DENSENESS OF MORSE-SMALE SYSTEMS ON SURFACES

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To My Family

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ABSTRACT

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Let M be a compact, connected 2-manifold without boundary. Morse-Smale fields are known to be dense in the space of all C^r vector fields on M when M is oriented or is one of $\mathbb{R}P^2$, Klein bottle or torus with a cross cap. In this work, we study the proofs of these facts. Furthermore, we exhibit a global picture of a C^r vector field X on a compact, connected 2-manifold without boundary when all the singularities of X are hyperbolic.

ÖZET

YÜZEYLER ÜZERİNDEKİ MORSE-SMALE SİSTEMLERİNİN YOĞUNLUĞU

Tıkız, bağlantılı ve kenarı olmayan iki boyutlu bir manifold M ile gösterilsin. Manifold M, yönlendirilebilen bir manifold ya da $\mathbb{R}P^2$, Klein şişesi veya torusla $\mathbb{R}P^2$'nin bağlantılı toplamı manifoldlarından biri olduğunda, Morse-Smale vektör alanlarının M üzerindeki tüm C^r vektör alanlarında yoğun olduğu bilinmektedir. Biz bu çalışmada, bu gerçeklerin kanıtlarını inceledik. Aynı zamanda, herhangi tıkız, bağlantılı ve kenarı olmayan iki boyutlu bir manifold üzerindeki herhangi bir C^r vektör alanı X'in tüm tekillikleri hiperbolik olduğunda, vektör alanı X'in global bir resmini çizen bir yöntem ve yapı geliştirdik.

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1. INTRODUCTION

We will assume all our manifolds, which we denote by M, to be compact and connected without boundary unless it is stated otherwise. For $r \geq 1$, let us denote by $\mathfrak{X}^r(M)$ the topological space of all C^r vector fields on M where $\mathfrak{X}^r(M)$ has the C^r topology. There is a notion of structural stability for a C^r vector field $X \in \mathfrak{X}^r(M)$. Vaguely speaking, when X has this nice property, its qualitative structure determined by its global flow is similar to the qualitative structures of vector fields that are close enough to X in $\mathfrak{X}^r(M)$. It is a curious question to determine which vector fields are structurally stable. In this regard, it has been shown that the Morse-Smale fields in $\mathfrak{X}^r(M)$ (see Definition 2.11) are structurally stable [1], [2]. Furthermore, in the case where the Morse-Smale fields are dense in $\mathfrak{X}^r(M)$, a C^r vector field $X \in \mathfrak{X}^r(M)$ is structurally stable if and only if X is a Morse-Smale field.

However, it is known that Morse-Smale fields are not dense in $\mathfrak{X}^r(M)$ when the dimension of M is greater than 2 (see [3]). Nevertheless, in 1962, M. Peixoto proved that Morse-Smale fields are dense on an orientable 2-manifold [4]. In Chapter 3, we present this fact where we will mainly follow the discussion in the text book by J. Palis and W. Melo [3]. Our presentation assumes the Kupka-Smale Theorem ([5] and [6]) which states that the Kupka-Smale fields (see Definition 2.9) are dense in $\mathfrak{X}^r(M)$ for any dimension of M. In particular, every Morse-Smale field is a Kupka-Smale field by definition. Furthermore, if the dimension of M is two and a Kupka-Smale field Y does not have any nontrivial recurrent orbit, then Y is a Morse-Smale field [4] (see Lemma 3.11). Peixoto gave the proof for any 2-manifold but it has been later understood that the proof is not valid for nonorientable 2-manifolds. His Connecting Lemma (Lemma 3.6) which can be shown for orientable manifolds is the only difficulty that prevents to generalize Peixoto's ideas to nonorientable cases. A positive answer to this obstacle is expected and sought in the literature (see e.g. [7], [8]).

In general, the density of Morse-Smale fields for nonorientable manifolds is still an open problem today but some partial results have been obtained about it since

Peixoto's work in 1962. There have been two main perspectives: For which 2-manifold M are Morse-Smale fields dense in $\mathfrak{X}^r(M)$ for any $r \geq 1$? For a given (nonorientable) 2-manifold M, for which $r \geq 1$ are Morse-Smale fields dense in $\mathfrak{X}^r(M)$? We remark for the last question that for any $k > j \geq 1$, the set $\mathfrak{X}^k(M)$ is a subset of $\mathfrak{X}^j(M)$ but $\mathfrak{X}^k(M)$ with the C^k topology does not have the subspace topology of $\mathfrak{X}^j(M)$ with the C^j topology.

The only known nonorientable 2-manifolds for which Morse-Smale fields are dense (for any $r \geq 1$) are the following: $\mathbb{R}P^2$, Klein bottle K [9] and a torus with a cross cap M_T [10]. In Chapter 4, we discuss these nonorientable cases following [10] by C. Gutierrez. The 2-manifolds S^2, $\mathbb{R}P^2$ and K have large Euler characteristics and their topologies put the following severe restriction: For any C^r vector field X on S^2, $\mathbb{R}P^2$ or K, the vector field X does not have a nontrivial recurrent orbit (see Lemma 3.1, Lemma 4.2 and Lemma 4.4). So, the proof of the density of Morse-Smale fields in $\mathfrak{X}^r(M)$ is relatively easier when M is S^2, $\mathbb{R}P^2$ or K. In [10], Gutierrez put very genuine ideas to take advantage of the large Euler characteristic of M_T. He showed that a nontrivial recurrent orbit on M_T exhibits the behavior of a nontrivial recurrent orbit on an orientable 2-manifold and he combined his ideas together with Peixoto's ones in [4] to prove the density of Morse-Smale fields in $\mathfrak{X}^r(M_T)$ (for any $r \geq 1$). In [11], he proves that his ideas for M_T cannot be extended to other nonorientable 2-manifolds the Euler characteristics of which are smaller than -1.

In 1967, C. C. Pugh proved in [12] that Morse-Smale fields are dense in $\mathfrak{X}^1(M)$ for any 2-manifold M. There, he proved the C^1 Closing Lemma and then, he followed Peixoto's ideas in [4] to show the density of Morse-Smale fields in $\mathfrak{X}^1(M)$. The Connecting Lemma in [4] implies that $X \in \mathfrak{X}^r(M)$ can be approximated by $Y \in \mathfrak{X}^r(M)$ arbitrarily close to X such that all singularities of Y are hyperbolic and Y does not have any nontrivial recurrent orbits. The absence of nontrivial recurrent orbits enables to yield a powerful lemma 3.10 [4]. Lemma 3.10 is essential to approximate $X \in \mathfrak{X}^r(M)$ by a Morse-Smale field $Z \in \mathfrak{X}^r(M)$ arbitrarily close to X although it is used in an indirect way (see the arguments in Lemma 3.1 before Case 2a). Pugh's C^1 Closing Lemma is an alternative to Peixoto's Connecting Lemma to eliminate problematic nontrivial recurrent orbits of $X \in \mathfrak{X}^1(M)$ by an arbitrarily small approximation (see [3] for this process).

The open problem whether Morse-Smale fields are dense in $\mathfrak{X}^r(M)$ $(r \geq 2)$ for any nonorientable 2-manifold M has given its way to the open problem whether the C^r Closing Lemma can be shown for any $r \geq 2$. In 1986, Gutierrez showed in [13] that the C^r Closing Lemma $(r \geq 1)$ holds for some flows on a torus and in 1992, C. R. Carroll generalized Gutierrez's result in [14]. A rotation number is defined relative to a transversal circle to X with a nontrivial recurrent orbit and those flows on the torus assumes specific rotation numbers. In 2009, S. Lloyd showed in [15] that the C^r Closing Lemma $(r \geq 4)$ holds for smooth vector fields on a torus that are area-preserving at all saddle points. Again in 2009, Gutierrez and B. Pires showed in [7] that C^r Closing Lemma $(r \geq 2)$ is valid for a large class of vector fields on any 2-manifold M. Their work assumes some contraction property of a small enough transversal section through a point of a nontrivial recurrent orbit of X. There, they state that the aforementioned results for C^r Closing Lemma $(r \geq 2)$ are the only ones in the literature but none of them are for a nonorientable M (except [7]). Earlier in 1987, Gutierrez questioned the validity of the C^r Closing Lemma in general [16]. He showed that if M contains a punctured torus, then there exists a smooth vector field X on M with a nontrivial recurrent orbit through $p \in M$ such that no $Y \in \mathfrak{X}^r(M - \{p\})$ $(r \geq 2)$ close enough to X has any closed orbits.

In Chapter 5, our own work describes an effective method to obtain a (almost) global picture of a C^r vector field $X \in \mathfrak{X}^r(M)$ on a 2-manifold M without boundary when all the singularities of X are hyperbolic. We start with the fact that a 2-manifold M can be represented by a fundamental polygon. Our work first constructs a particular fundamental polygon Γ for M. Each side of Γ is a C^r embedded circle in M and it is transversal to X except at finitely many points. It is also transversal to X at the single point that corresponds to the intersections of all sides of Γ. Then, we consider subsets of finitely many closed arcs of X to complete a global picture of X. We call such an exhibition a transversal base $\mathcal{T}^{\pm}$ for X and Γ (see Definition 5.10 and Theorem 5.3). After the exhibition of a transversal base for X and Γ, we prove Lemma 5.10 which gives some idea about how difficult it is to imagine a nontrivial recurrent orbit of X (if X has any). We hope that transversal base will be helpful to understand some global features of any C^r vector field with all hyperbolic singularities on any 2-manifold M.

2. PRELIMINARIES

In this chapter, we will introduce various basic notations, definitions, terminology and some lemmas. All the material in this chapter except *the First Intersection Assignment* are taken from [3] and the omitted proofs can be found there.

2.1. Global Flow

M will always denote a compact and connected manifold of dimension n without boundary unless it is stated otherwise. The positive integer r will denote C^r functions, vector fields or embeddings. An interval in M is the image of a C^r embedding of an ordinary (closed, open or half open) interval in $\mathbb{R}$ into M and a circle in M is the image of a C^r embedding of an ordinary circle S^1 into M.

The space $\mathfrak{X}^r(M)$ denotes the space of all C^r vector fields on M with C^r topology (see e.g [3]). Let $X \in \mathfrak{X}^r(M)$. It is known that for every point p of M, there exists a C^r function $\varphi : (-\epsilon, \epsilon) \times V \to M$ for some $\epsilon > 0$ and an open neighborhood V of p such that we have $\varphi(0, q) = q$ and $\frac{\partial \varphi}{\partial t}(t, q) = X(\varphi(t, q))$ for all q in V and for all t in $(-\epsilon, \epsilon)$ and the function φ is unique in the neighborhoods V and $(-\epsilon, \epsilon)$. The function φ is called *a local flow of X at p.* As M is compact, the neighborhoods V and $(-\epsilon, \epsilon)$ can be maximally extended to yield *the (unique) global flow $\varphi : \mathbb{R} \times M \to M$ for X* (with the same properties of the local flow). We will usually use the notation $X_t(q) = \varphi(t, q)$ for the flow of X. For any $t_1 \in \mathbb{R}$, the map $X_{t_1} : M \to M$ is a C^r diffeomorphism. For any t_1 and t_2 in $\mathbb{R}$, we have $X_{t_1} \circ X_{t_2} = X_{t_1+t_2}$.

The variable t of X_t will be called *the time variable.* For each p in M, the set $\{X_t(p) : t \in \mathbb{R}\}$ is called *the orbit of X through p. The positive semi trajectory and the negative semi trajectory of p* are the sets $p^+ := \{X_t(p) : \ t \geq 0\}$ and $p^- := \{X_t(p) : \ t \leq 0\}$ respectively. We will often use the letters X, Y and Z to denote C^r vector fields on M. The letters β, γ and τ will usually denote single orbits of X and the letters p, q, w and z will usually denote single points in M.

For any p in M, *the ω-limit set of p* which is denoted by $\omega(p)$ is the set of all points in M with the following property: q is in $\omega(p)$ if there exists a sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ such that we have $\lim\limits_{n\to\infty} t_n = \infty$ and $\lim\limits_{n\to\infty} X_{t_n}(p) = q$. Similarly, *the α-limit set of p* which is denoted by $\alpha(p)$ is the set of all points in M with the following property: q is in $\alpha(p)$ if there exists a sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ such that we have $\lim\limits_{n\to\infty} t_n = -\infty$ and $\lim\limits_{n\to\infty} X_{t_n}(p) = q$.

If q is a point in $\omega(p)$, then the orbit through q is contained in $\omega(p)$. Similarly, if q is a point in $\alpha(p)$, then the orbit through q is contained in $\alpha(p)$. For any orbit γ of X and any two points $w, z \in \gamma$, we have $\omega(w) = \omega(z)$ and $\alpha(w) = \alpha(z)$. So, we will sometimes use the notation $\omega(\gamma)$ to denote $\omega(w)$ and the notation $\alpha(\gamma)$ to denote $\alpha(w)$ as it does not depend on the choice of the point $w \in \gamma$. The w-limit sets are transitive in the following sense: if there exist orbits γ, β and τ of X with $\omega(\gamma) \supseteq \beta$ and $\omega(\beta) \supseteq \tau$, then we have $\omega(\gamma) \supseteq \tau$. Similarly, the α-limit sets are transitive.

Both $\omega(\gamma)$ and $\alpha(\gamma)$ share the following properties: they are nonempty, closed, connected and invariant by the flow of X. The last property means $X_t(\omega(\gamma)) = \omega(\gamma)$ and $X_t(\alpha(\gamma)) = \alpha(\gamma)$ for all t in $\mathbb{R}$. The compactness of M is essentially used to derive these properties. Later, we will consider linear vector fields on $\mathbb{R}^n$. Although $\mathbb{R}^n$ is not compact, the global flows for these linear vector fields can be given. The subsets $\alpha(\gamma)$ and $\omega(\gamma)$ can be defined there also but one of them can be the empty set which does not happen in compact manifolds. Also, connectedness might fail for some other vector fields on $\mathbb{R}^n$ the global flows for which can be given.

We remark that $(-X)_t(p) = X_{-t}(p)$ and q is in $\omega(p)$ for the flow of X if and only if q is in $\alpha(p)$ for the flow of $-X$. During our study, it is usually necessary to consider both $\omega(p)$ and $\alpha(p)$ in various settings but this dualism enables us to talk about claims for only one of them and claims for the other one can be derived analogously.

2.2. C^r Topology

We need a natural topology on $\mathfrak{X}^r(M)$ and on the space of C^r diffeomorphisms $\mathrm{Diff}^r(M)$ from M to itself. For this purpose, we first put a topology on $C^r(M, \mathbb{R}^s)$ which is the set of all C^r functions from M to $\mathbb{R}^s$. For any two closed C^r manifolds N_1 and N_2, the set $C^r(N_1, N_2)$ will similarly denote the set of all C^r functions from N_1 to N_2. Note that $C^r(M, \mathbb{R}^s)$ has a natural vector space structure defined as $(af+bg)(q) := af(q) + bg(q)$ where we have $f, g \in C^r(M, \mathbb{R}^s)$, $a, b \in \mathbb{R}$ and $q \in M$.

Let (U_j, ϕ_j) be a regular cover of M (see [17]). As M is compact, there exists a finite $\{U_1, \dots U_n\}$ open subcover of it. For every $f \in C^r(M, \mathbb{R}^s)$, let $\|f\|_r = \sup \{ \, \|d^j(f \circ \phi_i^{-1})(x)\| : \; 1 \leq i \leq n, \; 0 \leq j \leq r, \; x \in \overline{B(0,1)} \, \}$ where $B(0,1)$ denotes the open ball in $\mathbb{R}^s$ at the origin with radius 1. Then, $\|\cdot\|_r$ is a complete norm on $C^r(M, \mathbb{R}^s)$ and the topology induced from this norm is independent of the finite open subcover.

Let N_1 and N_2 be closed smooth manifolds and $\mathfrak{i}_1$ and $\mathfrak{i}_2$ be smooth embeddings of N_1 and N_2 into $\mathbb{R}^{s_1}$ and $\mathbb{R}^{s_2}$ respectively. Put subspace topologies on $C^r(M, \mathfrak{i}_1(N_1))$ and $C^r(M, \mathfrak{i}_2(N_2))$ that are induced from $C^r(M, \mathbb{R}^{s_1})$ and $C^r(M, \mathbb{R}^{s_2})$ respectively. For $l \geq r$ and a C^l map $\Phi : \mathfrak{i}_1(N_1) \to \mathfrak{i}_2(N_2)$, define $\Phi^* : C^r(M, \mathfrak{i}_1(N_1)) \to C^r(M, \mathfrak{i}_2(N_2))$ as $\Phi^*(f) = \Phi \circ f$. Then, Φ^* is continuous. By this continuity, the subspace topology on $C^r(M, N_1)$ which is induced by the embedding $\mathfrak{i}$ of N_1 into some $\mathbb{R}^s$ and by the the topology of $C^r(M, \mathbb{R}^s)$ is independent from the embedding $\mathfrak{i}$ and the space $\mathbb{R}^s$.

So, using a smooth embedding $\mathfrak{i}$ of M into some $\mathbb{R}^s$, one defines a C^r topology on $C^r(M, M)$ as explained above. The set $\mathrm{Diff}^r(M)$ then becomes an open subset of the topological space $C^r(M, M)$. Using this embedding $\mathfrak{i}$ of M into $\mathbb{R}^s$, the space $\mathfrak{X}^r(M)$ becomes a closed subset of $C^r(M, \mathbb{R}^s)$ when one naturally identifies $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ for all x in $\mathbb{R}^n$. As $\mathfrak{X}^r(M)$ is also a vector space itself, it admits the norm $\|\cdot\|_r$ of $C^r(M, \mathbb{R}^s)$. With this norm, $\mathfrak{X}^r(M)$ becomes a complete Banach space.

2.3. Structural Stability

Note that every orbit of X has a natural orientation that is induced from the positive time flow of X. We will now give some strong definitions which demand particularly nice vector fields on M.

Definition 2.1. *Let* $X, Y \in \mathfrak{X}^r$*. We say that* X *and* Y *are* topologically equivalent *if there exists a homeomorphism* $h : M \to M$ *such that:*

(i) For every orbit β_X *of* X*, the set* $h(\beta_X)$ *is an orbit of* Y*;*

(ii) h *preserves the orientation of every orbit* β_X *of* X*.*

Note that Definition 2.1 defines a relation between some of the vector fields in $\mathfrak{X}^r(M)$ and this relation is an equivalence relation.

Definition 2.2. *Let* $X \in \mathfrak{X}^r$*. We say that* X *is* structurally stable *if there exists a neighborhood* $\mathcal{V}$ *of* X *such that every* $Y \in \mathcal{Y}$ *is topologically equivalent to* X*.*

It can be shown that Morse-Smale fields that we will define in the sequel are structurally stable. Also, they are dense in $\mathfrak{X}^r(M)$ when M is an orientable, compact and connected 2-manifold without boundary. The density of Morse-Smale fields in $\mathfrak{X}^r(M)$ when M is a Klein Bottle, $\mathbb{R}P^2$ or a torus with one cross cap is also known. Note that their density implies that a vector field on M is structurally stable if and only if it is a Morse-Smale field.

After defining a natural topology on $\mathfrak{X}^r(M)$, the structural stability is the best behavior that we can expect from a C^r vector field X on M in the topological space $\mathfrak{X}^r(M)$. In regard to this phenomenum, we cannot expect h in Definition 2.1 to preserve the time parameter t for the flows of X and Y (i.e. $h(X_t(p)) = Y_t(h(p))$) nor we can expect h to be a diffeomorphism because these requirements are too restrictive so that very few or no vector fields in $\mathfrak{X}^r(M)$ would have been structurally stable otherwise. This fact, however, is difficult to explain and we refer the reader to the literature.

2.4. Hyperbolic Critical Elements and Local Study

A C^r vector field X on M is a C^r section of TM; i.e. the map $X : M \to TM$ satisfies $\Pi \circ X(q) = q$ where $\Pi : TM \to M$ is the natural projection map. A point p in M is called *a singularity of* X if $X(p)$ is the 0 vector in T_pM and it is called *a regular point of* X if $X(p)$ is a nonzero vector in T_pM. If p is a singularity, the orbit through p contains the point p only. If p is a regular point, then the orbit through p is a C^r immersion of $\mathbb{R}$ into M. For a regular point p, if $X_T(p) = p$ for some $T > 0$, then the orbit τ through p is called *a closed orbit* of X and the smallest such $T > 0$ is called *the period of* τ. As τ is a C^r immersion of $\mathbb{R}$, the period of τ is well defined. If the orbit through a regular point is not a closed orbit, then it is a C^r immersion of $\mathbb{R}$ that does not have self intersections. It is not necessarily a C^r embedding of $\mathbb{R}$.

If X and Y are topologically equivalent vector fields and $h : M \to M$ is the homeomorphism satisfying the condition in Definition 2.1, then h maps the singularities of X to the singularities of Y (if there are any) and it maps the closed orbits of X to the closed orbits of Y (if there are any).

The local study of X focuses on a small neighborhood of its regular point or its singularity. We now make a weaker definition than Definition 2.1.

Definition 2.3. *Let* X, Y *be in* $\mathfrak{X}^r$ *and* p, q *be points in* M. *We say that* X *and* Y *are* topologically equivalent at p and q *respectively if there exist neighborhoods* U_p *and* U_q *of* p *and* q *respectively and a homeomorphism* $h : U_p \to U_q$ *such that* $h(p) = q$ *and* h *maps the orbits of* X *in* U_p *to the orbits of* Y *in* U_q *preserving their orientations.*

The local study of regular points follows from an application of Implicit Function Theorem. For any regular point p of X, there exists a neighborhood F of p, some $a > 0$ and a C^r diffeomorphism $f : F \to [-a, a] \times [-1, 1]^{n-1}$ such that $f(p) = 0$ and $f_*(X_{|F})$ is the parallel unit vector field in the direction of e_1 where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Some authors consider $f : F \to [-1, 1] \times [-1, 1]^{n-1}$ instead of $f : F \to [-a, a] \times [-1, 1]^{n-1}$ but this choice assumes that the orbit γ through p is either not a closed orbit or the period of the closed orbit γ is greater than 2. This local characterization of regular points show that two C^r vector fields X and Y are topologically equivalent at any regular points of them. Before we continue with our local study of singularities, we will now focus on the behavior of orbits through regular points.

2.4.1. Flow Box, Tubular Flow Neighborhood and the Transversal Section

The pair (f, F) where $f : F \to [-a, a] \times [-1, 1]^{n-1}$ is as in the last section will be called *a flow box at p for the flow of X*. It has played a central role in the study of the dynamical systems because of its natural local characterization of the orbits through regular points. It is often used to perturb X only in F. Note that we have $\phi_*(t, (x, y)) = (x + t, y)$ where ϕ_* is a local flow of $f_*(X_{|F})$ at (x, y) and (x, y) is in $[-a, a] \times [-1, 1]^{n-1}$. The perturbations are usually required to be confined to small neighborhoods so that given a small neighborhood U of a regular point p, we would like F to be a subset of U. This requirement is possible as $a > 0$ can be taken arbitrarily small.

Let p be a regular point of X and let $\epsilon > 0$ be such that we have $p \notin A := \{X_t(p) : 0 < t \leq \epsilon\}$. Let $p_1 = X_\epsilon(p)$. By pp_1, we denote the set $\{p\} \cup A$ which is a closed subset of the orbit γ through p and it is called *a closed arc of X (or γ) with boundary points p and p_1*. Since pp_1 is compact, it can be finitely covered with flow boxes at points in pp_1. In this way, an open neighborhood V of pp_1 that is diffeomorphic to an open ball can be found together with a C^r diffeomorphism $f : V \to A_0 \subseteq \mathbb{R}^n$ such that $f_*(X_{|V})$ is the parallel unit vector field in the direction of e_1. The neighborhood V will be called *a tubular flow neighborhood of pp_1 for the flow of X*. Note that if the orbit τ through p is a closed orbit, then τ is not a subset of V. We will always assume that V is an open subset of M in contrast to a flow box which is a closed subset of M. The tubular flow neighborhoods have indispensably emanated in the text and we now state it as a theorem (see e.g.[3] for a proof).

Theorem 2.1 (Tubular Flow Theorem). *Suppose that pp_1 is a closed arc of X. Then,* *there exist an open neighborhood V of pp_1 in M and a C^r diffeomorphism $f : V \to A_0 \subseteq \mathbb{R}^2$ such that: $f_*(X_{|V})$ is the parallel unit vector field in the direction of e_1; the open set A_0 is the interior of the set $[-a, b] \times [-1, 1]^{n-1}$ for some $a, b > 0$ and; we have $f(pp_1) \subseteq (-a, b) \times \{0\} \times \cdots \times \{0\}$.*

The orbits through regular points are locally embedded submanifolds in M of dimension 1 so that certain embedded submanifolds in M of dimension $n-1$ play a central role as well. Let Σ be a C^r embedded, simply connected submanifold in M of dimension $n-1$, with or without boundary, the points of which are all regular points. For q in Σ, let $\epsilon_q > 0$ be such that the C^r embedded closed arc q_0q_1 can be defined with the points $q_0 = X_{-\epsilon_q}(q)$ and $q_1 = X_{\epsilon_q}(q)$. We say that Σ *is transversal to X at q* if q_0q_1 and Σ are transversal to each other at q. If Σ is transversal to X at all of its points, then we say that Σ *is a transversal section to X*. We have sometimes preferred to say "transversal section Σ" only when the unspecified vector field X can be understood from the context.

Suppose that Σ is transversal to X at p. Then, there exists a neighborhood U of p in Σ such that $U \cap \Sigma$ is transversal to X so that the set of points in Σ at which Σ is transversal to X is open in Σ. Also, there exist some other neighborhood V of p in Σ and some $\epsilon > 0$ such that for every distinct t_1 and t_2 in $(-\epsilon, \epsilon)$, we have $X_{t_1}(V \cap \Sigma) \cap X_{t_2}(V \cap \Sigma) = \varnothing$. If p is an interior point of Σ and V is simply connected and small enough, then $\{z \in M : z \in X_t(V \cap \Sigma), t \in (-\epsilon, \epsilon)\}$ is diffeomorphic to an open ball. If Σ is a transversal section to X, then any orbit of X intersects Σ at discrete times t if it ever does.

Let pq be a closed arc of X and Σ_p and Σ_q be disjoint transversal sections to X through p and q respectively. Here, when we say that they are through p and q, we mean that p and q are interior points of Σ_p and Σ_q respectively. Then, there exist a small tubular flow neighborhood V of pq such that the positive semi trajectory of each z in $V \cap \Sigma_p$ intersects $V \cap \Sigma_q$ at a unique point in $V \cap \Sigma_q$ within V. Define $P : \Sigma_p \cap V \to \Sigma_q \cap V$ in the following way: $P(z)$ is the first intersection of z^+ with Σ_q. Then, the map P is a diffeomorphism (if V is small enough). See Figure 2.1.

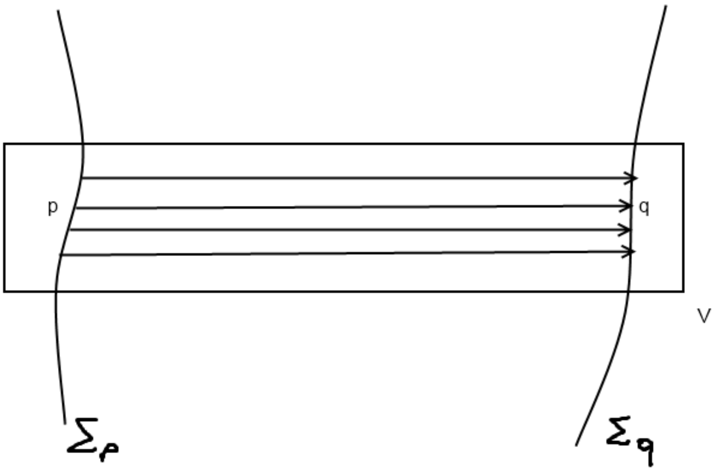

Figure 2.1. Tubular flow neighborhood V of pq.

The transversal sections Σ_p and Σ_q will be employed again and again and we need a convenient way to express the map P. Let Σ_1 and Σ_2 be any two transversal sections to X (whether if they are disjoint or not). *The First Intersection Assignment* $P_1 : \Sigma_1 \to \Sigma_2$ is defined as follows: for any point $z \in \Sigma_1$, the point $P_1(z) \in \Sigma_2$ is the first intersection of z^+ with Σ_2 if z^+ intersects Σ_2 at all. If z^+ does not intersect Σ_2, then P_1 at z is not defined. Note that P_1 is not actually a map because its domain (the points in Σ_1 at which P_1 is defined) is not necessarily Σ_1. Its domain can be very well the empty set. In the previous situation, The First Intersection Assignment $P : \Sigma_p \cap V \to \Sigma_q \cap V$ is defined on the whole $\Sigma_p \cap V$ and also for z in its domain, we have that $(zP(z) - \{z\}) \cap \Sigma_p = \varnothing$. On the other hand, if $P_1 : \Sigma_1 \to \Sigma_2$ is defined at some point $z \in \Sigma_1$, then we do not necessarily have $(zP_1(z) - \{z\}) \cap \Sigma_1 = \varnothing$. In other words, z^+ can intersect Σ_1 before it intersects Σ_2 and P_1 is not necessarily injective.

We caution the reader that P_1 is not necessarily a local diffeomorphism yet even a continuous function. Some nice situations we prefer to have are such that either the property $\Sigma_1 = \Sigma_2$ or the property $\Sigma_1 \cap \Sigma_2 = \varnothing$ holds. The worst scenario that one can possibly imagine is that $\Sigma_1 \cap \Sigma_2$ has infinitely many connected components. We will come back to these situations later in Chapter 5. Another nice situation that we will consider in Chapter 5 will be the following property: Σ_1 and Σ_2 are not disjoint and also, $\Sigma_1 \cup \Sigma_2$ is a C^r embedded submanifold of M.

For the applications in Chapter 3, it is sufficient for us to consider the properties $\Sigma_1 \cap \Sigma = \varnothing$ or $\Sigma_1 = \Sigma_2$ but even these properties are not good enough in general. We would like to consider the positive semi trajectories of interior points of Σ_1 which intersect the interior of Σ_2 but some of them might go through a boundary point of Σ_2 before they intersect the interior of Σ_2 and these semi trajectories cause problems for the continuity of the First Intersection Assignment $P_1 : \Sigma_1 \to \Sigma_2$. In Chapter 5, we will introduce *First Proper Intersection Assignment* which naturally copes with these problematic situations. For the moment, we will use the First Intersection Assignment $P_1 : \Sigma_1 \to \Sigma_2$ and we will be more specific about our choice of Σ_1 and Σ_2.

Consider the the closed arc situation once again. Let wz be a closed arc of X and and let V be a tubular flow neighborhood of wz. Let Σ_1 and Σ_2 be disjoint transversal sections to X through w and z respectively such that we have $\Sigma_1 \subseteq V$ and $\Sigma_2 \subseteq V$ and; Σ_1 is small enough so that the positive semi trajectory of every point in Σ_1 intersect Σ_2 within V. By the Tubular Flow Theorem, the First Intersection Assignment $P_1 : \Sigma_1 \to \Sigma_2$ is a local diffeomorphism. See Figure 2.2.

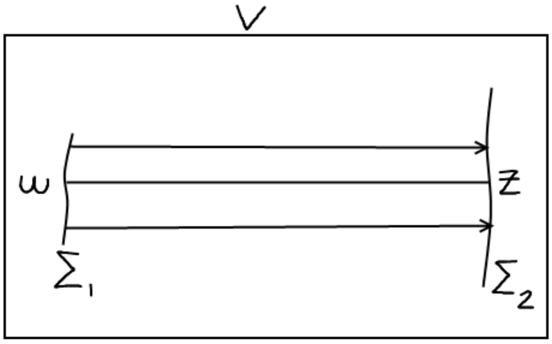

Figure 2.2. The First Intersection Assignment $P_1 : \Sigma_1 \to \Sigma_2$.

For the sake of applications in Chapter 3, we now assume that the dimension of M is 2. Suppose now that Σ_1 is a C^r embedded circle in M and we have $\Sigma_1 = \Sigma_2$. The nice property that Σ_1 possesses is that the C^r embedded circle Σ_1 does not have any (topological) boundary points in the manifold M. Let $P_1 : \Sigma_1 \to \Sigma_1$ be the First Intersection Assignment. Suppose that P_1 is defined at $p \in \Sigma_1$. We have two cases: either $P_1(p) \neq p$ or $P_1(p) = p$.

Assume that we have $P_1(p) \neq p$ and let V be a tubular flow neighborhood of $pP_1(p)$. Since the C^r embedded circle Σ_1 does not have any boundary points, the First Intersection Assignment P_1 is defined on $V \cap \Sigma_1$ and also $P_{1|V\cap\Sigma_1}$ is a diffeomorphism.

Assume that we have $P_1(p) = p$. Then, the orbit through p is a closed orbit τ. Let $z \in \tau - \{p\}$ and consider a transversal section Σ_3 through z such that Σ_1 and Σ_3 are disjoint and we also have $\bar{\Sigma}_3 \cap \tau = \{z\}$ where $\bar{\Sigma}_3$ is the closure of Σ_3. The last required property is possible because τ is a compact set. Let $P_2 : \Sigma_1 \to \Sigma_3$ and $P_3 : \Sigma_3 \to \Sigma_1$ be the First Intersection Assignments. Let V_2 be a tubular flow neighborhood of the closed arc zp such that $V_2 \cap \Sigma_3$ is connected and also, it does not include the boundary points of Σ_3. Let V_1 be a tubular flow neighborhood of pz such that we have $V_1 \cap \Sigma_3 \subseteq V_2 \cap \Sigma_3$. Then, P_1 is defined on $V_1 \cap \Sigma_1$ as it is equal to $P_{3|V_2\cap\Sigma_3} \circ P_{2|V_1\cap\Sigma_1}$. Also, $P_{1|V_1\cap\Sigma_1}$ is a diffeomorphism. See Figure 2.3.

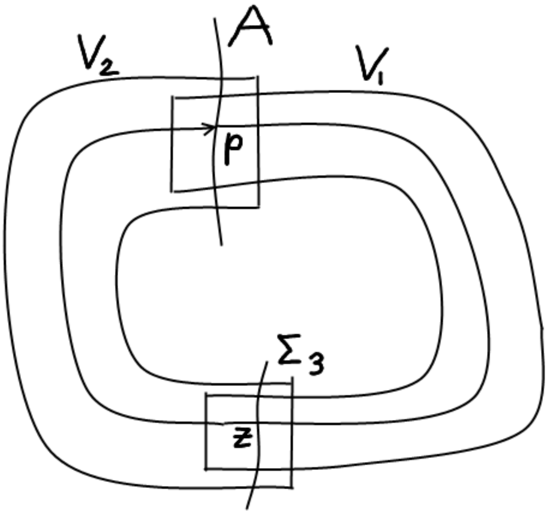

Figure 2.3. A is part of the C^r embedded circle Σ_1.

So, the First Intersection Assignment $P_1 : \Sigma_1 \to \Sigma_1$ is a local diffeomorphism in either case. Because distinct orbits of X cannot intersect each other, the First Intersection Assignment $P_1 : \Sigma_1 \to \Sigma_1$ is an injective local diffeomorphism whenever defined.

2.4.2. Hyperbolic Singularities

We continue our local study by considering a singularity q of X. We first need to exhibit the linear map $DX_q : T_qM \to T_qM$ which is yet to be defined (in contrast to $DX_q : T_qM \to T_{X(q)}TM$). Let (U, ϕ) and (V, φ) be two local charts at q. Let $f : \varphi(U \cap V) \to \phi(U \cap V)$ be the change of coordinates: $f = \phi \circ \varphi^{-1}$. Let $Y = \varphi_*(X_{|U \cap V})$ and $Z = \phi_*(X_{|U \cap V})$. For every x in $\mathbb{R}^n$, the tangent space $T_x\mathbb{R}^n$ can be naturally identified with $\mathbb{R}^n$. So, Y (respectively Z) is then a function from $\varphi(U \cap V)$ (respectively $\phi(U \cap V)$) to $\mathbb{R}^n$. We have $Z(y) = df_{f^{-1}(y)} \circ Y \circ f^{-1}(y)$ for all $y \in \phi(U \cap V)$. Now, both $df_{f^{-1}(y)}$ and $Y \circ f^{-1}(y)$ depend on y and their composition is the product of a matrix with a vector. So, we have $D(Z(y)) = D(df_{f^{-1}(y)}) \cdot (Y \circ f^{-1}(y)) + df_{f^{-1}(y)} \cdot D(Y \circ f^{-1}(y))$. As q is a singularity of X, we have $Y \circ f^{-1}(\phi(q)) = 0$ and $DZ_{\phi(q)} = df_{\varphi(q)} \cdot D(Y \circ f^{-1})_{\phi(q)}$ $= df_{\varphi(q)} \cdot DY_{\varphi(q)} \cdot df^{-1}_{\phi(q)}$. Therefore, DY evaluated at $\varphi(q)$ and DZ evaluated at $\phi(q)$ are similar matrices which represent the same linear transformation in different bases. Without loss of generality, we can assume $\varphi(q) = 0$. Then, we have $DY_0 : T_0\mathbb{R}^n \to T_0\mathbb{R}^n$. So, the map $DX_q : T_qM \to T_qM$ which is induced by the map DY_0 is well defined as it does not depend on the choice of the local chart. Now, we are ready to make the below definition.

Definition 2.4. *Let X be in $\mathfrak{X}^r(M)$ and p be a singularity of X. If all the eigenvalues of the linear transformation $DX_p : T_pM \to T_pM$ have nonzero real parts, then p is called* a hyperbolic singularity of X. The index of a hyperbolic singularity p *is defined to be the number of eigenvalues of DX_p with negative real parts.*

Lemma 2.1. *Suppose that X is in $\mathfrak{X}^r(M)$ and p is a hyperbolic singularity of X. Then, there exists a neighborhood U of p and a neighborhood $\mathcal{V}$ of X such that every Y in $\mathcal{V}$ has a unique singularity p_Y in U which is hyperbolic. Moreover, p_Y and p have the same index.*

The proof of Lemma 2.1 relies on two facts: a singularity p of X with a nonsingular linear map DX_p is an isolated singularity of X and every Y in $\mathcal{V}$ has a unique singularity p_Y in U such that DY_{p_Y} is also nonsingular (such singularities are called *simple singularities*) and; the eigenvalues of DY_{p_Y} depend continuously on $Y \in \mathcal{V}$

(when they are counted with their multiplicities). The proof of the first fact follows from an application of the Implicit Function Theorem for Banach Spaces. Observe that if the singularities of X are all hyperbolic (or simple), then their number is finite as they are isolated and M is compact.

Definition 2.5. *Let f be a C^r diffeomorphism of M and p be a fixed point of f. The fixed point p is called* a hyperbolic fixed point of f *if the absolute values of all eigenvalues of $df_p : T_pM \to T_pM$ are distinct from 1.* The index of a hyperbolic fixed point *is defined to be the number of eigenvalues of df_p the absolute values of which are smaller than 1.*

We remark that the following useful characterization of a singularity p of a C^r vector field X on M holds: p is a hyperbolic singularity of X if and only if p is a hyperbolic fixed point of the diffeomorphism X_1. The below lemma is similar to Lemma 2.1.

Lemma 2.2. *Suppose $f \in \mathrm{Diff}^r(M)$ and $p \in M$ is a hyperbolic fixed point of f. Then, there exist a neighborhood U of p and a neighborhood $\mathcal{V}$ of f such that every g in $\mathcal{V}$ has a unique fixed point p_g in U which is hyperbolic. Moreover, the p and p_g have the same index.*

The local study on a small neighborhood of a singularity of X has also been successfully completed in respect of structural stability. Let $\mathcal{G}_1$ denote the set of vector fields in $\mathfrak{X}^r(M)$ the singularities of which are all hyperbolic. The following important result is part of the success: $\mathcal{G}_1$ is open and dense in $\mathfrak{X}^r(M)$. Its proof involves two steps: first the openness and the density of $\mathcal{G}_0$ in $\mathfrak{X}^r(M)$ is shown where $\mathcal{G}_0$ denotes the set of vector fields in $\mathfrak{X}^r(M)$ the singularities of which are all simple. The second step is to show that $\mathcal{G}_1$ is open and dense in $\mathcal{G}_0$. The fact that $\mathcal{G}_1$ is dense in $\mathfrak{X}^r(M)$ and Grobman-Hartman Theorems in the next section lead together to the following important conclusion: if X is a structurally stable vector field, then all of its singularities are hyperbolic.

2.4.3. Grobman-Hartman Theorems

Not every hyperbolic singularity of X behaves in the same manner (the behavior of orbits in a small neighborhood of them). We will now state two important linearization theorems one of which relate the orbits in a neighborhood of a hyperbolic singularity to the orbits of a linear vector field on $\mathbb{R}^n$ near the origin (where the origin is a singularity of the linear vector field). These theorems have been credited to Grobman and Hartman who have independently found relevant results. See [18-21] for their original work. It should be noted that these linearization theorems locally classify hyperbolic singularities in the topological category in contrast to the differential category.

A linear vector field Y on $\mathbb{R}^n$ is defined as $Y(x) = A_Y \cdot x$ for some linear transformation A_Y from $\mathbb{R}^n$ to itself. A linear vector field Y on $\mathbb{R}^n$ is called *a hyperbolic linear vector field* if all the eigenvalues of A_Y have nonzero real parts. A linear isomorphism B of $\mathbb{R}^n$ is called *a hyperbolic linear isomorphism* if the absolute values of all the eigenvalues of B differ from 1.

Theorem 2.2 (Grobman-Hartman)**.** *Suppose $f \in \mathrm{Diff}^r(\mathbb{R}^n)$ and the point $0 \in \mathbb{R}^n$ is a hyperbolic fixed point of the diffeomorphism f. Then, f at 0 is locally topologically conjugate to the hyperbolic isomorphism df_0 at 0; i.e. there exist a neighborhood U of 0 in $\mathbb{R}^n$, a neighborhood V of 0 in $\mathbb{R}^n$ and a homeomorphism $h : U \to V$ such that we have $h \circ df_{0|W} = f \circ h_{|W}$ for $W = {df_0}^{-1}(U) \cap U$.*

Theorem 2.3 (Grobman-Hartman)**.** *Suppose that a point p in M is a hyperbolic singularity of X. Define $Y(v) := DX_p(v)$ for all $v \in T_pM$ which is a hyperbolic linear vector field on T_pM. Then, X and Y are topologically equivalent at p and 0 respectively; i.e. there exist a neighborhood U of p in M, a neighborhood V of 0 in T_pM and a homeomorphism $h : U \to V$ such that $h(p)$ is equal to 0 and h maps the orbits of X in U to the orbits of Y in V preserving their orientations.*

Definition 2.6 (Grobman-Hartman Neighborhood)**.** *Let p be a hyperbolic singularity of X. Let (V, φ) be a local chart at p with $\varphi(p) = 0$; let $A = D(\varphi_*(X_{|V}))(\varphi(p))$ and; let $Y(x) = A \cdot x$ for $x \in \mathbb{R}^n$ which is a hyperbolic linear vector field on $\mathbb{R}^n$. An open* *neighborhood U of p is said to be* a Grobman-Hartman neighborhood of p *if there exist an open ball $B(a,0)$ of some radius a at 0 and a homeomorphism $h : U \to B(a,0)$ such that $h(p)$ is equal to 0 and h maps the orbits of X in U to the orbits of Y in $B(a,0)$ preserving their orientations.*

2.4.4. Poincaré Return Map and Hyperbolic Closed Orbits

Suppose now that the orbit τ through p is a closed orbit of X and let Σ be a transversal section to X through p (recall that p is an interior point of Σ). Since τ is compact, $\tau \cap \Sigma$ is a finite set so that we can assume (by taking a smaller Σ) the equality $\bar{\Sigma} \cap \tau = \{p\}$ where $\bar{\Sigma}$ is the closure of Σ. Let q be a point in $\tau - \{p\}$ and Σ_q be a transversal section through q that is disjoint from Σ and also, we have $\bar{\Sigma}_q \cap \tau = \{q\}$. By considering the tubular flow neighborhoods of pq and qp, we can conclude that the map $P : \Sigma \to \Sigma$ is defined in a small neighborhood U of p in Σ where $P(z)$ (for z in U) denotes the first intersection of z^+ with Σ. For historical reasons, P is called the *Poincaré Return Map* in the literature although this notion is unfortunately misleading because P is not a map since it may be not defined on the whole Σ. So, for any $z \in \Sigma$, the point $P(z) \in \Sigma$ is defined to be the first intersection of z^+ with Σ if such an intersection exists and otherwise, $P(z)$ is not defined. Note that Poincaré Return Map is a First Intersection Assignment that we have introduced earlier. One important distinction between them is that Poincaré Return Map is always injective because the domain of P and the range of P are equal to each other. We remark that the eigenvalues of $DP_p : T_p\Sigma \to T_p\Sigma$ does not depend on the choice of the transversal section Σ through p (see e.g. [3]).

Definition 2.7. *Let τ be a closed orbit of X in $\mathfrak{X}^r(M)$; let Σ be a transversal section through a point p of τ such that we have $\bar{\Sigma} \cap \tau = \{p\}$. Let $P : \Sigma \to \Sigma$ be the Poincaré Return Map. If the point p is a hyperbolic fixed point of DP_p, then we say that τ is* a hyperbolic closed orbit of X. The index of a hyperbolic closed orbit τ *is defined to be the number of eigenvalues of DP_p with absolute values smaller than 1.*

Using the Poincaré Return Map, we can prove the following lemma (see e.g. [3]).

Lemma 2.3. *Suppose that τ is a hyperbolic closed orbit of X. Then, there exist a neighborhood U of τ and a neighborhood $\mathcal{V}$ of X such that every Y in $\mathcal{V}$ has a unique closed orbit τ_Y in U which is hyperbolic. Moreover, τ and τ_Y have the same index.*

2.5. Stable and Unstable Manifolds

A critical element of X is defined to be either a singularity of X or a closed orbit of X. If all the eigenvalues in the definition of a critical element σ have negative real parts, then σ is called *a hyperbolic attractor.* If all the eigenvalues in the definition of a critical element σ have positive real parts, then σ is called *a hyperbolic repellor.*

The ongoing discussions in the previous sections show that there exist neighborhoods U_σ and $\mathcal{V}_\sigma$ of σ and X respectively such that every Y in $\mathcal{V}_\sigma$ has a unique critical element in U_σ which is hyperbolic. Nevertheless, the situation for hyperbolic singularities and the situation for hyperbolic closed orbits are different. Suppose that the singularities of X are all hyperbolic and let $\sigma_1, \ldots, \sigma_n$ be the singularities of X. Let U_j be an open neighborhood of σ_j (for $1 \leq j \leq n$) and $\mathcal{V}_j$ be a neighborhood of X such that every Y in $\mathcal{V}_j$ has a unique singularity in U_j which is hyperbolic. Let $A = M - \bigcup_{1 \leq j \leq n} U_j$. Note that A is compact (even if it is the empty set) the points of which are all regular points so that there exists a neighborhood $\mathcal{U}$ of X such that no Y in $\mathcal{U}$ has a singularity in A. So, every Y in $\mathcal{U} \cap \mathcal{V}_1 \cap \cdots \cap \mathcal{V}_n$ has exactly n singularities all of which are hyperbolic. We do not have an analogous conclusion for hyperbolic closed orbits. The best we can do is as follows: if X has n closed orbits all of which are hyperbolic, then there exist a neighborhood $\mathcal{V}_0$ of X such that every Y in $\mathcal{V}_0$ has at least n hyperbolic closed orbits.

The stable and unstable manifolds can be defined, however, both for the hyperbolic singularities and the hyperbolic closed orbits of X. For a hyperbolic linear vector field Y on $\mathbb{R}^n$, there exists a unique decomposition $\mathbb{R}^n = E^s \oplus E^u$ such that for every x in E^s, we have $\omega(x) = 0$ and for every x in E^u, we have $\alpha(x) = 0$ (see e.g. [3]). The Grobman-Hartman Theorem says that orbits in a small neighborhood of a hyperbolic singularity p of $X \in \mathfrak{X}^r(M)$ behave like the orbits of the hyperbolic linear vector field DX_p on $T_pM \simeq \mathbb{R}^n$ which are in a small neighborhood of the origin. So, for some points close to p in M, we conclude that either their positive semi trajectories or their negative semi trajectories go to p.

Definition 2.8. *Let σ be a hyperbolic critical element of X.* The stable manifold of σ *is defined to be the set $W^s(\sigma) = \{q \in M : \omega(q) = \sigma\}$ and similarly,* the unstable manifold of σ *is defined to be the set $W^u(\sigma) = \{q \in M : \alpha(q) = \sigma\}$. For an open neighborhood V of σ, we define the set $W^s_V(\sigma)$ to be $W^s_V(\sigma) = \{q \in M : X_t(q) \in V$ for all $t \geq 0\}$. Similarly, we define the set $W^u_V(\sigma)$ to be $W^u_V(\sigma) = \{q \in M : X_t(q) \in V$ for all $t \leq 0\}$.*

The above definition suggests that $W^s(\sigma)$ and $W^u(\sigma)$ are submanifolds of M which, of course, requires proof. We now need some additional notion which is to be used only in the proof of Theorem 3.1. Let f be a diffeomorphism of M and p be a point of M. We define $\omega_d(p)$ as follows: a point q of M is in $\omega_d(p)$ if there exist a sequence of positive integers $\{j_n\}_{n\in\mathbb{N}}$ such that we have $j_n \to \infty$ as $n \to \infty$ and also $f^{j_n}(p) \to q$. Similarly, we define $\alpha_d(p)$ as follows: a point q of M is in $\alpha_d(p)$ if there exist a sequence of negative integers $\{j_n\}_{n\in\mathbb{N}}$ such that we have $j_n \to -\infty$ as $n \to \infty$ and also $f^{j_n}(p) \to q$.

For a hyperbolic fixed point z of f, define $W^s_d(z)$ to be $W^s_d(z) = \{q \in M : \omega_d(q) = z\}$. Similarly, define $W^u_d(z)$ to be $W^u_d(z) = \{q \in M : \alpha_d(q) = z\}$. For an open neighborhood U of z, define $W^s_{d,U}(z)$ to be $W^s_{d,U}(z) = \{q \in M : f^n(q) \in U$ for all $n \geq 0\}$. Similarly, define $W^u_{d,U}(z)$ to be $W^u_{d,U}(z) = \{q \in M : f^n(q) \in U$ for all $n \leq 0\}$. It can be shown that $W^u_{d,U}$ and $W^s_{d,U}$ are C^r embedded submanifolds of M if U is small enough and the sum of their dimensions is equal to n. Moreover, for a small enough U, we have $W^u_{d,U} \subseteq W^u_d$ and $W^s_{d,U} \subseteq W^s_d$. Also, $W^s_d(z)$ and $W^u_d(z)$ are C^r injectively immersed submanifolds of M.

We say that two diffeomorphic submanifolds N_1 and N_2 of M are ϵ-close if there exists a diffeomorphism $h : N_1 \to N_2$ such that $\mathfrak{i}_1$ is ϵ-close to $\mathfrak{i}_2 \circ h$ in the topological space $C^r(N_1, M)$ where $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are the inclusion functions of N_1 and N_2 into M

respectively. We are now ready to state the necessary theorem below (see [3] for a proof).

Theorem 2.4 (The Stable Manifold Theorem)**.** *Let* $f \in \mathrm{Diff}^r(M)$ *and* p *be a hyperbolic fixed point of* f*. Then, we have:*

a) $W^s_d(p)$ *is a* C^r *injectively submanifold of* M*;*

b) Let U *be an open neighborhood of* p *such that* $W^s_{d,U}(p)$ *is a* C^r *embedded submanifold of* M *which is diffeomorphic to an open ball. Let* V *and* $\mathcal{N}$ *be neighborhoods of* p *and* f *respectively such that each* g *in* $\mathcal{N}$ *has a unique fixed point* p_g *in* V *which is hyperbolic and also, the hyperbolic fixed points* p *and* p_g *have the same index. Then, there exist a neighborhood* $\tilde{\mathcal{N}} \subseteq \mathcal{N}$ *of* f *with the following property: for each* g *in* $\tilde{\mathcal{N}}$*, there exist an open neighborhood* U_g *of* p_g *such that* $W^s_{d,U_g}(p_g)$ *is a* C^r *embedded submanifold of* M *which is diffeomorphic to an open ball and also* $W^s_{d,U_g}(p_g)$ *is* ϵ*-close to* $W^s_{d,U}(p)$*.*

Note that the Stable Manifold Theorem can be analogously stated for the unstable manifold $W^u_d(p)$. Analogously suppose that σ is a hyperbolic critical element of a C^r vector field X on M. It can be shown that both $W^s(\sigma)$ and $W^u(\sigma)$ are C^r immersed submanifolds. Also, there exists a small open neigbhorhood V of σ such that we have $W^u_V(\sigma) \subseteq W^u(\sigma)$ and $W^s_V(\sigma) \subseteq W^s(\sigma)$ and both $W^u_V(\sigma)$ and $W^s_V(\sigma)$ are C^r embedded submanifolds of M. If σ is a hyperbolic singularity and U is a Grobman-Hartman neighborhood of σ, then $W^u_U(\sigma)$ or $W^s_U(\sigma)$ is not necessarily a C^r embedded submanifold of M because the homeomorphism h in the Grobman-Hartman Theorem is not necessarily a C^r diffeomorphism. Nevertheless, the properties $W^u_U(\sigma) \subseteq W^u(\sigma)$ and $W^s_U(\sigma) \subseteq W^s(\sigma)$ hold for a Grobman-Hartman neighborhood U of σ.

2.6. Kupka-Smale and Morse-Smale Fields

Definition 2.9. *A* C^r *vector field* X *on* M *is called* a Kupka-Smale *or shortly* K-S field *if it satisfies all the following conditions:*

(i) All critical elements of X are hyperbolic;

(ii) For any two (not necessarily distinct) critical elements σ_1 and σ_2 of X, we have $W^s(\sigma_1) \pitchfork W^u(\sigma_2)$, i.e. $W^s(\sigma_1)$ is transversal to $W^u(\sigma_2)$.

The Kupka-Smale Theorem (see [5] and [6]) states that the K-S fields are dense in $\mathfrak{X}^r(M)$. In other words, given $X \in \mathfrak{X}^r(M)$ and a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^r(M)$, there exists a Kupka-Smale field Y in $\mathcal{U}$. A classical example for a K-S field is the irrational flow X_0 on the torus (see [3]). It is trivially a K-S field since it does not have any singularities or closed orbits. For any neighborhood $\mathcal{U}_0$ of X_0, one can find a smooth vector field Y in $\mathcal{U}$ such that all orbits of Y are closed orbits (Y is actually the rational flow on the torus here). So, a K-S field on M is not necessarily structurally stable. On the other hand, it has been show that the Morse-Smale fields that are K-S fields with an extra condition are structurally stable.

Definition 2.10. *Let $X \in \mathfrak{X}^r(M)$ and $p \in M$. If there exist a neighborhood V of p and $t_0 > 0$ such that $X_t(V) \cap V = \varnothing$ for all $|t| > t_0$, then p is said to be* a wandering point *for X. Otherwise, it is called* a nonwandering point for X. *The set of all nonwandering points of X will be denoted by $\Omega(X)$.*

Observe that all critical elements of X is a subset of $\Omega(X)$.

Definition 2.11. *A C^r vector field X on M is called* a Morse-Smale *or shortly* M-S field *if it satisfies all the following conditions:*

(i) All critical elements of X are hyperbolic;

(ii) For any two (not necessarily distinct) critical elements σ_1 and σ_2 of X, we have $W^s(\sigma_1) \pitchfork W^u(\sigma_2)$;

(iii) The critical elements of X constitute the nonwandering set $\Omega(X)$.

3. DENSITY OF MORSE-SMALE FIELDS FOR ORIENTED 2-MANIFOLDS

It is known that Morse-Smale fields are not dense in $\mathfrak{X}^r(M)$ for dimensions greater than 2. However, M-S fields are dense in $\mathfrak{X}^r(M)$ for orientable, compact and connected 2-manifolds without boundary. The proof we will present is due to M. Peixoto [4] but we will again follow the discussion of it in [3].

The dimension of M will always be 2 unless it is stated otherwise. For this specific dimension, we will adopt some additional terminology. Suppose that C is a simple closed continuous curve in M. If there exists a simply connected open neighborhood U of C such that U is homeomorphic to an open cylinder and also C separates U into two disjoint open cylinders, then we say that *C is two sided.* If there exists a simply connected open neighborhood U of C such that U is homeomorphic to an open Möbius band and C is the 0-section of U, then we say that *C is one sided.* In either case, such a neighborhood U of C will be called *a circle neighborhood U of C.*

A closed transversal section to X means a transversal section to X that is diffeomorphic to $[0,1]$. An open transversal section to X means a transversal section to X that is diffeomorphic to $(0,1)$. Note that a transversal section to X is either a C^r embedded interval or a C^r embedded circle in M. Let Σ be a transversal section to X and let p_1 and p_2 be two distinct points in Σ. By *a closed interval in Σ between p_1 and p_2*, we mean a closed connected subset of Σ with boundary points p_1 and p_2. By *an open interval in Σ between p_1 and p_2*, we mean an open connected subset of Σ with boundary points p_1 and p_2.

For a hyperbolic singularity p of a C^r vector field X on M, DX_p has two eigenvalues λ_1 and λ_2. If the real parts of both λ_1 and λ_2 are positive, then p is called *a source*; if they are both negative, then p is called *a sink* and; if one of them is positive and the other one is negative, then p is called *a saddle.* Consider a saddle σ of X. For a point q in $W^s(\sigma)$ or $W^u(\sigma)$, the orbit through q of X is contained in either of them.

As they are both C^r immersed submanifolds of dimension 1, σ is an interior point of each of them and each of $W^s(\sigma)-\sigma$ and $W^u(\sigma)-\sigma$ contains two distinct orbits of X. Each of the two distinct orbits in $W^s(\sigma)-\sigma$ is called *a stable separatrix of* σ and each of the two distinct orbits in $W^u(\sigma)-\sigma$ is called *an unstable separatrix of* σ. Figure 3.1 illustrates some local pictures of rather simple hyperbolic singularities as they refer to eigenvalues λ_1 and λ_2 which have zero imaginary parts and also their absolute values $|\lambda_1|$ and $|\lambda_2|$ are equal to each other. *A saddle connection* is an orbit γ of X such that both $\omega(\gamma)$ and $\alpha(\gamma)$ are saddles.

Consider a C^r vector field X on M such that all critical elements of X are hyperbolic. Then, X is a K-S field if and only if X has no saddle connections.

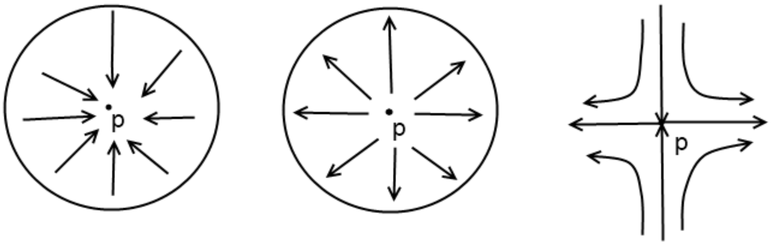

Figure 3.1. Some possible Grobman-Hartman neighborhoods of a sink, a source and a saddle.

3.1. Nontrivial Recurrent Orbit

Let $X \in \mathfrak{X}^r(M)$ and γ be an orbit of X. If $\gamma \subseteq \omega(\gamma)$, or $\gamma \subseteq \alpha(\gamma)$, then γ is called *a recurrent orbit.* If we have $\gamma \subseteq \omega(\gamma)$, then γ is called *a forward recurrent orbit.* If we have $\gamma \subseteq \alpha(\gamma)$, then it is called *a backward recurrent orbit.*

Note that all critical elements of X are recurrent. Since there might exist other recurrent orbits, a critical element is called a *trivial recurrent* orbit and a recurrent orbit that is not critical is called a *nontrivial recurrent* orbit. If Y is an M-S field, then Y does not have any nontrivial recurrent orbits because the existence of them would have violated the fact that $\Omega(Y)$ is equal to the union of critical elements of Y.

Lemma 3.1. *Suppose that a 2-manifold M (not necessarily compact) can be embedded* *into* S^2*. Then, for any* C^r *vector field* X *on* M*,* X *does not have a nontrivial recurrent orbit.*

Proof. Note that if X has finitely many hyperbolic singularities, then the lemma follows from the Poincaré-Bendixson Theorem (see [3]). We prove the general case. Assume γ is a nontrivial recurrent orbit of X. We will consider the case that γ is nontrivial forward recurrent as the other case is analogous.

Let p be a point of γ and Σ be a transversal section to X through p. Note that M is not necessarily compact and there may not exist a global flow for X. Nevertheless, the assumption that γ is nontrivial forward recurrent requires that p^+ is defined. As γ is nontrivial forward recurrent, p^+ intersects Σ at infinitely many points. Let p_1 be its first intersection. Note that $p_1 \neq p^+$ since γ is not a closed orbit. Let B be the connected interval in Σ with boundary points p_1 and p. Then, $pp_1 \cup B$ is a simple closed curve which bounds an open disc A in S^2 by the Jordan Curve Theorem (see [22]). As B is transversal to X and p^+ cannot intersect itself, either p_1^+ does not enter the region A or p_1^+ is trapped in the region A (see Figure 3.2). In either case, it cannot accumulate to p which contradicts that γ is nontrivial forward recurrent. Hence, γ does not exist. □

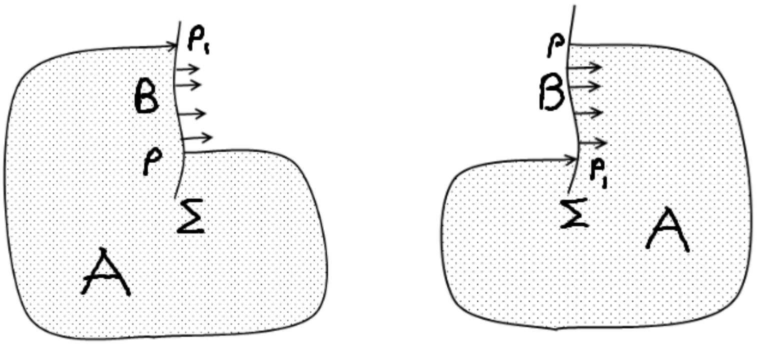

Figure 3.2. Two possible cases for the enclosed region A.

3.2. Lemmas About a Transversal Circle

We follow the proofs in [3] for the following lemmas.

Lemma 3.2. *Suppose that X has a nontrivial recurrent orbit γ. Then, there exists a simple closed C^r curve C^* on M through a point p in γ and transversal to X.*

Proof. We will consider the case $\gamma \subseteq \omega(\gamma)$ as the other case is analogous. Let $\varphi(t,q) = X_t(q)$ be the flow of X and $p \in \gamma$. Since γ is nontrivial recurrent, p is regular. Let (f, F) be a flow box centered at p. Say, $f(F) = [-a,a] \times [-1,1]$. Since γ is nontrivial forward recurrent, p^+ will leave F and return to it infinitely many times. Let p_j be its j^{th} leave of F at $f^{-1}(\{1\} \times [-1,1])$ and q_j be its j^{th} return to F at $f^{-1}(\{-1\} \times [-1,1])$. For $n = 1, 2$, let $\Pi_n : \mathbb{R}^2 \to \mathbb{R}$ be the projection map onto the n^{th} coordinate. Let V be a tubular flow neighborhood of p_1q_1 and $g : V \to A \subseteq \mathbb{R}^2$ be the C^r function such that $g_*(X_{|V})$ is the parallel unit vector field in the direction of e_1.

Let L be the straight line in $f(F)$ between $f(p_1)$ and $f(q_1)$. If the simple closed curve $p_1q_1 \cup f^{-1}(L)$ is two sided, then we can define the desired curve in the following way. Assume $\Pi_2(f(q_1)) > 0$. Let $w_1 \in V \cap F$ such that $\Pi_1(f(w_1)) = -1$ and $0 < \Pi_2(f(w_1)) < \Pi_2(f(q_1))$. Let s_1 be the straight line in $g(V)$ between $g(p_1)$ and $g(w_1)$; and $l_1 = {f_1}^{-1}(s_1)$. Let s be the straight line in $f(F)$ between $f(w_1)$ and $f(p_1)$; and $l = f^{-1}(s)$. Let $C^* = l \cup l_1$. Then, C^* is a simple closed continuous curve that is C^r except at p_1 and w_1 and also transversal to X except at these two points by the Tubular Flow Theorem (See Figure 3.3). We can redefine C^* in sufficiently small neighborhoods of these two points to make C^* a C^r simple closed curve and transversal to X. These two processes can be done with Lemma 5.1 at p_1 and w_1. If $\Pi_2(f(q_1)) < 0$, then we take w_1 in $V \cap F$ with $\Pi_1(f(w_1)) = -1$ and $0 > \Pi_2(f(w_1)) > \Pi_2(f(q_1))$ and define C^* similarly.

If $p_1q_1 \cup f^{-1}(L)$ is a one sided simple closed continuous curve and we define C^* as in the previous way, then the transversality condition will fail at w_1 or another point in its neighborhood. So, we will define C^* using different returning arcs. Let $I^a = f^{-1}(\,\{-1\} \times (0,1]\,)$ and $I^b = f^{-1}(\,\{-1\} \times [-1,0)\,)$. Let I_0 be one of them such that $f^{-1}((-1,0))$ is a limit point of $p^+ \cap I_0$. Let m, n be the smallest positive integers such that we have: $m < n$; $q_m, q_n \in I_0$ and; $\Pi_2 \circ f(q_n)$ is between 0 and $\Pi_2 \circ f(q_m)$.

Let L_1 be the straight line in $f(F)$ between q_m and p_1; let L_2 be the straight line in $f(F)$ between q_n and p_{m+1} and; let L_3 be the straight line in $f(F)$ between q_n and p_1 . By our choice of m and n, all $p_1q_m \cup f^{-1}(L_1)$, $p_{m+1}q_n \cup f^{-1}(L_2)$ and $p_1q_n \cup f^{-1}(L_3)$ are simple closed curves. If either one of $p_1q_m \cup f^{-1}(L_1)$ and $p_{m+1}q_n \cup f^{-1}(L_2)$ is two sided, then we can define C^* in the above way by taking a tubular flow neighborhood of p_1q_m or $p_{m+1}q_n$. If both of them are one sided, then $p_1q_n \cup f^{-1}(L_3)$ is two sided so that we can again define C^* as before. Observe in all the possible cases that when we define l and l_1 as before, $l \cap l_1$ contains only the two common boundary points of l and l_1 by our choice of m and n. Therefore, $l \cup l_1$ is a simple closed continuous curve and a C^r simple closed curve C^* transversal to X can be obtained from this union after two small perturbations at p_1 and q_n. □

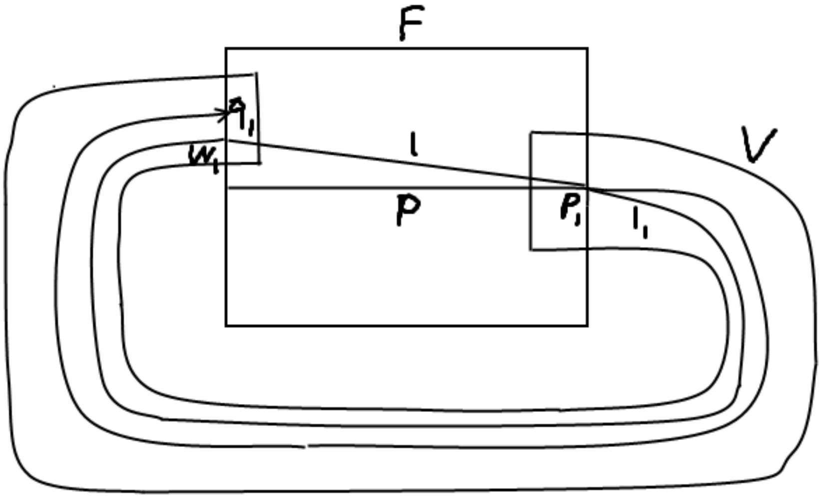

Figure 3.3. The case when $p_1q_1 \cup f^{-1}(L)$ is two sided.

Definition 3.1. *The simple closed C^r curve C^* in Lemma 3.2 will be called* a transversal circle (to X through p^*).

Lemma 3.3. *Suppose that all singularities of X are hyperbolic and C^* is a transversal circle. Then, the domain E of the Poincaré Return Map $P : C^* \to C^*$ is an open subset of C^*. If E is not equal to C^* and q is a boundary point of C^*, then $\omega(q)$ is a saddle.*

Proof. The set E is open in C^* because it is the domain of the Poincaré Return Map and also because the transversal section C^* does not have any boundary points in M.

Assume that E is not equal to C^* and q is a boundary point of E. As E is open, q is not in E so that P is not defined at q. Let I be a connected component (i.e. the maximal open interval) of E such that $q \in \bar{I}$. Assume $\omega(q)$ contains a regular point z_0. Let Σ be an open transversal section through z_0. Without loss of generality, we can assume that z_0 is not in C^* and also Σ and C^* are disjoint.

By the Tubular Flow Theorem, the positive semi trajectory of every point in a connected open neighborhood U of q intersects Σ. For any z in I, either $P(z)$ is unequal to z and the closed arc $zP(z)$ is defined or they are equal then the orbit through z is a closed orbit τ_z. Let A_z be the one of them which is defined. Because Σ can have at most two boundary points, there can exist at most two points p_1 and p_2 in I such that A_{p_1} and A_{p_2} include a boundary point of Σ. Let $\tilde{I}$ be the connected component of $I - \{p_1, p_2\}$ such that q is a boundary point of $\tilde{I}$ (if such points p_1 and p_2 exist at all).

Let w be in $\tilde{I}$. As A_w is compact, $A_w \cap \Sigma$ is a finite set. Let $m = |A_w \cap \Sigma|$. If A_w is equal to $wP(w)$, then we can consider a tubular flow neighborhood of it. If A_w is equal to τ_w, then we can take a point q_w from $\tau_w - w$ and consider the tubular flow neighborhoods of wq_w and $q_w w$. In either case, we can find an open neighborhood U_w of w in C^* such that for every z in U_w, we have $m = |A_z \cap \Sigma|$ because of all the following reasons: Σ is an open transversal section; A_z is disjoint from the boundary of Σ for all z in $\tilde{I}$ and; Σ and C^* are disjoint. Hence, the set $I_m := \{z \in \tilde{I} : \ |A_z \cap \Sigma| = m\}$ is open in $\tilde{I}$. By considering tubular flow neighborhoods of relevant closed arcs once again, we can deduce that I_m is closed in $\tilde{I}$. As $\tilde{I}$ is connected and I_m is a nonempty, closed and open set in $\tilde{I}$, we conclude $I_m = \tilde{I}$.

As z_0 is in $\omega(q)$, the positive semi trajectory of q intersects Σ infinitely many times. So, there exists a neighborhood V of q in C^* such that the positive semi trajectory of every point in V intersects Σ at least $m + 1$ times (again by considering relevant tubular flow neighborhoods) which contradicts $I_m \cap V \neq \varnothing$. Therefore, z_0 does not exist and $\omega(q)$ consists of only singularities.

As $\omega(q)$ is connected, it is a single singularity z. Clearly, z is not a source. It cannot be a sink either because otherwise an open neighborhood of q will go to this sink without returning to C^*. Hence, z is a saddle. □

Remark 3.1. Lemma 3.3 is a special case of Tubular Flow Extension which will be introduced in Chapter 5.

Lemma 3.4. *Suppose that* $X \in \mathfrak{X}^r(M)$*;* C^* *is a transversal circle and; the Poincaré Return Map* $P : C^* \to C^*$ *is defined on the whole* C^**. Then,* X *has no singularities.*

Proof. For any q in C^*, either $P(q)$ is not equal to q and the closed arc $qP(q)$ is defined or $P(q)$ is equal to q is equal to q and the orbit τ_q through q is a closed orbit. Let A_q denote one of them which is defined. Let $B = \{p \in M : p \in A_q, q \in C^*\}$. The Poincaré Return Map P is injective. Because the transversal section C^* does not have any boundary points in M, the Poincaré Return Map P is a local diffeomorphism. Now, the only subset of C^* which is diffeomorphic to it is C^* itself. Therefore, we have $P(C^*) = C^*$ and P is a diffeomorphism. Hence, $B - C^*$ is diffeomorphic to an open cylinder and B is diffeomorphic to either a Klein Bottle or a torus. As M is connected, B is equal to M so that X has no singularities in $B = M$. □

Lemma 3.5. *Suppose that:* X *has at least one singularity; all singularities of* X *are hyperbolic and;* γ *is a nontrivial recurrent orbit of* X*. Then, there exists a saddle, an unstable separatrix of which accumulates to* γ *forward and; there exists a saddle, a stable separatrix of which accumulates to* γ *backward.*

Proof. We will consider the case $\gamma \subseteq \omega(\gamma)$ as the other case is analogous. Let C^* be a transversal circle through a point p^* of γ by Lemma 3.2. Let $P : C^* \to C^*$ be the Poincaré Return Map. Because the transversal section C^* does not have any boundary points in M, the Poincaré Return Map P is a local diffeomorphism whenever defined. Let E be the domain of P. Because X has singularities, E is not equal to C^* by Lemma 3.4. Clearly, we have $p^* \in E$. As E is open (Lemma 3.3), let I be the connected component of E that contains p^*.

Assume that there does not exist a saddle with a stable separatrix which accumulates to p^* backward. Then, there exists a connected open neighborhood U of p^* in C^* such that it contains no points belonging to a stable separatrix of a saddle. Let $\tilde{U}$ be such a maximal connected open neighborhood of p^* in C^*.

By Lemma 3.3, $\tilde{U} \subseteq I$. For $1 \leq j$, assume $P^j(\tilde{U}) = \tilde{U}_j$ is defined. We claim that $\tilde{U}_j \subseteq E$ so that $P^{j+1}(\tilde{U})$ will be defined. Since P^j is a local diffeomorphism (whenever defined), $\tilde{U}_j$ is a connected open subset of C^*. Clearly, $P^j(p^*) = p_j$ is in E. Let I_j be the connected component of E containing p_j. If $\tilde{U}_j \nsubseteq I_j$, then $\tilde{U}_j$ contains a boundary point w of I_j which goes to a saddle by Lemma 3.3 and this contradicts the definition of $\tilde{U}$ because $P^{-j}(w) \in \tilde{U}$. Hence, $\tilde{U}_j \subseteq I_j \subseteq E$ as claimed.

As p^{*+} is nontrivial forward recurrent and C^* is transversal to X, it intersects C^* at infinitely many distinct points. Let $p_0 = p^*$ and p_j be its j^{th} return to C^*. Again by the nontrivial recurrence, let m be the smallest positive integer such that $p_m \in \tilde{U}$. By the above argument, $P^m(\tilde{U}) = \tilde{U}_m$ is defined and $\tilde{U}_m \subseteq I_m$. Moreover, $\tilde{U}_m \subseteq \tilde{U}$ by the same reasoning of $\tilde{U}_m \subseteq I_m$. Note that p_n is not in $\tilde{U}$ for $1 \leq n < m$ by the definition of m. Hence, $P^n(\tilde{U})$ is not subset of $\tilde{U}$. Therefore, $P^n(\tilde{U}) \cap \tilde{U}$ is the empty set because otherwise $P^n(\tilde{U})$ would have contained a boundary point w of $\tilde{U}$ which goes to a saddle and $P^{-n}(w)$ would have been in $\tilde{U}$. Hence, only $(j \cdot m)^{th}$ return of p^{*+} is in $\tilde{U}$ for any positive integer j. As P^m is a local diffeomorphism and $P^m(\tilde{U}) \subseteq \tilde{U}$, we conclude that p^{*+} can accumulate to at most two points in $\tilde{U}$ which contradicts that it is a nontrivial forward recurrent orbit (semi trajectory). Hence, $\tilde{U}$ does not exist and the existence of a backward accumulating stable separatrix is proven.

Assume that there does not exist a saddle with an unstable separatrix which accumulates to p^* forward. As before, let V be the maximal connected open neighborhood of p^* in C^* such that V contains no points belonging to an unstable separatrix of a saddle. Let $P^{-1} : C^* \to C^*$ be the Poincaré Return Map for the flow $-X$. Let n be the smallest integer such that $p_n \in V$. By applying the Tubular Flow Theorem to p^*p_n, P^{-n} is defined on a neighborhood of p_n in C^*.

To look for an unstable separatrix, we consider $-X$ on M and essentially repeat the argument for the stable separatrix case. The saddles of X are also saddles of $-X$ with the mere interchange of stable and unstable separatrices. Now, apply Lemma 3.3 to $-X$. Let $\hat{E}$ be the domain of $P^{-1} : C^* \to C^*$. Clearly, $p_n \in \hat{E}$. Let T be the connected component of $\hat{E}$ containing p_n. Then, $V \subseteq T$ by Lemma 3.3. Assume $P^{-j}(V) = V_j$ is defined for $0 \leq j$. We will show that $V_j \subseteq \hat{E}$ so that $P^{-j-1}(V)$ is defined. As P^{-1} is a local diffeomorphism (whenever defined), P^{-j} is a local diffeomorphism (whenever defined) and V_j is a connected open subset of C^*. If V_j is not a subset of $\hat{E}$, then V_j contains a boundary point w of $\hat{E}$ which goes to a saddle (for $-X$) so that $P^j(w)$ in V goes to a saddle which contradicts the definition of V. Hence, $V_j \subseteq \hat{E}$ as claimed. We have $V_n \cap V \neq \varnothing$ and by the same reasoning of $V_n \subseteq \hat{E}$, we conclude $V_n \subseteq V$. Also, V_j is not a subset of V for $1 \leq j < n$ by the definition of n. Hence, $V_j \cap V$ is the empty set because otherwise it would have contained a boundary point of V which goes to saddle. Therefore, only $(j \cdot n)^{th}$ return of p^{*+} can be in V_n for a possible positive integer j. As we have $P^n(V_n) = V \supseteq V_n$, p^{*+} can accumulate to at most two points in V_n which contradicts that p^{*+} is a nontrivial forward recurrent orbit (semi trajectory). Hence, V does not exist and the existence of a backward accumulating stable separatrix is also proven. This completes the proof. □

Lemma 3.6 (Connecting Lemma)**.** *Suppose that M is orientable; the C^r vector field $X \in \mathfrak{X}^r(M)$ has at least one singularity; all singularities of X are hyperbolic and; the orbit γ is a nontrivial recurrent orbit of X. Then, for all $\epsilon > 0$, $p^* \in \gamma$ and a neighborhood U of p^*, there exists some $Y \in \mathfrak{X}^r(M)$ such that:*

(i) $\|X - Y\|_r < \epsilon$;

(ii) X is equal to Y on U^c;

(iii) Y has one more saddle connection than X has.

Proof. Let $\tilde{F}$ be a neighborhood of p^* such that we have $\tilde{F} \subseteq U$ and $\tilde{F}$ contains no saddle connections. This last requirement is possible since the closure of saddle connections is a compact subset of M that is disjoint from $\{p^*\}$. Let C^* be a transversal circle through a point p_1^* in $\gamma \cap \tilde{F}$ by Lemma 3.2. Consider a point q in C^*. Because C^* is transversal to X at q, there exists an open neighborhood V_q of q in C^* and a $\delta_q > 0$ such that both $X_{-\delta_q}(C^* \cap V_q)$ and $X_{\delta_q}(C^* \cap V_q)$ are transversal to X and all $X_{-\delta_q}(C^* \cap V_q)$, $X_{\delta_q}(C^* \cap V_q)$ and C^* are pairwise disjoint. As C^* is compact, there exists a $\delta_0 > 0$ such that $C_1^* = X_{-\delta_0}(C^*)$ and $C_2^* = X_{\delta_0}(C^*)$ are transversal circles and all C_1^*, C_2^* and C^* are pairwise disjoint. Let $v_1 = X_{-\delta_0}(p_1^*)$ and $v_2 = X_{\delta_0}(p_1^*)$. Let V^* be a tubular flow neighborhood of $v_1 v_2$.

By Lemma 3.5, there exist a stable separatrix β^s of some saddle η_s of X and a unstable separatrix β^u of some saddle η_u of X that accumulate to p^* backward and forward respectively. Let U_{η_s} be a Grobman-Hartman neighborhood of η_s and let $w_{\eta_s} \in U_{\eta_s} \cap \beta^s$ be such that we have ${w_{\eta_s}}^+ \subseteq U_{\eta_s}$ and ${w_{\eta_s}}^+ \cap (C_1^* \cup C_2^*) = \varnothing$. Similarly, let U_{η_u} be a Grobman-Hartman neighborhood of η_u and let $w_{\eta_u} \in U_{\eta_u} \cap \beta^u$ be such that we have ${w_{\eta_u}}^- \subseteq U_{\eta_u}$ and ${w_{\eta_u}}^- \cap (C_1^* \cup C_2^*) = \varnothing$.

Let w_s be the first intersection of ${w_{\eta_s}}^-$ with C_2^* and similarly, let w_u be the first intersection of ${w_{\eta_u}}^+$ with C_1^*. The closures of w_s^+ and w_u^- are compact sets that are disjoint from $\{p^*\}$. Therefore, that there exists a flow box (f, F) at p_1^* such that we have: $F \subseteq (\tilde{F} \cap V^*)$; $F \cap (C_1^* \cup C_2^*) = \varnothing$ and; $F \cap (w_s^+ \cup w_u^-) = \varnothing$. Say, $f(F) = [-a, a] \times [-1, 1]$. Let $\delta > 0$ be such that for a C^r vector field Y on $f(F)$, the condition $\|Y\|_r < \delta$ implies that $\|\tilde{Y}\|_r < \epsilon$ where $\tilde{Y}$ is equal to 0 on F^c and $\tilde{Y}$ is equal to $f_*^{-1}(Y)$ of F. See Figure 3.4.

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonnegative smooth function such that we have: $\operatorname{supp}(g) = [-a, a] \times [-1, 1]$; g is positive on the interior of $[-a, a] \times [-1, 1]$ and; $\|g\|_r < \delta$. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$. Recall that $f_*(X_{|F})(q) = e_1$ for $q \in [-a, a] \times [-1, 1]$. For $z \in [-1, 1]$, let $\tilde{Y}_z : [-a, a] \times [-1, 1] \to \mathbb{R}^2$, $\tilde{Y}_z(q) = e_1 + zg(q) \cdot e_2$. Let Y_z be the vector field on M that is equal to X on F^c and it is equal to $f_*^{-1}(\tilde{Y}_z)$ on F. Since g is smooth and $\operatorname{supp}(g) = [-a, a] \times [-1, 1]$, the vector field Y_z is in $\mathfrak{X}^r(M)$. As we have $F \subseteq U$, the condition (ii) is satisfied for Y_z. The condition (i) is also satisfied by the choice of $\delta > 0$, g and $z \in [-1, 1]$. We will show that the condition (iii) is also satisfied for Y_z with some $z \in [-1, 1]$.

Let $A_- = [-a] \times [-1,1]$ and $A_+ = [a] \times [-1,1]$ both of which are transversal to $f_*(X_{|F})$. As $f_*(X_{|F})$ is the parallel vector field and g is equal to 0 on the boundary of $[-a,a] \times [-1,1]$, the First Intersection Assignment $P_1 : A_- \to A_+$ is a diffeomorphism for the flow of Y_1. Let $\Pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projection map onto the second coordinate. Let $k' = \inf\{\Pi_2 \circ P_1(q) - \Pi_2(q) : \ q \in \{-a\} \times [-1/2, 1/2]\}$. Because g is positive on the interior of $[-a,a] \times [-1,1]$, we have $k' > 0$. Let $k = \min\{k', 1/2\}$.

Let β^u_j be the j^{th} intersection of β^u (or equivalently ${w_u}^+$) with $f^{-1}(A_-)$ for the flow of X and similarly, let β^s_j be the j^{th} intersection of β^s (or equivalently ${w_s}^-$) with $f^{-1}(A_+)$ for the flow of X. Let $\tilde{\beta}^u_j = f(\beta^u_j)$ and $\tilde{\beta}^s_j = f(\beta^s_j)$. As the semi trajectories through $\tilde{\beta}^u_j$'s and $\tilde{\beta}^s_j$'s accumulate to $f(p^*) = (0,0)$ in $f(F)$, they will visit the stripe $[-a,a] \times [-k/2, k/2]$ infinitely many times. Let m, n be the smallest positive integers such that both $\tilde{\beta}^u_m$ and $\tilde{\beta}^s_n$ are in $[-a,a] \times [-k/2, k/2]$. We will consider the case for $\Pi_2 \circ \tilde{\beta}^u_m < \Pi_2 \circ \tilde{\beta}^s_n$ where we will take the parameter z from $[0,1]$. The other case requires to take the parameter z from $[-1,0]$ and it is analogous to the this case.

Let $P_z : A_- \to A_+$ be the First Intersection Assignment for the flow of $f_*(Y_{z|F})$. Note that P_z is a diffeomorphism. Let β^u_z and β^s_z be the separatrices of saddles of Y_z such that w_u^- and w_s^+ are subsets of β^u_z and β^s_z respectively. Let $\beta^u_{z,j}$ be the j^{th} intersection of β^u_z with $f^{-1}(A_-)$ and similarly, let $\beta^s_{z,j}$ be the j^{th} intersection of β^s_z with $f^{-1}(A_+)$. Let $\tilde{\beta}^u_{z,j} = f(\beta^u_{z,j})$ and $\tilde{\beta}^s_{z,j} = f(\beta^s_{z,j})$. Since g is positive on the interior of $[-a,a] \times [-1,1]$, we have $\Pi_2 \circ P_z(q) > \Pi_2 \circ P_0(q)$ for $z \in (0,1]$ and for $q \in {A_-}^\circ$. So, we have $\Pi_2 \circ P_z({\tilde{\beta}^u}_{z,1}) \geq \Pi_2 \circ P_0({\tilde{\beta}^u}_{0,1})$ for $z \in [0,1]$. From this relation and the fact that M is orientable, we conclude that $\Pi_2 \circ {\tilde{\beta}^u}_{z,2} \geq \Pi_2 \circ {\tilde{\beta}^u}_{0,2}$ if $\tilde{\beta}^u_{z,2}$ is defined and also if ${\tilde{\beta}^u}_{y,2}$ is an interior point of A_- for all y in $[0,z]$. As the "$\geq$" relations are preserved, we can inductively conclude that $\Pi_2 \circ {\tilde{\beta}^u}_{z,j} \geq \Pi_2 \circ {\tilde{\beta}^u}_{0,j}$ for $1 \leq j$ if ${\tilde{\beta}^u}_{z,j}$ is defined and also if ${\tilde{\beta}^u}_{y,x}$ is an interior point of A_- for all y in $[0,z]$ and for all x with $1 \leq x \leq j$. When we consider the negative semi trajectory of $\tilde{\beta}^s$, we can conclude in a similar way that $\Pi_2 \circ {\tilde{\beta}^s}_{z,j} \leq \Pi_2 \circ {\tilde{\beta}^s}_{0,j}$ for $1 \leq j$ if ${\tilde{\beta}^s}_{z,j}$ is defined and also if ${\tilde{\beta}^s}_{y,x}$ is an interior point of A_+ for all y in $[0,z]$ and for all x with $1 \leq x \leq j$.

As $F \cap (C_1^* \cup C_2^*)$ is the empty set, C_1^* and C_2^* are transversal circles for the flow of Y_z with any z in $[0,1]$. Let $T_{1,z} : C_1^* \to C_1^*$ be the Poincaré Return Map for the flow of Y_z and similarly, let $T_{2,z} : C_2^* \to C_2^*$ be the Poincaré Return Map for the flow of $-Y_z$. As both C_1^* and C_2^* do not have any boundary points in M, both $T_{1,z}$ and $T_{2,z}$ are injective local diffeomorphisms whenever defined.

Let $Q_1 : f^{-1}(A_-) \to C_1^*$ be the First Intersection Assignment for the flow of $-X$ (or for the flow of any $-Y_z$) and similarly, let $Q_2 : f^{-1}(A_+) \to C_2^*$ be the First Intersection Assignment for the flow of X (or for the flow of any Y_z). Because of $F \subseteq V^*$, both Q_1 is defined on the whole $f^{-1}(A_-)$ and Q_2 is defined one the whole $f^{-1}(A_+)$. Moreover, they are both injective local diffeomorphisms. Let c_n and c_m be the unique positive integers such that we have $T_{1,0}^{c_m}(w_u) = Q_1(\beta_m^u)$ and $T_{2,0}^{c_n}(w_s) = Q_2(\beta_n^s)$. See Figure 3.4.

Assume that both $T_{1,z}^{c_m}(w_u)$ and $T_{2,z}^{c_n}(w_s)$ are defined for all z in $[0,1]$. Possibly, $T_{1,1}^{c_m}(w_u)$ may not be in $Q_1(f^{-1}(A_-))$ or $T_{2,1}^{c_n}(w_s)$ may not be in $Q_2(f^{-1}(A_+))$. Nevertheless, by our choice of $k > 0$, β_m^u and β_n^s, there exists some z_0 in $[0,1]$ (by the Intermediate Value Theorem) such that we have $\Pi_2 \circ P_{z_0} \circ Q_1^{-1} \circ T_{1,z_0}^{c_m}(w_u) = \Pi_2 \circ Q_2^{-1} \circ T_{2,z_0}^{c_n}(w_s)$ and the extra saddle connection can be created in this way.

Assume that one of $T_{1,z}^{c_m}(w_u)$ and $T_{2,z}^{c_n}(w_s)$ is not defined for all z in $[0,1]$. Say, $T_{1,z_1}^{c_m}(w_u)$ is not defined for some z_1 in $(0,1]$. Note that $T_{1,z}^{c_m}(w_u)$ is defined for all z in $[0,\mu)$ for some μ in $[0,z_1)$ by the continuous dependence of the flows of Y_z on z. Let μ in $[0,z_1)$ be the greatest such possible μ. Then, there exists some j with $0 \leq j < c_m$ such that $T_{1,\mu}^{j}(w_u)$ is defined and $T_{1,\mu}$ is not defined at $T_{1,\mu}^{j}(w_u)$. The boundary points of the domain of $T_{1,z}$ go to saddles by Lemma 3.3 so that those boundary points in C_1^* depend continuously on the parameter z. Hence, $T_{1,\mu}^{j}(w_u)$ is the boundary point of the domain of $T_{1,\mu}$ so that $\omega(T_{1,\mu}^{j}(w_u))$ is a saddle and the extra saddle connection can be created in this way. The case that $T_{2,z}^{c_n}(w_s)$ is not defined for all z in $[0,1]$ is analogous and the proof is complete. □

Remark 3.2. Our proof of Lemma 3.6 is due to [3] but the proof in [3] has a subtle mistake. It considers the intersections of ${w_{\eta_s}}^-$ only with $f^{-1}(A_+)$ instead of considering

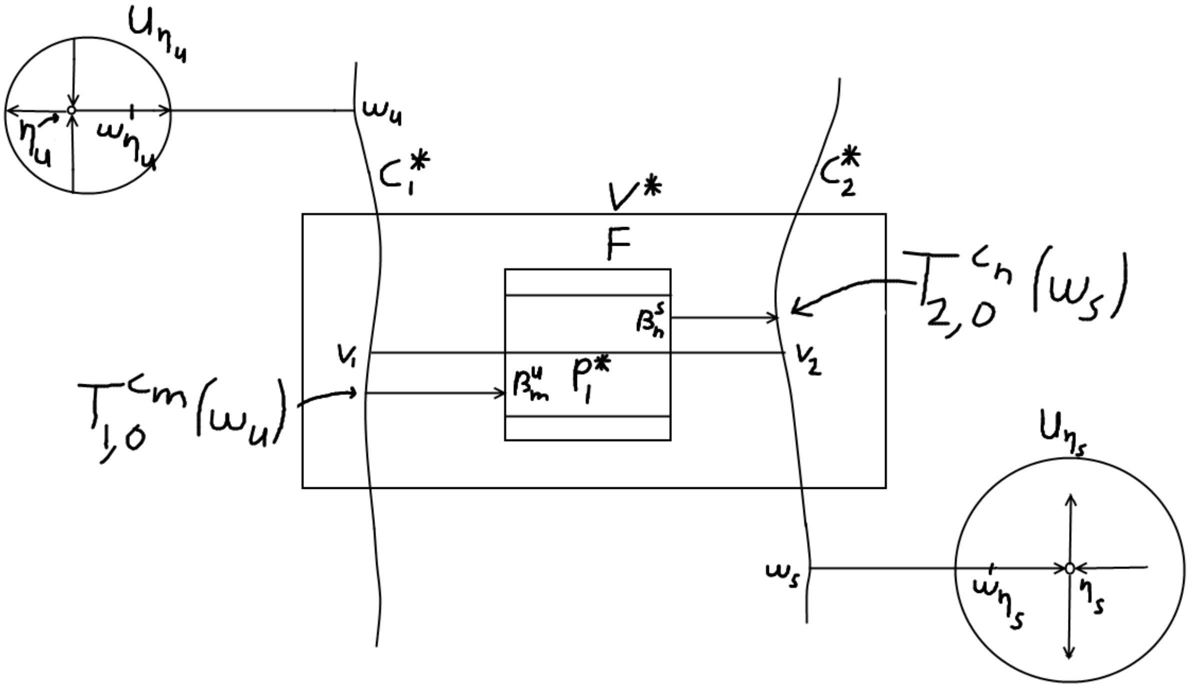

Figure 3.4. Some illustrations for the proof of Lemma 3.6.

their intersections with C_2^*. Similarly, it considers the intersections of ${w_{\eta_u}}^+$ only with $f^{-1}(A_-)$ instead of considering their intersections with C_1^*. When we make a perturbation on the flow box (f, F) with the parameter $z \in [0, 1]$ and when these perturbed separatrices go through a boundary point of $f^{-1}(A_+)$ or $f^{-1}(A_-)$, the arguments in [3] do not apply. This lemma will play a key role in the the proof of Theorem 3.1. Coincidentally, also Peixoto has made a mistake in [4] about this lemma by assuming that it holds for nonorientable manifolds as well.

3.3. Lemmas About Closed orbits

All the lemmas here except Lemma 3.7 are due to [3]. For convenience, Lemma 3.7 has been stated only for the case $\omega(p) \supseteq \tau$ but it can be analogously stated and proven for the case $\alpha \supseteq \tau$. In the latter case, one must consider the Poincaré Return Map $P : \Sigma \to \Sigma$ for the flow of $-X$.

Lemma 3.7. *Suppose that* p *is a point of* M *and* $\omega(p)$ *contains a closed orbit* τ *with* $p \notin \tau$. *Let* U_τ *be a circle neighborhood of* τ *such that* p *is not in* U_τ. *Let* q *be a point of* τ *and* Σ *be a transversal section through* q *in* U_τ. *Let* $\tilde{\Sigma}$ *be a connected open subset* *of* Σ *containing* q *such that the Poincaré Return Map* $P : \Sigma \to \Sigma$ *is defined on the whole* $\tilde{\Sigma}$ *and also* P *is defined again on the whole* $P(\tilde{\Sigma})$*. Let* $\tilde{\Sigma}^a, \tilde{\Sigma}^b$ *be the connected components of* $\tilde{\Sigma} - \{q\}$*. Suppose that* p^+ *accumulates to* q *from the side* $\tilde{\Sigma}^a$*; i.e.* q *is a limit point of* $p^+ \cap \Sigma^a$*. Then, there exist a neighborhood* U_q *of* q *in* Σ *such that:*

a) If U_τ *is a cylinder, then we have* $P(\,\tilde{\Sigma}^a \cap U_q\,) \subsetneq \tilde{\Sigma}^a \cap U_q$ *and* $P(x) \neq x$ *for all* x *in* $\tilde{\Sigma}^a \cap U_q$*;*

b) If U_τ *is a Möbius band, then we have* $P^2(\,\tilde{\Sigma}^a \cap U_q\,) \subsetneq \tilde{\Sigma}^a \cap U_q$ *and* $P^2(x) \neq x$ *for all* x *in* $\tilde{\Sigma}^a \cap U_q$*;*

c) $\omega(p) = \tau$*.*

Proof. Let Σ^a and Σ^b be the connected components of $\Sigma - \{q\}$ containing $\tilde{\Sigma}^a$ and $\tilde{\Sigma}^b$ respectively. Define a partial relation "$\leq_a$" on Σ^a such that for any x, y in Σ^a, we have $x \leq y$ if and only if $x \in [q, y]$ where $[q, y]$ denotes the closed connected subset of Σ with boundary points q and y. Define a similar partial relation "$\leq_b$ " on Σ^b. Let $\leq_{\tilde{a}} = \leq_a$ and $\leq_{\tilde{b}} = \leq_b$ be the induced partial relations on $\tilde{\Sigma^a}$ and $\tilde{\Sigma^b}$ respectively. Let all these connected sets have orientations induced from their partial relations. Note that Σ^a and Σ^b are oriented but they together with $\{q\}$ do not induce a consistent orientation for Σ. Yet, we need an orientation for it as well. Define any orientation on Σ and let $\tilde{\Sigma}$ have the induced orientation from Σ.

Let p_1 be a point in $p^+ \cap \tilde{\Sigma}^a$. Then, we have $p_1 \neq q$ because we have $p \notin \tau$. Again by $p \notin \tau$ and $\omega(p) = \tau$, it follows that p^+ is not a closed orbit. Hence, we have $P(p_1) \neq p_1$. Let $P(p_1) = p_2$.

Assume now that U_τ is homeomorphic to a cylinder. Then, P preserves the orientation of $\tilde{\Sigma}$ so that we have $P(\tilde{\Sigma}^a) \subseteq \Sigma^a$ and $p_2 \in \Sigma^a$. Let $[p_1, p_2]$ be the connected closed subset of Σ^a with boundary points p_1 and p_2. Then, $C_1 = p_1p_2 \cup [p_1, p_2]$ is a simple closed continuous curve. If p_1 is close enough to q, then C_1 separates U_τ into two disjoint cylinders because τ does so and the positive semi trajectories of points close to q in $\tilde{\Sigma}^a$ travel near τ before returning to Σ^a. We will assume that p_1 is close enough to q in this sense.

We claim that $p_1 >_a p_2$. Assume otherwise. Because both the disjoint circles C_1 and τ separate U_c into two disjoint cylinders, C_1 and τ bound an open region A which is homeomorphic to an open cylinder. Also, ${p_2}^+$ leaves A because $p_2 >_a p_1$. As, p^+ accumulates to τ from the $\tilde{\Sigma}^a$ side, ${p_2}^+$ must enter A but this is not possible since $\tilde{\Sigma}^a$ is transversal to X and the open region A is bounded by p_1p_2 and τ. This contradiction proves $p_1 >_a p_2$ (see Figure 3.5). As P is a continuous; $P(q) = q$ and; $P(p_1) = p_2$, we have $P([q, p_1]) = [q, p_2] \subsetneq [q, p, 1]$.

Let U_q be a neighborhood of q in Σ such that we have $U_q \cap \tilde{\Sigma}^a = (q, p_1]$. Assume that we have $P(x) = x$ for some x in $U_q \cap \tilde{\Sigma}^a$. Clearly, we have $x \neq p_1$. As $P(U_q \cap \tilde{\Sigma}^a) \subseteq U_q \cap \tilde{\Sigma}^a$ holds, $P^n(U_q \cap \tilde{\Sigma}^a)$ is defined for all positive integers n. The properties $P(x) = x$ and $p_1 >_a x$ imply together that the set $\{P^n(p_1)\}_{n \in \mathbb{N}}$ converges to a point in $[x, p_1]$, which contradicts that it does not accumulate to q so that p^+ does not accumulate to τ. Hence, x does not exist and part a) is proven for this choice of U_q. When U_τ is homeomorphic to a Möbius band, part b) can be proven in a similar way.

Whether U_τ is orientable or not, we see that the intersections of p^+ with $\tilde{\Sigma}^a$ are strictly decreasing in the sense of $\leq_a$. Hence, q is the only accumulation point of $p^+ \cap \tilde{\Sigma}^a$ and we have $\omega(p) \cap U_\tau = \tau$. Moreover, p^+ does not leave U_τ once it intersects a point of $\tilde{\Sigma}^a$. Therefore, we have $\omega(p) = \tau$ and the proof is complete. □

Remark 3.3. Lemma 3.7 does not imply that the closed orbit τ is hyperbolic because the single eigenvalue of DP_q might be equal to ± 1.

Lemma 3.8. *Suppose that X is in $\mathfrak{X}^r(M)$ and; τ is a closed orbit of X. Then, for all $\epsilon > 0$ and for all neighborhoods U of τ, there exists some Y in $\mathfrak{X}^r(M)$ such that:*

(i) $\|X - Y\|_r < \epsilon$;

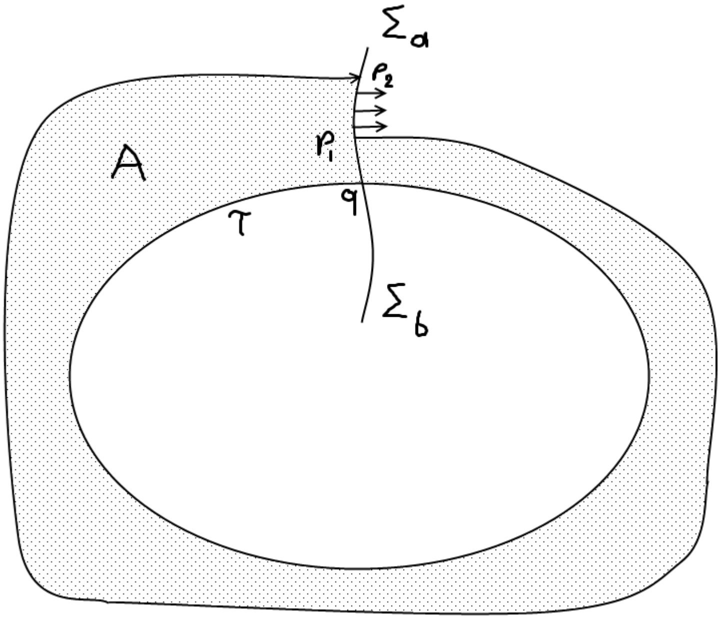

Figure 3.5. The contradiction in the cylindrical U_τ case.

(ii) Y *is equal to* X *on* U^c*;*

(iii) The closed set τ *of* M *is a hyperbolic closed orbit of* Y*.*

Proof. Let $p \in \tau$ and (f, F) be a flow box at p such that F is a subset of U and $\tau \cap F$ is connected. The last requirement is possible because τ is compact. Say, $f(F) = [-a, a] \times [-1, 1]$.

Let $g : [-a, a] \to \mathbb{R}$ be a smooth function with the following properties: $g = 0$ on $[-a, -a/2] \cup [a/2, a]$ and $g > 0$ on $(-a/2, a/2)$. Let $h : [-1, 1] \to \mathbb{R}$ be a smooth function with the following properties: $h(y) = 0$ for y with $|y| \geq 3/4$, $h(y) = 1$ for y with $|y| \leq 1/2$ and $h > 0$ for y with $1/2 \leq |y| \leq 3/4$. For any $s \in \mathbb{R}$, define the smooth vector fields Z_s on $[-a, a] \times [-1, 1]$ as $Z_s(x, y) = (\,1, sg(x)h(y)y\,)$. Note that the y-component of Z_s vanishes at the boundary of $F(f)$.

To exhibit the flow of Z_s we consider the differential equations, $\dfrac{dx}{dt} = 1$ and $\dfrac{dy}{dt} = sg(x)h(y)y$. Consider the initial conditions $x(0) = -a$ and $y(0) = y_0$ with $|y_0| \leq 1/4$. Clearly, $x(t) = t - a$ for $0 \leq t \leq 2a$. By the continuity of $y(t)$, there exist

l with $2a \geq l > 0$ such that we have $|y(l)| \leq 1/2$. So, $h(y(t)) = 1$ for $t \in [0, l]$. Let $\mu(t) = \exp(\, s \int_0^t g(z-a)dz\,)$ for $0 \leq t \leq l$. Then, $y(t) = y_0 \cdot \mu(t)$ for $0 \leq t \leq l$ as it satisfies the given differential equation of y for $0 \leq t \leq l$ and also the initial condition $y(0) = y_0$. Let $\epsilon_1 = (\,\log 2\,)/(\,\int_{-a}^{a} g(x)dx)$ and $s < \epsilon_1$. Then, $|y_0\mu(t)| \leq 2y_0 \leq 1/2$ for all $t \in [0, 2a]$. Hence, $y(t) = y_0 \cdot \mu(t)$ for all t in $[0, 2a]$ with this choice of s. We remark the inequality $\epsilon_1 > 0$ so that the last conclusion holds for any negative s. Note $y(t) = 0$ for $y_0 = 0$.

Let $s \in \mathbb{R}$ with $|s| < \epsilon_1$. As $|s|$ goes to 0, the norm $\|Z_s - f_*(X_{|F})\|_r$ goes to 0. So, if $|s|$ is small enough, then we have $\|\hat{Z}_s - X\|_r < \epsilon$ where the C^r vector field $\hat{Z}_s$ is equal to X outside F and $\hat{Z}_s$ is equal to $f_*^{-1}(Z_s)$ on F. Because F is a subset of U, the property (ii) is satisfied as well. As X and $\hat{Z}_s$ are equal outside F and $y(t)$ is equal to 0 when y_0 is equal to 0, we conclude that τ is a closed orbit of $\hat{Z}_s$. So, it remains to show that τ is hyperbolic for some small s.

Let $p_1 = X_a(p)$ and $I_+ = f^{-1}(\,\{a\} \times [-1, 1]\,)$ which is transversal to X. Let Σ be a connected open neighborhood of p_1 in I_+ such that the Poincaré Return Map $P_X : I_+ \to I_+$ is defined on the whole Σ for the flow of X. The Poincaré Return Map $P_{\hat{Z}_s} : I_+ \to I_+$ is also defined on a connected open subset Σ_s of Σ (with $p_1 \in \Sigma_s$) for the flow of $\hat{Z}_s$ because τ is still a closed orbit of $\hat{Z}_s$ through p_1. If we have $\Sigma_s \subseteq f^{-1}(\,\{a\} \times [-1/4, 1/4]\,)$, then we have $P_{\hat{Z}_s} = \mu(2a)P_X$ by comparing the flows X and $\hat{Z}_s$ in F. Note that $\mu(2a)$ depends smoothly on s and it goes to 1 as s goes to 0. By the equality $(DP_{\hat{Z}_s})_{p_1} = \mu(2a)(DP_X)_{p_1}$, we can find some small s such that we have $|(DP_{\hat{Z}_s})_{p_1}| \neq 1$. So, τ is a hyperbolic closed orbit of $\hat{Z}_s$. □

Suppose that: τ is a closed orbit of X; the point p is not in τ and; $\omega(p) = \tau$. We can find a small transversal section Σ through a point $q \in \tau$ such that the properties in Lemma 3.7 are satisfied. If τ is not a hyperbolic closed orbit of X, then at the point q, we can take a small flow box (f, F) with $\Sigma \subsetneq F$ and make a perturbation of X only in F as in the proof of Lemma 3.8. The below corollary is easy to deduce.

Corollary 3.1. *Suppose that p is in M and $\omega(p) = \tau$ ($\alpha(p) = \tau$) where τ is a closed* *orbit of* X. *Then, for all* $\epsilon > 0$ *and for all neighborhoods* U *of* τ, *there exists some* $Y \in \mathfrak{X}^r(M)$ *such that:*

(i) $\|X - Y\|_r < \epsilon$;

(ii) Y *is equal to* X *on* U^c;

(iii) The closed set τ *of* M *is a closed orbit of* Y *which is a hyperbolic attractor (repellor);*

(iv) $\omega(p) = \tau$ $(\,\alpha(p) = \tau\,)$.

3.4. Saddle Graph

We follow the proofs in [3] for the following lemmas.

Definition 3.2. *A nonempty subset* E *of* M *is called* an invariant set for X *if* $X_t(E)$ *is a subset* E *for all* t *in* $\mathbb{R}$. *A nonempty, closed, invariant set* E *for* X *is called* minimal *if there doesn't exist a proper subset of* E *with all these properties.*

Remark 3.4. Let E be a minimal set for a C^r vector field X on M and τ an orbit in E. Since E is closed and $\tau \subseteq E$, we have $\omega(\tau) \subseteq E$. As $\omega(\tau)$ is nonempty, closed and invariant and E is minimal, we have $\omega(\tau) = E \supseteq \tau$; i.e. τ is recurrent.

The existence of minimal sets naturally relies on the Zorn's Lemma. See [3] for the proof of the below lemma.

Lemma 3.9. *Let* $\hat{E}$ *be a nonempty, closed and invariant set for* X. *Then, there exists a subset* E *of* $\hat{E}$ *such that* E *is minimal.*

Definition 3.3. *A connected closed subset* $\mathcal{G}$ *of* M *that satisfies all the following:*

(i) $\mathcal{G}$ *consists of only saddles and saddle separatrices of* X;

(ii) For each separatrix τ *in* $\mathcal{G}$, $\omega(\tau)$ *and* $\alpha(\tau)$ *are saddles contained in* $\mathcal{G}$;

(iii) For each saddle σ in $\mathcal{G}$, at least one unstable separatrix and one stable separatrix of σ are contained in $\mathcal{G}$.

is called a saddle graph for X.

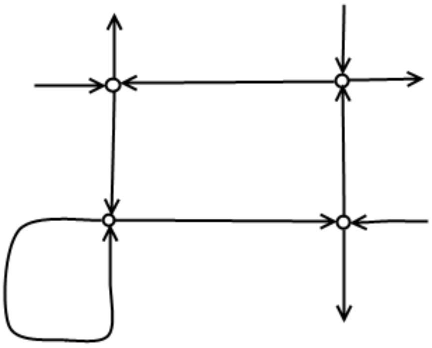

Figure 3.6. A saddle graph with four saddles and five distinct separatrices.

To express a saddle graph by stating all its saddles and specifying their separatrices which are the saddle connections in the graph is cumbersome. So, we omit such a general representation. Instead, we would like to obtain an intuitive and small neighborhood of a saddle graph $\mathcal{G}$ to which we can apply the Grobman-Hartman Theorem and Tubular Flow Theorem.

Let γ be a saddle connection in $\mathcal{G}$ and let $\eta = \alpha(\gamma)$ and $\sigma = \omega(\gamma)$. Let U_η and U_σ be the Grobman-Hartman neighborhoods of η and σ respectively. Let $p_\eta \in U_\eta \cap \gamma$ be such that we have $p_\eta^- \subseteq U_\eta$ and similarly, let $p_\sigma \in U_\sigma \cap \gamma$ be such that we have $p_\sigma^+ \subseteq U_\sigma$. As $p_\eta p_\sigma$ is compact and U_σ and U_η are Grobman-Hartman neighborhoods, each of the sets $p_\eta p_\sigma \cap U_\sigma$ and $p_\eta p_\sigma \cap U_\eta$ have finitely many connected components. So, if U_η and U_σ are small enough, then each of the sets $p_\eta p_\sigma \cap U_\sigma$ and $p_\eta p_\sigma \cap U_\eta$ are connected.

Assume $\mathcal{G}$ has a saddle σ_1 such that all η, σ and σ_1 are distinct. As $p_\eta p_\sigma$ is compact, there exists a Grobman-Hartman neighborhood U_{σ_1} of σ_1 such that we have $p_\eta p_\sigma \cap U_{\sigma_1} = \varnothing$. Also, any two closed arcs of any two distinct separatrices in $\mathcal{G}$ are disjoint compact sets so that they admit disjoint tubular flow neighborhoods. As $\mathcal{G}$ has finitely many saddle connections and saddles, the below definition is valid. See Figure 3.7.

Definition 3.4. *Let* $X \in \mathfrak{X}^r(M)$ *and let* $\mathcal{G}$ *be a saddle graph for* X*. Let* $\{\sigma_1, \dots, \sigma_n\}$ *be the set of all saddles in* $\mathcal{G}$ *and let* $\{\gamma_1, \dots, \gamma_m\}$ *be the set of all saddle connections in* $\mathcal{G}$*. Let* U_j *be a Grobman-Hartman neighborhood of* σ_j *for* $1 \le j \le n$*. For* $1 \le l \le m$*, let* $\sigma_{j_l} = \alpha(\gamma_l)$*,* $\sigma_{k_l} = \omega(\gamma_l)$*. Let* $a_l \in U_{j_l} \cap \gamma_l$ *be such that we have* ${a_l}^- \subseteq U_{j_l}$ *and similarly, let* $b_l \in U_{k_l} \cap \gamma_l$ *be such that we have* ${b_l}^+ \subseteq U_{k_l}$*. Let* V_l *be a tubular flow neighborhood of* $a_l b_l$*. The sets* U_j *and* V_l *are chosen to be so small so that they satisfy all the following properties:*

(i) All U_j*'s are pairwise disjoint;*

(ii) Both $a_l b_l \cap U_{j_l}$ *and* $a_l b_l \cap U_{k_l}$ *are connected for* $1 \le l \le m$*;*

(iii) For $1 \le l \le m$*, we have* $a_l b_l \cap U_j = \varnothing$ *for all* $j \in \{1, \dots, n\} - \{j_l, k_l\}$*;*

(iv) All V_l*'s are pairwise disjoint;*

(v) For $1 \le l \le m$*, we have* $V_l \cap U_j = \varnothing$ *for all* $j \in \{1, \dots, n\} - \{j_l, k_l\}$*.*

The set $U_{\mathcal{G}} = \Big(\bigcup_{1 \le l \le m} V_l\Big) \cup \Big(\bigcup_{1 \le j \le n} U_n\Big)$ *is called* a band neighborhood of $\mathcal{G}$*.*

Remark 3.5. A band neighborhood of a saddle graph $\mathcal{G}$ is a non-disjoint union of regions that are either homeomorphic to a cylinder or a Möbius band.

Lemma 3.10. *Suppose* $X \in \mathfrak{X}^r(M)$ *such that all all singularities of* X *are hyperbolic and* X *has no nontrivial recurrent orbits. Then,* ω*-limit and* α*-limit of any orbit* β *of* X *can be only one of the following list: a singularity, a closed orbit or a saddle graph.*

Proof. Let β be an orbit of X. We will consider only $\omega(\beta)$ since the argument for $\alpha(\beta)$ is analogous to the one of $\omega(\beta)$. Assume that $\omega(\beta)$ is not a singularity or a closed orbit. We will show that $\omega(\beta)$ is a saddle graph.

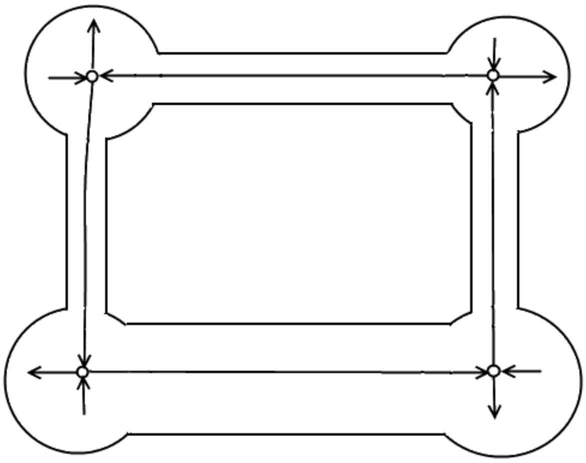

Figure 3.7. A band neighborhood of a saddle graph with four Grobman-Hartman neighborhoods and four tubular flow neighborhoods.

Let F be a minimal set of $\omega(\beta)$. By assumption, $\omega(\beta)$ is not a sink so that F cannot contain a sink. Assume that F contains a regular point q. By Remark 3.2, $\omega(q)$ is recurrent. By hypothesis, there are no nontrivial recurrent orbits so that $\omega(q)$ is a closed orbit τ. As $\tau \subseteq F \subseteq \omega(\beta)$, we have $\tau = \omega(\beta)$ by Lemma 3.7 which contradicts our initial assumption. Therefore, F cannot contain a regular point so that F is a single saddle σ_1. By our initial assumption, $\omega(\beta)$ is not a single singularity so that it contains an unstable separatrix γ^1 of σ_1.

Assume that for each unstable separatrix γ in $\omega(\beta)$, $\omega(\gamma)$ is a saddle in $\omega(\beta)$. Denote this assumption as $(*)$. For $j \geq 1$, inductively define $\sigma_{j+1} := \omega(\gamma^j)$ and let γ^{j+1} be an unstable separatrix of σ^{j+1} that is in $\omega(\beta)$. As the number of saddles is finite, there exists some $j \geq 1$ such that $\sigma_j = \sigma_1$ so that $\omega(\beta)$ contains a saddle graph $\mathcal{G}_0$. Note that $\mathcal{G}_0$ is not necessarily unique. Let $\mathcal{G}$ be a maximal saddle graph that is contained in $\omega(\beta)$; i.e. there doesn't exist a saddle graph in $\omega(\beta)$ that properly contains $\mathcal{G}$. Then, this maximal saddle graph $\mathcal{G}$ is the unique one in $\omega(\beta)$ because both $\omega(\beta)$ and a maximal saddle graph in $\omega(\beta)$ are connected sets so that $\omega(\beta)$ cannot contain more than one maximal saddle graph. So, the assumption $(*)$ leads to the conclusion $\omega(\beta) = \mathcal{G}$. We will prove $(*)$ but first, we need to prove some claim.

Assume that $\omega(\beta)$ contains a maximal saddle graph $\mathcal{G}_1$. We claim that we have

$\omega(\beta) = \mathcal{G}_1$ (even without the assumption $(*)$). Assume that $\omega(\beta) - \mathcal{G}_1$ is nonempty. Let $\mathcal{G}_1, \ldots, \mathcal{G}_n$ be the all maximal saddle graphs in $\omega(\beta)$. As their union is a disconnected set and $\omega(\beta)$ is connected, the set $\omega(\beta) - (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_n)$ is nonempty. Let $A = \omega(\beta) - (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_n)$. By considering a minimal set in A just as we did before, we can find a saddle η_1 and an unstable separatrix τ^1 of η_1 in A. Now, $\omega(\tau^1)$ cannot be a sink or a closed orbit because $\omega(\beta)$ would have reduced to one of these elements which contradicts that it contains $\mathcal{G}_1$. So, $\omega(\tau^1)$ contains a saddle, η_2. We can't have both "$\eta_2 = \eta_1$" and "$\eta_2 = \omega(\tau^1)$" because otherwise, $\eta_1 \cup \tau^1$ defines a saddle graph in $\omega(\beta)$ but A does not contain any saddle graph. So, at least one of "$\eta_2 = \eta_1$" and "$\eta_2 = \omega(\tau^1)$" is false. If the first one is false, then A contains an unstable separatrix τ^2 of η_2 that is distinct from τ^1. If the first one is true, then $\omega(\tau^1)$ contains the unstable separatrix τ^2 of η_2 that is different from τ^1 because there are no nontrivial recurrent orbits. In either case, A contains an unstable separatrix τ^2 of η_2 that is distinct from τ^1. We can continue our process by analyzing $\omega(\tau^2)$ and this analysis is analogous to the one of $\omega(\tau^1)$. We can continue this process inductively to find saddles η_j's and unstable separatrices τ^j's of η_j's in A for $j \geq 1$ such that η_{j+1} is in $\omega(\tau^j)$. In some cases, we might have that η_{j+1} is equal to $\omega(\tau^j)$ but A does not contain a saddle graph so that there exists at least one τ^k such that $\omega(\tau^k)$ is not a saddle because X has finitely many saddles. Note that this $\omega(\tau^k)$ contains all τ^m for $m > k$ by the transitivity of ω-limit sets. This process leads to the conclusion that eventually some $\omega(\tau^k)$ contains τ^k (by using the transitivity of ω-limit sets) because the number of unstable separatrices of saddles of X is finite but this conclusion contradicts that X does not have nontrivial recurrent orbits. Hence, our claim is proven.

Assume that $(*)$ is false. Then, there exists a saddle η_1 in $\omega(\beta)$ and an unstable separatrix τ^1 of η_1 in $\omega(\beta)$ such that $\omega(\tau^1)$ is not a saddle. It cannot be a sink or a closed orbit because otherwise $\omega(\beta)$ would have reduced to one of these elements which is against our initial assumption. We can conclude that $\omega(\tau^1)$ contains a saddle and an unstable separatrix of it again by considering a minimal set in $\omega(\tau^1)$ just as we did before. If the assumption $(*)$ is true for $\omega(\tau^1)$, then $\omega(\tau^1)$ is a single maximal saddle graph $\mathcal{G}_1$. Then, $\omega(\beta)$ contains $\mathcal{G}_1$ so that $\omega(\beta)$ is equal to $\mathcal{G}_1$ as we have shown above but this conclusion contradicts that $\omega(\beta)$ contains τ_u^1 which is not in $\mathcal{G}_1$. Hence, the assumption $(*)$ is false for $\omega(\tau^1)$ and $\omega(\tau^1)$ contains a saddle η_2 and an unstable separatrix τ^2 of η_2 such that $\omega(\tau^2)$ is not a saddle. Now, we can continue our process with $\omega(\tau^2)$ just as before to conclude in an analogous way that the assumption $(*)$ is false for $\omega(\tau^2)$. At each step (for $1 \leq j$), we have that: $\omega(\tau^j)$ contains a saddle η_{j+1} and an unstable separatrix τ^{j+1} such that $\omega(\tau^{j+1})$ is not a saddle. Also, $\omega(\tau^j)$ contains τ_u^k for $k > j$ by the transitivity of ω-limit sets. As the number of separatrices of saddles of X is finite, there exists some τ^m such that $\omega(\tau^m)$ contains τ^m which contradicts that X does not have nontrivial recurrent orbits. Hence, $(*)$ holds and the proof is complete. □

3.5. Final Steps Toward Theorem 3.1

Except Lemma 3.13, we follow the proofs in [3] for the following lemmas and Theorem 3.1. The important Lemma 3.11 is originally due to [4].

Lemma 3.11. *Suppose that X is a K-S field and all the recurrent orbits of X are trivial. Then, X is an M-S field.*

Proof. We need to show only that $\Omega(X)$ is the union of critical elements of X. Let $p \in M$ such that the orbit of p is not a critical element of X. Since all the recurrent orbits of X are trivial, $\omega(p)$ is either a singularity, a closed orbit or a saddle graph by Lemma 3.10. As X is a K-S field, there are no saddle connections; otherwise, the transversality condition in Definition 2.9 would have been violated. Hence, X does not have a saddle graph and $\omega(p)$ is a hyperbolic critical element σ.

Assume that σ is an attractor. Let U_σ be an open neighborhood of σ such that we have $W^s_{U_\sigma}(\sigma) \subseteq W^s(\sigma)$ and $W^s_{U_\sigma}(\sigma) = U_\sigma$. The last requirement is possible since σ is an attractor. So, for all z in U_σ, we have $X_t(z) \in U_\sigma$ for all nonnegative real t. Let $T > 0$ be such that $X_T(p)$ is in U_σ and let $q = X_T(p)$. As $W^s(\sigma)$ is an immersed C^r submanifold of dimension 2, for each point z in the closed arc pq, we can find a neighborhood U_z of z in M such that we have $U_z \subseteq W^s(\sigma)$. As pq is compact, there exists a tubular flow neighborhood V of pq such that every for every point z in V, we have $\omega(z) = \sigma$. Let U_p be a neighborhood of p such that both U_p and and $X_{t_0}(U_p)$ are in V for some $0 < t_0 < T$ and also they are disjoint. Hence, U_p and and $X_t(U_p)$ are disjoint for all $t \geq t_0$.

Assume now that σ is a saddle. Then, the orbit γ of p is a stable separatrix of σ. Let γ_1^u and γ_2^u be the unstable separatrices of σ. Because X does not have any saddle connections, $\omega(\gamma_1^u)$ and $\omega(\gamma_2^u)$ are hyperbolic attractors σ_1 and σ_2 respectively. Let U_σ be a small Grobman-Hartman neighborhood of σ such that we have $p \notin U_\sigma$. Let $p_1 \in \gamma_1^u \cap U_\sigma$ and $p_2 \in \gamma_2^u \cap U_\sigma$ be such that we have ${p_1}^- \subseteq U_\sigma$ and ${p_2}^- \subseteq U_\sigma$. Now, we repeat the previous part twice. For $j = 1$ or 2, let U_{σ_j}, be a neighborhood of σ_j which has the analogous properties of the previous U_σ. Let $T_j > 0$ be such that $q_j = X_{T_j}$ is in U_{σ_j}. Let V_j be tubular flow neighborhood of $p_j q_j$ which has the analogous properties of the previous V. Also, we can take V_j small enough to ensure that p is not in the closure of V_j. Let U_{p_j} be a neighborhood of p_j and let $t_j > 0$ such that they have the analogous properties of U_p and t_0.

Let $w \in \gamma \cap U_\sigma$ be such that we have $w^+ \subseteq U_\sigma$. By the Grobman-Hartman Theorem, there exists a neighborhood U_w of w in U_σ such that the positive semi trajectory of every point in $U_w - \gamma$ goes to either U_1 or U_2 within U_σ. Let $T > 0$ such that $X_T(p) = w$. By applying the Tubular Flow Theorem to pw, we can find a neighborhood U_p of p such that U_p and U_w are disjoint and also, we have $X_T(U_p) \subseteq U_w$ if U_w is small enough. As p is not in $\bar{V_j}$ (for $1 \leq j \leq 2$), we can take a smaller U_p such that we have $U_p \cap V_j = \varnothing$.

Now, every positive semi trajectory of a point in $U_p - \gamma$ goes to first U_w; then some U_{p_j}; then, it stays in V_j for some time and finally, it enters and stays in U_{σ_j} forever. Hence, we have $U_p \cap X_t(U_p - \gamma) = \varnothing$ for all $t \geq T$. Using this property, we can also conclude that $U_p \cap \gamma$ is connected because a positive semi trajectory of a point in $U_p - \gamma$ cannot return to U_p once it enters U_w so that this positive semi trajectory cannot belong to γ. As U_p and U_w are disjoint; $X_T(U_p) \subseteq U_w$ and; $U_p \cap \gamma$ is connected, we conclude that $U_p \cap X_t(U_p \cap \gamma) = \varnothing$ for all $t \geq T$. Therefore, we have $U_p \cap X_t(U_p) = \varnothing$ for all $t \geq T$. See Figure 3.8.

We have considered all the possible cases for $\omega(p)$. We can make analogous arguments for $\alpha(p)$. So, there exists a neighborhood V_p of p and some $L < 0$ such that $X_t(V_p) \cap V_p = \varnothing$ for all $t \leq L$. Let $A_p = V_p \cap U_p$ and $H = \max\{T, |L|\}$. Then, $X_t(A_p) \cap A_p = \varnothing$ for all t with $|t| \geq H$. Thus, p is a wandering point for X. Hence, $\Omega(X)$ does not contain p and as p was arbitrary, $\Omega(X)$ is equal to the union of the critical elements of X. □

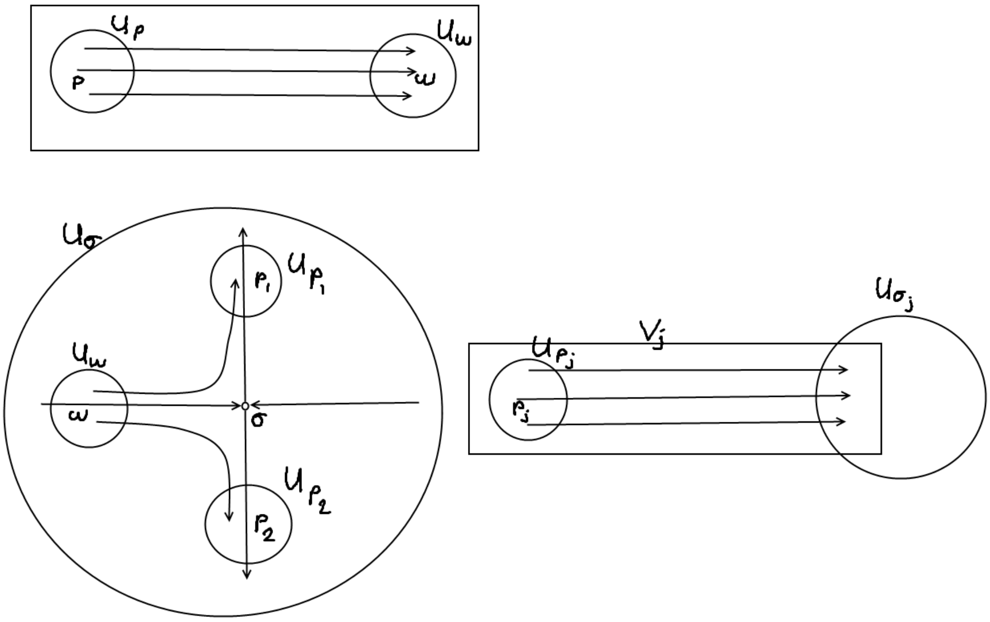

Figure 3.8. The local pictures when σ is a saddle. Note that U_{σ_j} is either a disc or a cylinder (annulus).

Lemma 3.12. *Suppose that M is a torus and that X in $\mathfrak{X}^r(M)$ has no singularities. Then, for any neighborhood $\mathcal{U}$ of X, there exists some Y in $\mathcal{U}$ such that Y has a closed orbit and Y has no singularities.*

Proof. If X has a closed orbit, then take $Y = X$. So, assume that X has no closed orbit. Let $p \in M$. If X does not have a nontrivial recurrent orbit, then $\omega(p)$ is either a closed orbit or a singularity or a saddle graph by Lemma 3.10 but X has none of these. Hence, X has a nontrivial recurrent orbit γ. We will consider that γ is nontrivial forward recurrent as the other case is analogous. Let C^* be a transversal circle through a point p^* of γ by Lemma 3.2. Let $P : C^* \to C^*$ be the Poincaré Return Map and let E be the domain of P. As X has no saddles, we have $E = C^*$ by Lemma 3.3.

Let q^* be in $p^*P(p^*) - \{p^*, P(p^*)\}$ and let (f, F) be a flow box at q^* such that we have $F \cap C^* = \varnothing$ and F is a subset of a tubular flow neighborhood V of $p^*P(p^*)$. Then, when the positive semi trajectories of points in F leave F, they intersect C^* before returning to F if they return at all. Say, $[-a, a] \times [-1, 1] = f(F)$. We will consider the case that q^{*+} accumulates to q^* from below in f(F); i.e. q^* is a limit point of $q^{*+} \cap f^{-1}(\{0\} \times [-1, 0])$ as the other case will amount to take $-\Delta Z$ instead of ΔZ below.

Let $\tilde{X} = f_*(X_{|F})$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $\text{supp}(g) = f(F)$ and $g > 0$ on the interior of $f(F)$. Let e_1, e_2 be the standard basis of $\mathbb{R}^2$. Note that $\tilde{X}(q) = e_1$ for $q \in f(F)$. Define the smooth vector field ΔZ on $f(F)$ as $\Delta Z(q) = g(q)\, e_2$. Let the C^r vector field Z be equal to X on F^c and let Z be equal to $f_*^{-1}(\tilde{X} + \Delta Z)$ on F. If $\|g\|_r$ is small enough, then Z will be in $\mathcal{U}$.

For $u \in [0, 1]$, let Z_u be equal to X on F^c and let Z_u be equal to $f_*^{-1}(u\Delta Z + \tilde{X})$ on F. Clearly, we have $Z_u \in \mathcal{U}$ for all $u \in [0, 1]$ and $Z_0 = X$. Also, Z_u does not have any singularities and we will show that Z_u has a closed orbit for some $u \in [0, 1]$. Because of $F \cap C^* = \varnothing$, the circle C^* is transversal to Z_u. The Poincaré Return Map $P_u : C^* \to C^*$ is defined on the whole C^* for the flow of Z_u by Lemma 3.3 because Z_u does not have any saddles.

Let $\Pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projection map onto the second coordinate. Let ϕ_u be the flow of Z_u. Let $k' = \inf\{\, \Pi_2 \circ f \circ \phi_1(2a, w) - \Pi_2 \circ f(w) : \ w \in f^{-1}(\,\{-a\} \times [-1/2, 1/2]\,)\,\}$. Note that we have $0 < k' < 1$. Let $k = \min\{k', 1/2\}$. Let $z = f^{-1}(a, 0)$ and $I_- = f^{-1}(\{-a\} \times [-1, 1])$. For $j \geq 1$, let q_j be the j^{th} intersection of z^+ with I_- for the flow of X. Let m be the smallest positive integer such that $f(q_m)$ is in $\{-a\} \times [-k, 0]$. Let n be the number of intersections of zq_m with C^* . Note that $n \geq m$. For $1 \leq j \leq n$, let $w_j(u)$ be the j^{th} intersection of z^+ with C^* for the flow of Z_u. Then, $w_j(u)$ depends continuously on u because Z_u depends continuously on u. Let I_c be the maximal closed connected subset of $C^* \cap V$ such that the positive semi trajectories of points in I_c go to I_- within V (for the flow of X or any Z_u). Let $P_c : I_c \to I_-$ be the First Intersection Assignment which is the same assignment for any of the flows. Note that P_c is a diffeomorphism because of $F \subseteq V$; $w_n(0)$ is in I_c and; $P_c(w_n(0)) = q_m$. See Figure 3.9.

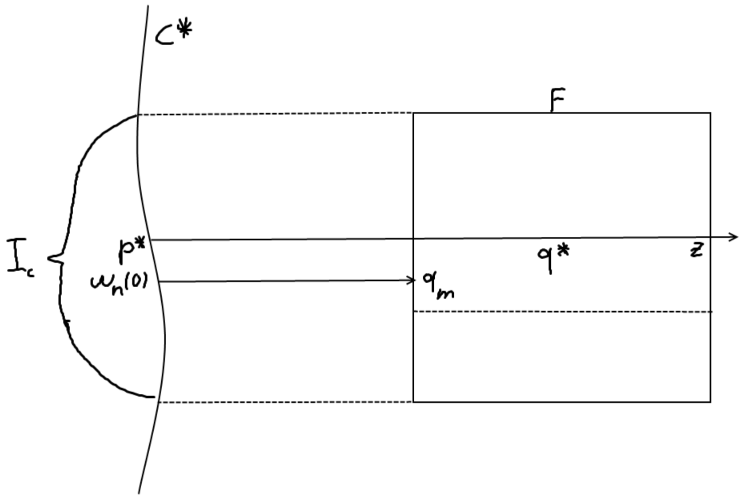

Figure 3.9. $w_n(0) \in I_c$ and $P_c(w_n(0)) = q_m$.

Let $v \in C^* - (\, I_c \cup \{w_1(0), \dots, w_n(0)\}\,)$. Define a partial relation "$\leq_c$" on $C^* - \{v\}$ such that for any two points r_1, r_2 in I_c, we have $r_1 \leq_c r_2$ if and only if $\Pi_2 \circ f \circ P_c(r_1) \leq \Pi_2 \circ f \circ P_c(r_2)$.

As M is orientable, we have $w_j(1) \geq_c w_j(0)$ for all $1 \leq j \leq n$ because of our choice of the direction of ΔZ provided that all $w_j(u)$'s are in $C^* - \{v\}$ for all $u \in [0,1]$. Let b_+ and b_- be the two boundary points of I_c such that we have $b_- <_c b_+$.

If $w_n(1)$ is not in I_c or $w_n(u)$ is equal to v for some u in $[0,1]$, then there exists some $u_1 \in [0,1]$ such that $w_n(u_1)$ is equal to b_+ by the continuous dependence of $w_n(u)$ on u and the Intermediate Value Theorem. Hence, we have $\Pi_2 \circ \phi_{u_1}(2a, P_c(w_n(u_1))\,) = 1$. Because of $\Pi_2 \circ \phi_0(2a, P_c(w_n(0))\,) < 0$, we conclude the equality $\Pi_2 \circ \phi_{u_2}(2a, P_c(w_n(u_2))\,) = 0$ for some u_2 in $[0, u_1]$. So, we have $\phi_{u_2}(2a, P_c(w_n(u_2))\,) = z$ and the orbit through z is a closed orbit.

Assume that we have $w_n(u) \in I_c$ for all $u \in [0,1]$. By our choice of $k > 0$ and the point q_m, we have $\Pi_2 \circ \phi_1(2a, P_c(w_n(1)\,)\,) > 0$. Again, there exists some $u_2 \in [0,1]$ such that we have $\Pi_2 \circ \phi_{u_2}(2a, P_c(w_n(u_2)\,)\,) = 0$. So, we have $\phi_{u_2}(2a, P_c(w_n(u_2)\,)\,) = z$ and the orbit through z is a closed orbit. $\square$

Lemma 3.13. *Suppose that X is a C^r vector field on M and τ is a closed orbit of X that bounds an open disc D in M. Then, D contains a singularity of X.*

Proof. For each closed orbit β of X in $\bar{D}$, let $g(\beta)$ denote its period. Let $E = \{g(\beta) \in \mathbb{R} : \beta \text{ is a closed orbit of } X \text{ in } \bar{D}\}$. Let $a = \inf E$. Assume $a > 0$. As τ is a closed orbit that bounds D, we have $X_{a/2}(\bar{D}) = \bar{D}$. By the Brouwer fixed-point theorem (see [22]), $X_{a/2}$ has a fixed point p in $\bar{D}$ and by our choice of a, the orbit through p is not a closed orbit. Hence, p is a singularity of X and the lemma is proven for the case $a > 0$.

Assume that we have $a = 0$. Then, there exist a set $\{\beta_n : n \in \mathbb{N}\}$ which is a set of distinct closed orbits of X in $\bar{D}$ such that the limit of the sequence $\{g(\beta_n)\}_{n\in\mathbb{N}}$ goes to 0 as n goes to ∞. Let p_n be a point of β_n. The compact $\bar{D}$ contains a limit point q of the infinite set $\{p_n : n \in \mathbb{N}\}$. Assume q is a regular point of X and let (f, F) be a flow box of at q such that F is a subset of D. Say, $f(F) = [-a, a] \times [-1, 1]$. Then, every closed orbit of D that intersects F has period greater than $2a$ which contradicts the definition of q. Hence, q is a singularity of X and the proof is complete. □

Theorem 3.1. *Suppose that M is an orientable, compact and connected 2-manifold without boundary. Then, M-S fields are dense in $\mathfrak{X}^r(M)$.*

Proof. Let $X \in \mathfrak{X}^r(M)$ and let $\mathcal{U}$ be a neighborhood of X. We are to find some M-S field Y in $\mathcal{U}$. Let $\tilde{X} \in \mathcal{U}$ be such that the singularities of $\tilde{X}$ are all hyperbolic (we may take a K-S field here) and let $\mathcal{U}_1 \subseteq \mathcal{U}$ be a neighborhood of $\tilde{X}$ such that the singularities of every Y in $\mathcal{U}_1$ are all hyperbolic and Y has the same number of sinks, sources and saddles that $\hat{X}$ has by Lemma 2.1. The proof will consist of several cases.

Case 1. $\tilde{X}$ has no singularities. Then, the Euler characteristic $\aleph(\tilde{X})$ is 0 and the orientable M is a torus. Let Y be in $\mathcal{U}_1$ such that Y has a closed orbit τ and Y has no singularities by Lemma 3.12. Since Y does not have any singularities, τ does not bound an open disc D in M by Lemma 3.13. So, $M - \tau$ is homeomorphic to an open cylinder. Let Y_1 be in $\mathcal{U}_1$ such that τ is a hyperbolic closed orbit by Lemma 3.8. There exist a neighborhood U_τ of τ a neighborhood $\mathcal{V}$ of Y_1 such that every Z in $\mathcal{V}$ has a unique hyperbolic closed orbit τ_Z in U_τ by Lemma 3.8. Assume U_τ and $\mathcal{V}$ are small enough such that each Z in $\mathcal{V}$ does not have any singularities in U_τ.

Let Z_0 be a K-S field in $\mathcal{V}$ and let τ_0 be its unique hyperbolic closed orbit in U_τ. Because Z_0 does not have any singularities in U_τ, τ_0 does not bound an open disc in M by Lemma 3.13 so that $M - \tau_z$ is homeomorphic to a cylinder R. Because R is invariant for the flow of Z_0 and R can be embedded into S^2, Z_0 does not have a nontrivial recurrent orbit by Lemma 3.1. Hence, Z is an M-S field by Lemma 3.11.

Case 2. $\tilde{X}$ has at least one singularity. Assume $\tilde{X}$ has no saddles and take a K-S field Y in $\mathcal{U}_1$. As also Y has no saddles, Y does not have a nontrivial recurrent orbit by Lemma 3.5. By Lemma 3.11, Y is then an M-S field.

From now on, we will assume that $\tilde{X}$ has saddles. We will say that an unstable (stable) separatrix γ of a saddle of X is stabilized if $\omega(\gamma)$ $(\,\alpha(\gamma)\,)$ is a hyperbolic attractor (hyperbolic repellor). Suppose that a separatrix γ of X is stabilized. We claim that this stabilization is persistent for the vector fields close to $\tilde{X}$. We will find neighborhoods $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ of $\tilde{X}$ to prove this claim.

We will consider the case that $\alpha(\gamma)$ is a saddle σ and $\omega(\gamma)$ is a hyperbolic attractor τ as the other case is analogous. Let B_σ be a Grobman-Hartman neighborhood of σ and let $z_\sigma \in \cap\gamma \cap B_\sigma$ such that we have $z_\sigma^- \subseteq B_\sigma$.

Assume that τ is a hyperbolic closed orbit. Let $q \in \tau$ and let Σ_τ be an open transversal section through q such that we have $\Sigma_\tau \cap \tau = \{q\}$. Say, the Poincaré Return Map $P : \Sigma_\tau \to \Sigma_\tau$ is defined on a connected closed neighborhood V_q of q which contains the closure of an open connected neighborhood U_q of q in Σ_τ. As Σ_τ is open, $P(V_q)$ is in the interior of Σ_τ. Let U_q be as small as that U_q is contained in $W_d^s(q)$ where $W_d^s(q)$ is stable manifold of the hyperbolic fixed point q of P. Note that the dimension of the submanifold U_q is 1 so that we have $W_{d,U_q}^s(q) = U_q$.

If $\mathcal{V}$ is a small enough neighborhood of $\tilde{X}$, then the Poincaré Poincaré Return

Map $P_Z : \Sigma_\tau \to \Sigma_\tau$ is defined on V_q for the flow of every Z in $\mathcal{V}$ because V_q is compact and $P(V_q)$ is in the interior of Σ_τ. Also, each P_Z has a unique fixed point q_Z in V_q which is hyperbolic (and has the same index of q) if $\mathcal{V}$ is small enough. The Poincaré Return Maps P_Z in the $C^r(V_q, \bar{\Sigma}_\tau)$ space depend continuously on Z. So, if both U_q and $\mathcal{V}$ are small enough, then we can use the Stable Manifold Theorem to conclude $U_q \subseteq W^s_d(q_Z)$ for every Z in $\mathcal{V}$. Let $\mathcal{V}_1$ denote this small $\mathcal{V}$ neighborhood. Let p_τ denote the point at which γ first intersects U_q. If necessary, we can take a smaller U_q to ensure that the closed arc $z_\sigma p_\tau$ does not include the boundary points of U_q.

If τ is not a hyperbolic closed orbit, then it is a sink. Assume this case. Let U_τ be a small open neighborhood of τ such that it is diffeomorphic to an open ball and we also have $W^s_{U_\tau}(\tau) = U_\tau$. Let $p_\tau \in \gamma \cap U_\tau$ and Σ_τ be an open transversal section through p_τ and in U_τ. If U_τ is small enough, then we can use the Stable Manifold Theorem to find a small neighborhood $\mathcal{V}_1$ of $\tilde{X}$ and make the following conclusion: every Z in $\mathcal{V}_1$ has a unique sink τ_Z in U_q and also there exists an open neighborhood U_{τ_Z} of τ_Z such that; U_{τ_Z} is diffeomorphic to an open ball; $W^s_{U_{\tau_Z}}(\tau_Z) = U_{\tau_Z}$ and; $\Sigma_\tau \subseteq U_{\tau_Z}$. Let p_τ be the point in Σ_τ at which γ first intersects Σ_τ.

Let U_σ be an open neighborhood of σ such that we have $W^u_{U_\sigma}(\sigma) \subseteq W^u(\sigma)$ and $W^u_{U_\sigma}$ is a C^r embedded open interval. Let $p_\sigma \in W^u_{U_\sigma}(\sigma) \cap \gamma$ and Σ_σ be an open transversal section through p_σ in U_σ. Let A_σ be a a small compact connected neighborhood of p_σ in Σ_σ such that A_σ is contained in a tubular flow neighborhood V of $p_\sigma p_\tau$. Here, V is so small so that V does not contain the boundary points of Σ_σ. Also, it is so small so that it does not include the boundary points of Σ_τ in the case that τ is a sink or it is so small so that it does not include the boundary points of U_q in the case that τ is a closed orbit. This last requirement is provided by the fact that $z_\sigma p_\tau$ does not include the boundary points of U_q.

If U_σ is small enough, then we can again use Stable Manifold Theorem to find a small neighborhood $\mathcal{V}_2$ of $\tilde{X}$ and make a similar conclusion as before: every Z in $\mathcal{V}_2$ has a unique saddle σ_Z in U_σ and there exists an open neighborhood U_{σ_Z} of σ_Z such that we have $W^u_{U_{\sigma_Z}}(\sigma_Z) \subseteq W^u(\sigma_Z)$; $W^u_{U_{\sigma_Z}}(\sigma_Z)$ is a C^r embedded open interval and;

$W^u_{U_{\sigma_Z}}(\sigma_Z) \cap A_\sigma \neq \varnothing$. So, each σ_Z has an unstable separatrix γ_Z which intersects A_σ.

If τ is a sink, define P_σ to be the First Intersection Assignment $P_\sigma : \Sigma_\sigma \cap V \to \Sigma_\tau \cap V$. If τ is a closed orbit, define P_σ to be the First Intersection Assignment $P_\sigma : \Sigma_\sigma \cap V \to U_q \cap V$. Note that P_σ is a diffeomorphism in either case. As A_σ and and $P_\sigma(A_\sigma)$ are compact transversal sections to $\tilde{X}$, they will be also compact transversal sections to vector fields that are close to $\tilde{X}$. Consider a point q in A_σ. Then, for the vector fields close to $\tilde{X}$, the positive semi trajectories of points in a small neighborhood of q in $\Sigma_\sigma \cap V$ intersect Σ_τ at a point in a small neighborhood of $P_\sigma(q)$ in $\Sigma_\tau \cap V$ or $U_q \cap V$. As A_σ is compact, there exists a neighborhood $\mathcal{V}_3$ of $\tilde{X}$ such that the positive semi trajectories of all points in A_σ intersect $\Sigma_\tau \cap V$ or $U_q \cap V$ for the flow of every vector field in $\mathcal{V}_3$.

Let $\mathcal{V}_0 = \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3$. Then, we have $\omega(\gamma_Z) = \tau_Z$ for any Z in V_0 and our claim has been proven. See Figure 3.10. Of course, we need to consider $\mathcal{V}_0 \cap \mathcal{U}_1$ so that it is contained in $\mathcal{U}_1$.

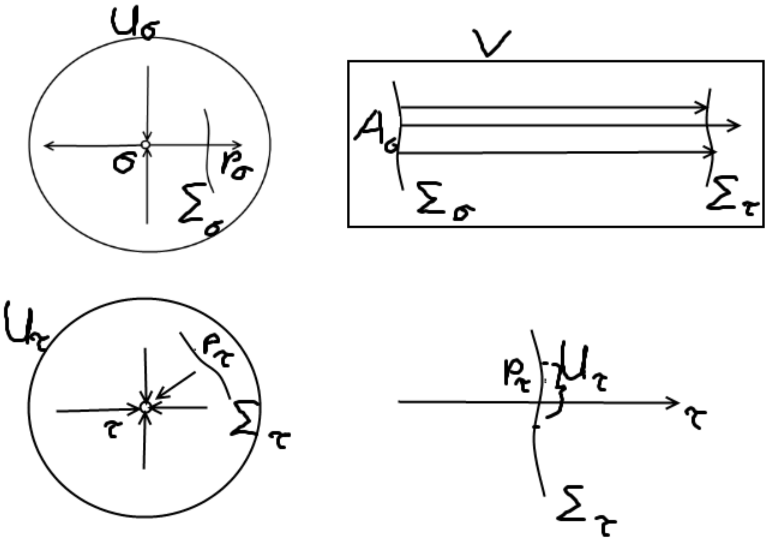

Figure 3.10. The neighborhoods U_σ and V. There are two cases for the neighborhood U_τ.

Now, let $\{\gamma_1, \ldots, \gamma_n\}$ be the set of all separatrices of $\tilde{X}$ which are stabilized. As we have shown above, we can find some neighborhood $\mathcal{U}_2$ of $\tilde{X}$ with $\mathcal{U}_2 \subseteq \mathcal{U}_1$ such that every Z in $\mathcal{U}_2$ has n stabilized separatrices each of which corresponds to a unique γ_j $(1 \leq j \leq n)$.

Assume now that all the finitely many separatrices of saddles of $\tilde{X}$ are stabilized. Let Z be a K-S field in $\mathcal{U}_2$. As all the separatrices of Z are stabilized, Z cannot have have nontrivial recurrent orbits by Lemma 3.5. By Lemma 3.11, Z is an M-S field.

Assume that not all separatrices of saddles of $\tilde{X}$ are stabilized. We claim that for any neighborhood $\mathcal{V} \subseteq \mathcal{U}_2$ of $\tilde{X}$, there exists some Y in $\mathcal{V}$ that has one more stabilized separatrix than $\tilde{X}$ has. Assume the claim for the moment. Since the number of separatrices of saddles of $\tilde{X}$ are finite and $\mathcal{V}$ can be taken arbitrarily small, we can stabilize all the separatrices of saddles of $\tilde{X}$ one by one with this claim within $\mathcal{U}_2$. By the previous argument, we can find some M-S field Z in $\mathcal{U}_2$. So, the proof of the theorem will be complete once we prove this claim.

As the number of separatrices of saddles of $\tilde{X}$ is finite, we can apply Lemma 3.6 to $\tilde{X}$ and to its successive perturbations finitely many times (if necessary) to find some Y in $\mathcal{V}$ such that Y has no nontrivial recurrent orbits. We will consider now the four possible cases for Y.

Case 2a. Y has no saddle connections. If all the separatrices of saddles of Y are stabilized, then we are done. Assume otherwise and let β be a separatrix of a saddle of Y which is not stabilized.

We will consider the case that β is an unstable separatrix as the other case is analogous. As Y has only trivial recurrent orbits and no saddle connections, $\omega(\beta)$ is a hyperbolic singularity or a closed orbit by Lemma 3.10. It cannot be a hyperbolic singularity because there are no saddle connections and β is not stabilized. Hence, $\omega(\beta)$ is a closed orbit τ that is not hyperbolic. Let Z be a vector field in $\mathcal{V}$ such that τ is a hyperbolic closed orbit of Z and also, we have $\omega(\beta) = \tau$ by Corollary 3.1. Thus, β has been stabilized.

Case 2b. Y has a saddle graph $\mathcal{G}$ that is $\omega(p)$ or $\alpha(p)$ for some orbit $p \in M$. Again, we will consider the case $\omega(p)$ as the other case is analogous.

Let $U_{\mathcal{G}}$ be a band neighborhood of $\mathcal{G}$. Note that the only singularities in $U_{\mathcal{G}}$ are the saddles of $\mathcal{G}$. Let γ be a separatrix of $\mathcal{G}$ and let $\sigma = \omega(\gamma)$. Let U_σ be the Grobman-Hartman neighborhood of σ where U_σ is one of the sets in the union $U_{\mathcal{G}}$. Let q be a point in γ and Σ be a transversal section through q in $U_{\mathcal{G}}$. Let Σ^a, Σ^b be the connected components of $\Sigma - \{q\}$. Let $\gamma_1^\sigma, \gamma_2^\sigma$ be the two unstable separatrices of σ. Let $z_1 \in \gamma_1^\sigma \cap U_\sigma$ and $z_2 \in \gamma_2^\sigma \cap U_\sigma$ be such that we have $z_1^- \subseteq U_\sigma$ and $z_2^- \subseteq U_\sigma$. Let Σ_1 and Σ_2 be the two transversal sections in $U_{\mathcal{G}}$ through z_1 and z_2 respectively. Let Σ_1^a and Σ_1^b be the connected components of $\Sigma_1 - \{z_1\}$ and let Σ_2^a and Σ_2^b be the connected components of $\Sigma_2 - \{z_2\}$. If Σ is small enough, then the First Intersection Assignment $P_a : \Sigma^a \to \Sigma_j^x$ is defined on the whole Σ^a for $j = 1$ or 2 and $x = a$ or b and P_a is a local C^r diffeomorphism. This last conclusion follows from the applications of Grobman-Hartman Theorem and Tubular Flow Theorem to the relevant neighborhoods of $U_{\mathcal{G}}$. Similarly, the Poincaré Map $P_b : \Sigma^b \to \Sigma_k^y$ is defined on the whole Σ^b for $k = 1$ or 2 and $y = a$ or b if Σ is small enough. Note that k and j are distinct.

Define a relation "$\leq$" on Σ^a as: For $c, d \in \Sigma^1$, $c \leq d$ if and only if $c \in [q, d]$ where $[q, d]$ denotes the connected closed interval in Σ with boundary points q and d. Then, this definition is reflexive and transitive; i.e. for all $c \in \Sigma$, $c \leq c$ and for $c, d, e \in \Sigma^a$, $c \leq d$ and $d \leq e$ imply that $c \leq e$. Define the partial relations "$\leq$" on $\Sigma^b, \Sigma_1^a, \Sigma_1^b, \Sigma_2^a, \Sigma_2^b$ in a similar way. Then, the First Intersection Assignment $P_a : \Sigma^a \to \Sigma_j^x$ preserves the "$\leq$" relation by the Grobman-Hartman Theorem and the Tubular Flow Theorem; i.e. for $c, d \in \Sigma^a$ with $c \leq d$, we have $P_a(c) \leq P_a(d)$. Of course, P_b preserves the partial relation as well.

Observe that the unstable separatrix γ of $\mathcal{G}$ with which we have started the above argument was arbitrary so that we can define the all the other relevant First Intersection Assignments and the "$\leq$" relations on respective transversal sections. Say, p^+ accumulates to γ from the side Σ^a; i.e. q is an accumulation point of $p^+ \cap \Sigma^a$. Then, the Poincaré Return Map $T_a : \Sigma^a \to \Sigma^a$ is defined on some $\tilde{\Sigma}^a$ where $\tilde{\Sigma}^a$ is a connected open subset of Σ^a such that q is a boundary point of $\tilde{\Sigma}^a$. This last conclusion about T_a follows from repeated applications (finitely many times) of our previous argument where T_a is the composition of finitely many First Intersection Assignments. Hence, T_a is a local C^r diffeomorphism if $\tilde{\Sigma}^a$ is small enough and it preserves the "$\leq$" relation as well since each relevant First Intersection Assignment preserves the relations "$\leq$".

Let $p_1 \in p^+ \cap \tilde{\Sigma}^a$ and $p_2 = T_a(p_1)$. The ideas in the following argument are similar to the ideas in the proof of Lemma 3.7 and we will omit a few complicated details here. We have $p_2 \neq p_1$ because otherwise it would have been a closed orbit and $\omega(p)$ would have been itself and not $\mathcal{G}$. If p_1 is close enough to q, then we have $p_2 < p_1$ and the sequence $\{T_a^j(p_1)\}_{j \in \mathbb{N}}$ is a strictly decreasing sequence in the sense of the relation"$\leq$" on $\tilde{\Sigma}^a$. Let $(q, p_1]$ denote the connected half open interval in Σ^a with boundary points q and p_1. Then, we have $T_a(\tilde{\Sigma}^a) \subsetneq \tilde{\Sigma}^a$ provided that $\tilde{\Sigma}^a$ is small enough; namely, $\tilde{\Sigma}^a \subseteq (q, p_1]$. We will assume the open transversal section $\tilde{\Sigma}^a$ to be as small as this. Then, we have $T_a(x) \neq x$ for all x in $T_a(\tilde{\Sigma}^a)$. So, the closed arc $wT_a(w)$ is well defined for w in $\tilde{\Sigma}^a$. Let $A = \displaystyle\bigcup_{w \in \tilde{\Sigma}^a} wT_a(w)$. Because A is a union of tubular flow neighborhoods of closed arcs $wT_a(w)$ for w in $\tilde{\Sigma}^a$, the connected set $A - \tilde{\Sigma}^a$ is diffeomorphic to an open disc.

We remark that the observations of T_a and $A - \tilde{\Sigma}^a$ applies to any M whether it is orientable or not and $A \cup \mathcal{G}$ can be complicated if M is not orientable. However, when M is orientable, we claim the following simple situation: each saddle in $\mathcal{G}$ has exactly one stable separatrix and one unstable separatrix that are in $\mathcal{G}$. In fact, let η_1 be a saddle in $\mathcal{G}$ and let τ_1 be an unstable separatrix of η_1 which is in $\mathcal{G}$. Inductively (for $j \geq 1$), define $\eta_{j+1} := \omega(\tau_1)$ and let τ_{j+1} be an unstable separatrix of η_2 that is in $\mathcal{G}$. The number of saddles are finite and for some $k \geq 1$, we have $\eta_k = \eta_1$. The union $\mathcal{G}_c = \{\eta_1, \ldots, \eta_k\} \cup \tau_1 \cup \cdots \cup \tau_k$ is a saddle graph which is a simple closed continuous curve in M. Because M is orientable, a circle neighborhood U_c of $\mathcal{G}_c$ can be homeomorphic to only an open cylinder. Therefore, p_1^+ which intersects Σ^a does not intersect Σ^b if Σ is small enough because $\mathcal{G}_c$ separates U_c into two disjoint regions and $\Sigma^a \cap U_c$ and $\Sigma^b \cap U_c$ lie in distinct separated regions. Hence, we have $\mathcal{G}_c = \mathcal{G}$ and our claim has been proven.

Let (f, F) be a Flow Box at q such that we have $F \subseteq U_{\mathcal{G}}$ and $\tilde{\Sigma}^a \nsubseteq F$. Because we have $F \subseteq U_{\mathcal{G}}$, the set $\gamma \cap F$ is connected and also, F does not intersect any other saddle connection in $\mathcal{G}$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that we have $\operatorname{supp}(g) = f(F)$ and $g > 0$ on the interior of $f(F)$. Let e_1, e_2 be the standard basis of $\mathbb{R}^2$. Recall that $f_*(Y_{|F})(x) = e_1$ for all x in $f(F)$. Let $\hat{Z}(q) = e_1 \pm g(q) \cdot e_2$ on $F(f)$ where the ± sign is to be determined later. Let Z be the C^r vector field on M such that Z is equal to Y on F^c and Z is equal to $f_*^{-1}(\hat{Z})$ on F. If $\|g\|_r$ is small enough, then Z is in $\mathcal{V}$. Let p_γ be a point of $\gamma - F$ such that we have ${p_\gamma}^- \cap F = \varnothing$ (for any of the flow of Y or Z). Now, choose ± sign such that ${p_\gamma}^+$ enters into the region A.

Observe the following four properties: A is invariant by the positive flow of Y; i.e. $Y_t(A) \subseteq A$ for $t \geq 0$; ${p_\gamma}^+$ enters A by our choice of ± and; the condition $\tilde{\Sigma}^a \nsubseteq F$. By these four properties, the Poincaré Return Map $\tilde{T}_a : \tilde{\Sigma}^a \to \tilde{\Sigma}^a$ is defined on the whole $\tilde{\Sigma}^a$ for the flow of Z. Because A is orientable, $\tilde{T}_a$ preserves the orientation induced from the relation "$\geq$". Again by the choice of ± sign and $\tilde{\Sigma}^a \nsubseteq F$, we have $\tilde{T}_a(\tilde{\Sigma}^a) \subseteq \tilde{\Sigma}^a$.

Let q_j be the j^{th} intersection of ${p_\gamma}^+$ with $\tilde{\Sigma}^a$ for the flow of Z. Clearly, q_1 is defined. Because of $\tilde{T}_a(\tilde{\Sigma}^a) \subseteq \tilde{\Sigma}^a$, q_j is defined for every positive integer j. We cannot have $q_2 = q_1$ because $\alpha(q_1)$ is a saddle. Since ${p_\gamma}^-$ is not in A and ${p_\gamma}^+$ enters the region A, we have $q_2 > q_1$. As $q_{j+1} = \tilde{T}_a(q_j)$ and $\tilde{T}_a$ preserves the relation"$\geq$", we conclude that q_j's are monotonically increasing (farther away from q).

Let $w \in \tilde{\Sigma}^a - F$. Recall that the Poincaré Return map $T_a : \Sigma^a \to \Sigma^a$ is defined for the flow of X. If $T_a(w)$ is not in F, then we have $wT_a(w) \cap F = \varnothing$ and $\tilde{T}_a(w) = T_a(w)$. In this case, $\tilde{T}_a(w) = T_a(w) < w$. If $T_a(w)$ is in F, then $\tilde{T}_a(w)$ is in F and we again have $\tilde{T}_a(w) < w$. So, we always have $\tilde{T}_a(w) < w$ (in either case). Let $w_n = \tilde{T}_a^n(w)$ for $n \in \mathbb{N}$. Then, $\{w_n\}_{n \in \mathbb{N}}$ is a decreasing sequence since $w_1 < w_0$ and $\tilde{T}^a$ preserves orientation. See Figure 3.11.

Recall that $\tilde{T}_a$ is a local diffeomorphism. As $q_1 < w_0$ and the sequences $\{q_n\}_{n \in \mathbb{N}}$

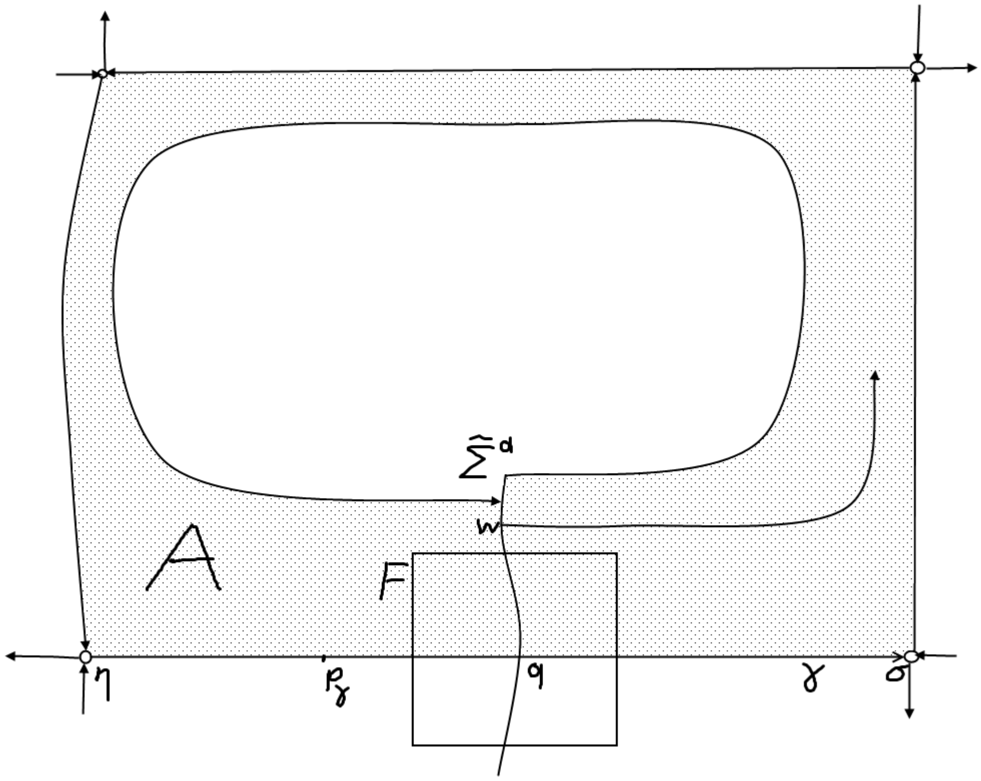

Figure 3.11. Case 2b in the proof of Theorem 3.1.

and $\{w_n\}_{n\in\mathbb{N}}$ are increasing and decreasing respectively, we conclude that the limit point of $\{q_n\}_{n\in\mathbb{N}}$ is a fixed point of $\tilde{T}^a$. So, ${p_\gamma}^+$ accumulates to a closed orbit τ of Z.

Let Z_1 be a C^r vector field in $\mathcal{V}$ such that τ is a hyperbolic closed orbit of Z_1 and also, we have $\omega(p_\gamma) = \tau$ by Corollary 3.1. Thus, we have stabilized one more separatrix of Y.

Case 2c. Y has a saddle graph $\mathcal{G}$ that is accumulated by closed orbits of Y. Let $U_{\mathcal{G}}$ be a band neighborhood of $\mathcal{G}$. This case will be similar to the previous Case 2b. As we have explained there, each saddle in $\mathcal{G}$ has only one unstable separatrix and only one unstable separatrix that are in $\mathcal{G}$ and $\mathcal{G}$ is a simple closed continuous curve the circle neighborhood of which is homeomorphic to an open cylinder.

Let γ be a separatrix in $\mathcal{G}$ and $q \in \gamma$. Let Σ be a transversal section through q in $U_{\mathcal{G}}$ and let Σ^a and Σ^b be the connected components of $\Sigma - \{q\}$. Let $\{\tau_n\}$ be the sequence of closed orbits of Y that accumulate to $\mathcal{G}$. Say, they accumulate from the side Σ^a; i.e. q is a limit point of the set $\Sigma^a \cap \Big(\bigcup_{n\in\mathbb{N}} \tau_n \Big)$. We can take a subset of them and assume that all τ_n's accumulate from the side Σ^a. Define "$\leq$" relation on Σ^a as in Case 2b. Note that $\tau_n \cap \Sigma^a$ is finite for each n because τ_n is compact and Σ^a is transversal to Y.

Let $T_a : \Sigma^a \to \Sigma^a$ be the Poincaré Return Map. Since $U_{\mathcal{G}}$ is orientable, T_a preserves the orientation of Σ^a induced from "$\leq$". Let $j \in \mathbb{N}$ and $z \in \Sigma^a \cap \tau_j$. If Σ is small enough, then we have $T^a(z) = z$ for z in any $\tau_j \cap \Sigma^a$ because otherwise, the intersections of τ_j with Σ^a would have been either strictly increasing or strictly decreasing so that τ_j would have been not a closed orbit. Let Σ be small in this sense. Hence, we have $|\Sigma^a \cap \tau_n| \in \{0, 1\}$ for all $n \in \mathbb{N}$. Note that γ in $\mathcal{G}$ was arbitrary and we could have applied this same argument by starting with another separatrix γ' in $\mathcal{G}$ and we could have deduced the same conclusions.

We can assume that the Grobman-Hartman neighborhoods and the tubular flow neighborhoods in the union $U_{\mathcal{G}}$ are small enough such that $U_{\mathcal{G}}$ is a circle neighborhood of $U_{\mathcal{G}}$. If τ_k is close enough to $\mathcal{G}$, then τ_k will be in $\mathcal{G}$ and also it will separate $U_{\mathcal{G}}$ into two disjoint open cylinders. So, $\mathcal{G}$ and τ_k bound an open region A in $U_{\mathcal{G}}$ that is homeomorphic to an open cylinder. So, A is invariant by the flow of Y. We can also assume that τ_k is close enough to $\mathcal{G}$ so that we have $\tau_k \cap \Sigma^a = \varnothing$. Then, $T_a^n(A \cap \Sigma^a) = A \cap \Sigma^a$ for every positive integer n. Say, $\{w\} = \tau_k \cap \Sigma^a$. See Figure 3.12.

Let (f, F) be a small flow box at q such that we have $F \subseteq U_{\mathcal{G}}$ and $F \cap \tau_k = \varnothing$. Because we have $F \subseteq U_{\mathcal{G}}$, $\gamma \cap F$ is connected and also, F does not intersect any other saddle connection in $\mathcal{G}$. We now mimic the perturbation in the Case 2b. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that we have $\mathrm{supp}(g) = f(F)$ and $g > 0$ on the interior of $f(F)$. Let e_1, e_2 be the standard basis of $\mathbb{R}^2$. Let $\hat{Z}(q) = e_1 \pm g(q) \cdot e_2$ on $F(f)$ where the $\pm$ sign is to be determined later. Let Z be the vector field on M such that Z is equal to Y on F^c and Z is equal to $f_*^{-1}(\hat{Z})$ on F. Then, Z is C^r. Moreover, if $\|g\|_r$ is small enough, then Z is in $\mathcal{V}$. Let p_γ be a point of $\gamma - F$ such that we have ${p_\gamma}^- \cap F = \varnothing$ (for any of the flow of Y or Z). Now, choose $\pm$ sign such that ${p_\gamma}^+$ enters

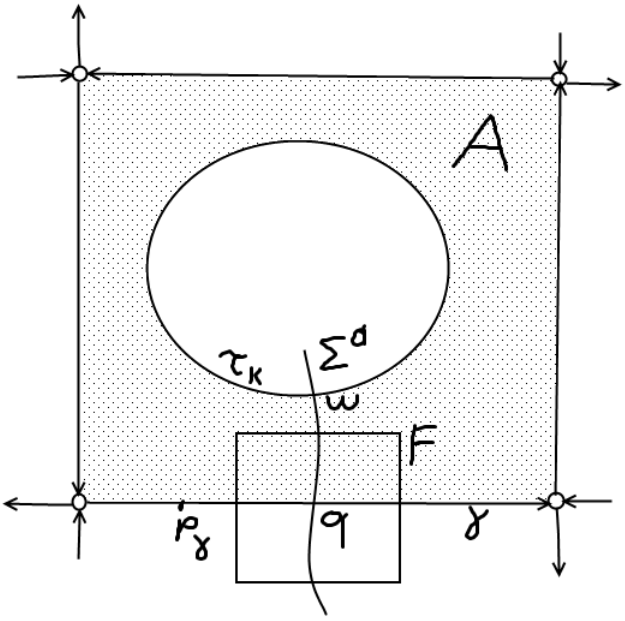

Figure 3.12. Case 2c in the proof of Theorem 3.1.

into the region A.

Let $\tilde{\Sigma}^a = (\Sigma^a \cap A) \cup \{w\}$. Then, the Poincaré Return Map $\tilde{T}_a : \tilde{\Sigma}^a \to \tilde{\Sigma}^a$ is defined on the whole $\tilde{\Sigma}^a$ for the flow of Z because of all the following properties: $Y_t(A) \subseteq A$ for all $t \geq 0$; $F \cap \tau_k = \varnothing$ and; also because of our choice of ± sign. Let q_j be the j^{th} intersection of ${p_\gamma}^+$ with $\tilde{\Sigma}^a$ for the flow of Z. Clearly, q_1 is defined so that q_j is defined for every positive integer j. We cannot have $q_2 = q_1$ because $\alpha(q_1)$ is a saddle. Because ${p_\gamma}^-$ is not in A and ${p_\gamma}^+$ enters the region A, we have $q_2 > q_1$. As $q_{j+1} = \tilde{T}_a(q_j)$ and $\tilde{T}_a$ preserves the relation"$\geq$", q_j's are monotonically increasing (farther away from q). Note that $\tilde{T}_a^n(w) = w$ for every positive integer n and also $q_1 < w$. Hence, the sequence $\{q_j\}_{j \in \mathbb{N}}$ converges to a fixed point of $\tilde{T}_a$ so that ${p_\sigma}^+$ accumulates to a closed orbit τ of Z.

Let Z_1 be a C^r vector field in $\mathcal{V}$ such that τ is a hyperbolic closed orbit of Z_1 and also, we have $\omega(p_\gamma) = \tau$ by Corollary 3.1. Thus, we have stabilized one more separatrix of Y.

Case 2d. When we will describe this last case, we will make some claims about some not possible situations. They are actually possible but we will exclude them as they have been studied in our previous cases. Note that Y has a saddle connection as we have dealt with the other situation in Case 2a.

We claim that Y does not have a saddle graph. Assume otherwise and let $\mathcal{G}$ be saddle graph for Y. Let $U_{\mathcal{G}}$ be a band neighborhood of $\mathcal{G}$. Let η be a saddle of $\mathcal{G}$ and let U_η be the Grobman-Hartman neighborhood of η which is one of the sets in the union $U_{\mathcal{G}}$. Let γ_η be an unstable separatrix of η such that γ_η is in $\mathcal{G}$. Let q be in $\gamma_\eta \cap U_{\mathcal{G}}$ and Σ be a transversal section through q in U_η. Let Σ^a and Σ^b be the connected components of $\Sigma - \{q\}$. Let $T_a : \Sigma^a \to \Sigma^a$ and $T_b : \Sigma^b \to \Sigma^b$ be the Poincaré Return Maps.

As we have explained before, $U_{\mathcal{G}}$ is a simple closed continuous curve the circle neighborhood of which is homeomorphic to a cylinder. So, there exists a point z in Σ^a or Σ^b such that: either T_a or T_b is defined at z; say, $T_a(z)$ is defined; the closed arc $zT_a(z)$ (for $z \neq T_a(z)$) or the closed orbit τ_z through z (for $T_a(z) = z$) is in $U_{\mathcal{G}}$. Note that we can argue these conclusions for any z in Σ^a that are close enough to q. The point q cannot be accumulated by closed orbits that intersect Σ^a because this situation refers to Case 2c which we exclude. So, there exist a neighborhood V_q of q in Σ such that T_a does not have any fixed points in $V_q \cap \Sigma^a$. Define the relation "$\leq$" on Σ^a as in the Case 2b. Let z be a point $V_q \cap \Sigma^a$ such that we have $T_a(z) \in V_q \cap \Sigma^a$. So, we have either $z < T_a(z)$ or $z > T_a(z)$. Because of $T_a(z) \in V_q \cap \Sigma^a$, either z^+ or z^- intersect Σ^a monotonically and their intersections will accumulate to q but this situation is Case 2b as either $\omega(z)$ or $\alpha(z)$ is equal to $\mathcal{G}$. As we exclude this situation, our claim has been proven.

As Y has a saddle connection and Y does not have a saddle graph, there exists a saddle connection γ of Y with the following property: the saddle $\sigma = \omega(\gamma)$ has an unstable separatrix γ_1 such that $\omega(\gamma_1)$ is not a saddle. So, $\omega(\gamma_1)$ is either a closed orbit or a sink by Lemma 3.10. Let U_σ be a Grobman-Hartman neighborhood of σ. Let $w \in \gamma \cap U_\sigma$ be such that we have $w^+ \subseteq U_\sigma$. Let $w_1 = Y_1(w)$ and Σ_1 be a transversal section through w_1 in U_σ. Let Σ_1^a and Σ_1^b be the connected components of $\Sigma_1 - \{w_1\}$.

Let $p \in \gamma_1 \cap U_\sigma$ such that we have $p^- \subseteq U_\sigma$.

Assume now that $\omega(\gamma_1)$ is a hyperbolic attractor η. Let U_η be a small open neighborhood of η such that we have $U_\eta \cap W^s(\eta) = U_\eta$ and also, w is not in $\bar{U}_\eta$. Let $p_\eta \in U_\eta \cap \gamma_1$ and let Σ_η be an open transversal section through η in U_η. Let V_0 be a small tubular flow neighborhood of pp_η such that V_0 does not include the boundary points of Σ_η and also, w is not in $\bar{V}_0$. So, we have $V_0 \subseteq W^s(\eta)$ because of $\Sigma_\eta \subseteq U_\eta$. Also, the positive semi trajectories of points either in Σ_1^a or in Σ_1^b enter V_0 and go to η provided that they are close enough to w_1. Say, the positive semi trajectories of points in Σ_1^a which are close to w_1 have this property.

Since w is not in $\bar{V}_0 \cup \bar{U}_\eta$, we can choose a small flow box (f, F) at w such that we have $F \subseteq U_\sigma$ and $F \cap (V_0 \cup U_\eta) = \varnothing$. We again mimic our previous perturbations in previous cases. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that we have $\text{supp}(g) = f(F)$ and $g > 0$ on the interior of $f(F)$. Let e_1, e_2 be the standard basis of $\mathbb{R}^2$. Let $\hat{Z}(q) = e_1 \pm g(q) \cdot e_2$ on $F(f)$ where the ± sign is to be determined later. Let Z be the vector field on M such that Z is equal to Y on F^c and Z is equal to $f_*^{-1}(\hat{Z})$ on F. Then, Z is C^r and it is also in $\mathcal{V}$ if $\|g\|_r$ is small enough. Now, choose the ± sign and make $\|g\|_r$ small enough such that w^+ intersects Σ_1^a. If $\|g\|_r$ is small enough, then w^+ enters V_0 because of $F \subseteq U_\sigma$ and the Grobman-Hartman Theorem. As we have $F \cap V_0 = \varnothing$, the trajectory w^+ enters U_η and as we have $F \cap U_\eta = \varnothing$, we conclude that $\omega(w) = \eta$. So, we can stabilize one more separatrix in this case.

If $\omega(\gamma_1)$ is not a hyperbolic attractor, then it is a closed orbit τ. Let U_τ be a neighborhood of τ such that it is disjoint from γ. This requirement is possible since $\bar{\gamma}$ and τ are disjoint compact sets. By Corollary 3.1, we can find some $Z \in \mathcal{V}$ such that: Z is equal to Y outside of U_τ; τ is a hyperbolic closed orbit of Z and; we have $\omega(\gamma_1) = \tau$. Now, we are in the previous situation as we have: γ is a saddle connection; $\omega(\gamma)$ has an unstable separatrix γ_1 such that $\omega(\gamma_1)$ is a hyperbolic attractor. This competes the proof for Case 2d. As we have considered all possible cases (1, 2a, 2b, 2c, 2d), the proof of the theorem is complete. □

3.6. The Reliance of Theorem 3.1 on Lemma 3.6

We make a few remarks about the density of M-S fields on orientable compact 2-manifolds without boundary before continuing onto the next chapter. The only obstacle in Theorem 3.1 that prevents to generalize this theorem to non-orientable manifolds is the application of Lemma 3.6. Whether this lemma can be proven on non-orientable manifolds has been an open problem. Nontrivial recurrent orbits exhibit complicated behavior to which we hope to convince our reader by proving Lemma 5.10. Note that their absence ensures the powerful Lemma 3.10 that has been used in the proof of Theorem 3.1.

We will prove in the next chapter that M-S fields are dense in $\mathfrak{X}^r(M)$ when M is Klein Bottle. Note that Klein Bottle and torus are the only compact 2-manifolds which admit vector fields without any singularities. Now, we explain that the Cases 2a, 2b, 2c, 2d in the above theorem can be generalized to non-orientable ones. Recall that the vector field Y in these cases does not have nontrivial recurrent orbits and Lemma 3.10 applies.

Case 2a does not rely on the fact that M is orientable at all. Case 2b does not immediately generalize because a band neighborhood $U_{\mathcal{G}}$ of a saddle graph $\mathcal{G}$ can have regions which are homeomorphic to Möbius bands and $\mathcal{G}$ is not necessarily a simple closed continuous curve. In the proof of Case 2b, we have relied on a simpler saddle graph $\mathcal{G}$ which is a simple closed continuous curve the circle neighborhood of which is homeomorphic to an open cylinder. Nevertheless, the ideas there and the perturbation on a small flow (f, F) do actually work out if one carefully thinks how the positive semi trajectories leave and come back to F. With this perturbation, one can still create a closed orbit τ to which γ accumulates. Then, one makes τ hyperbolic if necessary. The Case 2c is similar to the Case 2b so that we are left with the Case 2d.

Assume a saddle connection γ is accumulated by closed orbits $\{\tau_n\}_{n\in\mathbb{N}}$. Then, τ_n's do not accumulate to γ only but also to a saddle graph $\mathcal{G}$ which contains γ. This can be seen in the following way: let $\omega(\gamma) = \sigma$ and γ_1 be an unstable separatrix of σ.

Then τ_n's accumulate to γ_1 as well. $\omega(\gamma_1)$ cannot be a hyperbolic attractor because a hyperbolic attractor cannot be accumulated by closed orbits. $\omega(\gamma_1)$ cannot be a closed orbit because this closed orbits would have behaved like an attractor by Lemma 3.7. $\omega(\gamma_1)$ cannot be a saddle graph $\mathcal{G}_1$ because otherwise τ_n's would have accumulated to $\mathcal{G}_1$ but a saddle graph cannot be accumulated by both closed orbits and a single (or more) non-closed orbit. One sees this fact by taking a transversal section Σ through a point q of a separatrix γ_0 in $\mathcal{G}$, and then one analyzes the Poincaré Return Map $P : \Sigma \to \Sigma$. Let Σ^a and Σ^b be the connected components of $\Sigma - \{q\}$. The assumption that γ_0 is accumulated by closed orbits from the side Σ^a implies that P has fixed points on Σ^a that accumulate to q so that for any z in Σ^a, the sequence $\{P^n(z)\}_{n\in\mathbb{N}}$ cannot accumulate to q (if this sequence is defined at all). Hence, $\omega(\gamma_1)$ which is not a saddle graph, closed orbit or a sink is a saddle σ_1. σ_1 might be distinct from σ but one again finds an unstable separatrix of σ_1 to which τ_n's accumulate. Continuing this process by an inductive manner, one necessarily finds a saddle graph $\mathcal{G}_2$ to which τ_n's accumulate because the number of saddles of Y are finite. This situation, however, refers to case 2c which we exclude in case 2d. Therefore, one finds a saddle connection γ which is not accumulated by closed orbits. Also, it cannot be accumulated by a non-closed orbit β because then, either $\omega(\beta)$ or $\alpha(\gamma)$ would have been a saddle graph by Lemma 3.10 which refers to Case 2b. So in Case 2d, even when M is nonorientable, one can find a saddle connection γ with the following property: $\omega(\gamma)$ has an unstable separatrix γ_1 such that $\omega(\gamma_1)$ is either a closed orbit or a sink. The rest of the proof of Case 2d does not require M to be orientable.

4. SOME RESULTS FOR NONORIENTABLE 2-MANIFOLDS

4.1. $\mathbb{R}P^2$ and Klein Bottle K

This section will finally prove Corollary 4.1 which states that M-S vector fields are dense in $\mathfrak{X}^r(M)$ when M is $\mathbb{R}P^2$ or a Klein bottle K. All the material here except Lemma 4.2 is due to [10] but Lemma 4.2 easily follows from Lemma 4.1 which can be found in [10]. The below lemmas are stated according to our terminology and their original statements in [10] slightly differ from ours. The density of M-S fields in $\mathfrak{X}^r(K)$ follows from Lemma 4.4 which was first proven in [9] by N. Markley but Gutierrez gave a simple and elegant proof for it in [10].

The notation $\Lambda(Z)$ which is used in this section chapter will denote the sum of the indexes of the finitely many singularities of a C^r vector field Z on any relevant manifold. For the proofs of Lemma 4.1, Lemma 4.3 and Lemma 4.5, we will assume the following fact: If C is a two sided simple closed C^1 curve in M, then there exists a smooth vector field Z on M such that Z has finitely many singularities and C is transversal to Z. We omit a rigorous proof for this assumption but instead, we will try to justify it.

Let U be a circle neighborhood of the two sided simple closed C^1 curve C. Recall that U is diffeomorphic to an open cylinder and C separates U into two disjoint open cylinders. The set of vector fields on M the singularities of which are all hyperbolic is open and dense in $\mathfrak{X}^r(M)$. In particular, these vector fields have finitely many singularities because M is compact and hyperbolic singularities are isolated in M. For any $Z \in \mathfrak{X}^r(M)$, let C_Z denote the set of all points in C at which C is not transversal to Z. By standard transversality arguments, we can find a smooth vector filed Z_0 on M such that: Z_0 has finitely many singularities; Z_0 has no singularities in U and; all the points of C_{Z_0} are isolated in C. As C is compact, the set C_{Z_0} is a finite set. Assume that C_{Z_0} is nonempty (otherwise, we are done). For a point $w \in C_{Z_0}$, there are two possible cases that are illustrated in Figure 4.1. Assume the first case in Figure 4.1. Then, the local picture of Z_0 at w looks like Figure 4.2. By a local perturbation of Z_0 at w, we can solve the transversality failure of C to Z_0 at w. So, we may assume that all the points in C_{Z_0} refer to the second case of Figure 4.1. With this assumption, the number of points of C_{Z_0} is even.

Figure 4.1. Two cases for $w \in C_{Z_0}$.

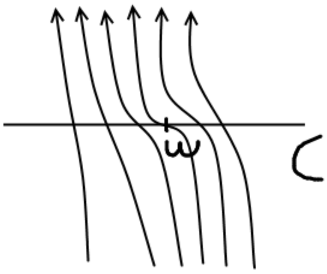

Figure 4.2. The first case of $w \in C_{Z_0}$ in Figure 4.1.

We will consider the case that C_{Z_0} has two points w_1 and w_2 as our argument can easily be generalized to the general case when C_{Z_0} has any even number of points all of which refer to the second case in Figure 4.1. Then, Z_0 in U looks like Figure 4.3 if U is small enough. Note the identifications a's of two sides in Figure 4.3 so that the depicted neighborhood U there is diffeomorphic to a cylinder. We need to locally perturb Z_0 only in the open cylinder U so that Z_0 does not change at the boundary of U after the perturbation and the obtained vector field Z_1 is smooth. Figure 4.4 shows such a perturbation of Z_0 in U where C is transversal to the obtained vector field Z_1. The vector field Z_1 has two saddles σ_1 and σ_2, a source η and a sink ζ in U. Note that the indexes of saddles are -1 and the indexes of a source and a sink are 1 so that their total sum is 0. So, we have $\Lambda(Z_0) = \Lambda(Z_1)$ and our perturbation of Z_0 in U has been indeed valid.

Lemma 4.1. *Suppose that C is a two sided simple closed C^1 curve in $\mathbb{R}P^2$. Then, C bounds an open disc in $\mathbb{R}P^2$.*

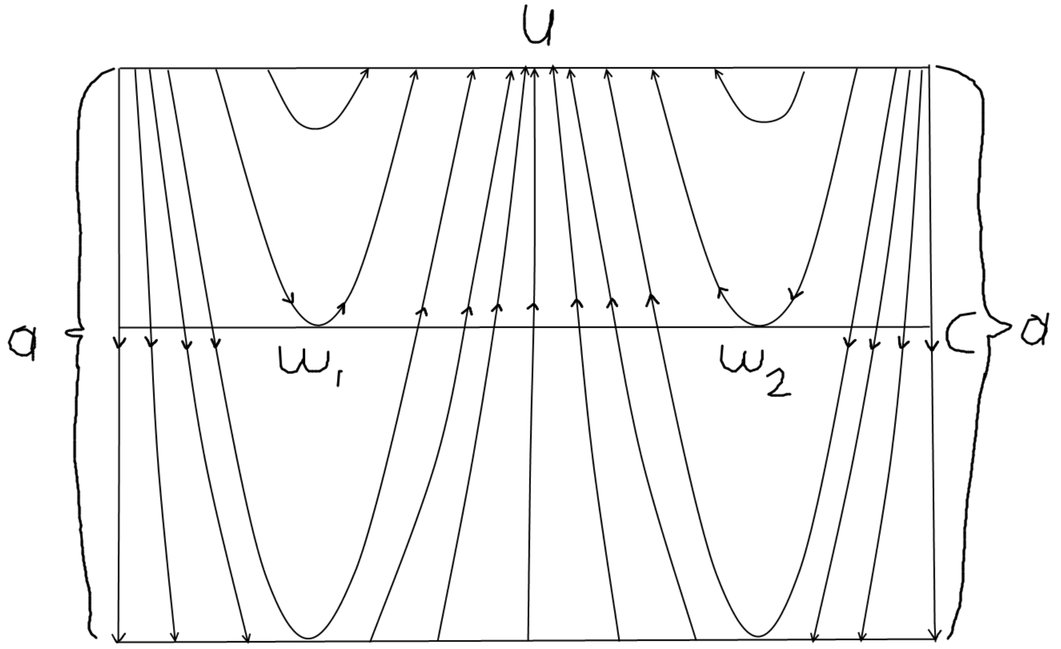

Figure 4.3. Local picture of Z_0 in the cylinder U when $C_{Z_0} = \{w_1, w_2\}$.

Proof. Let X be a smooth vector field on M such that X has finitely many singularities and C is transversal to X.

We claim that $M_c := \mathbb{R}P^2 - C$ is disconnected. Because C is two sided, there exists a circle neighborhood U of C such that $\bar{U}$ is diffeomorphic to a closed cylinder. Let U_1 and U_2 be the connected components of $U - C$ both of which are diffeomorphic to an open cylinder. For $j = 1$ or 2, let g_j be a smooth function on $\bar{U}_j$ such that g_j is equal to 1 on $\bar{U}_j - (U_j \cup C)$; g_j is equal to 0 on C and; g_j is positive on U_j.

Let Y be a vector field on M_c such that Y is equal to X on $(U - C)^c$ and Y is equal to $g_j \cdot X$ on U_j ($1 \leq j \leq 2$). By our choice of g_j's, the vector field Y is smooth. Moreover, the singularities of Y and $X_{|M_c}$ are the same and the corresponding singularities have the same index because multiplication by nonnegative g does not change the directions of the vectors.

Let $\mathfrak{i}$ be a smooth embedding of M_c into $\mathbb{R}^4$ such that $\overline{\mathfrak{i}(M_c)} - \mathfrak{i}(M_c)$ has two connected components C_1 and C_2 each of which is diffeomorphic to C. Assume that

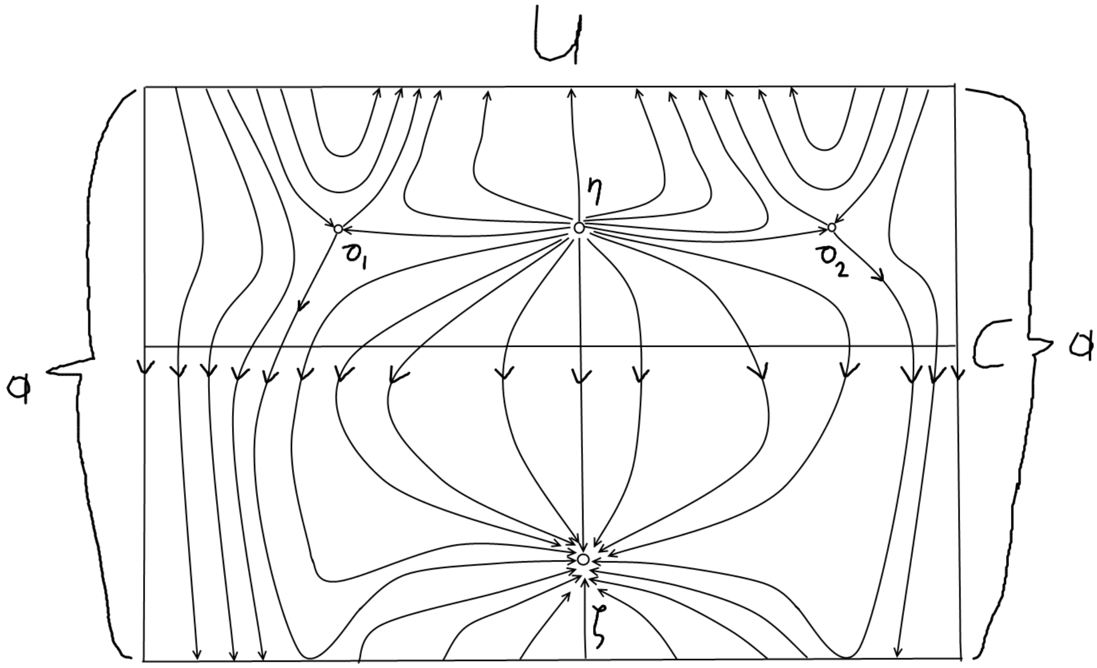

Figure 4.4. C is transversal to the shown vector field.

M_c is connected. Then, we can apply one point compactifications to the C_1 side and the C_2 side of $\mathfrak{i}(M_c)$ such that it becomes a compact manifold $\tilde{M}_c$; i.e. $\tilde{M}_c = \overline{\mathfrak{i}(M_c)} \,/ \sim_{1,2}$ where $\sim_{1,2}$ identifies all points of C_1 with a single point of C_1 and all points of C_2 with a single point of C_2 and it leaves the points of $\mathfrak{i}(M_c)$ unchanged. Clearly, $\tilde{M}_c$ is a compact topological space. Let ∞_j be the point coming from the compactification of side C_j. For simplicity, we will identify the topological space $\tilde{M}_c - \{\infty_1, \infty_2\}$ with the topological space $\mathfrak{i}(M_c)$. So, we have $\tilde{M}_c - \{\infty_1, \infty_2\} \simeq \mathfrak{i}(M_c)$ which is a smooth manifold. Also, $\mathfrak{i}(U_j) \cup \{\infty_j\}$ is a neighborhood of ∞_j such that $\mathfrak{i}(U_j)$ is diffeomorphic to $B(0,1) - \{0\}$ where $B(0,1)$ is an open ball of radius one at 0 in $\mathbb{R}^2$. Hence, $\mathfrak{i}(U_j) \cup \{\infty_j\}$ is diffeomorphic to $B(0,1)$ and $\tilde{M}_c$ is a compact smooth manifold. Also, $\tilde{M}_c$ is connected because we assume that M_c is connected.

Let $\tilde{Y}$ be a vector field on $\tilde{M}_c$ such that $\tilde{Y}$ is equal to $\mathfrak{i}_*(Y)$ on $\mathfrak{i}(M_c)$ and we have $\tilde{Y}(\infty_1) = \tilde{Y}(\infty_2) = 0$. As g_j's are equal to 0 on C, the vector field $\tilde{Y}$ is smooth. Note that one of ∞_j's is a sink and the other one is a source both of which have index 1.

We have, $\Lambda(\tilde{Y}) = \Lambda(Y) + 2 = \chi(\mathbb{R}P^2) + 2 = 3$ which is a contradiction because a compact connected 2-manifold cannot have Euler characteristic greater than 2. Therefore, M_c must be disconnected and our claim is proven.

To prove the lemma, we make a similar argument to the previous one. Let M_1 and M_2 be the connected components of M_c. For $1 \leq j \leq 2$, let X_j be a vector field on M_j such that X_j is equal to X outside of U_j and X_j is equal to $g \cdot X$ on U_j. Embed M_c with $\mathfrak{i}$ into $\mathbb{R}^4$ and compactify $\mathfrak{i}(M_j)$ into $\tilde{M}_j$ as before such that $\tilde{M}_c = \tilde{M}_1 \cup \tilde{M}_2$ is a disjoint union of two connected, compact and smooth 2-manifolds. Let $\tilde{X}_j = \mathfrak{i}_*(X_j)$.

Assume that C does not bound an open disc in $\mathbb{R}P^2$. Then, each $\tilde{M}_j$ is not homeomorphic to a sphere so that we have $\chi(\tilde{M}_j) \leq 1$. As we have $\Lambda(\tilde{X}_j) = 1 + \Lambda(X_j)$, we have $\Lambda(X_j) \leq 0$. So, we have $\Lambda(X) = \Lambda(X_1) + \Lambda(X_2) \leq 0 + 0 = 0$ which is a contradiction to $\Lambda(X) = \chi(\mathbb{R}P^2) = 1$. Therefore, C bounds an open disc in $\mathbb{R}P^2$. $\square$

Lemma 4.2. *Suppose that X is a C^r vector field on $\mathbb{R}P^2$. Then, X does not have any non-trivial recurrent orbits.*

Proof. Assume X has a nontrivial recurrent orbit γ. Let C^* be a transversal circle through a point $p^* \in \gamma$ by Lemma 3.2. Because C^* is transversal to X, it is two sided (as in the construction of Lemma 3.2). By Lemma 4.1, C^* bounds an open disc D in $\mathbb{R}P^2$. So, the vectors of X on C point either inward into D or outward away from D because C^* is transversal to X but then neither positive semi trajectory nor the negative semi trajectory of p^* can accumulate to p^* because those trajectories will intersect C^* at only p^*. This conclusion contradicts that γ is nontrivial recurrent. Hence, such a nontrivial recurrent orbit γ of X does not exist. $\square$

Lemma 4.3. *Suppose that C is a two sided simple closed C^1 curve in Klein bottle K and $K - C$ is connected. Then, $K - C$ is diffeomorphic to an open cylinder.*

Proof. The argument is analogous to the one of Lemma 4.1. Let $M_c = K - C$. Let Y and $\tilde{Y}$ be vector fields on M_c and $\tilde{M}_c$ respectively as in the proof of Lemma 4.1. We have $\Lambda(\tilde{Y}) = 2 + \Lambda(\tilde{Y}_{|\mathfrak{i}(K-C)}) = 2 + \Lambda(X_{|K-C}) = 2 + \Lambda(X) = 2 + \chi(K) = 2$. Hence, $\tilde{M}_c$ is a sphere and $K - C \approx \tilde{M}_c - \{\infty_1, \infty_2\}$ is diffeomorphic to an open cylinder. $\square$

Lemma 4.4. *Suppose that X is a C^r vector field on a Klein bottle K. Then, X does not have any non-trivial recurrent orbits.*

Proof. Assume X has a nontrivial recurrent orbit γ. We will consider the case that γ is nontrivial forward recurrent as the other case is analogous. Let C^* be a transversal circle through a point $p \in \gamma$ by Lemma 3.2. Because C^* is transversal to X, it is two sided. As p^+ is nontrivial forward recurrent, it intersects C^* at infinitely many distinct points and let p_1 be its first intersection. Then, $pp_1 - \{p, p_1\}$ joins the two sides of C^* in $K - C^*$ because C^* is transversal to X. Hence, $K - C^*$ is connected. By Lemma 4.3, $K - C^*$ is diffeomorphic to an open cylinder.

Let $\mathfrak{i}$ be a smooth embedding of $K - C^*$ into $\mathbb{R}^3$ such that $\overline{\mathfrak{i}(K - C^*)} - (K - C^*)$ has two connected components C_1 and C_2 each of which is diffeomorphic to C^*. Define an equivalence relation "$\sim$" on $C_1 \cup C_2$ such that $\overline{\mathfrak{i}(K - C^*)}/\sim$ is homeomorphic to a Klein bottle. So, an orientation on C_1 induces an orientation on C_2 with this relation. Let $P : C^* \to C^*$ be the Poincaré Return Map. Assume P is defined at z in C^*. Because C^* is transversal to X, the open arc $\mathfrak{i}(zP(z) - \{z, P(z)\})$ starts at, say, C_1 and ends at C_2.

Let p_2 be the second intersection of p^+ with C^*. Let $[p, p_1]$ denote the connected closed subset of C^* with boundary points p and p_1 such that we have $p_2 \in [p, p_1]$. Let $[p_1, p_2]$ be the connected closed interval in C^* with boundary points p_1 and p_2 such that we have $p \notin [p_1, p_2]$. Then, we have $[p_1, p_2] \subseteq [p, p_1]$ because of the opposite orientations of C_1 and C_2. Because $K - C^*$ is a cylinder and C^* is transversal to X, the arcs $pp_1 - \{p, p_1\}$ and $p_1p_2 - \{p_1, p_2\}$ separates $K - C^*$ into two disjoint connected open regions each of which is homeomorphic to an open disc. So, if P is defined at $z \in [p, p_1]$, we conclude $P(z) \in [p_1, p_2]$. By considering all the properties $[p_1, p_2] \subseteq [p, p_1]$ and $P([p, p_1]) \subseteq [p_1, p_2]$ and $p_2 \in [p, p_1]$ and $p \notin [p_1, p_2]$, we conclude that p^+ will not visit a neighborhood of p in C^* which contradicts that p^+ is nontrivial forward recurrent. Hence, such a nontrivial forward recurrent orbit γ of X does not exist. See Figure 4.5. $\square$

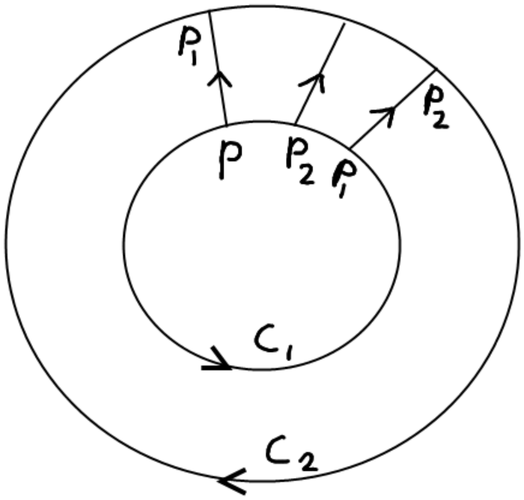

Figure 4.5. The positive semi trajectories of p, p_1 and p_2.

Corollary 4.1. *Suppose that* M *is a Klein bottle or* $\mathbb{R}P^2$*. Then, M-S fields are dense in* $\mathfrak{X}^r(M)$*.*

Proof. Let $X \in \mathfrak{X}^r(M)$ and let $\mathcal{U}$ be a neighborhood of X. Let Y be a K-S field in $\mathcal{U}$. By Lemma 4.2 or Lemma 4.4, the vector field Y does not have a nontrivial recurrent orbit so, the K-S field Y is an M-S field by Lemma 3.11. □

4.2. Torus With a Cross Cap

We will present Theorem 4.1 due to [10] which extends Lemma 3.6 to a torus with a cross cap $M_T \simeq T\#\mathbb{R}P^2$. Let X be a C^r vector field on M_T such that all of its singularities are hyperbolic and X has a nontrivial recurrent orbit γ. As we have $\chi(M_T) = -1$, the vector field X has singularities so that there exists a stable separatrix β_s of a saddle and an unstable separatrix β_u of a saddle such that β_s and β_u accumulate to γ by Lemma 3.5. [10] proves that their subsequent intersections (except possibly a single one) with a transversal circle C through a point $p^* \in \gamma$ behave like as if they are on an orientable manifold so that [10] extends the same ideas of Lemma 3.6, hence, Theorem 3.1 to the nonorienatable manifold M_T. We emphasize that Cases $2b$, $2c$ and $2d$ of Theorem 3.1 have been specifically explained for orientable manifolds but the ideas in those three cases can be extended to non-orientable manifolds as well as we explained at the end of Chapter 3.

In [11], Gutierrez discusses that the ideas of Lemma 3.6 cannot be extended anymore to other nonorientable, compact, connected 2-manifolds without boundary by giving a good example of a smooth vector field Y on a torus with two cross caps. In that example, Y has a nontrivial recurrent orbit γ, an unstable separatrix β_u with $\omega(\beta_u) \supseteq \gamma$ and a stable separatrix β_s with $\alpha(\beta_s) \supseteq \gamma$. In [11], it is proven that for any (arbitrarily small) transversal section Σ through some point of γ, the intersections of β_u and β_s with Σ exhibit the nonorientable behavior that one usually expects from nonorientable manifolds. So, the torus with a cross cap is an exception in regard to the nonorientable, compact, connected 2-manifolds without boundary the Euler characteristics of which are less than or equal to -1. Note that the nonorientable compact 2-manifolds $\mathbb{R}P^2$ and Klein bottle K have Euler characteristics 1 and 0 respectively and M-S fields are dense in the space of C^r vector fields on $\mathbb{R}P^2$ or K by Corollary 4.1. This corollary has been derived rather easily because any C^r vector field on $\mathbb{R}P^2$ or K does not have a nontrivial recurrent orbit.

We explain the notation $A_{xy}(M - C)$ in Lemma 4.5 which is nothing but the compactifications of $M - C$ as in the proof of Lemma 4.1. Suppose that C is a two-sided curve in M and let $M_c := M - C$. Let $\mathfrak{i}$ be a smooth embedding of M_c into $\mathbb{R}^4$ such that $\overline{\mathfrak{i}(M_c)} - \mathfrak{i}(M_c)$ has two connected components C_1 and C_2 each of which is diffeomorphic C. We can put an equivalence relation on $\sim_{1,2}$ on $\overline{\mathfrak{i}(M_c)}$ where $\sim_{1,2}$ identifies all points of C_1 with a single point of C_1 and all points of C_2 with a single point of C_2 and it leaves the points of $\mathfrak{i}(M_c)$ unchanged. Then, $\tilde{M}_c := \overline{\mathfrak{i}(M_c)}/\sim_{1,2} = \mathfrak{i}(M_c) \cup \{\infty_1, \infty_2\}$ is a compact manifold. Now, $A_{xy}(M_c)$ is the compact manifold $M_c \cup \{x, y\}$ that is diffeomorphic to $\tilde{M}_c$. Here, x corresponds to one of the ∞_j's and y corresponds to the other one.

Lemma 4.5. *Suppose that M is a torus with a cross cap; C is a two sided simple closed C^1 curve in M and; $M - C$ is connected. Then, $A_{xy}(M - C)$ is diffeomorphic to $\mathbb{R}P^2$.*

Proof. Let X be a smooth vector field on M with finitely many singularities such that C is a transversal to X. Let Y and $\tilde{Y}$ be vector fields on M_c and $\tilde{M}_c$ respectively as in the proof of Lemma 4.1. Then, we have $\chi(A_{xy}(M-C)) = \Lambda(\tilde{Y}) = 2 + \Lambda(Y) = 2 + \Lambda(X) = 2 + \chi(M) = 1$. Hence, $A_{xy}(M-C)$ is $\mathbb{R}P^2$. $\square$

Lemma 4.6. *Suppose that C is a one sided simple closed C^1 curve in M. Let U be the closure of a circle neighborhood of C such that U is diffeomorphic to a closed Möbius band. Let $R = [0,1] \times [-1,1]$ and let $\sim$ be an equivalence relation on R such that the relation $\sim$ identifies $(0,y)$ with $(1,-y)$ and leaves the other points of R unchanged. Let $f : U \to R/\sim$ be a diffeomorphism such that we have $f(C) = ([0,1] \times \{0\})/\sim$. Then, there exists a smooth vector field X on M such that X has finitely many singularities and $f_*(X_{|U})(q) = e_1$ where q is in $R/\sim$ and $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. So, the orbit through any point of U is a closed orbit of X and also, C is a closed orbit of X.*

For the proof of Lemma 4.5, we will assume Lemma 4.6. We again omit a rigorous proof of Lemma 4.6 and we justify it in the following way. Let U_0 be a circle neighborhood of C and let τ be a two sided simple closed smooth curve in U_0 as shown in Figure 4.6. Note the identifications a and a^{-1} of sides in Figure 4.6 so that U_0 is indeed diffeomorphic to a Möbius band and C is the 0-section of the Möbius band U_0. Let Z_0 be vector field on M such that Z_0 has finitely many singularities and τ is transversal to Z_0. Let $U_\tau \subseteq U_0$ be a circle neighborhood of τ. If U_τ is small enough, then Z_0 in U_τ looks like as in Figure 4.6 (the vectors on τ may point in the opposite direction). We can locally perturb Z_0 in U_0 to obtain a smooth vector field Z_1 where Z_1 in U_0 looks like Figure 4.7. The vector field Z_1 and the circle neighborhood U of C that is shown in Figure 4.7 satisfy the properties of Lemma 4.6.

We explain the notation $A_x(M-C)$ in Lemma 4.7 where it is originally used in [10]. Suppose that C is a one sided simple closed curve in M. Let U be a circle neighborhood of C which is diffeomorphic to an open Möbius band. Let $\sim_u$ be an equivalence relation on U such that $\sim_u$ identifies all points of C with a single point ∞ and leaves the other points of U unchanged. Since C is the 0-section of the Möbius band

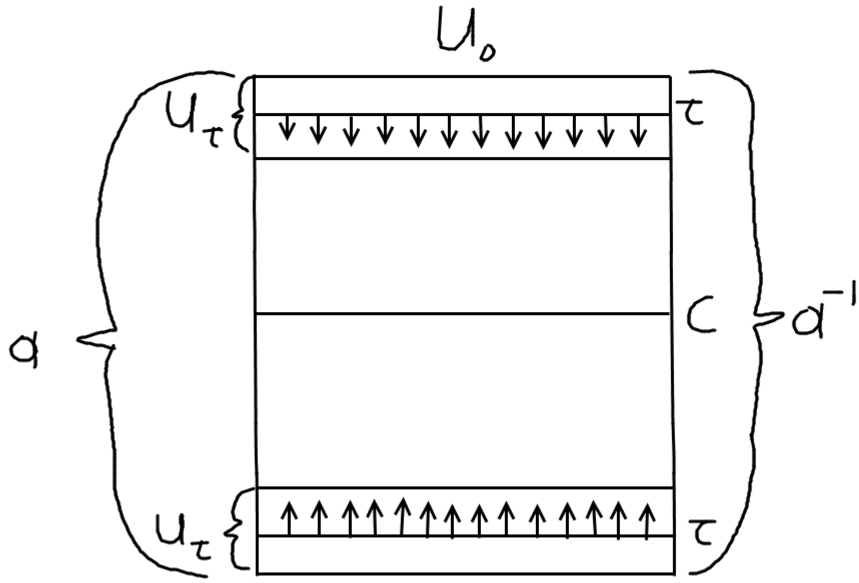

Figure 4.6. Two sided simple closed smooth curve τ in U_0 and the vector field Z_0 in U_τ.

U, the smooth manifold $U - C$ is diffeomorphic to an open cylinder. So, $U/\sim_u -\{\infty\}$ is also a smooth manifold that is diffeomorphic to $B(1,0)-\{(0,0)\}$ where $B(1,0)$ is an open ball at the origin with radius 1. Hence, $U/\sim_u$ is also a smooth manifold that is diffeomorphic to $B(1,0)$. So, $M/\sim_c$ is a compact smooth manifold without boundary where the equivalence relation $\sim_c$ identifies all points of C with a single point ∞ an leaves the other points of M unchanged. Now, we define $A_x(M-C) := M/\sim_c$.

Lemma 4.7. *Suppose M is a torus with a cross cap and C is a one sided simple closed C^1 curve in M. Then, we have $\chi(A_x(M-C)) = 0$*

Proof. Let X be the smooth vector field on M which is specified by Lemma 4.6. We will use the notations in that lemma. Let $g : U \to \mathbb{R}$ be a smooth function such that: g is equal to 1 on the boundary of U; g is positive on $U - C$ and; g is equal to 0 on C. Let Y be the vector field on M such that Y is equal to X on U^c and Y is equal to $g \cdot X$ on U. Then, Y is smooth by our choice of g. Let $\sim_c$ an equivalence relation on M such that $\sim_c$ identifies all points of C with a single point ∞ and leaves the other points of M unchanged. As $\sim_c$ leaves the points of $M - C$ unchanged and $M - C$ is an open subset of M, the smooth vector field $Y_{|M-C}$ on $M - C$ naturally induces a smooth vector field

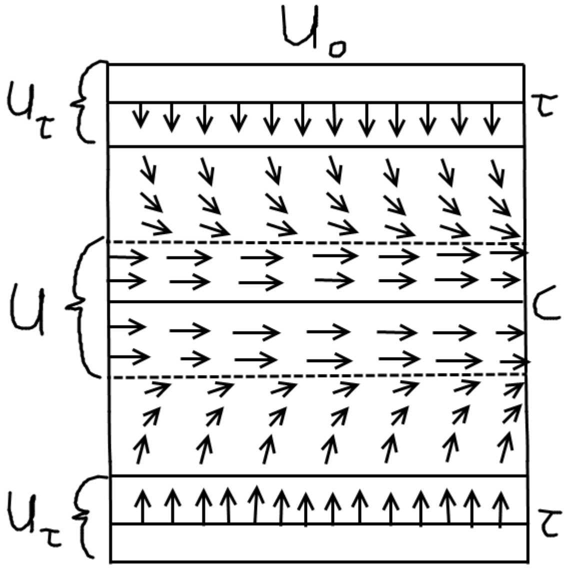

Figure 4.7. The vector field Z_1 in U_0 and the circle neighborhood U of C.

$\tilde{Y}_0$ on $M/\sim_c -\{\infty\}$. Let $\tilde{Y}$ be a smooth vector field on $M/\sim_c$ such that $\tilde{Y}$ is equal to $\tilde{Y}_0$ on $M/\sim_c -\{\infty\}$ and we have $\tilde{Y}(\infty) = 0$. By our choice of g, the vector field $\tilde{Y}$ is a smooth vector field. Note that the singularity ∞ of $\tilde{Y}$ is isolated and it is accumulated by closed orbits of $\tilde{Y}$ because of our choices of g and X so that the index of ∞ is 1. Therefore, we have $\chi(A_x(M-C)) = \Lambda(\tilde{Y}) = \Lambda(Y)+1 = \Lambda(X)+1 = \chi(M)+1 = 0$. $\square$

We now make some definitions that are used in Lemma 4.8 and Theorem 4.1. Suppose that X is a vector field on M and C^* is a transversal circle to X in M. Let $P : C^* \to C^*$ be the Poincaré Return Map. Suppose that P is defined at $z \in C^*$ and we also have $P(z) \neq z$. Let I_z be a closed interval in C^* with boundary points z and $P(z)$. Then, we call the closed arc $zP(z)$ *a one sided C^*-arc* if $zP(z) \cup I_z$ is a one sided curve simple closed continuous curve. Similarly, we call the closed arc $zP(z)$ *a two sided C^*-arc* if $zP(z) \cup I_z$ is a two sided curve simple closed continuous curve. Note that these two definitions are well defined as they do not depend on the choice of I_z.

Lemma 4.8. *Suppose that M is a torus with a cap; X is a C^r vector field on M; C^* is a transversal circle to X. Let $P : C^* \to C^*$ be the Poincaré Return Map. Suppose that p and q are distinct points in the domain of P such that both $pP(p)$ and $qP(q)$ are either one sided C^*-arcs or two sided C^*-arcs. Then, $pP(p)$, $qP(q)$ and C^* all together bound an open disc D in M (where $p(P)$ and $qP(q)$ are in the boundary of D).*

Proof. Because C^* is transversal to X, it is two sided. As the domain of P is nonempty, $M - C^*$ is connected. So, $A_{xy}(M - C^*)$ is diffeomorphic to $\mathbb{R}P^2$ by Lemma 4.5. Note that $\beta := \{x, y\} \cup (pP(p) - \{p, P(p)\}) \cup (qP(q) - q, P(q))$ is a simple closed continuous curve in $A_{xy}(M_c^2 - C^*)$. Since both $pP(p)$ and $qP(q)$ are either one sided C^*-arcs or two sided C^*-arcs, the simple closed continuous curve β is two sided. The continuous curve β can fail to be C^1 at the points x and y only so that we can still use Lemma 4.1 to conclude that β bounds an open disc D in $A_{xy}(M - C^*)$. Hence, the open disc D is bounded by $pP(p) \cup qP(q) \cup C^*$ in M. See Figure 4.8. ☐

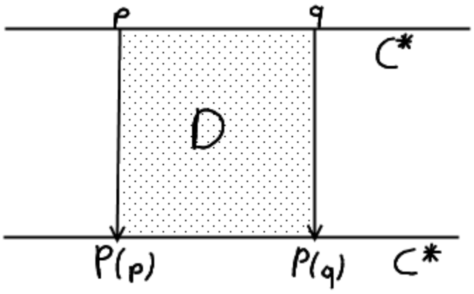

Figure 4.8. The bounded open disc D.

In [10], Gutierrez proves the following theorem which we state slightly differently according to our terminology. Recall that for any nontrivial recurrent orbit γ of X, there exists a transversal circle C to X through a point of γ by Lemma 3.2.

Theorem 4.1. *Suppose that: M is a torus with a cross cap; X is a C^r vector field on M; γ_1 is a nontrivial forward recurrent orbit of X and; γ is an orbit of X such that we have $\omega(\gamma) \supseteq \gamma_1$. Then, for any transversal circle C to X through a point of γ_1, all C-arcs of γ are two sided except possibly a single one.*

We will explain most of the proof in Theorem 4.1 and omit some details in Step C. Those details are actually quite important and our aim here is to give the reader a good idea about the claims, the properties and the proofs in Step C. Beforehand, we remark that the hypothesis about M to be a torus with a cross cap is very important and it will be used several times. It is really hard to tell which use of this hypothesis is more important than the other ones and it is difficult to apply any of the ideas below to other nonorientable manifolds the Euler characteristics of which are smaller than -1. We divide our arguments into three steps as in [10] : Step A, Step B and Step C.

Step A. If R is a Möbius band in M such that R is invariant either by the positive time flow of X (i.e. $X_t(R) \subseteq R$ for all $t \geq 0$) or by the negative time flow of X, then we have $R \cap \gamma_1 = \varnothing$. The proof of this claim is easy because if R is invariant by the positive time flow or by the negative time flow of X and if we also have $R \cap \gamma_1 \neq \varnothing$, then we can conclude $\gamma_1 \subseteq R$ but the invariant region R can be embedded into a Klein bottle K and any C^r vector field on K does not admit any nontrivial recurrent orbit by Lemma 4.4. So, the claim in Step A holds.

Step B. There doesn't exist an open disc R in M such that :

(i) R is bounded by two C^*-arcs q_1q_2 and w_1w_2, and two curves I_0 and I_1 where I_0 is a closed interval in C^* with boundary points q_1 and q_2 and I_1 is a closed interval in C^* with boundary points w_1 and w_2;

(ii) At least one of the following properties $I_0 \subseteq I_1$ or $I_1 \subseteq I_0$ holds;

(iii) The closure of R is a Möebius band that is invariant either by the positive time flow of X or by the negative time flow of X.

Assume the existence of such R. By Step A, we have $\gamma_1 \cap R = \varnothing$. We can find a C^r vector field Y on M with the following properties: Y is equal to X on R^c and Y has a closed orbit τ in R such that τ is a one sided simple closed continuous curve. The last requirement is possible as the closure of R is a Möbius band. Note that γ_1 is a nontrivial forward recurrent orbit of Y. By Lemma 4.7, $A_x(M-\tau)$ is either a torus or a Klein bottle. As the closure of R is a Möbius band, the simple closed continuous curve $(C^* - I_0) \cup q_1q_2$ in $A_x(M-\tau)$ is one sided so that $A_x(M-\tau)$ is a Klein bottle. This last conclusion is a contradiction because a Klein bottle does not admit a C^r vector field with a nontrivial recurrent orbit. So, the claim in Step B holds.

Assume that Theorem 4.1 is false. Then, *Step C* can be formulated in the following way:

Step C. There exists a transversal circle C through a point of γ_1 such that:

(i) There exist consecutive C-arcs $A_1 := p_0p_1, \ldots, A_m := p_{m-1}p_m$ of γ such that all A_m's are two sided except A_1 and A_m (here, we have $1 < m$) ;

(ii) If $\tilde{C}$ is an arbitrary transversal circle through a point of γ_1 and $B_1, \ldots B_k$ is a sequence of consecutive $\tilde{C}$-arcs of γ all of which are two sided except B_1 and B_k, then we have $k \geq m$.

We will argue that the inequality $m \leq 3$ holds. Assume $m > 3$. Then, A_1 and A_m are both one sided C-arcs and A_2 and $A_{m+1} := p_mp_{m+1}$ are both two sided C-arcs. So, A_1, A_m and C bound all together an open disc R_1 in M by Lemma 4.8 and A_2, A_{m+1} and C bound all together an open disc R_2 in M by the same lemma. Let I_0 be the closed interval in C between p_0 and p_{m-1} and let I_1 be the closed interval in C between p_1 and p_m such that $A_1 \cup A_m \cup I_0 \cup I_1$ bounds R_1. Similarly, let E_1 be the closed interval in C between p_1 and p_m and let E_2 be the closed interval in C between p_2 and p_{m+1} such that $A_2 \cup A_{m+1} \cup E_1 \cup E_2$ bounds R_2. In [10], the equality $I_1 = E_1$ is proven and also, it is shown that all I_0, I_1 and E_2 are pairwise disjoint. The proofs of these properties utilize the minimality condition (ii) in Step C. These properties yield two possible cases, say Ordering Case I and Ordering Case II, for the pairwise orderings of the points $\{p_1, p_2, p_{m-1}, p_m, p_{m+1}\}$ in $C - \{p_0\}$. The below discussion is only for one of them, say Ordering Case I.

For this part of the discussion, see Figure 4.9. Next, the region R_2 is used to define a C^r embedded closed interval θ in R_2 such that: θ is transversal to X; we have $p_0p_2 \cap \theta = \{p_1\}$ and $\theta \cap C = \{p_0, \tilde{p}_2\}$ for some point $\tilde{p}_2 \in E_2$. Let λ be the open interval between $\tilde{p}_2$ and p_1 in $C - \{p_0\}$. We can assume θ to be such that $\tilde{C} := (C - \lambda) \cup \theta$ is a C^r embedded circle in M. Because R_2 is bounded by two sided C-arcs A_2 and A_{m+1} and we have $\theta \subseteq R_2$, the simple closed continuous curve $\tilde{C}$ is two sided. Hence, $\tilde{C}$ is a transversal circle to X. Figure 4.9 shows $p_0p_2 \cap \theta = \{p_1\}$ but the closed intervals I_0, I_1 and E_2 have such properties so that the C-arcs $A_1, \ldots, A_{m+1}$ do not visit the open discs R_1 and R_2 and the condition $p_0p_{m+1} \cap \theta = \{p_1\}$ holds. The cross section θ and the open interval λ have been chosen according to the Ordering Case I which we haven't specified. If we had the Ordering Case II, then a different cross section $\tilde{\theta}$ can be defined in R_2 and a different open interval $\tilde{\lambda}$ can be defined in C and so that $(C - \tilde{\lambda}) \cup \tilde{\theta}$ is a transversal circle to X and the condition $p_0p_{m+1} \cap \tilde{\theta} = \{p_2\}$ holds.

We continue our argument for the transversal circle $\tilde{C}$. By the choice of λ and θ, both p_0p_1 and $p_{m-1}p_{m+1}$ are one sided $\tilde{C}$-arcs and also p_0p_{m+1} intersects $\tilde{C}$ less than m times. This last conclusion contradicts the minimality condition (ii) in Step C once we show the inequality $\gamma_1 \cap \tilde{C} \neq \varnothing$. As $\tilde{C}$ is a transversal circle to X and the closed arc p_0p_1 is a $\tilde{C}$-arc, the curve $p_0p_1 - \{p_0, p_1\}$ joins the two sides of $\tilde{C}$ in $M - \tilde{C}$, so that $M - \tilde{C}$ is connected. By Lemma 4.5, $A_{xy}(M - \tilde{C})$ is diffeomorphic to $\mathbb{R}P^2$ but any C^r vector field on $\mathbb{R}P^2$ does not admit any nontrivial recurrent orbit by Lemma 4.2. So, we must have $\gamma_1 \cap \tilde{C} \neq \varnothing$ and we reach the above contradiction. So, the inequality $m \leq 3$ holds.

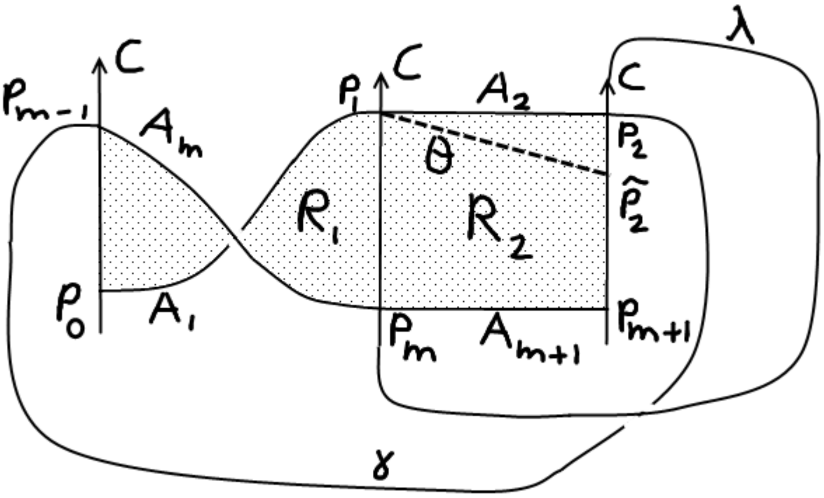

Figure 4.9. The case $m > 3$.

Assume $m = 2$. Then, the C-arcs p_0p_1 and p_1p_2 are one sided and they together with C bound an open disc J_1 by Lemma 4.8. Let I_0 be the closed in C between p_0 and p_1 and let I_1 be the closed interval in C between p_1 and p_2 such that $I_0 \cup I_1 \cup p_0p_1 \cup p_1p_2$

bound the open disc J_1. Since both p_0p_1 and p_1p_2 are one sided C-arcs and we have $p_1 \in p_0p_1 \cap p_1p_2$, we conclude that we have either $I_1 \subseteq I_0$ or $I_1 \supseteq I_0$ and also, the closure of J_1 is a Möbius band. If we have $I_1 \subseteq I_0$, then J_1 is invariant by the positive time flow of X. If we have $I_1 \supseteq I_0$, then J_1 is invariant by the negative time flow of X. Either conclusion contradicts the claim in Step B. So, we have $m \neq 2$.

Assume now $m = 3$. This case will be similar to the case $m = 2$ and our discussion here is omitted in [10]. As we have $m = 3$, the C-arcs p_0p_1 and p_2p_3 are one and they together with C bound an open disc J_1 by Lemma 4.8. Let I_0 be the closed interval in C between p_0 and p_2 and let I_1 be the closed interval in C between p_1 and p_3 such that $I_0 \cup I_1 \cup p_0p_1 \cup p_2p_3$ bound the open disc J_1. Also, p_1p_2 and p_3p_4 are two sided C-arcs and they together with C bound an open disc J_2 again by Lemma 4.8. Let E_1 be the closed interval in C between p_1 and p_3 and let E_2 be the closed interval in C between p_2 and p_4 such that $E_1 \cup E_2 \cup p_1p_2 \cup p_3p_4$ bound the open disc J_2. Define any orientation on C which induces orientations on I_0, I_1, E_1 and E_2. Note that the orientations of I_0 and I_1 are opposite to each other and E_1 and E_2 are oriented in the same direction (in regard to the orientation of C). Since p_1p_2 and p_3p_4 is are two sided arcs, they cannot be in J_1 and we have $p_1 \notin I_0$ and $p_3 \notin I_0$. So, we either have $I_1 \supseteq I_0$ or $I_1 \cap I_0 = \varnothing$. If we have $I_1 \supseteq I_0$, then the closure of J_1 will be an Möebius band that is invariant by the positive time flow of X but then, the positive semi trajectory ${p_1}^+$ will not go through p_2. Hence, we must have $I_1 \cap I_0 = \varnothing$. As E_1 is in the boundary of the open disc J_2 which contains two sided C-arcs only and I_0 is in the boundary of the open disc J_1 which contains one sided C-arcs only, we cannot have $I_0 \subseteq E_1$ and we must have $E_1 = I_1$. As E_1 and E_2 are oriented in the same direction and we have $p_2 \in E_2 \cap I_0$, we have either $E_2 \subseteq I_0$ or $E_2 \supseteq I_0$. If we have $E_2 \subseteq I_0$, then the closure of $J_1 \cup J_2$ is a Möbius band that is invariant by the positive time flow of X. If we have $E_2 \supseteq I_0$, then the closure of $J_1 \cup J_2$ is a Möbius band that is invariant by the negative time flow of X. Either conclusion contradicts the claim in Step B. Hence, $m \neq 2$. See Figure 4.10. The proof of Theorem 4.1 ends here.

We end this Chapter by showing that M-S fields are dense in the space of C^r vector

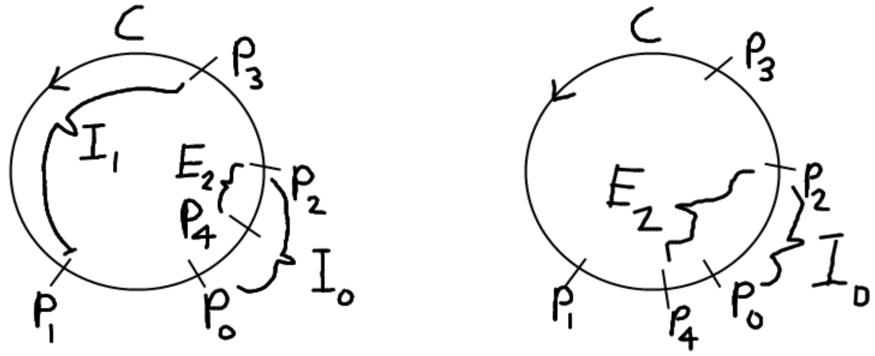

Figure 4.10. The case $m = 3$. There are two possible cases for the interval E_2.

fields on a torus with a cross cap M_T. Let M be any compact, connected 2-manifold without boundary. Suppose that X is a C^r vector field on M; X has singularities all of which are hyperbolic and; γ_1 is a non-trivial recurrent orbit X. By Lemma 3.5, there exist an unstable separatrix β_u of some saddle σ_u and a stable separatrix β_s of some saddle σ_s such that we have $\omega(\beta_u) \supseteq \gamma_1$ and $\alpha(\beta_s) \supseteq \gamma_1$. Let U_u be a Grobman-Hartman neighborhood of σ_u and let $p \in \beta_u \cap U_u$ such that we have $p^- \subseteq U_u$. Similarly, let U_s be a Grobman-Hartman neighborhood of σ_s and let $q \in \beta_s \cap U_s$ such that we have $q^+ \subseteq U_s$. Let C be transversal circle through a point of γ_1. Let the point p_j denote j^{th} intersection of p^+ with C and similarly, let the point q_j denote j^{th} intersection of q^- with C.

If M is orientable, then for every positive integer j, each closed arcs $p_j p_{j+1}$ and $q_{j+1} q_j$ are a two-sided C-arcs and the proof of Lemma 3.6 relies on the assumption that all these C-arcs are two-sided. Note that any C^r vector field on M_T has singularities and Theorem 4.1 shows that this assumption (except possibly for one or two C-arcs) holds as well when we have $M = M_T$ so that Lemma 3.6 can be proven in the same way when we have $M = M_T$. Hence, Theorem 3.1 the proof of which relies on Lemma 3.6 can be proven when we have $M = M_T$ and M-S fields are dense in $\mathfrak{X}^r(M)$.

5. TRANSVERSAL BASE

We would like to acquire a complete global picture of a C^r vector field X on a compact and connected, two manifold M without boundary. These manifolds have been classified and are represented by fundamental polygons. Given X in $\mathfrak{X}^r(M)$, our goal is to exhibit a special fundamental polygon for M that expresses X by means of finitely many transversal sections to X and finitely many closed arcs of X. A firsthand requirement is that X has finitely many singularities and we shall assume X to be so.

5.1. Tubular Flow Extension

Definition 5.1. *Suppose that:*

(i) Σ_1 and Σ_2 are closed disjoint transversal sections to X with boundary points $\{a_1, b_1\}$ and $\{a_2, b_2\}$ respectively such that: the positive semi trajectory of a_1 goes through a_2; we have $a_1a_2 \cap (\Sigma_1 \cup \Sigma_2) = \{a_1, a_2\}$; the positive semi trajectory of b_1 goes through b_2 and; $b_1b_2 \cap (\Sigma_1 \cup \Sigma_2) = \{b_1, b_2\}$.

(ii) a_1a_2, b_1b_2, Σ_1 and Σ_2 bound all together an open disc (open rectangle) D in M.

Then, the closed region $\bar{D}$ is called a section and orbit sided rectangle with sides $\Sigma_1, \Sigma_2, a_1a_2$ and b_1b_2.

Let $\bar{D}$ be a section and orbit sided rectangle with sides $\Sigma_1, \Sigma_2, a_1a_2$ and b_1b_2. By Tubular Flow Theorem, the positive semi trajectories of points in a neighborhood of a_1 in Σ_1 and the positive semi trajectories of points in a neighborhood of b_1 in Σ_1 go to Σ_2 without intersecting Σ_1. We would like to extend these two neighborhoods to the whole Σ_1 but without certain assumptions, this is not possible. The Figure 5.1 illustrates a situation where this extension fails. It has two saddles σ_1 and σ_2 and two sources η_1 and η_2. Here, the only positive semi trajectory of a point in Σ_1 which does not intersect Σ_2 is the one that goes to the saddle σ_1. The key assumption that we need to prevent this situation is the exclusion of saddles in $\bar{D}$ when the singularities of X are all hyperbolic.

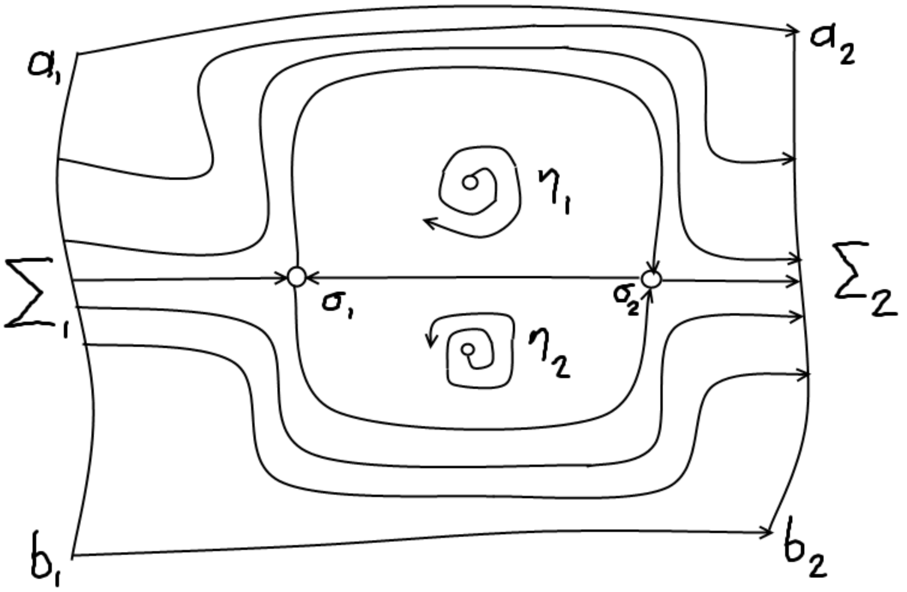

Figure 5.1. Not all the positive semi trajectories of all points in Σ_1 intersect Σ_2.

Suppose that the singularities of X are all hyperbolic and pq is a closed arc of X. Let Σ_p and Σ_q be disjoint transversal sections through p and q respectively. By the Tubular Flow Theorem, there exist a connected open neighborhood U_p of p in Σ_p and a connected open neighborhood U_q of q in Σ_q such that the positive semi trajectory of every point z_p in U_p goes to a point z_q in U_q and $z_pz_q \cap (\Sigma_p \cup \Sigma_q) = \{z_p, z_q\}$. Assume that U_p and U_q are initially as small as that they do not contain any boundary points of Σ_p or Σ_q so that we can talk about the extension of them. This described situation is specific in the sense that Σ_p and Σ_q are disjoint and we will later run into the situations when two transversal sections Σ_1 and Σ_2 are not disjoint. We need some limitations for Σ_1 and Σ_2 when we try to generalize the disjoint Σ_p and Σ_q situation. In such delicate situations, one needs to have proper notions at hand. The positive semi trajectories through boundary points of Σ_1 and Σ_2 cause problems and we exclude them in the definition of First Proper Intersection Assignment below. ${\Sigma_1}^\circ$ and ${\Sigma_2}^\circ$ denote there the interiors of Σ_1 and Σ_2 respectively and $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$ denote there the closures of Σ_1 and Σ_2 respectively. Also, we will make another definition to consider specific non-disjoint transversal sections Σ_1 and Σ_2.

Definition 5.2 (First Proper Intersection Assignment)**.** *Let* Σ_1 *and* Σ_2 *be transversal sections to* X*. For any point* $p \in {\Sigma_1}^\circ$ *the positive semi trajectory of which satisfies the following two conditions:*

> *(i)* $p^+ \cap {\Sigma_2}^\circ \neq \varnothing$*. Let the point* q *denote the first intersection of* p^+ *with* ${\Sigma_2}^\circ$*. If we have* $q \neq p$*, then define* A_p *to be the closed arc* pq*. If we have* $q = p$*, then define* A_p *to be the closed orbit through* p*.*
>
> *(ii)* $(A_p - \{p, q\}) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$.
>
> *we define* the First Proper Intersection Assignment $P : \Sigma_1 \to \Sigma_2$ *at* p *as* $P(p) = q$*. If the positive semi trajectory of* p *fails to satisfy any of the above conditions, then* P *at* p *is not defined. Also,* P *is not defined at any boundary point of* Σ_1*.*

The domain of the First Proper Intersection Assignment $P : \Sigma_1 \to \Sigma_2$ might be very well the empty set. Suppose that τ is a closed orbit of X and Σ is a small open transversal section through a point q of τ such that we have $\tau \cap \bar{\Sigma} = \{q\}$. Then, the First Proper Intersection Assignment $P : \Sigma_1 \to \Sigma_1$ is the Poincaré Return Map in a small neighborhood of q in Σ. The reason that the Poincaré Return Map and the First Proper Intersection Assignment are distinct relies on the second condition in the definition of the First Proper Intersection Assignment.

Definition 5.3. *Let* N_1 *and* N_2 *be connected,* C^r *embedded submanifolds of dimension 1, with or without boundary, in* M *such that* $N_1 \cap N_2$ *is a nonempty connected set. If* $N_1 \cup N_2$ *is a* C^r *embedded submanifold of* M*, then we say that* $N_1 \cap N_2$ *is* interval-like connected.

Figure 5.2 illustrates three examples of transversal sections Σ_1 and Σ_2 to X when $\Sigma_1 \cap \Sigma_2$ is nonempty and connected but not interval-like connected. In the first two examples, $\Sigma_1 \cup \Sigma_2$ is not even a topologically embedded submanifold of M. In the third example, $\Sigma_1 \cup \Sigma_2$ is a topologically embedded submanifold of M but it is not a C^r embedded submanifold of M. Figure 5.3 illustrates two examples of transversal sections Σ_1 and Σ_2 to X when $\Sigma_1 \cap \Sigma_2$ is interval-like connected.

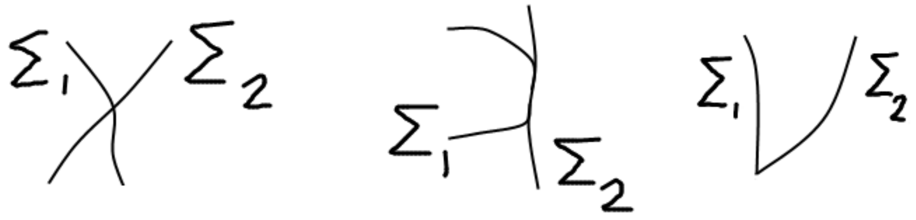

Figure 5.2. $\Sigma_1 \cap \Sigma_2$ is nonempty and connected but not interval-like connected.

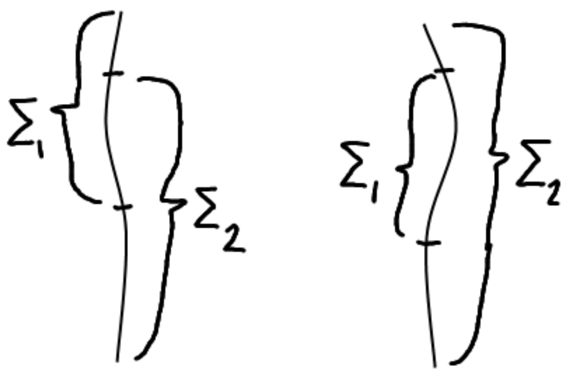

Figure 5.3. $\Sigma_1 \cap \Sigma_2$ is interval-like connected.

Theorem 5.1 (Tubular Flow Extension)**.** *Suppose that all the singularities of X are hyperbolic and Σ_1 and Σ_2 are two transversal sections to X such that $\Sigma_1 \cap \Sigma_2$ is an interval-like connected set or the empty set. Then, the domain E_0 of the First Proper Intersection Assignment $P : \Sigma_1 \to \Sigma_2$ is open in Σ_1 and P is an injective local diffeomorphism if $E_0 \neq \varnothing$. Suppose that E_0 has a boundary point p in Σ_1 such that p^+ does not intersect $\bar{\Sigma}_2$. Then, $\omega(p)$ is a saddle. Let $B = \{z \in M : \ w \in E_0, z \in A_w\}$ where A_w is as in Definition 5.2. Then, $B - (\Sigma_1 \cup \Sigma_2)$ is a disjoint union of open discs in M and there exist finitely many saddles $q_1, \dots, q_n$ in the boundary of B such that an unstable separatrix of q_j goes to q_{j+1} within the boundary of B for $1 \leq j < n$ and an unstable separatrix of q_n goes to a point of $\bar{\Sigma}_2$ within the boundary of A.*

Proof. For the sake of proof, assume $E_0 \neq \varnothing$.

Consider a point $w \in \Sigma_1{}^\circ$. As Σ_1 is transversal to X and we have $w \in \Sigma_1{}^\circ$, there exists a simply connected open neighborhood $\tilde{U}_1$ of w in M and some $a > 0$ such that

$\tilde{U}_1 \cap \Sigma_1$ does not include the boundary points of Σ_1 (if they exist at all) and we have $X_t(\tilde{U}_1 \cap \Sigma_1) \cap \bar{\Sigma}_1 = \varnothing$ for $0 < t \leq a$. We claim that there exist a smaller simply connected open neighborhood U_1 of w in M (with $U_1 \subseteq \tilde{U}_1$) and some (smaller) $a > 0$ such that we have $X_t(U \cap \Sigma_1) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ for $0 < t \leq a$.

Assume that we have $w \notin \Sigma_1 \cap \bar{\Sigma}_2$. Then, we can find a simply connected open neighborhood U_1 of w with $U_1 \subseteq \tilde{U}_1$ in M and some $a > 0$ such that we have $U_1 \cap \bar{\Sigma}_2 = \varnothing$ and $X_t(U \cap \Sigma_1) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ for $0 < t \leq a$. So, the claim is proven for this situation.

Assume that we have $w \in \Sigma_1 \cap \bar{\Sigma}_2$. If w is an interior point of the connected set $\Sigma_1 \cap \bar{\Sigma}_2$, then there exists a simply connected open neighborhood U_1 of w in M with $U_1 \subseteq \tilde{U}_1$ and some $a > 0$ such that we have $U_1 \cap \Sigma_1 = U_1 \cap \Sigma_2$ and $X_t(U_1 \cap \Sigma_1) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ for $0 < t \leq a$. Even when w is not an interior point of the connected set $\Sigma_1 \cap \bar{\Sigma}_2$ but a boundary point of it, we can still find a simply connected open neighborhood U_1 of w in M with $U_1 \subseteq \tilde{U}_1$ and some $a > 0$ such that we have $X_t(U \cap \Sigma_1) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ for $0 < t \leq a$ because $\Sigma_1 \cap \Sigma_2$ is interval-like connected. So, the claim is proven for these two situations. As we have considered all the possible situations, our claim has been proven for any point $w \in {\Sigma_1}^\circ$.

Note that there is no distinction between the sets Σ_1 and Σ_2 in terms of disjointness or interval-like connectedness and we could have applied the same arguments for interior points in Σ_2 and also, we can analogously consider the negative time flow of X instead of its positive time flow. Therefore, for any point $q \in {\Sigma_2}^\circ$, there exist a simply connected open neighborhood U_2 of q in M and some $b > 0$ such that $U_2 \cap \Sigma_2$ does not include the boundary points of Σ_2 (if they exist at all) and we have $X_t(U_2 \cap \Sigma_2) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ for $-b \leq t < 0$.

Let $w \in E_0$. Note that we have $w \in {\Sigma_1}^\circ$ and $P(w) \in {\Sigma_2}^\circ$ by the definition of the First Proper Intersection Assignment. We face two possible cases: either $P(w) \neq w$ or $P(w) = w$.

Assume that we have $P(w) \neq w$. Let $a, b > 0$ and let U_1 and U_2 be simply connected open neighborhoods of w and $P(w)$ in M respectively such that: they do not include the boundary points of Σ_1 and Σ_2 respectively; we have $X_t(U_1 \cap \Sigma_1) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ for $0 < t \leq a$ and; we have $X_t(U_2 \cap \Sigma_2) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ for $-b \leq t < 0$. Also, if a and b are small enough, then the points $p_1 := X_a(w)$ and $p_2 := X_{-b}(P(w))$ are in U_1 and U_2 respectively and the closed arc p_1p_2 is defined. We have $p_1p_2 \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ because w is in the domain of the First Proper Intersection Assignment. As all the sets $\bar{\Sigma}_1$, $\bar{\Sigma}_2$ and p_1p_2 are compact and we have $p_1p_2 \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$, there exists a simply connected open neighborhood U_3 of p_1p_2 in M such that we have $U_3 \cap (\, \bar{\Sigma}_1 \cup \bar{\Sigma}_2 \,) = \varnothing$. Let V be a tubular flow neighborhood of $wP(w)$ in $U_1 \cup U_2 \cup U_3$. Then, P is defined on $V \cap \Sigma_1$ by our choice of U_1, U_2 and U_3. Hence, the set $E_0 \cap V = \Sigma_1 \cap V$ is open in Σ_1. Also, P is a local diffeomorphism in $E_0 \cap V$ because V is a tubular flow neighborhood of p_1p_2. See Figure 5.4.

Assume that we have $P(w) = w$ and let τ be the closed orbit of w. As we have $w \in {\Sigma_1}^\circ$ and $w \in {\Sigma_2}^\circ$, we also have $w \in (\Sigma_1 \cap \Sigma_2)^\circ$ so that there exist a simply connected open neighborhood U_0 of w in M such that we have $U_0 \cap \Sigma_1 = U_0 \cap \Sigma_2$ and U_0 does not include the boundary points of Σ_1 and Σ_2. Let $a, b > 0$ and let U_1 be a simply connected open neighborhood of w in M such that: we have $U_1 \cap \Sigma_1 = U_1 \cap \Sigma_2$; U_1 does not include the boundary points of Σ_1 and Σ_2 and; we have $X_t(U_1 \cap \Sigma_1) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ for $t \in [-b, 0) \cup (0, a]$. Also, if a and b are small enough, then the points $p_1 := X_a(w)$ and $p_2 := X_{-b}(P(w))$ are in U_1 and the closed arc p_1p_2 is defined. We have $p_1p_2 \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$ because w is in the domain of the First Proper Intersection Assignment. As all $\bar{\Sigma}_1$, $\bar{\Sigma}_2$ and p_1p_2 are compact, there exists a simply connected open neighborhood U_3 of p_1p_2 in M such that we have $U_3 \cap (\, \bar{\Sigma}_1 \cup \bar{\Sigma}_2 \,) = \varnothing$.

Let z be an interior point of p_1p_2 and let Σ_3 be an open transversal section to X through z such that we have $\bar{\Sigma}_3 \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$; and $\bar{\Sigma}_3 \cap \tau = \{z\}$. So, the First Proper Intersection Assignment $Q_1 : \Sigma_3 \to \Sigma_1$ is defined at z with $Q_1(z) = w$. By the previous part, there exists an open tubular flow neighborhood V_1 of zw such that Q_1 is defined on $V_1 \cap \Sigma_3$. Let V_1 be small enough such that we have $V_1 \subseteq (U_1 \cup U_3)$. Similarly, the First Proper Intersection Assignment $Q_2 : \Sigma_1 \to \Sigma_3$ is defined at w with $Q_2(w) = z$ and; there exists an open tubular flow neighborhood V_2 of wz such that Q_2

is defined on $V_2 \cap \Sigma_1$. Let V_2 be small enough such that we have $V_2 \subseteq (U_1 \cup U_3)$ and $V_2 \cap \Sigma_3 \subseteq V_1 \cap \Sigma_3$. By our choice of U_1, U_3 V_1 and V_2, P is defined on $V_2 \cap \Sigma_1$ as it is equal to $Q_{1|V_1 \cap \Sigma_3} \circ Q_{2|V_2 \cap \Sigma_1}$. In this case, we also conclude that the set $E_0 \cap V_1 = \Sigma_1 \cap V_1$ is open in Σ_1. Also, P is a local diffeomorphism in $E_0 \cap V_1$ because we have $\Sigma_3 \cap \Sigma_1 = \varnothing$ and $\Sigma_3 \cap \Sigma_2 = \varnothing$ and; V_1 and V_2 are tubular flow neighborhoods of zw and wz respectively. See Figure 5.5.

Let $\{w_1, w_2\} \subseteq E_0$ be such that we have $P(w_1) = P(w_2)$. Define A_{w_1} and A_{w_2} as in Definition 5.2. Assume that A_{w_1} is a closed orbit of X. Then, we have $P(w_1) = w_1$. Since w_1 is in the domain of the First Proper Intersection Assignment, we have $A_{w_1} \cap \bar{\Sigma}_1 = \{w_1\}$. We conclude $w_1 = w_2$ because of all the following properties: we have $P(w_2) = P(w_1) = w_1$; $w_2 \in \Sigma_1$; A_{w_1} is a closed orbit and; $A_{w_1} \cap \Sigma_1 = \{w_1\}$. So, the injective property of P is proven for this case.

If A_{w_1} is not a closed orbit of X, then it is a closed arc of X and we have $A_{w_1} = w_1 P(w_1)$. The set A_{w_2} cannot be a closed orbit of X because otherwise, we would have concluded $w_1 = w_2$ as we have shown above and we would have $A_{w_1} = A_{w_2}$ which contradicts that A_{w_1} is not a closed orbit. So, A_{w_2} is a closed arc of X as well and we have $A_{w_2} = w_2 P(w_2)$. As both A_{w_1} and A_{w_2} are closed arcs of X with the same end points $P(w_1) = P(w_2)$ and distinct orbits of X cannot intersect each other, we conclude that we have either $A_{w_1} \subseteq A_{w_2}$ or $A_{w_2} \subseteq A_{w_1}$. Say, we have $A_{w_1} \subseteq A_{w_2}$. As w_2 is in the domain of the First Proper Intersection Assignment, we have $(A_{w_2} - \{w_2, P(w_2)\}) \cap \bar{\Sigma}_1 = \varnothing$. As we have $A_{w_1} \subseteq A_{w_2}$, $(A_{w_2} - \{w_2, P(w_2)\}) \cap \Sigma_1 = \varnothing$ and $w_1 \in \Sigma_1$, we conclude that $w_1 = w_2$ and the injective property of P holds for this case as well. Hence, P is injective in general.

Let E be a connected component of E_0. As we have shown above, E is open in Σ_1. Let $B_E = \{z \in M : \; q \in E, \, z \in A_q\}$. As the set E is connected and open in Σ_1 and B_E is a subset of a non-disjoint union of tubular flow neighborhoods of the closed arcs in B_E, we conclude that $B_E - (\Sigma_1 \cup \Sigma_2)$ is diffeomorphic to an open disc. As P is injective, the set $B - (\Sigma_1 \cup \Sigma_2)$ where B is defined in the statement of the theorem is a disjoint union of open discs in M.

Suppose that E_0 has a boundary point p in Σ_1 such that p^+ does not intersect $\bar{\Sigma}_2$. Let B_p be the closed interval in the boundary of B such that: the boundaries of B_p are the points p and p_e for some $p_e \in \bar{\Sigma}_2$ and; $(B_p - \{p, p_e\}) \cap (\bar{\Sigma}_1 \cup \bar{\Sigma}_2) = \varnothing$. Such a point p_e exists in $\bar{\Sigma}_2$ because p is the boundary point of E_0 in Σ_1. By the continuity of the flow of X, the semi trajectory of a regular point in B_p travels in B_p until it leaves B_p. If all the points of B_p are regular points, then we hav $B_p \subseteq p^+$ which contradicts that p^+ does not intersect $\bar{\Sigma}_2$. Hence, B_p contains a hyperbolic singularity. Let q be a singularity in B_p. Then, q is not a sink or source because for any simply connected neighborhood U_q of q in M with $U_q \cap (\Sigma_1 \cup \Sigma_2) = \varnothing$, we have $U_q \cap A \neq \varnothing$ so that there does not exist $T > 0$ such that $X_t(U_q) \subseteq U_q$ for all $t > T$ or for all $t < -T$. Hence, q is a saddle. Therefore, there exist finitely many saddles $q_1, \ldots, q_n$ in B_p such that an unstable separatrix of q_j goes to q_{j+1} in B_p for $1 \leq j < n$ and an unstable separatrix of q_n goes to p_e in B_p. This completes the proof. $\square$

Remark 5.1. We have stated the proof of Tubular Flow Extension in such a way that the proof points out clearly when the hypothesis about the interval-like connectedness of $\Sigma_1 \cap \Sigma_2$ is used. Let Σ_a and Σ_b be transversal sections of X such that $\Sigma_a \cap \Sigma_b$ is a nonempty connected set. The First Proper Intersection Assignment $P : \Sigma_a \to \Sigma_b$ excludes the positive semi trajectories through the boundary points of Σ_a and Σ_b with the only exception that P might be defined at a point in ${\Sigma_a}^\circ \cap \partial\Sigma_2$ where $\partial\Sigma_2$ is the boundary of Σ_2. As the positive semi trajectories through these boundary points may cause problems for the continuity of P, the trajectories through the boundary points of $\Sigma_1 \cap \Sigma_2$ may cause problems as well. Figure 5.6 gives an example of this. There, the (topological) boundary of the set $\{p\}$ is the point p.

So, the solution to this problem is obvious: just exclude the set of points $\{p_a, p_b\}$ from the domain of P where the ${p_a}^+$ and ${p_b}^+$ goes through the boundary points of the connected set $\Sigma_a \cap \Sigma_b$. If $\Sigma_a \cap \Sigma_b$ has finitely many connected components, then one can generalize Tubular Flow Extension by excluding the finitely many points from the domain of P the positive semi trajectories of which go through the boundary points of $\Sigma_a \cap \Sigma_b$ and it is best to refrain from situations when $\Sigma_a \cap \Sigma_b$ has infinitely many connected components.

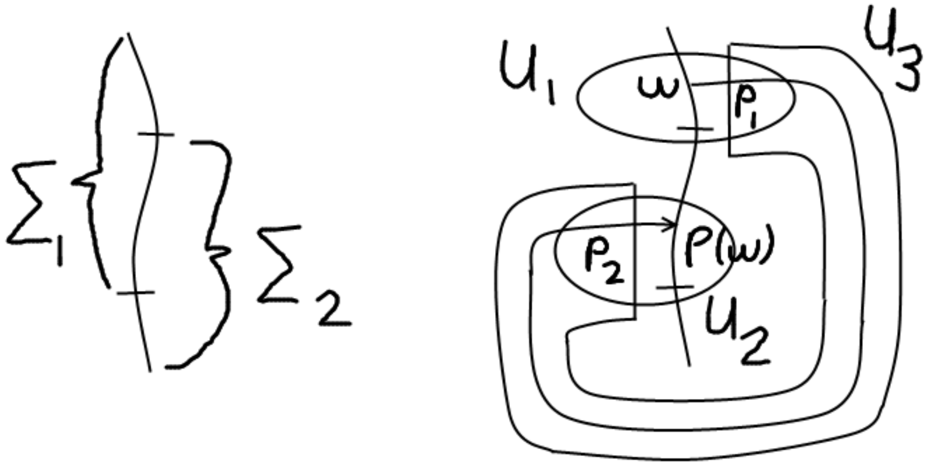

Figure 5.4. The case $P(w) \neq w$.

Corollary 5.1. *Suppose that: X is in $\mathfrak{X}^r(M)$ the singularities of which are all hyperbolic; $\bar{D}$ is a section and orbit sided rectangle with sides $\Sigma_1, \Sigma_2, a_1a_2$ and b_1b_2 and; $\bar{D}$ does not contain any saddles. Then, the positive semi trajectory of every point in Σ_1 intersects Σ_2.*

Proof. By applying the Tubular Flow Theorem to the closed arc a_1a_2, we conclude that the First Proper Intersection Assignment $P : \Sigma_1 \to \Sigma_2$ is defined in a neighborhood of a_1 in Σ_1. By the Tubular Flow Extension, P is defined on the whole ${\Sigma_1}^\circ$ because of all the following properties: D does not contain any saddles; the positive semi trajectory of a point in ${\Sigma_1}^\circ$ cannot intersect Σ_1 before it intersects ${\Sigma_2}^\circ$ first because $\bar{D}$ is a section and orbit sided rectangle; the positive semi trajectory of a point in ${\Sigma_1}^\circ$ cannot go to a boundary point of Σ_2 within D because of the closed arcs a_1a_2 and b_1b_2. So, the corollary follows. □

The Corollary 5.1 will be the key observation to exhibit the transversal base in Definition 5.10.

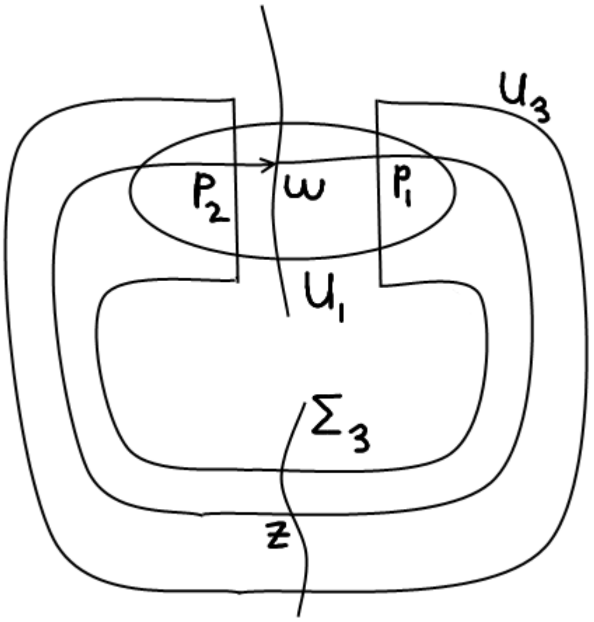

Figure 5.5. The case $P(w) = w$.

5.2. Weakly Transversal Fundamental Polygon

A fundamental polygon Γ for M is a quotient topological space of some even sided polygon $\hat{Q}$ in $\mathbb{R}^2$. For simplicity, we will omit the diffeomorphism from M to Γ and we will make no distinction between them. A subset of M will also be a subset of Γ and vice versa. Let $2n_r$ denote the number of sides of $\hat{Q}$ (we will assume that $n_r \geq 2$). Let $\Pi_r : \hat{Q} \to \Gamma$ be the quotient map which induces the quotient topology on Γ . A side S of Γ means that $\Pi_r^{-1}(S)$ is the union of two sides of $\hat{Q}$.

Let X be in $\mathfrak{X}^r(M)$ that has finitely many singularities. Let Γ be a fundamental polygon for M such that each side of it is a smoothly embedded circle in M and it does not contain the singularities of X. Let S be a side of Γ. We cannot expect S to be transversal to X in general but we can expect it to be transversal to X except at a finite number of points in S. Definition 5.5 has been formulated with slightly stronger conditions for this purpose. Our sides will satisfy that definition.

Our first idea is to perturb S to achieve this goal which we will abandon soon. We now need to change the situation a bit and take a C^r embedded closed interval L

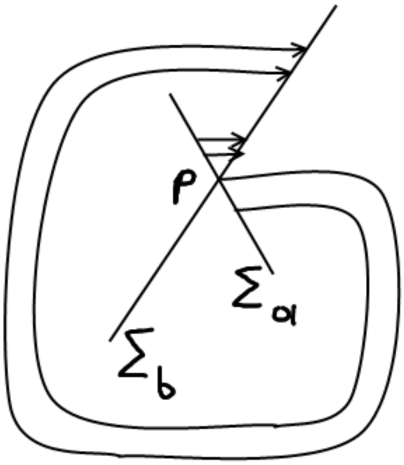

Figure 5.6. The First Proper Intersection Assignment $P : \Sigma_a \to \Sigma_b$ is not continuous at p.

in M instead of a circle. We give an example below to show how problematic L can be.

Example 5.1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4 \sin^2(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$. We have $f'(x) = 4x^3 \sin^2(\frac{1}{x}) - 2x^2 \sin(\frac{1}{x}) \cos(\frac{1}{x})$ for $x \neq 0$. The equality $f'(0) = 0$ can be shown directly and f is C^1. Note that $f(x) \geq 0$ and at $x = (n\pi)^{-1}$ for $n \in \mathbb{Z}$, we have $f(x) = f'(x) = 0$. So, the graph of $f(x)$ and the x-axis are not transversal to each other at the common points in $S = \{(x, 0) : x = (n\pi)^{-1}, n \in \mathbb{Z}\}$. If we take $L = [-1, 1] \times \{0\}$ and the set $\{(x, f(x)) : x \in [-1, 1]\}$ to be a closed arc of some C^1 vector field X on $\mathbb{R}^2$, then the transversality of L to X fails at the set S. Let $(x_0, 0) \in S - \{(0, 0)\}$. $S - \{(0, 0)\}$ is a discrete set and in a small connected neighborhood $[x_0 - \epsilon, x_0 + \epsilon]$ of x_0 in the x-axis, we have $f'(x) \neq 0$ for $x \in [x_0 - \epsilon, x_0) \cup (x_0, x_0 + \epsilon]$. So, there exists a small tubular flow neighborhood V_0 of $\{\, (x, f(x)) : x \in [x_0 - \epsilon, x_0 + \epsilon] \,\}$ such that L is transversal to X at the points in $(L \cap V_0) - \{(x_0, 0)\}$. Therefore, $(0, 0)$ is a limit point of L_T where L_T is the set of points in L at which L is transversal to X. So, the connected component of $L - L_T$ containing $(0, 0)$ is $\{(0, 0)\}$.

The Example 5.1 shows that $\{(0, 0)\}$ is not only a connected component of $L - L_T$ but also $(0, 0)$ is a limit point of $L - L_T$ since we have $S \subseteq L - L_T$ and $(0, 0)$ is a limit point of S. Our aim is to eliminate these infinitely many points in S. More precisely, we cannot find a small compact connected neighborhood N_0 of $(0, 0)$ in L (a closed interval) such that $N_0 \cap (L - L_T)$ is a finite set and our perturbation of L should achieve this goal. The first idea would be to take a small simply connected open neighborhood U of $(0,0)$ in $\mathbb{R}^2$ and perturb L in U. Let $\tilde{L}$ denote its perturbation. We would like to have $\tilde{L} \cap U$ is transversal to X except at finitely many points in it. We also prefer to have $\tilde{L} - U = L - U$. This procedure does not allow us to take advantage of the compactness of L because after an initial finite open cover of L with such U sets and a single perturbation on a single U, some of the remaining U's may no longer apply to $\tilde{L}$.

Henceforth, we would like to define a new $\tilde{L}$ that disregards L except a small neighborhood and boundary points of it. Let's come back to our manifold M and let L be a C^r embedded closed interval in M. In the involved long discussion below, we will eventually prove Theorem 5.2. The first part of this long discussion proves Lemma 5.2.

Let U be a neighborhood of L. For every q in L, let (f_q, F_q) be a flow box at q such that F_q is a subset of U and also $F_q \cap L$ is connected. The last requirement is possible since L is compact. Say, $f_q(F_q) = [-a_q, a_q] \times [-1, 1]$. The set $\bigcup_{q \in L} F_q$ is a cover of the compact set L. Let $\{F_{q_1}, \ldots, F_{q_m}\}$ be a finite subcover of it such that we have $F_{q_j} \cap F_{q_k} \neq \varnothing$ if and only if k and j are adjacent integers in $\{1, \ldots, m\}$. This last requirement is possible since any interval can be covered with finite open intervals in this way. Let $f_j = f_{q_j}$ and $F_j = F_{q_j}$ for $1 \leq j \leq m$. Let p_1 and p_{m+1} denote the boundary points of L in F_1 and F_m respectively. By taking smaller F_j's (before finding the finite subcover), we can also assume that we have $p_1 \in F_1 - F_2$ and $p_{m+1} \in F_m - F_{m-1}$. Note that the interior of $\bigcup_{1 \leq j \leq m} F_m$ is diffeomorphic to an open disc. See Figure 5.7.

For $s = 1, 2$, let $\Pi_s : \mathbb{R}^2 \to \mathbb{R}$ be the natural projection map onto the s^{th} coordinate of $\mathbb{R}^2$. For $1 \leq j < m$, let $p_{j+1} \in F_j \cap F_{j+1}$ be such that we have: $\Pi_1 \circ f_j(p_j) \neq \Pi_1 \circ f_j(p_{j+1})$; $\Pi_2 \circ f_j(p_j) \neq \Pi_2 \circ f_j(p_{j+1})$; $\Pi_1 \circ f_m(p_m) \neq \Pi_1 \circ f_m(p_{m+1})$ and; $\Pi_2 \circ f_m(p_m) \neq \Pi_2 \circ f_m(p_{m+1})$.

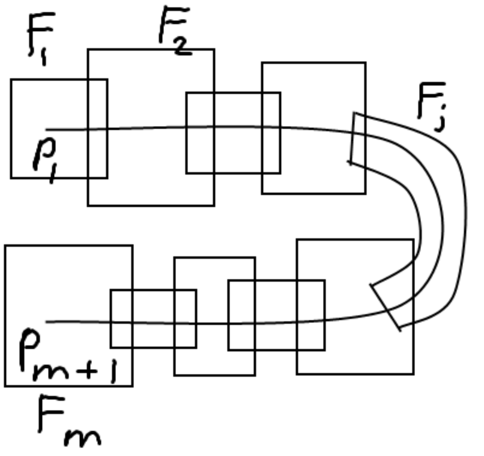

Figure 5.7. The cover of L with finitely many flow boxes.

Consider the straight line l_j in $f_j(F_j)$ between $f_j(p_j)$ and $f_j(p_{j+1})$. By our choice of p_j's, the straight line l_j is not parallel to the x or y-axis in $f_j(F_j)$ (see Figure 5.8). So, ${f_j}^{-1}(l_j)$ is transversal to X. The set $l_j \cap f_j(F_j \cap L)$ can have infinitely many connected components and this possible situation is very problematic. This possible problem can be solved by taking smaller flow boxes (f_q, F_q)'s at each $q \in L$ in the very beginning before finding the finite subcover. We will not revise our initial choice of flow boxes (f_q, F_q)'s but instead, we opt for the following choice of l_j. For $1 \leq j \leq m$, let l_j be a C^r embedded closed interval in $f_j(F_j)$ such that: the boundary points of l_j are $f_j(p_j)$ and $f_j(p_{j+1})$; the C^r embedded interval l_j is transversal to ${f_j}_*(X_{|F_j})$; the tangent line at each point of l_j is not parallel to the y-axis and; $l_j \cap f_j(F_j \cap L)$ is a finite union of isolated points or C^r embedded intervals in $f_j(F_j)$.

Note that by our choice of p_j's, each l_j in $f_j(F_j)$ can be parametrized in the following two ways: the y-coordinates of l_j is a C^r function of the x-coordinates of l_j and l_j is the graph of some C^r function $y = y(x)$; the x-coordinates of l_j is a C^r function of the y-coordinates of l_j and l_j is the graph of some C^r function $x = x(y)$.

For $1 \leq j \leq m$, let $L_j = {f_j}^{-1}(l_j)$. The set $L_j \cap L_{j+1}$ is not necessarily equal to $\{p_{j+1}\}$ yet even to a finite set. Let $h_j : L_j \to [0, 1]$ be a diffeomorphism such that we have $h_j(p_j) = 0$ and $h_j(p_{j+1}) = 1$. For any two points z_1 and z_2 in L_j, we say that z_1 is closer to p_j than z_2 is if and only if we have $h_j(z_1) \leq h_j(z_2)$. For $1 \leq j \leq m-1$, let

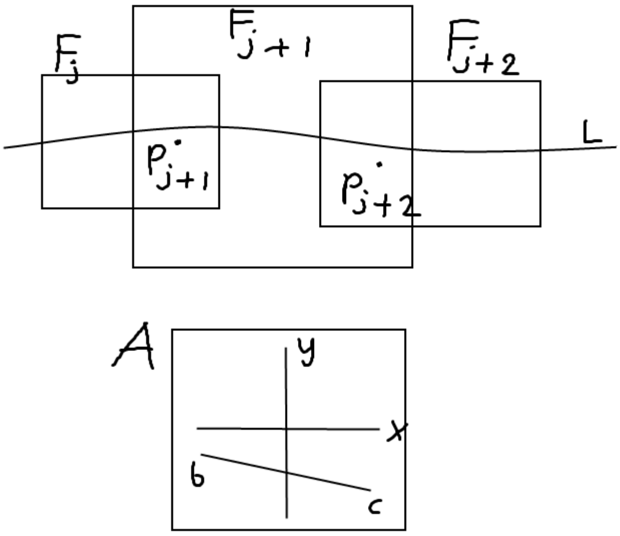

Figure 5.8. The straight line between b and c in $\mathbb{R}^2$ which is not parallel to the x and y axes. Here, we have $A = f_{j+1}(F_{j+1})$, $b = f_{j+1}(p_{j+1})$ and $c = f_{j+1}(p_{j+2})$.

w_j be the unique accumulation point of $L_j \cap L_{j+1}$ such that w_j is closer to p_j than any other point in $L_j \cap L_{j+1}$. By our choice of F_j's, we have $w_1 \neq p_1$ and $w_m \neq p_{m+1}$.

Let $\tilde{l}_1$ be the closed connected subset of l_1 with boundary points $f_1(p_1)$ and $f_1(w_1)$. For $1 < j < m$, let $\tilde{l}_j$ be the closed connected subset of l_j with boundary points $f_j(w_{j-1})$ and $f_j(w_j)$. Let $\tilde{l}_m$ be the connected closed subset of l_m with boundary points $f_m(w_{m-1})$ and p_{m+1}. Let $\tau_j = f_j{}^{-1}(\tilde{l}_j)$ for $1 \leq j \leq m$. Then, we have $\tau_j \cap \tau_{j+1} = \{w_{j+1}\}$ by the definitions of w_j's.

Let $\tau = \bigcup\limits_{1 \leq j \leq m} \tau_j$. Then, τ is homeomorphic to a closed interval; it is a piecewise C^r embedded submanifold of M; it is transversal to X except possibly at w_j's where it is most likely not C^r and; $\tau \cap L$ is a finite union of isolated points of M or C^r embedded intervals in M. We will perturb τ in some neighborhoods of w_j's to make it a C^r embedded submanifold of M such that w_j's will still belong to the perturbed τ and the perturbed τ will be transversal to X except possibly at those w_j's.

For $1 \leq j < m$, let $E_j = f_j(\tau_{j+1} \cap F_j)$. By our choice of l_j's and the fact that F_j's are flow boxes, the tangent line to E_j at a point of E_j is not parallel to the y-axis.

Hence, the y components of E_j is a C^r function k_j of the x components of E_j. As E_j is transversal to ${f_j}_*(X_{|F_j})$, we have ${k_j}'(x) \neq 0$ for $x \in (\Pi_1 \circ E_j)$. This inequality gives the converse conclusion that the x components of E_j is the C^r function ${k_j}^{-1}$ of the y components of E_j (by Inverse Function Theorem). For each $j \in \{1, \ldots, m\}$, translate the origin of $\mathbb{R}^2$ to $f_j(w_j)$ and let (x^j, y^j) denote the coordinates of $\mathbb{R}^2$ after such a translation. Note that $f_j(\tau_j) \cap f_j(E_j)$ is the origin in the $(x^j,\ y^j)$ coordinates. So, $f_j(\tau_j)$ and $f_j(E_j)$ are either on the same side of the x^j-axis or on the different sides of the x^j-axis. In either case, we need the below lemma to make τ a C^r curve at w_j.

Lemma 5.1. *Suppose that* $f : (-a, 0) \to \mathbb{R}$ *and* $g : (0, b) \to \mathbb{R}$ *are positive smooth functions for some* $a, b > 0$ *such that they satisfy:* $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} g(x) = 0$*;* $f'(x) \neq 0$ *for* $x \in (-a, 0)$*;* $g'(x) \neq 0$ *for* $x \in (0, b)$*;* $\lim_{x \to 0^-} f^{(n)}(x)$ *is bounded for each* $n \in \mathbb{N}$ *and;* $\lim_{x \to 0^+} g^{(n)}(x)$ *is bounded for each* $n \in \mathbb{N}$*. Then, there exists a smooth nonnegative function* $h : \mathbb{R} \to \mathbb{R}$ *such that* $h(x)$ *is equal to* 1 *outside of a connected open neighborhood* U_0 *of* 0 *with* $U_0 \subsetneq (-a, b)$ *and the function* $k : (-a, b) \to \mathbb{R}$ *which is defined as* $k(x) = f(x)h(x)$ *for* $x \in (-a, 0)$*,* $k(0) = 0$ *and* $k(x) = g(x)h(x)$ *for* $x \in (0, b)$ *is smooth. Moreover,* $k'(x)$ *is not equal to* 0 *for* $x \in (-a, 0) \cup (0, b)$*.*

Proof. Let $c = \frac{1}{2}\min\{a, b\}$. Define $h_1 : \mathbb{R} \to \mathbb{R}$ in the following way: $h_1(x) = 0$ for $|x| \geq c$; $h_1(x) = \exp(-(c + x)^{-1}(c - x)^{-1})$ for $x \in (-c, c)$. Define $h_2 : \mathbb{R} \to \mathbb{R}$ in the following way: $h_2(0) = 0$ and $h_2(x) = \exp(-|x|^{-1})$ for $x \neq 0$. Then, both h_1 and h_2 are smooth. Moreover, all the derivatives of $h_2(x)$ at $x = 0$ is 0. Let $h_3 : \mathbb{R} \to \mathbb{R}$ with $h_3(x) = (\,-h_1(x) + \exp(-c^{-2})\,)$. Let $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = (\,\exp(\exp(c^{-2})\,) \cdot (h_2 \circ h_3(x))$. Then, $h(x)$ is smooth and positive on $\mathbb{R} - \{0\}$; $h(0) = 0$; $h(x) = 1$ for $|x| \geq c$ and; all the derivatives of $h(x)$ is 0 at $x = 0$. Define the function $k : (-a, b) \to \mathbb{R}$ as in the statement of the lemma together with this choice of $h(x)$.

Clearly, k is smooth on $(-a, 0) \cup (0, b)$. Both $k(0+)$ (limit of $k(x)$ as x goes to 0 from the right side of 0) and $k(0-)$ are 0. So, k is continuous. Also, both $k^n(0+)$ and $k^n(0-)$ are 0 for each positive integer n because it is the limit of the sum of finitely many terms which come from the product rule for derivatives and the limit of each term in the summand goes to 0. Hence, all the derivatives of $k(x)$ is 0 at $x = 0$. Therefore,

k is smooth. It remains to check that $k'(x) \neq 0$ for $x \neq 0$.

We have $k'(x) = f'(x)h(x) + f(x)h'(x)$ for $x \in (-a, 0)$ and $k'(x) = g'(x)h(x) + g(x)h'(x)$ for $x \in (0, b)$. As $\lim_{x \to 0^+} g(x) = 0$, $g(x) > 0$ and $g'(x) \neq 0$, we conclude that g(x) is a strictly increasing function. Hence, $g'(x) > 0$. We also have that both $h_2'(x)$ and $h_3'(x)$ are positive for $c > x > 0$. Hence, $h'(x)$ is positive for $c > x > 0$. So, the terms in $g'(x)h(x) + g(x)h'(x)$ are all positive so that $k'(x)$ is nonzero for $0 < x < c$. As we have $k(x) = g(x)$ for $c \leq x < b$, we conclude that $k'(x)$ is nonzero for $c \leq x < b$ as well. We can analogously conclude that of $k'(x)$ is nonzero for $-a < x < 0$ as both $f'(x)$ and $h'(x)$ are negative for $-c < x < 0$ and both $f(x)$ and $h(x)$ are positive for $-c < x < 0$ so that $f'(x)h(x) + f(x)h'(x)$ is negative for $-c < x < 0$. As we have $k(x) = f(x)$ for $-a < x \leq -c$, $k'(x)$ is nonzero for $-a < x \leq c$ as well. The proof is complete. □

We continue on with our last discussion. Suppose that either both $f_j(\tau_j)$ and $f_j(E_j)$ or both $-f_j(\tau_j)$ and $-f_j(E_j)$ are above the x^j-axis. We have stated Lemma 5.1 for smooth f and g but the lemma can clearly be generalized for C^r functions instead of smooth ones. As both $f_j(\tau_j)$ and $f_j(E_j)$ are the graphs of some C^r functions of x^j, we can apply Lemma 5.1 to them in a small simply connected open neighborhood U_j of the origin. Note that Lemma 5.1 indeed applies because the limit of relevant derivatives do not go to $\pm\infty$ because each l_j was a C^r embedded closed interval in $f_j(F_j)$. Let U_j be as small as we have $U_j \subseteq f_j(F_j \cap F_{j+1})$. Let C_j be the perturbation of $f_j(\tau_j) \cup f_j(E_j)$ which is obtained with Lemma 5.1. So, we have $C_j - U_j = (f_j(\tau_j) \cup f_j(E_j)) - U_j$ and C_j is a graph of a C^r function $k(x^j)$ such that we have $k'(x^j) \neq 0$ for $x \neq 0$. Hence, the C^r embedded submanifold $f_j^{-1}(C_j)$ is transversal to X except at w_j. Moreover, if U_j is small enough, then $f_j^{-1}(C_j) \cap L$ is a finite union of isolated points in M or C^r embedded intervals in M.

Suppose that one of $f_j(\tau_j)$ and $f_j(E_j)$ is above the x-axis and the other one is below the x^j-axis. Then, $f_j(\tau_j) \cup f_j(E_j)$ is the graph of a continuous function $s(y^j)$ that is C^r except at $y^j = 0$. Substitute x in $h(x)$ with y^j to obtain $h(y^j)$ where $h(x)$ is as in the proof of Lemma 5.1. We can multiply $s(y^j)$ with $h(y^j)$ so that $s(y^j)h(y^j)$ is C^r and it is equal to $s(y^j)$ outside of a small connected open neighborhood 0. Again, let U_j be a small simply connected open neighborhood of the origin such that U_j is a subset of $f_j(F_j \cap F_{j+1})$ and the perturbation is done in U_j. As the derivative of $s(y^j)$ is defined everywhere in its domain, the tangent line to the graph K_j of $s(y^j)$ at a point of it is not parallel to the x^j-axis. Therefore, $f^{-1}(K_j)$ is a C^r embedded submanifold of M that is transversal to X. Also, if U_j is small enough, then $f^{-1}(K_j) \cap L$ is a finite union of isolated points in M or C^r embedded intervals in M.

So, we can apply one of the two above procedures in the neighborhood $f_j^{-1}(U_j)$ of each w_j to obtain a C^r embedded closed interval $\tilde{\tau}$ that is equal to τ except in those small neighborhoods of w_j's. Also, it is transversal to X except at some of the possible w_j's. We will summarize our results in the below lemma but we first emphasize an important fact about the behavior of the orbits of X near such w_j's at which the transversailty of $\tilde{\tau}$ to X fails.

Consider such a possible point w_j at which $\tilde{\tau}$ is not transversal to X. Intuitively, we think that $\tilde{\tau}$ shows parabolic behavior in F_j as if $f_j(\tilde{\tau} \cap F_j)$ is part of the graph of $y = x^2$ in $\mathbb{R}^2$. This parabola function is positive except at the origin and its derivative is nonzero except at the origin. The orbits of the parallel unit vector field ${f_j}_*(X_{|F_j})$ intersects $f_j(\tilde{\tau} \cap F_j)$ twice. Let $d_j = \Pi_2 \circ f_j(w_j)$ (in $f_j(F_j)$ coordinates and not in (x^j, y^j)) and let $\gamma_j = {f_j}^{-1}([-a_j, a_j] \times \{d_j\})$. Note that γ_j is a closed arc of X containing w_j. See Figure 5.9.

If we would like to look from the perspective of τ and take some other diffeomorphism $\tilde{f}_j : F_j \to A_j \subseteq \mathbb{R}^2$ such that we have $\tilde{f}_j(w_j) = (0,0)$ and $\tilde{f}_j(\tilde{\tau} \cap F_j)$ is a subset of the x-axis, then $\tilde{f}_j(\gamma_j)$ looks like a parabola in $\tilde{f}_j(F_j)$. Let V_j be a tubular flow neighborhood of γ_j and V_j^1 and V_j^2 be the connected components of $V_j - \gamma_j$. If V_j is small enough, then the orbits of one them does not intersect $\tilde{\tau}$ in V_j and the orbits in the other one intersect it twice in V_j. Let Γ be an orbit in V_1. Then, also $\tilde{f}_j(\Gamma)$ looks like a parabola in $\mathbb{R}^2$ which is either above $\tilde{f}_j(\gamma_j)$ or below it. We now make some appropriate definitions before stating Lemma 5.2.

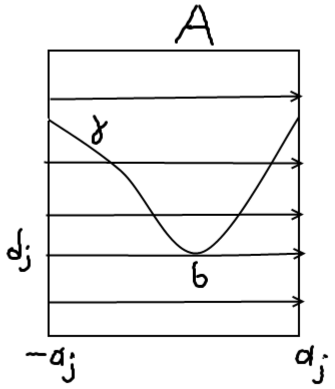

Figure 5.9. $A = f_j(F_j)$, $\gamma = f_j(\tilde{\tau} \cap F_j)$ and $b = f_j(w_j)$.

Definition 5.4. *Let X be in $\mathfrak{X}^r(M)$ and let L be a C^r embedded interval or circle in M that does not contain the singularities of X. For an interior point p of L, suppose that there exists a simply connected open neighborhood U_p of p in M such that $U_p \cap L$ is connected and $U_p \cap L$ is transversal to X except at p. Let $\epsilon_1, \epsilon_2 > 0$ be small enough such that we have $p_1p_2 \subseteq U_p$ and $p_1p_2 \cap L = \{p\}$ where we have $p_1 = X_{-\epsilon_1}(p)$ and $p_2 = X_{\epsilon_2}(p)$. Let U_1 and U_2 be the connected components of $U_p - L$. If $p_1p_2 - \{p\}$ is a subset of U_1 or U_2, we say that L has* a parabolic transversality failure at p.

Definition 5.5 (Weakly transversal)**.** *Let X be in $\mathfrak{X}^r(M)$ and L be a C^r embedded interval or circle in M that does not contain the singularities of X. Suppose that L is transversal to X except at finitely many interior points $p_1, \ldots, p_n$ of it. If L has a parabolic transversality failure at each p_j for $1 \leq j \leq n$, then we say that L is* weakly transversal to X.

Lemma 5.2. *Suppose that X is in $\mathfrak{X}^r(M)$ and L is a C^r embedded closed interval in M that does not contain the singularities of X. Then, for every neighborhood U_l of L, there exists a C^r embedded closed interval τ in M such that: τ and L have the same boundary points; we have $\tau \subseteq U_l$; the set $\tau \cap L$ is a finite union of isolated points in M or C^r embedded intervals in M and; τ is weakly transversal to X.*

Lemma 5.2 is the first step toward Theorem 5.2. We make some definitions below to work on a fundamental polygon properly.

Definition 5.6. *Let X be in $\mathfrak{X}^r(M)$ and let Γ be a fundamental polygon for M. The point c in Γ will denote the unique point in the intersection of all sides of Γ. The set*

$\mathcal{S}_r = \{S_1, \ldots, S_{n_r}\}$ *will denote the sides of* Γ. *The region* $\Gamma - \mathcal{S}_r$ *which is diffeomorphic to an open disc will be called* the polygon disc D_Γ of Γ.

The first thing that one should notice is that the polygon disc D_Γ of a fundamental polygon Γ can be embedded into S^2. We have powerful results for C^r vector fields on S^2. Jordan Curve Theorem (see [22]) and Poincaré-Bendixson Theorem (see [3]) have been stated for S^2. We have used Jordan Curve Theorem to derive Lemma 3.1 which states that a C^r vector field on S^2 does not have any nontrivial recurrent orbits.

Let X be in $\mathfrak{X}^r(M)$ that has finitely many singularities. Then, there exists a fundamental polygon Γ for M such that each side S_j of Γ is a smoothly embedded circle in M and it does not contain the singularities of X. Let $\hat{Q}$ be an even sided polygon and $\Pi_r : \hat{Q} \to \Gamma$ be the quotient map which defines the quotient topology on Γ. Let $D_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y \geq 0\}$. Let U_c be a small simply connected open neighborhood of c in M with the following natural property: U_c is a union of sets $\Pi_r(A_1), \ldots \Pi_r(A_{n_r})$ in Γ where each A_j (for $1 \leq j \leq n_r$) in $\hat{Q}$ is homeomorphic to D_2 and all A_j's are pairwise disjoint. So, each $S_j \cap U_c$ is connected. Let U_c be small enough so that each $S_j - U_c$ is a closed interval in S_j. See Figure 5.10.

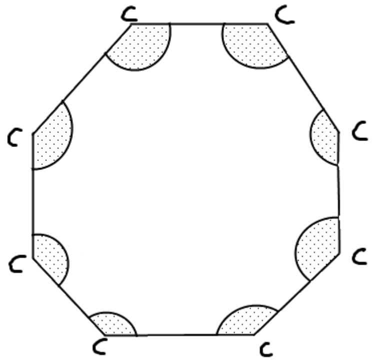

Figure 5.10. U_c is the union of shaded regions.

Let (f_c, F_c) be a flow box at c such that F_c is a subset of U_c. Say, $f_c(F_c) = [-a_c, a_c] \times [-1, 1]$. For each j with $1 \leq j \leq n_r$, let $T_j = f_c(S_j \cap F_c)$. Note that all T_j's include the origin and all $T_j - \{(0, 0)\}$'s are pairwise disjoint because S_j's are sides of Γ. For $z > 0$, let $C(z)$ denote the unit circle of radius z centered at the origin in $\mathbb{R}^2$

and let $B(z, 0)$ denote the open ball with the boundary $C(z)$.

Because T_j is an embedded C^1 interval in $\mathbb{R}^2$, the slopes of the tangent lines to T_j at points in T_j change continuously. So, there exists some $r_j > 0$ with $0 < r_j < a_c$ such that $C(z)$ is transversal to T_j for $0 < z \leq r_j$ and $C(z) \cap T_j$ has two points. Define any orientation on the curve T_j and let the set $\{\tilde{b}_j^1, \tilde{b}_j^2, \tilde{b}_j^3, \tilde{b}_j^4\}$ denote the ordered set $T_j \cap (C(r_j/2) \cup C(r_j))$ according to this orientation. So, $\tilde{b}_j^2$ and $\tilde{b}_j^3$ are in $C(r_j/2)$ and $\tilde{b}_j^1, \tilde{b}_j^4$ are in $C(r_j)$. Let $\tilde{l}_j^1$ be the straight line between $\tilde{b}_j^1$ and $\tilde{b}_j^2$ and let $\tilde{l}_j^2$ be the straight line between $\tilde{b}_j^3$ and $\tilde{b}_j^4$. Let $\tilde{H}_j = (T_j - B(r_j, 0)) \cup (T_j \cap B(r_j/2, 0)) \cup \tilde{l}_j^1 \cup \tilde{l}_j^2$. If r_j is small enough, then $\tilde{H}_j$ is a continuous curve that does not have self intersections. See Figure 5.11.

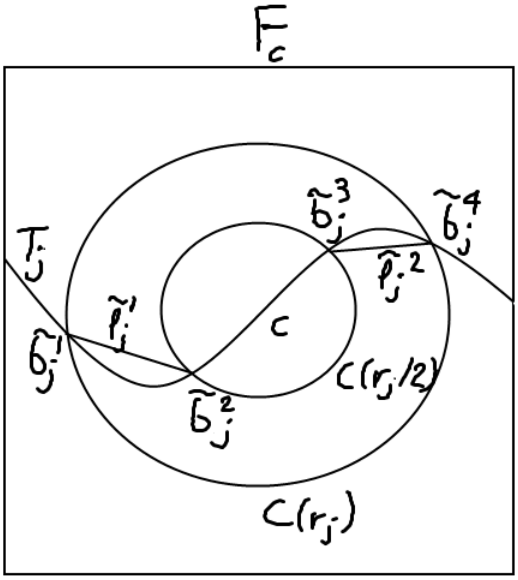

Figure 5.11. The straight line $\tilde{l}_j^1$ between $\tilde{b}_j^1$ and $\tilde{b}_j^2$ and the straight line $\tilde{l}_j^2$ between $\tilde{b}_j^3$ and $\tilde{b}_j^4$.

Consider two distinct sets $\tilde{H}_m$ and $\tilde{H}_k$ $(1 \leq m < k \leq n_r)$. If r_m and r_k are both small enough, then we have $(\tilde{H}_m - \{(0, 0)\}) \cap (\tilde{H}_k - \{(0, 0)\}) = \varnothing$. As there are finitely many $\tilde{H}_j$'s, we can ensure this property for any two distinct sets $\tilde{H}_m$ and $\tilde{H}_k$ if all r_j's are small enough. We will assume that all r_j's are small enough in this sense.

Let $\mathcal{R}_\theta$ denote the anticlockwise rotation of $\mathbb{R}^2$ by θ degrees at the origin. There exist some small $\theta_0 > 0$ (with $\theta_0 < \pi/2$) such that each $\mathcal{R}_{\theta_0}(T_j)$ is transversal to the x-

axis at the origin. So, there exist some $g_1, \dots, g_{n_r} > 0$ such that each $\mathcal{R}_{\theta_0}((B(g_j, 0) \cap T_j)$ is transversal to the x-axis for $1 \leq j \leq n_r$. Let $r_e = \frac{1}{2}\min\{g_1, \dots g_{n_r}, r_1, \dots, r_{n_r}\}$. Define any orientation on the curve T_j and let the set $\{b_j^1, b_j^2, b_j^3, b_j^4\}$ denote the ordered set $T_j \cap (C(r_e) \cup C(2r_e))$ according to this orientation. So, b_j^2 and b_j^3 are in $C(r_e)$ and b_j^1, b_j^4 are in $C(2r_e)$. Let l_j^1 be the straight line between b_j^1 and $\mathcal{R}_{\theta_0}(b_j^2)$ and let l_j^2 be the straight line between $\mathcal{R}_{\theta_0}(b_j^3)$ and b_j^4. Let $H_j = (T_j - B(2r_e, 0)) \cup (\mathcal{R}_{\theta_0}(\, T_j \cap B(r_e, 0)\,)) \cup l_j^1 \cup l_j^2$. If θ_0 is small enough, then all $H_j - \{(0,0)\}$'s are pairwise disjoint.

Let $Q_j = f_c^{-1} \circ \mathcal{R}_{\theta_0}(T_j \cap B(r_e, 0))$. Note that Q_j is a C^r embedded open interval interval in M which is transversal to X, in particular at c. Let $L_j = S_j - Q_j$. Let U_j be an open neighborhood of L_j such that: U_j is diffeomorphic to an open disc; $U_j \cap S_j$ is connected and for any other side S_k of Γ, we have $U_j \cap S_k = \varnothing$ and; all U_j's are pairwise disjoint. See Figure 5.12.

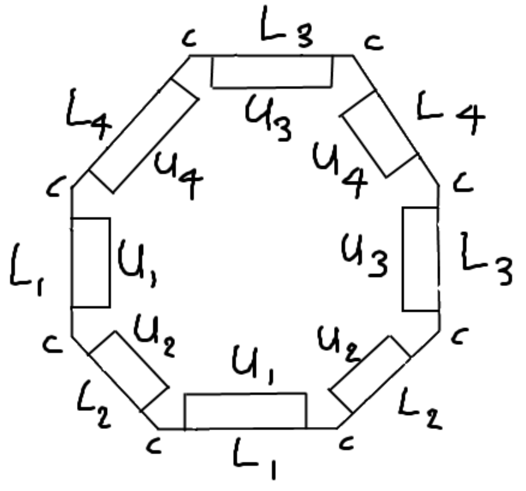

Figure 5.12. An arbitrary identification of the sides of the depicted fundamental polygon is used but it is irrelevant to the illustration of U_j's.

$L_j - \{\, f_c^{-1}(b_j^1), f_c^{-1}(b_j^4)\,\}$ is a C^r embedded interval in M so that we can still apply Lemma 5.2 to L_j to obtain a C^r embedded closed interval τ_j in M such that: τ_j and L_j have the same boundary points $f_c^{-1}(b_j^2)$ and $f_c^{-1}(b_j^3)$; we have $\tau_j \subseteq U_j$; the set $\tau_j \cap L_j$ is a finite union of isolated points in M or C^r embedded intervals in M and; τ_j is weakly transversal to X. We can apply Lemma 5.1 in small flow boxes at $f_c^{-1}(b_j^2)$ and $f_c^{-1}(b_j^3)$ where each flow box is in U_j to obtain perturbations $\tilde{\tau}_j$ and $\tilde{Q}_j$ of τ_j and Q_j respectively such that: $\tilde{\tau}_j \cup \tilde{Q}_j$ is a C^r embedded circle in M that is weakly transversal to X. Because the flow boxes at $f_c^{-1}(b_j^2)$'s and $f_c^{-1}(b_j^3)$'s are in U_j's, the point c is still the only common point of any two distinct $\tilde{\tau}_m \cup \tilde{Q}_m$ and $\tilde{\tau}_k \cup \tilde{Q}_k$ $(1 \leq m < k \leq n_r)$.

Since we have chosen small enough neighborhoods U_j's and U_c and also, the set $S_j \cap (\tilde{\tau}_j \cup \tilde{Q}_j)$ is a finite union of isolated points in M or C^r embedded intervals in M, a new fundamental polygon $\tilde{P}_r$ with the sides $\tilde{\tau}_j \cup \tilde{Q}_j$'s can be obtained from the fundamental polygon Γ with the sides S_j's. We state this result as a theorem now.

Theorem 5.2. *Suppose that X has finitely many singularities. Then, there exist a fundamental polygon Γ for M such that each side S of Γ is a C^r embedded circle in M that is weakly transversal to X and also, it is transversal to X at the point c where c is the common point of all sides of Γ.*

Definition 5.7 (Weakly Transversal Fundamental Polygon)**.** *Let X be a C^r vector field on M that has finitely many singularities. A fundamental polygon Γ for M which satisfies the properties in Theorem 5.2 is called* a weakly transversal fundamental polygon for M and X.

5.3. Positioning Saddles

At the very end of our program about the exposition of X with finitely many closed arcs and transversal sections to X, we will be in need of Tubular Flow Extension. So, we need to assume more than that X has finitely many singularities.

Let X be a C^r vector field on M the singularities of which are all hyperbolic. We will assume that X has saddles as the ongoing construction will trivially follow otherwise. Let $\sigma_1, \ldots, \sigma_m$ denote all the saddles of X. Let Γ be a weakly transversal fundamental polygon for M and X. Let U_c be the (natural) simply connected open neighborhood of c as before with the following additional property: U_c does not contain the singularities of X. For each positive integer k with $1 \leq k \leq m$, let V_k be a small Grobman-Hartman neighborhood of σ_k in the polygon disc D_Γ of Γ such that: each V_k is disjoint from U_c ; all V_k's are pairwise disjoint and; the stable set $W^s_{V_k}(\sigma)$ and the unstable set $W^u_{V_k}(\sigma)$ are both C^r embedded submanifolds.

By considering a local chart (U_ϕ^k, ϕ) at σ_k with $U_\phi^k \subseteq V_k$, we can obtain a C^r embedded open interval L_k in U_ϕ^k, ϕ such that: σ_k is in L_k; $L_k - \sigma_k$ is disjoint from the separatrices of σ_k; $L_k - \sigma_k$ is transversal to X and; $\bar{L}_k \cap \partial V_k$ consists of two points where ∂V_k is the topological boundary of V_k in M. We caution the reader that one cannot use the Grobman-Hartman Theorem to obtain such L_k because otherwise L_k would be a topologically embedded open interval and not necessarily a C^r embedded open interval. Define such C^r embedded open intervals L_k's for each positive integer k with $1 \leq k \leq m$. See Figure 5.13.

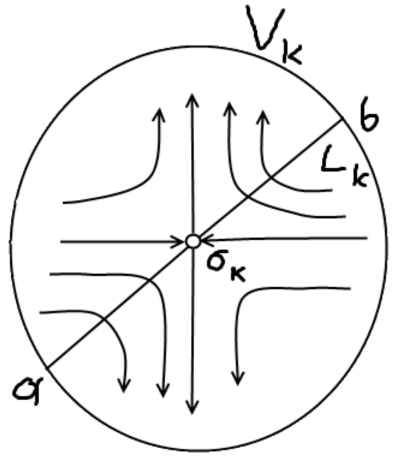

Figure 5.13. A local picture for L_k. Here, we have marked the two points in $\bar{L}_k \cap \partial V_k$ with a and b.

Let V_c be a simply connected open neighborhood of c in M such that we have $V_c \subseteq U_c$ and $V_c \cap \tilde{S}_1$ is transversal to X. Let $\tilde{S}_1$ be a C^r embedded circle in M such that: we have $\tilde{S}_1 \cap V_c = S_1 \cap V_c$; $\tilde{S}_1 - \bar{V}_c$ does not intersect any side of Γ; $\tilde{S}_1$ contains all L_k's ($1 \leq k \leq m$) and; $\tilde{S}_1$ does not contain any sinks or sources of X. Let $l_1, \ldots, l_{m+1}$ be the connected components of $\tilde{S}_1 - (V_c \cup V_1 \cup \cdots \cup V_m)$. Then, only two of the l_k's intersect S_1, say l_1 and l_{m+1} and the remaining l_k's do not intersect any side of Γ. As any two distinct l_{j_1} and l_{j_2} are disjoint and compact $(1 \leq j_1 < j_2 \leq m+1)$, we can find a simply connected open neighborhood U_k^s of l_k for each positive integer k with $1 \leq k \leq m+1$ such that all U_k^s's are pairwise disjoint and also, U_k^s does not intersect any side of Γ for $1 < k < m+1$.

We apply Lemma 5.2 to each l_k ($1 \leq k \leq m+1$) to obtain a C^r embedded closed interval τ_k such that: we have $\tau_k \subset U_k^s$; the C^r embedded intervals τ_k and l_k have the same boundary points; $\tau_k \cap l_k$ is a finite union of isolated points in M or C^r embedded intervals in M and; τ_k is weakly transversal to X. We can apply Lemma 5.1 in small flow boxes at the boundary points of τ_k to obtain a perturbation $\tilde{\tau}_k$ of τ_k and also a perturbation Q_1 of $\tilde{S}_1 \cap V_c$ to define a C^r embedded circle $\hat{S}_1 := Q_1 \cup \tilde{\tau}_1 \cup \cdots \cup \tilde{\tau}_{m+1}$ in M such that we have $\hat{S}_1 \cap N_c = S_1 \cap N_c$ for a small simply connected open neighborhood of c with $N_c \subseteq V_c$ and $\hat{S}_1 - \{\sigma_1, \ldots, \sigma_m\}$ is weakly transversal to X.

By using the fundamental polygon Γ for M, we can now define a new fundamental polygon $\tilde{G}_r$ for M such that $\hat{S}_1$ is a side of $\tilde{G}_r$ and also, both $\tilde{G}_r$ and Γ have the same sides except the sides $S_1 \subseteq \Gamma$ and $\tilde{S}_1 \subseteq \tilde{G}_r$. We state this result now as a lemma.

Lemma 5.3. *Suppose that the singularities of X are all hyperbolic. Let $A_\sigma = \{\sigma_1, \ldots, \sigma_m\}$ be the set of saddles of X where we might possibly have $A_\sigma = \varnothing$. Then, there exist a fundamental polygon Γ for M such that: each side S of Γ is a C^r embedded circle in M; each side S of Γ is transversal to X at the point c where c is the common point of all sides of Γ; the set A_σ is a subset of one of the sides of Γ and; for each side S of Γ, the C^r embedded submanifold $S - A_\sigma$ is weakly transversal to X.*

Definition 5.8. *Let X be in $\mathfrak{X}^r(M)$ the singularities of which are all hyperbolic and let Γ be a fundamental polygon for M that satisfies the properties of Lemma 5.3. The set $B_\Gamma = \{p_1, \ldots, p_{n_f}\}$ will denote the set of all points in all sides of Γ such that if the point $p_k \in B_\Gamma$ is in the side S_j of Γ, then S_j has a parabolic transversality failure at p_k.*

5.4. Focus on the Polygon Disc D_Γ

We continue to assume that all singularities of X are hyperbolic. Let Γ be a fundamental polygon for M that satisfies the properties of Lemma 5.3.

Suppose that the orbit γ through a point of the polygon disc D_Γ of Γ does not intersect any side of Γ. Although all the sides of Γ are weakly transversal to X when we remove any possible saddles from them, we cannot benefit from their useful structures to study the behavior of γ in general. This situation will be ameliorated by the integration of the Jordan Curve Theorem as D_Γ can be embedded into S^2. In regard to the possible existence of orbits of X that behave like γ, we now question how useful the weakly transversal sides of Γ can be to express the behavior of orbits of X.

Let $p_j \in B_\Gamma$. We have two situations to consider: either p_j^+ intersects a side of Γ or it does not. Assume X has a saddle σ and let V be a Grobman-Hartman neighborhood of σ such that V intersects a single side of Γ. Let q be a point in an unstable separatrix of σ in V such that we have $q^- \subseteq V$. Again we face two situations: either q^+ intersects a side of Γ or it does not. We have analogous situations for p_j^- and also for the semi trajectories of points in other separatrices of σ. Beforehand, it is not obvious that the study of these situations for p_j^+ or p_j^- will be analogous for the semi trajectories of separatrices of σ because for any given neighborhood U_1 of p_j we can find a smaller neighborhood U_2 of p_j in U_1 such that both p_j^+ and p_j^- eventually leave U_2 (it may come back to U_2 later). On the other hand, q^- does not leave the Grobman-Hartman neighborhood V. It will turn out, however, that the situations for p_j^+ and q^+ do not pose any difference when we try to exhibit X on Γ with finitely many closed arcs of X and finitely many transversal sections to X. Their alike situations have been encapsulated by Lemma 5.5.

For convenience, we state Lemma 5.4 and Lemma 5.5 for positive semi trajectories but they can analogously be stated and proven for the negative semi trajectories as well. Similarly, Lemma 3.10 can be analogously stated in the following way: when $\alpha(p)$ is a saddle graph, then $\omega(p)$ is either a sink or a closed orbit.

Lemma 5.4. *Suppose that: the singularities of X are all hyperbolic; Γ is a fundamental polygon for M which satisfies the properties of Lemma 5.3; p is a regular point of X in the polygon disc D_Γ of Γ such that p^+ does not intersect any side of Γ and; $\omega(p)$ contains a regular point. Then, $\omega(p)$ is either a saddle graph for X or a closed orbit of X.*

Proof. Case 1. $\omega(p)$ contains a saddle σ. We will show that $\omega(p)$ is a saddle graph in this case.

As $\omega(p)$ contains a regular point, we have $\omega(p) \neq \sigma$ so that $\omega(p)$ contains an unstable separatrix γ of σ. We claim that $\omega(\gamma)$ does not contain a regular point. Assume otherwise and let q be a regular point of $\omega(\gamma)$. Let A_σ be the set of all saddles of X; let $\mathcal{S}$ be the union of all sides of Γ and; let $B = \mathcal{S} - (A_\sigma \cup B_\Gamma)$. As B is transversal to X and the sides of Γ have a parabolic transversality failure at points in B_Γ, we may assume $q \in D_\Gamma$.

Let U_σ be a Grobman-Hartman neighborhood of σ such that U_σ intersects a single side of Γ. Let $w \in \gamma \cap U_\sigma$ be such that we have $w^- \subseteq U_\sigma$. As B is transversal to X and $w \in \omega(p)$, we have $w^+ \cap B = \varnothing$ because otherwise we would have $p^+ \cap B \neq \varnothing$ which contradicts the hypothesis. So, w^+ may intersect a side of Γ only at a point in B_Γ.

Let Σ be a transversal section through q in D_Γ. Let w_1 and w_2 be the first and second intersection of w^+ with Σ respectively. As $\alpha(w)$ is a saddle, we have $w_1 \neq w_2$ because otherwise the orbit through w would have been a closed orbit. Let I_w be the closed interval between w_1 and w_2 in Σ. As D_Γ can be embedded into S^2, we can use the Jordan Curve Theorem for the simple closed continuous curve $w_1w_2 \cup I_w$ even when we have $w^+ \cap B_\Gamma \neq \varnothing$ because B_Γ is a finite set the points of which are at the boundary of D_Γ. So, $w_1w_2 \cup I_w$ bounds an open disc A in D_Γ. As Σ is transversal to X, we have either $w_2^+ \subseteq A$ or $w_1^- \subseteq A$. As w^+ accumulates to q, we conclude $q \notin I_w$ and $w_1^- \subseteq A$ which is a contradiction because w_1^- goes to the saddle σ but A does not contain any saddles. So, our claim has been proven. See Figure 5.14.

So, $\omega(\gamma)$ contains singularities only and as the hyperbolic singularities are isolated and $\omega(\gamma)$ is connected, $\omega(\gamma)$ is a single hyperbolic singularity. It cannot be a sink because then, $\omega(p)$ would have reduced to this sink and it would not contain a regular point. Hence, $\omega(p)$ is a saddle and γ is a saddle connection.

Note that the saddle $\sigma \in \omega(p)$ and the unstable separatrix $\gamma \subseteq \omega(p)$ were arbitrary and we can apply the above argument to any saddles in $\omega(p)$ and any unstable separatrices in $\omega(p)$. Let $\mathcal{G}$ be the union of all saddles and all unstable separatrices in $\omega(p)$. As each unstable separatrix in $\omega(p)$ is a saddle connection, $\mathcal{G}$ is a saddle graph.

We complete the proof for the saddle graph case by showing the equality $\omega(p) = \mathcal{G}$. Assume otherwise and let $z \in \omega(p) - \mathcal{G}$. Then, z is not a sink because otherwise, $\omega(p)$ would have reduced to this sink and it will not contain $\mathcal{G}$. As $\mathcal{G}$ contains all the saddles in $\omega(p)$, z is a regular point. As we have $z \in \omega(p) - \mathcal{G}$, $\omega(z)$ is not a saddle. Hence, $\omega(z)$ contains a regular point w_z. As we have assumed $q \in D_\Gamma$ before, we can likewise assume $w_z \in D_\Gamma$.

We claim that the orbit through z is not a closed orbit. Assume otherwise and let τ_z denote the closed orbit through z. Then, we have $p \notin \tau_z$ because otherwise, the orbit through p would have been τ_z which does not accumulate to $\mathcal{G}$. As we have $\omega(p) \supseteq \tau_z$ and $p \notin \tau_z$, we conclude $\omega(p) = \tau_z$ by Lemma 3.7 which contradicts the fact $\mathcal{G} \subseteq \omega(p)$ and our claim has been proven.

Let Σ_z be a transversal section through w_z in D_Γ. Let z_1 and z_2 be the first and second intersections of z^+ with Σ_z respectively. As the orbit through z is not a closed orbit, we have $z_1 \neq z_2$. Let I_z be the closed interval in Σ_z between z_1 and z_2. As B is transversal to X, z^+ cannot intersect B because otherwise p^+ would have intersected B which contradicts our hypothesis. So, we have $z^+ \subseteq D_\Gamma \cup B_\Gamma$ so that we have $z_1 z_2 \subseteq D_\Gamma \cup B_\Gamma$. As B_Γ consists of finitely many boundary points of the open disc D_Γ, we can use the Jordan Curve Theorem to conclude that the simple closed continuous curve $z_1 z_2 \cup I_z$ bounds an open disc A_z in D_Γ. As Σ_z is transversal to X, we have either ${z_2}^+ \subseteq A_z$ or ${z_1}^- \subseteq A_z$. Either way, we reach a contradiction because p^+ can intersect I_z at most one time and it cannot accumulate to both z_1 and z_2. Therefore, the point z does not exist and we have $\omega(p) = \mathcal{G}$.

Case 2. $\omega(p)$ does not have any saddles. We will show that $\omega(p)$ is a closed orbit in this case.

Let q be a regular point in $\omega(p)$. Then, $\omega(q)$ is not a saddle because otherwise,

$\omega(p)$ would have contained this saddle. Also, $\omega(q)$ cannot be a sink because otherwise, $\omega(p)$ would have reduced to this sink and it will not contain q. As $\omega(q)$ does not contain saddles or a sink, it contains a regular point w. As we have explained in Case 1, we may assume $w \in D_\Gamma$.

Let B be the set which is defined in Case 1. As B is transversal to X, q^+ does not intersect B because otherwise, p^+ would have intersected B which contradicts the hypothesis. So, q^+ can intersect a side of Γ only at a point in B_Γ.

Let Σ be a transversal section through w in D_Γ. Let q_1 and q_2 be the first and second intersections of q^+ with Σ respectively. Assume that we have $q_1 \neq q_2$ and let I_q be the closed interval between q_1 and q_2 in Σ. As we have $q^+ \subseteq D_\Gamma \cup B_\Gamma$, we also have $q_1q_2 \subseteq D_\Gamma \cup B_\Gamma$. As D_Γ is an open disc and B_Γ consists of finitely many boundary points of D_Γ, we can use the Jordan Curve Theorem to conclude that the simple closed curve $I_q \cup q_1q_2$ bounds an open disc A_q in D_Γ. As Σ is transversal to X, we have either ${q_2}^+ \subseteq A_q$ or ${q_1}^- \subseteq A_q$. In either case, p^+ can intersect I_q at most one time and it cannot accumulate to both q_1 and q_2 which contradicts $q \in \omega(p)$. Therefore, we have $q_1 = q_2 = w$ and the orbit through q is closed orbit τ_q.

So, we have $\tau_q \subseteq \omega(p)$. If we have $p \in \tau_q$, then we have $\omega(p) = \tau_q$. If we have $p \notin \tau_q$, then we have $\omega(p) = \tau_q$ by Lemma 3.7. Either way, the proof is complete. □

Definition 5.9. *Suppose that all singularities of X are all hyperbolic; Γ is a fundamental polygon for M that satisfies the properties of Lemma 5.3 and; σ is a saddle in S_1. A Grobman-Hartman neighborhood U of σ is called* a fair Grobman-Hartman neighborhood of σ *if U does not intersect any side of Γ other than S_1 and also the set $U \cap S - \{\sigma\}$ is transversal to X.*

Lemma 5.5. *Suppose that the singularities of X are all hyperbolic; z is either a point of an unstable separatrix of a saddle in a fair Grobman-Hartman neighborhood U_σ of σ such that we have $z^- \subseteq U_\sigma$ or z is a point in $B_\Gamma \cup \{c\}$; $z^+ - \{z\}$ does not intersect any side of Γ. If $\omega(z)$ has a regular point w, then the orbit through w is a closed orbit.*

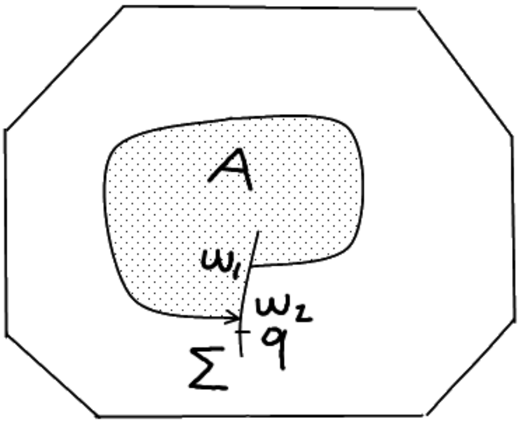

Figure 5.14. Recall that the closed arc w_1w_2 may touch a side of the fundamental polygon which is not so here.

Proof. By Lemma 5.4, $\omega(w)$ is either a saddle or the closed orbit through ω. Assume that it is a saddle σ_0. Let U_0 be a Grobman-Hartman neighborhood of σ_0 and let $w_1 \in U_0 \cap w^+$ such that we have $w_1^+ \subseteq U_0$. Let Σ be a transversal section through w_1 in the polygon disc D_Γ of Γ. Let z_1 and z_2 be the first and second intersection of z^+ with Σ. Because ${w_1}^+$ does not accumulate to w_1, we have $w_1 \notin z^+ \cap \Sigma$ and in particular, we have $w_1 \neq z_1$ or z_2. We also have $z_1 \neq z_2$ because otherwise, the orbit through z would have been a closed orbit which cannot accumulate to a stable separatrix of a saddle. Let I_z be the closed interval between z_1 and z_2 in Σ. As D_Γ can be embedded into S^2, $I_z \cup z_1z_2$ is a simple closed continuous curve which bounds an open disc A in D_Γ by the Jordan Curve Theorem. As Σ is transversal to X, we have either $z_2^+ \subseteq A$ or $z_2^- \subseteq A$. As z^+ accumulates to w_1, we conclude that w_1 is not in I_z and also $z_2^- \subseteq A$. See Figure 5.15.

Let S_r be the union of all sides of Γ. As we have $\bar{A} \cap S_r = \varnothing$ and $z_2^- \subseteq A$, we conclude that the intersection of the closure of z_2^- with S_r is the empty set which contradicts that z_2^- is either a saddle or a point in $B_\Gamma \cup \{c\}$ either of which lies in some side of Γ. Therefore, the orbit through w is a closed orbit. □

Lemma 5.6. *Suppose that: the singularities of X are all hyperbolic; Γ is a fundamental polygon for M which satisfies the properties of Lemma 5.3; p is a regular point of X in the polygon disc D_Γ of Γ such that the orbit γ through p does not intersect any side*

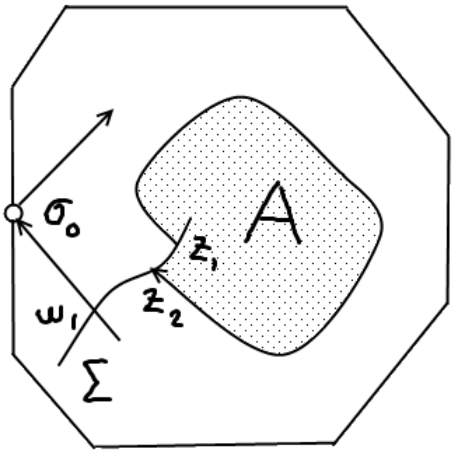

Figure 5.15. The contradiction in the proof of Lemma 5.5.

of Γ *and;* $\omega(p)$ *is a saddle graph. Then,* $\alpha(\gamma)$ *is a either a source or a closed orbit.*

Proof. Assume that $\alpha(\gamma)$ is not a source. We will show that $\alpha(\gamma)$ is a closed orbit.

Assume that $\alpha(\gamma)$ is a saddle σ. Let U_σ be a fair Grobman-Hartman neighborhood of σ and let $w \in U_\sigma \cap \gamma$ be such that we have $w^- \subseteq U_\sigma$. By Lemma 5.5, $\omega(w)$ is a closed orbit and not a saddle graph which contradicts our hypothesis. Therefore, $\alpha(\gamma)$ is not a saddle.

As $\alpha(\gamma)$ is neither a source nor a saddle, $\alpha(\gamma)$ contains a regular point q. By Lemma 5.4, $\alpha(\gamma)$ is either a saddle graph or a closed orbit.

Assume that $\alpha(\gamma)$ is a saddle graph $\mathcal{G}$. Let η be a saddle in $\mathcal{G}$ and let β be an unstable separatrix of η such that we have $\beta \subseteq \mathcal{G}$. Let U_η be a fair Grobman-Hartman neighborhood of η and let $z \in \beta \cap U_\eta$ such that we have $z^- \subseteq U_\eta$. As U_η is a fair Grobman-Hartman neighborhood of η, we have $z^- \subseteq D_\Gamma$. Let Σ be a transversal section through z in D_Γ. Let the points p_1 and p_2 denote the first and second intersections of p^- with Σ respectively. As $\omega(p)$ is a saddle graph, the orbit through p is not a closed orbit. Therefore, we have $p_1 \neq p_2$. Let I_p be the closed interval in Σ between p_1 and p_2. As γ does not intersect any side of Γ, we have $p_2p_1 \subseteq D_\Gamma$. By the Jordan Curve Theorem, the simple closed continuous curve $p_2p_1 \cup I_p$ bounds an open disc A. As Σ is transversal to X, we have either ${p_2}^- \subseteq A$ or ${p_1}^+ \subseteq A$. As p^- accumulates to z, we conclude $z \notin I_p$ and ${p_1}^+ \subseteq A$. As the saddles of X are at the sides of Γ, the set A does not contain any saddles. So, our conclusion ${p_1}^+ \subseteq A$ was a contradiction since $\omega(p_1)$ is a saddle graph. Therefore, $\alpha(\gamma)$ does not contain a saddle η and an unstable separatrix β of η. So, $\alpha(\gamma)$ is a closed orbit. See Figure 5.16. $\square$

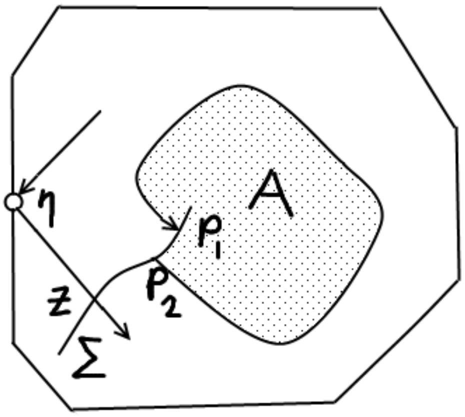

Figure 5.16. The contradiction in the proof of Lemma 5.6.

Remark 5.2. We summarize the facts in Lemma 5.4 and Lemma 5.6. Suppose that: the singularities of X are all hyperbolic; Γ is a fundamental polygon for M which satisfies the properties of Lemma 5.3 and; p is a regular point of X in the polygon disc D_Γ of Γ such that p^+ does not intersect any side of Γ. Then, $\omega(p)$ is only one of the following list: a sink, a saddle, a saddle graph or a closed orbit. Similarly, if p^- does not intersect any side of Γ, then $\alpha(p)$ is only one of the following list: a source, a saddle, a saddle graph or a closed orbit. Suppose now that both p^+ and p^- do not intersect any side of Γ. If one of the limit sets $\omega(p)$ and $\alpha(p)$ is a saddle graph, then the other limit set is neither a saddle graph nor a saddle.

We continue our focus on D_Γ. Lemma 5.8 and Lemma 5.9 will state some of the possible situations there.

Lemma 5.7. *Suppose that τ is a closed orbit of X and there exists some point p in $M - \tau$ such that $\omega(p) = \tau$ ($\alpha(p) = \tau$). Then, there exists a C^r embedded circle C_τ in M such that C_τ is transversal to X and for ever point q in C_τ, we have $\omega(p) = \tau$ ($\alpha(p) = \tau$). Moreover, C_τ and τ bound an open cylinder in M that does not contain any singularities or closed orbits of X.*

Proof. We will consider the case $\omega(p) = \tau$ as the other case is analogous. Let U be a circle neighborhood of τ which is diffeomorphic to either an open cylinder or Möbius band and does not contain the point p.

Case 1. U is an open cylinder. Let q be a point of τ and Σ be a transversal section through q in U. Let Σ^a and Σ^b be the connected components of $\Sigma - \{q\}$. Say, q is a limit point of $p^+ \cap \Sigma^a$. By Lemma 3.7, the Poincaré Return Map $P : \Sigma \to \Sigma$ is defined on the whole Σ^a such that we have $P(\Sigma^a) \subsetneq \Sigma^a$ and $P(x) \neq x$ for $x \in \Sigma^a$ if Σ is small enough. Let p_1 and p_2 be first and second intersections of p^+ with Σ^a respectively. Let B be the connected closed interval in Σ^a with boundary points p_1 and p_2. Let $A = \{z \in M : z \in wP(w), w \in B\}$. Then, $A - (\Sigma^a \cup p_1p_2)$ is diffeomorphic to an open disc since it is a subset of union of tubular flow neighborhoods of closed arcs in A. Let $f : [0,1] \to B$ be a C^r diffeomorphism with $f(0) = p_2$ and $f(1) = p_1$. Let $\phi(t,q)$ be the flow of X. Let $g : \Sigma^a \to \mathbb{R}^+$ be the unique function which satisfies the equality $\phi(g(w),w) = P(w)$. Then, g is C^r because Σ is a C^r submanifold of M that is transversal to X for $w \in \Sigma^a$ so that $\phi(g(w),w)$ is a C^r parametrization of $P(\Sigma^a)$. Define $h : [0,1] \to \mathbb{R}$ as $h(t) = t \cdot g \circ f(t)$ for $t \in [0,1]$. So, h is C^r and we have $h(0) = 0$, $h(1) = g(p_1)$ and $h(t) < g \circ f(t)$ for $0 \leq t < 1$. Define $\psi : [0,1] \to M$ as $\psi(t) = \phi(h(t), f(t))$. Then, we have $\psi(0) = \psi(1) = p_2$. Let $C_1 = \{\psi(t) : t \in [0,1]\}$. See Figure 5.17.

We claim that $C_1 - \{p_2\}$ is a C^r immersed submanifold of M of dimension 1. The linear map df_t is not trivial ($\neq 0$) for any t in $[0,1]$. We have $d\psi_t(1) = d\phi_{\psi(t)} \circ (dh_t(1), df_t(1)) = d\phi_{\psi(t)} \circ (dh_t(1), 0) + d\phi_{\psi(t)} \circ (0, df_t(1))$. Because Σ is transversal to X, the two vectors (terms) in the summand are linearly independent if they are both nonzero. As the latter one is not zero, we conclude $d\psi_t(1) \neq 0$ and our claim has been proven because we have $\psi(t) = p_2$ if and only if we have $t = 0$ or 1. This calculation also shows that $C_1 - \{p_2\}$ is transversal to X. We claim now that $C_1 - \{p_2\}$ is an embedded C^r submanifold of M. For any z in $(A - \Sigma^a)$, there exist a unique b_z in B and a unique $t_z > 0$ such that z is equal to $\phi(t_z, b_z)$. By our choice of $h(t)$, C_1 is a subset of A. Therefore, C_1 does not have self intersections. Let z_0 be in $C_1 - \{p_2\}$. As there are no self intersections, there exists a simply connected neighborhood U_0 of z_0

in M such that $U_z \cap C_1$ is homeomorphic to an interval. Hence, C_1 is a topologically embedded circle and $C_1 - \{p_2\}$ is a C^r embedded submanifold as we have claimed.

Let (f_1, F_1) be a flow box at p_2. We will apply Lemma 5.1 in F_1 to make C_1 also C^r at p_2. Because $C_1 - \{p_2\}$ is transversal to X, $f_1((C_1 - \{p_2\}) \cap F_1)$ can be parametrized as the graph of a C^r function $x = x(y)$. So, multiplying $x(y)$ with the smooth $h(y)$ which is provided by Lemma 5.1 makes the graph also C^r at p_2 by perturbing it in a small neighborhood of $f(p_2)$. As its derivative is defined, the tangent line to the perturbed graph at any point of it is not parallel to the x-axis. Let G denote the graph of $x(y)h(y)$ for $-1 \leq y \leq 1$. Let $C_\tau = ((F_1)^c \cap C_1) \cup f_1^{-1}(G)$. So, C_τ is a C^r embedded circle that is transversal to X. As the negative semi trajectories of points of C_τ intersect Σ^a, we conclude that for every point q of C_τ, we have $\omega(q) = \tau$.

We complete the proof for the cylindrical U case by showing the last assertion in the statement of the theorem. Let B_1 be the connected half open interval in Σ such that it includes its boundary point p_1 and not its other boundary point q. Let $A_1 = \{z \in M : z \in wP(w), w \in B_1\}$. Then, $A_1 - (\Sigma^a \cup p_1p_2)$ is diffeomorphic to an open disc because it is a subset of a union of tubular flow neighborhoods of closed arcs in A_1. As $P(B_1)$ is a subset of B_1 and we have $P(q) = q$, we conclude that $A_1 - p_1p_2$ is homeomorphic to an open cylinder with boundaries τ and p_1p_2. Also, it does not contain any singularities of X. As C_τ is a subset of it, either C_τ bounds an open disc there or it separates $A_1 - p_1p_2$ into two open cylinders. If it bounds an open disc D_τ, then it must contain a singularity in D_τ by Lemma 3.13 but A_1 does not contain any singularities. So, C_τ separates $A_1 - p_1p_2$ into two open cylinders and C_τ and τ are the boundaries of an open cylinder V in $A_1 \subseteq U$ that does not contain any singularities. As $P(x) \neq x$ for x in B_1 and $P(B_1) \subsetneq B_1$, the set V does not contain any closed orbits.

Case 2. U is an open Möbius band. This case is very similar to the cylindrical U case and we mimic it. Let q be a point of τ and Σ be a transversal section through q in U. Let Σ^a and Σ^b be the connected components of $\Sigma - \{q\}$. By Lemma 3.7, the Poincaré Return Map $P : \Sigma \to \Sigma$ is defined on the whole Σ such that we have $P^2(\Sigma^a) \subsetneq \Sigma^a$ and $P^2(x) \neq x$ for $x \in \Sigma^a$ if Σ is small enough. Note that we have

$P(\Sigma^a) \subseteq \Sigma^b$ and $P(\Sigma^b) \subseteq \Sigma^a$. Let p_1 and p_2 be first and second intersections of p^+ with Σ^a respectively. Note that p_2 is the second intersection of p_1^+ with Σ because U is homeomorphic to an Möbius band. Let B be the connected closed interval in Σ^a with boundary points p_1 and p_2. Let $f : [0,1] \to B$ be a C^r diffeomorphism with $f(0) = p_2$ and $f(1) = p_1$. Let $\phi(t,q)$ be the flow of X. Let $g : \Sigma^a \to \mathbb{R}^+$ be the unique C^r function which satisfies the equality $\phi(g(w), w) = P^2(w)$. Define $h : [0,1] \to \mathbb{R}$ as $h(t) = t \cdot g \circ f(t)$ for $t \in [0,1]$. Define $\psi : [0,1] \to M$ as $\psi(t) = \phi(h(t), f(t))$. Let $C_1 = \{\psi(t) : t \in [0,1]\}$. Then, $C_1 - \{p_2\}$ is a C^r embedded submanifold and C_1 is a topologically embedded circle as it has been proven in Case 1. We can apply a flow box perturbation at p_2 as in the Case 1 to obtain a C^r embedded circle C_τ that is transversal to X and also for every q in C_τ, we have $\omega(q) = \tau$. So, it remains to show the last assertion in the statement of the theorem. We once again mimic Case 1.

Let B_1 be the connected half open interval in Σ such that it includes its boundary point p_1 and not its other boundary point q. Let $A_1 = \{z \in M : z \in wP^2(w), w \in B_1\}$. Then, $A_1 - (\Sigma^a \cup p_1p_2)$ is diffeomorphic to an open disc because it is a subset of a union of tubular flow neighborhoods of closed arcs in A_1. As $P^2(B_1)$ is a subset of B_1 and we have $P^2(q) = P(q) = q$, we conclude that $A_1 - p_1p_2$ is homeomorphic to an open cylinder. Note that $(A_1 - p_1p_2) \cup \tau$ is homeomorphic to an open Möbius band the 0-section of which is τ. Also, $A_1 - p_1p_2$ does not contain any singularities of X. As C_τ is a subset of it, either C_τ bounds an open disc there or it separates $A_1 - p_1p_2$ into two open cylinders. If it bounds an open disc D_τ, then it must contain a singularity in D_τ by Lemma 3.13 but A_1 does not contain any singularities. So, C_τ separates $A_1 - p_1p_2$ into two open cylinders. As τ is the 0-section of the Möbius band $(A_1 - p_1p_2) \cup \tau$, C_τ and τ are the boundaries of an open cylinder V in A_1 that does not contain any singularities. As $P^2(x) \neq x$ for $x \in B_1$ and $P^2(B_1) \subsetneq B_1$, the set V does not contain any closed orbits. $\square$

Lemma 5.8. *Suppose that the singularities of X are all hyperbolic; τ is a closed orbit of X that bounds an open disc D_τ in M and; D_τ does not contain any closed orbit or any saddle of X. Then, D_τ contains a unique hyperbolic singularity η and for every point p in $D_\tau - \{\eta\}$, we have either $\omega(p) = \tau$ and $\alpha(p) = \eta$ or $\omega(p) = \eta$ and $\alpha(p) = \tau$.*

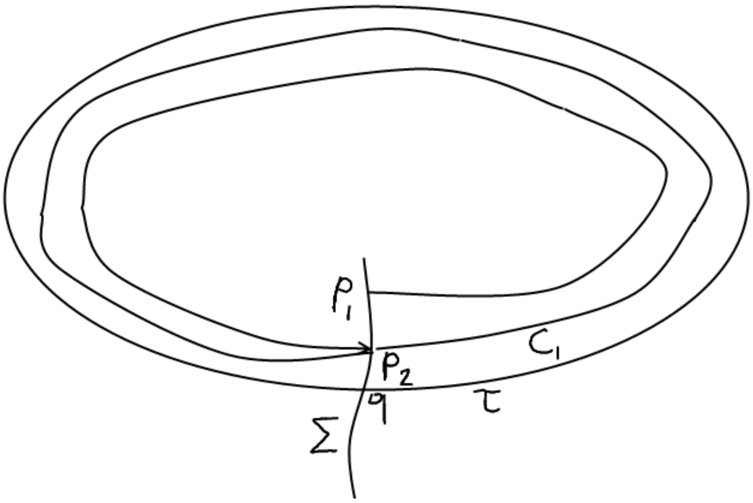

Figure 5.17. The cylindrical U case in the proof of Lemma 5.7.

Proof. Let $p \in D_\tau$. Note that $D_\tau \cup \tau$ is a closed set that is invariant by the flow of X and also, it can be embedded into S^2 so that we may use the Poincaré-Bendixson Theorem on D_τ.

We claim that we cannot have both $\omega(p) = \tau$ and $\alpha(p) = \tau$. A semi trajectory of p which accumulates to τ intersects a transversal section Σ through a point c of τ monotonically (in the sense that the subsequent intersections get closer to c in Σ) by Lemma 3.7 if Σ is small enough. Say, p_1 and p_2 are two subsequent intersections either in $p^- \cap \Sigma$ or $p^+ \cap \Sigma$. Let γ be the closed arc of the orbit of p with boundaries p_1 and p_2 and I be the connected interval between p_1 and p_2. Then, $I \cup \gamma$ is a simple closed curve that bounds an open disc D_p by the Jordan Curve Theorem. Hence, the other semi trajectory of p is confined to the region D_p so that it cannot accumulate to τ and our claim is proven.

We will consider the case $\omega(p) = \tau$ as the other case is analogous. By the Poincaré-Bendixson Theorem, $\alpha(p)$ is a source η because there are no closed orbits in D_τ and there are no saddles in D_τ so that $\alpha(p)$ cannot contain both a singularity of X and a regular point of X. Let U_η be a simply connected closed neighborhood of η in D_τ such that: we have $W^u_{U_\eta} = U_\eta$; $W^u_{U_\eta}(\eta)$ is a C^r embedded submanifold and; the boundary C_η of U_η is transversal to X. Let q be in C_η.

If $\omega(q)$ is not τ, then it must be a sink ζ by the Poincaré-Bendixson Theorem as there are no saddles or closed orbits in D_τ. Let U_ζ be a simply connected closed neighborhood of ζ in D_τ such that: we have $W^s_{U_\zeta} = U_\zeta$; $W^s_{U_\zeta}(\zeta)$ is a C^r embedded submanifold and; the boundary C_ζ of U_ζ is transversal to X. If both U_η and U_ζ are small enough, then we have $U_\eta \cap U_\zeta = \varnothing$ so that C_η and C_ζ are disjoint and none of them encircle the other one. Because of $W^s_{U_\zeta} = U_\zeta$, $q^+ \cap C_\zeta$ consists of a single point denoted by q_1. Because of $W^u_{U_\eta} = U_\eta$, we have $q_1^- \cap U_\eta = q$. Therefore, the First Proper Intersection Assignment $P : C_\eta \to C_\zeta$ is defined at q with $P(q) = q_1$ because the disjoint C^r embedded circles C_η and C_ζ do not have any boundary points. Hence, the domain of P is nonempty. By the Tubular Flow Extension, the First Proper Intersection Assignment $P : C_\eta \to C_\zeta$ is defined on the whole C_η because there are no saddles in D_τ and again, C_η and C_ζ are disjoint C^r embedded circles that do not have any boundary points. P is a local diffeomorphism and we can consider the First Proper Intersection Assignment $P^- : C_\zeta \to C_\eta$ for the flow of $-X$ and conclude that $P(C_\eta) = C_\zeta$. Therefore, the set $\{z \in D_\tau : z \in wP(w), w \in C_\eta\}$ is diffeomorphic to a closed cylinder which is a contradiction because neither of the circles C_η and C_ζ encircle the other one and there doesn't exist a closed cylinder (annulus) in D_τ with these boundaries. Therefore, the sink ζ does not exist and $\omega(q) = \tau$ because there are no closed orbits in D_τ. See Figure 5.18.

Let C_τ be an embedded C^r interval such that C_τ is transversal to X and for every point z in C_τ, we have $\omega(z) = C_\tau$ by Lemma 5.7. We may assume that we have $C_\tau \cap C_\eta = \varnothing$ and also if C_τ is near enough to τ, then C_τ encircles C_η because τ does so. In a similar way in the previous part, we conclude that the First Proper Intersection Assignment is $P : C_\eta \to C_\tau$ is defined on the whole C_η and $P(C_\eta) = C_\tau$. Therefore, the closed annulus in D_τ with boundaries C_τ and C_η does not contain any singularities. As the closed annulus in D_τ with boundaries τ and C_τ does not contain singularities by Lemma 5.7 and η is the unique singularity in U_η, we conclude that it is also the unique singularity in D_τ and the proof is complete. $\square$

Lemma 5.9. *Suppose that: the singularities of X are all hyperbolic; τ and β are distinct closed orbits that together bound an open cylinder (annulus) $A_{\tau\beta}$ in M and;*

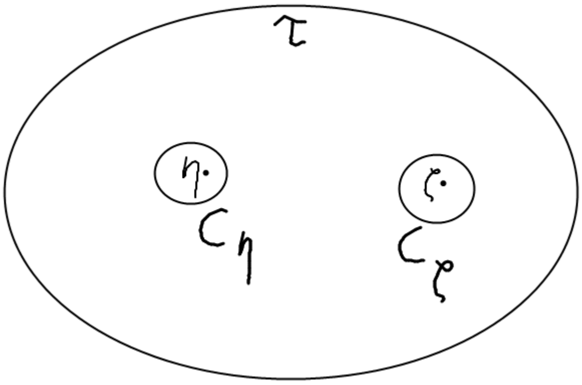

Figure 5.18. The circles C_η and C_ζ do not encircle each other which yields the contradiction.

$A_{\tau\beta}$ does not contain any closed orbit or any saddle of X. Then, $A_{\tau\beta}$ contains no singularities and for every point p in $A_{\tau\beta}$, we have that either $\omega(p) = \tau$ and $\alpha(p) = \beta$ or $\omega(p) = \beta$ and $\alpha(p) = \tau$.

Proof. The proof will be similar to the one of Lemma 5.8 and we will reach similar contradictions. Note that $A_{\tau\beta}$ is a closed set that is invariant by the flow of X and also, $A_{\tau\beta}$ can be embedded into S^2 so that we may use the Poincaré-Bendixson Theorem on $A_{\tau\beta}$.

Let $p \in A_{\tau\beta}$. By the Poincaré-Bendixson Theorem, $\omega(p)$ is either a hyperbolic singularity or a closed orbit because there are no saddles in $A_{\tau\beta}$ so that $\omega(p)$ cannot contain both a singularity and a regular point of X. Assume that $\omega(p)$ is a sink η in $A_{\tau\beta}$. Let U_η be a simply connected closed neighborhood of η in $A_{\tau\beta}$ such that: we have $W^s_{U_\eta} = U_\eta$; $W^s_{U_\eta}(\eta)$ is a C^r embedded submanifold and; the boundary C_η of U_η is transversal to X.

We can't have that $\alpha(p)$ is a source in $A_{\tau\beta}$ because there are no saddles in $A_{\tau\beta}$ and we will reach a similar contradiction to the one in the proof of Lemma 5.8. So, $\alpha(p)$ is one of the two closed closed orbits. Say, $\alpha(p) = \tau$. We apply Lemma 5.7 to find a C^r embedded circle C_τ in $A_{\tau\beta}$ such that: we have $C_\tau \cap C_\eta = \varnothing$; C_τ and τ bound together an open region A_{tc} which is homeomorphic to an open cylinder; C_τ is transversal to X

and; for every point q in C_τ, we have $\alpha(q) = \tau$. Since A_{tc} is homeomorphic to an open cylinder and C_τ is transversal to X, we conclude that the set $p^- \cap C_\tau$ has a single point which we denote by p_1. As we have $W^s_{U_\eta} = U_\eta$ and C_η is transversal to X, we conclude likewise that $p^+ \cap C_\eta$ has a single point which we denote by p_2. Therefore, the First Proper Intersection Assignment $P : C_\tau \to C_\eta$ is defined at p_1 with $P(p_1) = p_2$ as the disjoint C^r embedded circles C_η and C_τ do not have any boundary points. Hence, the domain of P is nonempty. By the Tubular Flow Extension, P is defined on the whole C_τ because there are no saddles in $A_{\tau\beta}$ and the disjoint embedded circles C_τ and C_η do not have any boundary points. Similarly, the First Proper Intersection Assignment $P^- : C_\eta \to C_\tau$ is defined on the whole C_η for the flow of $-X$. Therefore, we have $P(C_\tau) = C_\eta$ and the set $\{z \in A_{\tau\beta} : z \in wP(w), w \in C_\tau\}$ is diffeomorphic to a closed cylinder which is a contradiction because C_η bounds an open disc in $A_{\tau\beta}$ and also, C_τ separates $A_{\tau\beta}$ into two open cylinders so that there doesn't exist a closed cylinder (annulus) in $A_{\tau\beta}$ with these boundaries. Hence, the sink η does not exist and $\omega(p)$ is either τ or β. See Figure 5.19.

Similarly, $A_{\tau\beta}$ does not contain any source and $\alpha(p)$ is either τ or β. As we have explained in the proof of Lemma 5.8, we can't have both $\alpha(p)$ and $\omega(p)$ to be the same closed orbit and this completes the proof. $\square$

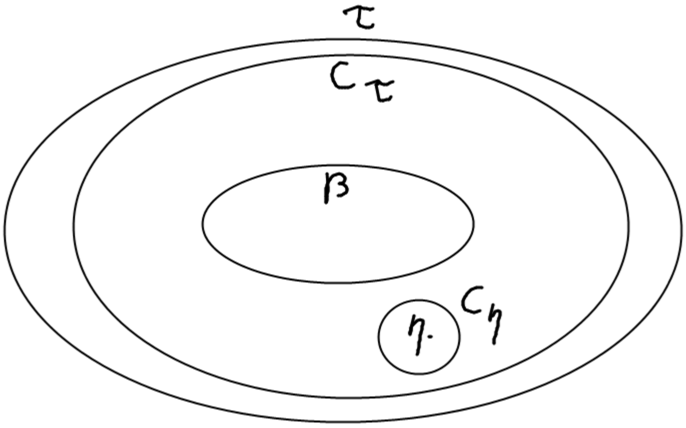

Figure 5.19. C_τ separates the open cylinder (annulus) $A_{\tau\beta}$ between τ and β into two open cylinders and C_η in $A_{\tau\beta}$ bounds an open disc.

Let p be a regular point in the polygon disc D_Γ of Γ such that at least one of the properties $p^+ \subseteq D_\Gamma$ and $p^- \subseteq D_\Gamma$ holds. Lemma 5.4, Lemma 5.6 and Remark 5.2 completely describe the possible cases for $\omega(p)$ or $\alpha(p)$. In addition to these facts, Lemma 5.5 specializes in the semi trajectories of points that belong to either $B_\Gamma \cup \{c\}$ or a separatrix of a saddle. When these semi trajectories (except their starting points) do not intersect any side of Γ, Lemma 5.5 produces stronger results than the previous general ones.

Closed orbits in the polygon disc D_Γ arise in some of the aforementioned possibilities. If X has infinitely many closed orbits in D_Γ, then Lemma 5.8 and Lemma 5.9 do not immediately apply. Nevertheless, one can elaborate them to select only a finite number of closed orbits of X in D_Γ to exhibit the semi trajectories of points in D_Γ that do not intersect any side of Γ. This elaboration can be carried out in a natural way when one uses Poincaré-Bendixson Theorem on relevant closed sets in D_Γ which are invariant by the flow of X. We will omit this discussion here because we are mainly interested in the behavior of a nontrivial recurrent orbit of X. As D_Γ can be embedded into S^2 the vector fields on which do not admit any nontrivial recurrent orbits by Lemma 3.1, a nontrivial recurrent orbit of X must intersect sides of Γ infinitely many times and it cannot be confined to an open disc in D_Γ which is bounded by a closed orbit of X. If X has finitely many closed orbits (such as a K-S field) in D_Γ, then 5.8 and Lemma 5.9 completely describe what happens in those open discs or open cylinders in D_Γ that are bounded by closed orbits of X.

5.5. Transversal Sections in the Polygon Disc D_Γ

We continue to assume that all singularities of X are hyperbolic. Let Γ be a fundamental polygon for M that satisfies the properties in Lemma 5.3. Let $S_\sigma = \{\sigma_1, \ldots, \sigma_{n_s}\}$ denote the set of all saddles of X. We might possibly have $S_\sigma = \varnothing$ but the existence of a transversal base for Γ in the Definition 5.10 can still be given with the construction that we are going to explain now. In this last discussion of our program, we finally prove Theorem 5.3.

To construct a general transversal base, we assume that S_σ is nonempty. For each positive integer j with $1 \leq j \leq n_s$, let V_j be a fair Grobman-Hartman neighborhood of σ_j such that all V_j's are pairwise disjoint. Let $\mathcal{S}_\sigma^+ = \{d_1, \ldots, d_{2n_s}\}$ be a set of points with the following property: for $1 \leq j \leq n_s$, the points d_j and d_{j+n_s} are points of distinct unstable separatrices of σ_j in V_j and; we have $d_j{}^- \subseteq V_j$ and $d_{j+n_s}{}^- \subseteq V_j$. Similarly, let $\mathcal{S}_\sigma^- = \{e_1, \ldots, e_{2n_s}\}$ be a set of points with the following property: for $1 \leq j \leq n_s$, the points e_j and e_{j+n_s} are points of distinct stable separatrices of σ_j in V_j and; we have $e_j{}^+ \subseteq V_j$ and $e_{j+n_s}{}^+ \subseteq V_j$. Let $\mathcal{E}^+ = \mathcal{S}_\sigma^+ \cup B_\Gamma \cup \{c\} = \{q_1, \ldots, q_T\}$ and $\mathcal{E}^- = \mathcal{S}_\sigma^- \cup B_\Gamma \cup \{c\} = \{w_1, \ldots, w_T\}$ where we have $T = n_f + 2n_s + 1$.

Let q_1 be in $\mathcal{E}^+$. If q_1^+ intersects a side of Γ, let $q_{1,+}$ denote its first intersection. Note that this first intersection is well defined because either the side containing $q_{1,+}$ is transversal at $q_{1,+}$ or $q_{1,+}$ is an element of the finite set B_Γ. Assume that q_1^+ does not intersect any side of Γ. We analyze all the possible situations now.

$\omega(q_1)$ might be a saddle so that there exists a point $w_j \in \mathcal{E}^-$ such that we have $w_j \in q_1{}^+$. If we have $q_1 \in \mathcal{S}_\sigma^+$, then $w_j{}^-$ does not intersect a side of Γ as well. If we have $q_1 \notin \mathcal{S}_\sigma^+$, then $w_j{}^-$ intersect a side of Γ for the first time at q_1 and we will later define $w_{j,-}$ to be q_1 in this case. In either case, we do not define $q_{1,+}$. See Figure 5.20.

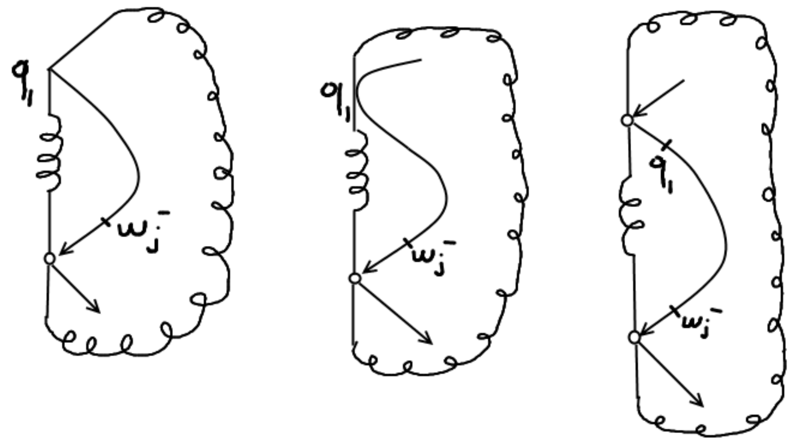

Figure 5.20. Three diagrams for the three possible cases are shown. They refer to the cases $q_1 = c$, $q_1 \in B_\Gamma$ and $q_1 \in \mathcal{S}_\sigma^+$ respectively.

Assume that $\omega(q_1)$ is not a saddle and let $\eta_1 = \omega(q_1)$. By Lemma 5.5, it has either a regular point and η_1 is a closed orbit in the polygon disc D_Γ of Γ or it is a single sink. In either case (by considering a C^r embedded local stable manifold of the sink η_1 or by applying Lemma 5.7), we can find a C^r embedded circle C_{η_1} in D_Γ such that C_{η_1} is transversal to X and for every z in C_{η_1}, we have $\omega(z) = \eta_1$. In this situation, define $q_{1,+}$ to be the unique point in $q_1^+ \cap C_{\eta_1}$.

We proceed now by induction. Assume that all q_j's have been analyzed for $1 \leq j < k \leq T$; i.e. either $q_{j,+}$ is defined or q_j^+ does not intersect a side of Γ and $\omega(q_j)$ is a saddle. We will try to define $q_{k,+}$ now in a similar manner. If q_k^+ does not intersect a side of Γ and $\omega(q_k)$ is a saddle, then we do not define $q_{k,+}$. If q_k^+ intersects a side of Γ, then let $q_{k,+}$ denote its first intersection. The last possible case is that q_k^+ does not intersect a side of Γ and $\omega(q_k) := \eta$ is a sink or a closed orbit. If η is equal to $\omega(q_j)$ for some q_j, then let m be the minimum number with $1 \leq m < k$ such that $\omega(q_m) = \eta$. In this case, we have $\eta = \eta_m$ and C_{η_m} has already been defined. Define $q_{k,+}$ to be the unique point in $q_k^+ \cap C_{\eta_m}$. If such q_m does not exist, then we define $\eta_k := \eta$ and C_{η_k} to be a C^r embedded circle in D_Γ that is transversal to X such that for every z in C_{η_k}, we have $\omega(z) = \eta_k$. We also define $q_{k,+}$ to be the unique point in $q_k^+ \cap C_{\eta_k}$.

So, we analyze the positive semi trajectories of all points in $\mathcal{E}^+$ and define $q_{j,+}$ if it is possible. We can analogously analyze (and we do so) the negative semi trajectories of all points in $\mathcal{E}^-$ and try to define $w_{j,-}$ to be the first intersection of w_j^- with either a side of Γ or some transversal section C_{ζ_n}. Here, C_{ζ_n} is a C^r embedded circle in D_Γ and for every z in C_{ζ_n}, we have $\alpha(z) = \zeta_n$ where ζ_n is either a source or a closed orbit in D_Γ.

Let $\mathcal{F}^- = \{z \in \Gamma : z = w_{j,-} \text{ if } w_{j,-} \text{ is defined for } 1 \leq j \leq T\}$. Let $\mathcal{T}_0^+$ be the union of all sides of Γ and all C_{ζ_j}'s for $1 \leq j \leq T$ whenever C_{ζ_j} is defined. Let $\mathcal{T}_1^+ = \mathcal{T}_0^+ - (B_\Gamma \cup S_\sigma \cup \{c\} \cup \mathcal{F}^-)$. Then, $\mathcal{T}_1^+$ has finitely many connected components each of which is an open transversal section to X. Say, $\mathcal{T}_1^+ = \bigcup_{1 \leq j \leq n_+} \Sigma_j^+$ for some positive integer n_+ where each Σ_j^+ is a connected component of $\mathcal{T}_1^+$. Let $\mathcal{T}^+ = \{\Sigma_1^+, \ldots, \Sigma_{n_+}^+\}$.

Similarly, let $\mathcal{F}^+ = \{z \in \Gamma : z = q_{j,+} \text{ if } q_{j,+} \text{ is defined for } 1 \leq j \leq T\}$. Let $\mathcal{T}_0^-$

be the union of all sides of Γ and all C_{η_j}'s for $1 \leq j \leq T$ whenever C_{η_j} is defined. Let $\mathcal{T}_1^- = \mathcal{T}_0^- - (B_\Gamma \cup S_\sigma \cup \{c\} \cup \mathcal{F}^+)$. Again, $\mathcal{T}_1^-$ has finitely many connected components each of which is an open transversal section to X. Say, $\mathcal{T}_1^- = \bigcup_{1\leq j\leq n_-} \Sigma_j^-$ for some positive integer n_- where each Σ_j^- is a connected component of $\mathcal{T}_1^-$. Let $\mathcal{T}^- = \{\Sigma_1^-, \ldots, \Sigma_{n_-}^-\}$.

Consider a boundary point z of Σ_j^+. Assume that z is equal to $w_{k,-}$ for some $1 \leq k \leq T$. If w_k is a point in B_Γ, then the positive semi trajectories of points close to z in Σ_j^+ travel along the closed arc $(w_{k,-})w_k$. Let $\epsilon > 0$ be as small as $w_k w_\epsilon - \{w_k\}$ does not intersect a side of Γ where we have $w_\epsilon = X_\epsilon(w_k)$. Recall that the side S of Γ which contains w_k has a parabolic transversality failure at w_k. So, the positive semi trajectories of points close to z in Σ_j^+ can travel along $(w_{k,-})w_\epsilon$ in two ways: either they intersect some S or they do not. If they intersect S, then they intersect some Σ_m^-. In either case, they continue to travel along w_k^+ (for a short time at least). If we have $w_k = c$, then the positive semi trajectories of points close to z in Σ_j^+ travel along the closed arc zc an intersect some Σ_m^- because all the sides of Γ are transversal to X at c. If w_k is a point of a stable separatrix of some saddle σ, then the positive semi trajectories of points close to z in Σ_j^+ travel along this stable separatrix and they either intersect some Σ_m^- or they continue to travel along an unstable separatrix of σ without intersecting a side in a neighborhood of this saddle. See Figure 5.21.

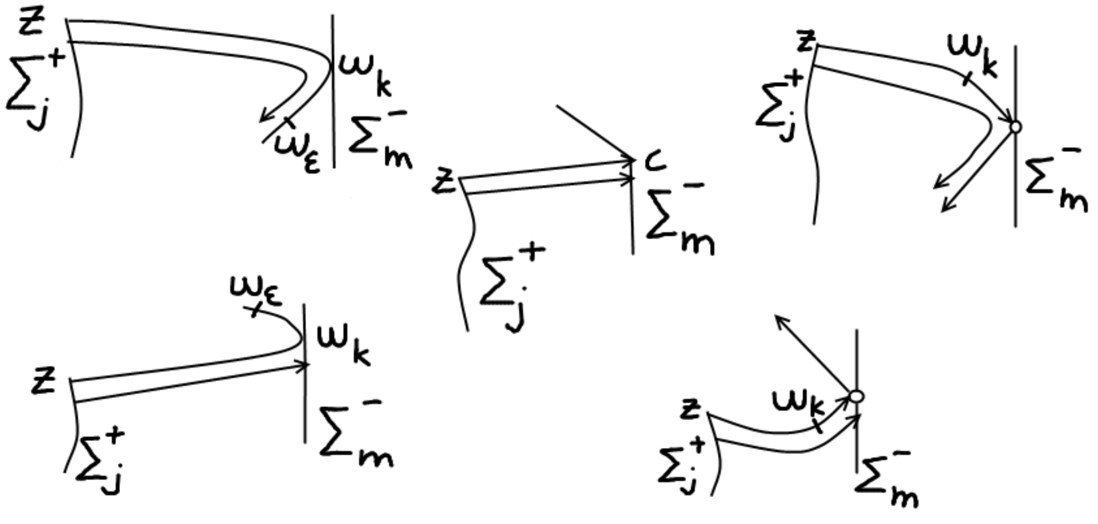

Figure 5.21. All the possible cases when $z = w_{k,-}$.

If z is not equal to any defined $w_{k,-}$ for $1 \leq k \leq T$, then z is in $B_\Gamma \cup S_\sigma \cup \{c\}$. Say, $z = q_n$. If q_n is in $B_\Gamma \cup \{c\}$, then the positive semi trajectories of points close to q_n in Σ_j^+ travel along q_n^+. If $q_{n,+}$ is defined, then they either intersect some Σ_m^- or $q_{n,+}$ is in B_Γ and they continue to travel along ${q_{n,+}}^+$ without intersecting a side of Γ (for a short time at least). If $q_{n,+}$ is not defined, then the positive semi trajectories of them travel along a stable separatrix of some saddle. They either intersect some Σ_m^- or they continue to travel along an unstable separatrix of that saddle without intersecting a side of Γ (for a short time at least). Finally, if q_n is a saddle, then the positive semi trajectories of points close to q_n in Σ_j^+ travel along an unstable separatrix γ_u of q_n. Let V_n be the Grobman-Hartman neighborhood of q_n which has been already defined and let $d_m \in V_n \cap \mathcal{S}_\sigma^+$ be such that we have $d_m \in \gamma_u$. In this last case, the possible situations for ${d_m}^+$ are similar to the ones of ${q_n}^+$ where q_n was in $B_\Gamma \cup \{c\}$.

We have considered all the possible situations when z is a boundary point of Σ_j^+. Analogously, we can consider a boundary point z of Σ_j^- and we can analogously analyze all the possible situations for the negative semi trajectories of points close to z in Σ_j^-. Lets go back. Let z be a boundary point of Σ_j^+. All in all, we claim that the positive semi trajectories of points close to z travel along some saddle connections or some closed arcs $(w_{k,-})w_k$'s or some closed arcs $q_k(q_{k,+})$'s and they eventually intersect some Σ_m^-. Assume the claim is false. Then, for any positive positive integer n, we can find a point p in Σ_j^+ close enough to z such that p^+ visits n small neighborhoods of n elements in $B_\Gamma \cup S_\sigma$. As the set $B_\Gamma \cup S_\sigma$ is finite, some of its elements will be repeated for large n. There might be some saddle connections or not. In any case, the relevant semi trajectories (separatrices of saddles or semi trajectories of points in B_Γ) and the relevant saddles bound altogether an open disc D_0 in D_Γ. This is a contradiction because p is not in D_0 and we can take some other point p_1 in Σ_j^+ that is closer to z than p is (if necessary) and conclude that p_1^+ intersects some Σ_m^- and our claim has been proven. Note that the open disc D_0 might be possibly defined in this way but even in this situation, the poistive semi trajectories of points close z in Σ_j^+ intersect some Σ_m^- within the polygon disc D_Γ. See Figure 5.22.

Let Σ_k^- be the first element in $\mathcal{T}_-$ which the positive semi trajectory of a point z_p

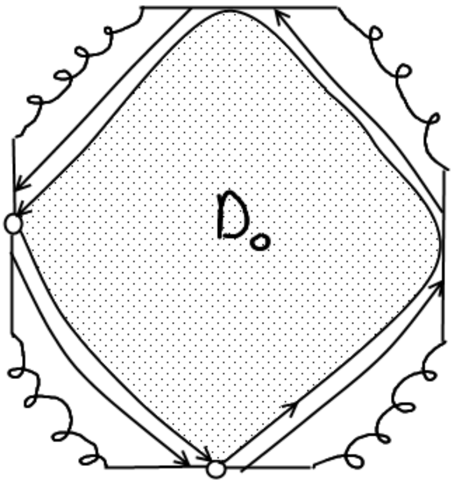

Figure 5.22. The bounded open disc D_0 in the polygon disc D_Γ.

close to z in Σ_j^+ intersects. Let the point z_q denote the first intersection of ${z_p}^+$ with Σ_k^-. As Σ_j^+ and Σ_k^- are both open transversal sections, z_p and z_q are interior points of Σ_j^+ and Σ_k^- respectively. Also, the closed arc $z_p z_q$ does not include any boundary points of Σ_j^+ and Σ_k^- by the definition of all Σ_n^+'s $(1 \leq n \leq n_+)$ and all Σ_m^-'s $(1 \leq m \leq n_-)$.

So, the First Proper Intersection Assignment $P : \Sigma_j^+ \to \Sigma_k^-$ is defined at z_p with $P(z_p) = z_q$ and the domain of P is nonempty. By the Tubular Flow Extension, P is defined on the whole Σ_j^+ because all the following properties have been satisfied: for any $\Sigma_n^+ \in \mathcal{T}^+$ and for any $\Sigma_m^- \in \mathcal{T}^-$, the set $\Sigma_n^+ \cap \Sigma_m^-$ is either an interval-like connected set or the empty set; the positive semi trajectory of a point of Σ_j^+ cannot go through a boundary point of an element in $\mathcal{T}^+ \cup \mathcal{T}^-$ within the polygon disc D_Γ because of the definitions of the elements in $\mathcal{T}^+ \cup \mathcal{T}^-$ and; the positive semi trajectory of a point in Σ_j^+ cannot go to a saddle within D_Γ again because of the definitions of the elements in $\mathcal{T}^+ \cup \mathcal{T}^-$.

So, for any given $\Sigma_j^+ \in \mathcal{T}^+$, there exists a unique $\Sigma_k^- \in \mathcal{T}^-$ such that the First Proper Intersection Assignment $P : \Sigma_j^+ \to \Sigma_k^-$ is defined on the whole Σ_j^+.

As we have considered the First Proper Intersection Assignment for the positive semi trajectories, we can consider it for the negative semi trajectories in the same manner and derive the analogous conclusions. Therefore, for any given $\Sigma_j^- \in \mathcal{T}^-$, there exists a unique $\Sigma_m^+ \in \mathcal{T}^+$ such that the First Proper Intersection Assignment $P_- : \Sigma_j^- \to \Sigma_m^+$ is defined on the whole Σ_j^- for the flow of $-X$. Therefore, the positive integers n_+ and n_- are equal and $P_- : \Sigma_j^- \to \Sigma_m^+$ is a diffeomorphism. See Figure 5.23 and Figure 5.24 which are in conjunction with each other. We now make a formal definition.

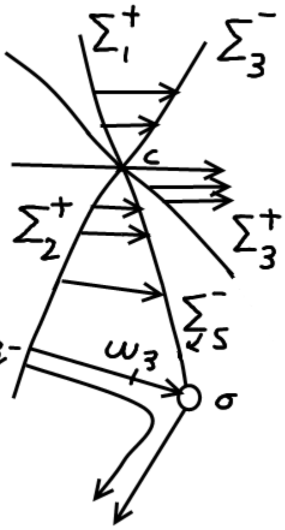

Figure 5.23. Local picture at the point c. Here, only three sides of Γ have been shown.

Definition 5.10 (Transversal Base). *Let X, Γ, $\mathcal{T}^+ = \{\Sigma_1^+, \ldots, \Sigma_{n_+}^+\}$ and $\mathcal{T}^- = \{\Sigma_1^-, \ldots, \Sigma_{n_+}^-\}$ be as above. Suppose that the enumeration of Σ_j^+'s and Σ_j^-'s is such that each First Proper Intersection Assignment $P_j : \Sigma_j^+ \to \Sigma_j^-$ is defined on the whole Σ_j^+ ($1 \le j \le n_+$) and P_j is a diffeomorphism. The set $\mathcal{T}^\pm := \mathcal{T}^+ \cup \mathcal{T}^-$ is called* a transversal base for Γ.

Theorem 5.3. *Suppose that X is a C^r vector field on a compact, connected, 2-manifold without boundary the singularities of which are all hyperbolic. Then, X admits a fundamental polygon Γ that satisfies the properties of Lemma 5.3 and also, X admits a transversal base $\mathcal{T}^\pm$ for Γ.*

Let X be a C^r vector field on M the singularities of which are all hyperbolic. Let Γ be a fundamental polygon for M that satisfies the properties of Lemma 5.3. Let $\mathcal{T}^\pm$ be a transversal base for Γ by Theorem 5.10. Say, we have $\mathcal{T}^\pm = \mathcal{T}^+ \cup \mathcal{T}^-$, $\mathcal{T}^+ = \{\Sigma_1^+, \ldots, \Sigma_{n_+}^+\}$ and $\mathcal{T}^- = \{\Sigma_1^-, \ldots, \Sigma_{n_+}^-\}$.

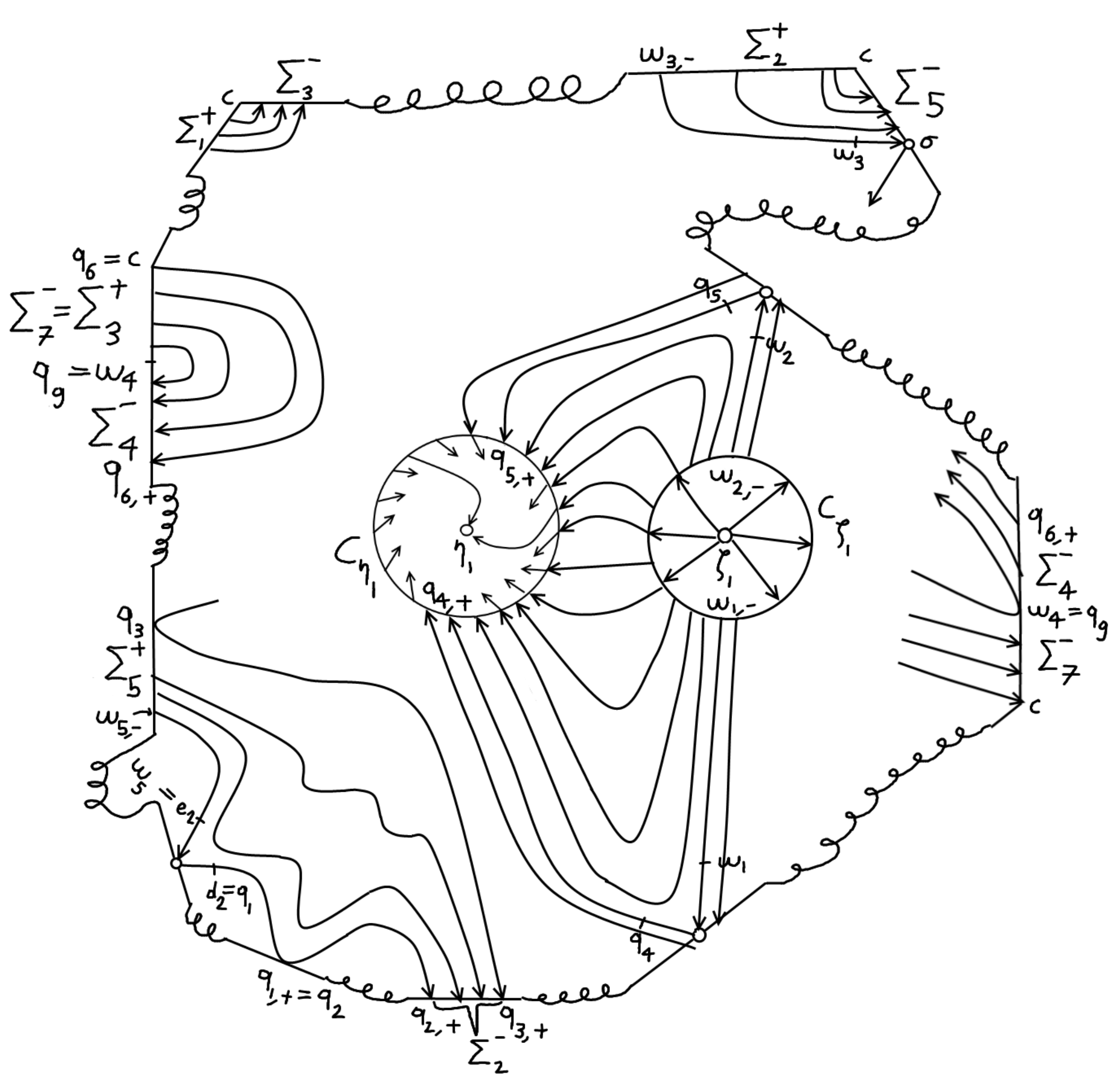

Figure 5.24. A few possible situations for a big fundamental polygon Γ.

It might possibly happen that Σ_j^+ and Σ_j^- are subsets of the same side of Γ for some j with $1 \leq j \leq n_+$ and also, $\Sigma_j^+ \cap \Sigma_j^-$ might be a nonempty, interval-like connected set. Consider an even sided polygon $\hat{Q}$ and a smooth quotient map $\Pi_r : \hat{Q} \to \Gamma$ which induces the quotient topology on Γ. Let $\hat{Q}^\circ$ denote the interior of $\hat{Q}$. The smooth map Π_r does not identify any point of $\hat{Q}^\circ$ with any other distinct point of $\hat{Q}$ and it is injective on the interior on $\hat{Q}^\circ$. Let $\hat{\Pi}_r$ be the restriction of Π_r to $\hat{Q}^\circ$. So, $\hat{\Pi}_r^{-1}$ is defined on the polygon disc D_Γ of Γ. The vector field $\hat{X} := (\hat{\Pi}_r^{-1})_*(X_{|D_\Gamma})$ is a C^r vector field on $\hat{Q}^\circ$ and $\hat{X}$ can be extended to a vector field $\hat{Y}$ on $\hat{Q} - {\Pi_r}^{-1}(c)$. Now, there exist two unique subsets $\hat{\Sigma}_j^+$ and $\hat{\Sigma}_j^-$ of the sides of $\hat{Q} - {\Pi_r}^{-1}(c)$ such that: we have $\hat{\Sigma}_j^+ \cap \hat{\Sigma}_j^- = \varnothing$; $\Pi_r(\hat{\Sigma}_j^+) = \Sigma_j^+$ and $\Pi_r(\hat{\Sigma}_j^-) = \Sigma_j^-$ and; the First Proper Intersection

Assignment $P : \hat{\Sigma}_j^+ \to \hat{\Sigma}_j^-$ is defined on the whole $\hat{\Sigma}_j^+$ for the flow of $\hat{Y}$ and P is a diffeomorphism. Loosely speaking, what was not disjoint before ($\Sigma_j^+ \cap \Sigma_j^- \neq \varnothing$) is now disjoint ($\hat{\Sigma}_j^+ \cap \hat{\Sigma}_j^- = \varnothing$). It might be also possible (but not necessarily) that the the closures of the open transversal sections $\hat{\Sigma}_j^+$ and $\hat{\Sigma}_j^-$ are the sides of some section and orbit sided rectangle $\bar{D}$ for the vector field $\hat{Y}$. The Tubular Flow Extension can be applied more easily when one considers disjoint transversal sections instead of non-disjoint ones and these ideas were always in the back of our mind when we were explaining our program toward the proof of Theorem 5.3.

5.6. Behavior of Nontrivial Recurrent Orbits

Let X be a C^r vector field on M the singularities of which are all hyperbolic. Let Γ be a fundamental polygon for M that satisfies the properties of Lemma 5.3. Let $\mathcal{T}^\pm$ be a transversal base for Γ by Theorem 5.10. Say, we have $\mathcal{T}^\pm = \mathcal{T}^+ \cup \mathcal{T}^-$, $\mathcal{T}^+ = \{\Sigma_1^+, \ldots, \Sigma_{n_+}^+\}$ and $\mathcal{T}^- = \{\Sigma_1^-, \ldots, \Sigma_{n_+}^-\}$.

Suppose that X has a nontrivial recurrent orbit γ. We will assume that γ is nontrivial forward recurrent as the study of the other case is analogous. Then, there does not exist a point $q \in \gamma$ such that q^+ is in the polygon disc D_Γ of Γ because D^r can be embedded into S^2 and the manifold S^2 does not admit any vector field with nontrivial recurrent orbits by Lemma 3.1. Hence, for any point $q \in \gamma$, the positive semi trajectory q^+ intersects sides of Γ infinitely many times. As the set $B_\Gamma \cup \{c\}$ is finite and γ is nontrivial forward recurrent, there exists a point $q_0 \in q^+$ such that we have $q_0^+ \cap (B_\Gamma \cup \{c\}) = \varnothing$.

To study the behavior of q_0^+, we define the real number v_+ with $0 < v_+ \leq 1$ by defining its decimal expansion in the base $(n_+ + 1)$. So, for a sequence $0.d_1d_2d_3\ldots$ where each d_j is an integer between 1 and n_+ , we have $0.d_1d_2d_3\cdots = d_1(n_+ + 1)^{-1} + d_2(n_+ + 1)^{-2} + d_3(n_+ + 1)^{-3} + \cdots$. Define $v_+ = 0.d_1d_2d_3\ldots$ in the following way: the j^{th} intersection of q_0^+ with an element of $\mathcal{T}^+$ is $\Sigma_{d_j}^+$.

Lemma 5.10. *The number v_+ which is defined as above is irrational.*

Proof. Assume that v_+ is rational. Then, there exist positive integers n and m such that after the first n digits in the decimal expansion $v_+ = 0.d_1d_2d_3\ldots$, the sequence becomes periodic of length m digits; i.e. $d_{n+1}\ldots d_{n+m}$ is a periodic cycle in the decimal expansion of v_+. For $1 \leq j \leq n_+$, let $P_j : \Sigma_j^+ \to \Sigma_j^-$ be the First Proper Intersection Assignment.

Let $A_{n+m-1} = P^{-1}_{d_{n+m-1}}(\Sigma^+_{d_{n+m}} \cap \Sigma^-_{d_{n+m-1}})$. The set of points in $\Sigma^+_{d_{n+m-1}}$ the positive semi trajectories of which go to $\Sigma^+_{d_{n+m}}$ within the polygon disc D_Γ of Γ is precisely the set A_{n+m-1}. Note that A_{n+m-1} is a connected set in $\Sigma^+_{d_{n+m-1}}$ because $\Sigma^+_{d_{n+m}} \cap \Sigma^-_{d_{n+m-1}}$ is connected and $P_{d_{n+m-1}}$ is a diffeomorphism. Also, A_{n+m-1} is nonempty because of the definition of v_+.

For $d_{n+m-1} > j \geq d_{n+1}$, inductively define $A_j := P_j^{-1}(A_{j+1} \cap \Sigma_j^-)$. So, $A_{d_{n+1}}$ is a nonempty connected set in $\Sigma^+_{d_{n+1}}$ because of the definition of v_+.

For simplicity, we now neglect the first n intersections of q_0^+ with elements of $\mathcal{T}^+$ and begin with the periodic cycle $d_{n+1}\ldots d_{n+m}$. Let $Q = P_{d_{n+m}} \circ \cdots \circ P_{d_{n+1}}\big|_{A_{d_{n+1}}}$. The set $Q(A_{d_{n+1}})$ is not necessarily a subset of $A_{d_{n+1}}$ but q_0^+ must always return to $A_{d_{n+1}}$ after following the intersection sequence $\Sigma_{d_{n+1}}\ldots\Sigma_{d_{n+m}}$ because of $q_0^+ \cap (B_\Gamma \cup \{c\}) = \varnothing$ and the definition of v_+.

If the diffeomorphism Q preserves orientation in $\Sigma_{d_{n+1}}$ (here, we mean that $A_{d_{n+1}}$ and $Q(A_{d_{n+1}}) \cap \Sigma_{d_{n+1}}$ have the same orientations), then each m^{th} intersection of q_0^+ with $\Sigma^+_{d_{n+1}}$ after its first intersection becomes monotonic. This monotonic sequence can accumulate to at most a single point in $\Sigma^+_{d_{n+1}}$. If the diffeomorphism Q reverses orientation in $\Sigma_{d_{n+1}}$, then each m^{th} intersection of q_0^+ with $\Sigma^+_{d_{n+1}}$ after its first intersection can accumulate to at most two points in $\Sigma^+_{d_{n+1}}$. In this latter case, note that Q has at least one fixed point. In this latter situation, we prefer to say that the intersections of q_0^+ flip around a point $p_q \in \Sigma_{d_{n+1}}$ where p_q is a fixed point of Q. Let $E_{d_{n+1}}$ denote the set of accumulation points that we have explained above. So, we have $|E_{d_{n+1}}| \in \{0, 1, 2\}$.

We have begun our analysis with the $d_{n+1} \dots d_{n+m}$ cycle but we can make the same argument for any cyclic permutation of the $d_{n+1} \dots d_{n+m}$ cycle. So, if $d_{n+1} = d_{n+j}$ for some $1 < j \leq m$, we can repeat our previous argument for the cycle $d_{n+j} \dots d_{n+m} d_{n+1} \dots d_{n+j-1}$ and define the set $E_{d_{n+j}}$ analogously.

Let E be the union of all possibly defined $E_{d_{n+j}}$'s ($1 \leq j \leq m$). Then, we have $E \subseteq \omega(q_0) \cap \Sigma_{d_{n+1}}$. As each m^{th} intersection of ${q_0}^{+}$ with $\Sigma_{d_{n+1}}$ after its k^{th} intersection ($1 \leq k \leq m$) is either monotonic or flips around a point of $\Sigma_{d_{n+1}}$, we conclude the equality $E = \omega(q_0) \cap \Sigma_{d_{n+1}}$. Hence, q_0^{+} accumulates to only a finite number of points in $\Sigma^{+}_{d_{n+1}}$ which contradicts that it is nontrivial forward recurrent. Therefore, v_{+} is irrational. $\square$

6. CONCLUSIONS

There has been a continuing progress about the density of M-S fields in $\mathfrak{X}^r(M)$ (for $r \geq 1$ or nonorientable M) since Peixoto's work in 1962. Especially, [7] is to note. We share the positive belief in the literature that M-S field are dense in $\mathfrak{X}^r(M)$ for any 2-manifold M and for any $r \geq 1$ and we also believe in that one can prove this result. We hope that the transversal base will be helpful in the study of the density of M-S fields and vector fields with nontrivial recurrent orbits. Our own work has been mostly inspired by the beautiful example of Gutierrez in [11]. His example X on a torus with two cross caps has two saddles and no other singularities. He gives his example by using eight arcs of X which are subsets of separatrices of X and four transversal sections to X. For every orbit γ of X, if $\omega(\gamma)$ is not a saddle, then γ is a nontrivial forward recurrent and also, if $\alpha(\gamma)$ is not a saddle, then γ is a nontrivial backward recurrent. The vector field X has no saddle connections, no closed orbits and an abundance of nontrivial recurrent orbits. In [11], Gutierrez gives this example in order to prove that his ideas in [10] cannot be extended to other nonorientable manifolds the Euler characteristics of which are smaller than -1. What is more important to us is that he obtains a smooth vector field Y arbitrarily close to X by a global perturbation of X on the whole M such that: Y has only two saddles and no other singularities and; every orbit of Y is either a saddle or a saddle connection or a closed orbit. So, Y has no nontrivial recurrent orbits. From here, one can first stabilize the four saddle connections of Y (see Theorem 3.1) and then, one can obtain an M-S field that is arbitrarily close to Y. This global perturbation of X in [11] is important because Gutierrez and Pires show in [8] that one cannot eliminate nontrivial recurrent orbits of X if one makes an arbitrary flow box perturbation of X.

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