ON THE INTEGRABILITY OF THE GENERALIZED DAVEY-STEWARTSON SYSTEM

by

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ABSTRACT

ON THE INTEGRABILITY OF THE GENERALIZED DAVEY-STEWARTSON SYSTEM

A method developed by V. E. Zakharov and E. I. Shul'man for understanding the integrable cases of a given system of differential equations having some certain Hamiltonian structure is represented. Then an application of this method to the Zakharov-Shul'man system which has been performed by Shul'man is explained in detail. Finally the same method is applied to the generalized Davey-Stewartson system and some conclusions on its integrability are made.

ÖZET

GENELLEŞTİRİLMİŞ DAVEY-STEWARTSON SİSTEMİ'NİN İNTEGRALLENEBİLİRLİĞİ ÜZERİNE

V.E. Zakharov ve E. I. Shul'man tarafından geliştirilmiş, belli bir Hamiltonyen yapıya sahip bir türevsel denklem sisteminin integrallenebildiği durumları anlamaya yarayan bir metod sunulacaktır. Ardından, Shul'man'ın bu metodu Zakharov-Shul'man sistemine uyguladığı çalışması detaylı bir şekilde anlatılacaktır. Son olarak da aynı metod, genelleştirilmiş Davey-Stewartson sistemine uygulanacak ve bu denklemin integrallenebilirliği üzerine bazı sonuçlar çıkarılacaktır.

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LIST OF SYMBOLS/ABBREVIATIONS

∂_{x_i}	First order partial differential operator with respect to the i-th space variable x_i
$\partial^2_{x_i x_j}$	Second order partial differential operator with respect to the i-th and j-th space variables x_i and x_j respectively
$\mathbf{A}$	d-dimensional constant vector in $\mathbb{R}^d$
C^2	Set of twice continuously differentiable functions
f'	First derivative of a single variable function f
f_t	Partial derivative of f with respect to t
f_{x_i}	Partial derivative of f with respect to the i-th space variable x_i
$\overline{f}$	Complex conjugate of f
$\widehat{f}$	Fourier transform of f
$\lvert f\rvert$	Modulus of f
$f * g$	Convolution of f and g
$\mathbf{k}$	d-dimensional Fourier domain variable in $\mathbb{R}^d$
$\mathbf{k}_i$	d-dimensional Fourier domain variable in $\mathbb{R}^d$, for each i
$\lvert t\rvert$	Absolute value of the real variable t
$\mathbf{x}$	d-dimensional variable in $\mathbb{R}^d$
$\dfrac{\delta F}{\delta f}$	The variational derivative of the functional F with respect to the function f
$\delta(x)$	The Dirac delta function

1. INTRODUCTION

It is difficult to give a general and universal definition for 'integrability', it may correspond to different meanings in different settings,[1]. In this thesis, integrability of a system of differential equations means that it is exactly solvable by the inverse scattering transform (IST) discovered by Gardner, Greene, Kruskal, and Miura,[2, 3].

Integrable nonlinear evolution equations show up in many applied areas of science. This is expected since these equations are obtained from large classes of nonlinear evolution equations by a procedure involving some certain scalings and expansions, and since this procedure expresses the nonlinear effects in the best way,[4]. Because the procedure usually preserves integrability of these equations, the widely applicable universal model equations obtained in this way is expected to be integrable if at least one of the many equations in that class is integrable. Keeping in mind that these classes contain so many equations, having integrable universal equations is not very unlikely though the fact that having an integrable equation is quite exceptional. Among these integrable nonlinear evolution equations, some of the very famous ones are the Korteveg-de Vries equation, the nonlinear Schrödinger equation, the sine-Gordon equation, the Kadomtsev-Petviashvili equation, and the Davey-Stewartson equations. The Davey-Stewartson equations, which we are interested in, has been studied on extensively,[5-8].

In this thesis, we represent a useful method for understanding the intagrable cases of a certain type nonlinear evolution equations and show some applications. In Chapter 2, we look closely at the method developed by Zakharov and Shul'man,[9-12], for testing the integrability of Hamiltonian wave systems. In the third chapter, we explain in detail the paper by Shul'man,[13], in which he applied this method to the (2+1)-dimensional Zakharov-Shul'man (ZS) system:

$$i\psi_t + L_1\psi + u\psi = 0 \ , \quad L_2 u = L_3|\psi|^2. \tag{1.1}$$

Here $u(\mathbf{x},t)$ is a real-valued function, $\psi(\mathbf{x},t)$ is a complex-valued function, $\mathbf{x}=(x_1,x_2)$, and

$$L_n(\partial_{\mathbf{x}}) = \sum_{i,j=1}^{2} c_{ik}^{n}\partial^2_{x_ix_j}, \quad n=1,2,3,$$

are second order differential operators with real constant coefficients. The Davey-Stewartson (DS) system

$$\begin{aligned} &iu_t + u_{xx} + c_0 u_{yy} = c_1|u|^2u + c_2 Q_x u, \\ &Q_{xx} + c_3 Q_{yy} = (|u|^2)_x, \end{aligned} \tag{1.2}$$

which is a special case of the ZS system, is known to be integrable when $(c_0,c_1,c_2,c_3) = (-1,1,-2,1)$ and $(c_0,c_1,c_2,c_3) = (1,-1,2,-1)$ which are called the DS-I and the DS-II respectively. We note that the method examined here has also been applied to the system of two coupled nonlinear Schrödinger equations,[14].

In Shul'man's paper that we studied on, there are two integrable equations which play important roles in the analysis for integrability. The first one is system (1.1) with the operators

$$\begin{aligned} L_1 = \pm\frac{1}{2}L_3 &= \lambda^2\partial^2_{x_2x_2} + 2(l-a)\lambda\partial^2_{x_1x_2} + (l^2-2la-a)\partial^2_{x_1x_1}, \\ L_2 &= \lambda^2\partial^2_{x_2x_2} + \lambda(2l+1)\partial^2_{x_1x_2} + l(l+1)\partial^2_{x_1x_1}, \end{aligned} \tag{1.3}$$

where l and a are real numbers and λ is a complex number. The other one is the system

$$i\psi_t + \psi_{\xi\eta} + u\psi = 0\ , \qquad u_\xi = \left(|\psi|^2\right)_\eta . \tag{1.4}$$

In Chapter 4, we apply the same method presented in the first chapter to the generalized Davey-Stewartson (GDS) system,[17], and derive and investigate the necessary conditons for the integrability of the system. In the last chapter, we finish by stating some conclusions and comments on the results we obtained.

2. A METHOD FOR TESTING INTEGRABILITY OF HAMILTONIAN WAVE SYSTEMS BY INVERSE SCATTERING TRANSFORM

As we mentioned above, we will point out the cases in which system (1.1) turns out to be integrable by the inverse scattering transform. For this purpose we are going to use a method developed by Zakharov and Shul'man. The method is applied to Hamiltonian wave systems in Fourier components of the form

$$ia_t(\mathbf{k}, t) = \frac{\delta H}{\delta \overline{a}(\mathbf{k}, t)}, \tag{2.1}$$

where $\mathbf{k} \in \mathbb{R}^d$, and the Hamiltonian H is a functional series in $a(\mathbf{k}, t)$ and $\bar{a}(\mathbf{k}, t)$ for small $|a(\mathbf{k}, t)|$. Here

$$H = H_0 + H_{int},$$

where

$$H_0 = \int \omega(\mathbf{k}) |a(\mathbf{k}, t)|^2 d\mathbf{k}$$

is the quadratic part of the Hamiltonian H, corresponding to the linear part of the system, and

$$\begin{aligned} H_{int} =& \frac{1}{3!} \sum_{s_0, s_1, s_2} \int_{\mathbb{R}^6} V(s_0, s_1, s_2; \mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2) a^{s_0}(\mathbf{k}_0, t) a^{s_1}(\mathbf{k}_1, t) a^{s_2}(\mathbf{k}_2, t) \\ & \times \delta(s_0\mathbf{k}_0 + s_1\mathbf{k}_1 + s_2\mathbf{k}_2) d\mathbf{k}_0 d\mathbf{k}_1 d\mathbf{k}_2 \\ & + \frac{1}{4!} \sum_{s_0, s_1, s_2, s_3} \int_{\mathbb{R}^8} W(s_0, \ldots, s_3; \mathbf{k}_0, \ldots, \mathbf{k}_3) a^{s_0}(\mathbf{k}_0, t) \cdots a^{s_3}(\mathbf{k}_3, t) \\ & \times \delta(s_0\mathbf{k}_0 + \cdots + s_3\mathbf{k}_3) d\mathbf{k}_0 d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \\ & + \cdots \end{aligned}$$

with $s_i = \pm 1$, and for each i

$$a^{+1}(\mathbf{k}_i, t) = a(\mathbf{k}_i, t), \quad a^{-1}(\mathbf{k}_i, t) = \overline{a}(\mathbf{k}_i, t),$$

$\mathbf{k}_i \in \mathbb{R}^d$. Physically, each choice of the s_i corresponds to a process of scattering of waves. $\omega(\mathbf{k})$ is the dispersion law of equation (2.1).

Definition 1 *Dispersion law (or dispersion relation) of a system of partial differential equations is the function $\omega : \mathbb{R}^d \to \mathbb{R}$ obtained by putting the plane wave solution*

$$a(\mathbf{x}, t) = m(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)}$$

for some spatial frequency $\mathbf{k}$ into the linear part of the system and solving for $\omega(\mathbf{k})$.

H_{int}, the interaction Hamiltonian corresponding to the nonlinear terms of the system, is treated as a perturbation, and to have solutions for large $|t|$, we consider the system

$$H = H_0 + H_{int} e^{-\epsilon|t|},$$

for $\epsilon > 0$. Then, as $t \to \pm\infty$, equation (2.1) becomes linear:

$$i a_t(\mathbf{k}, t) = \omega(\mathbf{k}) a(\mathbf{k}, t),$$

whose solution is

$$[a(\mathbf{k}, t)]^{\pm} = [c(\mathbf{k})]^{\pm} e^{-i\omega(\mathbf{k})t},$$

where $c(\mathbf{k})$ is constant in time. This suggests to replace $a(\mathbf{k}, t)$ by $b(\mathbf{k}, t) e^{-i\omega(\mathbf{k})t}$. Then equation (2.1) becomes

$$i b_t(\mathbf{k}, t) = \frac{\delta H_{int}}{\delta \bar{b}(\mathbf{k}, t)} e^{-\epsilon|t|},$$

where H_{int} here is written in $b(\mathbf{k},t)$. We can solve for $b(\mathbf{k},t)$ in the above equation, and obtain an integral equation

$$b(\mathbf{k},t) = [c(\mathbf{k})]^- - i\int_{-\infty}^{t} \frac{\delta H_{int}}{\delta \bar{b}(\mathbf{k},\tau)} e^{-\epsilon|\tau|} d\tau, \tag{2.2}$$

since as $t \to -\infty$, $b(\mathbf{k},t) = a(\mathbf{k},t)e^{i\omega(\mathbf{k})t} \to [c(\mathbf{k})]^-$. We can think of the above equation as a map from $[c(\mathbf{k})]^-$ to $b(\mathbf{k},t)$ and rewrite it using the notation

$$b(\mathbf{k},t) = S_t^\epsilon\left([c(\mathbf{k})]^-\right).$$

Then as $t \to +\infty$, we have

$$b(\mathbf{k},t) \to [c(\mathbf{k})]^+ \quad \text{and} \quad S_t^\epsilon\left([c(\mathbf{k})]^-\right) \to S^\epsilon\left([c(\mathbf{k})]^-\right),$$

so that

$$[c(\mathbf{k})]^+ = S^\epsilon\left([c(\mathbf{k})]^-\right),$$

where $S^\epsilon\left([c(\mathbf{k})]^-\right) = S_\infty^\epsilon\left([c(\mathbf{k})]^-\right)$.

Putting the series form of H_{int} in (2.2) gives a series for S^ϵ. Then letting $\epsilon \to 0$ in each term of this series, we obtain the classical scattering matrix S for the system (2.1). N-th term of this series is given by

$$S_N = 2\pi\delta\left(\sum_{i=0}^{N} s_i\omega(\mathbf{k}_i)\right) W_N,$$

where W_N is the N-th term of the series for H_{int} and has the form

$$W_N(s_0, s_1, \ldots, s_N; \mathbf{k}_0, \mathbf{k}_1, \ldots, \mathbf{k}_N)\delta\left(\sum_{i=0}^{N} s_i\mathbf{k}_i\right).$$

Here, δ is the Dirac delta function. System (2.1) can be symbolically represented in graphical form,[9], and each term W_N corresponds to a vertex, so that each term is called a vertex.

We are interested in motion invariants of equation (2.1). For establishing integrability conditions, we are going to use the fact that in order for the system (2.1) to be integrable, it needs to have infinitely many motion invariants. The Hamiltonian H (energy) and the momentum

$$P = \int_{\mathbb{R}^2} \mathbf{k}|a(\mathbf{k},t)|^2 d\mathbf{k}$$

are two invariants which can be verified by simply differentiating them with respect to t. We are looking for another motion invariant in the form of the Hamiltonian of equation (2.1):

$$\begin{aligned} I = & \int_{\mathbb{R}^2} f(\mathbf{k})|a(\mathbf{k},t)|^2 \, d\mathbf{k} \\ & + \frac{1}{3!} \sum_{s_0,s_1,s_2} \int_{\mathbb{R}^6} F(s_0,s_1,s_2;\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2) a^{s_0}(\mathbf{k}_0,t) a^{s_1}(\mathbf{k}_1,t) a^{s_2}(\mathbf{k}_2,t) \\ & \times \delta(s_0\mathbf{k}_0 + s_1\mathbf{k}_1 + s_2\mathbf{k}_2) d\mathbf{k}_0 d\mathbf{k}_1 d\mathbf{k}_2 \\ & + \frac{1}{4!} \sum_{s_0,s_1,s_2,s_3} \int_{\mathbb{R}^8} G(s_0,\ldots,s_3;\mathbf{k}_0,\ldots,\mathbf{k}_3) a^{s_0}(\mathbf{k}_0,t) \cdots a^{s_3}(\mathbf{k}_3,t) \\ & \times \delta(s_0\mathbf{k}_0 + \cdots + s_3\mathbf{k}_3) d\mathbf{k}_0 d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \\ & + \cdots \end{aligned}$$

for some $f : \mathbb{R}^d \to \mathbb{R}$.

Here we have an important theorem,[10], we will state without its proof.

Theorem 1 *If I is a motion invariant of system (2.1), then for any N, whenever*

$$\sum_{i=0}^{N} s_i \omega(\mathbf{k}_i) = \sum_{i=0}^{N} s_i \mathbf{k}_i = 0,$$

for some chioce of the $s_i \in \{-1, 1\}$*, either of the following alternatives is true:*

(i) *The term of the classical scattering matrix* S_N *corresponding to that process is zero so that, because of the form of* S_N*, the corresponding term of the Hamiltonian is zero:*

$$W_N(s_0, s_1, \dots, s_N; \mathbf{k}_0, \mathbf{k}_1, \dots, \mathbf{k}_N) = 0,$$

(ii) f *satisfies the functional equation*

$$\sum_{i=0}^{N} s_i f(\mathbf{k}_i) = 0.$$

So, for a system of differential equations which can be written of the form (2.1), if we can eliminate the second alternative of the theorem, then we have a condition on the Hamiltonian that lets the system to have an additonal motion invariant. If that condition is not satisfied, the system will not have another motion invariant other than the natural ones (energy and momentum), leading to the nonintegrability of the system. Indeed, the second alternative of the theorem can be eliminated when some certain conditions are satisfied. First we need to define the notion of degenerate and nondegenerate dispersion law.

Definition 2 *A dispersion law* $\omega(\mathbf{k})$ *is said to be degenerate with respect to a process of scattering of* m *waves into* n *waves*

$$\begin{aligned} \mathbf{k}_1 + \cdots + \mathbf{k}_m &= \mathbf{k}_{m+1} + \cdots + \mathbf{k}_{m+n}, \\ \omega(\mathbf{k}_1) + \cdots + \omega(\mathbf{k}_m) &= \omega(\mathbf{k}_{m+1}) + \cdots + \omega(\mathbf{k}_{m+n}), \end{aligned} \tag{2.3}$$

at a point P *of the manifold defined by the above equations if in a neighbourhood of* P *on the manifold there exists* $f : \mathbb{R}^d \to \mathbb{R}$ *satisfying*

$$f(\mathbf{k}_1) + \cdots + f(\mathbf{k}_m) = f(\mathbf{k}_{m+1}) + \cdots + f(\mathbf{k}_{m+n}),$$

but $f(\mathbf{k}) \neq A\omega(\mathbf{k}) + \mathbf{B}\cdot\mathbf{k} + C$*, for some scalar constants* A, C *and constant vector* $\mathbf{B}$*.* $\omega(\mathbf{k})$ *is called completely degenerate (or just 'degenerate') if it is degenrate at each point of the manifold. Finally, if such a point does not exist,* $\omega(\mathbf{k})$ *is called nondegenerate.*

Having defined these notions, we can state the following: If the dispersion law of an equation of the form (2.1) is nondegenerate with respect to a process (2.3), then I is not a new motion invariant unless the term of the Hamiltonian corresponding to that process vanishes on the manifold (2.3), leading to the nonintegrability of the equation.

3. APPLICATION TO THE ZAKHAROV-SCHUL'MAN EQUATIONS

Let us turn to the ZS system (1.1) with $d = 2$ and apply the method described above. We can take L_1, L_2 and L_3 to be symmetric since we assume u and ψ to be C^2. Then we can further assume

$$L_1 = \partial^2_{x_1x_1} + \sigma\partial^2_{x_2x_2}$$

by a coordinate transformation, where $\sigma = \pm 1$. We denote the corresponding coefficients of L_2 and L_3 by α_{ij} and β_{ij} respectively, for $i, j = 1, 2$. We will write $\alpha_{ii} = \alpha_i$ and $\beta_{ii} = \beta_i$ for $i = 1, 2$.

3.1. Finding the Vertex Function

Let us rewrite the system (1.1) with the 'new' operators in Fourier components of the functions

$$\widehat{u}(\mathbf{k}, t) = \int_{\mathbb{R}^2} u(\mathbf{x}, t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x},$$

expressing $\widehat{u}(\mathbf{k}, t)$ in terms of $\widehat{\psi}(\mathbf{k}, t)$. Under this transformation we have

$$\widehat{(u_{x_1})}(\mathbf{k}, t) = ip\ \widehat{u}(\mathbf{k}, t), \quad \widehat{(u_{x_2})}(\mathbf{k}, t) = iq\ \widehat{u}(\mathbf{k}, t),$$

where $\mathbf{k} = (p, q)$. From this point on, we will write $\widehat{u}(\mathbf{k})$ and $\widehat{\psi}(\mathbf{k})$ instead of $\widehat{u}(\mathbf{k}, t)$ and $\widehat{\psi}(\mathbf{k}, t)$ respectively, keeping an implicit dependence on the time variable t. Note that

$$\widehat{(L_2u)}(\mathbf{k}) = \left(-\alpha_1p^2 - 2\alpha_{12}pq - \alpha_2q^2\right)\widehat{u}(\mathbf{k}),$$
$$\widehat{(L_3u)}(\mathbf{k}) = \left(-\beta_1p^2 - 2\beta_{12}pq - \beta_2q^2\right)\widehat{u}(\mathbf{k}).$$

Thus, we have

$$
\begin{aligned}
L_2 u =& L_3|\psi|^2 \\
-L_2(\mathbf{k})\widehat{u}(\mathbf{k}) =& -L_3(\mathbf{k})\widehat{(|\psi|^2)}(\mathbf{k}) \\
L_2(\mathbf{k})\widehat{u}(\mathbf{k}) =& L_3(\mathbf{k})(\widehat{\psi} * \widehat{\overline{\psi}})(\mathbf{k}) \\
\widehat{u}(\mathbf{k}) =& \frac{L_3(\mathbf{k})}{L_2(\mathbf{k})}\int_{\mathbb{R}^2} \widehat{\overline{\psi}}(-\mathbf{k}_1)\widehat{\psi}(\mathbf{k}-(-\mathbf{k}_1))\ d(-\mathbf{k}_1) \\
\widehat{u}(\mathbf{k}) =& \int_{\mathbb{R}^2} \frac{L_3(\mathbf{k})}{L_2(\mathbf{k})}\overline{\widehat{\psi}}(\mathbf{k}_1)\widehat{\psi}(\mathbf{k}+\mathbf{k}_1)\ d\mathbf{k}_1,
\end{aligned} \tag{3.1}
$$

where $L_i(\mathbf{k})$ are the symbols of the operators L_i. Then we have

$$
\begin{aligned}
i\widehat{\psi}_t(\mathbf{k}) - L_1(\mathbf{k})\widehat{\psi}(\mathbf{k}) + \widehat{(u\psi)}(\mathbf{k}) =& 0 \\
i\widehat{\psi}_t(\mathbf{k}) - L_1(\mathbf{k})\widehat{\psi}(\mathbf{k}) + \int_{\mathbb{R}^2} \widehat{u}(\mathbf{k}-\mathbf{k}_3)\widehat{\psi}(\mathbf{k}_3)\ d\mathbf{k}_3 =& 0 \\
i\widehat{\psi}_t(\mathbf{k}) - L_1(\mathbf{k})\widehat{\psi}(\mathbf{k}) + \int_{\mathbb{R}^4} \frac{L_3(\mathbf{k}-\mathbf{k}_3)}{L_2(\mathbf{k}-\mathbf{k}_3)}\overline{\widehat{\psi}}(\mathbf{k}_1)\widehat{\psi}(\mathbf{k}-\mathbf{k}_3+\mathbf{k}_1)\widehat{\psi}(\mathbf{k}_3)\ d\mathbf{k}_1 d\mathbf{k}_3 =& 0, \text{ by (3.1)}
\end{aligned}
$$

so that system (1.1) becomes

$$
\begin{aligned}
i\widehat{\psi}_t(\mathbf{k}) - L_1(\mathbf{k})\widehat{\psi}(\mathbf{k}) + \int_{\mathbb{R}^6} & \frac{L_3(\mathbf{k}-\mathbf{k}_3)}{L_2(\mathbf{k}-\mathbf{k}_3)}\overline{\widehat{\psi}}(\mathbf{k}_1)\widehat{\psi}(\mathbf{k}_2)\widehat{\psi}(\mathbf{k}_3) \\
& \times \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\ d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 = 0
\end{aligned} \tag{3.2}
$$

by defining $\mathbf{k}_2 := \mathbf{k} - \mathbf{k}_3 + \mathbf{k}_1$ and using the fact $f(x) = (f * \delta)(x)$. While obtaining (3.2), we first introduced the variable $\mathbf{k}_3$ and then $\mathbf{k}_2$. We would have a similar but different expression if we did it the other way around. Adding those two expressions and dividing by two is symmetrizing equation (3.2) with respect to $(\mathbf{k}, \mathbf{k}_1)$ and $(\mathbf{k}_2, \mathbf{k}_3)$. After symmetrization, we introduce the function

$$
T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) := -\frac{1}{2}\left[\frac{L_3(\mathbf{k}-\mathbf{k}_3)}{L_2(\mathbf{k}-\mathbf{k}_3)} + \frac{L_3(\mathbf{k}-\mathbf{k}_2)}{L_2(\mathbf{k}-\mathbf{k}_2)}\right].
$$

We can now rewrite (3.2) as

$$\begin{aligned} i\widehat{\psi}_t(\mathbf{k}) =& L_1(\mathbf{k})\widehat{\psi}(\mathbf{k}) + \int_{\mathbb{R}^6} T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\overline{\widehat{\psi}}(\mathbf{k}_1)\widehat{\psi}(\mathbf{k}_2)\widehat{\psi}(\mathbf{k}_3) \\ & \times \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\ d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \end{aligned}$$

so that it is a Hamiltonian system of the form

$$i\widehat{\psi}_t(\mathbf{k}) = \frac{\delta H}{\delta \overline{\widehat{\psi}}(\mathbf{k})}, \tag{3.3}$$

where

$$\begin{aligned} H =& \int_{\mathbb{R}^2} L_1(\mathbf{k})\widehat{\psi}(\mathbf{k})\overline{\widehat{\psi}}(\mathbf{k}) + \frac{1}{2}\int_{\mathbb{R}^8} T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \\ & \times \overline{\widehat{\psi}}(\mathbf{k})\overline{\widehat{\psi}}(\mathbf{k}_1)\widehat{\psi}(\mathbf{k}_2)\widehat{\psi}(\mathbf{k}_3)\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\ d\mathbf{k}d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \\ =& H_0 + H_{int} \end{aligned}$$

which can be verified by calculating the variational derivative above.

Now, for testing the integrability of the system (1.1), we can use the method of Zakharov and Shul'man which says that: for system (3.3) to be integrable by IST, vertices for all processes of scattering of waves with respect to which the corresponding dispersion law is nondegenerate must vanish. We shall start checking the vanishing of these vertices from the ones contaning the least number of waves to the most. Indeed, the only identically nonvanishing vertices are the ones corresponding to scattering of m waves into m waves,[15].

3.2. Verifying the Nondegeneracy of the Dispersion Law

First we need to check the nondegeneracy of the dispersion law $\omega(\mathbf{k})$ of equation (1.1) with respect to the process of scattering of two waves into two waves

$$\begin{aligned} \mathbf{k}+\mathbf{k}_1 &= \mathbf{k}_2+\mathbf{k}_3, \\ \omega(\mathbf{k})+\omega(\mathbf{k}_1) &= \omega(\mathbf{k}_2)+\omega(\mathbf{k}_3), \end{aligned} \tag{3.4}$$

and then we can examine the vertex T.

Let us calculate the dispersion law of the system (1.1).

$$\begin{aligned} i\left(a(\mathbf{k})e^{i[\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t]}\right)_t &= -\,L_1\left(a(\mathbf{k})e^{i[\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t]}\right) \\ \omega(\mathbf{k})a(\mathbf{k})e^{i[\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t]} &= -\,(i^2p+\sigma i^2 q)a(\mathbf{k})e^{i[\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t]} \\ \omega(\mathbf{k}) &= p^2+\sigma q^2 \end{aligned}$$

where $\sigma=\pm 1$.

Having obtained the dispersion law explicitly, we can parametrize the manifold (3.4) to check the nondegeneracy of $\omega(\mathbf{k})$. We assume the parametrization,[13]

$$\begin{aligned} \mathbf{k} = \ (p,q) \ &= \left(P_1+\frac{1}{2}x(1-\sigma\tau s), P_2+\frac{1}{2}x(\tau+s)\right), \\ \mathbf{k}_1 = (p_1,q_1) &= \left(P_1-\frac{1}{2}x(1-\sigma\tau s), P_2-\frac{1}{2}x(\tau+s)\right), \\ \mathbf{k}_2 = (p_2,q_2) &= \left(P_1+\frac{1}{2}x(1+\sigma\tau s), P_2+\frac{1}{2}x(\tau-s)\right), \\ \mathbf{k}_3 = (p_3,q_3) &= \left(P_1-\frac{1}{2}x(1+\sigma\tau s), P_2-\frac{1}{2}x(\tau-s)\right), \end{aligned} \tag{3.5}$$

for some independent variables P_1, P_2, x, τ, s.

Let f be a function that satisfies the equation

$$f(\mathbf{k})+f(\mathbf{k}_1)=f(\mathbf{k}_2)+f(\mathbf{k}_3)$$

on the manifold (3.4). Then, using the parametrization (3.5), we rewrite this functional equation as

$$\begin{aligned} &f\left(P_1+\frac{x}{2}(1-\sigma\tau s), P_2+\frac{x}{2}(\tau+s)\right)+f\left(P_1-\frac{x}{2}(1-\sigma\tau s), P_2-\frac{x}{2}(\tau+s)\right)\\ =&f\left(P_1+\frac{x}{2}(1+\sigma\tau s), P_2+\frac{x}{2}(\tau-s)\right)+f\left(P_1-\frac{x}{2}(1+\sigma\tau s), P_2-\frac{x}{2}(\tau-s)\right). \end{aligned} \tag{3.6}$$

First we differentiate the above equation with respect to both τ and s, and then subtract one from the other and obtain

$$\begin{aligned} &\frac{\partial f(p,q)}{\partial p}\left(x\sigma\frac{\tau-s}{2}\right)+\frac{\partial f(p_1,q_1)}{\partial p_1}\left(x\sigma\frac{s-\tau}{2}\right)\\ =&\frac{\partial f(p_2,q_2)}{\partial p_2}\left(x\sigma\frac{s-\tau}{2}\right)+\frac{\partial f(p_2,q_2)}{\partial q_2}x+\frac{\partial f(p_3,q_3)}{\partial p_3}\left(x\sigma\frac{\tau-s}{2}\right)-\frac{\partial f(p_3,q_3)}{\partial q_3}x. \end{aligned}$$

Since this equation must hold for any τ and s, it must also hold when $s=\tau$, which yields

$$\frac{\partial f(p_2,q_2)}{\partial q_2}=\frac{\partial f(p_3,q_3)}{\partial q_3},$$

where now $p_2=P_1+\frac{x}{2}(1+\sigma\tau^2)$, $p_3=P_1-\frac{x}{2}(1+\sigma\tau^2)$, and $q_2=q_3=P_2$. Differentiating both sides of this last equation once again with respect to τ gives

$$\sigma x\tau\frac{\partial^2 f(p_2,q_2)}{\partial q_2\partial p_2}=-\sigma x\tau\frac{\partial^2 f(p_3,q_3)}{\partial q_3\partial p_3}$$

$$\frac{\partial^2 f(p_2,q_2)}{\partial q_2\partial p_2}=-\frac{\partial^2 f(p_3,q_3)}{\partial q_3\partial p_3}.$$

This equation holds for any x, hence, in particular it holds for $x=0$ as well so that now $p_2=p_3=P_1$ and we have

$$\frac{\partial^2 f(P_1,P_2)}{\partial P_1\partial P_2}=-\frac{\partial^2 f(P_1,P_2)}{\partial P_1\partial P_2},$$

which in turn implies

$$\frac{\partial^2 f(P_1, P_2)}{\partial P_1 \partial P_2} = 0$$

for independent variables P_1 and P_2. Hence, f is a sum of two functions of each of its variables:

$$f(p, q) = g(p) + h(q). \tag{3.7}$$

Now, rewrite (3.6) using (3.7) and differentiate with respect to P_1 to get

$$g'(p) + g'(p_1) = g'(p_2) + g'(p_3).$$

Differentiating once again with respect to $\sigma\tau s$ and setting $\tau = 0$, we obtain

$$g''\left(P_1 + \frac{x}{2}\right) = g''\left(P_1 - \frac{x}{2}\right),$$

which means, by the arbitrariness of P_1 and x, that g'' is constant and

$$g(p) = A_1 p^2 + B_1 p + C_1, \tag{3.8}$$

for some real constants A_1, B_1, C_1.

If we perform a similar analysis for finding the form of h, that is if we write (3.6) using (3.7), differentiate first with respect to P_2 and then with respect to x, and set $s = \tau$, we get h'' to be constant as well so that

$$h(q) = A_2 q^2 + B_2 q + C_2, \tag{3.9}$$

for some real constants A_2, B_2, C_2.

Finally, we rewrite (3.6) using (3.7), (3.8), and (3.9), do the simplifications and get

$$A_2 = \sigma A_1,$$

which establishes the nondegeneracy of the dispersion law:

$$\begin{aligned} f(\mathbf{k}) &= f(p, q) \\ &= A_1 p^2 + B_1 p + C_1 + A_2 q^2 + B_2 q + C_2 \\ &= A_1(p^2 + \sigma q^2) + (B_1, B_2) \cdot (p, q) + C_1 + C_2 \\ &= A\omega(\mathbf{k}) + \mathbf{B} \cdot \mathbf{k} + C. \end{aligned}$$

3.3. Pointing Out the Integrable Cases

Since the dispersion law is nondegenerate with respect to $\mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$, the only possibly completely integrable cases are given by the equation $T = 0$ to be solved on the manifold (3.4). We are trying to find the operators L_2 and L_3 for which

$$\frac{L_3(\mathbf{k} - \mathbf{k}_3)}{L_2(\mathbf{k} - \mathbf{k}_3)} + \frac{L_3(\mathbf{k} - \mathbf{k}_2)}{L_2(\mathbf{k} - \mathbf{k}_2)} = 0, \tag{3.10}$$

with L_2 and L_3 of the form

$$\begin{aligned} L_2(\mathbf{k}) &= -\alpha_1 p^2 - 2\alpha_{12} pq - \alpha_2 q^2, \\ L_3(\mathbf{k}) &= -\beta_1 p^2 - 2\beta_{12} pq - \beta_2 q^2, \end{aligned}$$

where $\alpha_{ii} = \alpha_i$ and $\beta_{ii} = \beta i$ for $i = 1, 2$.

On the manifold (3.4)

$$\begin{aligned} \mathbf{k} - \mathbf{k}_2 =& (-\sigma x \tau s, xs), \\ \mathbf{k} - \mathbf{k}_3 =& (x, x\tau), \end{aligned}$$

so that we have

$$
\begin{aligned}
L_2(\mathbf{k}-\mathbf{k_2}) &= -\,\alpha_1 x^2\tau^2 s^2 + 2\alpha_{12}\sigma x^2\tau s^2 - \alpha_2 x^2 s^2,\\
L_2(\mathbf{k}-\mathbf{k_3}) &= -\,\alpha_1 x^2 - 2\alpha_{12}x^2\tau - \alpha_2 x^2\tau^2,\\
L_3(\mathbf{k}-\mathbf{k_2}) &= -\,\beta_1 x^2\tau^2 s^2 + 2\beta_{12}\sigma x^2\tau s^2 - \beta_2 x^2 s^2,\\
L_3(\mathbf{k}-\mathbf{k_3}) &= -\,\beta_1 x^2 - 2\beta_{12}x^2\tau - \beta_2 x^2\tau^2.
\end{aligned}
$$

Putting these into (3.10), cancelling the common factor x^4s^2 of each term, and collecting similar terms together gives

$$
\begin{aligned}
&(\alpha_1\beta_2+\alpha_2\beta 1)\tau^4 + 2(\alpha_1\beta_{12}-\sigma\alpha_{12}\beta_2+\alpha_{12}\beta_1-\sigma\alpha_2\beta_{12})\tau^3\\
&+(2\alpha_1\beta_1-8\alpha_{12}\beta_{12}+2\alpha_2\beta_2)\tau^2 - 2\sigma(\alpha_1\beta_{12}-\sigma\alpha_{12}\beta_2+\alpha_{12}\beta_1-\sigma\alpha_2\beta_{12})\tau\\
&+\alpha_1\beta_2+\alpha_2\beta 1 = 0.
\end{aligned}
$$

The left-hand side of the above equation is a polynomial in τ. Thus, for the equation to hold, we need to have all the coefficients of powers of τ equal to zero, giving us three equations which can be considered to be a system of equations for $\alpha_1, \alpha_2, \alpha_{12}$:

$$
\begin{bmatrix} \beta_1 & \beta_2 & -4\sigma\beta_{12} \\ \beta_2 & \beta_1 & 0 \\ \beta_{12} & -\sigma\beta_{12} & \beta_1-\sigma\beta_2 \end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_{12} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.11}
$$

This system has nontrivial solutions for $(\alpha_1, \alpha_2, \alpha_{12})$ if the coefficient matrix is not invertible, that is, when its determinant is zero:

$$
\beta_2(\beta_1\beta_2 - \sigma{\beta_2}^2 - 4{\beta_{12}}^2) - \beta_1({\beta_1}^2 - \sigma\beta_1\beta_2 + 4\sigma{\beta_{12}}^2) = 0. \tag{3.12}
$$

We will now examine the cases $\sigma = +1$ and $\sigma = -1$ seperately.

Case 1: $\sigma = +1$. In this case, $L_1 = -p^2 - q^2$, and equation (3.12) becomes

$$
(\beta_1+\beta_2)\left[(\beta_1-\beta_2)^2 + 4{\beta_{12}}^2\right] = 0.
$$

Then there are two subcases:

(i) $\beta_1 = \beta_2 = \beta$, $\beta_{12} = 0$. Then equation (3.11) yields $\alpha_1 = -\alpha_2 = \alpha$ and α_{12} is free, and the symbols of the operators are

$$L_2(\mathbf{k}) = -\alpha(p^2 - q^2) - 2\alpha_{12}pq, \quad L_3(\mathbf{k}) = -\beta(p^2 + q^2) = \beta L_1. \tag{3.13}$$

(ii) $\beta_1 = -\beta_2 = \beta$ and β_{12} is free. Then (3.11) yields $\alpha_1 = \alpha_2 = \alpha$ and $\alpha_{12} = 0$, and the symbols of the operators are

$$L_2(\mathbf{k}) = -\alpha(p^2 + q^2) = \alpha L_1, \quad L_3(\mathbf{k}) = -\beta(p^2 - q^2) - 2\beta_{12}pq. \tag{3.14}$$

Case 2: $\sigma = -1$. In this case $L_1 = -p^2 + q^2$ and (3.12) becomes

$$(\beta_1 - \beta_2)\left[4{\beta_{12}}^2 - (\beta_1 + \beta_2)^2\right] = 0.$$

We again have two subcases:

(i) $\beta_1 = \beta_2 = \beta$. Then β_{12} is free, and (3.11) yields $\alpha_1 = -\alpha_2 = \alpha$ and $\alpha_{12} = 0$. The corresponding operators are

$$L_2(\mathbf{k}) = -\alpha(p^2 - q^2) = \alpha L_1, \quad L_3(\mathbf{k}) = -\beta(p^2 + q^2) - 2\beta_{12}pq. \tag{3.15}$$

(ii) $\beta_1 \neq \beta_2$. Then $4{\beta_{12}}^2 = (\beta_1 + \beta_2)^2$, that is, $2\beta_{12} = \pm(\beta_1 + \beta_2)$. We will split this case into two further subcases:

- $\beta_{12} = 0$. Then $\beta_1 = -\beta_2 = \beta$ and (3.11) gives $\alpha_1 = \alpha_2$ and α_{12} free. In this case we have

$$L_2(\mathbf{k}) = -\alpha(p^2 + q^2) - 2\alpha_{12}pq, \quad L_3(\mathbf{k}) = -\beta(p^2 - q^2) = \beta L_1. \tag{3.16}$$

– $\beta_{12} \neq 0$. Then $2\beta_{12} = \pm(\beta_1 + \beta_2)$. The matrix equation (3.11) gives $2\alpha_{12} = \mp(\alpha_1 + \alpha_2)$. Then we have for L_2

$$\begin{aligned} L_2(\mathbf{k}) =& -\alpha_1 p^2 \pm (\alpha_1 + \alpha_2)pq - \alpha_2 q^2 \\ =& (-\alpha_1 p \pm \alpha_2 q)(p \mp q) \\ =& -\alpha_2 \left(\frac{\alpha_1}{\alpha_2} p \mp q\right)(p \mp q), \end{aligned}$$

and for L_3

$$\begin{aligned} L_3(\mathbf{k}) =& -\beta_1 p^2 \mp (\beta_1 + \beta_2)pq - \beta_2 q^2 \\ =& (-\beta_1 p \mp \beta_2 q)(p \pm q) \\ =& -\beta_2 \left(\frac{\beta_1}{\beta_2} p \pm q\right)(p \pm q) \\ =& -\beta_2 \left(-\frac{\alpha_1}{\alpha_2} p \pm q\right)(p \pm q) \quad \text{by the second row of (3.11)} \\ =& -\beta_2 \left(\frac{\alpha_1}{\alpha_2} p \mp q\right)(p \mp q). \end{aligned}$$

Thus, we can have

$$L_2(\mathbf{k}) = (Mp \mp q)(p \mp q), \quad L_3(\mathbf{k}) = (Mp \mp q)(-p \mp q), \tag{3.17}$$

if we scale ψ by $\sqrt{\left|\frac{\alpha_2}{\beta_2}\right|}$ in (1.1). Here $M \neq \pm 1$ since otherwise we would have either $\beta_{12} = 0$ or $\beta_1 = \beta_2$, which are not the case.

Hence we get all the possible cases in which the system (1.1) is integrable. Now let us consider these cases in detail.

In the case with the operators (3.13), diagonalizing L_2 and L_3 simultaneously leaves L_3 unchanged since it is already diagonal and positive definite, whereas it reduces

L_2 to a difference of squares and by writing $\gamma = \sqrt{\alpha^2 + \alpha_{12}^2}$, we have

$$\begin{aligned} L_1(\partial_{\mathbf{x}}) =& \partial^2_{x_1 x_1} + \partial^2_{x_2 x_2}, \\ L_2(\partial_{\mathbf{x}}) =& \gamma\partial^2_{x_1 x_1} - \gamma\partial^2_{x_2 x_2}, \\ L_3(\partial_{\mathbf{x}}) =& \beta\partial^2_{x_1 x_1} + \beta\partial^2_{x_2 x_2}, \end{aligned}$$

and via a linear transformation

$$\mathbf{x} \mapsto 2\sqrt{\frac{2\alpha}{\beta}}\mathbf{x}, \quad t \mapsto \frac{2\alpha}{\beta}, \quad u \mapsto, \frac{\beta}{2\alpha},$$

they take the form (1.3) with $l = a = -1/2$ and $\lambda = 1/2$. So, system (1.1) with the operators (3.13) is integrable by IST.

For the system corresponding to the operators (3.14), we cannot find a linear transformation that transforms the system to a form of (3) since L_3 is not a multiple of L_1. So, we cannot decide whether this system is integrable or not just by considering the vertex corresponding to the process (3.4). In this case, we need to compute the next order vertex,[16], corresponding to the process

$$\begin{aligned} \mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 =& \mathbf{k}_3 + \mathbf{k}_4 + \mathbf{k}_5, \\ \omega(\mathbf{k}) + \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) =& \omega(\mathbf{k}_3) + \omega(\mathbf{k}_4) + \omega(\mathbf{k}_5). \end{aligned} \tag{3.18}$$

For this pupose, we need to find a rational parametrization for the manifold (3.18). The calculation of this vertex, even for specific forms of L_2 and L_3 given in (3.14), is very complicated, and it is shown in Shul'man's paper that it does not vanish on (3.18). Thus, the system corresponding to the case (3.14) is not integrable and the vanishing of T in this case is a consequence of integrability of the system with the operators in (3.13).

Similar to the previous case, the system with the operators (3.15) is not integrable and $T = 0$ due to the integability of the of the system corresponding to (3.16) which we will see below.

Considering the operators (3.16), we see that L_2 is elliptic if $\alpha^2 - \alpha_{12}^2 > 0$ and hyperbolic if $\alpha^2 - \alpha_{12}^2 < 0$. Reducing L_2 to its canonical form accordingly, we have the following two cases:

$$L_2(\mathbf{k}) = -p^2 + -q^2, \quad L_3(\mathbf{k}) = \mp 2(p^2 - q^2), \tag{3.19a}$$

$$L_2(\mathbf{k}) = -pq, \quad L_3(\mathbf{k}) = -2(p^2 - q^2), \tag{3.19b}$$

System (1.1) with operators (3.19a) is obtained from system (1.3) with $l = a = \frac{1}{2}$ and $\lambda = \frac{i}{2}$ via the linear transformation

$$t \mapsto \frac{1}{8}t, \quad u \mapsto \frac{1}{4}u, \quad \psi \mapsto \frac{1}{2}\psi.$$

Note that this is the Davey-Stewartson system. The system corresponding to (3.19b) is also obtained from (1.3) by the linear transformation

$$\begin{aligned} x_1 &\mapsto \frac{1}{\sqrt{l+1}}\, x_1 - \frac{l+1}{\lambda\sqrt{l+1}}\, x_2, \\ x_2 &\mapsto -\frac{1}{\sqrt{l}}\, x_1 + \frac{\sqrt{l}}{\lambda}\, x_2, \\ \psi &\mapsto \frac{1}{\sqrt[4]{l(l+1)}}\, \psi. \end{aligned}$$

In the last case in which we have the system corresponding to the operators (3.17), we are going to check the integrability of the system with first order differential operators

$$L_2(\mathbf{k}) = (p \mp q), \quad L_3(\mathbf{k}) = (-p \mp q), \tag{3.20}$$

since if there is a pair of solutions (ψ_0, u_0) for the system with the operators (3.20), then the same pair also solves the system in (3.17). And we see that under the change of variables

$$\xi = x_1 - x_2, \quad \eta = x_1 + x_2,$$

this system takes the form (1.4), which is integrable.

Hence, we can list all the integrable cases for the system (1.1) as the following: The Davey-Stewartson equations (corresponding to the case (3.19a)), the equations obtained from them by the substitution $x_2 \mapsto ix_2$ (corresponding to the case (3.13)), the equations (1.4) (corresponding to the case (3.17)), and the system in (3.19b).

3.4. A Remark on the Operator L_1

Quadratic form of the operator L_1 directly affects the analysis as it determines the dispersion law of the system. The analysis above is done if L_1 has an invertible quadratic form. Suppose that the quadratic form of L_1 is degenerate, that is we have $\omega(\mathbf{k}) = p^2 = -L_1(\mathbf{k})$. Then the interaction manifold is determined by the following three equations:

$$\begin{aligned} p + p_1 &= p_2 + p_3, \\ q + q_1 &= q_2 + q_3, \\ p^2 + p_1^2 &= p_2^2 + p_3^2. \end{aligned}$$

The first and the third equations yield

$$pp_1 = p_2 p_3. \tag{3.21}$$

Consider the function

$$f(\mathbf{k}) = f(p, q) = \ln |p|.$$

Then

$$
\begin{aligned}
f(\mathbf{k}) + f(\mathbf{k}_1) &= \ln|p| + \ln|p_1| \\
&= \ln|pp_1| \\
&= \ln|p_2p_3| \qquad\qquad \text{by (3.21)} \\
&= \ln|p_2| + \ln|p_3| \\
&= f(\mathbf{k_2}) + f(\mathbf{k_3}),
\end{aligned}
$$

but $f(k) \neq A\omega(\mathbf{k}) + \mathbf{B} \cdot \mathbf{k} + C$. Around the point $(p, p_1, p_2, p_3) = (0, 0, 0, 0)$, we can modify f to be $f(\mathbf{k}) = f(p, q) = \ln|1 + p|$. So, the dispersion law ω is degenerate with respect to the process (3.4). To find necessary conditions for integrability, we need to have a nondegenerate dispersion law with respect to the process being considered, so that the next order vertex of the perturbation analysis must be investigated with the general forms of the operators L_2 and L_3, which would surely be even more complicated then it is for the case (3.14).

4. AN ATTEMPT AT THE GENERALIZED DAVEY-STEWARTSON SYSTEM

Now we will apply the same method of testing integrability presented and used above to the GDS equations. The GDS system,[17], is given by the equations

$$iu_t + u_{xx} + \delta u_{yy} = \chi|u|^2 u + b(\varphi_{1,x} + \varphi_{2,y})u, \tag{4.1a}$$

$$\varphi_{1,xx} + m_2\varphi_{1,yy} + n\varphi_{2,xy} = (|u|^2)_x, \tag{4.1b}$$

$$\lambda\varphi_{2,xx} + m_1\varphi_{2,yy} + n\varphi_{1,xy} = (|u|^2)_y, \tag{4.1c}$$

where $\delta, \chi, b, m_1, m_2, n$ are real coefficients with m_1 and m_2 of the same sign, and δ, χ, b of arbitrary signs (without loss of generality δ can be taken to be ± 1). Also the condition

$$n^2 = (1 - \lambda)(m_1 - m_2) \tag{4.2}$$

must be satisfied by these coefficients. The actual physical interpretations of these dimensionless variables are given in [17].

The name for this system suggests a relation with the classical DS system. Under the reduction

$$1 - \lambda = m_1 - m_2, \tag{4.3}$$

the GDS system reduces to the DS system

$$\begin{aligned} &iu_t + u_{xx} + \delta u_{yy} = \left(\chi + \frac{1}{m_1}\right)|u|^2 u + b\left(1 - \frac{1}{m_1}\right)Q_x u, \\ &Q_{xx} + m_1 Q_{yy} = (|u|^2)_x, \end{aligned} \tag{4.4}$$

for $Q_x = \varphi_{1,x} + \varphi_{2,y} - \dfrac{1}{m_1}|u|^2$.

The GDS system (4.1) has been classified,[18], as EEE, EHH and EEH if $\delta > 0$, and HEE, HHH and HEH if $\delta < 0$ according to the respective signs $(+,+,+)$, $(-,-,+)$ and$(+,+,-)$ of the variables (m_1, m_2, λ), where 'E' and 'H' stand for the words 'elliptic' and 'hyperbolic', respectively. Of these classes, EHH with $1 < \lambda$ and EEH correspond to two different physical cases,[19].

4.1. Calculating the Vertex Function

Following the method presented in the second chapter, we rewrite the system (4.1) in terms of Fourier components of the functions u, φ_1, φ_2. To the find $\hat{\varphi}_1(\mathbf{k})$ and $\hat{\varphi}_2(\mathbf{k})$ explicitly, we need to solve the Fourier transforms of (4.1b) and (4.1c) simultaneously. Doing so gives

$$\begin{aligned}\widehat{\varphi}_1(\mathbf{k}) =& A(\mathbf{k})\widehat{(|u|^2)}_x,\\ \widehat{\varphi}_2(\mathbf{k}) =& B(\mathbf{k})\widehat{(|u|^2)}_y,\end{aligned}$$

where

$$A(\mathbf{k}) = \frac{i(n-m_1)pq^2 - i\lambda p^3}{\lambda p^4 + (\lambda m_2 + m_1 - n^2)p^2q^2 + m_1 m_2 q^4},$$

$$B(\mathbf{k}) = \frac{i(n-1)p^2q - im_2q^3}{\lambda p^4 + (\lambda m_2 + m_1 - n^2)p^2q^2 + m_1 m_2 q^4}.$$

Then we have

$$\begin{aligned}\widehat{\varphi}_{1,x}(\mathbf{k}) =& ipA(\mathbf{k})\widehat{(|u|^2)}(\mathbf{k}) = \tilde{A}(\mathbf{k})\widehat{(|u|^2)}(\mathbf{k}),\\ \widehat{\varphi}_{2,y}(\mathbf{k}) =& iqB(\mathbf{k})\widehat{(|u|^2)}(\mathbf{k}) = \tilde{B}(\mathbf{k})\widehat{(|u|^2)}(\mathbf{k}),\end{aligned}$$

where

$$\tilde{A}(\mathbf{k}) = \frac{\lambda p^4 + (m_1 - n)p^2q^2}{\lambda p^4 + (\lambda m_2 + m_1 - n^2)p^2q^2 + m_1m_2q^4},$$

$$\tilde{B}(\mathbf{k}) = \frac{(1-n)p^2q^2 + m_2q^4}{\lambda p^4 + (\lambda m_2 + m_1 - n^2)p^2q^2 + m_1m_2q^4}$$

are real-valued. Having found $\hat{\varphi}_{1,x}$ and $\hat{\varphi}_{2,y}$ explicitly, we can transform (4.1a) to find the vertex function corresponding to the process (3.4) once we calculate $\widehat{(|u|^2u)}$ and $b(\widehat{\varphi_{1,x} + \varphi_{1,x}})$. We have seen in the previous case of the ZS system that

$$\widehat{(|u|^2u)} = \iiint_{\mathbb{R}^6} \overline{\widehat{u}}(\mathbf{k}_1)\widehat{u}(\mathbf{k}_2)\widehat{u}(\mathbf{k}_3)\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2\mathbf{k}_3.$$

Similar calculations show that

$$\begin{aligned} b(\widehat{\varphi_{1,x} + \varphi_{1,x}}) = \int_{\mathbb{R}^6} b\left[\tilde{A}(\mathbf{k}-\mathbf{k}_2) + \tilde{B}(\mathbf{k}-\mathbf{k}_2)\right] \overline{\widehat{u}}(\mathbf{k}_1)\widehat{u}(\mathbf{k}_2)\widehat{u}(\mathbf{k}_3) \\ \times \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2\mathbf{k}_3. \end{aligned}$$

Then (4.1a) becomes

$$\begin{aligned} i\hat{u}_t(\mathbf{k}) - (p^2+\delta q^2)\hat{u}(\mathbf{k}) = \int_{\mathbb{R}^6} \left\{\chi + b\left[\tilde{A}(\mathbf{k}-\mathbf{k}_1) + \tilde{B}(\mathbf{k}-\mathbf{k}_2)\right]\right\} \\ \times \overline{\widehat{u}}(\mathbf{k}_1)\widehat{u}(\mathbf{k}_2)\widehat{u}(\mathbf{k}_3)\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2\mathbf{k}_3 \end{aligned}$$

Symmetrizing the expression $\left\{\chi + b\left[\tilde{A}(\mathbf{k}-\mathbf{k}_2) + \tilde{B}(\mathbf{k}-\mathbf{k}_2)\right]\right\}$ with respect to $(\mathbf{k},\mathbf{k}_1)$ and $(\mathbf{k}_2,\mathbf{k}_3)$ gives us the vertex function with respect to the process (3.4):

$$S = \chi + b\frac{\tilde{A}(\mathbf{k}-\mathbf{k}_2) + \tilde{B}(\mathbf{k}-\mathbf{k}_2) + \tilde{A}(\mathbf{k}-\mathbf{k}_3) + \tilde{B}(\mathbf{k}-\mathbf{k}_3)}{2}.$$

4.2. Deriving Necessary Conditions for Integrability

We have already shown that the dispersion law of the GDS system, which is the same as the dispersion law of the ZS sytem, is nondegenrate with respect to the process (3.4). So the necessary conditions for the integrability the GDS system by IST is given by the equation $S = 0$ on the manifold (3.4). Using the parametrization (3.5) for this manifold, we again have

$$\begin{aligned}\mathbf{k} - \mathbf{k}_2 =& (-\sigma x \tau s, xs),\\ \mathbf{k} - \mathbf{k}_3 =& (x, x\tau).\end{aligned}$$

Using these coordinates and cancelling the common factors '$x^4 s^4$' in the term $\tilde{A}(\mathbf{k} - \mathbf{k}_2) + \tilde{B}(\mathbf{k} - \mathbf{k}_2)$ and 'x^4' in $\tilde{A}(\mathbf{k} - \mathbf{k}_3) + \tilde{B}(\mathbf{k} - \mathbf{k}_3)$ we obtain

$$\begin{aligned}&2\chi + b\left[\tilde{A}(\mathbf{k} - \mathbf{k_2}) + \tilde{B}(\mathbf{k} - \mathbf{k_2}) + \tilde{A}(\mathbf{k} - \mathbf{k_3}) + \tilde{B}(\mathbf{k} - \mathbf{k_3})\right]\\ =& 2\chi + b\left[\frac{\lambda\tau^4 + (m_1 - 2n + 1)\tau^2 + m_2}{\lambda\tau^4 + (\lambda m_2 + m_1 - n^2)\tau^2 + m_1 m_2} + \frac{\lambda + (m_1 - 2n + 1)\tau^2 + m_2\tau^4}{\lambda + (\lambda m_2 + m_1 - n^2)\tau^2 + m_1 m_2 \tau^4}\right].\end{aligned}$$

We replace the terms $\lambda m_2 + m_1 - n^2$ in the above equation by $\lambda m_1 + m_2$ using the relation $n^2 = (1 - \lambda)(m_1 - m_2)$ so that we eliminate n^2. After performing the summation above, we equate it to zero, that is, the numerator of the sum must be zero. The numerator is a polynomial in τ, and for it to be zero, every coefficient of this polynomial must be zero. This gives us three linearly independent equations:

$$\lambda m_2(b + bm_1 + 2\chi m_1) = 0, \tag{4.5}$$

$$(\lambda m_1 + m_2)\left[(b + 2\chi)\lambda + (b + 2\chi m_1)m_2\right] + b(m_1 - 2n + 1)(m_1 m_2 + \lambda) = 0, \tag{4.6}$$

$$\begin{aligned}&(\lambda m_1 + m_2)\left[2\chi(\lambda m_1 + m_2) + 2b(m_1 - 2n + 1)\right]\\ &\qquad + (2b + 2\chi)\lambda^2 + (2b + 2\chi m_1)m_1 m_2^2 = 0.\end{aligned} \tag{4.7}$$

Since neither λ nor m_2 is zero, equation (4.8) further simplifies into

$$b + bm_1 + 2\chi m_1 = 0. \tag{4.8}$$

We will examine the integrability of the GDS system using these three equations in two cases as $1 < \lambda$ and $\lambda < 1$. We will further divide each case into two subcases as $m_1 = 1$ and $m_1 \neq 1$. First let us mention that in any of the cases, we are not looking for solutions with $b = 0$ or else we would be left with a single equation

$$iu_t + u_{xx} + \delta u_{yy} = \chi |u|^2 u,$$

which is integrable when $\chi = 0$.

4.2.1. The Case when $\lambda > 1$

Then we have $m_1 < m_2$ by (4.2).

Case 1: $m_1 = 1$. In this case, equation (4.8) gives $b + 2\chi = -b$. Using this in equation (4.6), we get

$$b(\lambda + m_2)(2 - 2n - m_2 - \lambda) = 0. \tag{4.9}$$

Here $b \neq 0$. Also $\lambda > 1$ and $m_2 > m_1 = 1$ so that $\lambda + m_2 \neq 0$. Hence

$$\begin{aligned} 2 - 2n - m_2 - \lambda =& 0 \\ \lambda + m_2 =& 2(1 - n). \end{aligned} \tag{4.10}$$

Putting (4.10) and $m_1 = 1$ into the condition $n^2 = (1 - \lambda)(m_1 - m_2)$, after some straightforward calculations we obtain

$$\lambda = m_2 \quad \text{and} \quad n = 1 - \lambda. \tag{4.11}$$

When (4.11) holds, equation (4.7) is trivially satisfied. Here we observe the following: When we have $m_1 = 1$ in (4.1), we see that

$$n = (1 - \lambda) = (m_1 - m_2),$$

and it is the case under which the system (4.1) reduces to the DS system. Hence for $m_1 = 1$, the GDS system is integrable whenever is the DS system (4.4) whose integrability conditions is well-known. We shall note here that this case does not correspond to a physical case the GDS system originated from.

Case 2: $m_1 \neq 1$. Using (4.8) and (4.6), we obtain

$$b(m_1 - 2n + 1) = \frac{[bm_1m_2 - (b + 2\chi)\lambda](\lambda m_1 + m_2)}{\lambda + m_1m_2}.$$

Using this equality and equation (4.8), equation (4.7) becomes

$$(\lambda m_1 + m_2)\left[2\chi(\lambda m_1 + m_2) + 2\frac{[bm_1m_2 - (b + 2\chi)\lambda](\lambda m_1 + m_2)}{\lambda + m_1m_2}\right] + (2b + 2\chi)\lambda^2 + bm_1m_2^2(1 - m_1) = 0.$$

Collecting similar terms together, we have

$$(2\chi + 2b)\left[(\lambda m_1 + m_2)^2\frac{m_1m_2 - \lambda}{m_1m_2 + \lambda} + \lambda^2\right] + bm_1m_2^2(1 - m_1) = 0. \tag{4.12}$$

From equation (4.8), we get

$$1 - m_1 = -\frac{(2b + 2\chi)m_1}{b}.$$

Using this equality, equation (4.12) becomes

$$(2\chi + 2b)\left[(\lambda m_1 + m_2)^2\frac{m_1m_2 - \lambda}{m_1m_2 + \lambda} + \lambda^2 - m_1^2m_2^2\right] = 0.$$

Since $m_1 \neq 1$, equation (4.8) yields $b + \chi \neq 0$. Thus we have

$$(\lambda m_1 + m_2)^2 \frac{m_1 m_2 - \lambda}{m_1 m_2 + \lambda} + \lambda^2 - m_1^2 m_2^2 = 0,$$

and after some straightforward calculations we obtain

$$\begin{aligned}(\lambda - m_1 m_1)(\lambda + m_1 m_2 - \lambda m_1 - m_2)(\lambda + m_1 m_2 + \lambda m_1 + m_2) &= 0 \\ (\lambda - m_1 m_1)(m_1 - 1)(m_2 - \lambda)(m_1 + 1)(m_2 + \lambda) &= 0.\end{aligned}$$

Since $m_1 \neq 1$, either of the following four conditions must hold:

$$m_1 = -1 \tag{4.13a}$$

$$\lambda = m_2 \tag{4.13b}$$

$$\lambda = -m_2 \tag{4.13c}$$

$$\lambda = m_1 m_2 \tag{4.13d}$$

Here we make the following comments:

- (4.13a) and (4.13b) cannot hold at the same time since m_1 and m_2 do not have different signs.
- (4.13a) and (4.13c) cannot be satisfied at the same time since otherwise we would have

$$\begin{aligned}\lambda = -m_2 &> 1 = -m_1 \\ m_1 &> m_2,\end{aligned}$$

  which contradicts with the condition $m_2 > m_1$.
- (4.13a) and (4.13d) cannot hold simultaneously or else we would again have

$$\begin{aligned}m_1 m_2 = -m_2 = \lambda &> 1 = -m_1 \\ m_1 &> m_2,\end{aligned}$$

  which is not the case.

- (4.13b) and (4.13c) clearly cannot both hold.
- (4.13b) and (4.13d) cannot hold at the same time since $m_1 \neq 1$.
- (4.13c) and (4.13d) cannot be satisfied simultaneously which would again require m_1 and m_2 to be of different signs, which is not the case.

Therefore, all of the above four cases are seperate. Let us investigate each case further.

(i) When $m_1 = -1$, equation (4.8) yields $\chi = 0$. Then equation (4.6) gives

$$\begin{aligned}(-\lambda + m_2)(b\lambda + bm_2) - 2bn(-m_2 + \lambda) =& 0 \\ b(\lambda - m_2)(\lambda + m_2 + 2n) =& 0 \\ \lambda + m_2 + 2n =& 0 \quad \text{since } b \neq 0 \text{ and } \lambda \neq m_2 \\ n + \frac{\lambda + m_2}{2} =& 0.\end{aligned}$$

Then condition (4.2) yields

$$\begin{aligned}n^2 = \frac{(\lambda + m_2)^2}{4} =& (1 - \lambda)(1 + m_2) \\ \lambda^2 + 2\lambda m_2 + m_2^2 =& 4\lambda m_2 + 4\lambda - 4m_2 - 4 \\ (\lambda - m_2)^2 =& 4(\lambda - m_2) - 4 \\ \lambda - m_2 - 2 =& 0 \\ \lambda =& 2 + m_2.\end{aligned}$$

Then we have

$$1 - \lambda = 1 - m_2 - 2 = -1 - m_2 = m_1 - m_2,$$

so that the GDS system again reduces to the DS system, whose integrable cases is well-known.

(ii) In the $\lambda = m_2$ case, equation (4.6) gives

$$\begin{aligned}(m_1m_2 + m_2)(2bm_2 + 2\chi m_2 + 2\chi m_1m_2) + b(m_1 - 2n + 1)(m_1m_2 + m_2) =&0\\ (b + 2\chi)m_2 + (b + 2\chi m_1) + b(m_1 - 2n + 1) =&0,\end{aligned}$$

since $m_1 \neq -1$. Remembering $b \neq 0$, we get

$$n = \frac{1 + m_1 - \dfrac{m_2}{m_1} - m_1m_2}{2} \quad \text{by (4.8).}$$

Checking with condition (4.2) shows

$$\begin{aligned}n^2 = \frac{\left(1 + m_1 - \dfrac{m_2}{m_1} - m_1m_2\right)^2}{4} =&(1 - m_2)(m_1 - m_2)\\ \left[m_1^2(1 - m_2) + (m_1 - m_2)\right]^2 =&4m_1^2(1 - m_2)(m_1 - m_2)\\ \left[m_1^2(1 - m_2) - (m_1 - m_2)\right]^2 =&0\\ m_1^2 - m_1^2m_2 - m_1 + m_2 =&0\\ (1 - m_1)\left[m_2(1 + m_1) - m_1\right] =&0,\end{aligned}$$

which cannot be true since $m_1 \neq 1$ and $m_2 > 1$. So, the case $\lambda = m_2$ does not lead to an integrable case of the GDS system, indeed there is no GDS system in this case.

(iii) Putting $\lambda = -m_2$ in (4.6) gives

$$\begin{aligned}(m_2 - m_1m_2)\left[-(b + 2\chi)m_2 - bm_1m_2\right] + b(m_1 - 2n + 1)(m_1m_2 - m_2) =&0\\ (m_1m_2 - m_2)\left[b(m_1 - 2n + 1) + (b + 2\chi)m_2 + bm_1m_2\right] =&0,\end{aligned}$$

as $m_1 \neq -1$. Then by (4.8) and since $b \neq 0$, we obtain

$$n = \frac{m_1^2 + m_1 - m_2 + m_1^2m_2}{2m_1}.$$

Using this result in condition (4.2) with $\lambda = -m_2$ yields

$$\begin{aligned} n^2 = \frac{(m_1^2 + m_1 - m_2 + m_1^2 m_2)^2}{4m_1^2} =& (1+m_2)(m_1 - m_2) \\ \left[m_1^2(1+m_2) + (m_1 - m_2)\right]^2 =& 4m_1^2(1+m_2)(m_1-m_2) \\ \left[m_1^2(1+m_2) - (m_1 - m_2)\right]^2 =& 0 \\ m_1^2(1+m_2) - m_1 + m_2 =& 0, \end{aligned} \tag{4.14}$$

and we have

$$m_1 = \frac{1 \pm \sqrt{1 - 4m_2(1+m_2)}}{2(1+m_2)}. \tag{4.15}$$

From (4.14) we see that the condition for reducing the GDS system to the DS system is satisfied only if $m_1 = \pm 1$, neither of which is the case. Hence in this case, system (4.1) is not reducable to system (4.4) unless equation (4.15) does not hold for any (m_1, m_2).

In order to have a real value for m_1, we need to have

$$\begin{aligned} 0 < 4m_2(1+m_2) < 1 \\ 1 < 4m_2 + 4m_2^2 + 1 < 2 \\ 1 < (1+2m_2)^2 < 2 \\ -\sqrt{2} < 1 + 2m_2 < -1 \qquad & \text{since } 1 + 2m_2 < 0 \\ -\frac{1+\sqrt{2}}{2} < m_2 < -1. \end{aligned}$$

Under these conditions, $m_1 < m_2$ is also satisfied. Therefore the GDS system cannot be reduced to the DS system and to establish the integrability of the GDS system with the restrictions imposed in this case, first the vertex function corresponding to the process (3.18) must be investigated. If we still do not get satisfactory information, we should continue analyzing higher order vertices at each time.

(iv) Similar calculations with $\lambda = m_1 m_2$ as in the previous cases give

$$m_1 < -1, \quad -2 < m_2 < 0,$$

and

$$m_2 = \pm \frac{2\left[\pm m_1^3 + \frac{\sqrt{P(m_1)}}{Q(m_1)}\left(2m_1^4 + 8m_1^3 + 20m_1^2 + 2\right) \pm m_1\right]}{Q(m_1)},$$

where

$$\begin{aligned} P(m_1) =& - m_1^3(m_1+1)^2, \\ Q(m_1) =& (m_1+1)^4 + 4m_1^2. \end{aligned}$$

The equation $1 - \lambda = m_1 - m_2$ which enables the reduction of the GDS system to the DS system is satisfied if and only if $m_1 = -(3 + 2\sqrt{2})$ and $m_2 = -1$, but for the rest of the (m_1, m_2), again the vertices corresponding to higher order processes must be inverstigated.

4.2.2. The Case when $\lambda < 1$

This time we have $m_2 < m_1$ by (4.2).

Case 1: $m_1 = 1$. Then $0 < m_2 < 1 = m_1$. From equation (4.9) we have two subcases:

(i) $\lambda = -m_2$. Equation (4.7) is trivially satisified but the GDS system cannot be reduced to the DS system since

$$1 - \lambda = 1 + m_2 \neq 1 - m_2 = m_1 - m_2,$$

and to decide integrability for this case, again higher order vertices must be investigated.

(ii) $\lambda \neq -m_2$. In this case we again have $\lambda = m_2$ as in (4.11), and the GDS system reduces to the DS system so that its integrability coincides with that of the DS system. However, this case does not correspond to a physical case since $0 < \lambda < 1$.

Case 2: $m_1 \neq 1$. We again have the four conditions (4.13) which are still mutually exclusive.

(i) $m_1 = -1$. Then we have $m_2 < -1 = m_1$. We cannot have $\lambda = m_2$ since they cannot be negative at the same time. Hence, as in the case for $\lambda > 1$, we have $\lambda = 2 + m_2$ for $-2 < m_2 < -1$ so that λ and m_2 are not both negative, and the GDS system reduces to the DS system. We remind that this case does not correspond to a physical setting case.

(ii) $\lambda = m_2$. This can only hold when $0 < \lambda < 1$ (hence is not a physical case) and $0 < m_2 < m_1$. Thus, $m_1 m_2 + m_2 \neq 0$ and we again have the equation

$$[m_2(1 + m_1) - m_1] = 0$$

to be satisfied since $m_1 \neq 1$. This yields

$$m_1 = \frac{m_2}{1 - m_2},$$

and reduction of the GDS system to the DS system is not allowed since the condition (4.3) is satisfied if and only if $m_1 = 1$, which is not the case here.

(iii) $\lambda = -m_2$. We again have to satisfy equation (4.14) since $m_2(1 - m_1) \neq 0$ as $m_1 \neq 1$. Hence reduction to DS is not possible as $m_1 \neq \pm 1$, and there are values

$$m_1 > -1 \quad \text{and} \quad -1 < m_2 = \frac{m_1 - m_1^2}{1 + m_1^2} < \frac{\sqrt{2} - 1}{2}$$

to be investigated further via applying the method for higher vertices. Here, only systems with $0 < m_2 < \dfrac{\sqrt{2}-1}{2}$ correspond to a physical case.

(iv) $\lambda = m_1 m_2$. In this case we have $0 < m_1 m_2 = \lambda < 1$ (so that this case is not a physical case either). After performing the calculations, we see that the same m_2 in the case for $\lambda > 1$ with $-1 < m_1 < 0$ satisfy the necessary conditions for the vertex S to be zero. The GDS system reduces to the DS system only when

$$m_1 = 2\sqrt{2} - 3 \quad \text{and} \quad m_2 = -1,$$

and the integrability of the GDS system for the rest of the (m_1, m_2) in this case is to be investigated by analyzing higher order terms of its classical scattering matrix.

5. CONCLUSIONS

The method presented here is actually a method for finding the *nonintegrable* cases of a given Hamiltonian wave system with nonzero quadratic part. We rather extract the integrable cases by showing that the possibly integrable equations, which are found through the analysis, are obtained by linear transformations from equations which are already known to be integrable. This may not always be possible, especially when an integrable equation to transform to the equation being studied on is unknown. We have seen in the ZS case that the method really works as a tool to deduce all the integrable cases of a system. However, with the part of the analysis performed in this thesis, we could not establish the integrability of the GDS system in full. We need to perform higher steps of the method as high as necessary for the cases which are already mentioned in the thesis. We could not perform those steps here because of the lack of time, but we wish to do it thoroughly in some other future work.

REFERENCES

1. H. Flaschka, A. C. Newell and M. Tabor, Integrability, *What Is Integrability*, Springer-Verlag (1991) 73-114.

2. C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Phys. Rev. Lett.* **19** (1967) 1095.

3. C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Comm. Pure Appl. Math.* **27** (1974) 97.

4. F. Calogero, Why are certain nonlinear PDEs both widely applicable and integrable?, *What Is Integrability*, Springer-Verlag (1991) 1-62.

5. D. Anker and N. C. Freeman, On the soliton solutions of the Davey-Stewartson equation for long waves, *Proc. R. Soc. Lond.* A. **360** (1978) 529-540.

6. P. M. Santini and A. S. Fokas, The initial-boundary value problem for the Davey-Stewartson 1 equation; how to generate and drive localized coherent structures in multidimensions, *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* **310** (1990) 223-259.

7. V. A. Arkadiev, A. K. Pogrebkov and M. C. Polianov, Inverse scattering transform method and soliton solutions for Davey-Sewartson II equation, *Physica* D **36** (1989) 189-197.

8. A.S. Fokas and P.M. Santini, Dromions and a boundary value problem for the Davey-Stewartson 1 equation, *Physica* D **44** (1990) 99-130.

9. V. E. Zakharov and E. I. Shul'man, On the additional motion invariants of classical Hamiltonian wave systems, *Physica* D **29** (1988) 283-320.

10. V. E. Zakharov and E. I. Shul'man, Integrability of nonlinear systems and perturbation theory, *What Is Integrability*, Springer-Verlag (1991) 185-250.

11. V. E. Zakharov and E. I. Shul'man, Degenerative dispersion laws, motion invariants and kinetic equations, *Physica* D **1** (1980) 192-202.

12. V. Zakharov, A. Balk and E. Shul'man, Conservation and scattering in nonlinear wave systems, *Springer Ser. Nonlinear Dynam.* (1993) 375-404.

13. E. I. Shul'man, On the integrability of equations of Davey-Stewartson type, *Theoretical and Mathematical Physics* **56** (1983) 720-724.

14. V. E. Zakharov and E. I. Shul'man, To the integrability of the system of two coupled nonlinear Schrödinger equations, *Physica* D **4** (1982) 270-274.

15. V. E. Zakharov and E. I. Shul'man, The scattering matrix and the integrability of classical wave systems with an additional integral of motion, *Sov. Phys. Dokl.* **30** (1985) 671-672.

16. V. E. Zakharov, The Hamiltonian formalism for waves in nonlinear media having dispersion, *Radiophysics and Quantum Electronics* **17** (1974) 326-343.

17. C. Babaoglu and S. Erbay, Two-dimensional wave packets in an elastic solid with couple stresses, *Int. J. Non-Linear Mech.* **39** (2004) 941-949.

18. A. Eden and S. Erbay, On travelling wave solutions of a generalized Davey-Stewartson system, *IMA Journal of Appl. Math.* **70** (2005) 15-24.

19. C. Babaoglu, A. Eden and S. Erbay, Global existence and nonexistence results for a generalized Davey-Stewartson system, *J. Phys. A: Math. Gen.* **37** (2004) 11531-11546.