at the point x. When there is no boundary, the odd terms in the expansion, i.e, $k = 1, 3, 5, ...$ vanishes [28]. Unless otherwise stated, we always assume that the manifolds in this thesis have no boundary, so that

$$K_t(x,x;g) \sim \frac{1}{(4\pi\hbar t/2m)^{D/2}} \sum_{n=0}^{\infty} u_n(x,x)(\hbar t/2m)^n \ . \tag{3.54}$$

This expansion is more or less plausible since every Riemannian manifold looks locally like $\mathbb{R}^D$ and for $t \simeq 0$ the heat flow does not have enough time to distribute itself over the whole manifold, so that all the information about heat flow can be locally computable from the well-known Euclidean result as a first approximation. The first coefficient is then $u_0(x,x) = 1$, whereas the explicit form of the others can be calculated recursively up to $u_5(x,x)$ and they are rather long and complicated [31].

The asymptotic expansion of the diagonal heat kernel can be integrated over the manifold to give the asymptotic expansion of the trace of the heat kernel as $t \to 0^+$

$$\begin{aligned} \mathrm{Tr}_{L^2} e^{\frac{t}{\hbar}\left(\frac{\hbar^2}{2m}\nabla_g^2\right)} = \int_{\mathcal{M}} \mathrm{d}_g^D x K_t(x,x;g) &= \sum_{l=0}^{\infty} e^{-\frac{\hbar t}{2m}\sigma_l} \\ &\sim \frac{1}{(4\pi\hbar t/2m)^{D/2}} \sum_{n=0}^{\infty} U_n(\hbar t/2m)^n \end{aligned} \tag{3.55}$$

where the coefficients $U_n = \int_{\mathcal{M}} \mathrm{d}_g^D x \ u_n(x,x)$ are spectral invariants of the operator $-\nabla_g^2$ and we have used (3.30) and the orthogonality of the eigenfunctions. Integration of an asymptotic expansion is still an asymptotic expansion since they can be integrated term by term [69]. This expansion (3.55) is known as Minakshisundaram-Pleijel formula [70, 71] in mathematics literature and Schwinger-DeWitt expansion in physics literature and the coefficients are called Hadamard - Minakshisundaram - DeWitt - Seeley (HMDS) coefficients. The short-time expansion (3.54) is even valid for self-adjoint elliptic pseudo differential operators P of any order d [29]

$$K_t(x,x;g) \sim \frac{1}{(4\pi\hbar t/2m)^{D/d}} \sum_{n=0}^{\infty} e_n(x,x)(\hbar t/2m)^{n/d} \ , \tag{3.56}$$

where $e_n(x,x)$ depends functionally on a finite number of jets of the symbol $p(x,\xi)$. The asymptotic expansion of the heat kernel for a formally self-adjoint, elliptic, second order operator for the points x near y is also given [72] as

$$K_t(x,y;g) \sim \frac{e^{-\frac{md^2(x,y)}{2\hbar t}}}{(4\pi\hbar t/2m)^{D/2}} \sum_n a_n(x,y)(\hbar t/2m)^n \ . \tag{3.57}$$

as $t \to 0^+$.

Indeed, we have also short-time asymptotic of the heat kernel for any x and y under the various assumptions about the structure of the set of geodesics which join the points x and y [73]. It is shown that (see Theorem 2.1 and 2.2 in [73]) for all y sufficiently close to x (so that x and y can be joined by a unique shortest geodesic $\gamma_{x,y}$ along which x and y are non-conjugate) then,

$$K_t(x,y;g) \sim \frac{e^{-\frac{md^2(x,y)}{2\hbar t}}}{(4\pi\hbar t/2m)^{D/2}} d^{(D-1)/2}(x,y)\Psi_\gamma^{-1/2}(x,y) \ , \tag{3.58}$$

$\Psi_\gamma(x,y)$ characterizes the divergence of the geodesic flow near γ, that is, if we emit a beam of geodesics from x along γ in the solid angle $\mathrm{d}\varphi$ illuminating a hypersurface of area $\mathrm{d}S$ at y orthogonal to γ, then $\Psi(x,y) = \mathrm{d}S/\mathrm{d}\varphi$. The function $\Psi(x,y)$ can also be written in terms of the Jacobi fields orthogonal to the geodesic $\gamma_{x,y}(s)$, where $0 \le s \le d(x,y)$. If the number of shortest geodesics joining x and y is greater than 1, or, if x and y are conjugate along some of them, then the result takes the following form (up to a bounded factor)

$$K_t(x,y;g) = O\left((\hbar t/2m)^{-\frac{(D+k)}{2}} e^{-\frac{md^2(x,y)}{2\hbar t}} \right) \tag{3.59}$$

where the index $k = k(x,y)$ depends on the character of the degeneracy of the geodesic flow between x and y and the symbol O stands for the Big O notation in asymptotic analysis. Since all our calculations essentially give the same physical results for all cases and subcases defined in [73], we will just consider a generic case (Case 3.1 in

[73]), in which we have:

$$K_t(x,y;g) \sim \frac{e^{-\frac{md^2(x,y)}{2\hbar t}}}{(4\pi\hbar t/2m)^{D/2}} d^{(D-1)/2}(x,y) \sum_i \Psi_i^{-1/2}(x,y) \ , \tag{3.60}$$

where each of the shortest geodesics γ_i gives an independent contribution to the sum.

3.6. Upper and Lower Bound Estimates of the Heat Kernel

The Gaussian upper bound of the heat kernel was first given by [74] for the case of complete Riemannian manifold of a bounded sectional curvature. The sharper results were later obtained by [75] and known as Li-Yau estimates: Let $\mathcal{M}$ be a geodesically complete manifold without boundary. If the Ricci curvature of $\mathcal{M}$ is nonnegative, then for all $\delta > 0$, there exists $c_\delta > 0$ such that

$$K_t(x,y;g) \leq c_\delta \left(V(x, \sqrt{\hbar t/2m}) \ V(y, \sqrt{\hbar t/2m}) \right)^{-1/2} \ \exp\left(-\frac{md(x,y)^2}{4(1+\delta)\hbar t} \right) \ , \tag{3.61}$$

for all $t > 0$ and $x, y \in \mathcal{M}$. A similar estimate is also given for lower bound and one can think of the above inequality as two sided. If one assumes that the Ricci curvature of the manifold is bounded below by a negative constant, the above results on the bounds are still valid. One point should be emphasized. Most of the information about the geometry of the manifold is hidden in the volume terms in front of the Gaussian factor of the above estimate and this is a general characteristic. Moreover, the studies of E.B. Davies [76] and B. Simon [77] show that the upper bounds of the heat kernel can be also deduced from the logarithmic Sobolev inequality [78]. Several alternative functional inequalities, e.g., Sobolev's, Nash's, and the Faber-Krahn's inequalities have been shown to be related on-diagonal upper bounds of the heat kernel [61]. The common achievement of the above inequalities are all equivalent to the statement that on-diagonal bound of the heat kernel is

$$K_t(x,x;g) \leq c/t^{D/2} \ , \quad c > 0 \ , \tag{3.62}$$

for all $t > 0$ and $x \in \mathcal{M}$ and then the off-diagonal estimates can be done by modifying it with a Gaussian exponential form. We will show in the next section that under certain conditions an upper bound of the heat kernel can be deduced from its on-diagonal upper bound. At the same time there is another direct way of obtaining on-diagonal estimates of the heat kernel [79] and we will follow this approach in the next section by considering topologically two different class of manifolds, namely compact and Cartan-Hadamard manifolds.

3.6.1. Heat Kernel Bounds for Compact Manifolds

(i) On-diagonal Upper Bound of the Heat Kernel: The following result is a simplified version of the corollary 3.6 given in [80], which assumes that some geometrical conditions must hold on the boundary. The global upper bound estimate of the diagonal heat kernel in [80] includes whole boundary information via an explicitly calculable strictly positive constant $A \equiv A(\mathrm{diam}(\mathcal{M}), H_1, H_2, K, V(\mathcal{M}))$, where $\mathrm{diam}(\mathcal{M})$ is the diameter of the manifold [8], and K is the lower bound on the Ricci curvature, and also H_1 and H_2 are parameters related to boundary conditions. We then state the following corollary by safely removing the boundary effects in corollary 3.6 in [80] since A is strictly positive:

Let $\mathcal{M}$ be a compact manifold. Suppose that the Ricci curvature of $\mathcal{M}$ satisfies $\mathrm{Ric}_{\mathcal{M}} \geq -K, K \geq 0$. Then $\forall t > 0$ and $x \in \mathcal{M}$

$$K_t(x, x; g) \leq \frac{1}{V(\mathcal{M})} + A'(\hbar t/2m)^{-D/2} \,, \tag{3.63}$$

where $A' \equiv A'(\mathrm{diam}(\mathcal{M}), K, V(\mathcal{M}))$.

(ii) Upper Bound of the Heat Kernel: The natural question is whether we could extend the above result to the off-diagonal of the heat kernel. If the answer is affirmative, how? In order to answer these questions, we must first observe the

[8] $\mathrm{diam}(\mathcal{M}) = \sup\{d(x, y) : x, y \in \mathcal{M}\}$

following identity

$$K_t(x,x;g) = \int_{\mathcal{M}} \mathrm{d}_g^D z \; K_{t/2}(x,z;g) K_{t/2}(z,x;g) = \int_{\mathcal{M}} \mathrm{d}_g^D z \; K_{t/2}^2(x,z;g) \;, \qquad (3.64)$$

where we have used the semigroup (3.36) and symmetry property of heat kernel (3.37). Using the semigroup property again with Cauchy-Schwarz inequality, we have

$$\begin{aligned} K_t(x,y;g) &= \int_{\mathcal{M}} \mathrm{d}_g^D z \; K_{t/2}(x,z;g) K_{t/2}(y,z;g) \\ &\leq \left[\int_{\mathcal{M}} \mathrm{d}_g^D z \; K_{t/2}^2(x,z;g) \right]^{1/2} \left[\int_{\mathcal{M}} \mathrm{d}_g^D y \; K_{t/2}^2(y,z;g) \right]^{1/2} . \end{aligned} \qquad (3.65)$$

Then, as a result of (3.64), we obtain

$$K_t(x,y;g) \leq K_t^{1/2}(x,x;g) K_t^{1/2}(y,y;g) \; . \qquad (3.66)$$

This inequality immediately gives us an upper bound on the off-diagonal heat kernel if we know an upper bound estimate of the diagonal heat kernel. For example, if $K_t(x,x;g) \leq F(t)$ which is valid for any point $x \in \mathcal{M}$ and $t > 0$, then (3.66) implies $K_t(x,y;g) \leq F(t)$ for all $x \in \mathcal{M}$ with $t > 0$. However, this is a poor estimate, that is, it does not take into account the information of the distance between the points x and y so this would not be a sharp bound, which we need for our calculations. Therefore, we will search for the upper bound of the off-diagonal heat kernel estimates which does not ignore the distance between the points and it will be a correction to the upper bound of the diagonal heat kernel, based on the diagonal estimate. It has been established that the corrections to the upper bounds do not depend on the specific geometry of the manifold. Instead, the upper bounds are characteristic of heat equation which implies that we must have a Gaussian factor somehow in the estimate. The following corollary given in [81] constrains the off-diagonal elements of the heat kernel from above.

Assume that for some points $x, y \in \mathcal{M}$ ($\mathcal{M}$ is any Riemannian manifold) and

$\forall t > 0$,

$$K_t(x,x;g) \le \frac{C}{f(t)} \text{ and } K_t(y,y;g) \le \frac{C}{g(t)} , \tag{3.67}$$

where f and g are increasing positive functions on $(0,\infty)$ satisfying the regularity condition given below. Then, for any $C_2 > 2$ and for all $t > 0$

$$K_t(x,y;g) \le \frac{4A}{\sqrt{f(\varepsilon\, t)g(\varepsilon\, t)}} \exp\left(-\frac{md^2(x,y)}{\hbar C_2 t}\right) , \tag{3.68}$$

where $\varepsilon = \varepsilon(C_2, a)$, A and a are the constants coming from the regularity condition below.

Regularity Condition: There are numbers $A \ge 1$ and $a > 1$ such that

$$\frac{f(as)}{f(s)} \le A\frac{f(at)}{f(t)} , \tag{3.69}$$

for all $0 < s < t$.

By comparing the equations (3.63) and (3.67), we realized that the right hand side of (3.63) can be an explicit candidate for the functions f or g in the theorem above. Hence, we could have

$$f(t) = g(t) = \left[\frac{1}{V(\mathcal{M})} + A'(\hbar t/2m)^{-D/2}\right]^{-1} , \tag{3.70}$$

by choosing $C = 1$. It is easy to check that these functions are positive and increasing. We can also verify that they satisfy the regularity condition (3.69) with $A = a^{D/2}$. Therefore, we have obtained the upper bound for the off-diagonal elements of the heat kernel

$$K_t(x,y;g) \le 4A\left[\frac{1}{V(\mathcal{M})} + B(\varepsilon)(\hbar t/2m)^{-D/2}\right] \exp\left(-\frac{md^2(x,y)}{\hbar C_2 t}\right) , \tag{3.71}$$

where $B(\varepsilon) = A'\varepsilon^{-D/2}$ and recall that $C_2 > 2$.

iii) Lower Bound of the Heat Kernel:

We have a direct theorem about the lower bound on the heat kernel (cf. theorem 5.6.1 in [33]). Let $\mathcal{M}$ be a complete Riemannian manifold with $\mathrm{Ric}_{\mathcal{M}} \geq 0$. Then, we have

$$K_t(x,y;g) \geq (4\pi\hbar t/2m)^{-D/2} \exp\left(-\frac{md^2(x,y)}{2\hbar t}\right) , \tag{3.72}$$

for all $x, y \in \mathcal{M}$ and $t > 0$. In particular, on-diagonal lower bound is

$$K_t(x,x;g) \geq (4\pi\hbar t/2m)^{-D/2} . \tag{3.73}$$

3.6.2. Heat Kernel Bounds for Cartan-Hadamard Manifolds

i) Upper Bound of the Heat Kernel: In order to give an upper bound for Cartan-Hadamard manifolds, we need to give the some definitions and related theorems in the literature.

Isoperimetric Inequalities: Isoperimetric inequalities are the relations between the boundary area of regions and their volume. We say that manifold $\mathcal{M}$ admits the isoperimetric function I if for any precompact open set $\Omega \subset \mathcal{M}$ with smooth boundary

$$A(\partial\Omega) \geq I(v) , \tag{3.74}$$

where $A(\partial\Omega)$ is the area of boundary of the region Ω, and $v = V(\Omega)$ is the volume of the region. Any Cartan-Hadamard manifold $\mathcal{M}$ of dimension D admits the isoperimetric function $I(v) = \kappa v^{\frac{D-1}{D}}$, $\kappa > 0$ [82]. We have an important theorem [83] given below:

Assume that the manifold $\mathcal{M}$ admits a non-negative continuous isoperimetric function $I(v)$ such that $I(v)/v$ is non-increasing. Let us define the function $f(t)$ by

$$t = 4\int_0^{f(t)} \mathrm{d}v \frac{v}{I^2(v)} \,, \tag{3.75}$$

assuming that the integral is not divergent at $t = 0$. Then for all $x \in \mathcal{M}$, $t > 0$ and $\varepsilon > 0$,

$$K_t(x, x; g) \leq \frac{2\varepsilon^{-1}}{f((1-\varepsilon)t)} \,. \tag{3.76}$$

Moreover, if the function f satisfies in addition the regularity condition (3.69) then, for all $x, y \in \mathcal{M}, t > 0, C_2 > 2$ and some $\varepsilon > 0$,

$$K_t(x, y; g) \leq \frac{4A}{f(\varepsilon t)} \exp\left(\frac{-md^2(x, y)}{\hbar C_2 t}\right) \,. \tag{3.77}$$

The isoperimetric function for Cartan-Hadamard manifolds given above satisfies all the requirements above, that is, it is a non-negative continuous function and $I(v)/v$ is non-increasing. Substituting the above isoperimetric function for Cartan-Hadamard manifolds into (3.75) we obtain the function $f(t)$ by a simple integration

$$f(t) = \left(\frac{\kappa^2}{2D} t\right)^{D/2} \,, \tag{3.78}$$

and it meets all the requirements of this theorem including the regularity condition. Hence, the upper bound on the off-diagonal elements of the heat kernel on Cartan-Hadamard manifolds is given by

$$K_t(x, y; g) \leq \frac{C(\varepsilon, \kappa)}{(4\pi\hbar t/2m)^{D/2}} \exp\left(\frac{-md^2(x, y)}{\hbar C_2 t}\right) \,, \tag{3.79}$$

where $C(\varepsilon, \kappa) = \frac{4A}{(\kappa^2\varepsilon/2D)^{D/2}}$. This upper bound estimate of the heat kernel is also valid for minimal submanifolds, which are submanifolds of $\mathbb{R}^D$ whose normal mean curvature vector $H(x) = (H_1(x), H_2(x), ..., H_D(x))$ vanishes for all $x \in \mathcal{M}$, because minimal submanifolds admit the same form of isoperimetric function [84] with the

Cartan-Hadamard manifolds.

ii) The Lower Bound of the Diagonal Heat Kernel: On Cartan-Hadamard manifolds, the lower bound of the diagonal elements of the heat kernel are obtained in [61]. The lower bound of the diagonal heat kernel is given also for any manifold given as a proposition (5.14) in [61]: For any manifold $\mathcal{M}$, for any $x \in \mathcal{M}$ and $\delta > 0$, there exists $c = c_x > 0$ such that

$$K_t(x, x; g) \geq \frac{c}{(4\pi\hbar t/2m)^{D/2}} \exp\left[-(\sigma_1(\mathcal{M}) + \delta)\frac{\hbar t}{2m}\right] \, , \tag{3.80}$$

where $\sigma_1(\mathcal{M})$ is the spectral radius corresponding to the Laplacian and it is restricted to the following range [34, 85]

$$\frac{1}{4}(D-1)^2 K_{max}^2 \geq \sigma_1(\mathcal{M}) \geq \frac{1}{4}(D-1)^2 K_{min}^2 \, , \tag{3.81}$$

for a Cartan-Hadamard manifold whose sectional curvature is bounded from above by $-K_{min}^2$.

4. FINITELY MANY POINT INTERACTIONS ON TWO AND THREE DIMENSIONAL RIEMANNIAN MANIFOLDS

4.1. Quantization on Manifolds

The classical Hamiltonian of a free particle moving in a curved space with metric g is given by $H = \frac{1}{2} g^{ij}(x) p_i p_j$, where g^{ij} is the inverse of the metric g_{ij}. If we quantize the system with the usual canonical quantization rule, we are confronted with ordering ambiguity. The difficulty is basically due to the reason that the corresponding principle and hermiticity condition of the Hamiltonian do not alone determine how the classical variables have to be ordered before being replaced by operators in canonical quantization scheme. The problem still can not be resolved in path integral quantization scheme and various path integral formulations of the problem suggest [86, 87, 88, 89, 90, 91, 92] us that the Hamiltonian for a particle in curved space must be in a local coordinate representation

$$\langle x | H_0 = \left(- \frac{\hbar^2}{2m} \nabla_g^2 + \xi \frac{\hbar^2}{m} R(x) \right) \langle x| \; , \tag{4.1}$$

where ξ can take several possible values and R is the Ricci scalar curvature of the manifold. Therefore, we have inequivalent Hamiltonians which differ from each other by a factor of $\frac{\hbar^2 R}{m}$. This issue has been first raised by DeWitt [86, 87, 88, 89] and there is still no consensus regarding the quantization in curved spaces. On the other hand, one can ask naturally whether this extra term is experimentally observable since only experiment can decide what the correct value of ξ must be. Up to now, no experimental result has shown the presence of this extra curvature term [91]. Therefore, we shall choose $\xi = 0$ for simplicity for our calculations and assume that there is no physical observable effects by removing this term. This ensures us that the spectrum of the free Hamiltonian is nonnegative. Otherwise, we will need a more detailed analysis which takes into account the negative eigenvalues of the free Hamiltonian. This ambiguity is not only encountered in non-relativistic quantum mechanics, but also exists in quantum field theory [72].

One more point should also be emphasized about the quantization in curved spaces or manifolds: In quantum mechanics, there is an alternative approach, called confining potential approach [93, 94, 95, 96], which investigate the dynamics of the particle constrained to a manifold. In this way, a particle constrained on a hypersurface which is embedded in a Euclidean space $\mathbb{R}^D$ is studied and embedding is assumed a priori. Then one can find an effective Hamiltonian approach for the particle moving on the hypersurface by freezing the motion normal to the surface in a low state of excitation of the confining potential. This effective Hamiltonian depends on the intrinsic geometry and on how the hypersurface is embedded in $\mathbb{R}^D$. However, we will pursue the first approach and postpone studying our problem in the second approach for future studies, that is, we shall assume that the motion is constrained to the manifold a priori so that we do not have to worry about the embedding space. All the properties of the system depend only on the geometry intrinsic to the manifold.

4.2. A Heuristic Renormalization of the Model on Riemannian Manifolds

We consider a non-relativistic point particle of mass m living in D dimensional Riemannian manifold $(\mathcal{M}, g)$ without boundary which interacts with finite number of Dirac delta interactions located on the manifold. For simplicity, we shall only deal with the bound state problem. Unless otherwise stated, we always consider two and/or three dimensional problems, that is $D = 2$ and $D = 3$. Following the arguments given in the previous Section 4.1, the time-independent Schrödinger equation on $\mathcal{M}$ for the bound states of a particle under the influence of N attractive Dirac delta interactions reads

$$\left[-\frac{\hbar^2}{2m} \nabla_g^2 - \sum_{i=1}^{N} \lambda_i \, \delta_g^D(x, a_i) \right] \psi(x) = -\nu^2 \psi(x) \; , \tag{4.2}$$

where $\lambda_i \in \mathbb{R}^+$ is the strength of the i-th Dirac delta interaction at point $a_i \in \mathcal{M}$ and we have parametrized the bound state energy E of the system by $-\nu^2$. We again assume that $a_i \neq a_j$ for $i \neq j$. Although we show the calculations for only compact manifolds, they are still true for some non-compact manifolds by appropriately replacing the sums by integrals. If we apply the spectral theorem (3.11), it yields

$$\sum_{l=0}^{\infty} \left[\frac{\hbar^2}{2m} \sigma_l \, C_l - \sum_{i=1}^{N} A_i \, \lambda_i \, f_l(a_i; g) + \nu^2 C_l \right] f_l(x; g) = 0, \tag{4.3}$$

where $A_i \equiv \psi(a_i)$ for simplicity of notation. The fact that f_l's form a complete orthonormal system allows us to solve C_l

$$C_l = \frac{1}{\frac{\hbar^2}{2m} \sigma_l + \nu^2} \sum_{i=1}^{N} A_i \, \lambda_i \, f_l(a_i; g). \tag{4.4}$$

Substituting (4.4) into the definition of A_i

$$A_i = \sum_{j=1}^{N} A_j \, \lambda_j \sum_{l=0}^{\infty} \frac{f_l(a_i; g) f_l(a_j; g)}{\frac{\hbar^2}{2m} \sigma_l + \nu^2} \, , \tag{4.5}$$

and grouping the A_i terms we find

$$\left[\lambda_i^{-1} - \sum_{l=0}^{\infty} \frac{|f_l(a_i; g)|^2}{\frac{\hbar^2}{2m} \sigma_l + \nu^2} \right] A_i - \sum_{\substack{j=1 \\ j \neq i}}^{N} \left[\frac{\lambda_j}{\lambda_i} \sum_{l=0}^{\infty} \frac{f_l(a_i; g) f_l(a_j; g)}{\frac{\hbar^2}{2m} \sigma_l + \nu^2} \right] A_j = 0 \, . \tag{4.6}$$

The observation that the preceding equation is linear in A_i permits us to write it naturally as a matrix equation

$$\sum_{j=1}^{N} \Phi_{ij}(E = -\nu^2) A_j = 0 \, , \tag{4.7}$$

where

$$\Phi_{ij}(E) = \begin{cases} \lambda_i^{-1} - \sum_{l=0}^{\infty} \frac{|f_l(a_i;g)|^2}{\frac{\hbar^2}{2m}\sigma_l + \nu^2} & \text{if } i = j \\ \\ -\frac{\lambda_j}{\lambda_i} \sum_{l=0}^{\infty} \frac{f_l(a_i;g) f_l(a_j;g)}{\frac{\hbar^2}{2m}\sigma_l + \nu^2} & \text{if } i \neq j \; . \end{cases} \tag{4.8}$$

This is nothing but a generalization of the principal matrix given (2.39) for flat spaces. We can similarly find the principal matrix for non-compact manifolds as well

$$\Phi_{ij}(E) = \begin{cases} \lambda_i^{-1} - \int \mathrm{d}\mu(l) \frac{|f_l(a_i;g)|^2}{\frac{\hbar^2}{2m}\sigma(l) + \nu^2} & \text{if } i = j \\ \\ -\frac{\lambda_j}{\lambda_i} \int \mathrm{d}\mu(l) \frac{f_l(a_i;g) f_l(a_j;g)}{\frac{\hbar^2}{2m}\sigma(l) + \nu^2} & \text{if } i \neq j \; . \end{cases} \tag{4.9}$$

Comparing (4.8) and (4.9) with (2.39), one can easily see the following similar structure

$$\begin{array}{ccc} \mathbb{R}^D & & \mathcal{M} \\ \\ \int_{\mathbb{R}^D} \frac{\mathrm{d}^D p}{(2\pi\hbar)^D} & \longleftrightarrow & \sum_{l=0}^{\infty} \text{ or } \int \mathrm{d}\mu(l) \\ \exp\left(\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{a}_i\right) & \longleftrightarrow & f_l(a_i;g) \end{array} \tag{4.10}$$

where the double arrow means that the expression in the right hand side can be replaced by the left hand side or vice versa. Because of the above association, one heuristically expects that the diagonal term in (4.8) or (4.9) diverges since their associated term in the flat space diverges for large values of momentum corresponding to short distances (so-called ultra-violet divergence). This expectation can also be explained by the following argument: As we have seen in the chapter 2, the divergent structure of our problem in flat spaces is due to the interaction of the particle with the potential at extremely short distances. The form or the character of the divergence must be of the same form with flat case since every Riemannian manifold can be considered locally flat. However, this is just a naive expectation since we have no explicit information about the eigenvalues and eigenfunctions nor their estimates for a general class of Riemannian manifolds. It is only the Weyl's asymptotic formula for the eigenvalues (3.18) that may help us to understand the structure of the divergences in our problem but it will not be sufficient since we have no information about the asymptotic behavior of the eigenfunctions on a general class of manifolds. What we can do at the moment may be to consider the special cases. Let us consider two dimensional sphere $\mathbb{S}^2$ as an explicit example. Spherical harmonics Y_l^m are the eigenfunctions of the Laplacian $-\nabla^2_{\mathbb{S}^2}$ with the eigenvalues $l(l+1)/R^2$. The principal matrix $\Phi_{ij}(-\nu^2)$ is then

$$\Phi_{ij}(-\nu^2) = \begin{cases} \lambda_i^{-1} - \dfrac{1}{4\pi R^2} \displaystyle\sum_{l=0}^{\infty} \dfrac{2l+1}{\frac{\hbar^2}{2mR^2} l(l+1) + \nu^2} & i = j \\ \\ -\dfrac{\lambda_j}{\lambda_i} \dfrac{1}{4\pi R^2} \displaystyle\sum_{l=0}^{\infty} \dfrac{2l+1}{\frac{\hbar^2}{2mR^2} l(l+1) + \nu^2} \; P_l \left(1 - \dfrac{\hat{d}_{ij}^2}{2}\right) & i \neq j; , \end{cases} \tag{4.11}$$

where $\hat{d}_{ij} = \frac{d_{ij}}{R} = |\hat{r}_i - \hat{r}_j| \in [0, 2]$ being rescaled distance between point centers with radius of the sphere R. Now, it follows easily from the Cauchy-MacLaurin integral test [38] that the infinite sum

$$\frac{1}{4\pi R^2} \sum_{l=0}^{\infty} \frac{2l+1}{\frac{\hbar^2}{2mR^2} \; l(l+1) + \nu^2} \tag{4.12}$$

is divergent, which is the expected result. Our heuristic idea can also be tested for hyperbolic manifolds [35] but we postpone discussing the problem on these special model manifolds to the end of this chapter and continue with our general discussion.

The question we address now is the following: Could we renormalize our problem by following a similar procedure developed for the flat spaces? Let us see what happens if we go through the analogous steps of the same problem in flat spaces. We introduce a cut-off to the upper bound of the infinite sum and choose the bare coupling constant in analogy with the flat case

$$\frac{1}{\lambda_i(\Lambda)} = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi\hbar)^2} \frac{\Theta_\Lambda(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} + \mu_i^2} \longleftrightarrow \frac{1}{\lambda_i(\Lambda)} = \sum_{l=0}^{\Lambda} \frac{|f_l(a_i;g)|^2}{\frac{\hbar^2}{2m}\sigma_l + \mu_i^2} , \tag{4.13}$$

where $-\mu_i^2$ in this case is the experimentally measured binding energy to a single Dirac delta interaction in $\mathcal{M}$. Then, we take the limit $\Lambda \to \infty$

$$\begin{aligned} &\lim_{\Lambda\to\infty} \left[\sum_{l=0}^{\Lambda} \frac{|f_l(a_i;g)|^2}{\frac{\hbar^2}{2m}\sigma_l + \mu_i^2} - \sum_{l=0}^{\Lambda} \frac{|f_l(a_i;g)|^2}{\frac{\hbar^2}{2m}\sigma_l + \nu^2} \right] \\ &= \left[\sum_{l=0}^{\infty} \frac{(\nu^2 - \mu_i^2)|f_l(a_i;g)|^2}{\left(\frac{\hbar^2}{2m}\sigma_l + \mu_i^2\right)\left(\frac{\hbar^2}{2m}\sigma_l + \nu^2\right)} \right] , \end{aligned} \tag{4.14}$$

and this should give us a finite result in two and three dimensions. Hence, we find the principal matrix at this limit

$$\Phi_{ij}(E) = \begin{cases} \displaystyle\sum_{l=0}^{\infty} \frac{(\nu^2 - \mu_i^2)|f_l(a_i;g)|^2}{\left(\frac{\hbar^2}{2m}\sigma_l + \mu_i^2\right)\left(\frac{\hbar^2}{2m}\sigma_l + \nu^2\right)} & \text{if } i = j \\ \\ \displaystyle -\sum_{l=0}^{\infty} \frac{f_l(a_i;g) f_l(a_j;g)}{\frac{\hbar^2}{2m}\sigma_l + \nu^2} & \text{if } i \neq j \ . \end{cases} \tag{4.15}$$

where we have used the fact $\lim_{\Lambda\to\infty} \lambda_j(\Lambda)/\lambda_i(\Lambda) = 1$. Although the final form of the principal matrix can be shown to be finite for special manifolds (such as $\mathbb{S}^2, \mathbb{H}^2, \mathbb{H}^3$), it is still not so clear that the infinite sums in the principal matrix are convergent. Therefore, this is not a smart way of doing renormalization since we do not know the convergence of the sums in a general setting so we need a new tool in order to achieve the renormalization rigorously.

There is a very elegant method, called heat kernel, which helps us to understand the singular structure of the Green's function of a given self-adjoint elliptic differential operator. It is a useful approach to calculate the one-loop effective action, which describes the quantum effects due to the background fields in the one-loop approximation of quantum field theory [68] and it can also be applied to the case of background gravitational field, i.e., the quantum field theory on curved space-times [31, 97, 98, 99]. With the help of the asymptotic expansion of the heat kernel, one can then find the singular structure of the problem and renormalize the problem in curved space-times. As introduced in the first section of this chapter, our approach is to renormalize the systems in Riemannian manifolds by finding a well-defined finite resolvent of the formal Hamiltonian. Therefore, our motivation in the next section will be to find a relation between the resolvent of the Hamiltonian and the heat kernel in order to extract the singular structure of the problem.

4.3. Construction of the Model with the Heat Kernel

The resolvent and heat kernel play an essential role in establishing the connection between spectral properties of a differential operator and the geometrical properties of the space on which this operator is defined. Therefore, it seems reasonable to look for a relation between the resolvent of the Hamiltonian and the heat kernel.

Let us consider the separable Hamiltonians $H^{\epsilon} = H_0 - \sum_{i=1}^{N} |f_i^{\epsilon}\rangle\langle f_i^{\epsilon}|$, where $|f_i^{\epsilon}\rangle$ is a particular set of normalized vectors specified by its coordinate wave function $f_i^{\epsilon}(x)$. We shall try to work out the resolvent formula of H in terms of the resolvent of the free Hamiltonian H_0. For this reason, let us assume that the two Dirac kets $|\psi\rangle$ and $|\rho\rangle$ are related in such a way that the equality $(H^{\epsilon} - E)|\psi\rangle = |\rho\rangle$ is satisfied, that is,

$$\left[H_0 - E - \sum_{j=1}^{N} |f_j^{\epsilon}\rangle\langle f_j^{\epsilon}| \right] |\psi\rangle = |\rho\rangle \; , \tag{4.16}$$

assuming complex number $E \notin \text{Spec}(H_0)$ and $\Re(E) < 0$. Acting the operator $(H_0 - E)^{-1}$ on both sides and projecting it onto $\langle f_i^{\epsilon}|$, we obtain

$$\langle f_i^{\epsilon}|\psi\rangle \;\; = \;\; \langle f_i^{\epsilon}|(H_0 - E)^{-1}|\rho\rangle + \sum_{j=1}^{N} \langle f_i^{\epsilon}|(H_0 - E)^{-1}|f_j^{\epsilon}\rangle\langle f_j^{\epsilon}|\psi\rangle \; , \tag{4.17}$$

or

$$\sum_{j=1}^{N} A^{\epsilon}_{ij}(E)\langle f^{\epsilon}_j|\psi\rangle = \langle f^{\epsilon}_i| (H_0 - E)^{-1} |\rho\rangle \ , \tag{4.18}$$

where we define a matrix $A^{\epsilon}_{ij}(E)$ as

$$A^{\epsilon}_{ij}(E) = \begin{cases} 1 - \langle f^{\epsilon}_i| (H_0 - E)^{-1} |f^{\epsilon}_i\rangle & \text{if } i = j \\ -\langle f^{\epsilon}_i| (H_0 - E)^{-1} |f^{\epsilon}_j\rangle & \text{if } i \neq j. \end{cases} \tag{4.19}$$

After a little algebra, it is evident that

$$(H^{\epsilon} - E)^{-1} = (H_0 - E)^{-1} + (H_0 - E)^{-1} \left[\sum_{i,j=1}^{N} |f^{\epsilon}_i\rangle \left[A^{\epsilon}(E)\right]^{-1}_{ij} \langle f^{\epsilon}_j| \right] (H_0 - E)^{-1} \ , \tag{4.20}$$

as long as $[A^{\epsilon}(E)]^{-1}_{ij}$ exists. We now choose the functions $\langle x|f^{\epsilon}_i\rangle = f^{\epsilon}_i(x)$'s as a weighted natural Dirac delta sequence defined by the heat kernels $\sqrt{\lambda_i(\epsilon)}K_{\epsilon/2}(x, a_i; g)$, centered at $x = a_i$. We now use $|f^{\epsilon}_i\rangle = \sqrt{\lambda_i(\epsilon)}|\tilde{f}^{\epsilon}_i\rangle$, where $\langle x|\tilde{f}^{\epsilon}_i\rangle = K_{\epsilon/2}(a_i, x; g)$, and write everything in terms of these pure Dirac delta sequences. Note that we get the original Hamiltonian given in Section 4.2 if we take the limit $\epsilon \to 0^+$:

$$\langle x|f^{\epsilon}_i\rangle\langle f^{\epsilon}_i|\psi\rangle \longrightarrow \delta^D_g(x, a_i)\psi(x) \ . \tag{4.21}$$

By defining new operators $D_{ij}(\epsilon) = \sqrt{\lambda_i(\epsilon)}\delta_{ij}$, $F_{ij}(\epsilon) = |f^{\epsilon}_i\rangle\langle f^{\epsilon}_j|$ and similarly for $\tilde{F}_{ij}(\epsilon)$, we can write the operator in (4.20) in a more transparent way:

$$\begin{aligned} \sum_{i,j=1}^{N} |f^{\epsilon}_i\rangle \left[A^{\epsilon}(E)\right]^{-1}_{ij} \langle f^{\epsilon}_j| &= \mathrm{Tr}\left[F(\epsilon)\left[A^{\epsilon}(E)\right]^{-1}\right] = \mathrm{Tr}\left[D(\epsilon)\tilde{F}(\epsilon)D(\epsilon)\left[A^{\epsilon}(E)\right]^{-1}\right] \\ &= \mathrm{Tr}\left[\tilde{F}(\epsilon)(D^{-1}(\epsilon)A^{\epsilon}(E)D^{-1}(\epsilon))^{-1}\right] \\ &= \sum_{i,j=1}^{N} |\tilde{f}^{\epsilon}_i\rangle \left[D^{-1}(\epsilon)A^{\epsilon}(E)D^{-1}(\epsilon)\right]^{-1}_{ij} \langle \tilde{f}^{\epsilon}_j| \\ &= \mathrm{Tr}\left[\tilde{F}(\epsilon)\left[\Phi^{\epsilon}(E)\right]^{-1}\right] \ , \end{aligned} \tag{4.22}$$

where we have defined a new operator $\Phi^{\epsilon}_{ij}(E)$

$$\Phi^{\epsilon}_{ij}(E) = \sum_{k,l=1}^{N} D^{-1}_{ik}(\epsilon) A^{\epsilon}_{kl}(E) D^{-1}_{lj}(\epsilon) = \begin{cases} \lambda_i^{-1}(\epsilon) - \langle \tilde{f}^{\epsilon}_i | (H_0 - E)^{-1} | \tilde{f}^{\epsilon}_i \rangle & \text{if } i = j \\ -\langle \tilde{f}^{\epsilon}_i | (H_0 - E)^{-1} | \tilde{f}^{\epsilon}_j \rangle \,, & \text{if } i \neq j. \end{cases} \tag{4.23}$$

One may ask: "Why do we have to choose these combinations of the operators?" Because this is the only way to get a symmetric combination of operators so that we have a symmetric integral kernel. Recall that this has been also the way we followed in Section 2.2 in order to preserve the hermiticity of the operators. Inserting the identity $1 = \int_{\mathcal{M}} \mathrm{d}^D_g x \; |x\rangle\langle x|$ between the operator $(H_0 - E)^{-1}$ and $\langle \tilde{f}^{\epsilon}_i|$ and the pair $|\tilde{f}^{\epsilon}_j\rangle$, we find that

$$\Phi^{\epsilon}_{ij}(E) = \begin{cases} \lambda_i^{-1}(\epsilon) - \int_{\mathcal{M}^2} \mathrm{d}^D_g x \, \mathrm{d}^D_g y \; K_{\epsilon/2}(a_i, x; g) R_0(x, y|E) K_{\epsilon/2}(y, a_i; g) & \text{if } i = j \\ - \int_{\mathcal{M}^2} \mathrm{d}^D_g x \, \mathrm{d}^D_g y \; K_{\epsilon/2}(a_i, x; g) R_0(x, y|E) K_{\epsilon/2}(y, a_j; g) & \text{if } i \neq j. \end{cases} \tag{4.24}$$

If we use the formula for the free resolvent expressed in terms of the heat kernel (3.52), we get

$$\Phi^{\epsilon}_{ij}(E) = \begin{cases} \lambda_i^{-1}(\epsilon) - \int_0^{\infty} \frac{\mathrm{d}t}{\hbar} \int_{\mathcal{M}^2} \mathrm{d}^D_g x \, \mathrm{d}^D_g y \; K_{\epsilon/2}(a_i, x; g) K_t(x, y; g) e^{tE/\hbar} K_{\epsilon/2}(y, a_i; g) \\ \qquad \text{if } i = j; \\ - \int_0^{\infty} \frac{\mathrm{d}t}{\hbar} \int_{\mathcal{M}^2} \mathrm{d}^D_g x \, \mathrm{d}^D_g y \; K_{\epsilon/2}(a_i, x; g) K_t(x, y; g) e^{tE/\hbar} K_{\epsilon/2}(y, a_j; g) \\ \qquad \text{if } i \neq j \,. \end{cases} \tag{4.25}$$

Then, the semi-group property of the heat kernel (3.36) leads to the following regularized matrix whose elements consist of time integrals

$$\Phi^{\epsilon}_{ij}(E) = \begin{cases} \lambda_i^{-1}(\epsilon) - \int_0^{\infty} \frac{\mathrm{d}t}{\hbar} K_{t+\epsilon}(a_i, a_i; g) \; e^{tE/\hbar} & \text{if } i = j \\ - \int_0^{\infty} \frac{\mathrm{d}t}{\hbar} K_{t+\epsilon}(a_i, a_j; g) \; e^{tE/\hbar} & \text{if } i \neq j. \end{cases} \tag{4.26}$$

Taking the limit $\epsilon \to 0^+$, we realize that the time integral in the diagonal part of the matrix is divergent due to the first term in the asymptotic expansion of diagonal heat kernel (3.54) in two and three dimensions. Hence, the divergence is controlled only by the first term in the asymptotic expansion (3.54). One may choose the bare coupling constant as

$$\frac{1}{\lambda_i(\epsilon)} = \int_{\epsilon}^{\infty} \frac{\mathrm{d}t}{\hbar}\, K_t(a_i, a_i; g)\, e^{-t\mu_i^2/\hbar}\,, \tag{4.27}$$

then obtain

$$\Phi_{ij}(E) = \begin{cases} \displaystyle\int_0^{\infty} \frac{\mathrm{d}t}{\hbar}\, K_t(a_i, a_i; g) \left[e^{-t\mu_i^2/\hbar} - e^{tE/\hbar}\right] & \text{if } i = j \\ \displaystyle -\int_0^{\infty} \frac{\mathrm{d}t}{\hbar}\, K_t(a_i, a_j; g)\, e^{tE/\hbar} & \text{if } i \neq j . \end{cases} \tag{4.28}$$

after we have taken the short-time limit $\epsilon \to 0^+$. Note that the diagonal part of the matrix (4.28) becomes convergent with the choice of substraction for the bare coupling constant (4.27). We can also show that the off-diagonal part of the matrix (4.28) is convergent by using the upper bounds of the heat kernel on manifolds given in (3.71) and (3.79) thanks to the exponential factor $e^{-md^2(x,y)/\hbar t}$. Due to the symmetry property of the heat kernel (3.37), the principal matrix is Hermitian for real values of energy.

One can naturally ask whether the renormalization performed with heat kernel is compatible with the one introduced in Section 4.2. The answer is affirmative and one can easily check that the momentum cut-off Λ for the infinite sum introduced in Section 4.2 corresponds to the short-time cut-off ϵ for the lower bound of time integral in the heat kernel method. This can be realized easily by using the spectral theorem in (4.28) for $E = -\nu^2$

$$\begin{aligned} \int_0^{\infty} \frac{\mathrm{d}t}{\hbar}\, e^{\frac{tE}{\hbar}} K_t(a_i, a_j; g) &= \sum_{l=0}^{\infty} f_l(a_i; g) f_l(a_j; g) \int_0^{\infty} \frac{\mathrm{d}t}{\hbar}\, e^{-\frac{t}{\hbar}\left(\frac{\hbar^2}{2m}\sigma_l - E\right)} \\ &= \sum_{l=0}^{\infty} \frac{f_l(a_i; g) f_l(a_j; g)}{\frac{\hbar^2}{2m}\sigma_l - E}\,, \end{aligned} \tag{4.29}$$

where the sum and integral can be interchanged. In other words, the series may be integrated term by term since summation converges uniformly [100]. Therefore the matrix is indeed the principal matrix of our original problem proposed in Section 4.2.

Finally, if we take the matrix element of (4.20) by projecting on to position Dirac bra-kets $\langle x|$ and $|y\rangle$ and using the equation (4.22), we have found the regularized resolvent kernel in coordinate space $R^\epsilon(x,y|E) \equiv \langle x|(H^\epsilon - E)^{-1}|y\rangle$

$$R^\epsilon(x,y|E) = R_0(x,y|E) + \sum_{i,j=1}^{N} \int_{\mathcal{M}^2} \mathrm{d}_g^D x' \; \mathrm{d}_g^D y' \; R_0(x,x'|E) K_{\epsilon/2}(x',a_i) \times [\Phi^\epsilon(E)]_{ij}^{-1} \, K_{\epsilon/2}(a_j,y') R_0(y',y|E) \; . \quad (4.30)$$

If we take the limit as $\epsilon \to 0^+$, we obtain an analog expression of the resolvent given in Section 2.2 for flat spaces:

$$\lim_{\epsilon \to 0^+} R^\epsilon(x,y|E) = R(x,y|E) = R_0(x,y|E) + \sum_{i,j=1}^{N} R_0(x,a_i|E) \, \Phi_{ij}^{-1}(E) R_0(a_j,y|E) \; , \quad (4.31)$$

where $R_0(x,y|E)$ is the free resolvent kernel. The equation (4.31) gives the relation between the resolvent defined on an infinite dimensional space and the principal matrix defined on a finite dimensional space. Although the resolvent formula above looks formally same with the one for the flat space, one must be aware of the fact that the free resolvent kernel (3.52) and the principal matrix (4.28) contain all the geometrical information through the heat kernel.

The resolvent essentially includes all the information about the spectrum. We will restrict ourselves only to the bound state problem since the scattering problem requires a deeper analysis. The discrete spectrum of the Hamiltonian is the set of numbers E at which the resolvent does not exist. Then, similar to the flat case the condition $\det \Phi(E) = 0$ determines the bound state spectrum of our system and it is so complicated that we can not solve exactly. Nevertheless, we will be able to apply the approximation methods in order to investigate model in detail in the subsequent sections.

Although we can not solve the eigenvalue problem exactly, we can express the wave function for the bound state in terms of the eigenvector of the principal matrix. For the moment, let us go back to our heuristic approach

$$\psi^{\Lambda}(x) = Z(\Lambda) \sum_{l=0}^{\Lambda} f_l(x;g) C_l(\Lambda) \tag{4.32}$$

where $Z(\Lambda)$ is the normalization constant to be determined later and

$$C_l(\Lambda) = \frac{1}{\frac{\hbar^2}{2m}\, \sigma_l + \nu^2} \; \sum_{i=1}^{N} A_i^{\Lambda}\, \lambda_i(\Lambda)\, f_l(a_i;g) \; . \tag{4.33}$$

and suppose that $\int_{\mathcal{M}} \mathrm{d}_g^D x \; |\psi^{\Lambda}(x)|^2 < \infty$ and

$$\lim_{\Lambda \to \infty} \int_{\mathcal{M}} \mathrm{d}_g^D x \; |\psi^{\Lambda}(x)|^2 = 1 \; , \tag{4.34}$$

for $D = 2, 3$. In spectral theorem in Section 3.2, the wave function $\psi(x)$ is assumed to be in $L^2(\mathcal{M})$ and we formulate our problem under this assumption. Here we shall check that this assumption is consistent. The square integrability leads to the following result

$$|Z(\Lambda)|^{-2} = \sum_{l=0}^{\Lambda} |C_l(\Lambda)|^2 = \sum_{i,j=1}^{N} \lambda_i(\Lambda)\, \lambda_j(\Lambda)\, A_i^{\Lambda}\, A_j^{\Lambda} \sum_{l=0}^{\Lambda} \frac{f_l(a_i;g) f_l(a_j;g)}{\left(\frac{\hbar^2}{2m}\, \sigma_l + \nu^2\right)^2} \; . \tag{4.35}$$

Moreover, it is useful to express the summation over the eigenmodes as the derivative of $\Phi(-\nu^2)$ with respect to ν, hence we get:

$$|Z(\Lambda)|^{-2} = \frac{1}{2\nu} \sum_{i,j=1}^{N} \lambda_i(\Lambda) \lambda_j(\Lambda) A_i^{\Lambda} \, \frac{\partial \Phi_{ij}(-\nu^2;\Lambda)}{\partial \nu} \, A_j^{\Lambda} \; . \tag{4.36}$$

Performing the limit $\Lambda \to \infty$, the properly normalized wave function of k-th state becomes

$$\psi_k(x) = \sqrt{2\nu_k}\left[\sum_{i,j=1}^{N} A_i(-\nu_k^2)\ \frac{\partial\Phi_{ij}(-\nu^2)}{\partial\nu}\bigg|_{\nu=\nu_k} A_j(-\nu_k^2)\right]^{-\frac{1}{2}}$$
$$\sum_{l=0}^{\infty}\sum_{i=1}^{N} A_i(-\nu_k^2)\ \frac{f_l(a_i;g)f_l(x;g)}{\left(\frac{\hbar^2}{2m}\,\sigma_l+\nu_k^2\right)}\ , \tag{4.37}$$

where ν_k is the k-th root of the energy equation $\det\Phi(-\nu^2)=0$. We may present a simplified expression in terms of the heat kernel

$$\begin{aligned}\psi_k(x) &= \sqrt{2\nu_k}\left[\sum_{i,j=1}^{N} A_i(-\nu_k^2)\ \frac{\partial\Phi_{ij}(-\nu^2)}{\partial\nu}\bigg|_{\nu=\nu_k} A_j(-\nu_k^2)\right]^{-\frac{1}{2}}\\ &\quad\times\sum_{l=0}^{\infty}\sum_{i=1}^{N} A_i(-\nu_k^2)\ f_l(a_i;g)f_l(x;g)\int_0^\infty\frac{\mathrm{d}t}{\hbar}\ e^{-\frac{t}{\hbar}\left(\frac{\hbar^2}{2m}\,\sigma_l+\nu_k^2\right)}\\ &= \sqrt{2\nu_k}\left[\sum_{r,s=1}^{N} A_r(-\nu_k^2)\ \frac{\partial\Phi_{rs}(-\nu^2)}{\partial\nu}\bigg|_{\nu=\nu_k} A_s(-\nu_k^2)\right]^{-\frac{1}{2}}\\ &\quad\times\int_0^\infty\frac{\mathrm{d}t}{\hbar}\ e^{-\frac{t\nu_k^2}{\hbar}}\sum_{i=1}^{N} A_i(-\nu_k^2)K_t(a_i,x;g)\ ,\end{aligned} \tag{4.38}$$

and using the identity from equation (4.28)

$$\frac{\partial\Phi_{ij}(-\nu^2)}{\partial\nu}\bigg|_{\nu=\nu_k} = \int_0^\infty\frac{\mathrm{d}t}{\hbar}\ \left(\frac{2\nu_k t}{\hbar}\right)K_t(a_i,a_j;g)e^{-\frac{t\nu_k^2}{\hbar}}\ , \tag{4.39}$$

we eventually get the wave function for the kth state

$$\psi_k(x) = \alpha\int_0^\infty\frac{\mathrm{d}t}{\hbar}\ e^{-\frac{t\nu_k^2}{\hbar}}\sum_{i=1}^{N} A_i(-\nu_k^2)K_t(a_i,x;g)\ , \tag{4.40}$$

where

$$\alpha = \left[\sum_{i,j=1}^{N} A_i(-\nu_k^2)\int_0^\infty\frac{\mathrm{d}t}{\hbar}\ \left(\frac{t}{\hbar}\right)K_t(a_i,a_j;g)e^{-\frac{t\nu_k^2}{\hbar}}\ A_j(-\nu_k^2)\right]^{-\frac{1}{2}}\ . \tag{4.41}$$

Note that $\psi_k(x)$ is finite except at $x = a_i$. It is instructive to show that the wave functions are square integrable to verify our earlier assumption.

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \, |\psi_k(x)|^2 = \alpha^2 \int_{\mathcal{M}} \mathrm{d}_g^D x \int_0^\infty \frac{\mathrm{d}t_1}{\hbar} \int_0^\infty \frac{\mathrm{d}t_2}{\hbar} \, e^{-\nu_k^2 (t_1+t_2)/\hbar} \times \sum_{i,j=1}^N A_i(-\nu_k^2) A_j(-\nu_k^2) K_{t_1}(a_i, x; g) K_{t_2}(x, a_j; g) \; . \tag{4.42}$$

Using the semi-group property of the heat kernel (3.51) and making the change of variables $u = t_1 + t_2$ with $v = t_1 - t_2$,we get

$$\begin{aligned} & \alpha^2 \int_0^\infty \frac{\mathrm{d}u}{\hbar} \left(\frac{1}{2} \int_{-u}^{u} \frac{\mathrm{d}v}{\hbar} \right) e^{-\nu_k^2 u/\hbar} \sum_{i,j=1}^N A_i(-\nu_k^2) A_j(-\nu_k^2) K_u(a_i, a_j; g) \\ = & \, \alpha^2 \int_0^\infty \frac{\mathrm{d}u}{\hbar} \, u \, e^{-\nu_k^2 u/\hbar} A_i^2(-\nu_k^2) K_u(a_i, a_i; g) \\ & + \alpha^2 \int_0^\infty \frac{\mathrm{d}u}{\hbar} \, u \, e^{-\nu_k^2 u/\hbar} \sum_{\substack{i,j=1 \\ i \neq j}}^N A_i(-\nu_k^2) A_j(-\nu_k^2) K_u(a_i, a_j; g) \; . \end{aligned} \tag{4.43}$$

One can now easily see that both terms are convergent due to the short-time asymptotic expansion of the diagonal heat kernel (3.54) and the upper bounds of the heat kernel (3.71) and (3.79) for $D < 4$. Hence we check that $\psi_k(x) \in L^2(\mathcal{M})$ for two and three dimensional manifolds. Since all the information about the spectrum lies in the resolvent, we must have expected that the bound state wave function must be derivable from the resolvent formula. In fact, we will derive the wave function (4.40) in Section 4.6 by a more elegant way.

It is well known that the same problem with a single delta potential in flat spaces is a good example of a dimensional transmutation in quantum mechanics. Our problem in $D = 2$ realizes a generalized dimensional transmutation [20, 21]: In our case, the coupling constants λ_i have the same dimension as $\frac{\hbar^2}{m}$ by dimensional analysis. In contrast to the flat case, we also have intrinsic scales coming from the geometry of the space, such as curvature and the geodesic distance between centers d_{ij}. However, after the renormalization procedure, we obtain a set of new dimensional parameters, that is, dimensionless coupling constant in two dimensions is traded for a dimensional parameter μ_i^2 from the relation (4.80). Hence, the energy is not determined by naive dimensional analysis. However, in the case of single delta attractor for the flat case there is no combination of dimensional parameters to come up with an energy scale, whereas in the case of a manifold we have geometric length scales which already may define an energy scale. The dimensional transmutation is most striking in cases where there is no intrinsic energy scale.

4.4. Self-Adjoint Extension and Krein's Formula

As far as the rigorous mathematical considerations are concerned, the Dirac delta function potential in Euclidean spaces is not a well defined operator in Hilbert space $\mathcal{H}$, that is, the way of writing the Hamiltonian of this system as $H = -\frac{\hbar^2}{2m}\nabla^2 - \lambda\delta(\mathbf{x})$ is only a formal expression. Nevertheless, there exists a proper treatment to this type of problem and it is generally known as self-adjoint extensions based on the studies of Weyl and von Neumann [101, 102, 103] and the detailed analysis of point interactions in the context of self-adjoint extensions is given in the monographs [5, 23]. What is more interesting is that, it can be shown that one can prove that the result obtained by the self-adjoint extension method is consistent with the result of the renormalization method for the point interactions [22] and [11]. In order to understand in a little more detail, we will give some definitions and theorems without going into details. In order to define properly the operators in infinite dimensional spaces, we must also specify their domains, while it is usually disregarded in standard quantum mechanics textbooks [104, 105]. Let us consider an operator H in a dense domain $\mathcal{D}(H)$ of a Hilbert space $\mathcal{H}$. Let $(\cdot, \cdot)$ represents the inner product of any two elements in $\mathcal{H}$. Then, the adjoint of H is defined by

$$(\phi, H\psi) = (H^\dagger \phi, \psi) \ , \tag{4.44}$$

for all ϕ and $\psi \in \mathcal{D}(H)$. The operator H is called symmetric (or Hermitian) if its action is the same as the action of its adjoint $H^\dagger$. If we also impose the condition on their domains such that $\mathcal{D}(H) = \mathcal{D}(H^\dagger)$, then the symmetric operators are called self-adjoint. In general, $\phi \in \mathcal{D}(H^\dagger)$ belongs to more general space than the one defined by

$\mathcal{D}(H)$ so we have $\mathcal{D}(H^\dagger) \supseteq D(H)$. Then, one can ask naturally whether we can make a given symmetric operator self adjoint by extending its domain and restricting its adjoint domain such that $\mathcal{D}(H) = \mathcal{D}(H^\dagger)$. If the answer is yes, then the next question is how. In order to answer these, let us first consider the equations

$$H^\dagger \phi_+ = +iE\phi_+ \,, \tag{4.45}$$

$$H^\dagger \phi_- = -iE\phi_- \,, \tag{4.46}$$

where E is the real parameter introduced for dimensional reasons, it has no connection with the energy whatsoever. Let $n_\pm$ be the number of linearly independent square integrable solutions of (4.45) and (4.46), respectively. The pair (n_+, n_-) is called deficiency index for H, and one can think that it is a kind of measure for the deviation from the self-adjointness of the operator H. Then, we have the following the theorem stating that [101, 102]:

(i) H is (essentially) self-adjoint if and only if $(n_+, n_-) = (0, 0)$. An operator which has a unique self-adjoint extension is said to be essentially self-adjoint.

(ii) If $n_+ = n_- \geq 1$, H admits infinitely many self-adjoint extensions, parametrized by a unitary $N \times N$ matrix.

(iii) H has no self-adjoint extensions if $n_+ \neq n_-$.

If we have the second case, say $n_+ = n_- = N$, then we can find the suitable domain in which H is self-adjoint according to

$$D_U(H) = \left\{ \phi + \phi_+ + U\phi_- | \phi \in D(H) \quad \text{and} \quad U \text{ is a unitary} \quad N \times N \quad \text{matrix} \right\} . \tag{4.47}$$

The rigorous proof of this theorem can also be found in [103]. For our understanding, we shall consider a definite example instead of the rigorous technical details in the subject. Let us consider a free particle moving in the positive real line [106] in coordinate space representation

$$H_0 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ , \tag{4.48}$$

with the domain

$$\mathcal{D}(H_0) = \left\{ \phi(x)|\phi(x) \in S, \phi(0) = \left.\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right|_{x=0} = 0 \right\} , \tag{4.49}$$

where $S = \{\phi(x)|\phi(x) \in L^2(\mathbb{R}^+), -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} \in L^2(\mathbb{R}^+), \frac{\mathrm{d}\phi(x)}{\mathrm{d}x}$ is absolutely continuous$\}$. We can easily show that H_0 is a symmetric operator, that is,

$$\begin{aligned} \Delta_{H_0} &= (\phi, H_0\psi) - (H_0\phi, \psi) \\ &= \frac{\hbar^2}{2m}\left(\left.\frac{\mathrm{d}\phi^*(x)}{\mathrm{d}x}\right|_0 \psi(0) - \phi^*(0)\left.\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right|_{x=0} \right) = 0 \end{aligned} \tag{4.50}$$

for any finite value of $\psi(0)$ and $\left.\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right|_0$ and $\phi(x) \in \mathcal{D}(H_0)$. This tells us that the domain of the adjoint of the operator is larger that the domain of the operator, which means that the operator H_0 is symmetric in $\mathcal{D}(H_0)$. In order to find the domain in which the operator is self-adjoint, let us use above theorem. The square integrable solutions to

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi_\pm}{\mathrm{d}x^2} = \pm iE\phi_\pm \ , \quad \phi_\pm \in D(H_0^\dagger) \ , \tag{4.51}$$

are given by

$$\begin{aligned} \phi_+(x) &= (2\eta)^{1/4} e^{(i-1)\sqrt{\eta/2}\,x} \ , \\ \phi_-(x) &= (2\eta)^{1/4} e^{-(i+1)\sqrt{\eta/2}\,x} \ , \end{aligned} \tag{4.52}$$

where $\eta = 2mE/\hbar^2$. Therefore the deficiency index is $(n_+, n_-) = (1, 1)$ so that $U(1) = e^{i\theta}$ with $\theta \in \mathbb{R}$ and the domain in which H_0 is self-adjoint can be found as

$$D_\theta(H_0) = \left\{ \psi = \phi + \phi_+ + e^{i\theta}\phi_- | \phi \in \mathcal{D}(H_0) \right\} . \tag{4.53}$$

One can also write the following expression as

$$\phi_+ + e^{i\theta}\phi_- = 2(2\eta)^{1/4} e^{-\sqrt{\eta/2}\,x+i\theta/2} \cos\left(\sqrt{\frac{\eta}{2}} - \frac{\theta}{2}\right) \tag{4.54}$$

and it is useful to have single condition relating the $\psi(0)$ and $\mathrm{d}\psi(x)/\mathrm{d}x$ at $x = 0$. We can find it by calculating the derivative of ψ at $x = 0$ by dividing the value $\psi(0)$ to get

$$\frac{\mathrm{d}\psi(x)/\mathrm{d}x|_{x=0}}{\psi(0)} = \frac{\psi'(0)}{\psi(0)} = -\sqrt{\eta/2}\,(1 + \tan\theta/2)\ ,. \tag{4.55}$$

We can reparametrize the right hand side so that we have

$$\psi'(0) + \alpha\psi(0) = 0\ , \tag{4.56}$$

which must hold for all functions in the operator domain. Now, the most interesting case is that the free Hamiltonian H_0 with the above boundary condition that we have found (4.56) for the self-adjointness

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} = -\nu^2\psi(x)\ , \quad \psi'(0) + \alpha\psi(0) = 0\ , \tag{4.57}$$

leads to the bound state solution

$$\psi(x) = \sqrt{2\alpha}\, e^{-\alpha x} \quad E = -\alpha^2\ , \tag{4.58}$$

which is similar to the bound state of the one dimensional delta potential given in (2.5). If we neglect some subtleties, we can identify the one dimensional Dirac delta potential problem with the extension of the free Hamiltonian on the real half line described above if do the following identification the self-adjoint parameter with the coupling constant by comparing the solutions

$$\frac{m\lambda^2}{2\hbar^2} = \alpha^2 \tag{4.59}$$

Similarly, one can also show that the point interaction located at $x = 0$ can be considered as a self-adjoint extension of the free Hamiltonian on $\mathbb{R}^D \setminus \{0\}$ and the idea can also be applied to the finitely many point interactions in Euclidean spaces and the detailed analysis is given in [5]. The rigorous treatment of the point interactions in two and three dimensions allows us to avoid the infinities and the renormalization procedures. In analogy with the above extension for one dimensional delta potential, the result of the self-adjoint extension method for two and three dimensional point interactions is identical to the renormalization method if certain relation between the extension parameter and the renormalized (or bare) coupling constant is satisfied [11]. There is another approach to the self-adjoint extension theory, so-called Krein's theory [23, 107]. Berezin and Faddeev [22] pointed out that the renormalization approach to the singular perturbations is equivalent to searching for self-adjoint extensions of a symmetric operator related to the unperturbed operator in question via Krein's theory of self-adjoint extensions. Krein's formula describes the relation between the resolvents of the two different self-adjoint extensions of one symmetric operator. The results of these two approaches are equivalent but the first one (von Neumann) allows an immediate definition of the domain of these operators whereas the second one makes immediately available their spectral properties. Let us now digress a little on this formula for completeness, all details can be found in [23].

We assume that H is a densely defined, closed symmetric operator in $\mathcal{H}$ with deficiency indices (N, N). If H^U and H^V are two self-adjoint extensions of H then an operator $\dot{H}$ exists such that $\dot{H} \subseteq H^U$, $\dot{H} \subseteq H^V$ and $\dot{H}$ extends any operator B that fulfills $B \subseteq H^U$, $B \subseteq H^V$, $\dot{H}$ is called the maximal common part of H^U and H^V. The deficiency indices of $\dot{H}$ are (M, M) with $0 < M \leq N$. A set $\{\phi_1^z, \ldots, \phi_M^z\}$ of independent solutions of

$$\dot{H}^\dagger \phi^z = z\, \phi^z \, , \qquad \phi^z \in D(\dot{H}^\dagger) \, , \quad z \in \mathbb{C} \setminus \mathbb{R} \, , \tag{4.60}$$

is a basis for $\mathcal{N}^z(\dot{H})$. The Krein's formula relates the resolvents of H^U and H^V by

$$(H^U - z)^{-1} = (H^V - z)^{-1} + \sum_{i,j=1}^{M} \lambda_{ij}^{-1}(z)(\phi_j^{z^*}, \cdot)\phi_i^z \quad z \in \rho(H^U) \cap \rho(H^V) \ , \tag{4.61}$$

where $\rho(H^U)$ and $\rho(H^V)$ denotes the resolvent set of H^U and H^V, respectively. $\lambda^{-1}(z)$ is a non singular matrix for $z \in \rho(H^U) \cap \rho(H^V)$ satisfying $\lambda^{\dagger}(z) = \lambda(z^*)$. The functions $\lambda_{ij}(z)$ and ϕ_i^z may be chosen to be analytic for $\rho(H^U) \cap \rho(H^V)$.

In fact ϕ_i^z may be defined as $\phi_i^z = \phi_i^{z_0} + (z - z_0)(H^V - z)^{-1}\phi_i^{z_0}$, $i = 1, \ldots, m$ and $z \in \rho(H^V)$, where $\phi_i^{z_0}$, $i = 1, \ldots, m$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$, are linearly independent solutions of equation (4.60) with $z = z_0$ and matrix $\lambda(z)$ satisfies

$$\lambda_{ij}(z) = \lambda_{ij}(z') - (z - z')(\phi_j^{z^*}, \phi_i^{z'}) \ \ z, z' \in \rho(H^U) \cap \rho(H^V) \tag{4.62}$$

Hence, the formula that we have obtained for the resolvent of the N Dirac delta centers (4.20) or (4.31) is connected to the Krein's formula defined above (4.61) in the mathematics literature. Due to this similarity with the resolvent formulae, we expect that our problem defined in Riemannian manifolds can also be considered as a kind of self-adjoint extension, which we will only be able show it heuristically in the Section 4.8.

4.5. An Alternative Construction of the Model including n Bosons

Before we continue discussing the problem in more detail, we will give an alternative construction of the problem, hoping to extend it to many body or non-relativistic field theoretical models. For this reason, let us consider the simplest possible many body extension of the problem in which non-relativistic bosons interact with N external attractive Dirac delta potentials in a Riemannian manifold of dimension $D = 2, 3$. In the second quantized language, the Hamiltonian of the system in the Schrödinger picture is given by

$$H = \int_{\mathcal{M}} \mathrm{d}_g^D x \left[\phi_g^\dagger(x) \left(-\frac{\hbar^2}{2m} \nabla_g^2 \right) \phi_g(x) - \sum_{i=1}^N \lambda_i \phi_g^\dagger(x) \delta_g^D(x, a_i) \phi_g(x) \right] , \qquad (4.63)$$

where $\phi_g^\dagger(x)$, $\phi_g(x)$ is defined as the bosonic creation and annihilation operators on the Riemannian manifold with metric structure g, respectively. It is easy to show that the number of bosons $\int_{\mathcal{M}} \mathrm{d}_g^D x \; \phi_g^\dagger(x) \, \phi_g(x)$ is conserved so that we can consider a sector with fixed number n of bosons. Since the previous problem is the one for the restricted sector $n = 1$, we must also expect that the many body version of it for any n gives divergent results. Hence, the first step we must do is to regularize the model. Motivated by the restricted problem corresponding to $n = 1$, the natural regularization of the Hamiltonian can be chosen as

$$H^\epsilon = H_0 - \sum_{i=1}^N \lambda_i(\epsilon) \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; K_{\epsilon/2}(x, a_i; g) K_{\epsilon/2}(y, a_i; g) \phi_g^\dagger(x) \phi_g(y) \; , \qquad (4.64)$$

where H_0 is the free Hamiltonian operator. In the limit $\epsilon \to 0^+$, one can see that we recover the original Hamiltonian we are interested in.

Now, we will consider the resolvent of the regularized Hamiltonian in a Fock space formalism with arbitrary number of bosons. Following the same methodology developed for the model in the plane [26], we shall extend the bosonic Fock space $\mathcal{F}_{\mathcal{B}}$ that we have started with, to $\tilde{\mathcal{F}_{\mathcal{B}}} = \mathcal{F}_{\mathcal{B}} \oplus \mathcal{F}_{\mathcal{B}} \otimes \mathbb{C}^N$ by defining new creation and annihilation operators at the locations of the Dirac delta interactions. These operators are called angels, which is first introduced in [26]. The angel states allow us rewrite the model in such a way that the coupling constant appears additively rather than multiplicatively. As a result, we can renormalize the model nonperturbatively by simply normal ordering. We assume that the angel operators obey the orthofermionic algebra [108] defined by the following product relations (not with commutators):

$$\chi_i \, \chi_j^\dagger + \delta_{ij} \sum_{k=1}^N \chi_k^\dagger \, \chi_k = \mathbf{1} \delta_{ij}, \quad \chi_i \, \chi_j = 0 = \chi_i^\dagger \, \chi_j^\dagger \; , \qquad (4.65)$$

where $\mathbf{1}$ is the identity operator and $i, j, k = 1, 2, \ldots, N$. It is more convenient for our purposes to write the angel algebra in terms of projection operators:

$$\chi_i \, \chi_j^\dagger = \delta_{ij} \Pi_0, \quad \chi_i \, \chi_j = 0 = \chi_i^\dagger \, \chi_j^\dagger \; , \tag{4.66}$$

where

$$\Pi_1 = \sum_{k=1}^{N} \chi_k^\dagger \, \chi_k, \quad \Pi_0 = \mathbf{1} - \Pi_1 \; , \tag{4.67}$$

are the projection operators onto the 1-angel and no-angel states, respectively. It should be emphasized that we have at most one angel in any state. Now we define the augmented regularized Hamiltonian $\tilde{H}_\epsilon$ on $\tilde{\mathcal{F}}_\mathcal{B}$

$$\tilde{H}_\epsilon = H_0 \Pi_0 + \left[\sum_{i=1}^{N} \int_\mathcal{M} \mathrm{d}_g^D x \; K_{\epsilon/2}(x, a_i; g) \phi_g(x) \chi_i^\dagger + h.c. \right] + \sum_{i=1}^{N} \frac{1}{\lambda_i(\epsilon)} \chi_i^\dagger \, \chi_i \; . \tag{4.68}$$

If we split the Hilbert space according to the angel number, the corresponding operator $\tilde{H}_\epsilon - E\Pi_0$ can be written in the following matrix form:

$$\tilde{H}_\epsilon - E\Pi_0 = \begin{pmatrix} a & b_\epsilon^\dagger \\ b_\epsilon & d_\epsilon \end{pmatrix} \; , \tag{4.69}$$

with $a : \mathcal{F}_\mathcal{B} \to \mathcal{F}_\mathcal{B}$, $b_\epsilon^\dagger : \mathcal{F}_\mathcal{B} \otimes \mathbb{C}^N \to \mathcal{F}_\mathcal{B}$, $d_\epsilon : \mathcal{F}_\mathcal{B} \otimes \mathbb{C}^N \to \mathcal{F}_\mathcal{B} \otimes \mathbb{C}^N$. Here,

$$\begin{aligned} a &= H_0 - E \; , \qquad\qquad d_\epsilon = \sum_{i=1}^{N} \frac{1}{\lambda_i(\epsilon)} \chi_i^\dagger \, \chi_i \\ b_\epsilon^\dagger &= \sum_{i=1}^{N} \int_\mathcal{M} \mathrm{d}_g^D x \; K_{\epsilon/2}(x, a_i; g) \phi_g^\dagger(x) \, \chi_i \; . \end{aligned} \tag{4.70}$$

Then, one can construct the augmented regularized resolvent

$$\tilde{R}^\epsilon(E) = \frac{1}{\tilde{H}_\epsilon - E\Pi_0} = \begin{pmatrix} \alpha_\epsilon & \beta_\epsilon^\dagger \\ \beta_\epsilon & \delta_\epsilon \end{pmatrix} \; . \tag{4.71}$$

One can find $\alpha_\epsilon, \beta_\epsilon, \delta_\epsilon$ in terms of $a, b_\epsilon, d_\epsilon$ by direct computation. This could be done apparently different but equivalent ways and the formulas were obtained in the appendix of [26]

$$\alpha_\epsilon = \left[a - b_\epsilon^\dagger \, d_\epsilon^{-1} \, b_\epsilon\right]^{-1} = \frac{1}{H^\epsilon - E} = R^\epsilon(E) \; . \tag{4.72}$$

This means that $\tilde{R}_\epsilon(E)$ projected to $\mathcal{F}_\mathcal{B}$ is just the resolvent of the operator H^ϵ. We have also another formula for α_ϵ [26]

$$\alpha_\epsilon = a^{-1} + a^{-1} \, b_\epsilon^\dagger \left[d_\epsilon - b_\epsilon \, a^{-1} \, b_\epsilon^\dagger\right]^{-1} b_\epsilon \, a^{-1} \; , \tag{4.73}$$

or

$$R^\epsilon(E) = \frac{1}{H_0 - E} + \frac{1}{H_0 - E} \, b_\epsilon^\dagger \left[\Phi^\epsilon(E)\right]^{-1} b_\epsilon \, \frac{1}{H_0 - E} \; , \tag{4.74}$$

where $\Re(E) < 0$ and

$$\Phi^\epsilon(E) = \sum_{i=1}^{N} \frac{1}{\lambda_i(\epsilon)} \chi_i^\dagger \, \chi_i - \sum_{i,j=1}^{N} \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; K_{\epsilon/2}(x, a_i; g) K_{\epsilon/2}(y, a_j; g) \\ \times \phi_g(y) \left(\frac{1}{H_0 - E}\right) \phi_g^\dagger(x) \, \chi_i^\dagger \, \chi_j \; . \tag{4.75}$$

The operator $\Phi^\epsilon(E)$ is now called the regularized principal operator. Note that writing the resolvent of H^ϵ in this way allows us to write the coupling constant additively. The renormalization procedure can then be done if we can separate the singular part of the regularized principal operator at the short-time limit. We will see that this is possible by normal ordering of the operators in the principal operator. In analogy with the plane wave mode expansion of the field operators in quantum field theory in Minkowski space-time, we have

$$\begin{aligned} \phi_g^\dagger(x) &= \sum_{l=0}^{\infty} \phi_g^\dagger(l) f_l(x; g) \\ \phi_g(x) &= \sum_{l=0}^{\infty} \phi_g(l) f_l(x; g) \; , \end{aligned} \tag{4.76}$$

due to the spectral theorem given in Section 3.2. Here $\phi_g^\dagger(l)$ and $\phi_g(l)$ are the creation and annihilation operators corresponding to the mode l. Using this result and the eigenfunction expansions of the heat kernel (3.30), one can shift the operator $\phi_g^\dagger(x)$ in (4.75) to the left

$$\frac{1}{H_0 - E}\phi_g^\dagger(x) = \int_{\mathcal{M}} \mathrm{d}_g^D x' \; \phi_g^\dagger(x') \int_0^\infty \frac{\mathrm{d}t}{\hbar} \; e^{-\frac{t}{\hbar}(H_0-E)} \, K_t(x,x';g) \; , \tag{4.77}$$

and shift the operator $\phi_g(x)$ to the right

$$\phi_g(x)\frac{1}{H_0 - E} = \int_{\mathcal{M}} \mathrm{d}_g^D x' \; \int_0^\infty \frac{\mathrm{d}t}{\hbar} \; e^{-\frac{t}{\hbar}(H_0-E)} \, K_t(x,x';g)\phi_g(x') \; , \tag{4.78}$$

Then, the normal ordered principal operator can be written by using the properties of heat kernel and separating the $i=j$ term from the sum

$$\begin{aligned}\Phi^\epsilon(E) = \sum_{i=1}^N \frac{1}{\lambda_i(\epsilon)} \chi_i^\dagger \chi_i - \sum_{i,j=1}^N \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; \int_0^\infty \frac{\mathrm{d}t}{\hbar} K_{(t+\epsilon/2)}(x,a_i;g) K_{(t+\epsilon/2)}(y,a_j;g) \\ \phi_g^\dagger(x) e^{-\frac{t}{\hbar}(H_0-E)} \phi_g(y) \chi_i^\dagger \chi_j - \sum_{i=1}^N \int_0^\infty \frac{\mathrm{d}t}{\hbar} K_{(\epsilon+t)}(a_i,a_i;g) e^{-\frac{t}{\hbar}(H_0-E)} \chi_i^\dagger \chi_i \\ - \sum_{\substack{i=1 \\ i\neq j}}^N \int_0^\infty \frac{\mathrm{d}t}{\hbar} K_{(\epsilon+t)}(a_i,a_j;g) e^{-\frac{t}{\hbar}(H_0-E)} \chi_i^\dagger \chi_j \; .\end{aligned} \tag{4.79}$$

Due to the short-time asymptotic expansion of the diagonal heat kernel (3.54), the third term is divergent as $\epsilon \to 0^+$. One can also show that remaining terms are finite by the upper bound of the heat kernel (3.63) and (3.79). Therefore, if we choose the coupling constant

$$\frac{1}{\lambda_i(\epsilon)} = \int_\epsilon^\infty \frac{\mathrm{d}t}{\hbar} K_t(a_i,a_i;g) e^{-\frac{t}{\hbar}\mu_i^2} \; , \tag{4.80}$$

where $-\mu_i^2$ corresponds to experimentally measured bound state energy of the individual i-th Dirac delta center, we find the principal operator

$$\Phi(E) = \lim_{\epsilon\to 0^+} \Phi^\epsilon(E) = \sum_{i=1}^N \int_0^\infty \frac{\mathrm{d}t}{\hbar} K_t(a_i, a_i; g) \left(e^{-\frac{t}{\hbar}\mu_i^2} - e^{-\frac{t}{\hbar}(H_0 - E)} \right) \chi_i^\dagger \chi_i$$
$$- \sum_{i,j=1}^N \int_0^\infty \frac{\mathrm{d}t}{\hbar} \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; K_t(a_i, x; g) K_t(a_j, y; g) \phi_g^\dagger(x) e^{-\frac{t}{\hbar}(H_0 - E)} \phi_g(y) \chi_i^\dagger \chi_j \quad (4.81)$$
$$- \sum_{\substack{i=1 \\ i\neq j}}^N \int_0^\infty \frac{\mathrm{d}t}{\hbar} K_t(a_i, a_j; g) e^{-\frac{t}{\hbar}(H_0 - E)} \chi_i^\dagger \chi_j \; .$$

The choice for the bare coupling constant (4.80) is exactly equal to the one for the restricted problem (4.27), as expected. The above formula of the principal operator can be written in a more compact way $\Phi(E) = \sum_{i,j=1}^N \Phi_{ij}(E) \, \chi_i^\dagger \chi_j$, where $\Phi_{ij}(E)$ can be read from (4.81). Once we have a proper definition of the principal operator, the divergence is completely removed since the spectrum of the problem can be found from the resolvent. We are now in a position to get the full resolvent of our problem in terms of the principal operator

$$R(E) = \lim_{\epsilon\to 0^+} R^\epsilon(E) = \frac{1}{H_0 - E} + \frac{1}{H_0 - E} \sum_{k=1}^N \phi_g^\dagger(a_k) \, \chi_k \, \Phi^{-1}(E) \sum_{l=1}^N \phi_g(a_l) \chi_l^\dagger \, \frac{1}{H_0 - E}$$
$$= \frac{1}{H_0 - E} + \frac{1}{H_0 - E} \sum_{i,j=1}^N \phi_g^\dagger(a_i) \, \Phi_{ij}^{-1}(E) \phi_g(a_j) \, \frac{1}{H_0 - E} \; , \quad (4.82)$$

where we have used $\Phi^{-1}(E) = \sum_{i,j=1}^N \Phi_{ij}^{-1}(E) \, \chi_i^\dagger \chi_j$ and the algebra of angel operators (4.66). Incidentally, there are projection operators Π_0 onto the zero angel states in the above expression but it can be omitted due to the fact that $R(E) : \mathcal{F}_\mathcal{B} \to \mathcal{F}_\mathcal{B}$. Let us return to the one boson problem

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \; \psi(x) \phi_g^\dagger(x) |0\rangle \otimes |\Omega\rangle \; , \quad (4.83)$$

where $|\Omega\rangle$ is the vacuum for the angel state and $\psi(x)$ is the wave function for the boson. Then, the resolvent kernel corresponding to this state satisfies the following equation after a straightforward calculation

$$R(x,y|E) = R_0(x,y|E) + \sum_{i,j=1}^{N} R_0(x,a_i|E)\Phi^{-1}_{ij}(E)R_0(a_j,y|E) \; , \tag{4.84}$$

which is formally equivalent to the formula given in the restricted problem (4.31). In fact, one can also check that the matrix $\Phi_{ij}(E)$ in (4.84) is exactly (4.28). To see this, let us study the bound state problem. The poles corresponding to bound states must be due to $\Phi^{-1}(E)$ in the resolvent formula. In other words, the roots of

$$\Phi(E)|\Psi\rangle = 0 \; , \tag{4.85}$$

determine the bound state spectrum of the model. Since $\Phi(E) : \mathcal{F}_{\mathcal{B}} \otimes \mathbb{C}^N \to \mathcal{F}_{\mathcal{B}} \otimes \mathbb{C}^N$, let us try to consider $|\Psi\rangle$ as a direct product of no-boson with one angel state:

$$|\Psi\rangle = |0\rangle \otimes \sum_{k=1}^{N} A_k |e_k\rangle \; , \tag{4.86}$$

where $|e_k\rangle \equiv \chi_k^{\dagger}|\Omega\rangle$ is a set of complete orthonormal basis for $\mathbb{C}^N$. Then, the equation (4.85) yields

$$\begin{aligned} \sum_{i=1}^{N} \int_0^{\infty} \frac{\mathrm{d}t}{\hbar} K_t(a_i,a_i;g) \left(e^{-\frac{t}{\hbar}\mu_i^2} - e^{\frac{t}{\hbar}E} \right) A_i |e_i\rangle \\ - \sum_{\substack{i=1 \\ i\neq j}}^{N} \int_0^{\infty} \frac{\mathrm{d}t}{\hbar} K_t(a_i,a_j;g) e^{\frac{t}{\hbar}E} A_j |e_i\rangle = 0 \; , \end{aligned} \tag{4.87}$$

and the result can be written as a matrix equation

$$\sum_{j=1}^{N} \Phi_{ij}(E) A_j = 0 \; , \tag{4.88}$$

where $\Phi_{ij}(E)$ is nothing but (4.28), as expected. The construction of the relativistic extension of this model in two dimensions is also possible [109].

4.6. Interlacing Theorem

A mathematically satisfactory calculation of the wave function should proceed from the resolvent equation. Since the eigenvalues are isolated we can find the projection operator to the subspace corresponding to this eigenvalue by a contour integral [46]:

$$\langle x|\mathbb{P}_k|y\rangle = \psi_k(x)\psi_k^*(y) = -\frac{1}{2\pi i}\oint_{\Gamma_k} \mathrm{d}E\ R(x,y|E), \tag{4.89}$$

where Γ_k is a small contour enclosing the isolated eigenvalue $-\nu_k^2$. We note that the free Green's functions $R_0(x,y|E)$ will not contain any poles on the negative real axis, so all the poles on the negative real axis will come from the poles of inverse principal matrix $\Phi^{-1}(E)$. For simplicity we can assume that the eigenvalues are nondegenerate and let us denote the k-th eigenvalue of the principal matrix as ω^k so eigenvalue problem is

$$\sum_{j=1}^{N} \Phi_{ij}(-\nu_k^2)A_j^k(-\nu_k^2) = \omega_k(-\nu_k^2)A_j^k(-\nu_k^2)\ . \tag{4.90}$$

Since the principal matrix is Hermitian on the real line and

$$\Phi_{ij}^{\dagger}(E) = \Phi_{ij}(E^*)\ , \tag{4.91}$$

on the complex plane, there exists a holomorphic family of projection operators on the complex plane [56], so that we can apply the spectral theorem for the principal matrix $\Phi(E)$

$$\Phi_{ij}(E) = \sum_k \omega^k(E)\mathbb{P}_k(E)_{ij}\ , \tag{4.92}$$

here $\mathbb{P}_k(E)_{ij} = A_i^{k*}(E)A_j^k(E)$, $A_i^k(E)$ is the normalized eigenvector corresponding to the eigenvalue $\omega^k(E)$. Similarly, we can write the spectral resolution of the inverse principal matrix,

$$\Phi_{ij}^{-1}(E) = \sum_k \frac{1}{\omega^k(E)} \mathbb{P}_k(E)_{ij} \; . \tag{4.93}$$

The residue can then be found

$$\begin{aligned}
&\lim_{E\to-\nu_k^2} R_0(x,a_i|E)(E+\nu_k^2)\Phi_{ij}^{-1}(E)R_0(a_j,y|E) \\
&= R_0(x,a_i|-\nu_k^2)\left[\left.\frac{\partial\omega^k(E)}{\partial E}\right|_{E=-\nu_k^2}\right]^{-1} \mathbb{P}_k(-\nu_k^2)_{ij} R_0(a_j,y|-\nu_k^2) \; .
\end{aligned} \tag{4.94}$$

Now we will look at the variations of the eigenvalues of Φ as we change the parameters ν. Using

$$\omega^k(-\nu^2) = \left(A^k(-\nu^2), \Phi(-\nu^2)A^k(-\nu^2)\right) \; , \tag{4.95}$$

and as a consequence of Feynman-Hellman theorem [110] in the non-degenerate case, we have

$$\begin{aligned}
\frac{\partial\omega^k(-\nu^2)}{\partial\nu} &= \left(A^k(-\nu^2), \frac{\partial\Phi(-\nu^2)}{\partial\nu}A^k(-\nu^2)\right) \\
&= \sum_{i,j=1}^N A_i^{k*}(-\nu^2)\frac{\partial\Phi_{ij}(-\nu^2)}{\partial\nu}A_j^k(-\nu^2) \; .
\end{aligned} \tag{4.96}$$

Taking the derivative of the principal matrix with respect to ν from (4.28)

$$\left.\frac{\partial\Phi_{ij}(-\nu^2)}{\partial\nu}\right|_{\nu=\nu_k} = \int_0^\infty \frac{\mathrm{d}t}{\hbar}\left(\frac{2\nu_k t}{\hbar}\right) K_t(a_i,a_j;g)e^{-\frac{t\nu_k^2}{\hbar}} \; , \tag{4.97}$$

and inserting the equation (4.97) into the equation (4.96), we obtain

$$\begin{aligned}
&\frac{\partial\omega^k(E)}{\partial E} = -\frac{1}{2\nu}\frac{\partial\omega^k(-\nu^2)}{\partial\nu} \\
&= -\frac{1}{2\nu}\sum_{i,j=1}^N A_i^{k*}(-\nu^2)\int_0^\infty \frac{\mathrm{d}t}{\hbar}\left(\frac{2\nu t}{\hbar}\right) K_t(a_i,a_j;g)e^{-\frac{t\nu^2}{\hbar}} A_j^k(-\nu^2) \; .
\end{aligned} \tag{4.98}$$

If we evaluate (4.98) at $E = -\nu_k^2$, or at $\nu = \nu_k$, it yields

$$\frac{\partial \omega^k(E)}{\partial E}\bigg|_{E=-\nu_k^2} = -\sum_{i,j=1}^{N} A_i^{k*}(-\nu_k^2) \int_0^\infty \frac{\mathrm{d}t}{\hbar} \left(\frac{t}{\hbar}\right) K_t(a_i, a_j; g) e^{-\frac{t\nu_k^2}{\hbar}} A_j^k(-\nu_k^2) \; . \quad (4.99)$$

Note that the integral is finite in two and three dimensions due to upper bounds on the heat kernel. If we combine all these results, we get

$$\begin{aligned} \psi_k(x)\psi_k^*(y) \;&=\; -\frac{1}{2\pi i}(2\pi i) R_0(x, a_i | -\nu_k^2) \Bigg[-\sum_{i,j=1}^{N} A_i^*(-\nu_k^2) \int_0^\infty \frac{\mathrm{d}t}{\hbar} \left(\frac{t}{\hbar}\right) \\ &\times K_t(a_i, a_j; g) e^{-\frac{t\nu_k^2}{\hbar}} A_j(-\nu_k^2) \Bigg]^{-1} A_i(-\nu_k^2)^* A_j(-\nu_k^2) R_0(a_j, y | -\nu_k^2). \quad (4.100) \end{aligned}$$

Note that we have written $A_i^k(-\nu_k^2)$ as $A_i(-\nu_k^2)$ for simplicity. Then, we can directly read off the bound state wave function from the equation above and get the same result (4.40). Incidentally the equation (4.99) implies an interesting result for the variation of eigenvalues,

$$\begin{aligned} \frac{\partial \omega^k(-\nu^2)}{\partial \nu}\bigg|_{\nu=\nu_k} &= \sum_{i,j=1}^{N} A_i^*(-\nu_k^2) \int_0^\infty \frac{\mathrm{d}t}{\hbar} \left(\frac{2\nu_k t}{\hbar}\right) K_t(a_i, a_j; g) e^{-\frac{t\nu_k^2}{\hbar}} A_j(-\nu_k^2) \\ &= \sum_{i,j=1}^{N} A_i^*(-\nu_k^2) \int_0^\infty \frac{\mathrm{d}t}{\hbar} \left(\frac{2\nu_k t}{\hbar}\right) \int_{\mathcal{M}} \mathrm{d}_g x \; K_{t/2}(a_i, x; g) K_{t/2}(x, a_j; g) e^{-\frac{t\nu_k^2}{\hbar}} A_j(-\nu_k^2) \\ &= \int_0^\infty \frac{\mathrm{d}t}{\hbar} \left(\frac{2\nu_k t}{\hbar}\right) e^{-\frac{t\nu_k^2}{\hbar}} \int_{\mathcal{M}} \mathrm{d}_g x \sum_{i=1}^{N} \left| K_{t/2}(a_i, x; g) A_i(-\nu_k^2) \right|^2 \; , \quad (4.101) \end{aligned}$$

where we have used the fact that $\nu_k \in \mathbb{R}^+$ for all k and properties of heat kernel (3.36) and (3.37) and the order of the integration and the finite sum can be interchanged. We can easily see that the above equation (4.101) is strictly positive due to the positivity of heat kernel, so

$$\frac{\partial \omega^k(-\nu^2)}{\partial \nu}\bigg|_{\nu=\nu_k} > 0 \; . \quad (4.102)$$

Energy eigenvalues $E = -\nu^2$ are obtained from the zeros of the eigenvalues of the principal matrix, that is, $\omega^k(-\nu_k^2) = 0$, and there is a unique solution for each $\omega^k(-\nu^2)$. We also know that for sufficiently small values of ν, the matrix $\Phi(-\nu^2)$ becomes negative, hence no zeros exists beyond some critical point.

With this result in mind, let us see what can be said about the comparison of the energy eigenvalues for the different number of delta centers. In order to see this, we need the Cauchy interlacing theorem in mathematics literature [111], which states that if we delete the last row and column of an Hermitian $(N+1)\times(N+1)$ matrix, the eigenvalues of the original matrix is interlaced by the eigenvalues of the new matrix, i.e. if $\omega^1(-\nu^2) \leq \omega^2(-\nu^2) \leq \cdots \leq \omega^{N+1}(-\nu^2)$ lists the eigenvalues of the original $(N+1)\times(N+1)$ matrix and if $\tilde{\omega}^1(-\nu^2) \leq \tilde{\omega}^2(-\nu^2) \leq \cdots \leq \tilde{\omega}^N(-\nu^2)$ lists the eigenvalues of the reduced matrix (any $N \times N$ principal submatrix of the $(N+1)\times(N+1)$ matrix), then we have

$$\omega^1(-\nu^2) \leq \tilde{\omega}^1(-\nu^2) \leq \omega^2(-\nu^2) \leq \tilde{\omega}^2(-\nu^2) \leq \cdots \leq \tilde{\omega}^N(-\nu^2) \leq \omega^{N+1}(-\nu^2) \; . \quad (4.103)$$

We assume that $(N+1)\times(N+1)$ matrix in the above-mentioned theorem is the principal matrix $\Phi^{N+1}(-\nu^2)$ corresponding to a certain arrangement of $N+1$ delta potentials. If we now delete the last row and the column it means that we remove the $(N+1)$th delta center from the system. The bound state problem of N centers corresponds to zero eigenvalue of the principal matrix Φ^N, and let us denote that bound state energy as $\tilde{E}$:

$$\tilde{\omega}^k(-\tilde{\nu}_k^{*2}) = 0, \quad \tilde{E}_k = -\tilde{\nu}_k^{*2} \; , \quad (4.104)$$

if it exists. By the Cauchy interlacing result, we then expect the following inequality

$$\cdots < \omega^k(-\tilde{\nu}_k^{*2}) < \tilde{\omega}^k(-\tilde{\nu}_k^{*2}) = 0 < \omega^{k+1}(-\tilde{\nu}_k^{*2}) < \cdots \; , \quad (4.105)$$

From the positivity of the derivative of the eigenvalues with respect to the argument (4.102), the eigenvalues ω are monotonically increasing functions. Hence, in order to get a zero root of $\omega^k(-\tilde{\nu}_k^{*2})$, we should increase $\tilde{\nu}_k^*$ to a higher value ν_k^*. As a result, $\omega^k(-\nu_k^{*2}) = 0$ if $\nu_k^{*2} > \tilde{\nu}_k^{*2}$, i.e. $E_k = -\nu_k^{*2} < \tilde{E}_k = -\tilde{\nu}_k^{*2}$. Thus, the energies also interlace in the same manner–this is a nonlinear analog of Sturm's comparison theorem of the eigenvalues. Moreover, $E_{gr}^{N+1} < E_{gr}^{N} < E_{gr}^{1} = -\mu_k^2$, that is, the ground state is always negative and approaches the bound state energy for the one delta center as N gets smaller. We can also generalize these results to the degenerate cases but the proof is more cumbersome.

4.7. Perturbation Theory

It is worth noting that we do not have to solve for the energy eigenvalues of the bound state while performing our non-perturbative renormalization method. In other words, although our renormalization method makes the problem well defined and finite, the energy eigenvalues must be found after this finite formulation has been constructed. If we cannot solve the problem exactly after the renormalization, we must apply the standard approximation methods, such as perturbation theory and variational techniques. An interesting estimate for our problem can be given by perturbation theory. For simplicity, we assume that all binding energies $-\mu_k^2$'s are different and the minimum binding energy of the k-th singular potential is much larger than the correlation energy between the k-th fixed center and the l-th center, that is to say, we assume $\frac{\hbar^2}{2md^2(a_k,a_l)} \ll \mu_{k,min}^2$ (on a noncompact manifold we may assume that the geodesic distance between the centers is large). This assumption makes the off-diagonal elements of the principal matrix much smaller than its diagonal elements. For this reason, let us separate the principal matrix for $E_k = -\nu_k^2$ as the sum of a diagonal matrix and an off-diagonal matrix, which is very small compared to the diagonal part:

$$\Phi(-\nu_k^2) = \Phi_D(-\nu_k^2) + \delta\Phi(-\nu_k^2) \ . \tag{4.106}$$

Since $\Phi(-\nu_k^2)$ is Hermitian, we can apply standard perturbation techniques to our problem. The eigenvalue problem for the principal matrix we wish to solve is given in (4.90). We again suppose that there is no degeneracy for simplicity. The energy eigenvalue changes to $E_k = E_k^{(0)} + \delta E_k$ or

$$\nu_k = \nu_k^{(0)} + \delta\nu_k \ . \tag{4.107}$$

Following the basic idea of perturbation theory in finite dimensional spaces, we assume the following expansions for the eigenvalues and eigenvectors:

$$\begin{aligned} \omega^k &= \omega^{k(0)} + \omega^{k(1)} + \omega^{k(2)} + \dots \ , \\ A_i^k &= A_i^{k(0)} + A_i^{k(1)} + A_i^{k(2)} + \dots \ , \end{aligned} \tag{4.108}$$

and the solution to the related unperturbed eigenvalue problem

$$\sum_{j=1}^{N} \left[\Phi_D(-\nu_k^2)\right]_{ij} A_j^{k(0)}(-\nu_k^2) = \omega^{k(0)}(-\nu_k^2) A_i^{k(0)}(-\nu_k^2) \ , \tag{4.109}$$

is given by

$$\omega^{k(0)}(-\nu_k^2) = \int_0^\infty \frac{\mathrm{d}t}{\hbar} K_t(a_k, a_k; g)[e^{-t\mu_k^2/\hbar} - e^{-t\nu_k^2/\hbar}] \ . \tag{4.110}$$

Then, the energy eigenvalues can easily be found from the condition $\omega^{k(0)}(-\nu_k^2) = 0$:

$$E_k^{(0)} = -\mu_k^2 \qquad \text{or} \qquad \nu_k^{(0)} = \mu_k \ , \tag{4.111}$$

and eigenvectors are

$$A^{k(0)}(\mu_k) \equiv A^{k(0)} \equiv \mathbf{e}^k \equiv \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \ , \tag{4.112}$$

where 1 is located in the kth position of the column and other elements of it are zero or we can write $A_i^{k(0)} = e_i^k = \delta_{ki}$. Here e_i^k s form a complete orthonormal set of basis vectors.

$$\sum_{i=1}^{N} e_i^k e_i^l = \delta_{kl} \ . \tag{4.113}$$

We must emphasize that there is no distinction between upper and lower indices for our purposes. Once we have found the solution of the diagonal part of the principal matrix or unperturbed eigenvalue problem, we can perturbatively solve the whole problem. The standard perturbation theory gives us the first and second order eigenvalues:

$$\begin{aligned}
\omega^{k(1)}(-\nu_k^2) &= \sum_{i,j=1}^{N} e_i^k \left[\delta\Phi(-\nu_k^2)\right]_{ij} e_j^k = \left[\delta\Phi(-\nu_k^2)\right]_{kk} = 0 \;, \\
\omega^{k(2)}(-\nu_k^2) &= \sum_{\substack{l=1 \\ l\neq k}}^{N} \frac{\left|\sum_{i,j=1}^{N} e_i^l \, [\delta\Phi(-\nu_k^2)]_{ij} \, e_j^k\right|^2}{\omega^{k(0)}(-\nu_k^2) - \omega^{l(0)}(-\nu_k^2)} \\
&= \sum_{\substack{l=1 \\ l\neq k}}^{N} \frac{\Phi_{lk}(-\nu_k^2)\Phi_{kl}(-\nu_k^2)}{\omega^{k(0)}(-\nu_k^2) - \omega^{l(0)}(-\nu_k^2)} ,
\end{aligned} \tag{4.114}$$

respectively. Hence the energy eigenvalues of the whole problem can be determined from

$$\omega^k(-\mu_k^2 + \delta E_k) = \omega^{k(0)}(-\mu_k^2 + \delta E_k) + \omega^{k(2)}(-\mu_k^2 + \delta E_k) + \ldots = 0 \;, \tag{4.115}$$

and $\omega^{k(0)}(-\nu_k^2)$ and $\Phi_{kl}(-\nu_k^2)$ for $k \neq l$ can be expanded around $\nu_k = \mu_k$

$$\begin{aligned}
\omega^{k(0)}(-\mu_k^2 + \delta E_k) &= \left.\frac{\partial \omega^{k(0)}(-\nu_k^2)}{\partial \nu_k}\right|_{\nu_k=\mu_k} \delta\nu_k + O(\delta^2\nu_k) \;, \\
\Phi_{kl}(-\mu_k^2 + \delta E_k) &= \Phi_{kl}(-\mu_k^2) + \left.\frac{\partial \Phi_{kl}(-\nu_k^2)}{\partial \nu_k}\right|_{\nu_k=\mu_k} \delta\nu_k + O(\delta^2\nu_k) \;,
\end{aligned} \tag{4.116}$$

where we have used the fact $\omega^{k(0)}(-\mu_k^2) = 0$. If we substitute (4.116) into (4.115) and (4.114), and use Feynman-Hellman theorem (4.96), we obtain

$$\begin{aligned}
0 = &\left.\frac{\partial \Phi_{kk}(-\nu_k^2)}{\partial \nu_k}\right|_{\nu_k=\mu_k} \delta\nu_k - \sum_{\substack{l=1 \\ l\neq k}}^{N} \frac{1}{\Phi_{ll}(-\mu_k^2)} \Bigg[\Phi_{kl}(-\mu_k^2)\Phi_{lk}(-\mu_k^2) \\
&+ \left(\Phi_{kl}(-\mu_k^2) \left.\frac{\partial \Phi_{lk}(-\nu_k^2)}{\partial \nu_k}\right|_{\nu_k=\mu_k} + \Phi_{lk}(-\mu_k^2) \left.\frac{\partial \Phi_{kl}(-\nu_k^2)}{\partial \nu_k}\right|_{\nu_k=\mu_k} \right) \delta\nu_k \Bigg] \\
\times &\left[1 + \frac{1}{\Phi_{ll}(-\mu_k^2)} \left(\left.\frac{\partial \Phi_{ll}(-\nu_k^2)}{\partial \nu_k}\right|_{\nu_k=\mu_k} - \left.\frac{\partial \Phi_{kk}(-\nu_k^2)}{\partial \nu_k}\right|_{\nu_k=\mu_k} \right) \delta\nu_k \right]^{-1} + O(\delta^2\nu_k) \;.
\end{aligned} \tag{4.117}$$

If we also expand the last factor in powers of $\delta\nu_k$ and ignore the second order terms and combine the terms using the symmetry property of principal matrix, we find

$$\left[\frac{\partial \Phi_{kk}(-\nu_k^2)}{\partial \nu_k}\bigg|_{\nu_k=\mu_k} + \sum_{\substack{l=1 \\ l\neq k}}^{N} \frac{\Phi_{kl}(-\mu_k^2)\Phi_{lk}(-\mu_k^2)}{\Phi_{ll}^2(-\mu_k^2)} \left(\frac{\partial \Phi_{ll}(-\nu_k^2)}{\partial \nu_k}\bigg|_{\nu_k=\mu_k} \right.\right.$$
$$\left.\left. - \frac{\partial \Phi_{kk}(-\nu_k^2)}{\partial \nu_k}\bigg|_{\nu_k=\mu_k} \right) - 2 \sum_{\substack{l=1 \\ l\neq k}}^{N} \frac{\Phi_{kl}(-\mu_k^2)}{\Phi_{ll}(-\mu_k^2)} \frac{\partial \Phi_{lk}(-\nu_k^2)}{\partial \nu_k}\bigg|_{\nu_k=\mu_k} \right] \delta\nu_k$$
$$= \sum_{\substack{l=1 \\ l\neq k}}^{N} \frac{\Phi_{kl}(-\mu_k^2)\Phi_{lk}(-\mu_k^2)}{\Phi_{ll}(-\mu_k^2)} + O(\delta^2 \nu_k) \ . \tag{4.118}$$

Ignoring the second and third terms on the left hand side of the equality due to the fact that $\Phi_{kk}(-\nu_k^2) \gg |\Phi_{kl}(-\nu_k^2)|$, we get the change in ν_k

$$\delta\nu_k \approx \left(\frac{\partial \Phi_{kk}(-\nu_k^2)}{\partial \nu_k}\bigg|_{\nu_k=\mu_k} \right)^{-1} \sum_{\substack{l=1 \\ l\neq k}}^{N} \frac{\Phi_{kl}(-\mu_k^2)\Phi_{lk}(-\mu_k^2)}{\Phi_{ll}(-\mu_k^2)} + O(\delta^2 \nu_k) \ , \tag{4.119}$$

so the change in the energy is $\delta E_k \approx -2\mu_k \delta\nu_k + O(\delta^2 \nu_k)$. Let us now consider how the bound state energy changes in the tunnelling regime for our problem in which $\frac{\hbar^2}{2md_{ij}^2} \ll \mu_{k,min}^2$. We must first calculate asymptotic behavior of the off-diagonal element of the principal matrix in this regime. In order to see this, we make the scaling transformation $t = u/B$, where $B = \hbar/2md_{ij}^2$ and use the scaling property of the heat kernel (3.50). In two dimensions, we have

$$\Phi_{ij}(-\mu_k^2) = -\int_0^\infty \frac{\mathrm{d}u}{\hbar} \ K_u(a_i, a_j; Bg) e^{-\frac{u\mu_k^2}{\hbar B}} \ . \tag{4.120}$$

In tunnelling regime, the most significant contribution to the integral comes from the region $u = 0$ due to the fact that integrand is suppressed by the exponential term for large values of u. Hence we can use the short time asymptotic of the heat kernel given in (3.60). The result is an integral representation of the modified Bessel function of the third kind [43]

$$\Phi_{ij}(-\mu_k^2) \sim -\frac{2d_{ij}^{1/2}}{(4\pi\hbar^2/2m)} K_0 \left(\sqrt{\frac{2md_{ij}^2\mu_k^2}{\hbar^2}} \right) \sum_i \Psi_i^{-1/2}(x,y) \; , \tag{4.121}$$

where we have used $d_{ij} \to B^{1/2} d_{ij}$ and $\Psi_i \to \Psi_i / B^{1/4}$ (for two dimensions) under the scaling transformation $g \to Bg$. The asymptotic expansion of $K_0(x)$ for large values of x

$$K_0(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \tag{4.122}$$

leads to

$$\Phi_{ij}(-\mu_k^2) \sim -\sqrt{\frac{\pi}{2}} \frac{\sum_i \Psi_i^{-1/2}(x,y)}{(\pi\hbar^2/m)} \left(\frac{\hbar^2}{2m\mu_k^2} \right)^{1/4} e^{-\frac{\sqrt{2m} d_{ij}\mu_k}{\hbar}} \; . \tag{4.123}$$

Here, large values of x corresponds to tunnelling regime in our problem. For three dimensional case, the idea is the same and the result would be

$$\Phi_{ij}(-\mu_k^2) \sim -\sqrt{\frac{2\pi}{m\hbar}} \frac{\sum_i \Psi_i^{-1/2}(x,y)}{(4\pi\hbar/2m)^{3/2}} e^{-\frac{\sqrt{2m} d_{ij}\mu_k}{\hbar}} \; . \tag{4.124}$$

Therefore, in the tunnelling regime, we can find the change in the bound state energy δE_k in the presence of other Dirac delta interactions by substituting (4.123) and (4.124) into (4.119). By using the positivity of the diagonal part of the principal matrix and derivative of it with respect to ν_k at μ_k.

$$\begin{aligned} \Phi_{ii}(-\mu_k^2) &> 0 \\ \left. \frac{\partial \Phi_{ii}(-\nu_k^2)}{\partial \nu_k} \right|_{\nu_k=\mu_k} &> 0 \; , \end{aligned} \tag{4.125}$$

we show that the bound state energy in the tunnelling regime gets exponentially smaller with increasing the geodesic distance between the centers, which is in agreement with the naive expectation in the standard quantum mechanics.

4.8. Pointwise Bounds on Wave function

There is extensive amount of literature on the exponential decays of the wave functions of the Schrödinger operators, which states that L^2 solutions of $(-\nabla^2+V)\psi = E\psi$ obey pointwise bounds of the form

$$|\psi(r)| \leq C_a e^{-ar} \ , \tag{4.126}$$

if the potential energy V is continuous and bounded below and E is in the discrete spectrum of $-\nabla^2 + V$ (see [46] for a review of the subject). The proofs given in the literature do not include the potentials which require renormalization and they are valid only for $\mathbb{R}^D$. We shall prove that it is still possible to get exponential pointwise bounds for our problem.

It is easy to see the upper bound of the wave function (4.40) by applying Cauchy Schwartz inequality

$$\begin{aligned} |\psi_k(x)| &\leq \alpha \left| \sum_{i=1}^N A_i(-\nu_k^2) \int_0^\infty \frac{\mathrm{d}t}{\hbar} \, e^{-\frac{t\nu_k^2}{\hbar}} K_t(a_i, x; g) \right| \\ &\leq \alpha \left[\sum_{i=1}^N \left| \int_0^\infty \frac{\mathrm{d}t}{\hbar} \, e^{-\frac{t\nu_k^2}{\hbar}} K_t(a_i, x; g) \right|^2 \right]^{1/2} \\ &\leq \alpha \sum_{i=1}^N \int_0^\infty \frac{\mathrm{d}t}{\hbar} \, e^{-\frac{t\nu_k^2}{\hbar}} K_t(a_i, x; g) \ , \end{aligned} \tag{4.127}$$

where we have used the fact that $\sum_{i=1}^N |A_i(-\nu_k^2)|^2 = 1$. Thanks to the upper bound on the heat kernel given in (3.63) and (3.79), we show that the wave function is pointwise bounded on $\mathcal{M}$. For compact manifolds, the upper bound (3.63) of heat kernel gives

$$\begin{aligned} |\psi_k(x)| \leq \ & 8\alpha A \sum_{i=1}^N \left[\frac{1}{V(\mathcal{M})} \sqrt{\frac{m d^2(a_i,x)}{\nu_k^2 \hbar^2 C_2}} K_1 \left(2 \sqrt{\frac{m d^2(a_i,x)\nu_k^2}{\hbar^2 C_2}} \right) \right. \\ & \left. + \frac{B(\varepsilon)}{\hbar \left(\hbar/2m\right)^{D/2}} \left(\frac{m d^2(a_i,x)}{\nu_k^2 C_2} \right)^{\frac{1}{2}-\frac{D}{4}} K_{\frac{D}{2}-1} \left(2 \sqrt{\frac{m d^2(a_i,x)\nu_k^2}{\hbar^2 C_2}} \right) \right] , \end{aligned} \tag{4.128}$$

where we use the following integral representation of $K_\nu(z)$ [43]

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \mathrm{d}s\; e^{-s-(z^2/4s)} s^{-\nu-1} \qquad |\arg z| < \frac{\pi}{4},\; \mathrm{Re}(\nu) > -\frac{1}{2}\,. \tag{4.129}$$

For $D = 2$ and $D = 3$, we can also find the upper bounds on Bessel functions K_0 and K_1 with another useful integral representation [43]:

$$K_\nu(z) = \frac{\sqrt{\pi} z^\nu}{2^\nu \Gamma(\nu + \frac{1}{2})} \int_0^\infty \mathrm{d}s\; e^{-z\cosh s} \sinh^{2\nu} s\,, \qquad \mathrm{Re}(z) > 0,\; \mathrm{Re}(\nu) > -\frac{1}{2}\,. \tag{4.130}$$

Using the inequality $\cosh s = \frac{e^s + e^{-s}}{2} > \frac{e^s}{2}$ for all $s \geq 0$ in (4.130), we have an upper bound $K_0(x)$ for $x \in \mathbb{R}^+$

$$K_0(x) \;<\; \int_0^\infty \mathrm{d}s\; e^{-\frac{x}{2}e^s}\,. \tag{4.131}$$

By subsequent change of variables $\xi = e^s$ and $\eta = \xi - 1$, we get

$$K_0(x) \;<\; \int_0^\infty \mathrm{d}\eta\; \frac{e^{-\frac{x(\eta+1)}{2}}}{\eta + 1}\,. \tag{4.132}$$

If we also define a new variable $z = x\eta$, we have

$$\begin{aligned} K_0(x) \;&<\; e^{-\frac{x}{2}} \int_0^\infty \mathrm{d}z\; \frac{e^{-\frac{z}{2}}}{z + x} \leq \frac{e^{-\frac{x}{2}}}{x} \int_0^\infty \mathrm{d}z\; e^{-\frac{z}{2}} \\ &\leq\; \frac{2}{x}\, e^{-\frac{x}{2}}\,. \end{aligned} \tag{4.133}$$

Alternatively, we can find a sharper bound for K_0 if we divide the integral above into two parts as follows:

$$\begin{aligned} K_0(x) \;&<\; e^{-\frac{x}{2}} \left[\int_0^1 \mathrm{d}z\; \frac{e^{-\frac{z}{2}}}{z + x} + \int_1^\infty \mathrm{d}z\; \frac{e^{-\frac{z}{2}}}{z + x}\right] \\ &\leq\; e^{-\frac{x}{2}} \left[\int_0^1 \mathrm{d}z\; \frac{1}{z + x} + \frac{1}{1 + x} \int_1^\infty \mathrm{d}z\; e^{-\frac{z}{2}}\right]\,. \end{aligned} \tag{4.134}$$

Hence, we have the following bound for K_0 which shows the logarithmic singularity near $x = 0$

$$\begin{aligned} K_0(x) &\leq e^{-\frac{x}{2}} \left[\ln\left(\frac{x+1}{x}\right) + \frac{2(1-e^{-1/2})}{1+x} \right] \\ &\leq \frac{2}{1+x} e^{-\frac{x}{2}} + e^{-\frac{x}{2}} \ln\left(\frac{x+1}{x}\right) . \end{aligned} \tag{4.135}$$

Using $\sinh^2 s = (\frac{e^s - e^{-s}}{2})^2 < \frac{e^{2s}}{4}$ for all $s \geq 0$ and following the same steps above for K_1, we find

$$K_1(x) \leq e^{-\frac{x}{2}} \left(\frac{1}{x} + \frac{1}{2} \right) . \tag{4.136}$$

Substituting (4.135) and (4.136) into (4.128) for $D = 2$, we get

$$\begin{aligned} |\psi_k(x)| \leq 8\alpha A \sum_{i=1}^{N} \Bigg[& \frac{1}{2V(\mathcal{M})} \sqrt{\frac{md^2(a_i,x)}{\nu_k^2 \hbar^2 C_2}} e^{-\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} \left(\frac{1}{\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} + 1 \right) \\ & + \frac{B(\varepsilon)}{\hbar\,(\hbar/2m)} e^{-\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} \left(\ln\left(\frac{2\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2} + 1}{2\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} \right) \right. \\ & \left. + \frac{2}{1 + 2\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} \right) \Bigg] , \end{aligned} \tag{4.137}$$

and for $D = 3$, we have

$$\begin{aligned} |\psi_k(x)| \leq\ & 8\alpha A \sum_{i=1}^{N} \left[\frac{1}{2V(\mathcal{M})} \sqrt{\frac{md^2(a_i,x)}{\nu_k^2 \hbar^2 C_2}} e^{-\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} \right. \\ & \left. \times \left(\frac{1}{\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} + 1 \right) + \frac{B(\varepsilon)\sqrt{\pi}}{\hbar\,(\hbar/2m)^{3/2}} \frac{e^{-2\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}}}{2\sqrt{md^2(a_i,x)/\hbar C_2}} \right] , \end{aligned} \tag{4.138}$$

where have used the explicit exact expression for $K_{\frac{1}{2}}$

$$K_{\frac{1}{2}}(u) = \sqrt{\frac{\pi}{2u}} e^{-u} . \tag{4.139}$$

We can repeat the same steps for Cartan-Hadamard manifolds by using the upper bounds of heat kernel given in (3.79) and the result is

$$|\psi_k(x)| \leq 2\alpha \sum_{i=1}^{N} \left[\frac{C(\varepsilon,\kappa)}{\hbar(4\pi\hbar/2m)} e^{-\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} \left(\ln\left(\frac{2\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}+1}{2\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} \right) + \frac{2}{1+2\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}} \right) \right], \tag{4.140}$$

for $D = 2$ and

$$|\psi_k(x)| \leq \alpha \sum_{i=1}^{N} \left[\frac{C(\varepsilon,\kappa)\sqrt{\pi}}{\hbar(4\pi\hbar/2m)^{3/2}} \frac{e^{-2\sqrt{md^2(a_i,x)\nu_k^2/\hbar^2 C_2}}}{\left(md^2(a_i,x)\nu_k^2/\hbar^2 C_2\right)^{1/4}} \right], \tag{4.141}$$

for $D = 3$. In standard quantum mechanics, the pointwise exponential bounds do not take into account singular interactions. Nevertheless, we prove that they are still valid for our problem in two and three dimensions.

In order to understand heuristically why our problem can be considered as a self-adjoint extension, which is also suggested by Krein's formula mentioned in Section 4.4, it is interesting to calculate the expectation value of the free energy for the bound state. The result (4.40) permits us to write the expectation value as

$$\langle\psi_k|H_0|\psi_k\rangle = \left[\sum_{i,j=1}^{N} A_i^*(-\nu_k^2) \int_0^\infty \frac{\mathrm{d}t}{\hbar} \left(\frac{t}{\hbar}\right) K_t(a_i,a_j;g) e^{-\frac{t\nu_k^2}{\hbar}} A_j(-\nu_k^2) \right]^{-1} \int_{\mathcal{M}} \mathrm{d}_g^D x \left[\int_0^\infty \frac{\mathrm{d}t_1}{\hbar} e^{-\frac{t_1\nu_k^2}{\hbar}} \sum_{i=1}^{N} A_i(-\nu_k^2) K_{t_1}(a_i,x;g) \right] \left[\int_0^\infty \frac{\mathrm{d}t_2}{\hbar} e^{-\frac{t_2\nu_k^2}{\hbar}} \sum_{j=1}^{N} A_j(-\nu_k^2) \left(-\frac{\hbar^2}{2m} \nabla_g^2 K_{t_2}(a_j,x;g) \right) \right], \tag{4.142}$$

where $H_0 = -\frac{\hbar^2}{2m}\nabla_g^2$.

Using (3.34) and applying an integration by parts to t_2 integral, we have

$$\langle\psi_k|H_0|\psi_k\rangle = -\left[\sum_{i,j=1}^{N} A_i(-\nu_k^2)\int_0^\infty \frac{\mathrm{d}t}{\hbar}\left(\frac{t}{\hbar}\right) K_t(a_i,a_j;g)e^{-\frac{t\nu_k^2}{\hbar}}\, A_j(-\nu_k^2)\right]^{-1}$$
$$\int_{\mathcal{M}} \mathrm{d}_g^D x \left[\int_0^\infty \frac{\mathrm{d}t_1}{\hbar}\, e^{-\frac{t_1\nu_k^2}{\hbar}} \sum_{i=1}^{N} A_i(-\nu_k^2) K_{t_1}(a_i,x;g)\right]\left[-\sum_{j=1}^{N} A_j(-\nu_k^2)\delta_g(x,a_j)\right.$$
$$\left. + \sum_{j=1}^{N} A_j(-\nu_k^2)\int_0^\infty \frac{\mathrm{d}t_2}{\hbar}\,\nu_k^2\, e^{-\frac{t_2\nu_k^2}{\hbar}} K_{t_2}(a_j,x;g)\right], \tag{4.143}$$

where we have used the initial condition of heat kernel (3.35). Integrating with respect to x and using the semigroup property of heat kernel (3.36), we obtain

$$\langle\psi_k|H_0|\psi_k\rangle = \left[\sum_{i,j=1}^{N} A_i^*(-\nu_k^2)\int_0^\infty \frac{\mathrm{d}t}{\hbar}\left(\frac{t}{\hbar}\right) K_t(a_i,a_j;g)e^{-\frac{t\nu_k^2}{\hbar}}\, A_j(-\nu_k^2)\right]^{-1}$$
$$\left[\int_0^\infty \frac{\mathrm{d}t_1}{\hbar}\, e^{-\frac{t_1\nu_k^2}{\hbar}} \sum_{i,j=1}^{N} A_i(-\nu_k^2)\; A_j(-\nu_k^2) K_{t_1}(a_i,a_j;g)\right.$$
$$\left. - \int_0^\infty \frac{\mathrm{d}t_1}{\hbar}\int_0^\infty \frac{\mathrm{d}t_2}{\hbar}\, \sum_{i,j=1}^{N} A_i(-\nu_k^2) A_j(-\nu_k^2) K_{t_1+t_2}(a_i,a_j;g) e^{-\frac{(t_1+t_2)\nu_k^2}{\hbar}}\,\nu_k^2\right]. \tag{4.144}$$

By change of variables $u = t_1 + t_2$ and $v = t_1 - t_2$, we find

$$\langle\psi_k|H_0|\psi_k\rangle = \left[\sum_{i,j=1}^{N} A_i(-\nu_k^2)\int_0^\infty \frac{\mathrm{d}t}{\hbar}\left(\frac{t}{\hbar}\right) K_t(a_i,a_j;g)e^{-\frac{t\nu_k^2}{\hbar}}\, A_j(-\nu_k^2)\right]^{-1}$$
$$\left[\int_0^\infty \frac{\mathrm{d}t_1}{\hbar}\, e^{-\frac{t_1\nu_k^2}{\hbar}} \sum_{i,j=1}^{N} A_i(-\nu_k^2)\; A_j(-\nu_k^2) K_{t_1}(a_i,a_j;g)\right.$$
$$\left. -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}u}{\hbar^2}\left(\int_{-u}^{u} \mathrm{d}v\right) \sum_{i,j=1}^{N} A_i(-\nu_k^2) A_j(-\nu_k^2) K_u(a_i,a_j;g) e^{-\frac{u\nu_k^2}{\hbar}}\,\nu_k^2\right]. \tag{4.145}$$

One can easily see that the $i = j$ term of the sum for the first term in the second line

$$\int_0^\infty \frac{\mathrm{d}t_1}{\hbar}\, e^{-\frac{t_1\nu_k^2}{\hbar}} |A_i(-\nu_k^2)|^2 K_{t_1}(a_i,a_i;g) \tag{4.146}$$

is divergent due to diagonal short time asymptotic of heat kernel (3.54) for $D \geq 2$ while the off- diagonal terms of this sum are convergent by upper bound on the heat kernel given in (3.63) and (3.79). The diagonal term in the last line is also convergent due to the term coming from v integral. Hence we find that the expectation value of the free Hamiltonian is divergent in the bound state energy wave function $\psi_k(x)$.

$$\langle \psi_k | H_0 | \psi_k \rangle \to \infty \; . \tag{4.147}$$

It is a well known fact that point interactions on $\mathbb{R}^D$ can be considered as a self-adjoint extension of the free Hamiltonian [5, 11]. We may heuristically think of our problem as a kind of self-adjoint extension since the wave function $\psi_k(x) \in L^2(\mathcal{M})$, that we have found, does not belong to the domain of the free Hamiltonian, whereas it lies in the domain of the full Hamiltonian H. Therefore, the self-adjoint extension of the formal free Hamiltonian extends the domain of it such that the states corresponding to the eigenfunctions $\psi_k(x)$ are included. The following section also supports our idea.

4.9. Existence of the Hamiltonian

Let Δ be a subset of the complex plane. A family $J(E)$, $E \in \Delta$ of bounded linear operators on the Hilbert space $\mathcal{H}$ under consideration, which satisfies the resolvent identity

$$J(E_1) - J(E_2) = (E_1 - E_2) J(E_1) J(E_2) \tag{4.148}$$

for E_1 , $E_2 \in \Delta$ is called a pseudo resolvent on Δ [112]. The following corollary (Corollary 9.5 in [112]) gives the condition for which there exists a densely defined closed linear operator A such that $J(E)$ is the resolvent family of A: Let Δ be a unbounded subset of $\mathbb{C}$ and $J(E)$ be a pseudo resolvent on Δ. If there is a sequence $E_n \in \Delta$ such that $|E_n| \to \infty$ as $n \to \infty$ and

$$\lim_{n \to \infty} -E_n J(E_n) x = x \; , \tag{4.149}$$

for all $x \in \mathcal{H}$, then $J(E)$ is the resolvent of a unique densely defined closed operator A. In order to show the resolvent kernel that we have found in (4.84) corresponds to a unique densely defined closed operator H, we need to prove that it satisfies the resolvent identity, i.e,

$$R(x,y|E_1) - R(x,y|E_2) = (E_1 - E_2) \int_{\mathcal{M}} \mathrm{d}_g^D z \; R(x,z|E_1) R(z,y|E_2) \; . \tag{4.150}$$

Substituting (4.84) into (4.150), we obtain

$$\begin{aligned}
R_0(x,y|E_1) - R_0(x,y|E_2) + \sum_{i,j=1}^{N} R_0(x,a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j,y|E_1) \\
- \sum_{i,j=1}^{N} R_0(x,a_i|E_2)\Phi_{ij}^{-1}(E_2)R_0(a_j,y|E_2) \\
= (E_1 - E_2) \int_{\mathcal{M}} \mathrm{d}_g^D z \; \Bigg[R_0(x,z|E_1)R_0(z,y|E_2) \\
+ \sum_{i,j=1}^{N} R_0(x,z|E_1)R_0(z,a_i|E_2)\Phi_{ij}^{-1}(E_2)R_0(a_j,y|E_2) \\
+ \sum_{i,j=1}^{N} R_0(x,a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j,z|E_1)R_0(z,y|E_2) \\
+ \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} R_0(x,a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j,z|E_1) \\
\times R_0(z,a_k|E_2)\Phi_{kl}^{-1}(E_2)R_0(a_l,y|E_2) \Bigg] \; .
\end{aligned} \tag{4.151}$$

Using the formula (3.52), it is easy to see that the free resolvent satisfies the resolvent identity

$$\begin{aligned}
&(E_1 - E_2) \int_{\mathcal{M}} \mathrm{d}_g^D z \; R_0(x,z|E_1)R_0(z,y|E_2) \\
&= (E_1 - E_2) \int_{\mathcal{M}} \mathrm{d}_g^D z \int_0^\infty \frac{\mathrm{d}t_1}{\hbar} K_{t_1}(x,z;g) e^{t_1 E_1/\hbar} \int_0^\infty \frac{\mathrm{d}t_2}{\hbar} K_{t_2}(z,y;g) e^{t_2 E_2/\hbar} \\
&= (E_1 - E_2) \int_0^\infty \int_0^\infty \frac{\mathrm{d}t_1 \, \mathrm{d}t_2}{\hbar^2} K_{t_1+t_2}(x,y;g) e^{t_1 E_1/\hbar} e^{t_2 E_2/\hbar} \\
&= \frac{(E_1 - E_2)}{2} \int_0^\infty \frac{\mathrm{d}u}{\hbar} \left[\int_{-u}^{u} \frac{\mathrm{d}v}{\hbar} K_u(x,y;g) e^{(u+v)E_1/2\hbar} e^{(u-v)E_2/2\hbar} \right] \\
&= \int_0^\infty \frac{\mathrm{d}u}{\hbar} K_u(x,y;g) \left(e^{uE_1/\hbar} - e^{uE_2/\hbar} \right) = R_0(x,y|E_1) - R_0(x,y|E_2) \; ,
\end{aligned} \tag{4.152}$$

where we have used the semigroup property of heat kernel (3.36) and made the change of variables $u = t_1 + t_2$, $v = t_1 - t_2$. Then, the equation (4.151) becomes

$$
\begin{aligned}
&\sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1)R_0(a_j, y|E_1) - \sum_{i,j=1}^{N} R_0(x, a_i|E_2)\Phi^{-1}_{ij}(E_2)R_0(a_j, y|E_2) \\
&= (E_1 - E_2) \int_{\mathcal{M}} \mathrm{d}^D_g z \left[\sum_{i,j=1}^{N} R_0(x, z|E_1)R_0(z, a_i|E_2)\Phi^{-1}_{ij}(E_2)R_0(a_j, y|E_2) \right. \\
&\qquad + \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1)R_0(a_j, z|E_1)R_0(z, y|E_2) \\
&\qquad + \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1)R_0(a_j, z|E_1) \\
&\qquad\qquad \left. \times R_0(z, a_k|E_2)\Phi^{-1}_{kl}(E_2)R_0(a_l, y|E_2) \right] .
\end{aligned} \tag{4.153}
$$

If we add and subtract the terms $\sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1)R_0(a_j, y|E_2)$, $\sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_2)R_0(a_j, y|E_2)$ on the left hand side of (4.153), and rearrange all the terms, we obtain

$$
\begin{aligned}
&\sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1) \left[R_0(a_j, y|E_1) - R_0(a_j, y|E_2)\right] \\
&\qquad + \sum_{i,j=1}^{N} R_0(x, a_i|E_1) \left[\Phi^{-1}_{ij}(E_1) - \Phi^{-1}_{ij}(E_2)\right] R_0(a_j, y|E_2) \\
&\qquad + \sum_{i,j=1}^{N} \left[R_0(x, a_i|E_1) - R_0(x, a_i|E_2)\right] \Phi^{-1}_{ij}(E_2)R_0(a_j, y|E_2) \\
&= (E_1 - E_2) \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1) \int_{\mathcal{M}} \mathrm{d}^D_g z \; R_0(a_j, z|E_1)R_0(z, y|E_2) \\
&\qquad + \sum_{i,j=1}^{N} R_0(x, a_i|E_1) \left[\Phi^{-1}_{ij}(E_1) - \Phi^{-1}_{ij}(E_2)\right] R_0(a_j, y|E_2) \\
&\quad + (E_1 - E_2) \sum_{i,j=1}^{N} \int_{\mathcal{M}} \mathrm{d}^D_g z \; R_0(x, z|E_1)R_0(z, a_i|E_2)\Phi^{-1}_{ij}(E_2)R_0(a_j, y|E_2) ,
\end{aligned} \tag{4.154}
$$

where we have used the result (4.152) in the first and third terms. The second term can be written as

$$\sum_{i,j=1}^{N} \sum_{k,l=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ik}(E_1) \left[\Phi_{kl}(E_2) - \Phi_{kl}(E_1)\right] \Phi^{-1}_{lj}(E_2) R_0(a_j, y|E_2) \; . \quad (4.155)$$

It is important to notice that difference of the principal matrix equals to the difference in free resolvent kernel, that is,

$$\begin{aligned} \Phi_{ij}(E_2) - \Phi_{ij}(E_1) &= R_0(a_i, a_j|E_1) - R_0(a_i, a_j|E_2) \\ &= (E_1 - E_2) \int_{\mathcal{M}} \mathrm{d}^D_g z \; R_0(a_i, z|E_1) R_0(z, a_j|E_2) \; , \end{aligned} \quad (4.156)$$

for any i and j. After substituting (4.156) into (4.155), the equation (4.154) becomes

$$\begin{aligned} &(E_1 - E_2) \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1) \int_{\mathcal{M}} \mathrm{d}^D_g z \; R_0(a_j, z|E_1) R_0(z, y|E_2) \\ &\qquad + (E_1 - E_2) \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1) \\ &\qquad \times \int_{\mathcal{M}} \mathrm{d}^D_g z \; R_0(a_j, z|E_1) R_0(z, a_k|E_2) \Phi^{-1}_{kl}(E_2) R_0(a_l, y|E_2) \\ &+ (E_1 - E_2) \sum_{i,j=1}^{N} \int_{\mathcal{M}} \mathrm{d}^D_g z \; R_0(x, z|E_1) R_0(z, a_i|E_2) \Phi^{-1}_{ij}(E_2) R_0(a_j, y|E_2) \; , \end{aligned} \quad (4.157)$$

This is exactly equal to (4.153) so resolvent identity is satisfied. We must now impose the following condition in L^2 norm

$$||E_n R(E_n) f + f|| \to 0 \; , \quad (4.158)$$

as $n \to \infty$ and f belongs to the Hilbert space $\mathcal{H}$ under consideration and the norm is taken with respect to $\mathcal{H}$. Let us choose the sequence $E_n = -nE_0$ since the resolvent is well defined in this resolvent set, in which we have no spectrum below the absolute value of the bound E_0 that we have found for the ground state energy. Without loss of generality, we can set $E_0 = c|E_*|$, where $c > 2$. Then, we have

$$||nE_0 R(-nE_0)f - f|| \to 0 \; , \tag{4.159}$$

as $n \to \infty$. Using (4.84) and separating the free part, we get

$$\begin{aligned} ||nE_0 R(-nE_0)f - f|| &\leq ||nE_0 R_0(-nE_0)f - f|| \\ &+ nE_0 ||R_0(-nE_0)\Phi^{-1}(-nE_0)R_0(-nE_0)f|| \; . \end{aligned} \tag{4.160}$$

Since it is well known that the first part of the sum converges to zero as $n \to \infty$, that is, free resolvent defines a self-adjoint densely defined closed operator (Laplacian) for the manifolds that we are interested in, we are going to investigate only the second term

$$\begin{aligned} & nE_0 ||R_0(-nE_0)\Phi^{-1}(-nE_0)R_0(-nE_0)f|| \\ & \qquad \leq nE_0 \Bigg[\sum_{i,j,k,l=1}^{N} \int_{\mathcal{M}} \mathrm{d}_g^D x \; R_0(a_i, x| - nE_0) R_0(x, a_l| - nE_0) \\ & \times \int_{\mathcal{M}} \mathrm{d}_g^D y \; R_0(a_j, y| - nE_0) R_0(y, a_k| - nE_0) |\Phi_{ij}^{-1}(-nE_0)||\Phi_{kl}^{-1}(-nE_0)| \Bigg]^{1/2} \; , \end{aligned} \tag{4.161}$$

where we have used the fact that the Hilbert space norm of an operator is smaller than its Hilbert-Schmidt norm: $||Af|| \leq \mathrm{Tr}^{1/2}(A^\dagger A)$ with $A = R_0(-nE_0)\Phi^{-1}(-nE_0)$ $R_0(-nE_0)$. By using (3.52) and the change of variables $u = t_1 + t_2$, $v = t_1 - t_2$, we get

$$\begin{aligned} & \int_{\mathcal{M}} \mathrm{d}_g^D x \; R_0(a_i, x| - nE_0) R_0(x, a_l| - nE_0) \\ & \qquad = \int_0^\infty \int_0^\infty \frac{\mathrm{d}t_1 \, \mathrm{d}t_2}{\hbar^2} \; K_{t_1+t_2}(a_i, a_l; g) e^{-n\frac{(t_1+t_2)E_0}{\hbar}} \\ & \qquad = \int_0^\infty \frac{\mathrm{d}t}{\hbar^2} \; t \; K_t(a_i, a_l; g) e^{-n\frac{tE_0}{\hbar}} \; . \end{aligned} \tag{4.162}$$

Let us first consider the diagonal case $i = l$ and $k = j$ in the above. Then, the equation (4.161) becomes

$$nE_0 \Bigg[\sum_{i,j=1}^{N} \int_{\mathcal{M}} \mathrm{d}_g^D x \; R_0(a_i, x| - nE_0) R_0(x, a_i| - nE_0) \times \int_{\mathcal{M}} \mathrm{d}_g^D y \; R_0(a_j, y| - nE_0) R_0(y, a_j| - nE_0) |\Phi_{ij}^{-1}(-nE_0)| |\Phi_{ji}^{-1}(-nE_0)| \Bigg]^{1/2} . \tag{4.163}$$

In this case, the upper bound of the equation (4.162) can be found from the upper bound of the heat kernel for compact manifolds

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \; R_0(a_i, x| - nE_0) R_0(x, a_l| - nE_0) \leq \frac{4A}{V(\mathcal{M})} \frac{1}{E_0^2 n^2} + \frac{4AB(\varepsilon)}{\hbar^2 (\hbar/2m)^{D/2}} \left(\frac{nE_0}{\hbar} \right)^{\frac{D}{2}-2} \Gamma\left(2 - \frac{D}{2}\right) , \tag{4.164}$$

and for Cartan-Hadamard manifolds

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \; R_0(a_i, x| - nE_0) R_0(x, a_l| - nE_0) \leq \frac{C(\varepsilon, \kappa)}{\hbar^2 (4\pi\hbar/2m)^{D/2}} \left(\frac{nE_0}{\hbar} \right)^{\frac{D}{2}-2} \Gamma\left(2 - \frac{D}{2}\right) . \tag{4.165}$$

In order to give the upper bound for the inverse principal matrix, we decompose the principal matrix into two positive matrices

$$\Phi = D - K \tag{4.166}$$

where D and K stand for the diagonal and off diagonal parts of the principal matrix. Then, it is easy to see $\Phi = D(\mathbf{1} - D^{-1}K)$. The principal matrix is invertible if and only if $(\mathbf{1} - D^{-1}K)$, and $(\mathbf{1} - D^{-1}K)$ has an inverse if the norm $||D^{-1}K|| < 1$. Then, we write the inverse of Φ as a geometric series

$$\begin{aligned} \Phi^{-1} &= (\mathbf{1} - D^{-1}K)^{-1} D^{-1} \\ &= \left(1 + (D^{-1}K) + (D^{-1}K)^2 + ...\right) D^{-1} , \end{aligned} \tag{4.167}$$

where we must have $||D^{-1}K|| < 1$. Since we are not concerned with the sharp bounds on Φ^{-1} for this problem, we can choose $||D^{-1}K|| < 1/2$ by adjusting nE_0 sufficiently large without loss of generality and get

$$|\Phi^{-1}| \leq 2|D^{-1}| \, . \tag{4.168}$$

The lower bound of the diagonal principal matrix for compact (4.198) and Cartan-Hadamard manifolds (4.220) gives the upper bound of the inverse principal matrix. Hence, we find

$$|\Phi_{ii}^{-1}(-nE_0)| \leq \begin{cases} \left(4\pi\hbar^2/2m\right) \ln^{-1}\left(nE_0/\mu^2\right) & \text{if } D = 2 \\ \dfrac{\hbar\,(4\pi\hbar/2m)^{3/2}}{2\sqrt{\pi}} \left(\sqrt{\dfrac{nE_0}{\hbar}} - \sqrt{\dfrac{\mu^2}{\hbar}}\right)^{-1} & \text{if } D = 3 \, , \end{cases} \tag{4.169}$$

for compact manifolds and

$$|\Phi_{ii}^{-1}(-nE_0)| \leq \begin{cases} \dfrac{(4\pi\hbar^2/2m)}{c} \ln^{-1}\left(\dfrac{\frac{nE_0}{\hbar} + \xi}{\frac{\mu^2}{\hbar} + \xi}\right) & \text{if } D = 2 \\ \dfrac{\hbar\,(4\pi\hbar/2m)^{3/2}}{2\sqrt{\pi}c} \left(\sqrt{\dfrac{nE_0}{\hbar} + \xi} - \sqrt{\dfrac{\mu^2}{\hbar} + \xi}\right)^{-1} & \text{if } D = 3 \, , \end{cases} \tag{4.170}$$

for Cartan-Hadamard manifolds. If we substitute the results (4.164) and (4.165) into (4.163) for $D = 2$, and take the limit $n \to \infty$, the result goes to zero. Since the norm is always positive, we prove

$$||nE_0 R(-nE_0)f - f|| \to 0 \tag{4.171}$$

as $n \to \infty$. It is almost evident that the off diagonal terms in the sum also vanishes in this limit primarily because these terms are exponentially damped $e^{-\sqrt{n}}$ due to the upper bound of the heat kernel. The flat case can be done by Fourier transform [32]. Unfortunately, the proof for $D = 3$ is more subtle and the volume growth conditions of the manifolds are very delicate in the analysis so we can not prove it for three dimensional case by the same approach. Instead we follow the following approach: In three dimensions, estimating the Hilbert space norm by Hilbert-Schmidt norm does not lead to zero. Instead we will first show that the last term

$$|E_n| \Bigg[\int_{\mathcal{M}} \mathrm{d}_g^3 x \sum_{i,j,k,l=1}^{N} R_0(x, a_i|E_n) \Phi_{ij}^{-1}(E_n) \int_{\mathcal{M}} \mathrm{d}_g^3 z \; R_0(a_j, z|E_n) f^*(z) \\ R_0(x, a_k|E_n) \Phi_{kl}^{-1}(E_n) \int_{\mathcal{M}} \mathrm{d}_g^3 y \; R_0(a_l, y|E_n) f(y) \Bigg]^{1/2} \tag{4.172}$$

goes to zero as $E_n \to -\infty$ for any $f \in L^2(\mathcal{M})$. From our previous argument, we know that the inverse of the principal matrix Φ satisfies:

$$\max_{ij} |\Phi_{ij}^{-1}(E_n)| \leq \frac{A_2}{|E_n|^{1/2}} \;, \tag{4.173}$$

where we define all the constant terms coming from the bounds of the heat kernel as A_2 (exact form of the constants is not important here) and ignore the term in the denominator for large values of n for simplicity, which can be read from (4.169) and (4.170). We shall use the notation for the constants coming from the bounds of the heat kernel combined with the other constants factors as $A_1, A_2, A_3, \ldots$ for simplicity. Moreover, we can combine the two resolvents with the common variable x, and as a result, we can express this combination as

$$\left[\int_0^\infty \frac{\mathrm{d}t}{\hbar} \, \frac{t}{\hbar} e^{-\frac{|E_n| t}{\hbar}} K_t(a_i, a_k; g) \right]^{1/2} , \tag{4.174}$$

and pull it out of the square root. Using similar arguments as before, we can show that this term in three dimensions, for both Cartan-Hadamard type manifolds and Compact manifolds (bounded Ricci), including the identical beginning and end points, is smaller than

$$\frac{A_3}{|E_n|^{1/4}} \;, \tag{4.175}$$

where A_3 can easily be read from the upper bound of the heat kernel. Hence we end up with the fact that the expression (4.172) is smaller than,

$$N A_4 |E_n|^{1/4} \sum_{j,l=1}^{N} \left[\int_{\mathcal{M}} \mathrm{d}_g^3 y \; R_0(a_j, z|E_n)|f(z)| \int_{\mathcal{M}} \mathrm{d}_g^3 z \; R_0(y, a_l)|f(y)| \right]^{1/2} . \qquad (4.176)$$

Hence we should show that the term

$$\int_{\mathcal{M}} \mathrm{d}_g^3 y \; R_0(a_j, y|E_n)|f(y)| \qquad (4.177)$$

decays faster than $|E_n|^{-1/4}$.

To do this, we will pick any one of the centers and choose Riemannian normal coordinates around it, we assume that the injectivity radius of the manifold is $\delta > 0$. But first we reexpress this term in terms of the heat kernel and use some bounds,

$$\begin{aligned} R_0(a_j, y|E_n) &= \int_0^\infty \frac{\mathrm{d}t}{\hbar} e^{-\frac{|E_n|t}{\hbar}} K_t(a_j, y; g) \\ &\leq A_5 \int_0^\infty \mathrm{d}t \; \frac{1}{t^{3/2}} e^{-\frac{m d^2(a_j, y)}{\hbar C_2 t} - \frac{|E_n| t}{\hbar}} , \end{aligned}$$

for Cartan-Hadamard manifolds, and for compact manifolds we have a similar term with an inverse volume term, $\frac{1}{V(\mathcal{M})}$, added. In the case of compact manifolds volume contribution term goes to zero faster than $|E_n|^{-1/4}$ as can be checked easily, so it causes no problems. In both cases we will concentrate on the least convergent part. If we evaluate the integral over t now we find,

$$\int_{\mathcal{M}} \mathrm{d}_g^3 y \; R_0(a_j, y|E_n)|f(y)| \leq A_6 \int_{\mathcal{M}} \mathrm{d}_g^3 y \; e^{-2\sqrt{\frac{m d^2(a_j, y)|E_n|}{\hbar^2 C_2}}} \frac{|f(y)|}{d(a_j, y)} . \qquad (4.178)$$

We divide the right hand side as,

$$\int_{B_\delta(a_j)} \mathrm{d}_g^3 y \; e^{-2\sqrt{\frac{md^2(a_j,y)|E_n|}{\hbar^2 C_2}}} \frac{|f(y)|}{d(a_j,y)} + \int_{\mathcal{M}\backslash B_\delta(a_j)} \mathrm{d}_g^3 y \; e^{-2\sqrt{\frac{md^2(a_j,y)|E_n|}{\hbar^2 C_2}}} \frac{|f(y)|}{d(a_j,y)} \; , \tag{4.179}$$

here the last term is smaller than

$$\begin{aligned} & \frac{e^{-\sqrt{\frac{m\delta^2|E_n|}{\hbar^2 C_2}}}}{\delta} \int_{\mathcal{M}\backslash B_\delta(a_j)} \mathrm{d}_g^3 y \; e^{-\sqrt{\frac{md^2(a_j,y)|E_n|}{\hbar^2 C_2}}} |f(y)| \\ & \leq \frac{e^{-\sqrt{\frac{m\delta^2|E_n|}{\hbar^2 C_2}}}}{\delta} \int_{\mathcal{M}} \mathrm{d}_g^3 y \; e^{-\sqrt{\frac{md^2(a_j,y)|E_n|}{\hbar^2 C_2}}} |f(y)| \\ & \leq \frac{e^{-\sqrt{\frac{m\delta^2|E_n|}{\hbar^2 C_2}}}}{\delta} \left[\int_{\mathcal{M}} \mathrm{d}_g^3 y \; e^{-2\sqrt{\frac{md^2(a_j,y)|E_n|}{\hbar^2 C_2}}} \right]^{1/2} ||f||_2 \; . \end{aligned}$$

By a theorem of Gaffney [64], for a stochastically complete manifold, for any $\alpha > 0$ (with the inverse length dimension) and any point a on the manifold, the integrals satisfy,

$$\int_{\mathcal{M}} \mathrm{d}_g^3 y \; e^{-\alpha d(a,y)} \leq \infty \; . \tag{4.180}$$

This establishes that the last term decays faster than $|E_n|^{-1/4}$, so we should look at the first part. As has been said we go to the Riemann normal coordinates of the geodesic ball of radius δ and write the integral in terms of Gaussian spherical representation:

$$\int_{\mathbb{S}^2} \mathrm{d}\Omega \int_0^\delta \mathrm{d}r \; r^2 J(r,\theta) e^{-2\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}} \frac{|f(r,\theta)|}{r} \; . \tag{4.181}$$

Let us recall that in Gaussian spherical coordinates, the integral of a function f on an D-dimensional Riemannian manifold $\mathcal{M}$ becomes,

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \; f(x) = \int_{\mathbb{S}^{D-1}} \mathrm{d}\Omega \int_0^{\rho_\Omega} \mathrm{d}r \; r^{D-1} f(r,\theta) J(r,\theta). \tag{4.182}$$

Here Ω denotes the direction in the tangent space around a point that we choose, and ρ_Ω refers to distance to the cut locus of the point in the direction Ω. We will now divide the integral over r to two parts,

$$\int_{\mathbb{S}^2} \mathrm{d}\Omega \int_0^{\Delta} \mathrm{d}r\ r^2 J(r,\theta) e^{-2\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}} \frac{|f(r,\theta)|}{r} + \int_{\mathbb{S}^2} \mathrm{d}\Omega \int_{\Delta}^{\delta} \mathrm{d}r\ r^2 J(r,\theta) e^{-2\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}} \frac{|f(r,\theta)|}{r}\ , \tag{4.183}$$

where $0 < \Delta < \delta$. Let us now introduce the following function:

$$\mathrm{sn}_k(r) = \begin{cases} \frac{\sin(\sqrt{k}r)}{\sqrt{k}} & \text{if } k > 0 \\ r & \text{if } k = 0 \\ \frac{\sinh(\sqrt{-k}r)}{\sqrt{-k}} & \text{if } k > 0\ . \end{cases} \tag{4.184}$$

This function is very useful for the Bishop-Gunther volume comparison theorems. We assume that $\mathcal{M}$ has Ricci tensor bounded from below by k_1, i.e. $\mathrm{Ric}(.,.) > k_1 g(.,.)$ and sectional curvature K bounded from above by k_2. Then, the Jacobian factor of the Gaussian spherical coordinates satisfies an inequality as follows [113]

$$\frac{\mathrm{sn}^2_{k_2}(r)}{r^2} < J(r,\theta) < \frac{\mathrm{sn}^2_{k_1}(r)}{r^2}. \tag{4.185}$$

In the second integral, we use

$$\begin{aligned} &\int_{\Delta}^{\delta} \mathrm{d}r\ r\ e^{-2\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}} \int_{\mathbb{S}^2} \mathrm{d}\Omega |f(r,\theta)| J^{1/2}(r,\theta)\ J^{1/2}(r,\theta) \\ &\quad \leq \left[\int_{\Delta}^{\delta} \mathrm{d}r\ r^2 \left(\int_{\mathbb{S}^2} \mathrm{d}\Omega |f(r,\theta)| J^{1/2}(r,\theta) \right)^2 \right]^{1/2} \left[\int_{\Delta}^{\delta} \mathrm{d}r\ \frac{\mathrm{sn}^2_{k_1}(r)}{r^2} e^{-4\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}} \right]^{1/2}, \end{aligned} \tag{4.186}$$

where we use the Bishop-Gunther volume comparison theorem again, for J. Let us now make the observation that there are constants A_+, A_-, which depend only on δ and k_i's such that

$$A_-(k_i,k_j) < \frac{\mathrm{sn}_{k_i}(r)}{\mathrm{sn}_{k_j}(r)} < A_+(k_i,k_j)\ , \tag{4.187}$$

for $r \in [0,\delta]$. This can now be invoked at the second piece, giving us,

$$\begin{aligned}
&\int_{\Delta}^{\delta} \mathrm{d}r\; r\; e^{-2\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}} \int_{\mathbb{S}^2} \mathrm{d}\Omega |f(r,\theta)| J^{1/2}(r,\theta)\; J^{1/2}(r,\theta) \\
&\leq \left[\left(\int_{\Delta}^{\delta} \mathrm{d}r\; r^2 \int_{\mathbb{S}^2} \mathrm{d}\Omega |f(r,\theta)|^2 J(r,\theta)\right)\left(\int_{\mathbb{S}^2} \mathrm{d}\Omega\right)\right]^{1/2} A_+(k_1,0)\left[\int_{\Delta}^{\delta} \mathrm{d}r e^{-4\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}}\right]^{1/2} \\
&\quad \leq ||f||_2 (4\pi)^{1/2} A_+(k_1,0) \frac{1}{2(m/\hbar^2 C_2)^{1/4} |E_n|^{1/4}} e^{-2\sqrt{\frac{m\Delta^2|E_n|}{\hbar^2 C_2}}} \; .
\end{aligned} \tag{4.188}$$

If we choose $\Delta = (\hbar^2 R/m)^{1/3} |E_n|^{-1/3}$, the exponent goes to zero as $|E_n| \to \infty$. For the first part of the integral we use the following characterization of essential supremum: let us define

$$\Lambda(\epsilon) = \mu(\{r \in [0,\Delta] |\; |r^{3/2} F(r)| > \epsilon\}), \tag{4.189}$$

then we have

$$\operatorname*{Essup}_{[0,\Delta]} |r^{3/2} F(r)| = \inf_{\epsilon} \{\epsilon | \Lambda(\epsilon) = 0\} \; . \tag{4.190}$$

Let us use now $F(r) = \int_{\mathbb{S}^2} \mathrm{d}\Omega f(r,\theta)$, and using Bishop-Gunther bound for the first part as,

$$\int_0^{\Delta} \mathrm{d}r\; r^{3/2} \int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)| \frac{e^{-2\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}}}{r^{1/2}} \frac{\mathrm{sn}_{k_1}^2(r)}{r^2} \tag{4.191}$$

which is smaller than;

$$\begin{aligned}
&A_+^2(k_1,0) \int_0^{\Delta} \mathrm{d}r\; r^{3/2} \int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)| \frac{e^{-2\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}}}{r^{1/2}} \\
&\quad \leq A_+^2(k_1,0) \Big(\operatorname*{Essup}_{[0,\Delta]} |r^{3/2} F(r)| \Big) \left(\int_0^{\Delta} \mathrm{d}r\; \frac{e^{-2\sqrt{\frac{mr^2|E_n|}{\hbar^2 C_2}}}}{r^{1/2}} \right) \\
&\quad \leq A_+^2(k_1,0) \Big(\operatorname*{Essup}_{[0,\Delta]} |r^{3/2} F(r)| \Big) \frac{1}{2(m/\hbar^2 C_2)^{1/4} |E_n|^{1/4}} \; .
\end{aligned}$$

If we know take the limit $\Delta = (\hbar^2/mR^2)^{1/3}|E_n|^{-1/3} \to 0$, we claim that the essential-supprempum goes to zero. To see this we observe by Markov inequality [114] that

$$\begin{aligned}
\Lambda(\epsilon) &\leq \frac{1}{\epsilon}\int_0^{\Delta} \mathrm{d}r\, |r^{3/2}F(r)| \\
&\leq \frac{1}{\epsilon}\left(\int_0^{\Delta} \mathrm{d}r\; r\right)^{1/2}\left(\int_0^{\Delta} \mathrm{d}r\; r^2\left(\int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)|\right)^2\right)^{1/2} \\
&\leq \frac{1}{\epsilon}\frac{\Delta}{\sqrt{2}}\left(\int_0^{\Delta} \mathrm{d}r\; \frac{r^2}{\mathrm{sn}_{k_2}^2(r)}\mathrm{sn}_{k_2}^2(r)\int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)|^2 \int_{\mathbb{S}^2} \mathrm{d}\Omega\right)^{1/2} \\
&\leq \frac{1}{\epsilon}\frac{\Delta}{\sqrt{2}}(4\pi)^{1/2}A_+(0,k_2)\left(\int_0^{\Delta} \mathrm{d}r\; \mathrm{sn}_{k_2}^2(r)\int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)|^2\right)^{1/2} \\
&\leq \frac{1}{\epsilon}\frac{\Delta}{\sqrt{2}}(4\pi)^{1/2}A_+(0,k_2)\left(\int_0^{\Delta} \mathrm{d}r\; r^2\int_{\mathbb{S}^2} \mathrm{d}\Omega\; J(r,\theta)|f(r,\theta)|^2\right)^{1/2} \\
&\leq \frac{1}{\epsilon}\frac{\Delta}{\sqrt{2}}(4\pi)^{1/2}A_+(0,k_2)||f||_2 \,.
\end{aligned}$$

Hence for any $\epsilon > 0$, as $\Delta \to 0$ we can make $\Lambda(\epsilon) = 0$, and the infimum goes to zero in this limit. As a result we see that the equation (4.172) is smaller than

$$\begin{aligned}
&\frac{A_+^2(k_1,0)}{2(m/\hbar^2 C_2)^{1/4}}\left(\operatorname*{Essup}_{[0,\Delta]}|r^{3/2}F(r)|\right) + ||f||_2(4\pi)^{1/2}\frac{A_+(k_1,0)}{2(m/\hbar^2 C_2)^{1/4}}e^{-|E_n|^{1/6}(mR^2/\hbar^2)^{1/6}} \\
&+ \frac{e^{-\sqrt{\frac{m\delta^2|E_n|}{\hbar^2 C_2}}}}{\delta}\left[\int_{\mathcal{M}} \mathrm{d}_g^3 y\; e^{-2\sqrt{\frac{md^2(a_j,y)|E_n|}{\hbar^2 C_2}}}\right]^{1/2} ||f||_2 \to 0 \quad \text{as} \quad |E_n| \to \infty \,,
\end{aligned} \tag{4.192}$$

and as a result it goes to zero as desired. This completes the proof.

4.10. Lower Bound of the Ground State Energy

Although we renormalize the model, we have not completely proven that the energy is bounded from below. A well-known theorem in matrix analysis, called Geršgorin theorem [115] states that all the eigenvalues ω of a matrix $\Phi \in M_N$ are located in the union of N discs

$$\bigcup_{i=1}^{N}\{|\omega - \Phi_{ii}| \leq R_i'(\Phi)\} \equiv G(\Phi) \; , \tag{4.193}$$

where $R_i'(\Phi) \equiv \sum_{i \neq j=1}^{N} |\Phi_{ij}|$ is the deleted absolute row sums. Since the matrix $\Phi_{ij}(E)$ is Hermitian due to the symmetry property of heat kernel, we have $\omega \in \mathbb{R}$. Indeed, all interesting eigenvalues are zero in our problem. When $\omega(E) = 0$ we get an energy eigenvalue E. If there is a lower bound on energy, that is, a bound on ground state energy, then we expect that there would be no solution at all beyond this lower bound, say $E_* = -\nu_*^2$. Then, we want $\omega = 0$ not to be an eigenvalue as we change E, thereby none of the discs defined above should contain the zero eigenvalue when $E < E_*$. This means that we should impose

$$| - \Phi_{ii}(E)| = |\Phi_{ii}(E)| > \sum_{\substack{j=1 \\ i \neq j}}^{N} |\Phi_{ij}(E)| \; , \tag{4.194}$$

for all i, that is, the principal matrix must be strictly diagonally dominant in order not to have a zero eigenvalue. However, before imposing this condition we can simplify the problem. We note that

$$\begin{aligned} |\Phi_{ii}(E)| &\geq |\Phi_{ii}(E)|^{\min} \\ \sum_{\substack{j=1 \\ i \neq j}}^{N} |\Phi_{ij}(E)| &\leq (N-1)|\Phi_{ij}(E)|^{\max} \; , \end{aligned} \tag{4.195}$$

so the above condition (4.194) is implied by the stronger requirement

$$|\Phi_{ii}(E)|^{\min} > (N-1)|\Phi_{ij}(E)|^{\max} \; . \tag{4.196}$$

Once we obtain a solution to this inequality, it is satisfied for all $E < E_*$ since the diagonal part of the principal matrix (4.28) is a decreasing function of E and the off-diagonal part of it is an increasing function for given a_i's and N. This means that there is no solution beyond this critical value E_*. Hence, the ground state energy must be larger than the critical value E_*:

$$E_{gr} \geq E_* \ . \tag{4.197}$$

The basic idea of the proof was given for special manifolds $\mathbb{S}^2$, $\mathbb{H}^2$ and $\mathbb{H}^3$ in our previous work [35] and more generally in [36].

4.10.1. Energy Bound for Compact Manifolds

For compact manifolds, the upper bound for the off-diagonal elements of heat kernel and the lower bound for on-diagonal part of the heat kernel has been given in (3.63) and (3.73), respectively. Using these bound estimates in (4.28), we find a lower bound for the principal matrix

$$\Phi_{ii}(E) \geq \begin{cases} \dfrac{1}{(4\pi\hbar^2/2m)} \ln\left(\nu^2/\mu^2\right) & \text{if } D=2 \\ \dfrac{2\sqrt{\pi}}{\hbar\left(4\pi\hbar/2m\right)^{3/2}} \left(\sqrt{\dfrac{\nu^2}{\hbar}} - \sqrt{\dfrac{\mu^2}{\hbar}}\right) & \text{if } D=3 \ , \end{cases} \tag{4.198}$$

and an upper bound for it

$$|\Phi_{ij}(E)| \leq \begin{cases} 4A\left[\sqrt{\dfrac{2}{C_2}} \dfrac{K_1\left(\sqrt{\frac{2}{C_2}\frac{\nu}{\mu_d}}\right)}{V(\mathcal{M})\nu\mu_d} + \dfrac{2B(\varepsilon)K_0\left(\sqrt{\frac{2}{C_2}\frac{\nu}{\mu_d}}\right)}{(4\pi\hbar^2/2m)}\right] & \text{if } D=2 \\ 4A\left[\sqrt{\dfrac{2}{C_2}} \dfrac{K_1\left(\sqrt{\frac{2}{C_2}\frac{\nu}{\mu_d}}\right)}{V(\mathcal{M})\nu\mu_d} + \dfrac{\sqrt{2\pi C_2}B(\varepsilon)\mu_d \exp\left(-\sqrt{\frac{2}{C_2}\frac{\nu}{\mu_d}}\right)}{\hbar^{3/2}\left(4\pi\hbar/2m\right)^{3/2}}\right] & \text{if } D=3 \ , \end{cases} \tag{4.199}$$

where $i \neq j$ and we have used the monotonic behavior of the functions in $\Phi_{ij}(E)$ so that we could maximize the principal matrix in which we defined $d \equiv \min_{i\neq j} d_{ij}$ and $\mu \equiv \max_i \mu_i$. We also introduced a natural energy scale $\mu_d^2 \equiv \frac{\hbar^2}{2md^2}$ for simplicity. In order to solve the inequality analytically, we must estimate the bounds on the Bessel and logarithm functions. We can estimate the lower bound of logarithmic function

[116]

$$\ln u > \frac{u-1}{u} \qquad \text{for } u > 0 \,, u \neq 1 \,. \tag{4.200}$$

Let us first consider the two dimensional case. As a result of bounds (4.133), (4.136) given for Bessel functions and the one for logarithmic function given above, we find

$$\Phi_{ii}(E) \geq \frac{2m}{\pi\hbar^2}\left(1 - \frac{1}{\nu/\mu}\right) \,, \tag{4.201}$$

and

$$\begin{aligned} |\Phi_{ij}(E)| \;&\leq\; 4A \exp\left(-\sqrt{\frac{1}{2C_2}}\frac{\nu}{\mu_d}\right)\left[\frac{1}{V(\mathcal{M})}\left(\frac{1}{\nu^2} + \frac{1}{\sqrt{2C_2}\nu\mu_d}\right)\right. \\ &+\; \left.\frac{2\sqrt{2C_2}B(\varepsilon)}{(4\pi\hbar^2/2m)\,\nu/\mu_d}\right] \,. \end{aligned} \tag{4.202}$$

Since $\nu > \mu_d$, we may have

$$\begin{aligned} |\Phi_{ij}(E)| \;&\leq\; 4A\frac{\exp\left(-\sqrt{\frac{1}{2C_2}}\frac{\nu}{\mu_d}\right)}{\nu/\mu_d} \\ &\quad \times\left[\frac{1}{V(\mathcal{M})\mu_d^2}\left(1 + \frac{1}{\sqrt{2C_2}}\right) + \frac{2\sqrt{2C_2}B(\varepsilon)}{(4\pi\hbar^2/2m)}\right] \,. \end{aligned} \tag{4.203}$$

Therefore, there will be no solution to the eigenvalue equation for values of the ground state energy below a critical value $\nu > \nu_*$ if the following inequality is satisfied

$$\begin{aligned} \frac{2m}{\pi\hbar^2}\left(1 - \frac{1}{\nu/\mu}\right) \;&>\; 4A(N-1)\frac{\exp\left(-\sqrt{\frac{1}{2C_2}}\frac{\nu}{\mu_d}\right)}{\nu/\mu_d}\left[\frac{1}{V(\mathcal{M})\mu_d^2}\left(1 + \frac{1}{\sqrt{2C_2}}\right)\right. \\ &+\; \left.\frac{2\sqrt{2C_2}B(\varepsilon)}{(4\pi\hbar^2/2m)}\right] \,. \end{aligned} \tag{4.204}$$

Hence, we can solve this and get the lower bound for the ground state energy

$$E_{gr} \geq -\nu_*^2 = -\mu_d^2\left[\frac{\mu}{\mu_d} + \sqrt{2C_2}\, W\Big((N-1)A_1\Big)\right]^2 \,, \tag{4.205}$$

where

$$A_1 \equiv \frac{\exp\left(-\sqrt{\frac{1}{2C_2}\frac{\mu}{\mu_d}}\right)}{\sqrt{2C_2}} \left[\frac{1}{V(\mathcal{M})\mu_d^2}\left(1+\frac{1}{\sqrt{2C_2}}\right) + \frac{2\sqrt{2C_2}B(\varepsilon)}{(4\pi\hbar^2/2m)}\right] 4A\left(\pi\hbar^2/2m\right) \,. \quad (4.206)$$

Here W is called Lambert-W function (also called the Omega or the ProductLog function [117]) and it is defined as the inverse function of xe^x. In other words,

$$y = xe^x \Longleftrightarrow x = W(y) \,. \quad (4.207)$$

As for the three dimensional case, the off-diagonal part of the principal matrix have the following upper bound

$$\begin{aligned} |\Phi_{ij}(E)| &\leq 4A\exp\left(-\sqrt{\frac{1}{2C_2}\frac{\nu}{\mu_d}}\right)\left[\frac{1}{V(\mathcal{M})\mu_d^2}\left(1+\frac{1}{\sqrt{2C_2}}\right)\right. \\ &+ \left.\frac{\sqrt{2\pi C_2}B(\varepsilon)\mu_d}{\hbar\,(4\pi\hbar^2/2m)}\right] , \end{aligned} \quad (4.208)$$

where we use weaker upper bound by using $\nu > \mu_d$ in order to solve the inequality. Therefore, we conclude that there exists a critical value of bound state energy $\nu > \nu_*$ for a given N and d such that $\omega \neq 0$ so the ground state energy cannot be less than $-\nu^{*2}$

$$E_{gr} \geq -\nu_*^2 = -\mu_d^2\left[\frac{\mu}{\mu_d} + \sqrt{\frac{C_2}{2}}\, W\left((N-1)A_2\right)\right]^2 , \quad (4.209)$$

where

$$\begin{aligned} A_2 &\equiv \frac{\exp\left(-\sqrt{\frac{2}{C_2}\frac{\mu}{\mu_d}}\right)}{\sqrt{2\pi C_2}}\left[\frac{1}{V(\mathcal{M})\mu_d^2}\left(1+\frac{1}{\sqrt{2C_2}}\right) + \frac{\sqrt{2\pi C_2}B(\varepsilon)\mu_d}{\hbar^{3/2}\,(4\pi\hbar/2m)^{3/2}}\right] \\ &\times \frac{\hbar^{3/2}4A\,(4\pi\hbar/2m)^{3/2}}{\mu_d} \,. \end{aligned} \quad (4.210)$$

Special Example $\mathbb{S}^2$: We remind that the principal matrix for $\mathbb{S}^2$ is given in Section 4.2. Although we have found a lower bound of the ground state energy for rather a large class of manifolds, we can consider the special cases as well. In fact, one can directly use the upper and lower bound formulas of the heat kernel obtained for general class of manifolds (compact manifolds or Cartan-Hadamard manifolds) by choosing the parameters (like lower bound of the Ricci scalar curvature of a manifold) associated with these special manifolds. We recall that the principal matrix is either expressed by the eigenfunctions and eigenvalues of the Laplacian or the heat kernel: (4.15) and (4.28). Therefore, if we can find an explicit expression of the principal matrix and estimate the eigenfunctions for a given manifold, then the bound state energy can be alternatively proven to be bounded from below by applying Geršgorin theorem, and this is what is basically done in [35]. We are not going to repeat this procedure since the general results have been already given including the special cases so all details can be found in [35].

For the first special case, we consider $\mathbb{S}^2$ as a concrete example for compact manifolds. Suppose that point interactions are located at the points given by the local coordinates $(\theta_i, \phi_i)_{i=1}^N$ on a sphere of radius R. Then, the Schrödinger equation for the bound states of a particle living on the sphere under the influence of N attractive delta interactions becomes

$$\left[-\frac{\hbar^2}{2m}\nabla^2_{g_{\mathbb{S}^2}} - \sum_{i=1}^{N} \lambda_i \, \delta^2(\theta-\theta_i, \phi-\phi_i)\right]\psi = -\nu^2\psi, \tag{4.211}$$

where $\nabla_{g_{\mathbb{S}^2}}$ is Laplacian on the sphere in spherical coordinates

$$\nabla^2_{g_{\mathbb{S}^2}} = \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \, , \tag{4.212}$$

and $\delta^2(\theta-\theta_i, \phi-\phi_i) = \frac{\delta(\theta-\theta_i)\,\delta(\phi-\phi_i)}{R^2\sin\theta}$ is the two dimensional delta function on the sphere centered at (θ_i, ϕ_i). It is well known that spherical harmonics Y_l^m are eigenfunctions of the Laplacian $-\nabla^2_{g_{\mathbb{S}^2}}$ with the eigenvalues $l(l+1)/R^2$ and form a complete orthonormal basis on $\mathbb{S}^2$. The principal matrix $\Phi_{ij}(-\nu^2)$ has been given in (4.11) in Section 4.2. We then choose the bare coupling constant λ_i as a function of the cut-off parameter Λ.

$$\lambda_i^{-1}(\Lambda) = \frac{1}{4\pi R^2} \sum_{l=0}^{\Lambda} \frac{2l+1}{\frac{\hbar^2}{2mR^2}\, l(l+1) + \mu_i^2} \,, \tag{4.213}$$

where $-\mu_i^2$ is experimentally measured value of bound state energy for the single delta interaction and taking the limit $\Lambda \to \infty$ of the difference, we have obtained

$$\begin{aligned} \lim_{\Lambda\to\infty} \left[\frac{1}{4\pi R^2} \sum_{l=0}^{\Lambda} \frac{2l+1}{\frac{\hbar^2}{2mR^2}\, l(l+1) + \mu_i^2} - \frac{1}{4\pi R^2} \sum_{l=0}^{\Lambda} \frac{2l+1}{\frac{\hbar^2}{2mR^2}\, l(l+1) + \nu^2} \right] \\ \longrightarrow \frac{1}{4\pi R^2 \mu_R^2} \left[h\left(\frac{\mu_i}{\mu_R}\right) - h\left(\frac{\nu}{\mu_R}\right) \right] \,, \end{aligned} \tag{4.214}$$

where $\mu_R^2 \equiv \frac{\hbar^2}{2mR^2}$. The function $h(x)$ here is defined as

$$h(x) \equiv \frac{1}{x^2} - H_{\frac{1}{2}-\sqrt{\frac{1}{4}-x^2}} - H_{\frac{1}{2}+\sqrt{\frac{1}{4}-x^2}} \,, \qquad x \in \mathbb{R}^+ \,, \tag{4.215}$$

where H's are the harmonic numbers, commonly defined on integers as $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and can be extended by analytical continuation to its largest domain in the entire complex plane as $H_z = \psi(z+1) + \gamma$, where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ being the digamma function [43] and γ being the Euler-Mascheroni constant. Due to the Schwarz reflection principle of harmonic numbers ($\bar{H}_z = H_{\bar{z}}$), the function $\phi(x)$ is real valued for all $x \in \mathbb{R}^+$. It is also easy to check $\lim_{\Lambda\to\infty} \frac{\lambda_j(\Lambda)}{\lambda_j(\Lambda)} \to 1$ in the non-diagonal part of (4.11), simply because they have the same form of divergence. Then, the principal matrix $\Phi(-\nu^2)$ for bound states can be eventually written as

$$\Phi_{ij}(-\nu^2) = \frac{1}{4\pi R^2 \mu_R^2} \begin{cases} h\left(\frac{\mu_i}{\mu_R}\right) - h\left(\frac{\nu}{\mu_R}\right) & i = j \\ \\ -\sum_{l=0}^{\infty} \frac{2l+1}{l(l+1) + \frac{\nu^2}{\mu_R^2}} \, P_l\left(1 - \frac{\hat{d}_{ij}^2}{2}\right) & i \neq j \,. \end{cases} \tag{4.216}$$

Hence, we have obtained a well-defined formulation of our problem, that is, the infinities have been removed. We then apply the Geršgorin theorem and estimate the upper bound of the off-diagonal part of the principal matrix with the help of the upper bound on the heat kernel based on the work by Li and Yau given in (3.61) and the lower bound of the diagonal part of the principal matrix with some analytical techniques given in [35]. Hence, we estimate the lower bound for the ground state energy

$$E_{gr} \geq -\nu^{*2} = -\left\{ \mu + \frac{\mu_R}{2\sqrt{2}} + 5\mu_d W \left[28\pi^2 \left(N-1\right) e^{-\frac{\left(\mu+\frac{\mu_R}{2\sqrt{2}}\right)}{5\mu_d}} \right] \right\}^2 . \tag{4.217}$$

We can also consider the large N behavior of the ground state energy. The asymptotic expansion of product-log function W [117] for large z is given as

$$W(z) \sim \ln z - \ln \ln z . \tag{4.218}$$

Hence, this leads to

$$E_{gr} \sim -\mu_d^2 \left[\ln \left(\frac{N}{\ln N} \right) \right]^2 . \tag{4.219}$$

4.10.2. Energy Bound for Cartan-Hadamard Manifolds

Similarly, using the upper and lower bounds of the heat kernel for Cartan-Hadamard manifolds, we have obtained the upper and lower bound on the off and on diagonal part of the principal matrix, respectively:

$$\Phi_{ii}(E) \geq \begin{cases} \dfrac{c}{(4\pi\hbar^2/2m)} \ln \left(\dfrac{\frac{\nu^2}{\hbar}+\xi}{\frac{\mu^2}{\hbar}+\xi} \right) & \text{if } D=2 \\ \dfrac{2\sqrt{\pi}c}{\hbar\,(4\pi\hbar/2m)^{3/2}} \left(\sqrt{\dfrac{\nu^2}{\hbar}+\xi} - \sqrt{\dfrac{\mu^2}{\hbar}+\xi} \right) & \text{if } D=3 \, , \end{cases} \tag{4.220}$$

where $\xi \equiv \frac{\hbar(\sigma_1(\mathcal{M})+\delta)}{2m} \geq 0$ and

$$\Phi_{ij}(E) \leq \begin{cases} \dfrac{2C(\varepsilon,\kappa)}{(4\pi\hbar^2/2m)} K_0\left(\sqrt{\dfrac{2}{C_2}}\dfrac{\nu}{\mu_d}\right) & \text{if } D=2 \\ \dfrac{\sqrt{2\pi C_2}C(\varepsilon,\kappa)\mu_d}{\hbar^{3/2}\left(4\pi\hbar/2m\right)^{3/2}} \exp\left(-\sqrt{\dfrac{2}{C_2}}\dfrac{\nu}{\mu_d}\right) & \text{if } D=3\ , \end{cases} \tag{4.221}$$

for $i \neq j$. For $D=2$, we have

$$\ln\left(\frac{\frac{\nu^2}{\hbar}+\xi}{\frac{\mu^2}{\hbar}+\xi}\right) \geq \ln\left(\frac{\xi}{\frac{\mu^2}{\hbar}+\xi}\right) \tag{4.222}$$

and if we use the same bounds for the Bessel function and the fact that $\nu > \mu_d$, we obtain the lower bound for the ground state energy

$$E_{gr} \geq -2C_2\mu_d^2\, W^2\left(\frac{2(N-1)C(\varepsilon,\kappa)}{\ln\left(\frac{\xi}{\frac{\mu^2}{\hbar}+\xi}\right)}\right)\ . \tag{4.223}$$

For three dimensional case, we must do some additional assumption in order to get an analytic solution. We will assume that $\nu \geq \mu^2+\hbar\xi$ and this assumption should be checked whether it is consistent or not after we have found the solution. Hence,

$$\left(\sqrt{\frac{\nu^2}{\hbar}+\xi}-\sqrt{\frac{\mu^2}{\hbar}+\xi}\right) \geq \left(\frac{\nu}{\hbar^{1/2}}-\sqrt{\frac{\mu^2}{\hbar}+\xi}\right)\ , \tag{4.224}$$

and finally

$$E_{gr} \geq -\mu_d^2\left[\frac{1}{\mu_d}\sqrt{\mu^2+\hbar\xi}+\sqrt{\frac{C_2}{2}}W\left(A_3(N-1)\right)\right]^2\ , \tag{4.225}$$

where

$$A_3 \equiv \frac{C(\varepsilon,\kappa)}{c}\exp\left(-\frac{1}{\mu_d}\sqrt{\frac{2}{C_2}\left(\mu^2+\hbar\xi\right)}\right)\ . \tag{4.226}$$

Special Examples: Hyperbolic Spaces $\mathbb{H}^2$ and $\mathbb{H}^3$

The hyperbolic space $\mathbb{H}^D$ is defined as maximally symmetric and simply connected complete D-dimensional Riemannian manifold with a constant negative sectional curvature $-1/R^2$, which is also in some sense considered to be the negative curvature analog of the sphere $\mathbb{S}^D$. For the hyperbolic spaces the explicit form of the heat kernel is known (3.27). Therefore, we can calculate the principal matrix (4.28) explicitly in this case.

D dimensional hyperbolic spaces $\mathbb{H}^D$ are defined in the upper half plane model as $\mathbb{H}^D = \{x \in \mathbb{R}^D | x_D > 0\}$, and the geodesic distance $d(x, y)$ is given as

$$\cosh \frac{d(x,y)}{R} = 1 + \frac{|x-y|^2}{2\, x_D\, y_D}, \tag{4.227}$$

where R is the scaling parameter. For $\mathbb{H}^2$, the Laplacian $\nabla^2_{g_{\mathbb{H}^2}}$ in polar coordinates (θ, ϕ) is given by

$$\nabla^2_{g_{\mathbb{H}^2}} = \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + \frac{2\coth\theta}{R^2}\frac{\partial}{\partial\theta} + \frac{1}{R^2 \sinh^2\theta}\frac{\partial^2}{\partial\phi^2} \; . \tag{4.228}$$

Then the principal matrix for $\mathbb{H}^2$ in the $\epsilon \to 0$ limit,

$$\Phi_{ij}(E) = \frac{1}{4\pi R^2}\frac{1}{\mu_R^2}\begin{cases} \sqrt{2}\left[\psi\left(\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{E}{\mu_R^2}}\right) - \psi\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu_i^2}{\mu_R^2}}\right)\right] & \text{if } i=j \\ -\int_{\frac{d_{ij}}{R}}^{\infty} \mathrm{d}r \, \frac{e^{-\frac{1}{2}r\sqrt{1-\frac{4E}{\mu_R^2}}}}{\sqrt{\cosh r - \cosh\frac{d_{ij}}{R}}} & \text{if } i \neq j, \end{cases} \tag{4.229}$$

where ψ is the digamma function. Using the Geršgorin Theorem (4.193) for this matrix, the lower bound of the ground state energy can be estimated [35] as

$$E_{gr} \geq -\nu^{*2} = -\mu_R^2\left[A + \frac{W\left(2A\frac{d}{R}(N-1)e^{\left(\frac{1}{2}-A\right)\ln\left(\cosh\frac{d}{R}+\frac{1}{2}\sinh\frac{d}{R}\right)}\right)}{\ln\left(\cosh\frac{d}{R}+\frac{1}{2}\sinh\frac{d}{R}\right)}\right]^2 , \tag{4.230}$$

where we define $A \equiv \frac{2}{1-\psi\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu^2}{\mu_R^2}}\right)}$ for simplicity of notation. For large values of N as long as the ratio $\frac{\mu}{\mu_R}$ and $\frac{d}{R}$ is finite, the behavior of the bound state energy is given by

$$E_{gr} \sim -\mu_R^2 \left[\frac{\ln N - \ln\ln N}{\ln\left(\cosh\frac{d}{R} + \frac{1}{2}\sinh\frac{d}{R}\right)} \right]^2 . \tag{4.231}$$

As for the hyperbolic space $\mathbb{H}^3$, the Laplacian $\nabla^2_{g_{\mathbb{H}^3}}$ in polar coordinates (r, θ, ϕ)

$$\nabla^2_{g_{\mathbb{H}^3}} = \frac{1}{R^3}\frac{\partial^2}{\partial r^2} + \frac{2\coth r}{R^3}\frac{\partial}{\partial r} + \frac{1}{R\sinh^2 r}\nabla^2_{g_{\mathbb{S}^2}} . \tag{4.232}$$

Similarly, we find the explicit principal matrix for $\mathbb{H}^3$

$$\Phi_{ij}(E) = \frac{1}{4\pi}\frac{1}{\mu_R^2 R^3} \begin{cases} \sqrt{1-\frac{E}{\mu_R^2}} - \sqrt{1+\frac{\mu_i^2}{\mu_R^2}} & \text{if } i = j \\ \\ -\frac{\mu_{d_{ij}}}{\mu_R}\frac{\frac{d_{ij}}{R}}{\sinh\frac{d_{ij}}{R}} \exp\left(-\frac{\mu_R}{\mu_{d_{ij}}}\sqrt{1-\frac{E}{\mu_R^2}}\right) & \text{if } i \neq j . \end{cases} \tag{4.233}$$

Using the Geršgorin Theorem (4.193), we obtain

$$\begin{aligned} E_{gr} \geq -\nu^{*2} = -\mu^2 - 2\mu_d\sqrt{\mu_R^2+\mu^2}\, W &\left[\frac{e^{-\frac{\mu_R}{\mu_d}\sqrt{1+\frac{\mu^2}{\mu_R^2}}}\, d(N-1)}{R\sinh\frac{d}{R}} \right] \\ &- \mu_d^2\, W\left[\frac{e^{-\frac{\mu_R}{\mu_d}\sqrt{1+\frac{\mu^2}{\mu_R^2}}}\, d(N-1)}{R\sinh\frac{d}{R}} \right]^2 . \end{aligned} \tag{4.234}$$

For the large N behavior of the ground state energy, the estimate becomes

$$E_{gr} \sim -2\mu_d\sqrt{\mu_R^2+\mu^2}\,[\ln N - \ln\ln N] - \mu_d^2\,[\ln N - \ln\ln N]^2 . \tag{4.235}$$

4.11. Non-degeneracy and Positivity of the Ground State

The rigorous proof of non-degeneracy and positivity of the ground state in standard quantum mechanics is given in [46, 118], which does not include the singular potentials. Therefore, it is necessary to check whether this is still valid for our problem. The proof in our case is based on Perron - Frobenius theorem [115]: It states that if $A \in M_N$ and suppose that $A > 0$ (that is, all $A_{ij} > 0$). Let $\rho(A) = \max\{|\omega| : \omega \text{ is an eigenvalue of } A\}$, called spectral radius. Then

(i) $\rho(A) > 0$;

(ii) $\rho(A)$ is an eigenvalue of A;

(iii) There is an $x \in \mathbb{C}^N$ with $x > 0$ and $Ax = \rho(A)x$;

(iv) $\rho(A)$ is an algebraically (and hence geometrically) simple eigenvalue of A;

(v) $|\omega| < \rho(A)$ for every eigenvalue $\omega \neq \rho(A)$, that is, $\rho(A)$ is the unique eigenvalue of maximum modulus. A simple proof of this theorem for the positive symmetric matrices is given in [119] which we reproduce below:

Proof: (i) Since $A_{ij} > 0$, $\mathrm{Tr}(A) = \sum_{i=1}^{N} \lambda_i > 0$, where λ_i is the eigenvalue of the matrix A. Then, we have $|\sum_{i=1}^{N} \lambda_i| > 0$, which immediately gives $\sum_{i=1}^{N} |\lambda_i| > 0$. Here the eigenvalues of the symmetric matrix A are real. Let us suppose that $\rho(A) = \max |\lambda_i|$, where λ_i is the i-th eigenvalue of A. Then, we get

$$n\rho(A) + \sum_{\substack{j=1 \\ j\neq i}}^{N-n} |\lambda_j| > 0 \tag{4.236}$$

where n is the degeneracy. If $\rho(A) = 0$, it means that we must have $\sum_{\substack{j=1 \\ j\neq i}}^{N-n} |\lambda_j| > 0$, which is inconsistent with the assumption $\rho(A) = 0$. We then obtain $\rho(A) \neq 0$ and from there, $\rho(A) > 0$.

(ii) Using the result of (a), that is, $\rho(A) > 0$ and $\sum_{i=1}^{N} \lambda_i > 0$, it follows that $\rho(A)$ is an eigenvalue of A since

$$\rho(A) = \max\{|\lambda_i|, \lambda_i \in \sigma(A)\} \tag{4.237}$$

(iii) Let u_j be any real normalized eigenvector belonging to $\rho(A)$.

$$\rho(A)u_i = \sum_{j=1}^{N} A_{ij}u_j \tag{4.238}$$

Let us also define $x_j = |u_j|$. Then it follows that

$$0 < \rho(A) = \sum_{i,j=1}^{N} A_{ij}u_iu_j = \left|\sum_{i,j=1}^{N} A_{ij}u_iu_j\right| \leq \sum_{i,j=1}^{N} |A_{ij}u_iu_j| = \sum_{i,j=1}^{N} A_{ij}x_ix_j \tag{4.239}$$

By the Rayleigh-Ritz variational theorem, the right-hand side is less than or equal to $\rho(A)$, with the equality if and only if x_j is an eigenvector belonging to $\rho(A)$. We then get

$$\rho(A)x_i = \sum_{j=1}^{N} A_{ij}x_j \tag{4.240}$$

for all i. Now if $x_i = 0$ for some i, then due to the fact $A_{ij} > 0$ for all j, it follows every $x_j = 0$, which is inconsistent. Thus every $x_j > 0$.

(iv) If $\rho(A)$ is degenerate, we can find two real orthonormal eigenvectors u_i and v_j corresponding to the eigenvalue $\rho(A)$. Suppose that $u_i < 0$ for some i. Then, we must have

$$\rho(A)u_i + \rho(A)x_i = \sum_{j=1}^{N} A_{ij}u_j + A_{ij}x_j = \sum_{j=1}^{N} A_{ij}(u_j + |u_j|) \tag{4.241}$$

or

$$0 = \rho(A)(u_i + |u_i|) = \sum_{j=1}^{N} A_{ij}(u_j + |u_j|) \tag{4.242}$$

which leads to $u_j + |u_j| = 0$ for every j. In other words, we have either $u_j = |u_j| > 0$ for every j or $u_j = -|u_j|$ for every j. The same method can be applied to v_j, so we have

$$\sum_{j=1}^{N} v_j u_j = \pm \sum_{j=1}^{N} |v_j u_j| \neq 0 \tag{4.243}$$

that is, u_j and v_j can not be orthogonal, which is inconsistent with our assumption. Thus, $\rho(A)$ is non- degenerate.

(v) Let ω_j be a normalized eigenvector belonging to $\mu < \rho(A)$,

$$\sum_{j=1}^{N} A_{ij}\omega_j = \mu\omega_i \tag{4.244}$$

The variational principle and non-degeneracy of $\rho(A)$ gives

$$\rho(A) > \sum_{j=1}^{N} A_{ij}|\omega_i||\omega_j| \geq |\sum_{i,j=1}^{N} A_{ij}\omega_i^*\omega_j| = |\mu| \tag{4.245}$$

so that $\rho(A) > |\mu| > \mu$. □

Since the principal matrix (4.28) is not a positive matrix, we cannot directly apply Perron-Frobenius theorem. Nevertheless, we can make the principal matrix strictly positive by subtracting the maximum of the diagonal part of it corresponding to the lower bound of energy $E = E_*$, which is found in Section 4.10, and reversing the overall sign:

$$\Phi'(E) = -\left[\Phi(E) - (1+\varepsilon)\,\mathbf{1}\,\Phi_{ii}^{\max}(E_*)\right] > 0 \;, \qquad \varepsilon > 0 \;, \tag{4.246}$$

where $\mathbf{1}$ is the $N \times N$ identity matrix. Hence, considering the transformed principal matrix Φ' in the light of this theorem, we conclude that there exist a strictly positive eigenvector which corresponds to the unique eigenvalue of maximum modulus.

$$\sum_{j=1}^{N} \Phi'_{ij}(E) A_j(E) = \rho(\Phi') A_i(E) \;. \tag{4.247}$$

We note that Φ' has the same eigenvector with Φ so it guarantees that there exist a strictly positive eigenvector for the principal matrix Φ. Using the eigenvalue problem, we find

$$\sum_{j=1}^{N} \Phi_{ij}(E) A_j(E) = \omega(E) A_i(E) \;, \tag{4.248}$$

where $\rho(\Phi') = -\omega(E) + (1+\varepsilon)\Phi_{ii}^{\max}(E_*)$. For a given $E = E_k$ or $\nu = \nu_k$, there is a unique corresponding minimum $\omega(E)$ and this minimum flows to zero at $\nu = \nu^{\max} = \nu_*$ (bound states energies are the zeros of the eigenvalues $\omega(E) = 0$). This means that the positive eigenvector A_i corresponds to the ground state energy so we prove that the ground state energy is unique and the associated eigenvector A_i is strictly positive. Due to the positivity property of heat kernel, it is easy to see that the ground state wave function is strictly positive from the equation (4.40).

$$\begin{aligned}
\psi_k(x) \;\; &= \;\; \left[\sum_{i,j=1}^{N} \underbrace{A_i(-\nu_k^2)}_{>0} \int_0^\infty \frac{\mathrm{d}t}{\hbar} \; \underbrace{\left(\frac{t}{\hbar}\right) K_t(a_i, a_j; g) e^{-\frac{t\nu_k^2}{\hbar}}}_{\geq 0} \; \underbrace{A_j(-\nu_k^2)}_{>0} \right]^{-\frac{1}{2}} \\
&\quad \times \int_0^\infty \frac{\mathrm{d}t}{\hbar} \; \underbrace{e^{-\frac{t\nu_k^2}{\hbar}}}_{>0} \sum_{i=1}^{N} \underbrace{A_i(-\nu_k^2)}_{>0} \underbrace{K_t(a_i, x; g)}_{>0} > 0 \;,
\end{aligned} \tag{4.249}$$

Hence, we prove that despite the singular character of the interaction, the ground state is still non-degenerate and unique.

4.12. Renormalization Group Equations

Renormalization group is just the set of transformations of the physical parameters associated with change in scale, say M, by keeping the physics unchanged. Under the infinitesimal change in the scale, the variation of physical parameters is described by so-called β function. In most cases, it can only be calculated perturbatively. It would be interesting to find explicit examples where it can be calculated non-perturbatively and exactly. It has been shown that a single Dirac delta potential in two dimensional Euclidean space admits a exact solution to the β function [16]. Here we shall consider our problem in the light of renormalization group and find the exact solution to the β function.

4.12.1. Two Dimensional Case

In this section, we shall choose the natural units $\hbar = 2m = 1$ for simplicity. It is useful to work with the dimensionless coupling constant. One possible way for the renormalization scheme in order to determine how the coupling constant changes with the energy scale is to define the following renormalized coupling constant $\lambda_i^R(M_i)$ in terms of the bare coupling constant $\lambda_i(\epsilon)$

$$\frac{1}{\lambda_i^R(M_i)} = \frac{1}{\lambda_i(\epsilon)} - \int_{\epsilon}^{\infty} \mathrm{d}t \frac{e^{-M_i^2 t}}{4\pi t} \; , \tag{4.250}$$

where M_i is a renormalization scale (it is of dimension $[E]^{1/2}$). Then, the renormalized principal matrix in terms of renormalized coupling constant in natural units is given

$$\Phi_{ij}^R(E) = \begin{cases} \dfrac{1}{\lambda_i^R(M_i)} - \displaystyle\int_0^{\infty} \mathrm{d}t \left(K_t(a_i, a_i; g) e^{tE} - \dfrac{e^{-M_i^2 t}}{4\pi t} \right) & \text{if } i = j \\ - \displaystyle\int_0^{\infty} \mathrm{d}t \; K_t(a_i, a_j; g) e^{tE} & \text{if } i \neq j \; , \end{cases} \tag{4.251}$$

and the bound state energy is determined from the condition $\det \Phi^R_{ij}(E) = 0$ which determines the relation between $\lambda^R_i(M_i)$ and M_i. Explicit dependence on M_i cancels the implicit dependence on M_i through $\lambda^R_i(M_i)$. Physics is determined by the value of $\lambda^R_i(M_i)$ at an arbitrary value of the renormalization point M_i. However, this is not a proper way to look at our problem since we have to deal with several renormalized coupling constants with the same kind of interaction, which essentially differs from one another with arbitrary constants. These constants can be determined by deciding the excited energy levels. We shall instead prefer one renormalized coupling constant by redefining the meaning of the renormalized coupling constant without altering physics. This could be done in the following way: As an external input, we decide about the relative strengths of individual delta interactions and do not use the ground state energy to fix the flow. We know that $-\mu_i^2$ is the bound state energy of the individual i-th Dirac delta center so it corresponds to the solution $\Phi^R_{ii}(-\mu_i^2) = 0$. Without loss of generality, let us assume that $\Phi^R_{11}(-\mu_1^2) = 0$ and this allows us to choose the renormalized coupling constant

$$\frac{1}{\lambda^R(M)} = \frac{1}{\lambda_1(\epsilon)} - \int_\epsilon^\infty \mathrm{d}t \frac{e^{-M^2 t}}{4\pi t} \; , \tag{4.252}$$

at some scale M. Once the renormalized coupling constant is fixed under this condition, we must also satisfy $\Phi^R_{ii}(-\mu_i^2) = 0$ for $i \neq 1$ with this choice at the same scale M. This is always possible if we add a constant term to the definition of renormalized coupling constant. Let us consider $i = 2$ case

$$\begin{aligned} \Phi^R_{22}(-\mu_2^2) &= \frac{1}{\lambda^R(M)} + \lim_{\epsilon \to 0} \left[\int_\epsilon^\infty \mathrm{d}t \; \frac{e^{-M^2 t}}{4\pi t} - \int_\epsilon^\infty \mathrm{d}t \; K_t(a_2, a_2; g) e^{-t\mu_2^2} \right] - \Sigma_2 \\ &= \int_0^\infty \mathrm{d}t \left[K_t(a_1, a_1; g) e^{-t\mu_1^2} - K_t(a_2, a_2; g) e^{-t\mu_2^2} \right] - \Sigma_2 = 0 \; , \end{aligned} \tag{4.253}$$

where we have used (4.252) and $\Phi^R_{11}(-\mu_1^2) = 0$. This means that there always exists a constant Σ_i depending only on μ_i with $\Sigma_1 = 0$ and $\Sigma_i \neq 0$ for $i \neq 1$ such that the condition $\Phi^R_{ii}(-\mu_i^2) = 0$ can be satisfied. Hence, the renormalized coupling constant becomes

$$\frac{1}{\lambda^R(M)} = \frac{1}{\lambda_i(\epsilon)} - \int_{\epsilon}^{\infty} \mathrm{d}t \, \frac{e^{-M^2 t}}{4\pi t} + \Sigma_i \; , \tag{4.254}$$

and the choice of Σ_i s refer to the relative strengths of delta interactions in this renormalization scheme. As a result, the renormalized principal matrix is

$$\Phi_{ij}^R(E) = \begin{cases} \dfrac{1}{\lambda_R(M)} - \displaystyle\int_0^{\infty} \mathrm{d}t \, \left(K_t(a_i, a_i; g) e^{tE} - \frac{e^{-M^2 t}}{4\pi t} \right) - \Sigma_i & \text{if } i = j \\ - \displaystyle\int_0^{\infty} \mathrm{d}t \; K_t(a_i, a_j; g) e^{tE} & \text{if } i \neq j . \end{cases} \tag{4.255}$$

The renormalization condition is given by

$$M \frac{\mathrm{d}\Phi_{ij}^R(M, \lambda_R(M), E; g)}{\mathrm{d}M} = 0 \; , \tag{4.256}$$

or

$$\left(M \frac{\partial}{\partial M} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} \right) \Phi_{ij}^R(M, \lambda_R(M), E; g) = 0 \; , \tag{4.257}$$

where

$$\beta(\lambda_R) = M \frac{\partial \lambda_R}{\partial M} \tag{4.258}$$

is called the β function and the equation (4.257) is the renormalization group (RG) equation. In [16], the renormalization condition (4.256) corresponding to the problem in flat space has been written in terms of the T-matrix. Using (4.255) in (4.257), we can find β function exactly

$$\beta(\lambda_R) = -\frac{\lambda_R^2}{2\pi} < 0 \; . \tag{4.259}$$

This result is the same as the one in flat spaces given in the literature [7, 15, 16, 17] so our problem is asymptotically free. From the explicit expression of the renormalized principal matrix, one can easily see the scaling property of it under a change of energy and metric scale γ using the scaling property of the heat kernel (3.50)

$$\Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = \Phi^R_{ij}(\gamma^{-1} M, \lambda_R(M), E; g) \; . \tag{4.260}$$

It is important to note that we need to scale the metric as well and the idea of the metric scaling in deriving the renormalization group equation was motivated by [120] in the context of renormalization group in quantum field theory on curved spaces. Hence we have

$$\gamma \frac{\mathrm{d}}{\mathrm{d}\gamma} \left[\Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = \Phi^R_{ij}(\gamma^{-1} M, \lambda_R(M), E; g) \right] \; . \tag{4.261}$$

It leads to the renormalization group equation for $\Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g)$

$$\gamma \frac{\mathrm{d}}{\mathrm{d}\gamma} \Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) + M \frac{\partial}{\partial M} \Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = 0 \; , \tag{4.262}$$

or

$$\left[\gamma \frac{\mathrm{d}}{\mathrm{d}\gamma} - \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} \right] \Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = 0 \; . \tag{4.263}$$

If we postulate the following functional form for the principal matrix

$$\Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = f(\gamma) \Phi^R_{ij}(M, \lambda_R(\gamma M), E; g) \; , \tag{4.264}$$

and substitute into (4.263) we obtain an ordinary differential equation for f

$$\gamma \frac{\mathrm{d} f(\gamma)}{\mathrm{d}\gamma} = 0 \; . \tag{4.265}$$

This gives the solution $f(\gamma) = 1$ using the initial condition at $\gamma = 1$. Therefore, we get

$$\Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = \Phi^R_{ij}(M, \lambda_R(\gamma M), E; g) \; , \tag{4.266}$$

which means that there is no anamolous scaling. After integrating

$$\beta(\lambda_R) = \bar{M}\frac{\partial \lambda_R(\bar{M})}{\partial \bar{M}} = -\frac{\lambda_R^2(\bar{M})}{2\pi} \tag{4.267}$$

between $\bar{M} = M$ to $\bar{M} = \gamma M$ we can find the flow equation for the coupling constant

$$\lambda_R(\gamma M) = \frac{\lambda_R(M)}{1 + \frac{1}{2\pi}\lambda_R(M)\ln\gamma} \ . \tag{4.268}$$

One can explicitly check the relation (4.266) if the coupling constant evolves according to (4.268). First, we add and subtract a term in the time integral:

$$\begin{aligned}
&\Phi^R_{ii}(M, \lambda_R(\gamma M), E; g) \\
&= \frac{1}{\lambda_R(M)} + \frac{1}{2\pi}\ln\gamma - \int_0^\infty \mathrm{d}t \left(K_t(a_i, a_i; g)e^{tE} - \frac{e^{-M^2 t}}{4\pi t} \right) - \Sigma_i \\
&= \frac{1}{\lambda_R(M)} + \frac{1}{2\pi}\ln\gamma - \int_0^\infty \mathrm{d}t \left(K_t(a_i, a_i; g)e^{tE} - \frac{e^{-M^2 t}}{4\pi t} + \frac{e^{-M^2\gamma^{-2} t}}{4\pi t} - \frac{e^{-M^2\gamma^{-2} t}}{4\pi t} \right) \\
&- \Sigma_i \\
&= \frac{1}{\lambda_R(M)} - \int_0^\infty \mathrm{d}t \left(K_t(a_i, a_i; g)e^{tE} - \frac{e^{-M^2\gamma^{-2} t}}{4\pi t} \right) - \Sigma_i
\end{aligned} \tag{4.269}$$

and then using the scaling property of heat kernel (3.50), we get

$$\begin{aligned}
&\frac{1}{\lambda_R(M)} - \int_0^\infty \mathrm{d}t \left(\gamma^{-2} K_{\gamma^{-2}t}(a_i, a_i; \gamma^{-2}g)e^{tE} - \frac{e^{-M^2\gamma^{-2} t}}{4\pi t} \right) - \Sigma_i \\
&\qquad = \frac{1}{\lambda_R(M)} - \int_0^\infty \mathrm{d}s \left(K_s(a_i, a_i; \gamma^{-2}g)e^{s\gamma^2 E} - \frac{e^{-M^2 s}}{4\pi s} \right) - \Sigma_i \\
&\qquad = \Phi^R_{ii}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) \ .
\end{aligned} \tag{4.270}$$

The off-diagonal term can be directly checked using just the scaling property of heat kernel (3.50). Alternatively, one can find how the coupling constant evolves, given (4.268) from the scaling relation (4.266).

4.12.2. Three Dimensional Case

Since it is convenient to work with the dimensionless coupling constant, we define a dimensionless coupling constant in three dimensions

$$\hat{\lambda}_R(M) = M\lambda_R(M) \, . \tag{4.271}$$

Then, by similar arguments developed for two dimensions, the renormalized principal matrix in the natural units is

$$\Phi^R_{ij}(E) = \begin{cases} \dfrac{M}{\hat{\lambda}_R(M)} - \displaystyle\int_0^\infty \mathrm{d}t \, \left(K_t(a_i, a_i; g) e^{tE} - \dfrac{e^{-M^2 t}}{(4\pi t)^{3/2}} \right) - \Sigma_i & \text{if } i = j \\ - \displaystyle\int_0^\infty \mathrm{d}t \; K_t(a_i, a_j; g) e^{tE} & \text{if } i \neq j. \end{cases} \tag{4.272}$$

Renormalization condition (4.257) in this case leads to the following β function

$$\beta(\hat{\lambda}_R) = M \frac{\partial \hat{\lambda}_R(M)}{\partial M} = \hat{\lambda}_R(M) - \frac{1}{4\pi} \hat{\lambda}^2_R(M) \, , \tag{4.273}$$

which is in agreement with the result for flat space [16]. From the explicit expression of the renormalized principal matrix for three dimensions, one can easily see the scaling property of it under a change of scale γ using the scaling property of the heat kernel (3.50)

$$\Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g) = \gamma \Phi^R_{ij}(\gamma^{-1} M, \hat{\lambda}_R(M), E; g) \, , \tag{4.274}$$

so we have

$$\gamma \frac{\mathrm{d}}{\mathrm{d}\gamma} \left[\Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g) = \gamma \Phi^R_{ij}(\gamma^{-1} M, \hat{\lambda}_R(M), E; g) \right] \, . \tag{4.275}$$

This leads to the renormalization group equation for $\Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g)$

$$\left[\gamma \frac{\mathrm{d}}{\mathrm{d}\gamma} - 1 + M \frac{\partial}{\partial M} \right] \Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g) = 0 \, , \tag{4.276}$$

or

$$\left[\gamma\frac{\mathrm{d}}{\mathrm{d}\gamma} - 1 - \beta(\hat{\lambda}_R)\frac{\partial}{\partial\hat{\lambda}_R}\right]\Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g) = 0 \ . \tag{4.277}$$

If we again postulate (4.264) and substitute it into (4.277), we obtain an ordinary differential equation for the function f

$$\gamma\frac{\mathrm{d}f(\gamma)}{\mathrm{d}\gamma} = f \ . \tag{4.278}$$

The solution is $f(\gamma) = \gamma$ by using the initial condition at $\gamma = 1$. Therefore, we have

$$\Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g) = \gamma\Phi^R_{ij}(M, \hat{\lambda}_R(\gamma M), E; g) \ . \tag{4.279}$$

This means that there is also no anamolous scaling in three dimensions. After integrating

$$\beta(\hat{\lambda}_R) = \bar{M}\frac{\partial\hat{\lambda}_R(\bar{M})}{\partial\bar{M}} = \hat{\lambda}_R(\bar{M}) - \frac{1}{4\pi}\hat{\lambda}^2_R(\bar{M}) \ , \tag{4.280}$$

between $\bar{M} = M$ to $\bar{M} = \gamma M$ we can find similarly the flow equation for the coupling constant

$$\hat{\lambda}_R(\gamma M) = \frac{\gamma\hat{\lambda}_R(M)}{1 - \frac{1}{4\pi}\hat{\lambda}_R(M)(1-\gamma)} \ . \tag{4.281}$$

One can similarly check the relation (4.279) if the coupling constant evolves according to (4.281). In this case, we have

$$\begin{aligned}
&\gamma\Phi^R_{ii}(M,\hat{\lambda}_R(\gamma M),E;g) \\
&= \frac{M}{\hat{\lambda}_R(M)} + \frac{1}{4\pi}M(1-\gamma) - \gamma\int_0^\infty \mathrm{d}t \left(K_t(a_i,a_i;g)e^{tE} - \frac{e^{-M^2 t}}{(4\pi t)^{3/2}}\right) - \Sigma_i \\
&= \frac{M}{\hat{\lambda}_R(M)} + \frac{1}{4\pi}M(1-\gamma) \\
&- \gamma\int_0^\infty \mathrm{d}t \left(K_t(a_i,a_i;g)e^{tE} - \frac{e^{-M^2 t}}{(4\pi t)^{3/2}} + \frac{e^{-M^2\gamma^{-2} t}}{(4\pi t)^{3/2}} - \frac{e^{-M^2\gamma^{-2} t}}{(4\pi t)^{3/2}}\right) - \Sigma_i \\
&= \frac{M}{\hat{\lambda}_R(M)} - \gamma\int_0^\infty \mathrm{d}t \left(K_t(a_i,a_i;g)e^{tE} - \frac{e^{-M^2\gamma^{-2} t}}{(4\pi t)^{3/2}}\right) - \Sigma_i \ ,
\end{aligned} \tag{4.282}$$

and then using the scaling property of heat kernel (3.50), we get

$$\begin{aligned}
\frac{M}{\hat{\lambda}_R(M)} &- \gamma\int_0^\infty \mathrm{d}t \left(\gamma^{-3} K_{\gamma^{-2}t}(a_i,a_i;\gamma^{-2}g)e^{tE} - \frac{e^{-M^2\gamma^{-2} t}}{(4\pi t)^{3/2}}\right) - \Sigma_i \\
&= \frac{M}{\hat{\lambda}_R(M)} - \int_0^\infty \mathrm{d}s \left(K_s(a_i,a_i;\gamma^{-2}g)e^{s\gamma^2 E} - \frac{e^{-M^2 s}}{(4\pi s)^{3/2}}\right) - \Sigma_i \\
&= \Phi^R_{ii}(M,\hat{\lambda}_R(M),\gamma^2 E;\gamma^{-2}g) \ .
\end{aligned} \tag{4.283}$$

One can similarly find how the coupling constant evolves, given in (4.281) from the scaling relation (4.279).

5. NONPERTURBATIVE RENORMALIZATION OF POINT INTERACTIONS IN MANY BODY THEORIES

5.1. Non-relativistic Lee Model on Two and Three Dimensional Riemannian Manifolds

5.1.1. A Short Introduction to the Original Lee model in Three Dimensional Flat Space

The Lee model, originally introduced in [121], is an exactly soluble (in principle eigenstates and eigenvalues can be exactly found) and renormalizable model that describes the interaction between a relativistic neutral bosonic field "pions" and two neutral fermionic fields "nucleons". Nucleons are assumed to be able to exist in two different intrinsic states and they can be transformed from one state to the other. The particle corresponding to the Bose field is called θ and the particles corresponding to the intrinsic states of the nucleon are called V and N particles. The fermionic field corresponding to the V and N particles are assumed to be spinless for simplicity. This may be the case when the only one spin species of the fermions is relevant for V and N particles. The V particle can emit a θ particle and transform into an N particle. The N particle can not emit a θ particle whereas it can absorb a θ particle and transform into a V particle. The V particle can not absorb a θ particle. We can summarize the allowable process as

$$V \leftrightharpoons N + \theta \tag{5.1}$$

and the following process is not allowed

$$N \leftrightharpoons V + \theta \, , \tag{5.2}$$

which makes the model rather simple. Here it is important to notice that the last reaction would be possible, in local theory, due to the identification of antiparticle of θ with itself. The interaction in the model includes only the positive or negative frequency part of the bosonic field operator as opposed to the "normal" field theory. The energies of the fermions V and N are assumed to be independent of their momentum. The Hamiltonian describing this system in the Schrödinger picture is given by (in the natural units $\hbar = c = 1$)

$$H = H_0 + H_I \ , \tag{5.3}$$

where

$$H_0 = m_{V_0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} V^\dagger(\mathbf{p}) V(\mathbf{p}) + m_{N_0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} N^\dagger(\mathbf{p}) N(\mathbf{p}) + \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} \omega(\mathbf{p}) \phi^\dagger(\mathbf{p}) \phi(\mathbf{p}) \tag{5.4}$$

$$H_I = \lambda_0 \int_{\mathbb{R}^3} \frac{\mathrm{d}^3q}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} \left[V^\dagger(\mathbf{p}) N(\mathbf{p}-\mathbf{q}) \phi(\mathbf{q}) + N^\dagger(\mathbf{p}-\mathbf{q}) \phi^\dagger(\mathbf{q}) V(\mathbf{p}) \right] \tag{5.5}$$

where the $V^\dagger(\mathbf{p})(N^\dagger(\mathbf{p}))$, $V(\mathbf{p})(N(\mathbf{p}))$ are respectively the creation and annihilation operators for an $V(\mathbf{p})(N(\mathbf{p}))$ quantum of momentum $\mathbf{p}$ while $\phi^\dagger(\mathbf{p}), \phi(\mathbf{p})$ are the creation and annihilation operators for θ quantum of momentum $\mathbf{p}$. These creation and annihilation operators obey the usual commutation and anticommutation rules. The parameters m_{V_0}, m_{N_0} are the bare masses of the quanta of V and N fields and $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_0^2}$ is the energy of a θ quantum of momentum $\mathbf{p}$ with its bare mass m_0. The parameter λ_0 is called the bare coupling constant. Total momentum operator

$$\mathbf{P} = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3q}{(2\pi)^3} \mathbf{p} \left[N^\dagger(\mathbf{p}) N(\mathbf{p}) + V^\dagger(\mathbf{p}) V(\mathbf{p}) + \phi^\dagger(\mathbf{p}) \phi(\mathbf{p}) \right] \tag{5.6}$$

is conserved since $[H, \mathbf{P}] = 0$. We introduce a suitable cut-off function $f_\Lambda(\mathbf{p})$ to make all momentum integrals finite and the regularized Hamiltonian becomes

$$H_{0,\Lambda} = m_{V_0}(\Lambda) \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} f_\Lambda(\mathbf{p}) V^\dagger(\mathbf{p}) V(\mathbf{p}) + m_{N_0}(\Lambda) \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} f_\Lambda(\mathbf{p}) N^\dagger(\mathbf{p}) N(\mathbf{p})$$

$$+\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega(\mathbf{p}) f_\Lambda(\mathbf{p}) \phi^\dagger(\mathbf{p}) \phi(\mathbf{p}) \tag{5.7}$$

and

$$H_{I,\Lambda} = \lambda_0(\Lambda) \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} f_\Lambda(\mathbf{q}) \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} f_\Lambda(\mathbf{p}) \Big[V^\dagger(\mathbf{p}) N(\mathbf{p}-\mathbf{q}) \phi(\mathbf{q}) + N^\dagger(\mathbf{p}-\mathbf{q}) \phi^\dagger(\mathbf{q}) V(\mathbf{p}) \Big] \tag{5.8}$$

One possible interpretation is to consider the N particle as a proton, the V particle as a neutron, and θ particle as a π^- meson. Then the allowed reaction is

$$n \leftrightharpoons p + \pi^- \tag{5.9}$$

and the reaction $p \leftrightharpoons n + \pi^-$ is not allowed due to the charge conservation. Since we have considered the positive and negative frequency part of the bosonic field operator, interaction is not causal and local. Although the model is not very realistic, it reflects important features of nucleon-pion system, and presents a powerful aspect that one can do the coupling constant, mass and wave function renormalizations. This is also an interesting model in the sense that it can be exactly soluble and renormalization can be performed non-perturbatively. For a detailed discussion of the Lee model, one may refer to Schweber's book on quantum field theory [122]. Since we restrict ourselves to simple models in this thesis, we shall consider the simplified non-relativistic version of the Lee model and study it on Riemannian manifolds.

5.1.2. Introduction to Non-relativistic Lee Model in Three Dimensional Euclidean Space

The complete non-relativistic version of the Lee model that describes one heavy particle sitting at some fixed point interacting with a field of non-relativistic bosons is as important as its relativistic counterpart. In this case the dispersion relation is $\omega(\mathbf{p}) \simeq \frac{\mathbf{p}^2}{2m_0} + m_0$ in the non-relativistic approximation and fermionic creation and annihilation operators are independent of momentum since they are assumed to be heavy sitting at an arbitrary fixed point in space ($x = 0$) so that the recoil effects are ignored. The Hamiltonian of the model in Schrödinger picture is then ($\hbar = c = 1$)

$$H_0 = m_{V_0} V^\dagger V + m_{N_0} N^\dagger N + \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\frac{\mathbf{p}^2}{2m_0} + m_0 \right) \phi^\dagger(\mathbf{p}) \phi(\mathbf{p}) \tag{5.10}$$

and

$$H_I = \lambda_0 \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} \left[V^\dagger N \phi(\mathbf{q}) + N^\dagger \phi^\dagger(\mathbf{q}) V \right] \tag{5.11}$$

where $V = V(\mathbf{p} = 0)$ and $N = N(\mathbf{p} = 0)$ and

$$\begin{aligned} [\phi(\mathbf{p}), \phi^\dagger(\mathbf{q})] &= (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) \\ \left\{ V, V^\dagger \right\} &= 1 \\ \left\{ N, N^\dagger \right\} &= 1 \end{aligned} \tag{5.12}$$

with all other pairs of the operators commute(or anti-commute). Note that the momentum integrals for these particles disappear since only the zero eigenvalue of the momentum contributes to the integrals (one first simply considers the system in a finite box and take the continuum limit at the end). As a consequence of disregarding the recoil effects, the total momentum of the system $\mathbf{P} = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} \mathbf{p} \phi^\dagger(\mathbf{p}) \phi(\mathbf{p})$ is not conserved, i.e, $[H, \mathbf{P}] \neq 0$. Although we loose the momentum conservation, this modified version of the Lee model is much simpler than its relativistic version because it is possible to renormalize the system with only an additive renormalization of the mass (energy) difference of the N particle and the V particle. It has been studied in a textbook by Henley and Thirring for small number of bosons from the point of view of scattering matrix [123] and there are other attempts in the literature [124, 125, 126, 127]. Exact solubility of the model is due to the conserved quantities which can be directly derived from equations of motion for the field operators. These conserved quantities are total number of fermion operator

$$Q_1 = V^\dagger V + N^\dagger N \tag{5.13}$$

and the difference of the total quanta of θ and N particles.

$$Q_2 = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} \phi^\dagger(\mathbf{q})\phi(\mathbf{q}) - N^\dagger N \tag{5.14}$$

that is, $[H, Q_1] = [H, Q_2] = 0$. We will study the system subject to the constraint that the conserved quantities Q_1 and Q_2 are chosen to take certain values so that the problem can be exactly solved. In order to find the exact physical states for bound states and scattering problem it is important to notice that the bare and physical (or sometimes called dressed) vacuum states coincide as well as bare and physical one θ and one N particle states. The bare vacuum is defined as

$$N(\mathbf{p})|0\rangle = V(\mathbf{p})|0\rangle = \phi(\mathbf{p})|0\rangle = 0 \, , \tag{5.15}$$

for all $\mathbf{p}$. Here the bare vacuum states must be understood as direct product of the bosonic and fermionic bare vacuum states in Fock space. The bare vacuum corresponds to the zero eigenvalue of the free Hamiltonian H_0, i.e, $H_0|0\rangle = 0$. This state is also the eigenvector of the full Hamiltonian with zero eigenvalue $H|0\rangle = 0$, that is, bare vacuum is also the physical vacuum. Similarly, one can also show that

$$H_0 N^\dagger(\mathbf{p})|0\rangle = m_{N_0} N^\dagger(\mathbf{p})|0\rangle \, , \tag{5.16}$$

and

$$H N^\dagger(\mathbf{p})|0\rangle = m_{N_0} N^\dagger(\mathbf{p})|0\rangle \, . \tag{5.17}$$

Similarly, we have also

$$H_0 \phi^\dagger(\mathbf{p})|0\rangle = \omega(\mathbf{p}) \phi^\dagger(\mathbf{p})|0\rangle \, , \tag{5.18}$$

and

$$H\phi^\dagger(\mathbf{p})|0\rangle = \omega(\mathbf{p})\phi^\dagger(\mathbf{p})|0\rangle \ . \tag{5.19}$$

Since N and θ particle corresponds to physical states, we can identify their bare masses with physical masses, that is, $m_{N_0} = m_N$ and $m_0 = m$. Therefore, we have

$$\begin{aligned} |0\rangle_d &= |0\rangle \\ |N(\mathbf{p})\rangle_d &= N^\dagger(\mathbf{p})|0\rangle \\ |\theta(\mathbf{p})\rangle_d &= \phi^\dagger(\mathbf{p})|0\rangle \ , \end{aligned} \tag{5.20}$$

where the subscript d stands for dressed or physical state. Conserved quantities Q_1 and Q_2 allows us to calculate other eigenstates as well. Since Q_1 and Q_2 commute with the Hamiltonian, the eigenstates of H can be labelled by the eigenvalues of the operators Q_1 and Q_2. Any physical state can be expanded as a superposition of all possible bare states

$$\begin{aligned} |\ \rangle_d = \psi^{(0,0,0)}|0\rangle + \psi^{(1,0,0)}V^\dagger|0\rangle + \ldots + \frac{1}{\sqrt{n!m!l!}}\int_{\mathbb{R}^{3l}} \frac{\mathrm{d}^3 q_1}{(2\pi)^3}\frac{\mathrm{d}^3 q_2}{(2\pi)^3}\cdots\frac{\mathrm{d}^3 q_l}{(2\pi)^3} \\ \times\psi^{(n,m,l)}(\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_l)\ V^{\dagger\, n}N^{\dagger\, m}\phi^\dagger(\mathbf{q}_1)\phi^\dagger(\mathbf{q}_2)\ldots\phi^\dagger(\mathbf{q}_l)|0\rangle \ , \end{aligned} \tag{5.21}$$

then only the states with the same eigenvalues of Q_1 and Q_2 can occur in the expansion above. In order to find the physical one V particle state, we must first guarantee the following limit

$$|V\rangle_d \to V^\dagger|0\rangle \ , \tag{5.22}$$

as λ_0 goes to zero. Hence, the physical one V particle state must include the one bare V particle state, $V^\dagger|0\rangle$. It is easy to see that the eigenvalues of Q_1 and Q_2 in this state are 1 and 0, respectively. However, this eigenstate is not the only one having these quantum numbers. One can show that the other bare state must be $|N,\theta\rangle = N^\dagger\phi^\dagger(\mathbf{p})|0\rangle$ with

$$
\begin{aligned}
Q_1|N,\theta\rangle &= 1|N,\theta\rangle \\
Q_2|N,\theta\rangle &= 0\,.
\end{aligned} \tag{5.23}
$$

Then the state of one physical V particle which is an eigenvector of H with the eigenvalue m_V (the observed mass of the V particle) can be represented by the superposition of the bare states having the same eigenvalue of Q_1 and Q_2 (sectors 1 and 0).

$$
|V\rangle_d = uV^\dagger|0\rangle + \int_{\mathbb{R}^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\psi(\mathbf{q})N^\dagger\phi^\dagger(\mathbf{q})|0\rangle\;, \tag{5.24}
$$

where $u = \psi^{(1,0,0)}$ and $\psi(\mathbf{p}) = \psi^{(0,1,1)}(\mathbf{p})$ for simplicity of notation. Taking the scalar product of stationary Schrödinger equation

$$
H|V\rangle_d = m_V|V\rangle_d\;, \tag{5.25}
$$

with $\langle 0|Vu^*$ we get

$$
(m_{V_0} - m_V)\,u + \lambda_0\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\psi(\mathbf{p}) = 0\;, \tag{5.26}
$$

and applying $\int_{\mathbb{R}^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\langle 0|\psi^*(\mathbf{q})N\phi(\mathbf{q})$ on the left, we get

$$
(\omega(\mathbf{p}) + m_N - m_V)\,\psi(\mathbf{p}) + \lambda_0 u = 0\;. \tag{5.27}
$$

We can solve $\psi(\mathbf{p})$ from (5.27)

$$
\psi(\mathbf{p}) = \frac{u\lambda_0}{(m_V - m_N - \omega(\mathbf{p}))}\;, \tag{5.28}
$$

as long as $m_V - m_N - \omega(\mathbf{p}) \neq 0$. This condition is always satisfied if we restrict the problem for the stable V particle, that is, the situation in which V particle can not decay into N particle and θ particle. In this case stability condition $m_V < m_N + m$

satisfies the condition above. Substituting the solution (5.28) in (5.26), we obtain

$$m_{V_0} + \lambda_0^2 \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{m_V - m_N - \omega(\mathbf{p})} = m_V \ . \tag{5.29}$$

The momentum integral above is divergent at large values of momentum, that is, it is an ultra-violet divergence. This is due to the fact that the approximation of no recoil is broken down for large enough momentum. The first step is regularization, i.e., we make all the momentum integrals finite by introducing a momentum cut-off Λ and make the bare parameters depend on Λ. Then, we choose the bare parameters in such a way that all the physical quantities are finite and independent of Λ. The regularized Hamiltonian is

$$\begin{aligned} H_{0,\Lambda} = {} & m_{V_0}(\Lambda) V^\dagger V + m_{N_0}(\Lambda) N^\dagger N \\ & + \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} f_\Lambda(\mathbf{p}) \left(\frac{\mathbf{p}^2}{2m_0(\Lambda)} + m_0(\Lambda) \right) \phi^\dagger(\mathbf{p}) \phi(\mathbf{p}) \end{aligned} \tag{5.30}$$

and

$$H_{I,\Lambda} = \lambda_0(\Lambda) \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} f_\Lambda(\mathbf{q}) \left[V^\dagger N \phi(\mathbf{q}) + N^\dagger \phi^\dagger(\mathbf{q}) V \right] \ . \tag{5.31}$$

If we repeat the same procedures we obtain the regularized version of the equations above

$$m_{V_0}(\Lambda) + \lambda_0^2 \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{f_\Lambda(\mathbf{p})}{m_V - m_N - \omega(\mathbf{p})} = m_V \tag{5.32}$$

It is enough to choose the bare mass of V particle at some arbitrary renormalization scale μ

$$m_{V_0}(\Lambda) = \mu - \lambda_0^2 \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{f_\Lambda(\mathbf{p})}{\mu - m_N - \omega(\mathbf{p})} \tag{5.33}$$

where the bare coupling constant is just the physical coupling constant $\lambda_0 = \lambda$, that is, we do not need a coupling constant renormalization for the non-relativistic case. Once we choose the bare coupling constant by the above renormalization condition, all other physical quantities stay finite. In order to show this, we can look at the scattering of θ particles from a static N particle. We must first find the "incoming" and the "outgoing" eigenstates of H. They are defined by the equation below

$$H|N,\theta\rangle_{d\pm} = (m_N + \omega(\mathbf{p}))|N,\theta\rangle_{d\pm} \tag{5.34}$$

where $|N,\theta\rangle_{d+}$ is the "incoming" and $|N,\theta\rangle_{d-}$ is the "outgoing" eigenstates of the Hamiltonian H. The eigenstate $|N,\theta\rangle_{d+}$ describes a θ particle of momentum $\mathbf{p}$ impinging on an N particle sitting at $x = 0$ and which then scatter off from each other.

$$|N,\theta\rangle_{d+} = N^\dagger\phi^\dagger(\mathbf{p})|0\rangle + |\chi\rangle_+ \tag{5.35}$$

where $N^\dagger\phi^\dagger(\mathbf{p})|0\rangle$ describes the incoming state of a θ particle of momentum $\mathbf{p}$ and of the N particle, and $|\chi\rangle$ corresponds to the outgoing scattered wave. The solutions are

$$|N,\theta\rangle_{d+} = |N,\theta\rangle + \frac{\lambda}{m_N + \omega(\mathbf{p}) - H + i\epsilon} V^\dagger|0\rangle \tag{5.36}$$

and similarly for $|N,\theta\rangle_{d-}$

$$|N,\theta\rangle_{d-} = |N,\theta\rangle + \frac{\lambda}{m_N + \omega(\mathbf{p}) - H - i\epsilon} V^\dagger|0\rangle \tag{5.37}$$

Using the distributional formula

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x) \, , \tag{5.38}$$

we find

$$|N,\theta\rangle_{d+} = |N,\theta\rangle_{d-} - 2\pi i\lambda\delta(m_N + \omega(\mathbf{p}) - H)V^\dagger|0\rangle \, , \tag{5.39}$$

where $\mathcal{P}$ refers to the principal value of the integral. One can then find S matrix

$$ {}_{d-}\langle N,\theta(\mathbf{p}')|N,\theta\rangle_{d+} = (2\pi)^3\delta(\mathbf{p}'-\mathbf{p}) - 2\pi i\lambda\delta(\omega(\mathbf{p})-\omega(\mathbf{p}'))_{d-}\langle N,\theta(\mathbf{p}')|V^\dagger|0\rangle \;, \tag{5.40} $$

or from this, one reads the T matrix

$$ \begin{aligned} \langle N,\theta(\mathbf{p}')|T|N,\theta(\mathbf{p})\rangle &= \lambda \; {}_{d-}\langle N,\theta(\mathbf{p}')|V^\dagger|0\rangle \\ &= \lambda^2\langle 0|V\frac{1}{m_N+\omega(\mathbf{p})-H+i\epsilon}V^\dagger|0\rangle \;, \end{aligned} \tag{5.41} $$

where we have used (5.37) and we are on the energy shell $\omega(\mathbf{p}) = \omega(\mathbf{p}')$. It is useful to express the bare V particle states in terms of the dressed V particle states from the relation (5.24). Then, we get

$$ \begin{aligned} \frac{1}{m_N+\omega(\mathbf{p})-H+i\epsilon}V^\dagger|0\rangle &= \frac{1}{u(m_N+\omega(\mathbf{p})-m_V+i\epsilon)}|V\rangle_d \\ -\lambda\int_{\mathbb{R}^3}\frac{\mathrm{d}^3q}{(2\pi)^3}&\frac{1}{m_V-m_N-\omega(\mathbf{q})}\frac{1}{m_N+\omega(\mathbf{p})-H+i\epsilon}\phi^\dagger(\mathbf{q})N^\dagger|0\rangle \;. \end{aligned} \tag{5.42} $$

Next, we use the operator identity

$$ \frac{1}{z-H} = \frac{1}{z-H_0} + \frac{1}{z-H}H_I\frac{1}{z-H_0} \;, \tag{5.43} $$

and apply it to the second term in (5.42)

$$ \begin{aligned} \frac{1}{m_N+\omega(\mathbf{p})-H+i\epsilon}\phi^\dagger(\mathbf{q}) = \Bigg[&\frac{1}{m_N+\omega(\mathbf{p})-H_0+i\epsilon} \\ &+\frac{1}{m_N+\omega(\mathbf{p})-H+i\epsilon}H_I\frac{1}{m_N+\omega(\mathbf{p})-H_0+i\epsilon}\Bigg]\phi^\dagger(\mathbf{q}) \;. \end{aligned} \tag{5.44} $$

Using

$$\begin{aligned}\frac{1}{m_N+\omega(\mathbf{p})-H_0+i\epsilon}\phi^\dagger(\mathbf{q}) &= -i\int_0^\infty \mathrm{d}t\; e^{i(m_N+\omega(\mathbf{p})-H_0+i\epsilon)t}\phi^\dagger(\mathbf{q}) \\ &= i\int_0^\infty \mathrm{d}t\; e^{i(m_N+\omega(\mathbf{p})+i\epsilon)t}e^{-iH_0t}\phi^\dagger(\mathbf{q})e^{iH_0t}e^{-iH_0t} \\ &= i\int_0^\infty \mathrm{d}t\; e^{i(m_N+\omega(\mathbf{p})-\omega(\mathbf{q})+i\epsilon)t}\phi^\dagger(\mathbf{q})e^{-iH_0t} \\ &= \phi^\dagger(\mathbf{q})\frac{1}{m_N+\omega(\mathbf{p})-\omega(\mathbf{q})-H_0+i\epsilon}\;, \end{aligned} \tag{5.45}$$

we get

$$\begin{aligned}\frac{1}{m_N+\omega(\mathbf{p})-H+i\epsilon}\phi^\dagger(\mathbf{q}) &= \phi^\dagger(\mathbf{q})\frac{1}{m_N+\omega(\mathbf{p})-\omega(\mathbf{q})-H+i\epsilon} \\ &+\frac{1}{m_N+\omega(\mathbf{p})-H+i\epsilon}[H_I,\phi^\dagger(\mathbf{q})]\frac{1}{m_N+\omega(\mathbf{p})-\omega(\mathbf{q})-H+i\epsilon}\;. \end{aligned} \tag{5.46}$$

Since $[H_I,\phi^\dagger(\mathbf{q})]=\lambda V^\dagger N$, it is easy to see that

$$\begin{aligned}&\frac{1}{m_N+\omega(\mathbf{p})-H+i\epsilon}N^\dagger\phi^\dagger(\mathbf{q})|0\rangle \\ &=\frac{1}{\omega(\mathbf{p})-\omega(\mathbf{q})+i\epsilon}\left[\phi^\dagger(\mathbf{q})N^\dagger|0\rangle+\frac{\lambda}{m_N+\omega(\mathbf{p})-H+i\epsilon}V^\dagger|0\rangle\right]\;. \end{aligned} \tag{5.47}$$

As a result of this, we have

$$\begin{aligned}&\frac{1}{m_N+\omega(\mathbf{p})-H+i\epsilon}V^\dagger|0\rangle \\ &=\frac{1}{G(\mathbf{p})}\frac{1}{m_N+\omega(\mathbf{p})-m_V}\left[V^\dagger|0\rangle+\lambda\int_{\mathbb{R}^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\omega(\mathbf{p})-\omega(\mathbf{q})+i\epsilon}N^\dagger\phi^\dagger(\mathbf{q})\right]\;, \end{aligned} \tag{5.48}$$

where

$$G(\mathbf{p})=1+\lambda^2\int_{\mathbb{R}^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{m_V-m_N-\omega(\mathbf{q})}\frac{1}{\omega(\mathbf{p})-\omega(\mathbf{q})+i\epsilon}\;. \tag{5.49}$$

Hence, we obtain the exact finite expression of the T matrix

$$\langle N,\theta(\mathbf{p}')|T|N,\theta(\mathbf{p})\rangle=\frac{1}{G(\mathbf{p})}\frac{\lambda^2}{m_N+\omega(\mathbf{p})-m_V}(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})\;. \tag{5.50}$$

Unfortunately, this approach becomes cumbersome for large number of bosons. Later on, we will give a more suitable way to deal with the problem for arbitrary number of bosons.

5.1.3. Two Level Atom - Field System and Non-relativistic Lee Model

The operator of a single mode electromagnetic field in a cavity of volume V in the Coulomb gauge is given by

$$\mathbf{E} = i\hat{\epsilon}\sqrt{\frac{2\pi\hbar\omega}{V}}\left(ae^{-i\mathbf{k}.\mathbf{r}} - a^{\dagger}e^{i\mathbf{k}.\mathbf{r}}\right) \ , \tag{5.51}$$

where $a^{\dagger}, a$ are the creation and annihilation operators of single mode photons with the wave vector $\mathbf{k}$, the frequency ω and the polarization vector $\hat{\epsilon}$. The free field Hamiltonian is just

$$H_{field} = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right) \ . \tag{5.52}$$

Let us consider a single two level atom with the upper energy level $|1\rangle$ and the lower energy level $|0\rangle$. Then the basis for linear operators acting in the Hilbert space of the states of a two level atom can be chosen as $|i\rangle\langle j|$, where $i, j = 0, 1$. It is well known that the realization of Pauli spin matrices can be put into the following form

$$\begin{aligned}
\sigma_1 &= |0\rangle\langle 1| + |1\rangle\langle 0| \\
\sigma_2 &= i(|0\rangle\langle 1| - |1\rangle\langle 0|) \\
\sigma_3 &= |1\rangle\langle 1| - |0\rangle\langle 0| \\
\mathbf{1} &= |1\rangle\langle 1| + |0\rangle\langle 0| \ ,
\end{aligned} \tag{5.53}$$

where the ground state and the excited state is defined as

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|1\rangle \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{5.54}$$

Then, Hamiltonian corresponding to the free atom

$$H_{atom} = \sum_{i=0}^{1} E_i |i\rangle\langle i| \;, \tag{5.55}$$

where E_0 is the bound state energy of the atom corresponding to the lower level $|0\rangle$ and E_1 is the bound state energy corresponding to the upper level $|1\rangle$. One can now write the free atomic Hamiltonian H_{atom} in terms of Pauli spin matrices

$$\begin{aligned} H_{atom} &= E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1| \\ &= E_0|1\rangle\langle 1| + (E_1 - E_0)|1\rangle\langle 1| + E_0|0\rangle\langle 0| \\ &= E_0\mathbf{1} + (E_1 - E_0)|1\rangle\langle 1| \\ &= E_0\mathbf{1} + (E_1 - E_0)\left(\frac{\mathbf{1} + \sigma_3}{2}\right) . \end{aligned} \tag{5.56}$$

If we define $E_1 - E_0 = \hbar\omega_0$ to simplify the notation, we have

$$H_{atom} = \hbar\omega_0 \left(\frac{\mathbf{1} + \sigma_3}{2}\right) , \tag{5.57}$$

where we have omitted the overall shift in the energy E_0 since it has no physical consequence (we always measure the energy differences).

If the wavelength of the field is large compared to the size of the atomic system, the interaction of the field with the atom is given by (dipole approximation)

$$H_{atom-field} = -\mathbf{d}\cdot\mathbf{E} \;, \tag{5.58}$$

where $\mathbf{E}$ is given by

$$\mathbf{E} = i\hat{\epsilon}\sqrt{\frac{2\pi\hbar\omega}{V}}\left(a - a^{\dagger}\right) , \tag{5.59}$$

in the dipole approximation ($\mathbf{k}.\mathbf{r} \ll 1$) and $\mathbf{d} = e\ \mathbf{r}$ is the electric dipole operator. Then, we have

$$H_{atom-field} = -ier\sqrt{\frac{2\pi\hbar\omega}{V}}\left(a - a^{\dagger}\right) \ , \tag{5.60}$$

where the coordinate operator $r = \mathbf{r}\cdot\hat{\epsilon}$. Due to the completeness, r can be decomposed as

$$er \ = \ e\sum_{i,j}|i\rangle\langle i|r|j\rangle\langle j| = \sum_{i,j}\varrho_{ij}|i\rangle\langle j| \ , \tag{5.61}$$

where $\varrho_{ij} = e\langle i|j\rangle$ is the electric dipole transition element. It is convenient to introduce the atomic transition operators defined by

$$\begin{aligned} \sigma_+|0\rangle &= |1\rangle \qquad \sigma_+|1\rangle = 0 \ , \\ \sigma_-|0\rangle &= 0 \qquad \sigma_-|1\rangle = |0\rangle \ . \end{aligned} \tag{5.62}$$

Then, we have

$$\begin{aligned} \sigma_+ &= |1\rangle\langle 0| \\ \sigma_- &= |0\rangle\langle 1| \ . \end{aligned} \tag{5.63}$$

For $\varrho_{ij} = \varrho_{ji}$ without loss of generality, we obtain

$$\begin{aligned} H_{atom-field} &= g\left(\sigma_+ + \sigma_-\right)\left(a - a^{\dagger}\right) \\ &= g\sigma_+a - g\sigma_+a^{\dagger} + g\sigma_-a - g\sigma_-a^{\dagger} \\ &= g\sigma_+a + g^*\sigma_-a^{\dagger} - g^*\sigma_-a - g\sigma_+a^{\dagger} \ , \end{aligned} \tag{5.64}$$

where $g = -i\varrho_{10}\sqrt{\frac{2\pi\hbar\omega}{V}}$. Let us consider the interaction Hamiltonian consisting of two separate terms: $H_{atom-field} = H_r - H_a$, where $H_r = g\sigma_+a + g^*\sigma_-a^{\dagger}$ and $H_a = g^*\sigma_-a + g\sigma_+a^{\dagger}$. The term $\sigma_-a^{\dagger}$ in the Hamiltonian H_r allows the atomic transition from the excited state to the ground state with the simultaneous emission of a photon while the other term $\sigma_+ a$ allows the transition from the ground state to the excited state with the absorption of a photon. On the other hand, the term in the Hamiltonian H_a allows the atomic excitation with a simultaneous creation of a photon, while the other term causes the absorbtion process. It is easy to see that the number of excitations

$$N = -\left(\frac{\mathbf{1}-\sigma_3}{2}\right) + a^\dagger a \ , \tag{5.65}$$

is conserved if we take the interaction Hamiltonian as H_r, that is,

$$[N, H_r] = 0 \ , \qquad \text{whereas} \qquad [N, H_a] \neq 0 \ . \tag{5.66}$$

This suggests us to work only with H_r but there is another reason for ignoring H_a. It is very convenient and crucial to study the perturbative quantum field theory in the interaction picture. Interaction Hamiltonian in the interaction picture is

$$H_{atom-field;I}(t) = e^{\frac{i}{\hbar}H_0 t} H_{atom-field} e^{-\frac{i}{\hbar}H_0 t} \ , \tag{5.67}$$

where

$$H_0 = H_{field} + H_{atom} = \hbar\omega_0 \left(\frac{\mathbf{1}+\sigma_3}{2}\right) + \hbar\omega a^\dagger a \ , \tag{5.68}$$

Moreover, we have ignored the zero-point energy coming from the field Hamiltonian. Using the Campbell-Baker-Hausdorff formula $e^{tA}Be^{-tA} = B + t[A,B] + \frac{t^2}{2!}[A,[A,B]] + \ldots$, we get

$$\begin{aligned} e^{i\omega a^\dagger a t} a e^{-i\omega a^\dagger a t} &= a e^{-i\omega t} \\ e^{i\omega a^\dagger a t} a^\dagger e^{-i\omega a^\dagger a t} &= a^\dagger e^{i\omega t} \\ e^{i\omega_0(\frac{\mathbf{1}+\sigma_3}{2})t} \sigma_\pm e^{-i\omega_0(\frac{\mathbf{1}+\sigma_3}{2})t} &= \sigma_\pm e^{\pm i\omega t} \ . \end{aligned} \tag{5.69}$$

Hence, we obtain

$$H_{atom-field;I} = g\sigma_+ a e^{-i(\omega-\omega_0)t} + g^*\sigma_- a^\dagger e^{i(\omega-\omega_0)t} \\ - g\sigma_+ a^\dagger e^{i(\omega+\omega_0)t} - g^*\sigma_- a e^{-i(\omega+\omega_0)t} \ . \tag{5.70}$$

In the case $\omega \simeq \omega_0$, the first two exponentials is almost unity while the other two exponentials oscillate rapidly. We may therefore neglect the rapidly oscillating terms (called rotating wave approximation (RWA)) and the resulting simplified Hamiltonian is

$$H = \hbar\omega_0 \left(\frac{\mathbf{1} + \sigma_3}{2} \right) + \hbar\omega a^\dagger a + g\sigma_+ a + g^*\sigma_- a^\dagger \ . \tag{5.71}$$

Usually, the zero-point energy in the first term is omitted in the literature so we have

$$H = \hbar\omega_0 \frac{\sigma_3}{2} + \hbar\omega a^\dagger a + g\sigma_+ a + g^*\sigma_- a^\dagger \ , \tag{5.72}$$

where the conserved quantity in this case $N = a^\dagger a + \sigma_3/2$. This Hamiltonian describes the interaction of a single two-level atom with a single mode field and it is known as Jaynes-Cummings model [128]. The case of many atoms corresponds to the Dicke model [129]. The Hamiltonian (5.72) acts on a tensor product of the infinite dimensional bosonic Fock space $\mathcal{F}_B$ with the two dimensional Hilbert space $\mathbb{C}^2$. The basis vectors in this space is

$$|n\rangle \otimes |\chi\rangle \ , \tag{5.73}$$

where $\chi = 0, 1$ and $a^\dagger a|n\rangle = n|n\rangle$. These states $|n\rangle$ and $|\chi\rangle$ are called bare states since they are eigenstates of the free Hamiltonian. Note that $\sigma_3|\chi\rangle = (2\chi - 1)|\chi\rangle$. Since N is conserved we can fix its value without loss of generality and study the model with the fixed N sector, which simplifies the problem. The eigenvector of N is given

$$|n, \chi\rangle_d = |n - \chi\rangle \otimes |\chi\rangle \ , \tag{5.74}$$

where d stands for dressed states (they are eigenstates of the full Hamiltonian) and its eigenvalue equation is

$$N|n,\chi\rangle_d = \left(n - \frac{1}{2}\right)|n,\chi\rangle_d \ . \tag{5.75}$$

For each fixed value of the eigenvalue of the number operator N, there is a two dimensional subspace for each value of χ except for the one dimensional subspace $n = 0$. Since $[H, N] = 0$, any physical eigenvector of the Hamiltonian must be a superposition of the basis vectors of this two dimensional subspace, which simplifies and allows one to solve the model exactly. Let us suppose that the eigenvalue of N is chosen as $n - \frac{1}{2}$ without loss of generality. This means that the state corresponding to this chosen N consists of either the state for n photons with the atom in the ground state $|0\rangle$ or $n-1$ photon with the atom in its excited state $|1\rangle$. The interaction is then responsible for excitation or absorbtion processes and mixes these states. Therefore, we have

$$|\psi\rangle_d = \sum_{n=1}^{\infty} \alpha_n |n\rangle \otimes |0\rangle + \beta_n |n-1\rangle \otimes |1\rangle \ . \tag{5.76}$$

The stationary Schrödinger equation $H|\psi\rangle_d = E|\psi\rangle_d$ gives the following coupled equations written in matrix form

$$\begin{pmatrix} \frac{\hbar\omega_0}{2} + \hbar\omega(n-1) & g\sqrt{n} \\ g^*\sqrt{n} & -\frac{\hbar\omega_0}{2} + \hbar\omega n \end{pmatrix} \begin{pmatrix} \beta_n \\ \alpha_n \end{pmatrix} = E \begin{pmatrix} \beta_n \\ \alpha_n \end{pmatrix} , \tag{5.77}$$

where we have used $\langle n|m\rangle = \delta_{nm}$. The eigenvalues of the Hamiltonian reads

$$E_\pm = \left(n - \frac{1}{2}\right)\hbar\omega \pm \Omega \ , \tag{5.78}$$

where $\Omega = \frac{1}{2}\sqrt{(\hbar\omega_0 - \hbar\omega)^2 + 4|g|^2 n}$. For each value of n, there are two eigenstates and the splitting between the eigenvalues is Ω. The normalized eigenstates are then

$$\begin{pmatrix} -\frac{e^{i\phi/2}|g|\sqrt{n}}{\sqrt{\left(\frac{\hbar(\omega_0-\omega)}{2}-\Omega\right)^2+n|g|^2}} \\ \frac{e^{-i\phi/2}(\frac{\hbar(\omega_0-\omega)}{2}-\Omega)}{\sqrt{\left(\frac{\hbar(\omega_0-\omega)}{2}-\Omega\right)^2+n|g|^2}} \end{pmatrix} \tag{5.79}$$

for $E = E_+$ and

$$\begin{pmatrix} \frac{e^{i\phi/2}(\frac{\hbar(\omega_0-\omega)}{2}+\Omega)}{\sqrt{\left(\frac{\hbar(\omega_0-\omega)}{2}+\Omega\right)^2+n|g|^2}} \\ -\frac{e^{-i\phi/2}|g|\sqrt{n}}{\sqrt{\left(\frac{\hbar(\omega_0-\omega)}{2}+\Omega\right)^2+n|g|^2}} \end{pmatrix} \tag{5.80}$$

for $E = E_-$. The further detailed analysis of the system can be found in quantum optics textbooks [130].

Let us return to our simplified non-relativistic Lee model in coordinate space.

$$H_0 = m_{V_0} V^\dagger V + m_{N_0} N^\dagger N + \int_{\mathbb{R}^3} \mathrm{d}^3x \phi^\dagger(\mathbf{x}) \left(-\frac{\nabla^2}{2m} + m\right) \phi(\mathbf{x}) \ , \tag{5.81}$$

and

$$H_I = \lambda_0 \left[V^\dagger N \phi_0 + N^\dagger \phi_0^\dagger V\right] \ , \tag{5.82}$$

where $\phi_0 = \phi(\mathbf{x} = 0) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3p}{(2\pi)^3} \phi(\mathbf{p})$. In the previous section, we only consider the sectors $Q_1 = 1, Q_2 = 0$. Previous discussion on the two level atom model motivates us to ask whether we can extend the restricted sector $Q_1 = 1, Q_2 = 1$ to the sector $Q_1 = 1, Q_2 = n$ or not. Indeed, the Lee model is very similar to the two level optical model except that the bosons are allowed to have many possible energies and momenta. The Hamiltonian acts on a tensor product of the infinite dimensional bosonic Fock space $\mathcal{F}_B$ with the four dimensional Hilbert space whose bare basis are $|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle$. We have $V^\dagger|0,0\rangle = |1,0\rangle, N^\dagger|0,0\rangle = |0,1\rangle$. Since only the states $|0,1\rangle, |1,0\rangle$ are the eigenstates of Q_1 with eigenvalue 1, the problem is restricted to the two dimensional subspaces. In analogy with the two level atom model, let us suppose that the eigenvalue of Q_2 is n, then any physical state consists of either the state for $(n+1)$ θ particles with one N particle or n θ particles with no N particle. It is the interaction that mixes these states into each other but keep the states in this two sector subspace. Then, we can find the representation of the Hamiltonian in two sector subspace. It is well known that there is a isomorphism between the algebra of the Pauli sigma matrices, $su(2)$ and bilinear products of fermionic creation and annihilation operators.

$$\begin{aligned} V^\dagger N &= \sigma_+ \\ V N^\dagger &= \sigma_- \\ N^\dagger N - V^\dagger V &= \sigma_3 \ , \end{aligned} \tag{5.83}$$

where

$$\begin{aligned} [\sigma_3, \sigma_\pm] &= \pm 2\sigma_\pm \\ [\sigma_+, \sigma_-] &= \sigma_3 \\ \sigma_\pm &= \frac{1}{2}(\sigma_1 \pm i\sigma_2) \ . \end{aligned} \tag{5.84}$$

We now reexpress the Hamiltonian

$$\begin{aligned} m_{V_0} V^\dagger V + m_{N_0} N^\dagger N &= m_{V_0} V^\dagger V + m_{N_0} N^\dagger N - m_{N_0} V^\dagger V + m_{N_0} V^\dagger V \\ &= \hat{\mu}_0 V^\dagger V + m_{N_0}(V^\dagger V + N^\dagger N) \ , \end{aligned} \tag{5.85}$$

where $\hat{\mu}_0 = m_{V_0} - m_{N_0}$ and the second term is just an overall shift in the energy since it is proportional to the conserved quantity Q_1, thus we omit this term. As a consequence of these, we may rewrite the Hamiltonian in the following form

$$\begin{aligned} H_0 &= \hat{\mu}_0 \left(\frac{1-\sigma_3}{2} \right) + \int_{\mathbb{R}^3} \mathrm{d}^3 x \phi^\dagger(\mathbf{x}) \left(-\frac{\nabla^2}{2m} + m \right) \phi(\mathbf{x}) \\ H_I &= \lambda \left(\sigma_+ \phi_0 + \sigma_- \phi_0^\dagger \right) \ , \end{aligned} \tag{5.86}$$

where we have ignored some overall constant terms. Two level optical model Hamiltonian is equivalent to this Hamiltonian above by the unitary transformation $\sigma_1 H \sigma_1$ except for the fact that the former model deals with only single mode massless bosons.

5.1.4. Construction of Non-relativistic Lee model in Two and Three Dimensional Riemannian Manifolds

It is possible to look at the same problem from the point of view of the resolvent of the Hamiltonian in a Fock space formalism with arbitrary number of bosons (in fact there is a conserved quantity which allows us to restrict the problem to the direct sum of n and $n+1$ boson sectors, as we have shown in the previous section). This will be achieved by the same method which we have used for delta potentials in the previous chapter.

We start with the regularized Hamiltonian of the non-relativistic Lee model on two or three dimensional Riemannian manifold $(\mathcal{M}, g)$ with a cut-off ϵ. Adopting the natural units ($\hbar = c = 1$), one can write down the regularized Hamiltonian on the local coordinates $x = (x_1, x_2, ..., x_D) \in \mathcal{M}$

$$H^{\epsilon} = H_0 + H_{I,\epsilon} \ , \tag{5.87}$$

where

$$H_0 = \int_{\mathcal{M}} \mathrm{d}_g^D x \; \phi_g^{\dagger}(x) \left(-\frac{1}{2m} \nabla_g^2 + m \right) \phi_g(x) \ , \tag{5.88}$$

$$H_{I,\epsilon} = \hat{\mu}(\epsilon) \frac{1 - \sigma_3}{2} + \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x \; \rho_\epsilon(x, a) \left(\phi_g(x) \, \sigma_- + \phi_g^{\dagger}(x) \, \sigma_+ \right) \ . \tag{5.89}$$

Here $\phi_g^{\dagger}(x)$, $\phi_g(x)$ is the bosonic creation-annihilation operators defined on the manifold with the metric structure g_{ij} and λ is the coupling constant. Also, $\rho_\epsilon(x, a)$ is a family of functions which converge to the Dirac delta function $\delta_g(x, a)$ around the point a on $\mathcal{M}$ as we take the limit $\epsilon \to 0^+$. Similar to the flat case, $\hat{\mu}(\epsilon)$ is defined as a bare mass difference between the V particle (neutron) and the N particle (proton). Its explicit form will be determined later on. Although the number of bosons is not conserved in the model, one can derive from the equations of motion that there exists a conserved quantity

$$Q_2 = -\left(\frac{1+\sigma_3}{2}\right) + \int_{\mathcal{M}} \mathrm{d}_g^D x \; \phi_g^\dagger(x)\phi_g(x)$$

In analogy with the flat space, if $Q_2 = n$, we have one bare V particle state (which we can identify as spin-down state (or neutron)) with n θ particles or one bare N particle state (interpreted as spin-up state (proton)) with $n+1$ θ particles.

$$\begin{aligned} V^\dagger|0,0\rangle &= |1,0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |n\rangle \\ N^\dagger|0,0\rangle &= |0,1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |p\rangle \; . \end{aligned} \tag{5.90}$$

If one considers the Hamiltonian without the cut-off ϵ, it can be shown that the bound state energy diverges as we have discussed for the flat space case. Before discussing how to deal with the infinities, we must see how the infinities emerge in our model on Riemannian manifolds . The simplest way to realize this is just to look at the sector that contains the neutron or the proton with one boson, which corresponds to $Q_2 = 1$, that is, we propose the following eigenstate ansatz [131]:

$$|u,\psi\rangle = \begin{pmatrix} \int_{\mathcal{M}} \mathrm{d}_g^D x \; \psi(x)\phi_g^\dagger(x)|0\rangle \\ u|0\rangle \end{pmatrix} \; . \tag{5.91}$$

For simplicity we explicitly perform our calculations for compact manifolds here, but our result is also valid for non-compact manifolds, which we are interested in.

From the eigenvalue equation $H|u,\psi\rangle = E|u,\psi\rangle$, we find the set of equations in terms of the bosonic wave function $\psi(x) = \sum_{l=0}^{\infty} \psi(l) f_l(x;g)$

$$\psi(l) = \frac{u\lambda f_l^*(a;g)}{E - \left(\frac{\sigma_l}{2m} + m\right)} , \tag{5.92}$$

$$u(E - \hat{\mu}) = \lambda \sum_{l=0}^{\infty} f_l(a;g)\psi(l) . \tag{5.93}$$

If we substitute the equation (5.92) into the equation (5.93), we obtain

$$\hat{\mu} = E - \lambda^2 \sum_{l=0}^{\infty} \frac{|f_l(a;g)|^2}{E - \left(\frac{\sigma_l}{2m} + m\right)} . \tag{5.94}$$

Expressing this equation in terms of heat kernel $K_s(x, x'; g) = \langle x | e^{s\nabla_g^2/2m} | x' \rangle$, we get

$$\hat{\mu} = E + \lambda^2 \int_0^\infty \mathrm{d}s \; K_s(a, a; g) \, e^{-s(m-E)} . \tag{5.95}$$

The integral in equation (5.95) diverges due to the asymptotic expansion of the diagonal part of heat kernel near $s = 0$ for any two and three dimensional manifolds (3.54). As a result of this, the bound state energy becomes divergent. Recall that the momentum integral in flat space case blows up at large values of momentum in two and three dimensions. The problem was basically taking the integral over all momenta because our no-recoil approximation breaks down for large enough momenta. So, we have introduced an ultra-violet cut-off Λ in the upper bound of the momentum integrals. Since large momenta means small distances, this cut-off corresponds to putting a small distance cut-off in coordinate space. Performing the calculations in coordinate space one sees that small distance cut-off can be replaced with a short "time" cut-off ϵ in the lower limit of the integral (5.95). Here we show that the idea of short "time" cut-off will work on Riemannian manifolds, whereas the momentum cut-off is not a natural method to use.

Therefore, we first regularize the Hamiltonian, that is, introduce the cut-off ϵ on the lower bound of the integral and make the parameters in the Hamiltonian depend on ϵ such that all physical quantities are independent of it. So we define

$$\hat{\mu}(\epsilon) = \hat{\mu} + \lambda^2 \int_{\epsilon}^{\infty} \mathrm{d}s \; K_s(a,a;g)\, e^{-s(m-\hat{\mu})} \;, \tag{5.96}$$

where, as we will see, the bound state energy E is traded with $\hat{\mu}$ which is defined as the physical energy of the composite state which consists of a boson and the attractive heavy neutron at the center. Now, using (5.92), (5.93) and (5.96), we get the finite expression for the bound state energy

$$E = \hat{\mu} + \lambda^2 \int_0^{\infty} \mathrm{d}s \; K_s(a,a;g) \left[e^{-s(m-\hat{\mu})} - e^{-s(m-E)} \right] \;. \tag{5.97}$$

Here it is easy to see that $E = \hat{\mu}$ is a possible solution to this equation. Then, one can calculate the bosonic wave function $\psi(x)$ for $E = \hat{\mu}$

$$\psi(x) = -u\lambda \int_0^{\infty} \mathrm{d}s \; K_s(x,a;g) e^{-s(m-\hat{\mu})} \;, \tag{5.98}$$

if $x \neq a$. Although the wave function is divergent as $x \to a$, it is square integrable as we will see. If we substitute (5.98) into our ansatz (5.91), we get

$$|u\psi\rangle = u \begin{pmatrix} -\frac{\lambda}{H_0 - \hat{\mu}} \phi_g^{\dagger}(a)|0\rangle \\ |0\rangle \end{pmatrix} \;. \tag{5.99}$$

Normalizability of (5.99) can be easily seen by using the properties of heat kernel [9] :

$$\begin{aligned} & \int_{\mathcal{M}} \mathrm{d}_g^D x \int_0^{\infty} \mathrm{d}s_1 \left[\int_0^{\infty} \mathrm{d}s_2 K_{s_1}(x,a;g) K_{s_2}(x,a;g) \; e^{-(s_1+s_2)(m-\hat{\mu})} \right] \\ = & \int_0^{\infty} \mathrm{d}s \left(\frac{1}{2} \int_{-s}^{s} \mathrm{d}t \right) K_s(a,a;g) \; e^{-s(m-\hat{\mu})} = \int_0^{\infty} s \, \mathrm{d}s \; e^{-s(m-\hat{\mu})} \; K_s(a,a;g) \;\;, \end{aligned} \tag{5.100}$$

[9] The same normalization can also be found by writing the operators $\phi(a)$ in the eigenbasis $f_l(a;g)$ of the Laplacian.

and as a result we find the normalization to be

$$\left[1+\lambda^2 \int_0^\infty s \, \mathrm{d}s \; e^{-s(m-\hat{\mu})} K_s(a,a;g)\right]^{-1/2} \; . \tag{5.101}$$

This integral is finite due to the short and long time behavior of heat kernel.

One can also consider the scattering of a boson from the N particle (or proton) at rest on a noncompact manifold with infinite volume [10] . The inhomogeneous Schrödinger equation $(H-E)|u,\psi\rangle = |v,\chi\rangle$ leads to

$$\psi(l) = \frac{\chi(l) - \lambda u f_l^*(a;g)}{\frac{\sigma_l}{2m} + m - E} \; , \tag{5.102}$$

and

$$\lambda \sum_{l=0}^{\infty} f_l(a;g)\psi(l) + u(\hat{\mu} - E) = v \; . \tag{5.103}$$

If we substitute (5.102) into (5.103), we find

$$-\lambda^2 u \sum_{l=0}^{\infty} \frac{|f_l(a;g)|^2}{\frac{\sigma_l}{2m} + m - E} + u(\hat{\mu} - E) = v - \lambda \sum_{l=0}^{\infty} \frac{f_l(a;g)\chi(l)}{\frac{\sigma_l}{2m} + m - E} \; . \tag{5.104}$$

If we express the above equation in terms of heat kernel, we immediately see that the integral is divergent due to singularity near $s=0$. So we must do the regularization by introducing a cut-off ϵ in the lower limit of the integral and choose $\hat{\mu}(\epsilon)$ as above. Taking the limit $\epsilon \to 0^+$ we get the following finite expression:

$$\lim_{\epsilon \to 0^+} \left(u(\hat{\mu} - E) + \lambda^2 u \int_\epsilon^\infty \mathrm{d}s \; K_s(a,a;g)\, e^{-s(m-\hat{\mu})} - \lambda^2 u \int_\epsilon^\infty \mathrm{d}s \; K_s(a,a;g)\, e^{-s(m-E)} \right)$$
$$= v - \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x \; \chi(x) \left(\int_0^\infty \mathrm{d}s \; K_s(x,a;g)\, e^{-s(m-E)} \right) \; . \tag{5.105}$$

[10] Compact manifolds have only discrete spectrum.

From the above equation, it follows that

$$u \equiv u[v,\chi] = \left[\hat{\mu} - E + \lambda^2 \int_0^\infty \mathrm{d}s\; K_s(a,a;g)\, \left(e^{-s(m-\hat{\mu})} - e^{-s(m-E)}\right)\right]^{-1} \times \left[v - \lambda \int_0^\infty \mathrm{d}s\; \left(\int_{\mathcal{M}} \mathrm{d}_g^D x\; K_s(x,a;g)\chi(x)\right)\, e^{-s(m-E)}\right] .$$

We can also read off $\psi(x)$ from the equation (5.102),

$$\psi(x) = \int_0^\infty \mathrm{d}s\; e^{-s(m-E)} \int_{\mathcal{M}} \mathrm{d}_g^D y\, K_s(x,y;g)\, \chi(y) - \lambda\, u[v,\chi] \int_0^\infty \mathrm{d}s\; K_s(x,a;g)\, e^{-s(m-E)} \, . \tag{5.106}$$

It is important to remind the reader that these expressions should be understood in the sense of analytic continuation in the complex E-plane to their largest domain of definition. Indeed if the real part of E is smaller than m these integrals all make sense, and the resulting expression is just the Green's function, or the resolvent for the Laplace operator, which exists away from the positive real axis. In fact, in order to get the analytic continuations it will be instructive to rewrite these expressions in terms of Green's functions:

$$\begin{aligned} u &= \left[\hat{\mu} - E + \lambda^2 \lim_{z\to a}(R_0(z,a|\hat{\mu}) - R_0(z,a|E))\right]^{-1} \left[v - \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x\, R_0(a,x|E)\chi(x)\right] \\ \psi(x) &= \int_{\mathcal{M}} \mathrm{d}_g^D y\, R_0(x,y|E)\chi(y) - \lambda u[v,\chi] R_0(x,a|E) \, . \end{aligned} \tag{5.107}$$

Here the limit of the difference of the two Green's functions is defined through the heat kernel.

Up to now, we have shown that non-relativistic Lee model on a manifold is divergent and can be renormalized with the help of heat kernel by considering the problem for $Q_2 = 1$ sector. However, it is not clear in this formulation, how we can extend this method to the case that have arbitrary number of particles, say $Q_2 = n$ sectors and show that the ground state energy is bounded from below. Nevertheless, we have an alternative and powerful method which is developed by Rajeev [26]. From now on, we will follow his approach in order to construct and develop the model on a manifold for any sector.

Let us first express the regularized Hamiltonian as a 2×2 block split according to $\mathbb{C}^2$:

$$H^{\epsilon} - E = \begin{pmatrix} H_0 - E & \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x \; \rho_\epsilon(x,a)\, \phi_g^\dagger(x) \\ \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x \; \rho_\epsilon(x,a)\, \phi_g(x) & H_0 - E + \hat{\mu}(\epsilon) \end{pmatrix} . \tag{5.108}$$

Then, one can construct the regularized resolvent of this Hamiltonian using an explicit formula given by Rajeev [26]

$$R^{\epsilon}(E) = \frac{1}{H^{\epsilon} - E} = \begin{pmatrix} \alpha_\epsilon & \beta_\epsilon^\dagger \\ \beta_\epsilon & \delta_\epsilon \end{pmatrix} = \begin{pmatrix} a_\epsilon & b_\epsilon^\dagger \\ b_\epsilon & d_\epsilon \end{pmatrix}^{-1} , \tag{5.109}$$

where

$$\begin{aligned} \alpha_\epsilon &= \frac{1}{H_0 - E} + \frac{1}{H_0 - E}\, b_\epsilon^\dagger \, \Phi_\epsilon^{-1}(E)\, b_\epsilon \, \frac{1}{H_0 - E} \\ \beta_\epsilon &= -\Phi_\epsilon^{-1}(E)\, b_\epsilon \, \frac{1}{H_0 - E} \\ \delta_\epsilon &= \Phi_\epsilon^{-1}(E) \\ b_\epsilon &= \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x \; \rho_\epsilon(x,a)\, \phi_g(x) \; . \end{aligned} \tag{5.110}$$

Here E should be considered as a complex variable. Most importantly, the operator $\Phi_\epsilon(E)$, called principal operator, is given as

$$\Phi_\epsilon(E) = H_0 - E + \hat{\mu}(\epsilon) - \lambda^2 \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; \rho_\epsilon(x,a) \rho_\epsilon(y,a) \; \phi_g(x) \frac{1}{H_0 - E} \phi_g^\dagger(y) \; , \tag{5.111}$$

Once we have a proper definition of the principal operator, all the divergences are removed since the resolvent is expressed in terms of it. We can extend the principal operator by analytic continuation to its largest domain of definition in the complex energy plane. For our purposes, we will assume that $\Re(E) < nm + \hat{\mu}$ in the formulae below. In fact, the energy of bound states are interesting and they satisfy the required conditions as we will see. Now, we will do the normal ordering of the operators in

(5.111) by using the commutation relations of the operators $\phi_g(x)$ and $\phi_g^\dagger(x)$

$$\frac{1}{H_0 - E}\phi_g^\dagger(x) = \int_{\mathcal{M}} \mathrm{d}_g^D x' \; \phi_g^\dagger(x') \int_0^\infty \mathrm{d}s \; e^{-s(H_0 - E + m)} \, K_s(x, x'; g) \; . \tag{5.112}$$

This can be proven simply by using an eigenfunction expansion. Then, the principal operator can be written in terms of heat kernel

$$\begin{aligned} \Phi_\epsilon(E) &= H_0 - E - \lambda^2 \int_0^\infty \mathrm{d}s \int_{\mathcal{M}^4} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \, \mathrm{d}_g^D x' \, \mathrm{d}_g^D y' \; \rho_\epsilon(x,a)\rho_\epsilon(y,a) \\ &\quad \times K_s(x,x';g)\, K_s(y,y';g)\, \phi_g^\dagger(y')\, e^{-s(H_0+2m-E)}\, \phi_g(x') \\ + \hat{\mu}(\epsilon) - \lambda^2 &\int_0^\infty \mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D y \, \mathrm{d}_g^D y' \; K_s(y,y';g)\rho_\epsilon(y,a)\rho_\epsilon(y',a)\, e^{-s(H_0+m-E)} \; . \end{aligned} \tag{5.113}$$

Since the heat kernel is a natural delta sequence, we can set $\rho_\epsilon(x,a) = K_{\epsilon/2}(x,a;g)$ without loss of generality. Hence,

$$\begin{aligned} \Phi_\epsilon(E) &= H_0 - E - \lambda^2 \int_0^\infty \mathrm{d}s \int_{\mathcal{M}^4} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \, \mathrm{d}_g^D x' \, \mathrm{d}_g^D y' \; K_{\epsilon/2}(x,a;g)K_{\epsilon/2}(y,a;g) \\ &\quad \times K_s(x,x';g)\, K_s(y,y';g)\, \phi_g^\dagger(y')\, e^{-s(H_0+2m-E)}\, \phi_g(x') + \hat{\mu}(\epsilon) \\ -\lambda^2 &\int_0^\infty \mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D y \, \mathrm{d}_g^D y' \; K_s(y,y';g)K_{\epsilon/2}(y,a;g)K_{\epsilon/2}(y',a;g)\, e^{-s(H_0+m-E)} \; . \end{aligned} \tag{5.114}$$

Using the semigroup or reproducing property of the heat kernel (3.36), one can rewrite the above equation as

$$\begin{aligned} \Phi_\epsilon(E) &= H_0 - E - \lambda^2 \int_0^\infty \mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; K_{(2s+\epsilon)/2}(x,a;g)K_{(2s+\epsilon)/2}(y,a;g) \\ \times \phi_g^\dagger(x)\, e^{-s(H_0+2m-E)}\, &\phi_g(y) + \hat{\mu}(\epsilon) - \lambda^2 \int_0^\infty \mathrm{d}s \; K_{(s+\epsilon)}(a,a;g)\, e^{-s(H_0+m-E)} \; . \end{aligned} \tag{5.115}$$

Shifting the variable s in the first integral by $\epsilon/2$ and in the second integral by ϵ, we get

$$\begin{aligned} \Phi_\epsilon(E) = H_0 - E - \lambda^2 &\int_{\epsilon/2}^\infty \mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; K_s(x,a;g)K_s(y,a;g) \\ &\times \; \phi_g^\dagger(x)\, e^{-(s-\epsilon/2)(H_0+2m-E)}\, \phi_g(y) + \hat{\mu}(\epsilon) \\ &- \lambda^2 \int_\epsilon^\infty \mathrm{d}s \; K_s(a,a;g)\, e^{-(s-\epsilon)(H_0+m-E)} \; . \end{aligned} \tag{5.116}$$

If we take the limit $\epsilon \to 0^+$, the only divergence is coming from the lower limit of the second integral term due to singular behavior of the diagonal part, for two and three dimensions ($k = 0$ term in the sum) of heat kernel near $s = 0$, see equation (3.54).

One can also see that the first interaction term is actually finite due to the quite sharp bounds on the heat kernel for various classes of manifolds [61, 75]. In fact we will explicitly show later on that this term is really a finite expression by working out a bound on the spectrum of the model. Since the principal operator or resolvent is not well defined in this limit, we must now regularize the model by choosing $\hat{\mu}(\epsilon)$ exactly the same as in (5.96). Then, we find

$$\begin{aligned}\Phi_\epsilon(E) = H_0 - E + \hat{\mu} - \lambda^2 \int_{\epsilon/2}^{\infty} \mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D x\, \mathrm{d}_g^D y\; K_s(x,a;g) K_s(y,a;g) \\ \times\, \phi_g^\dagger(x) e^{-(s-\epsilon/2)(H_0+2m-E)} \phi_g(y) \\ +\lambda^2 \int_{\epsilon}^{\infty} \mathrm{d}s\; K_s(a,a;g) \left[e^{-s(m-\hat{\mu})} - e^{-(s-\epsilon)(H_0+m-E)} \right] \; . \qquad (5.117)\end{aligned}$$

Here the limit $\epsilon \to 0^+$ is now well-defined so we have

$$\begin{aligned}\Phi(E) \;&=\; H_0 - E + \hat{\mu} + \lambda^2 \int_0^{\infty} \mathrm{d}s\; K_s(a,a;g) \left[e^{-s(m-\hat{\mu})} - e^{-s(H_0+m-E)} \right] \\ -\lambda^2 \int_0^{\infty} \mathrm{d}s & \int_{\mathcal{M}^2} \mathrm{d}_g^D x\, \mathrm{d}_g^D y\; K_s(x,a;g) K_s(y,a;g)\, \phi_g^\dagger(x) e^{-s(H_0+2m-E)} \phi_g(y) \; . \qquad (5.118)\end{aligned}$$

This is the renormalized form of the principal operator so that we have a well-defined explicit formula for the resolvent of the Hamiltonian in terms of the inverse of the principal operator $\Phi^{-1}(E)$:

$$R(E) = \frac{1}{H-E} = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix} , \qquad (5.119)$$

where

$$\begin{aligned}\alpha \;&=\; \frac{1}{H_0 - E} + \frac{1}{H_0 - E}\, b^\dagger\, \Phi^{-1}(E)\, b\, \frac{1}{H_0 - E} \\ \beta \;&=\; -\Phi^{-1}(E)\, b\, \frac{1}{H_0 - E}\end{aligned}$$

$$\begin{aligned} \delta &= \Phi^{-1}(E) \\ b &= \lambda\phi_g(a) \; . \end{aligned} \tag{5.120}$$

This can be thought of as a kind of Krein's formula for the resolvent of our field theory. The spectrum of the Hamiltonian is the set of numbers E at which the resolvent does not exist (discrete spectrum) or exist but is unbounded (continuous spectrum). Thus, on a noncompact manifold, the continuous spectrum is that of H_0 and the values of E where $\Phi(E)$ does not have a bounded inverse. Since, on a noncompact manifold, there are no poles in $\frac{1}{H_0 - E}$, the poles corresponding to bound states must arise from those of $\Phi^{-1}(E)$, that is, the roots of the equation

$$\Phi(E)|\Psi\rangle = 0 \; , \tag{5.121}$$

determine the poles in the resolvent, which means that the principal operator $\Phi(E)$ determines the bound state spectrum of the theory. For compact manifolds only the discrete spectrum exists and bound states again appear as poles of $\Phi(E)$ below the values of the free Hamiltonian. After we have found a root, we can determine the corresponding eigenstate of the Hamiltonian. A trivial example is the bosonic vacuum state,

$$\begin{aligned} \Phi(E)|0\rangle = \Bigg\{ H_0 - E + \hat{\mu} + \lambda^2 \int_0^\infty \mathrm{d}s \; K_s(a,a;g) \left[e^{-s(m-\hat{\mu})} - e^{-s(H_0+m-E)} \right] \\ - \lambda^2 \int_0^\infty \mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; K_s(x,a;g) K_s(y,a;g) \, \phi_g^\dagger(x) e^{-s(H_0+2m-E)} \phi_g(y) \Bigg\} |0\rangle = 0 \; , \end{aligned} \tag{5.122}$$

where the root can be easily found to be

$$E = \hat{\mu} \, . \tag{5.123}$$

We remark here that the linear eigenvalue problem is converted into a nonlinear problem for an operator family parametrized through the energy eigenvalues after the renormalization.

The residue of the pole in the resolvent is the projection operator to the corresponding eigenspace of H [46]

$$\mathbb{P}_{\hat{\mu}} = -\frac{1}{2\pi i} \oint_{\Gamma_{\hat{\mu}}} \mathrm{d}E \; R(E) \; , \tag{5.124}$$

where $\Gamma_{\hat{\mu}}$ is a small contour enclosing the isolated eigenvalue $\hat{\mu}$ in the complex plane. We again use the same method given in the section for point interactions, that is, there exists a holomorphic family of projection operators on the complex plane [56] so that we can apply spectral theorem. By inserting a resolution of identity for the bosonic Fock space, we see that we need to evaluate the residue

$$\mathrm{Res}_{E=\hat{\mu}}(\Phi^{-1}(E)|0\rangle\langle 0|) = (\Phi'(\hat{\mu}))^{-1}|0\rangle\langle 0| \; . \tag{5.125}$$

As a result of this calculation, we find the projection operator to be

$$\begin{aligned} \mathbb{P}_{\hat{\mu}} &= \left[1 + \lambda^2 \int_0^\infty s \, \mathrm{d}s \; e^{-s(m-\hat{\mu})} K_s(a,a;g)\right]^{-1} \\ &\quad \times \begin{pmatrix} \frac{\lambda}{H_0-\hat{\mu}}\phi_g^\dagger(a)|0\rangle\langle 0|\phi_g(a)\frac{\lambda}{H_0-\hat{\mu}} & -\frac{\lambda}{H_0-\hat{\mu}}\phi_g^\dagger(a)|0\rangle\langle 0| \\ -|0\rangle\langle 0|\phi_g(a)\frac{\lambda}{H_0-\hat{\mu}} & |0\rangle\langle 0| \end{pmatrix} \; . \end{aligned} \tag{5.126}$$

Thus, one can read off the eigenvector of H corresponding to the root $E = \hat{\mu}$ from the projection operator and then find the normalizable eigenstate (5.99) with the correct normalization factor (5.101). This eigenstate (5.99) is the first excited state (eigenstate of neutron), not the vacuum in the whole Hilbert space $\mathcal{F}_{\mathcal{B}} \otimes \mathbb{C}^2$. Also, it is easy to see that the zero eigenvalue of the Hamiltonian corresponds to the proton state $\begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}$. Here $m > \hat{\mu}$ for bound states since we want this model to describe the attractive interaction of such a two state system with bosons. However, it is not clear whether the proton is the state of lowest energy. There may be states which contain many bosons that have a lower energy. These questions will be answered in studying the principal operator as we will do in the next section.

One can also generalize these ideas into the relativistic regime with an additional coupling constant and a wave function renormalization, but this exceeds the scope of this thesis and it has been done in [132].

5.1.5. A Lower Bound on the Ground State Energy for Two and Three Dimensions

After the renormalization of our model, we must look at the spectrum of the problem because there are many theories in which even after the renormalization there are still divergences that make the spectrum unbounded from below [26]. In this section we will restrict E to the real axis. In order to give the proof that the energy E is bounded from below, following the same idea as in [26], we first split the principal operator as

$$\Phi(E) = K(E) - U(E) , \tag{5.127}$$

such that

$$K(E) = H_0 - E + \hat{\mu} , \tag{5.128}$$

and

$$\begin{aligned} U(E) \;&=\; -\lambda^2 \int_0^\infty \mathrm{d}s \; K_s(a,a;g) \left[e^{-s(m-\hat{\mu})} - e^{-s(H_0+m-E)} \right] \\ +\lambda^2 \int_0^\infty \mathrm{d}s & \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; K_s(x,a;g) \, K_s(y,a;g) \, \phi_g^\dagger(x) \, e^{-s(H_0+2m-E)} \, \phi_g(y) \; . \end{aligned} \tag{5.129}$$

It follows immediately that $K(E) \geq nm - E + \hat{\mu}$, so it is a positive definite operator from our assumption $E < nm + \hat{\mu}$. Due to the positivity of the heat kernel (3.38) and the difference of the two exponentials is a positive operator, the first integral term in $U(E)$ is a negative operator and we remark that

$$U(E) < U'(E) , \tag{5.130}$$

where

$$U'(E) = \lambda^2 \int_0^\infty \mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; K_s(x,a;g) \, K_s(y,a;g) \, \phi_g^\dagger(x) \, e^{-s(H_0+2m-E)} \, \phi_g(y) \; . \tag{5.131}$$

This clearly forces

$$\Phi(E) > K(E) - U'(E) \; , \tag{5.132}$$

or rewriting it as

$$\Phi(E) > K(E)^{1/2} \; (1 - V(E)) \; K(E)^{1/2} \; , \tag{5.133}$$

where $V(E) = K(E)^{-1/2} \; U'(E) \; K(E)^{-1/2}$ and $K(E)$, $U'(E)$ are positive operators (so is $V(E)$). It must be emphasized that the unique square root of the positive self-adjoint operator $K(E)$ are well defined for real values of E. We will now show that by choosing E sufficiently small it is always possible to make the operator $\Phi(E)$ strictly positive, hence it becomes invertible, and has no zeros beyond this particular value of E. Therefore, if we impose

$$||V(E)|| < 1 \; , \tag{5.134}$$

then the principal operator $\Phi(E)$ becomes strictly positive. In order to do some estimates, we will rewrite the interaction term in terms of eigenfunctions $f_l(x;g)$ and shift the operator $\phi_g^\dagger(x)$ to the leftmost side and the operator $\phi_g(x)$ to the rightmost side

$$\begin{aligned} V(E) = \lambda^2 \sum_{l_1, l_2 \geq 0} & \phi_g^\dagger(l_1) \, f_{l_1}(a;g) [H_0 + \hat{\mu} + \sigma_{l_1}/2m + m - E]^{-1/2} \\ & \times \; [H_0 + \sigma_{l_1}/2m + \sigma_{l_2}/2m + 2m - E]^{-1} \\ & \times [H_0 + \hat{\mu} + \sigma_{l_2}/2m + m - E]^{-1/2} \, \phi_g(l_2) \, f_{l_2}(a;g) \; , \end{aligned} \tag{5.135}$$

where $\phi_g(l) = \int_{\mathcal{M}} \mathrm{d}_g^D x \; f_l^*(x;g) \, \phi_g(x)$. In order to convert the product of operators in the above formula into a summation of them, we will use the Feynman parametrization

[133], which was first introduced for calculating the loop integrals in quantum field theory

$$\frac{1}{A_1^{\alpha_1}A_2^{\alpha_2}A_3^{\alpha_3}} = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_0^1 \Pi_i^3 \mathrm{d}u_i \frac{\delta\left(\sum_i u_i - 1\right)\, u_1^{\alpha_1-1}u_2^{\alpha_2-1}u_3^{\alpha_3-1}}{[u_1A_1+u_2A_2+u_3A_3]^{\alpha_1+\alpha_2+\alpha_3}} \;, \quad (5.136)$$

so that,

$$\begin{aligned} V(E) = \lambda^2 \sum_{l_1,l_2} \phi_g^\dagger(l_1)\, f_{l_1}(a;g)\, \frac{\Gamma(1/2+1/2+1)}{\Gamma(1/2)\Gamma(1/2)\Gamma(1)} \\ \times \int_0^1 \mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\; u_1^{\frac{1}{2}-1}u_2^{\frac{1}{2}-1}u_3^{1-1}\delta(u_1+u_2+u_3-1) \\ \times \frac{1}{[H_0+m+\hat{\mu}-E+(u_1+u_3)\frac{\sigma_{l_1}}{2m}+(u_2+u_3)\frac{\sigma_{l_2}}{2m}+u_3(m-\hat{\mu})]^2}\,\phi_g(l_2)\, f_{l_2}(a;g)\;, \end{aligned} \quad (5.137)$$

or one can rewrite it as

$$\begin{aligned} V(E) \;=\; \lambda^2 \sum_{l_1,l_2} \phi_g^\dagger(l_1) f_{l_1}(a;g) \frac{\Gamma(2)}{\Gamma(1/2)^2} \int_0^1 \frac{\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\;\delta(u_1+u_2+u_3-1)}{(u_1\,u_2)^{1/2}} \\ \times \int_0^\infty s\,\mathrm{d}s\; e^{-s(H_0+m+\hat{\mu}-E+u_3(m-\hat{\mu})+(u_1+u_3)(\sigma_{l_1}/2m)+(u_2+u_3)(\sigma_{l_2}/2m))}\,\phi_g(l_2)\, f_{l_2}(a;g)\;. \end{aligned} \quad (5.138)$$

Let us express this equation in terms of the heat kernel:

$$\begin{aligned} V(E) \;=\; \frac{\lambda^2\,\Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s\,\mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D x\, \mathrm{d}_g^D y \int_0^1 \frac{\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\;\delta(u_1+u_2+u_3-1)}{(u_1\,u_2)^{1/2}} \\ \times\, \phi_g^\dagger(x)\, K_{s(u_1+u_3)}(x,a;g)\, K_{s(u_2+u_3)}(y,a;g)\, e^{-s(H_0+\hat{\mu}+m-E)}\, e^{-su_3(m-\hat{\mu})}\, \phi_g(y)\;. \end{aligned} \quad (5.139)$$

In order to give an upper bound estimate on the norm of the operator $V(E)$, we apply the Cauchy-Schwartz inequality in the norm and get

$$\begin{aligned} ||V(E)|| \;<\; n\, \frac{\lambda^2\,\Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s\,\mathrm{d}s\; e^{-s(nm+\hat{\mu}-E)} \int_0^1 \frac{\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\;\delta(u_1+u_2+u_3-1)}{(u_1\,u_2)^{1/2}} \\ \times \left[\int_{\mathcal{M}} \mathrm{d}_g^D x\; K^2_{s(u_1+u_3)}(x,a;g)\right]^{1/2} \left[\int_{\mathcal{M}} \mathrm{d}_g^D y\; K^2_{s(u_2+u_3)}(y,a;g)\right]^{1/2} . \end{aligned}$$

Here we have replaced the term $H_0 + m + \hat{\mu} - E$ in the exponent by its minimum value $nm + \hat{\mu} - E$ and dropped the term $e^{-su_3(m-\hat{\mu})} < 1$. By using the reproducing property of the heat kernel, we have

$$\begin{aligned} ||V(E)|| &< n \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s \, \mathrm{d}s \, e^{-s(nm+\hat{\mu}-E)} \int_0^1 \frac{\mathrm{d}u_1 \, \mathrm{d}u_2 \, \mathrm{d}u_3}{(u_1 \, u_2)^{1/2}} \delta(u_1 + u_2 + u_3 - 1) \\ & \times \left[K_{2s(u_1+u_3)}(a,a;g)\right]^{1/2} \left[K_{2s(u_2+u_3)}(a,a;g)\right]^{1/2} . \end{aligned} \tag{5.140}$$

For each class of manifolds, there are different upper bounds on the heat kernel so we will consider them separately. We will first consider Cartan-Hadamard manifolds. Using the resulting upper bound of the heat kernel for Cartan-Hadamard manifolds given in (3.79), we have

$$K_s(x,x;g) \leq \frac{C}{(s/2m)^{D/2}} \,, \tag{5.141}$$

for all $x \in \mathcal{M}$ and $s > 0$, and C is a dimensionless constant. Then, after performing u_3 integral, we get

$$\begin{aligned} ||V(E)|| &< n \, C m^{D/2} \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s \, \mathrm{d}s \, \frac{e^{-s(nm+\hat{\mu}-E)}}{s^{D/2}} \\ & \int_0^1 \mathrm{d}u_1 \frac{1}{u_1^{1/2}} \frac{1}{(1-u_1)^{D/4}} \int_0^{1-u_1} \mathrm{d}u_2 \, \frac{1}{u_2^{1/2}} \frac{1}{(1-u_2)^{D/4}} \\ & \leq n \, C m^{D/2} \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s \, \mathrm{d}s \, \frac{e^{-s(nm+\hat{\mu}-E)}}{s^{D/2}} \left[\int_0^1 \mathrm{d}u \frac{1}{u^{1/2}} \frac{1}{(1-u)^{D/4}} \right]^2 . \end{aligned} \tag{5.142}$$

Evaluating the integrals, we have

$$||V(E)|| < n \, C m^{D/2} \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} (nm + \hat{\mu} - E)^{\frac{D}{2}-2} \Gamma(2 - \frac{D}{2}) \left[\frac{\sqrt{\pi} \Gamma(1 - \frac{D}{4})}{\Gamma(\frac{3}{2} - \frac{D}{4})} \right]^2 \tag{5.143}$$

Then the strict positivity of the principal operator (5.134) leads to the following inequality for the parameter E to stay away from the spectrum

$$E < nm + \hat{\mu} - \left(n \tilde{C} \lambda^2 m^{D/2} \right)^{\frac{1}{2-\frac{D}{2}}} . \tag{5.144}$$

If we choose E below this value we avoid the spectrum, so the inequality above implies a lower bound for the ground state energy

$$E_{gr} \geq nm + \hat{\mu} - \left(n\tilde{C}\lambda^2 m^{D/2} \right)^{\frac{1}{2-\frac{D}{2}}} , \tag{5.145}$$

where

$$\tilde{C} = C \, \frac{\pi\Gamma(2)\Gamma(1-\frac{D}{4})^2\Gamma(2-\frac{D}{2})}{\Gamma(\frac{1}{2})^2\Gamma(\frac{3}{2}-\frac{D}{4})^2} \, . \tag{5.146}$$

Secondly, as an explicit application, we consider the 3 dimensional hyperbolic space $\mathbb{H}^3$. The heat kernel in $\mathbb{H}^3$ is explicitly known [57, 58, 59] and it is in natural units

$$K_s(x,y;g) = \frac{1}{R^3} \frac{d(x,y)/R}{\sinh(d(x,y)/R)} \frac{e^{-\frac{s}{2mR^2}-\frac{md^2(x,y)}{2s}}}{\left(4\pi \frac{s}{2mR^2}\right)^{3/2}} , \tag{5.147}$$

where R is the length scale and $d(x,y)$ is the geodesic distance between two points on $\mathbb{H}^3$. Again we impose the strict positivity condition on the principal operator and get

$$\begin{aligned} ||V(E)|| < n \, \frac{\lambda^2 \, \Gamma(2)}{\Gamma(1/2)^2} \int_0^1 \frac{\mathrm{d}u_1 \, \mathrm{d}u_2}{(u_1 \, u_2)^{1/2}(1-u_1)^{3/4}(1-u_2)^{3/4}} \\ \times \int_0^\infty \frac{s \, \mathrm{d}s \; e^{-s(nm+\hat{\mu}-E)} e^{-s(1-u_1)/2mR^2} e^{-s(1-u_2)/2mR^2}}{(4\pi s/m)^{3/2}} < 1 \, . \end{aligned} \tag{5.148}$$

Using the fact that $e^{-s(1-u_{1(2)})/2mR^2} < 1$ and then taking the integrals, one can immediately see that this implies

$$E_{gr} \geq nm + \hat{\mu} - n^2\lambda^4 D^2 m^3 , \tag{5.149}$$

where the constant D is defined as

$$D = \frac{\Gamma(2)\Gamma(1/4)^2}{\Gamma(1/2)^2\Gamma(3/4)^2 2^3} \, . \tag{5.150}$$

Finally, we apply our method to the compact manifolds with Ricci curvature bounded from below by $-K \geq 0$. Using the diagonal upper bound of the heat kernel given in (3.63), we obtain

$$\begin{aligned} ||V(E)|| &< n\, \frac{\lambda^2\,\Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s\,\mathrm{d}s\; e^{-s(nm+\hat{\mu}-E)} \Bigg\{ \int_0^1 \frac{\mathrm{d}u_1}{(u_1)^{1/2}} \\ &\quad \times \left[\frac{1}{V(\mathcal{M})^{1/2}} + A'^{1/2}(s(1-u_1)/m)^{-3/4} \right] \\ &\quad \times \int_0^{1-u_1} \frac{\mathrm{d}u_2}{(u_2)^{1/2}} \left[\frac{1}{V(\mathcal{M})^{1/2}} + A'^{1/2}(s(1-u_2)/m)^{-3/4} \right] \Bigg\} \,. \end{aligned} \tag{5.151}$$

One can even simplify the integrals, that is, the upper bound of the u_2 integral is replaced with 1 and the square roots of the sums are replaced with the sums of the square roots at the cost of getting less sharp bound on the norm of $V(E)$. Integrating with respect to u_1, u_2 and s, we have

$$\begin{aligned} ||V(E)|| &< n\, \frac{\lambda^2\,\Gamma(2)}{\Gamma(1/2)^2} \Bigg[\frac{4}{V(\mathcal{M})} \frac{1}{(nm+\hat{\mu}-E)^2} + \frac{4A'^{1/2} m^{D/4} \pi^{1/2} \Gamma(2-\frac{D}{4})\Gamma(1-\frac{D}{4})}{V(\mathcal{M})^{1/2}\Gamma(\frac{3}{2}-\frac{D}{4})} \\ &\times \frac{1}{(nm+\hat{\mu}-E)^{2-\frac{D}{4}}} + \frac{A' m^{D/2} \pi \Gamma(2-\frac{D}{2})\Gamma(1-\frac{D}{4})^2}{\Gamma(\frac{3}{2}-\frac{D}{4})^2} \frac{1}{(nm+\hat{\mu}-E)^{2-\frac{D}{2}}} \Bigg] \,. \end{aligned} \tag{5.152}$$

In order to get explicit solution of this inequality, let us put some further natural assumption $nm + \hat{\mu} - E > \hat{\mu}$, then, we find

$$\begin{aligned} ||V(E)|| &< n\, \frac{\lambda^2\,\Gamma(2)}{\Gamma(1/2)^2} \Bigg[\frac{4}{V(\mathcal{M})\hat{\mu}^{\frac{D}{2}}} + \frac{4A'^{1/2} m^{D/4} \pi^{1/2} \Gamma(2-\frac{D}{4})\Gamma(1-\frac{D}{4})}{\hat{\mu}^{\frac{D}{4}} V(\mathcal{M})^{1/2}\Gamma(\frac{3}{2}-\frac{D}{4})} \\ &\quad + \frac{A' m^{D/2} \pi \Gamma(2-\frac{D}{2})\Gamma(1-\frac{D}{4})^2}{\Gamma(\frac{3}{2}-\frac{D}{4})^2} \Bigg] \frac{1}{(nm+\hat{\mu}-E)^{2-\frac{D}{2}}} \,. \end{aligned} \tag{5.153}$$

Now if we impose the strict positivity of the principal operator we find similarly

$$E_{gr} \geq nm + \hat{\mu} - \left(n\lambda^2 F\right)^{\frac{1}{2-\frac{D}{2}}} \;, \tag{5.154}$$

where

$$F = \frac{\Gamma(2)}{\Gamma(1/2)^2}\left[\frac{4}{V(\mathcal{M})\hat{\mu}^{\frac{D}{2}}} + \frac{4A'^{1/2}m^{D/4}\pi^{1/2}\Gamma(2-\frac{D}{4})\Gamma(1-\frac{D}{4})}{\hat{\mu}^{\frac{D}{4}}V(\mathcal{M})^{1/2}\Gamma(\frac{3}{2}-\frac{D}{4})} + \frac{A'm^{D/2}\pi\Gamma(2-\frac{D}{2})\Gamma(1-\frac{D}{4})^2}{\Gamma(\frac{3}{2}-\frac{D}{4})^2}\right] . \tag{5.155}$$

Therefore, lower bounds on the ground state energies for different classes of manifolds (5.145), (5.149) and (5.154) are of almost the same form up to a constant factor so the form of the lower bound has a general character. It is also worth pointing out that the form of the lower bound on the ground state energy for three dimensional Riemannian manifolds is the same as in the case for the three dimensional flat space $\mathbb{R}^3$ [26, 134]. From the general form of the lower bounds, we conclude that for each sector with a fixed number of bosons, there exists a ground state. However, in three dimensions, the ground state energy bound that we have found diverges quadratically as the number of bosons increases. In other words, these estimates with our present analysis is not good enough to prove the existence of the thermodynamic limit in three dimensions. To attack this problem we will study the thermodynamic limit of the model by a kind of mean field approximation.

5.1.6. Mean Field Approximation of the Model in Three Dimensions

In the limit of large number of bosons $n \to \infty$, one expects that all the bosons have the same wave function $u(x)$ and mean field approximation is valid, as in the case of flat spaces. Therefore, one can introduce the following mean field ansatz for n bosonic particle state on a Riemannian manifold

$$|u\rangle = \frac{1}{\sqrt{n!}}\int_{\mathcal{M}^n} \mathrm{d}_g^3 x_1\, \mathrm{d}_g^3 x_2 \cdots \mathrm{d}_g^3 x_n \; u(x_1)\, u(x_2)\cdots u(x_n)\, \phi_g^\dagger(x_1)\, \phi_g^\dagger(x_2)\cdots \phi_g^\dagger(x_n)|0\rangle \; , \tag{5.156}$$

with the normalization

$$||u||^2 = \int_{\mathcal{M}} \mathrm{d}_g^3 x \; |u(x)|^2 = 1 \; , \tag{5.157}$$

where $|0\rangle$ denotes the bosonic vacuum state. In the mean field approximation, the operators are usually approximately replaced by their expectation values in this state i.e., $\langle f(u)\rangle \approx f(\langle u\rangle)$, so the expectation value of the principal operator by applying the mean field ansatz becomes

$$\begin{aligned}\phi_E[u] &= nh_0[u] - E + \hat{\mu} + \lambda^2 \int_0^\infty \mathrm{d}s\; K_s(a,a;g)\,[e^{-s(m-\hat{\mu})} - e^{-s(nh_0[u]+m-E)}] \\ - n\lambda^2 \int_0^\infty \mathrm{d}s & \int_{\mathcal{M}^2} \mathrm{d}_g^3x\,\mathrm{d}_g^3y\; K_s(x,a;g)\,K_s(y,a;g)\,u^*(x)\,e^{-s(nh_0[u]+2m-E)}\,u(y)\;, \quad (5.158)\end{aligned}$$

called principal functional and we have defined

$$h_0[u] = \int_{\mathcal{M}} \mathrm{d}_g^3x\, \left(\frac{|\nabla_g u(x)|^2}{2m} + m|u(x)|^2\right)\;. \quad (5.159)$$

We can also express it in terms of kinetic energy functional $h_0[u] = m + K[u]$. However, the exact value of the expectation value of the principal operator is given in terms of cummulant expansion theorem if it converges [135]. Therefore, in order to write the above formula (5.158), we have to further assume that the corrections coming from the higher order cummulants are negligibly small and indeed we will see that this assumption is justified for the particular solution we will find.

Now, we must solve the functional equation $\phi_E[u] = 0$ (giving the spectrum of the problem (5.121)), that is, we solve E as a functional of $u(x)$, and then find the smallest possible value of E with the constraint (5.157). One can try to write E as a functional of $u(x)$ from the equation $\phi_E[u] = 0$ and apply the variational methods to minimize $E = E[u]$. However, there is no simple way to solve exactly this functional equation since E is a complicated functional of $u(x)$. Moreover, we have no explicit expression of the heat kernel on any Riemannian manifold to solve E. So, we follow an approximation method which is essentially suggested in [26]. Since $nh_0[u]$ and E come together in equation (5.158), it turns out to be convenient to introduce a new variable $\chi = \chi[u]$

$$\chi \equiv nh_0[u] - E\;. \quad (5.160)$$

Then, the condition $\phi_E[u] = 0$ gives

$$\begin{aligned}
&\chi + \hat{\mu} + \lambda^2 \int_0^\infty \mathrm{d}s\; K_s(a,a;g)\,[e^{-s(m-\hat{\mu})} - e^{-s(\chi+m)}] \\
&\qquad = n\lambda^2 \int_0^\infty \mathrm{d}s \left| \int_{\mathcal{M}} \mathrm{d}^3_g x\; K_s(x,a;g)\, u(x) \right|^2 e^{-s(\chi+2m)} = nU[u]\;. \qquad (5.161)
\end{aligned}$$

The explicit χ dependence can be removed to the left hand side of (5.161) by first defining a new dimensionless parameter $s' = 2m(2m+\chi)s$ and scaling the metric $\tilde{g}_{ij} = [2m(2m+\chi)]g_{ij}$. Using the scaling property of heat kernel (3.50) and then defining new dimensionless wave function $v(x)$

$$v(x) \;\equiv\; [2m(2m+\chi)]^{-3/4} u(x)\;, \qquad (5.162)$$

all explicit χ dependence becomes shifted to the left hand side of (5.161)

$$\begin{aligned}
&[2m(2m+\chi)]^{-1/2}\Big(\chi + \hat{\mu} + \lambda^2 \int_0^\infty \mathrm{d}s\; K_s(a,a;g)\,\big[e^{-s(m-\hat{\mu})} - e^{-s(\chi+m)}\big]\Big) \\
&= n\lambda^2 (2m)^{3/2} \int_0^\infty \mathrm{d}s' \left| \int_{\mathcal{M}} \mathrm{d}^3_{\tilde{g}} x\; K_{s'}(x,a;\tilde{g})\, v(x) \right|^2 e^{-s'/2m}\;. \qquad (5.163)
\end{aligned}$$

Incidentally, the new wave functions $v(x)$ are normalized with respect to the new metric $\tilde{g}_{ij}$

$$\int_{\mathcal{M}} \mathrm{d}^3_{\tilde{g}} x\; |v(x)|^2 = \int_{\mathcal{M}} \mathrm{d}^3_g x\; |u(x)|^2 = 1\;. \qquad (5.164)$$

Let us denote the left hand side of (5.163) as $F(\chi)$. One can easily prove that $F(\chi)$ is an increasing function and the right hand side of (5.163) is nonnegative. Therefore, the left hand side is minimum when $\chi = -\hat{\mu}$, which is attained when the right hand side becomes zero (so we have $\chi \geq -\hat{\mu}$). Let us denote the inverse function of the left hand side as $f_1(nU)$, that is,

$$\begin{aligned}
F(\chi) &= nU[v] \\
\chi &= F^{-1}(nU) = f_1(nU)\;, \qquad (5.165)
\end{aligned}$$

where $U = U[v]$ denotes the nonnegative function on the right hand side. We can write χ in terms of new wave function $v(x)$ and new metric $\tilde{g}$ from its definition (5.160),

$$\chi = n[\chi + 2m]K[v] + nm - E, \tag{5.166}$$

and substitute (5.165) into the equation above we obtain

$$E = nm + 2mnK[v] + (nK[v] - 1)f_1(nU) \ , \tag{5.167}$$

where $K[v] = \int_{\mathcal{M}} \mathrm{d}^3_{\tilde{g}}x \ |\nabla_{\tilde{g}} v(x)|^2$, which is considered to be the parameter of the model because it is the variable we can control and we may use many trial functions $v(x)$ and they can be scaled to any desired value. If we assume that $nK[v] > 1$, then the energy E in (5.167) is minimized when $f_1(nU)$ is minimized. As a direct consequence of the chain rule $\frac{dF(\chi)}{d\chi}\frac{df_1(nU)}{d(nU)} = 1$, the inverse function $f_1(nU)$ is an increasing function so its minimum will be at the point $U[v] = 0$ and this corresponds to the solution $F(\chi) = 0$ which happens at $\chi = f_1(nU) = -\hat{\mu}$. Since $\chi = f_1(nU) \geq -\hat{\mu}$, we have

$$E \geq nm + \hat{\mu} \qquad \text{if} \qquad nK[v] > 1 \ . \tag{5.168}$$

On the other hand, if $K[v]$ is small enough, i.e., $nK[v] < 1$, we also see that the minimum of the energy is attained with the reversed sign of the last term. In that case, we should find an upper bound for $f_1(nU)$ which is expressed in terms of the kinetic energy functional $K[v]$. In order to discuss the case $nK[v] < 1$ properly, we will separate our calculations for compact and non-compact manifolds.

Let us first consider the case for compact manifolds. In order to achieve our aim, we will go back to the original variable $u(x)$ and the parameter s in the equation (5.161). If we assume that the mean field approximation gives us a reliable equality, we can find the solution for χ given all the other parameters. Note that the right hand side of (5.161) is a decreasing function of χ, whereas the left hand side is an increasing one. Hence there is always a unique solution which defines the inverse function for a given $u(x)$. It is quite easy to see by a graphical construction that if we replace the left hand side by a smaller function of χ, and the right hand side by a larger function of χ, then we get an upper bound for the inverse function, and this corresponds to a lower bound for the energy. In order to find an upper bound on the right hand side, one may first think that Cauchy Schwartz inequality can be applied to the term

$$\left| \int_{\mathcal{M}} \mathrm{d}_g^3 x \; K_s(x,a;g) \, u(x) \right|^2 \leq K_{2s}(a,a;g) \; , \tag{5.169}$$

by using the semigroup property of the heat kernel (3.36) and the normalization condition of the wave function $u(x)$. However, this immediately leads to divergent result due to the short time asymptotic behavior of the heat kernel when we take the integral with respect to s. Therefore, we must develop a nontrivial method to find an upper bound on the right hand side of (5.161). We first note that

$$\int_{\mathcal{M}} \mathrm{d}_g^3 x \; K_s(x,a;g) \, u(x) = \sum_{l=0}^{\infty} e^{-\frac{s\sigma_l}{2m}} f_l(a;g) \, u(l) \; , \tag{5.170}$$

where $u(l) = \int_{\mathcal{M}} \mathrm{d}_g^3 x \; f_l^*(x;g) \, u(x)$. If we rescale the eigenfunction $f_l(a;g)$ and $u(l)$ in the following way

$$f_l'(a;g) = \begin{cases} \sqrt{\dfrac{2m}{\sigma_l}} f_l(a;g) & \text{if } l \neq 0 \\[2em] f_0(a;g) & \text{if } l = 0, \end{cases} \tag{5.171}$$

and

$$u'(l) = \begin{cases} \sqrt{\dfrac{\sigma_l}{2m}} \, u(l) & \text{if } l \neq 0 \\[2em] u(0) & \text{if } l = 0, \end{cases} \tag{5.172}$$

we can write

$$
\begin{aligned}
\left|\int_{\mathcal{M}} \mathrm{d}_g^3 x\; K_s(x,a;g)\, u(x)\right|^2 &= \left|\sum_{l=0}^{\infty} e^{-\frac{s\sigma_l}{2m}} f_l'(a;g) u'(l)\right|^2 \leq \sum_{l=0}^{\infty} e^{-\frac{s\sigma_l}{m}} |f_l'(a;g)|^2 \sum_{l=0}^{\infty} |u'(l)|^2 \\
&\leq \left(|f_0(a;g)|^2 + \sum_{l=1}^{\infty} \frac{e^{-\frac{s\sigma_l}{m}}}{\sigma_l/2m} |f_l(a;g)|^2 \right) \left(|u(0)|^2 + K[u] \right) \;, \qquad (5.173)
\end{aligned}
$$

where we have used Cauchy-Schwartz inequality, as well as the observation that $K[u] = \sum_{l=0}^{\infty} \frac{\sigma_l}{2m} |u(l)|^2$ and $\sigma_0 = 0$. The important point to be emphasized is that we have defined the zero modes separately while defining the rescaled functions above. Otherwise, we would get infinity due to $\sigma_0 = 0$. Using $|f_0(a;g)|^2 = 1/V(\mathcal{M})$ and $|u(0)|^2 \leq 1$ thanks to the normalization condition, the upper bound on $U[u]$ after taking the s integral yields

$$
U[u] \leq (1 + K[u])\; \Omega_c \;, \qquad (5.174)
$$

where

$$
\Omega_c = \lambda^2 \left[\frac{1}{V(\mathcal{M})} + \sum_{l=1}^{\infty} \frac{1}{\sigma_l/2m} \left(\frac{1}{\chi + 2m + \sigma_l/m} \right) |f_l(a;g)|^2 \right] . \qquad (5.175)
$$

Here the subscript c stands for compact manifolds. Let us now write the fraction in front of $|f_l(a;g)|^2$ as a sum of two partial fractions and express them as an integral of exponential function:

$$
\begin{aligned}
\frac{1}{\sigma_l/2m} \left(\frac{1}{\chi + 2m + \sigma_l/m} \right) &= \frac{1}{\chi + 2m} \left(\frac{1}{\sigma_l/2m} - \frac{2}{\chi + 2m + \sigma_l/m} \right) \\
&= \frac{1}{\chi + 2m} \int_0^{\infty} \mathrm{d}s \left(e^{-s\sigma_l/2m} - 2 e^{-s(\chi + 2m + \sigma_l/m)} \right) . \qquad (5.176)
\end{aligned}
$$

Substituting this result into (5.175) and using (3.30), we get

$$
\begin{aligned}
\Omega_c = \frac{\lambda^2}{\chi + 2m} \Bigg[\frac{1}{V(\mathcal{M})} &+ \int_0^{\infty} \mathrm{d}s\; \left(1 - e^{-s(\chi+2m)/2}\right) \\
&\times \left(K_s(a,a;g) - \lim_{s\to\infty} K_s(a,a;g) \right) \Bigg] . \qquad (5.177)
\end{aligned}
$$

Here the sum which excludes the zero mode corresponds to substraction of the large s behavior of heat kernel. Using $K[u] = (2m + \chi)K[v]$, the inequality becomes

$$\chi + \hat{\mu} \leq \frac{n\lambda^2}{\chi + 2m}\left[\frac{1}{V(\mathcal{M})} + \int_0^\infty \mathrm{d}s \left(1 - e^{-s(\chi+2m)/2}\right)\right.$$
$$\left.\times \left(K_s(a, a; g) - \lim_{s\to\infty} K_s(a, a; g)\right)\right]\left[1 + (\chi + 2m)K[v]\right] , \quad (5.178)$$

where we have replaced the left hand side by its smaller value $\chi + \hat{\mu}$ due to fact that $\chi \geq -\hat{\mu}$ and the positivity of the heat kernel (3.38). Using the upper bound estimate of the heat kernel for compact manifolds (3.63) and the large time asymptotics of the heat kernel (3.32), and taking the integral with respect to s, we find that

$$\chi + \hat{\mu} \quad \leq \quad \frac{n\lambda^2}{\chi + 2m}\left[1 + (\chi + 2m)\, K[v]\right]\left[\sqrt{2\pi}(2m)^{3/2}A'(\chi + 2m)^{1/2}\right] . \quad (5.179)$$

If we now introduce the variables $z = \chi + 2m$ and $A'' = \sqrt{2\pi}(2m)^{3/2}A'$ for simplicity of notation, we find the following inequality

$$\begin{aligned} z - (2m - \hat{\mu}) \quad &\leq \quad \frac{\lambda^2}{z}\left(n + znK[v]\right) A'' z^{1/2} \\ &\leq \quad \frac{\lambda^2 A''}{\sqrt{z}}\left(n + z\right) , \end{aligned} \quad (5.180)$$

where we have used the fact in the last step that we are interested in the region $nK[v] < 1$. We now look for a systematic expansion of z in n by allowing a fractional power. In this case we see that if we substitute the following asymptotic expansion as $n \to \infty$, $z \sim B_1 n^{\nu_1} + B_2 n^{\nu_2} + B_3 n^{\nu_3} + \cdots$, where the consecutive powers $\nu_1, \nu_2, \ldots$ decrease, we find that this asymptotic expansion is an upper bound for this inequality as long as the coefficients are to be chosen as

$$B_1 = (A''\lambda^2)^{2/3} , \quad (5.181)$$

$$B_2 = \frac{2}{3}(A''\lambda^2)^{4/3} , \quad (5.182)$$

and

$$B_3 \;=\; \frac{2}{3}(2m-\hat{\mu})+\frac{1}{3}(A''\lambda^2)^2 \;, \tag{5.183}$$

where $\nu_1 = 2/3$, $\nu_2 = 1/3$, and $\nu_3 = 0$. Hence we get an upper bound on the inverse function $f_1(nU)$ as an asymptotic series in powers of n, in the spirit of mean field approximation. This upper bound can be put back into the energy equation and then we find the following lower bound on the bound state energy in the mean field approximation,

$$E \geq nm + 2mnK[v] - (1-nK[v])\left(B_1 n^{2/3} + B_2 n^{1/3} + B_3 - 2m + \cdots\right) \; . \tag{5.184}$$

This means that the ground state energy goes asymptotically

$$E_{gr} \sim nm - \left(B_1 n^{2/3} + B_2 n^{1/3} + B_3 - 2m + \cdots\right) \; , \tag{5.185}$$

since we can consider the functional $nK[v]$, which is restricted to the region $0 \leq nK[v] < 1$, as a parameter so that $nK[v] \to 0$ gives us an absolute lower bound. We note now that the behavior of $K[u]$ for large n is found from the scaling law, $K[u] = (2m+\chi)K[v] \sim n^{-1/3}$. This in turn justifies our use of the mean field approximation since higher order derivatives can be considered heuristically negligible, and the solution approaches to an essentially constant function on the manifold as n gets larger. This proves that for a compact manifold the energy is actually bounded from below by a much milder behavior and there is a nice thermodynamic limit since the energy per particle E/n will approach to the mass (rest mass energy) as $n \to \infty$. The same conclusion can also be drawn for non-compact manifolds as we will see.

Now we will consider the mean field approximation of the model for non-compact manifolds. We again assume that the eigenfunction expansion in compact manifolds can be generalized to the non-compact manifolds. Then, one can use the above method for the non-compact manifolds as well. Using (3.33) and $u(x) = \int \mathrm{d}\mu(l) f_l^*(x;g)u(l)$, we obtain

$$\int_{\mathcal{M}} \mathrm{d}_g^3 x \; K_s(x,a;g)\, u(x) = \int \mathrm{d}\mu(l) e^{-\frac{s\sigma(l)}{2m}} f_l(a;g)\, u(l) \; . \tag{5.186}$$

Similar to the compact case, we define

$$\begin{aligned} f_l'(x;g) &= \sqrt{\frac{2m}{\sigma(l)}} f_l(x;g) \\ u'(l) &= \sqrt{\frac{\sigma(l)}{2m}} u(l) \; . \end{aligned} \tag{5.187}$$

Applying the Cauchy Schwartz inequality to the modulus square of (5.186), we get

$$\begin{aligned} \left| \int_{\mathcal{M}} \mathrm{d}_g^3 x \; K_s(x,a;g)\, u(x) \right|^2 &= \int \mathrm{d}\mu(l) e^{-\frac{s\sigma(l)}{2m}} f_l'(a;g)\, u'(l) \\ &\leq K[u] \int \mathrm{d}\mu(l) \frac{e^{-s\sigma(l)/m}}{\sigma(l)/2m} |f_l(a;g)|^2 \; . \end{aligned} \tag{5.188}$$

We now take the s integral in the right hand side of (5.161) use the above result and apply (5.176) to obtain

$$U[u] \leq \frac{\lambda^2}{\chi + 2m} \left[\int_0^\infty \mathrm{d}s \left(1 - e^{-s(\chi+2m)/2} \right) K_s(a,a;g) \right] K[u] \; . \tag{5.189}$$

If we use the upper bound on the diagonal heat kernel for Cartan-Hadamard manifolds used in the previous Section (5.141), we finally find

$$\begin{aligned} \chi + \hat{\mu} &\leq \lambda^2 C \sqrt{2\pi} (2m)^{3/2} \sqrt{\chi + 2m} n K[v] \\ &\leq \lambda^2 C \sqrt{2\pi} (2m)^{3/2} \sqrt{\chi + 2m} \; , \end{aligned} \tag{5.190}$$

from which we can write an upper bound on χ or a lower bound on the energy

$$E \geq nm - \frac{1}{2} \left[B_4 - 2\hat{\mu} + \sqrt{B_4^2 + B_4(8m - 4\hat{\mu})} \right] \; , \tag{5.191}$$

where

$$B_4 = \lambda^4 C^2 2\pi (2m)^3 \ , \tag{5.192}$$

that is,

$$E_{gr} \sim nm + C_1 \ , \tag{5.193}$$

as $n \to \infty$. Here C_1 is just second term in (5.191).

We have also an alternative method to find the ground state energy in the mean field approximation. Let us first go back to the equation (5.163) and using the generalized eigenfunction expansions, we find

$$U[v] \leq K[v] \, \Omega_{nc} \ , \tag{5.194}$$

where

$$\Omega_{nc} = \lambda^2 (2m)^{3/2} \int_0^\infty \mathrm{d}s' \, \left(1 - e^{-s'/2}\right) K_{s'}(a, a; \tilde{g}) \ . \tag{5.195}$$

The explicit form of the inverse function $f_1(nU)$ is too difficult to find so one can estimate it. In order to do this, let us first notice that

$$f_2^{-1}(\chi) < f_1^{-1}(\chi) \quad \text{then} \quad f_1(nU) < f_2(nU). \tag{5.196}$$

For this purpose, we use the simplest possible function as $f_2^{-1}(\chi)$

$$f_2^{-1}(\chi) = \frac{\chi + \hat{\mu}}{(\chi + 2m)^{1/2}} \ . \tag{5.197}$$

Then we can replace $f_1(nU)$ with something bigger, and its argument with something even bigger. Moreover, $f_2(u)$ is dominated by a simpler function

$$f_2(u) < u^2 + 2m - 2\hat{\mu}, \tag{5.198}$$

which is a very crude bound, but easy to work with. Using the upper bound for $U[v]$, we get

$$\chi = f_1(nU) < f_2(u) < n^2U[v]^2 + 2m - 2\hat{\mu} < n^2(K[v])^2\Omega^2 + 2m - 2\hat{\mu}\ . \tag{5.199}$$

Hence,

$$E \geq nm + 2mnK[v] - (1 - nK[v])\left(n^2K[v]^2\,\Omega^2 + 2m - 2\hat{\mu}\right)\ , \tag{5.200}$$

where the polynomial $(4m - 2\hat{\mu})y - \Omega^2 y^2 + \Omega^2 y^3$ never becomes negative within the range $0 \leq y = nK[v] \leq 1$ if

$$\Omega^2 < 16m - 8\hat{\mu}\ . \tag{5.201}$$

In this case, the minimum is achieved when $y = 0$. This means that the functional $nK[v] \to 0$ by a proper family of functions. This is why it is consistent to ignore the higher order cummulants if we choose an arbitrarily slowly varying family for the function $v(x)$. Therefore one can conclude that the ground state energy for non-compact manifolds when $nK[v] < 1$

$$E_{gr} \sim nm - 2(m - \hat{\mu})\ . \tag{5.202}$$

The condition on Ω may imply some restrictions on the coupling constant λ. If we go back to the definition of Ω and scaling back again to the usual geometric variables, we find

$$\Omega = \lambda^2(\chi + 2m)^{-1/2}\int_0^\infty \mathrm{d}s\ \left(1 - e^{-s(\chi+2m)/2}\right)K_s(a,a;g)\ . \tag{5.203}$$

Since there is a nice sharp upper bound on the heat kernel for Cartan-Hadamard manifolds and minimal submanifolds of $\mathbb{R}^3$ (5.141), we can find an upper bound on Ω and the restriction (5.201) gives the following upper bound on the coupling constant.

$$\lambda < \frac{(16m - 8\hat{\mu})^{1/4}}{(2\pi)^{1/4} C^{1/2} (2m)^{3/4}} \,. \tag{5.204}$$

Now, let us calculate explicitly the function Ω for the hyperbolic manifold $\mathbb{H}^3$, which belongs to the class of Cartan-Hadamard manifolds. Using the result (5.147), we obtain

$$\Omega = \lambda^2 \left(\chi + 2m\right)^{-1/2} \left(4\pi/2m\right)^{-3/2} 2\sqrt{\pi} \left\{ \sqrt{\frac{\chi + 2m}{2} + \frac{1}{2mR^2}} - \sqrt{\frac{1}{2mR^2}} \right\} \,. \tag{5.205}$$

Then, one can easily find the upper bound on the coupling constant from the restriction on Ω

$$\lambda < 2\sqrt{4\pi(2m - \hat{\mu})}(2m)^{-3/4} \left\{ \sqrt{\frac{2m - \hat{\mu}}{2} + \frac{1}{2mR^2}} - \sqrt{\frac{1}{2mR^2}} \right\}^{-1/2} \,. \tag{5.206}$$

Therefore, similar to the case for compact manifolds, we have shown that the leading behavior of the system varies linearly with the number of bosons n, which leads to a nice thermodynamic limit on non-compact manifolds.

5.1.7. Mean Field Approximation of the Model in Two Dimensions

The condition $\phi_E[u] = 0$ in two dimensions gives

$$\begin{aligned} &\Big(\chi + \hat{\mu} + \lambda^2 \int_0^\infty \mathrm{d}s \; K_s(a,a;g) \left[e^{-s(m-\hat{\mu})} - e^{-s(\chi+m)} \right] \Big) \\ &= n\lambda^2 (2m) \int_0^\infty \mathrm{d}s' \left| \int_{\mathcal{M}} \mathrm{d}^2_{\tilde{g}} x \; K_{s'}(x,a;\tilde{g})\, v(x) \right|^2 e^{-s'/2m} \,, \end{aligned} \tag{5.207}$$

where we have written the right hand side in the scaled variables defined by

$$\begin{aligned}\tilde{g}_{ij} &= 2m(\chi + 2m)g_{ij} \\ s' &= 2m(\chi + 2m)s \\ v(x) &= [2m(\chi + 2m)]^{-1/2}u(x) \ . \end{aligned} \tag{5.208}$$

The main idea of the analysis for $nK[v] > 1$ is exactly the same as in three dimensions. However, in the analysis for $nK[v] < 1$, the integral

$$\int_0^\infty \mathrm{d}s \ (1 - e^{-s(\chi+2m)/2m})K_s(a, a; g) \tag{5.209}$$

is divergent since the lower bounds of the heat kernel goes like C/s in both two dimensional compact and noncompact manifolds while estimating the right hand side of (5.161). Therefore, we must develop a different method to handle the two dimensional problem. For the case $nK[v] < 1$, we will again consider the problem for compact and noncompact manifolds separately. Using the eigenfunction expansion for the heat kernel (3.30) and for $v(x)$ and taking the integral with respect to s', we find the right hand side of (5.207)

$$n(2m)\lambda^2 \sum_{l_1,l_2=0}^{\infty} v^*(\bar{\sigma}_{l_1}) \frac{1}{\bar{\sigma}_{l_1} + \bar{\sigma}_{l_2} + 1} v(\bar{\sigma}_{l_2}) f_{l_1}(a;\tilde{g}) f^*_{l_2}(a;\tilde{g}) \ . \tag{5.210}$$

Since this is always positive, it is smaller than the following bound that we can separate the zero mode terms

$$\begin{aligned} \leq n(2m)\lambda^2 \Bigg\{ |f_0(a;\tilde{g})|^2 |v(0)|^2 + 2 \Bigg| \sum_{\substack{l_1 \neq 0 \\ (l_2=0)}} \frac{f^*_0(a;\tilde{g}) v^*(0) f_{l_1}(a;\tilde{g}) v(l_1)}{(1 + \bar{\sigma}_{l_1})} \Bigg| \\ + \Bigg| \sum_{\substack{l_1 \neq 0 \\ l_2 \neq 0}} \frac{f^*_{l_1}(a;\tilde{g}) v^*(l_1) f_{l_2}(a;\tilde{g}) v(l_2)}{(1 + \bar{\sigma}_{l_1} + \bar{\sigma}_{l_2})} \Bigg| \Bigg\} \ . \end{aligned} \tag{5.211}$$

Using $f_0(a;\tilde{g}) = \frac{1}{\sqrt{V(\mathcal{M}(\tilde{g}))}}$ and $|v(0)| \leq 1$ we get

$$\leq n(2m)\lambda^2 \Bigg\{ \frac{1}{V(\mathcal{M}(\tilde{g}))} + \frac{2}{\sqrt{V(\mathcal{M}(\tilde{g}))}} \Bigg| \sum_{\substack{l_1 \neq 0 \\ (l_2=0)}} \frac{f_{l_1}(a;\tilde{g})v(l_1)}{(1+\bar{\sigma}_{l_1})} \Bigg| + \Bigg| \sum_{\substack{l_1 \neq 0 \\ l_2 \neq 0}} \frac{f^*_{l_1}(a;\tilde{g})v^*(l_1)f_{l_2}(a;\tilde{g})v(l_2)}{(1+\bar{\sigma}_{l_1}+\bar{\sigma}_{l_2})} \Bigg| \Bigg\} . \tag{5.212}$$

Let us first consider the second term and apply the Cauchy Schwartz inequality with the fact that $\left[\sum_{\substack{l_1 \neq 0 \\ (l_2=0)}} |v(l_1)|^2 \right]^{1/2} < 1$ to find that the second term is smaller than

$$\frac{2}{\sqrt{V(\mathcal{M}(\tilde{g}))}} \left[\sum_{\substack{l_1 \neq 0 \\ (l_2=0)}} \frac{|f_{l_1}(a;\tilde{g})|^2}{(1+\bar{\sigma}_{l_1})^2} \right]^{1/2} . \tag{5.213}$$

One can express the factor $\frac{1}{(1+\bar{\sigma}_{l_1})^2}$ as an integral of an exponential

$$\frac{1}{(1+\bar{\sigma}_{l_1})^2} = \int_0^\infty \frac{\mathrm{d}s'}{2m}(s'/2m)e^{-s'(1+\bar{\sigma}(l_1))/2m} , \tag{5.214}$$

so that we find (5.213) is smaller than

$$\frac{2}{\sqrt{V(\mathcal{M}(\tilde{g}))}} \left[\int_0^\infty \frac{\mathrm{d}s'}{2m}(s'/2m)e^{-s'/2m} \left(K_{s'}(a,a;\tilde{g}) - \frac{1}{V(\mathcal{M}(\tilde{g}))} \right) \right]^{1/2} , \tag{5.215}$$

by expressing it in terms of the heat kernel and removing the zero mode in the sum corresponds to the substraction of the large time asymptotic of the heat kernel (3.32). Using the diagonal upper bound of the heat kernel (3.63), we can find an upper bound on the above equation

$$\frac{2}{\sqrt{V(\mathcal{M}(\tilde{g}))}} \Bigg| \sum_{\substack{l_1 \neq 0 \\ (l_2=0)}} \frac{f_{l_1}(a;\tilde{g})v(l_1)}{(1+\bar{\sigma}_{l_1})} \Bigg| \leq \frac{2}{\sqrt{V(\mathcal{M}(g))}} \sqrt{\frac{A'}{2m(\chi+2m)}} , \tag{5.216}$$

where we have scaled back to the original variables. Let us now multiply both numerator and denominator of third term in (5.212) with the factor $\bar{\sigma}_{l_1}^{\frac{1-\epsilon}{2}} \bar{\sigma}_{l_2}^{\frac{1-\epsilon}{2}}$ and apply Cauchy Schwartz inequality and get that it is smaller than

$$\left[\sum_{l_1 \neq 0} |v(l_1)|^2 \bar{\sigma}_{l_1}^{1-\epsilon}\right] \left[\sum_{\substack{l_1 \neq 0 \\ l_2 \neq 0}} \frac{|f_{l_1}(a;\tilde{g})|^2 |f_{l_2}(a;\tilde{g})|^2}{(1+\bar{\sigma}_{l_1}+\bar{\sigma}_{l_2})^2 \bar{\sigma}_{l_1}^{1-\epsilon} \bar{\sigma}_{l_2}^{1-\epsilon}}\right]^{1/2}, \tag{5.217}$$

where we choose $0 < \epsilon < 1/2$. For simplicity we can use the following inequality in the second integral $\frac{1}{(1+\bar{\sigma}_1+\bar{\sigma}_2)^2} \leq \frac{1}{(1+\bar{\sigma}_1)(1+\bar{\sigma}_2)}$ and obtain an upper bound on (5.217)

$$\leq \left[\sum_{l_1 \neq 0} |v(l_1)|^2 \bar{\sigma}_{l_1}^{1-\epsilon}\right] \left[\sum_{l_1 \neq 0} \frac{|f_{l_1}(a;\tilde{g})|^2}{(1+\bar{\sigma}_{l_1}) \bar{\sigma}_{l_1}^{1-\epsilon}}\right]. \tag{5.218}$$

In order to convert the products $(1+\bar{\sigma}_{l_1})\bar{\sigma}_{l_1}^{1-\epsilon}$ in the denominator into a summation of them, we use Feynman parametrization (5.136) and get

$$\frac{1}{(1+\bar{\sigma}_{l_1})\bar{\sigma}_{l_1}^{1-\epsilon}} = \frac{\Gamma(2-\epsilon)}{\Gamma(1-\epsilon)} \int_0^1 \mathrm{d}u_1 \frac{1}{(1-u_1)^{\epsilon}(u_1+\bar{\sigma}_{l_1})^{2-\epsilon}} . \tag{5.219}$$

Then, we rewrite the factor $\frac{1}{(u_1+\bar{\sigma}_{l_1})}$ as an exponential integral by a similar integral (5.214) and get

$$\begin{aligned} \sum_{l_1 \neq 0} \frac{|f_{l_1}(a;\tilde{g})|^2}{(1+\bar{\sigma}_{l_1})\bar{\sigma}_{l_1}^{1-\epsilon}} &= \frac{1}{\Gamma(1-\epsilon)} \int_0^1 \mathrm{d}u_1 \frac{1}{(1-u_1)^{\epsilon}} \int_0^{\infty} \frac{\mathrm{d}s'}{2m} (s'/2m)^{1-\epsilon} \\ &\quad \times \sum_{l_1 \neq 0} e^{-s'(u_1+\bar{\sigma}_{l_1})/2m} |f_{l_1}(a;\tilde{g})|^2 . \end{aligned} \tag{5.220}$$

The last expression can be expressed in terms of the heat kernel due to (3.30) but it is important to emphasize that excluding the zero mode corresponds to removing the large time asymptotics of the heat kernel, which is inverse of the volume of the manifold (3.32).

$$\frac{(2m)^{\epsilon-2}}{\Gamma(1-\epsilon)} \int_0^1 \mathrm{d}u_1 \frac{1}{(1-u_1)^{\epsilon}} \int_0^{\infty} \mathrm{d}s' (s')^{1-\epsilon} e^{-s' u_1/2m} \left(K_{s'}(a,a;\tilde{g}) - \frac{1}{V(\mathcal{M}(\tilde{g}))} \right) . \tag{5.221}$$

Scaling back to the original variables and using the diagonal upper bound of the heat kernel (3.63) for compact manifolds we can take the s integral and get an upper bound of (5.220)

$$A' \int_0^1 \mathrm{d}u_1 \frac{u_1^{\epsilon-1}}{(1-u_1)^{\epsilon}} \; . \tag{5.222}$$

Taking the u_1 integral with the assumption that $0 < \epsilon < 1$, we obtain

$$\sum_{l_1 \neq 0} \frac{|f_{l_1}(a;\tilde{g})|^2}{(1+\bar{\sigma}_{l_1})\bar{\sigma}_{l_1}^{1-\epsilon}} \leq \frac{\pi A'}{\sin \pi\epsilon} \; . \tag{5.223}$$

We now come to the crucial point. The upper bound of the first sum in (5.218) can be found by using the following inequality, which was first used in [134] for a similar reason

$$\bar{\sigma}_1^{1-\epsilon} < \delta + \left(\frac{\epsilon}{\delta}\right)^{\epsilon/1-\epsilon} \bar{\sigma}_1 \; , \tag{5.224}$$

where $\delta > 0$ and $0 < \epsilon < 1/2$. Note that this inequality applies only to dimensionless variables $\bar{\sigma}_1$, so that is why we use the scaling transformation at the beginning of the problem as opposed to the three dimensional case. The proof of this inequality is given in Appendix A. If we use this inequality in the first sum of (5.218), we obtain

$$\sum_{l_1 \neq 0} |v(l_1)|^2 \bar{\sigma}_{l_1}^{1-\epsilon} \leq \delta + \left(\frac{\epsilon}{\delta}\right)^{\epsilon/1-\epsilon} K[v] \; , \tag{5.225}$$

where excluding the zero mode from the sum again gives the kinetic energy functional since $\bar{\sigma}_0 = 0$. Then, we finally get

$$n(2m)\lambda^2 \left| \sum_{\substack{l_1 \neq 0 \\ l_2 \neq 0}} \frac{f_{l_1}^*(a;\tilde{g}) v^*(l_1) f_{l_2}(a;\tilde{g}) v(l_2)}{(1+\bar{\sigma}_{l_1}+\bar{\sigma}_{l_2})} \right| \leq n(2m)\lambda^2 \left[\delta + \left(\frac{\epsilon}{\delta}\right)^{\epsilon/1-\epsilon} K[v] \right] \frac{\pi A'}{\sin \pi\epsilon} \tag{5.226}$$

Since $\sin \pi\epsilon \geq 2\epsilon$ for $0 \leq \pi\epsilon \leq \pi/2$ (a useful inequality in complex analysis: $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$) and $nK[v] < 1$, the last expression is smaller than

$$m\lambda^2 \pi A' \left[n \left(\frac{\delta}{\epsilon} \right) + \frac{1}{\epsilon} \left(\frac{\epsilon}{\delta} \right)^{\epsilon/1-\epsilon} \right] . \tag{5.227}$$

One can think that this can be taken as a solution to our problem by choosing arbitrary constants ϵ and δ and finally get the large n behavior of the energy of the order $O(n)$. However, one can even find a better solution to the large n behavior of the energy. In order to control the energy as $n \to \infty$, we can assume that the parameters ϵ and δ are sequences in n. Without loss of generality we can assume that $\epsilon(n)$ go to zero as $n \to \infty$ (recall that $0 < \epsilon < 1/2$). If we want to find a better large n behavior of the energy, we must choose the sequence $\delta(n)$ such that

$$\frac{n\delta(n)}{\epsilon(n)} = O(1) . \tag{5.228}$$

We are now looking for an optimal solution for the energy and tell how fast the sequences $\epsilon(n)$ and $\delta(n)$ must change with n. In order to see this, let us write (5.227) in the following way

$$m\lambda^2 \pi A' \left[\frac{n\delta(n)}{\epsilon(n)} + e^{\frac{\epsilon(n)}{1-\epsilon(n)} \ln(\epsilon(n)/\delta(n)) - \ln \epsilon(n)} \right] \tag{5.229}$$

An optimal solution of the sequences can be found in such a way that the second term goes asymptotically

$$e^{\frac{\epsilon(n)}{1-\epsilon(n)} \ln(\epsilon(n)/\delta(n)) - \ln \epsilon(n)} = O(\ln n) . \tag{5.230}$$

This implies that we can choose

$$\epsilon(n) = \frac{1}{\ln n} , \tag{5.231}$$

and as a consequence of (5.228)

$$\delta(n) = \frac{1}{n \ln n} \, . \tag{5.232}$$

Therefore, we obtain the upper bound of (5.226) as $n \to \infty$

$$n(2m)\lambda^2 \left| \sum_{\substack{l_1 \neq 0 \\ l_2 \neq 0}} \frac{f^*_{l_1}(a;\tilde{g}) v^*(l_1) f_{l_2}(a;\tilde{g}) v(l_2)}{(1 + \bar{\sigma}_{l_1} + \bar{\sigma}_{l_2})} \right| \leq 2mA' \lambda^2 e \ln n \tag{5.233}$$

Combining the terms (5.216) and (5.233) together in (5.212), we finally get

$$\chi + \hat{\mu} \leq n \left[\frac{\lambda^2}{(2m+\chi)V(\mathcal{M})} \right] + n \left[2\lambda^2 \sqrt{\frac{2mA'}{(2m+\chi)V(\mathcal{M})}} \right] + (\ln n) \left[2mA'\lambda^2 e \right] . \tag{5.234}$$

To simplify the inequality, we use $\frac{1}{2m+\chi} \leq \frac{1}{2m-\hat{\mu}}$ since $\chi \geq -\hat{\mu}$. Following the same steps for the three dimensional problem, we obtain

$$E_{gr} \sim nm + \hat{\mu} - n \left[\frac{\lambda^2}{(2m-\hat{\mu})V(\mathcal{M})} + 2\lambda^2 \sqrt{\frac{2mA'}{(2m-\hat{\mu})V(\mathcal{M})}} + \left(\frac{\ln n}{n} \right) (2mA'\lambda^2 e) \right] . \tag{5.235}$$

As for the noncompact manifolds, the analog expression of (5.210) is

$$n(2m)\lambda^2 \int \mathrm{d}\mu(\bar{\sigma}_{l_1}) \mathrm{d}\mu(\bar{\sigma}_{l_2}) v^*(\bar{\sigma}_{l_1}) \frac{1}{\bar{\sigma}_{l_1} + \bar{\sigma}_{l_2} + 1} v(\bar{\sigma}_{l_2}) f_{l_1}(a;\tilde{g}) f^*_{l_2}(a;\tilde{g}) \, . \tag{5.236}$$

where one can think that the sums are replaced with the integrals in the case for noncompact cases and the analysis is basically same as the one for compact manifolds except that we do not have to bother for extracting the zero mode. Following the same steps for the analog expression of the compact cases, and using the diagonal upper bound of the heat kernel on Cartan Hadamard manifolds (5.141), the ground state energy goes like

$$E_{gr} \sim nm + \hat{\mu} - 2mC\lambda^2 e \ln n \; . \tag{5.237}$$

5.2. Non-relativistic $\lambda\phi^4$ Model in Two dimensional Riemannian Manifolds

The formal nonrelativistic limit of $\lambda\phi^4$ scalar field theory in (2+1) and (3+1) dimensions is considered as the many body model of bosons interacting with delta function interactions in two and three dimensions, respectively [11, 136]. In this paper, we consider this many body problem as a nonrelativistic quantum field theory in (2+1) dimensional Riemannian manifolds. The naive Hamiltonian of the system on $\mathbb{R}^2$ is

$$\begin{aligned} H = -\frac{\hbar^2}{2m} \int_{\mathbb{R}^2} \mathrm{d}^2x \; \phi^\dagger(\mathbf{x}) \nabla^2 \phi(\mathbf{x}) \\ - \frac{\lambda}{2} \int_{\mathbb{R}^4} \mathrm{d}^2x \; \mathrm{d}^2x' \; \phi^\dagger(\mathbf{x}) \phi^\dagger(\mathbf{x}') \delta^2(\mathbf{x}-\mathbf{x}') \phi(\mathbf{x}') \phi(\mathbf{x}) \; , \end{aligned} \tag{5.238}$$

where λ is the positive coupling constant and d^2x is the two dimensional volume element in coordinate space. $\phi^\dagger(\mathbf{x})$ and $\phi(\mathbf{x})$ are the creation and annihilation operators. Although this model in one dimension is exactly soluble [10, 137], the two dimensional version gives ultra-violet divergent results and this has been removed by regularization and renormalization procedures [11]. This model is extensively discussed first by J. Hoppe's thesis [138] for two and three particle sectors in two and three dimensions. It has been also formulated in terms of the principal operator by S. G. Rajeev [26] and a rigorous version of the nonrelativistic $\lambda\phi^4$ model in the context of [26] has been studied in [139]. We will not review the ideas for the renormalization of the model in $\mathbb{R}^2$, but strongly suggest the reader to read through the paper [26] to make the reading of this part of the thesis easier.

The model that we shall construct is the extension of the many body problem of non-relativistic bosons interacting with Dirac-delta function potential on the plane, which is indeed nonrelativistic limit of $\lambda\phi^4$ model, to the Riemannian manifolds. The

Hamiltonian of the model in the local coordinates $x \equiv (x^1, x^2)$ on a Riemannian manifold $(\mathcal{M}, \mathrm{g})$ is

$$H = H_0 + H_I \;, \tag{5.239}$$

where

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2m} \int_{\mathcal{M}} \mathrm{d}_g^2 x \; \phi_g^\dagger(x) \, \nabla_g^2 \, \phi_g(x) \;, \\ H_I &= -\frac{\lambda}{2} \int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \; \phi_g^\dagger(x') \, \phi_g^\dagger(x) \, \delta_g^2(x, x') \, \phi_g(x) \, \phi_g(x') \;, \end{aligned} \tag{5.240}$$

where λ is a positive coupling constant (it corresponds to an attractive potential), $\phi_g^\dagger(x)$, $\phi_g(x)$ are defined as the bosonic creation-annihilation operators on the Riemannian manifold with metric structure g. It is easy to show that the number of bosons $\int_{\mathcal{M}} \mathrm{d}_g^2 x \; \phi_g^\dagger(x) \, \phi_g(x)$ is conserved. In the natural units ($\hbar = m = 1$), $\phi(x)$ has dimension of $[E]^{-1/2}$ so that the coupling constant λ must be dimensionless. This means that the lowest eigenvalue of the Hamiltonian H in a sector with fixed number of bosons is either zero or negative infinity. In other words, the energy of the ground state is not bounded from below: $E \to -\infty$. We will use the units in which $\hbar = 1$ from now on. The regularized Hamiltonian with a cut-off ϵ can be naturally chosen as

$$\begin{aligned} H^\epsilon = H_0 - \frac{\lambda(\epsilon)}{2} \int_{\mathcal{M}^5} & \mathrm{d}_g^2 x_1 \mathrm{d}_g^2 x_1' \mathrm{d}_g^2 x_2 \mathrm{d}_g^2 x_2' \mathrm{d}_g^2 y \; \phi_g^\dagger(x_1) \phi_g^\dagger(x_2) K_\epsilon(x_1, y; g) K_\epsilon(x_2, y; g) \\ & \times K_\epsilon(x_1', y; g) K_\epsilon(x_2', y; g) \phi_g(x_1') \phi_g(x_2') \;, \end{aligned} \tag{5.241}$$

where $K_\epsilon(x, y; g)$ is the heat kernel defined on the manifold which converge to Dirac-Delta function as $\epsilon \to 0^+$. In this limit, one can see that we recover the original Hamiltonian we are interested in.

Now, we will consider the resolvent of the Hamiltonian (5.239) in Fock space $\mathcal{F}_{\mathcal{B}}$ with arbitrary number of bosons. Following the same methodology developed for the model in the plane [26], we shall extend the bosonic Fock space $\mathcal{F}_{\mathcal{B}}$ that we have started with to $\tilde{\mathcal{F}}_{\mathcal{B}} = \mathcal{F}_{\mathcal{B}} \oplus \mathcal{F}_{\mathcal{B}} \otimes \mathcal{L}^2(\mathcal{M})$ by defining new creation and annihilation operators.

These are continuous analog of the angels operators defined in (4.66), which is first introduced in [26]. The reason why the angel states are introduced is based on the fact that it allows us nonperturbatively renormalize the model by just normal ordering. Their algebra is defined as

$$\chi_g(x)\chi_g^\dagger(y) = \delta_g^2(x,y)\Pi_0, \quad \chi_g(x)\chi_g(y) = 0 = \chi_g^\dagger(x)\chi_g^\dagger(y) \;, \tag{5.242}$$

where

$$\Pi_1 = \int_{\mathcal{M}} \mathrm{d}_g^2 x \, \chi_g^\dagger(x)\chi_g(x), \quad \Pi_0 = 1 - \Pi_1 \tag{5.243}$$

are the projection operators onto the 1-angel and no-angel states, respectively. Now we define the augmented regularized Hamiltonian $\tilde{H}^\epsilon$ on $\tilde{\mathcal{F}}_{\mathcal{B}}$ as

$$\begin{aligned}\tilde{H}^\epsilon &= H_0\Pi_0 + \Bigg[\frac{1}{\sqrt{2}}\int_{\mathcal{M}^3} \mathrm{d}_g^2 x_1 \mathrm{d}_g^2 x_2 \mathrm{d}_g^2 y \; \phi_g^\dagger(x_1)\, \phi_g^\dagger(x_2) K_\epsilon(x_1, y; g) \\ &\qquad \times K_\epsilon(x_2, y; g)\chi_g(y) + h.c.\Bigg] + \frac{\Pi_1}{\lambda(\epsilon)}\end{aligned} \tag{5.244}$$

If we split the Hilbert space according to the angel number, the corresponding splitting of the operator $\tilde{H}^\epsilon - E\Pi_0$ can be written in the following matrix form

$$\tilde{H}^\epsilon - E\Pi_0 = \begin{pmatrix} a & b_\epsilon^\dagger \\ b_\epsilon & d_\epsilon \end{pmatrix} \;, \tag{5.245}$$

with $a : \mathcal{F}_{\mathcal{B}} \to \mathcal{F}_{\mathcal{B}}$, $b_\epsilon^\dagger : \mathcal{F}_{\mathcal{B}} \otimes \mathcal{L}^2(\mathcal{M}) \to \mathcal{F}_{\mathcal{B}}$, $d_\epsilon : \mathcal{F}_{\mathcal{B}} \otimes \mathcal{L}^2(\mathcal{M}) \to \mathcal{F}_{\mathcal{B}} \otimes \mathcal{L}^2(\mathcal{M})$. Here,

$$\begin{aligned} a &= H_0 - E \;, \qquad\qquad d_\epsilon = \frac{1}{\lambda(\epsilon)} \\ b_\epsilon^\dagger &= \frac{1}{\sqrt{2}}\int_{\mathcal{M}^3} \mathrm{d}_g^2 x_1 \mathrm{d}_g^2 x_2 \mathrm{d}_g^2 y \; \phi_g^\dagger(x_1)\, \phi_g^\dagger(x_2) K_\epsilon(x_1, y; g) K_\epsilon(x_2, y; g)\chi_g(y) \end{aligned} \tag{5.246}$$

Then, one can construct the augmented regularized resolvent

$$\tilde{R}^{\epsilon}(E) = \frac{1}{\tilde{H}^{\epsilon} - E\Pi_0} = \begin{pmatrix} \alpha_\epsilon & \beta_\epsilon^\dagger \\ \beta_\epsilon & \delta_\epsilon \end{pmatrix} , \tag{5.247}$$

Here E should be considered as a complex variable and using an explicit formula given in [26]

$$\alpha_\epsilon = \left[a - b_\epsilon^\dagger \, d_\epsilon^{-1} \, b_\epsilon\right]^{-1} = \frac{1}{H^\epsilon - E} = R^\epsilon(E) \tag{5.248}$$

This means that $\tilde{R}_\epsilon(E)$ projected to $\mathcal{F}_{\mathcal{B}}$ is just the resolvent of the operator H^ϵ. We have also another formula for the resolvent $R^\epsilon(E)$

$$\alpha_\epsilon = a^{-1} + a^{-1} \, b_\epsilon^\dagger \left[d_\epsilon - b_\epsilon \, a^{-1} \, b_\epsilon^\dagger\right]^{-1} b_\epsilon \, a^{-1} \; . \tag{5.249}$$

This will give

$$\alpha_\epsilon = R^\epsilon(E) = a^{-1} + a^{-1} \, b_\epsilon^\dagger \, [\Phi^\epsilon(E)]^{-1} \, b_\epsilon \, a^{-1} \; , \tag{5.250}$$

where

$$\Phi^\epsilon(E) = \frac{\Pi_1}{\lambda(\epsilon)} - \frac{1}{2} \int_{\mathcal{M}^6} \mathrm{d}_g^2 x_1 \mathrm{d}_g^2 x_2 \, \mathrm{d}_g^2 y \, \mathrm{d}_g^2 x_1' \, \mathrm{d}_g^2 x_2' \, \mathrm{d}_g^2 y' \; K_\epsilon(x_1, y; g) K_\epsilon(x_2, y; g)$$
$$\times K_\epsilon(x_1', y'; g) K_\epsilon(x_2', y'; g) \chi_g^\dagger(y) \left[\phi_g(x_1)\phi_g(x_2) \frac{1}{H_0 - E} \phi_g^\dagger(x_1') \, \phi_g^\dagger(x_2')\right] \chi_g(y') \; , \tag{5.251}$$

and the operator $\Phi^\epsilon(E)$ is called the regularized principal operator. Once we have a proper definition of the principal operator, all the divergences are removed since the resolvent is expressed in terms of it. The principal operator can be extended to its largest domain of the definition in the complex energy plane by analytic continuation.

Now, in order to remove the divergent part, we will do the normal ordering of the operators in (5.251) by using the commutation relations of the field operators. For simplicity we explicitly perform our calculations for compact manifolds here, but our result is also valid for non-compact manifolds similar to procedure that we have done for nonrelativistic Lee model. By using (4.77) and (4.78), the normal ordered principal operator can be written in terms of heat kernel

$$\begin{aligned}
\Phi^{\epsilon}(E) = \frac{\Pi_1}{\lambda(\epsilon)} - \frac{1}{2} \int_{\mathcal{M}^6} \mathrm{d}_g^2 x_1 \, \mathrm{d}_g^2 x_2 \, \mathrm{d}_g^2 y \, \mathrm{d}_g^2 x_1' \, \mathrm{d}_g^2 x_2' \, \mathrm{d}_g^2 y' \; K_\epsilon(x_1, y; g) K_\epsilon(x_2, y; g) \\
\times K_\epsilon(x_1', y'; g) K_\epsilon(x_2', y'; g) \chi_g^{\dagger}(y) \Bigg[\int_{\mathcal{M}^4} \mathrm{d}_g^2 z_1 \, \mathrm{d}_g^2 z_2 \, \mathrm{d}_g^2 z_1' \, \mathrm{d}_g^2 z_2' \; \phi_g^{\dagger}(z_1') \, \phi_g^{\dagger}(z_2') \\
\times \int_0^\infty \mathrm{d}t \; K_t(z_1, x_1; g) \, K_t(x_2, z_2; g) \\
\times K_t(x_1', z_1'; g) \, K_t(z_2', x_2'; g) \, e^{-t(H_0 - E)} \phi_g(z_1) \, \phi_g(z_2) + \int_{\mathcal{M}^2} \mathrm{d}_g^2 z_1 \, \mathrm{d}_g^2 z_2 \; \phi_g^{\dagger}(z_1) \\
\times \int_0^\infty \mathrm{d}t \; \Bigg(K_t(x_2', x_1; g) \, K_t(z_2, x_2; g) \, K_t(z_1, x_1'; g) \\
+ \; K_t(z_2, x_1; g) \, K_t(x_1', x_2; g) \, K_t(x_2', z_1; g) \\
+ \; K_t(z_2, x_1; g) \, K_t(x_2', x_2; g) \, K_t(x_1', z_1; g) \\
+ \; K_t(x_1', x_1; g) \, K_t(z_2, x_2; g) \, K_t(x_2', z_1; g) \Bigg) \, e^{-t(H_0 - E)} \phi_g(z_2) \\
+ \int_0^\infty \mathrm{d}t \; \Bigg(K_t(x_1, x_1'; g) K_t(x_2, x_2'; g) + K_t(x_1, x_2'; g) K_t(x_1', x_2; g) \Bigg) \\
\times e^{-t(H_0 - E)} \Bigg] \chi_g(y') \; . \qquad (5.252)
\end{aligned}$$

Since heat kernel is a continuous function of coordinates and of the variable t, we can change the order of integrations. From now on, we always assume that this is possible. Therefore, using the semigroup property of the heat kernel (3.36), we obtain

$$\begin{aligned}
\Phi^{\epsilon}(E) = \frac{\Pi_1}{\lambda(\epsilon)} - \frac{1}{2} \int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \; \chi_g^{\dagger}(x) \Bigg[\int_{\mathcal{M}^4} \mathrm{d}_g^2 x_1 \, \mathrm{d}_g^2 x_2 \, \mathrm{d}_g^2 x_1' \, \mathrm{d}_g^2 x_2' \; \phi_g^{\dagger}(x_1') \, \phi_g^{\dagger}(x_2') \\
\times \int_0^\infty \mathrm{d}t \; K_{t+\epsilon}(x_1, x; g) \, K_{t+\epsilon}(x_2, x; g) \, K_{t+\epsilon}(x_1', x'; g) \, K_{t+\epsilon}(x_2', x'; g) \\
\times e^{-t(H_0 - E)} \phi_g(x_1) \, \phi_g(x_2) + 4 \int_{\mathcal{M}^2} \mathrm{d}_g^2 x_1 \, \mathrm{d}_g^2 x_2 \; \phi_g^{\dagger}(x_1) \\
\times \int_0^\infty \mathrm{d}t \; K_{t+2\epsilon}(x', x; g) \, K_{t+\epsilon}(x_2, x; g) \, K_{t+\epsilon}(x_1, x'; g) \, e^{-t(H_0 - E)} \phi_g(x_2) \\
+ 2 \int_0^\infty \mathrm{d}t \; K_{t+2\epsilon}^2(x, x'; g) \, e^{-t(H_0 - E)} \Bigg] \chi_g(x') \; . \qquad (5.253)
\end{aligned}$$

We expect that as $\epsilon \to 0^+$ the last integral is divergent since it is this term that corresponds to the divergence in $\mathbb{R}^2$ [26]. In fact, we can also naively show that divergence of the principal operator is due to the singular behavior of heat kernel if we write expectation values between the n-bosonic and one-angel states

$$|\Psi\rangle = |\psi_b\rangle \otimes \int_{\mathcal{M}} \mathrm{d}_g^2 x \, \chi_g^\dagger(x) F(x) |0\rangle \; , \tag{5.254}$$

so

$$\begin{aligned}
&\langle\Psi| \int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \; \chi_g^\dagger(x') \int_0^\infty \mathrm{d}t \; K_t^2(x,x';g) e^{-t(H_0-E)} \chi_g(x) |\Psi\rangle \\
&\leq \int_0^\infty \mathrm{d}t \; \langle\psi_b| e^{-t(H_0-E)} |\psi_b\rangle \int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \; K_{2t}^2(x,x';g) |F(x)|^2 \; , \\
&\leq \int_{\mathcal{M}} \mathrm{d}_g^2 x \; \int_0^\infty \mathrm{d}t \; \langle\psi_b| e^{-t(H_0-E)} |\psi_b\rangle K_{2t}(x,x;g) |F(x)|^2 \; ,
\end{aligned} \tag{5.255}$$

where we have used the Cauchy-Schwartz inequality and reproducing property of heat kernel. Therefore the integral with respect to t in the right hand side of (5.255) is divergent due to the $1/t$ term in the asymptotic expansion of heat kernel (3.54) as $t \to 0^+$. This means that if the left hand side of (5.255) is divergent, this is basically due to the singular behavior of heat kernel near $t = 0$ (short "time") in the last term of the principal operator. It can also be shown in a similar fashion that the other terms in the principal operator are finite as $\epsilon \to 0^+$. All these suggest us to choose the bare coupling constant as

$$\frac{1}{\lambda(\epsilon)} = \int_\epsilon^\infty \mathrm{d}t \; \frac{e^{-t\mu^2}}{4\pi t/m} \tag{5.256}$$

where $-\mu^2$ is to be related to the bound state energy of a two particle system. If we take the limit $\epsilon \to 0^+$, we obtain the principal operator

$$\Phi(E) = \int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \chi_g^\dagger(x) \int_0^\infty \mathrm{d}t \, \left[\frac{e^{-t\mu^2}}{4\pi t/m} \delta_g^2(x,x') - K_t^2(x,x';g) e^{-t(H_0-E)} \right] \chi_g(x')$$
$$-\frac{1}{2} \int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \, \chi_g^\dagger(x)$$
$$\times \left[\int_{\mathcal{M}^4} \mathrm{d}_g^2 x_1 \, \mathrm{d}_g^2 x_2 \, \mathrm{d}_g^2 x_1' \, \mathrm{d}_g^2 x_2' \, \phi_g^\dagger(x_1') \, \phi_g^\dagger(x_2') \int_0^\infty \mathrm{d}t \, K_t(x_1,x;g) \, K_t(x_2,x;g) \right.$$
$$\times K_t(x',x_1';g) \, K_t(x',x_2';g) \, e^{-t(H_0-E)} \phi_g(x_1) \, \phi_g(x_2) + 4 \int_{\mathcal{M}^2} \mathrm{d}_g^2 x_1 \, \mathrm{d}_g^2 x_2 \, \phi_g^\dagger(x_1)$$
$$\left. \times \int_0^\infty \mathrm{d}t \, K_t(x_2,x;g) \, K_t(x',x;g) \, K_t(x',x_1;g) \, e^{-t(H_0-E)} \phi_g(x_2) \right] \chi_g(x') \; . \tag{5.257}$$

This is a finite form of principal operator and we can show that the choice for the coupling constant (5.256) is sufficient to remove the divergence. We must first note that the behavior of the off diagonal term of heat kernel near $t = 0$ is intimately related to the small distance behavior due to the initial condition given for the heat kernel. In fact one can show the choice for the coupling constant (5.256) is the appropriate one by writing the square of the heat kernel in the following tricky way near $t = 0$:

$$\Phi(E) = \int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \chi_g^\dagger(x) \int_0^\infty \mathrm{d}t \, \left[\frac{e^{-t\mu^2}}{4\pi t/m} \delta_g^2(x,x') \right.$$
$$\left. -K_{2t}(x,x';g) \delta_g^2(x,x') e^{-t(H_0-E)} \right] \chi_g(x') + \text{Finite terms} \; , \tag{5.258}$$

where the following heuristic argument can be given to justify this choice. Here what we mean by "finite terms" are the other terms in (5.257) and the ignored terms that is coming from the region $t \neq 0$. The heat kernel becomes a delta function as we take the limit $t \mapsto 0^+$. We can replace the angel operators by some functions, and look at the matrix element of the expression in principal operator:

$$\int_{\mathcal{M}} \mathrm{d}_g^2 x \; \psi_a^*(x) \, K_t(x,y;g) \, K_t(x,y;g) \, \psi_b(y), \tag{5.259}$$

as $t \mapsto 0^+$. In this limit, it is possible to replace the function $\psi_a^*(x)$ by $\psi_a^*(y)$, so that we have

$$\lim_{t\to 0^+} \int_{\mathcal{M}} \mathrm{d}_g^2 x \; \psi_a^*(x)\, K_t(x,y;g)\, K_t(x,y;g)\, \psi_b(y)$$
$$\approx \psi_a^*(y) \int_{\mathcal{M}} \mathrm{d}_g^2 x \; K_t(x,y;g)\, K_t(x,y;g)\, \psi_b(y)$$
$$\approx \psi_a^*(y)\, K_{2t}(y,y;g)\, \psi_b(y) \;, \tag{5.260}$$

where we use the semi-group property of the heat kernel (3.36). Let us try to give a better justification of this choice: we will again assume that the angel operators act on some smooth functions, since the set of smooth functions are dense in the Hilbert space norm, this is allowed. We will write one of the heat kernels as a distributional solution, and use the fact that $-\nabla_g^2$ is a self adjoint operator,

$$\int_{\mathcal{M}^2} \mathrm{d}_g^2 y \mathrm{d}_g^2 x \; \psi_a^*(x)\, K_t(x,y;g) e^{\frac{t}{2m}\nabla^2_{g,x}}\, \delta_g^2(x,y) e^{-t(H_0-E)} \psi_b(y)$$
$$= \int_{\mathcal{M}^2} \mathrm{d}_g^2 y \, \mathrm{d}_g^2 x \; e^{\frac{t}{2m}\nabla^2_{g,x}} \left[\psi_a^*(x)\, K_t(x,y;g) \right] \delta_g^2(x,y) e^{-t(H_0-E)}\, \psi_b(y), \tag{5.261}$$

If we expand the exponential into a formal power series we will see that there will be some derivatives acting on $\psi_a^*(x)$ and some derivatives acting on $K_t(x,y;g)$. If we define

$$(\nabla_g)^k := \begin{cases} (\nabla_g^2)^{k/2}, & \text{if } k = 2,4,6,\dots\,; \\ \nabla_g (\nabla_g^2)^{(k-1)/2}, & \text{if } k = 3,5,7,\dots, \end{cases} \tag{5.262}$$

then we get terms of the following form

$$(t/2m)^k \left[(\nabla_{g,x})^k \, \psi_a^*(x) \right] \, (t/2m)^{n-k} \, \left[(\nabla_{g,x})^{n-k}\, K_t(x,y;g) \right]. \tag{5.263}$$

As $t \mapsto 0^+$, the most singular terms in this expansion will come from the terms with the highest number of derivatives, thanks to the following theorem (Lemma 1.7.7 in [29]): If D_x^α is a differential operator of order α, then the asymptotic expansion of the kernel of the operator $D_x^\alpha e^{\frac{t}{2m}\nabla^2_{g,x}}$ on the diagonal in D dimensions

$$D_x^\alpha K_t(x,y)|_{x=y} \sim \sum_{k=0}^{\infty} (t/2m)^{-(D+\alpha-k)/2} e_k(x, D_x^\alpha, \frac{1}{2m}\nabla^2_{g,x}) \;, \tag{5.264}$$

where e_k are smooth local invariants of the jets of the symbols of the operators D_x^α and $\frac{1}{2m}\nabla^2_{g,x}$. Also e_k are zero if $k+\alpha$ is odd. Thus, the most singular terms will come from the highest powers of the Laplacian acting on the heat kernel when we formally expand the exponential operator. This means that the dominant contribution will be given by

$$\int_{\mathcal{M}^2} \mathrm{d}_g^2 y\, \mathrm{d}_g^2 x\; \psi_a^*(x) \left[e^{\frac{t}{2m}\nabla^2_{g,x}}\, K_t(x,y;g) \right] \delta_g^2(x,y)\, e^{-t(H_0-E)} \psi_b(y) \;. \tag{5.265}$$

Recalling that $\frac{1}{2m}\nabla^2_{g,x} K_t(x,y;g) = \frac{\partial}{\partial t} K_t(x,y;g)$, we have

$$\int_{\mathcal{M}^2} \mathrm{d}_g^2 y\, \mathrm{d}_g^2 x\; \psi_a^*(x) \left(e^{t\frac{\partial}{\partial t}}\, K_t(x,y;g) \right) \delta_g^2(x,y)\, e^{-t(H_0-E)}\, \psi_b(y). \tag{5.266}$$

We note now that $e^{t\frac{\partial}{\partial t}}$ generates a time translation by an amount t, which is again true in the sense of distributions:

$$\lim_{t' \to t} e^{t'\frac{\partial}{\partial t}} K_t(x,x';g) = \lim_{t' \to t} K_{t+t'}(x,x';g) = K_{2t}(x,x';g) \;. \tag{5.267}$$

As a result, we see that the most singular part of the full integral as $t \mapsto 0^+$ becomes

$$\int_{\mathcal{M}} \mathrm{d}_g^2 y\; \psi_a^*(y)\, K_{2t}(y,y;g)\, e^{-t(H_0-E)}\, \psi_b(y), \tag{5.268}$$

after executing the integral with respect to the local coordinates x. This justifies our choice of the coupling constant (5.256).

Therefore, if we integrate (5.258) with respect to x' and write the asymptotic expansion of the diagonal heat kernel as $t \to 0^+$ by keeping the first term in the expansion, we have

$$\Phi(E) = \int_{\mathcal{M}} \mathrm{d}_g^2 x \; \chi_g^\dagger(x) \int_0^\infty \mathrm{d}t \left[\frac{e^{-t\mu^2}}{4\pi t/m} - \frac{e^{-t(H_0 - E)}}{4\pi t/m} \right] \chi_g(x) + \text{Finite terms}$$
$$= \frac{m}{4\pi} \int_{\mathcal{M}} \mathrm{d}_g^2 x \; \chi_g^\dagger(x) \, \ln \left(\frac{H_0 - E}{\mu^2} \right) \chi_g(x) + \text{Finite terms} \; . \tag{5.269}$$

Although this is not a rigorous proof that the principal operator is renormalized with the choice of the coupling constant (5.256), we can explicitly show that our idea can be confirmed for the manifold $\mathbb{R}^2$ by writing the principal operator in momentum space that has already been calculated in [26]. For this purpose, let us consider the first part of the equation (5.257) written in the plane.

$$\Phi(E) = \int_{\mathbb{R}^4} \mathrm{d}^2 x \, \mathrm{d}^2 x' \chi^\dagger(\mathbf{x}) \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \mathrm{d}t \left[\frac{e^{-t\mu^2}}{4\pi t/m} \delta^2(\mathbf{x}, \mathbf{x}') - K_t^2(\mathbf{x}, \mathbf{x}') e^{-t(H_0 - E)} \right] \chi(\mathbf{x}') \; . \tag{5.270}$$

Substituting the heat kernel for $\mathbb{R}^2$, we find

$$\Phi(E) = \int_{\mathbb{R}^4} \mathrm{d}^2 x \, \mathrm{d}^2 x' \; \chi^\dagger(\mathbf{x}) \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \mathrm{d}t \left[\frac{e^{-t\mu^2}}{4\pi t/m} \delta^2(\mathbf{x}, \mathbf{x}') - \frac{e^{-m|\mathbf{x}-\mathbf{x}'|^2/2t}}{(4\pi t/2m)} \frac{e^{-m|\mathbf{x}-\mathbf{x}'|^2/2t}}{(4\pi t/2m)} e^{-t(H_0 - E)} \right] \chi(\mathbf{x}') \; . \tag{5.271}$$

If we write the heat kernel as a Fourier transform of a function $e^{-t\mathbf{p}^2/2m}$ and then change the integration order, we find

$$\int_{\mathbb{R}^2} \mathrm{d}^2 x \, \mathrm{d}^2 x' \chi^\dagger(\mathbf{x}) \int_\epsilon^\infty \mathrm{d}t \; \frac{e^{-m|\mathbf{x}-\mathbf{x}'|^2/2t}}{(4\pi t/2m)} \frac{e^{-m|\mathbf{x}-\mathbf{x}'|^2/2t}}{(4\pi t/2m)} e^{-t(H_0 - E)} \chi(\mathbf{x}')$$
$$= \int_{\mathbb{R}^4} \mathrm{d}^2 x \, \mathrm{d}^2 x' \; \chi^\dagger(\mathbf{x}) \int_\epsilon^\infty \mathrm{d}t \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{e^{i\mathbf{p}.(\mathbf{x}-\mathbf{x}') - t\mathbf{p}^2/4m}}{(4\pi t/m)} e^{-t(H_0 - E)} \chi(\mathbf{x}')$$
$$= \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi)^2} \chi^\dagger(\mathbf{p}) \int_\epsilon^\infty \mathrm{d}t \; \frac{e^{-t(H_0 - E + \mathbf{p}^2/4m)}}{(4\pi t/m)} \chi(\mathbf{p}) \; . \tag{5.272}$$

Then, we get

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi)^2} \chi^\dagger(\mathbf{p}) \int_\epsilon^\infty \mathrm{d}t \left[\frac{e^{-t\mu^2}}{4\pi t/m} - \frac{e^{-t(H_0 - E + \mathbf{p}^2/4m)}}{4\pi t/m} \right] \chi(\mathbf{p})$$
$$= \frac{m}{4\pi} \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 p}{(2\pi)^2} \chi^\dagger(\mathbf{p}) \ln \left(\frac{H_0 - E + \mathbf{p}^2/4m}{\mu^2} \right) \chi(\mathbf{p}) \; . \tag{5.273}$$

This is exactly the same result that was already calculated for flat space $\mathbb{R}^2$ [26]. As a consequence of this, we find the renormalized resolvent

$$\lim_{\epsilon \to 0^+} R^\epsilon(E) = R(E)$$
$$= \frac{1}{H_0 - E} + \frac{1}{H_0 - E} \frac{1}{\sqrt{2}} \int_{\mathcal{M}} \mathrm{d}_g^3 y \; \phi_g^\dagger(y) \phi_g^\dagger(y) \chi_g(y)$$
$$\times \Phi^{-1}(E) \frac{1}{\sqrt{2}} \int_{\mathcal{M}} \mathrm{d}_g^3 y \; \phi_g(y) \phi_g(y) \chi_g^\dagger(y) \, \frac{1}{H_0 - E} \; , \tag{5.274}$$

which is the analogue of the Krein's formula in the case of the many body problem. Therefore, the roots of

$$\Phi(E)|\Psi\rangle = 0 \; , \tag{5.275}$$

determines the bound state spectrum of the model. The same method does not work even in three dimensional Euclidean space as explained in [26]. Although we can find a finite and well defined resolvent and principal operator, the spectrum is not bounded from below for more than two particles in $\mathbb{R}^3$ [26].

6. CONCLUSIONS

This thesis was intended as an attempt to renormalize some simple quantum mechanical and field theoretical models on two and three dimensional Riemannian manifolds, hoping to understand the idea of the renormalization in curved spaces better.

We first studied the system in which the particles live in two or three dimensional Riemannian manifolds and they interact with the finitely many point interactions located on the manifold. Although a heuristic approach could be given in analogy with the flat case, the rigorous construction of the model was possible after we introduce a new mathematical tool, called heat kernel. This is very useful for subtracting the singular part of the problem due to the short-time asymptotic expansion of the heat kernel. After that the resolvent of the problem was completely determined by a finite well defined principal operator and the energy eigenvalues of the system was determined from the poles of the resolvent, i.e., from the condition $\det \Phi(E) = 0$. Then the wave function of the bound states was given in terms of the heat kernel and it was emphasized that the problem exhibits a kind of dimensional transmutation in the sense of [17]. It was noticed that the formula that we found is a kind of Krein's formula which show that we can think of our problem as a self-adjoint extension of the formal free hamiltonian. We also proposed an alternative construction of the many body version of the problem by extending the Fock space with the help of new hypothetical particles, called angels as a mathematical device. This allowed us to make the coupling constant appear in the equations additive so that we were able to do the renormalization by just normal ordering the operators. When we reduced the problem onto the one boson sector, we reproduced the same result obtained using the previous method. After that, we investigated the spectrum of the problem in detail. We found the wave function of the bound states more elegantly and then proved the comparison theorem which relates the ground state energy of $N+1$ center case with the N center. Furthermore, we considered a single delta center which is separated far from the other centers, and applied a perturbation theory. Using the short-time asymptotic expansion formula of the heat kernel which is valid for any points x and y, we estimated that the binding energy decreases exponentially as we increased the geodesic distance between the fixed center and other centers. Moreover, using the lower bounds on the heat kernel, the pointwise bounds on the wave function is explicitly given as an exponential functions of the geodesic distance. At the end, we heuristically demonstrated that the point interactions on manifolds can also be considered as a kind of self-adjoint extension as in the flat case [5]. We also proved the existence of the self-adjoint densely defined closed hamiltonian operator from the resolvent formula that we have found. The lower bound of the ground state energy due to the sharp upper bound estimates on the heat kernel for several classes of manifolds was given and this also showed that the ground state energy is finite. We also rigorously proved that the ground state is non-degenerate and positive. Finally, we studied the renormalization group equations and the β function is exactly calculated and two dimensional delta potential has been shown to be asymptotically free as in the case of flat spaces.

We conclude that many well known theorems given in standard quantum mechanics are still valid when we include the point interactions. The form of the pointwise bounds on the wave functions and the non-degeneracy and uniqueness of the ground state remains valid in standard quantum mechanics in flat spaces. Therefore we extend all these theorems to the cases on manifolds including singular interactions. The renormalization procedure does not radically change these well-known results given in standard quantum mechanics.

Although we have investigated the point interaction problem in detail, we can also extend the same problem by adding a smooth potential into the Hamiltonian. This could be done by using the heat kernel method except that the heat kernel now corresponds to a elliptic second order differential operator rather than the Laplace-Beltrami operator. The asymptotic expansion formula and the most of the bounds on the heat kernel are still applicable in this case as well.

Another question that we should address about the point interactions is to formulation of our method in the context of confining potential quantization. This remains as a challenging problem that we should try to solve in the future.

What we did not consider in this thesis is the scattering problem since the detailed analysis requires the distinction the scattering due to the geometry from the potential itself. We also postpone this issue for future studies.

The non-relativistic Lee model in two and three dimensional manifolds are similarly constructed. The resolvent is expressed in terms of a principal operator (operator valued function) rather than finite dimensional matrix. This surely makes the spectral investigation of the problem more complicated. Nevertheless, we proved that the ground state energy is bounded from below. In the large number of bosons, the mean field approximation has been obtained. Two and three dimensional cases needed a separate analysis.

Finally, we constructed the non-relativistic $\lambda\phi^4$ model in two dimensional manifolds. Unfortunately, the rigorous lower bound of the ground state energy and the mean field approximation remain as open problems.

We hope that our viewpoint is not restricted to the above mentioned models and has wider applicability and it may shed some new light on the non-perturbative renormalization on manifolds as well.

APPENDIX A: The Proof of the Inequality

In order to prove the following inequality

$$x^{1-\epsilon} < \delta + \left(\frac{\epsilon}{\delta}\right)^{\frac{\epsilon}{1-\epsilon}} x \; , \tag{A.1}$$

where x is a dimensionless variable, let us consider the following polynomial function

$$f(x) = \delta + \left(\frac{\epsilon}{\delta}\right)^{\frac{\epsilon}{1-\epsilon}} x - x^{1-\epsilon} \tag{A.2}$$

We assume that $\delta > 0$ and $0 < \epsilon < 1/2$. This function has one extremum point at x_*

$$x_*^{\epsilon} = (1-\epsilon)\left(\frac{\delta}{\epsilon}\right)^{\frac{\epsilon}{1-\epsilon}} \tag{A.3}$$

and this location x_* corresponds to the minimum. One can easily see that

$$f(x_*) = \delta\left(1 - (1-\epsilon)^{\frac{1-\epsilon}{\epsilon}}\right) \tag{A.4}$$

Since $\epsilon < 1/2$, we have $\frac{1-\epsilon}{\epsilon} > 1$. We can also show that $(1-\epsilon)^{\alpha} < 1-\epsilon$ for $\alpha > 1$ since $1-\epsilon < 1$ or

$$(1-\epsilon)^{\frac{1-\epsilon}{\epsilon}} < 1-\epsilon \tag{A.5}$$

Then, we find

$$f(x_*) > \delta\epsilon \tag{A.6}$$

and it is always positive. Since this is a global minimum point, we obtain $f(x) > 0$ for all x, which completes the proof.

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