INVENTORY POLICIES UNDER ADVANCE CAPACITY INFORMATION

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ABSTRACT

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One of the most important challenges the inventory managers face is the uncertainty on both sides of the demand and supply. Exchange of information on both demand and capacity processes can decrease the uncertainty and it can be beneficial to all parties involved. But the quantification of the benefits of information sharing is not easy. In this thesis our purpose is to see how managers utilize and integrate available advance information on capacity into the replenishment decisions, and identify the types of operating environments under which Advance Capacity Information (ACI) is most valuable. We consider a production/inventory system that faces stochastic demand, with a supplier whose capacity is limited and uncertain. But the supplier has agreed on sharing the capacity information for a certain number of future periods. We consider three different problems under these conditions. We first model the rationing problem of a production/inventory system that serves customers from two different classes, which are distinguished by the penalty costs charged for unsatisfied demand. Then we study the ordering policies under average cost criterion. We propose heuristic approaches to calculate ACI-dependent order-up-to levels. The third problem we consider is of a firm that uses outsourcing whenever the in-house capacity is not adequate. We propose an ACI-dependent order-up-to level policy, where ACI is defined in terms of the distribution of the in-house capacity. Through numerical studies we derive managerial insights with respect to the benefits of using ACI and the policies we propose.

ÖZET

ERKEN KAPASİTE BİLGİSİ ALTINDA STOK POLİTİKALARI

Stok yöneticilerinin karşılaştığı en önemli zorluklardan biri talep ve tedarikteki belirsizliklerdir. Talep ve tedarik ile ilgili bilgi paylaşımı belirsizliği azaltabilir ve ilgili taraflar açısından karlı olabilir. Ama bilgi paylaşımının karlılığını hesaplamak genel olarak kolay değildir. Bu tezde amacımız yöneticilerin elde edilen kapasite bilgisini stok politikalarında nasıl kullanabileceğini görmek, ve hangi işletme koşullarında erken kapasite bilgisinin daha değerli olduğunu belirlemektir. Bu tezde belli bir dönemde rassal olan talebi karşılamaya çalışan ve tedarikçisinin kapasitesi limitli ve rassal olan bir üretim/stok sistemi çalışmaktayız. Tedarikçinin kısıtlı sayıda gelecek dönem için kapasite bilgisini paylaştığını varsayarak, bu koşullar altında üç farklı problemi analiz ediyoruz. İlk olarak müşterilerin talebi karşılayamama cezalarına göre sınıflandırıldığı, iki müşteri sınıfı olan bir üretim/stok sisteminin tayın problemini inceliyoruz. Daha sonra ortalama maliyet ölçütü altında erken kapasite bilgisine bağlı sipariş politikalarının bulunması için sezgisel yöntemler öneriyoruz. Son olarak kapasitesi yeterli gelmediğinde dış kaynaklar kulanan bir üretim/stok sistemi için, erken kapasite bilgisinin kapasitenin dağılımı olduğu durumda, bir sipariş politikası tanımlıyoruz. Önerdiğimiz bu yöntemleri ve erken kapasite bilgisini kulanmanın karlılığını sayısal çalışmalarla analiz ediyoruz.

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LIST OF SYMBOLS

ACI_n	signal for capacity distribution at the beginning of period n
BS_n	β-adjusted base-stock level
b	unit backorder cost
$\mathbb{C}_1$	stochastically larger regular capacity
$\mathbb{C}_2$	stochastically smaller regular capacity
C_n	capacity available at the beginning of period n
$\underline{\mathfrak{C}}_n$	capacity information vector including periods n to $n+M$
$\mathfrak{C}_\mathfrak{n}$	capacity information vector including periods $n+1$ to $n+M$
$C(n,n+j)$	sum of the capacities over periods n to n+j
D_n^i	random demand of customer class i in period n
$\mathfrak{D}_\mathfrak{n}$	demand vector for period n
$D(n,n+j)$	sum of random demands over periods n to $n+j$
d_n^i	realization of D_n^i
$F_X(.)$	cumulative distribution function of random variable X
$f_X(\cdot)$	density function of random variable X
h	unit holding cost
I_n	on hand inventory at the beginning of period n
IA_n	interarrival time for the n^{th} customer
M	number of periods capacity information is available
N	planning horizon length
p	probability of stochastically larger regular capacity
p_i	penalty cost for not satisfying unit demand from customer class i
R_n	inventory left at the end of period n
R_n^L	lower bound for R_n
R_n^U	upper bound for R_n
S_1^{UB}	upper bound for S_1
S_i	order-up-to level when i type of signal is observed
S_m	base-stock level for the policy with no ACI

ST_n	service time of the n^{th} customer
S_w	base-stock level for the weighted cost heuristic
s	constant term in the β-adjusted base-stock policy
v_n^i	amount allocated to customer class i in period n
W_n	waiting time of the n^{th} customer
X_n	step size of the reflected random walk that defines Z_n
y_n	inventory level in period n after the order is received
Z_n	shortfall of the inventory level for period n

β	capacity weight for the β-adjusted base-stock policy
β_i	fill rate for customer class i
Δ	difference between S_1 and S_2
Δ_{UB}	upper bound for Δ
γ	unit cost of outsourcing
κ	rationed amount
μ_C	mean capacity
μ_i	mean of demand from customer class i
μ_D	mean demand
σ_C	standard deviation of C
σ_i	standard deviation of D^i
ω_i	weight of the i^{th} cost term for the weighted cost heuristic
ρ	utilization of capacity

LIST OF ACRONYMS/ABBREVIATIONS

ACI	Advance Capacity Information
G/G/1	Single-server queue with a general distribution of inter-arrival and service times
VACI	Value of ACI
VO	Value of Outsourcing
VOR	Value of Rationing

1. INTRODUCTION

With the globalization of economies, many manufacturers have global presence through exports and strategic alliances in foreign markets. As global transactions increase, purchasing, inventory management, and logistics, become more and more significant. Proper management of these functions is necessary to compete more effectively in such a marketplace. Specifically, the total investment in inventories is very large. The capital is tied up in the inventories of raw material, work-in-progress, and finished goods. The objective of inventory control is often to balance conflicting goals. One goal is to keep stock levels low to make cash available for other purposes, while a high stock of finished goods can be desired to be able to provide a high service level. With conflicting goals effective inventory control can give a significant competitive advantage.

One of the most important challenges the inventory managers face is the uncertainty on both sides of the demand and supply. There is an increasing understanding that exchange of information in the logistics process can decrease the uncertainty and that it can be beneficial to all parties involved. With the available information one can design and operate supply chains more effectively.

In general, information sharing

- can reduce variability in the supply chain,
- help suppliers make better forecasts,
- enables the coordination of manufacturing and distribution systems strategies,
- enables retailers to react and adapt to supply problems more rapidly,
- enables lead time reductions (Simchi-Levi *et al.*, 2003).

Specifically a manufacturer could make use of its supplier's production or delivery schedule to improve its own production schedule. For example, US auto companies have access to the production schedule for their orders at steel suppliers and semiconductor foundries share their capacity status with the buyers (Lee and Whang, 2000). Information about availability helps a buyer revise his own production schedule, and by sharing planned capacity information with the downstream partners in advance, supply chain partners can coordinate and prepare against possible shortages.

Unfortunately, using the available information may make the design and management of the supply chain more complex, since more issues are to be considered. Sharing information is not a simple issue. The first challenge is related to the incentives of different partners. Information sharing and cooperation do not always increase individual profits. Even when each partner is guaranteed a positive gain in return for information sharing, how much is still an issue. Confidentiality of the shared information and who is going to invest in the technology needed for information sharing are other concerns.

Another important concern associated with information sharing is that the quantification of the benefits of information exchange, which is not easy in general. In the literature the consequences of sharing advance demand information has been investigated extensively. However, there is limited work on advance capacity/supply information sharing. This discrepancy between academic literature and practice gave us the motivation for this study.

In this thesis our main purpose is to see how the managers utilize and integrate available advance information on capacity into the replenishment decisions, and identify the types of operating environments under which Advance Capacity Information (ACI) is most valuable. We consider a production/inventory system that faces uncertain demand and capacity. The supplier of this system provides ACI for a certain number of future periods. We assume that the supplier has already agreed on sharing the capacity information truthfully, hence we do not consider incentive issues.

The capacity of a production system can be uncertain due to breakdowns or unplanned maintenance activities of machinery. Also, the resource can be shared by several items, and due to randomness in yield and rework activities, the capacity allocated to an item may exhibit random fluctuations. The information provided by the upstream supplier may be affected by different events. For example, the supplier may be serving multiple downstream buyers, and it may be the case that a portion of the capacity is already promised to other buyers (as a result of priority contracts). The supply information may also vary due to the planned downtime for maintenance or inspection.

We consider three different problems under the defined conditions. In Chapter 3, we consider the rationing problem of a production/inventory system that serves customers from two different classes, which are distinguished by the penalty costs charged for unsatisfied demand. We assume that the unsatisfied demand is lost independent of the customer class. Customer differentiation is increasingly used as a production/inventory management strategy. Inventory-related cost performance can be improved by effective allocation of production capacity and/or inventory. There are cases where different demand classes with different stockout costs and/or required service levels arise. For example, by contractual agreements with a supplier, a customer can have priority among other customers the supplier is serving. An example is the spare parts industry, where some parts can be crucial, and the customer may be willing to make a binding contract with the supplier, with which a high fill rate is required.

Another example where several demand classes may be distinguished is an assemble-to-order system where a common component is shared by several end-products with different values to the firm. In such a case, a firm may be able to prioritize its products using several criteria, such as the annual sales volume of a product, its marginal profit contribution, and/or the long-term nature of the customers' association with the firm. In case of a make-to-stock (MTS) firm, the allocation of production capacity is considered. From a modeling point of view, rationing inventory among customer classes and rationing capacity for different products can be treated similarly. Multi-echelon inventory system where the highest echelon may face demand both from external customers and from lower echelon stocking points is another example with multiple demand classes.

With the rationing policy the available inventory is allocated according to the customer type and hence priority classes are considered. The customer with the higher penalty cost has priority over the customer with lower penalty cost. Demand from the customer class with lower priority can be rejected even though the on hand inventory level is non-zero, with the expectation of future higher class demands in the upcoming periods. This kind of policy is meaningful if the cost of holding inventory plus the penalty cost for one type of customer is smaller than the penalty cost of the other customer.

In an environment with uncertain capacity, advance information has the potential to improve decision making both in terms of ordering and rationing. For example, the order can be inflated to avoid shortages in future periods, or the allocated amounts can be adjusted, in favor of the priority class in case of low future capacity.

We model the inventory/production environment incorporating information on capacity, and for a two-class problem we show the optimality of an ACI dependent base-stock policy and we characterize the optimal allocation of available inventory. Through a numerical study we investigate the conditions under which ACI and rationing policy are more valuable, and we provide managerial insights.

In Chapter 4, we consider the ordering policies for an inventory system that faces stochastic demand and capacity, where the unsatisfied demand is backordered. We propose two types of heuristics to calculate ACI-dependent order-up-to levels to minimize the average cost per period. The first type of heuristics are based on asymptotic results. First one is based on the logarithmic asymptotic for the steady-state waiting-time distribution for a single-server queue with unlimited waiting space and first-in first-out service discipline (without any explicit independence conditions on the interarrival and service times). The second one is based on a heavy traffic asymptotic approximation for the tail distribution of a reflected random walk with independent step sizes. Then, we present a second type of heuristic that considers weighing the costs of the planning periods for which the capacity information is available plus a cost term associated with the periods for which the capacity information is not available. We evaluate these methods via simulation and discuss their efficiency in reducing costs.

In Chapter 5, we consider the inventory/production problem of a firm that uses operational level outsourcing to hedge against uncertainty. In planning and managing inventory/production systems, outsourcing is one of the two main strategies that can be used to manage uncertainty besides gradually building inventory. The ability to adjust the total capacity temporarily by acquiring extra resources, such as subcontracting, overtime production, hiring temporary workers, leads to capacity flexibility. In the existence of capacity flexibility, inventory/production related costs may be reduced by managing the capacity and inventory in a joint fashion.

We consider an inventory/production system under ACI with flexible capacity, where whenever the target base-stock level cannot be reached using the currently available capacity at a given planning period, the short amount can be bought from an exogenous supplier. The capacity of the inventory system we consider is the maximum amount that can be produced in a period, which we call the *regular capacity* of the system. We assume that the regular capacity for the inventory process is uncertain and non-stationary, but the manufacturer observes the distribution of the capacity in advance, for a number of future periods.

Since obtaining the optimal policy for a system facing volatile demand in the existence of outsourcing option is difficult, even for a system where the regular capacity is deterministic and stationary, we propose an ACI-dependent order-up-to level policy. We model the uncertainty in the regular capacity, and present an ACI process that tracks and updates the capacity information. Then, we discuss in detail the operating characteristics of the inventory system. We develop an expression for the average cost of the system, and characterize the optimal solution under the order-up-to level policy. Through a numerical study we derive managerial insights with respect to the benefits of using outsourcing option and ACI.

2. LITERATURE SURVEY

The aim of this study is to see effects of integrating ACI into the inventory replenishment decisions of a production/inventory system that faces stochastic demand. Our work is basically related with the literature of inventory problems with limited capacity and also with multiple demand classes and flexible capacity. The research on the modeling of advance information on the supply/capacity is very recent. In this section we first give examples of studies that model limited capacity. Then we give a review of multi-class demand models with a special emphasis on rationing models, and then we briefly discuss some important studies on capacity flexibility.

2.1. Uncertain Capacity

In this section, we review studies that model uncertainty in the production/ordering process. Although sources of uncertainty in production/ordering processes due to production and supply resources are numerous (such as limited capacity, random supply disruption or unavailability due to unexpected breakdowns, uncertain repair duration, rework of defective items), they can be divided into two main categories: random capacity, and random yield (Wang and Gerchak, 1996). When capacity is random, maximum amount that can be produced in any planning period is random, and when yield is random, only a random fraction of the actually processed quantity is usable, which is also referred to as *multiplicative yield*. The literature on random yields is summarized in (Yano and Lee, 1995), and a review of models that consider multiple lot-sizing in production-to-order is given in (Grosfeld-Nir and Gerchak, 2004).

Modified base-stock policy is shown to be optimal for constant production capacity in (Federgruen and Zipkin, 1986a) and (Federgruen and Zipkin, 1986b) under average cost criterion and discounted cost criterion, respectively. The policy states that the inventory level must be brought up to the base-stock level, as much as the capacity allows. (Ciarallo *et al.*, 1994) show that an order-up-to type policy is optimal for the random capacity model. (Henig and Gerchak, 1990) study a model with random yield. They analyze stochastically proportional yield model with convex cost structure, and prove the existence of an optimal reorder point for the finite horizon problem. (Wang and Gerchak, 1996) extend models of (Henig and Gerchak, 1990) and (Ciarallo *et al.*, 1994) to incorporate random yield as well as random capacity. They show for the finite-horizon problem that the optimal policy is characterized by a single critical point for the initial stock level at each period. With this policy a production order is given only if the inventory level is less than the critical inventory level. Expressions for determining the critical point and the optimal planned production are obtained. (Iida, 2002) studies a non-stationary production/inventory system with uncertain production capacity. (Iida, 2002) introduces an equivalent problem formulation of (Ciarallo *et al.*, 1994) in order to develop bounds for optimal policies for the finite horizon problem (under discounted expected total cost criterion), and shows that bounds for the optimal order-up-to levels for the finite horizon counterparts converge as the planning horizons considered get longer. Furthermore, under mild conditions the differences between the upper and the lower bounds converge exponentially to zero.

(Gullu, 1998) considers a discrete time, stochastic demand production/inventory system that operates under a stationary modified base-stock policy. The objective is to obtain the optimal base stock level that minimizes the long term expected average cost. For this purpose (Gullu, 1998) establishes the analogy between the class of base-stock production/inventory policies that operate under demand/capacity uncertainty, and the G/G/1 queues. By using this analogy he obtains the stationary probability distribution of the inventory position as a transformation of the stationary waiting time distribution in the queue, and shows that this stationary distribution is sufficient to characterize the expected average costs.

In some studies capacity is modeled as a Markov process. (Parlar *et al.*, 1995) and (Ozekici and Parlar, 1999) consider a periodic review setting with setup costs. In a given period supply is either available or not, where the availability is modeled as a Markov Chain. (Yang *et al.*, 2006) also model the Markovian uncertain production capacity and show that the optimal production and order acceptance policies are both of critical point type.

Our work is also related to the studies that model advance capacity information. This line of work is more recent and scarce. (Altug and Muharremoglu, 2011) and (Jaksic *et al.*,

2011) incorporate advance information on capacity in the inventory replenishment problem. For an inventory/production system, ACI-dependent ordering policies was first introduced by (Jaksic *et al.*, 2011). They present the dynamic program for a single product inventory system, and show that for a finite horizon problem the optimal ordering policy is a state-dependent modified base-stock policy. They consider an ACI structure where the ACI for a number of future periods is fixed after it has been observed. They show that most benefits of using ACI can be reached with only limited future visibility, but the computation of the base-stock levels is very complicated.

In (Altug and Muharremoglu, 2011) advance supply information is available in terms of capacity forecasts provided by the supplier. They model the evolution of the capacity availability forecasts via the Martingale Method of Forecast Evolution, which was introduced by (Heath and Jackson, 1994), and show that state-dependent base-stock policies are optimal under the average cost criterion. They develop heuristics assuming a functional form for the relationship between the forecast vector and the state-dependent base-stock level. Different from (Jaksic *et al.*, 2011) where ACI is fixed once it is announced, in their model ACI for all the future periods is updated every period.

2.2. Rationing

In this section we review studies from the literature of inventory problems with multiple demand classes. To the best of our knowledge, an inventory problem with multi-class customers is first studied by (Veinott, 1965). (Veinott, 1965) introduced the concept of a critical-level rationing policy for a system with periodic review. Critical level policy suggests not to satisfy demand from a lower priority class, if on hand stock is positive but below a critical level, with the expectation of a higher priority demand. This policy was proven to be optimal by (Topkis, 1968) and (Kaplan, 1969) for lost sales and backorder cases, respectively.

In general, approaches such as admission control, revenue management, due-date quotation and rationing are used to solve inventory problems with multiple demand classes. Rationing is a well-studied problem. A summary of the rationing literature can be found in (Teunter and Klein Haneveld, 2008). They provide a list in terms of the following properties of the system: system type (make-to-stock production/queueing systems or inventory ordering systems), shortage treatment (backorder or lost sales), number of classes, time (discrete or continuous), and rationing policy (static critical level policies or dynamic rationing policies).

Rationing models can be classified according to how they treat capacity: unlimited, fixed and uncertain capacity. Unlimited capacity models assume positive lead times, which is another source of randomness for the availability. In the models with fixed capacity either make-to-order problems are studied or capacity is perishable. In this case capacity is rationed rather than the inventory. An example is (Kapuscinski and Tayur, 2007), where a production/inventory system with two customer classes and deterministic processing times are studied in a make-to-order setting. In this study the optimal policy is characterized for due-date quotation. In some other studies with fixed capacity, backorders are managed by giving incentives to wait, or equivalently, under revenue management, receiving extra for giving backorder clearing priority (e.g., (Gans and Savin, 2007), (Ding *et al.*, 2006). Most of the time these studies assume an ordering policy and a rationing policy.

Our work is more related to rationing models that consider uncertain capacity explicitly. Most of these models consider production/inventory problem as a queueing system. (Ha, 1997a) models the rationing problem with lost sales as an M/M/1 queue with several demand classes and shows that for Poisson demands and exponential production times, base-stock policy is the optimal policy for production and that the optimal allocation is characterized by monotone rationing levels. (Ha, 1997b) extends this model to the backordering case with two demand classes to show that the optimal production control and rationing policies can be characterized by a single monotone switching curve. With this policy it is always optimal to produce for class 1 backorders and when there are only class 2 backorders, it is optimal to produce for more inventory if the inventory level is below a certain level and to produce to satisfy class 2 backorders otherwise. As to the allocation policy, it is optimal to satisfy an incoming class 2 customer from on-hand inventory if the inventory level is above a certain level and to backorder for this customer otherwise.

(de Vericourt *et al.*, 2002) is the extension of (Ha, 1997b) to include multiple demand classes. Formulating the problem as a Markov decision process and by exploiting the nested structure between an n-class problem and a related $n-1$ class problem, (de Vericourt *et al.*, 2002) show that a base-stock type production policy and a critical level type allocation policy are optimal. With this allocation policy there are thresholds for each product such that it is optimal to satisfy the arriving demand from the on-hand stock if the stock level is above the threshold for that customer and to backorder otherwise.

(Ha, 1997c) considers the capacity rationing problem of a single server MTS queue with two products and backorders, and characterizes the optimal production for Poisson demands and exponential production times. (Ha, 2000) also considers an MTS system, but with several demand classes and lost sales, and shows that when processing time is not exponential, a single-state variable, which is the number of completed stages, can be used to completely capture the information of system state.

(Benjaafar and ElHafsi, 2006) study a model similar to the ones in (Ha, 1997a), and (de Vericourt *et al.*, 2002) , considers multiple components and multiple production facilities. They model the system as a Markov decision process and characterize the structure of an optimal policy under both the total expected discounted cost and the average cost criteria. They show that the optimal production policy for each component is a state-dependent base-stock production policy, and that optimal inventory allocation for each component is a rationing policy, where rationing levels for each component are dynamic and also nondecreasing in the inventory level of all other components.

2.3. Capacity Flexibility

Our work on flexible capacity is related to the studies in the capacity management literature. Capacity management problems have been studied at different levels of decision making. Research on tactical outsourcing has mostly been concentrated on the strategic questions related to price setting, capacity investing, and contract writing (Yang *et al.*, 2005). We first give some examples of papers that consider capacity investing, where outsourcing is a strategic decision leading to the ownership of the capacity. Then, we review some papers that analyze outsourcing decisions in a contract environment. Lastly we mention the papers that are closest to our work, that focus on the interactions between capacity planning and inventory/production decisions.

Some studies that deal with capacity investing are (Rocklin *et al.*, 1984), (Eberly and Van Mieghem, 1997) and (Angelus and Porteus, 2002). (Rocklin *et al.*, 1984) consider a make-to-order system where capacity can be reduced or increased at exogenously set unit prices. For a system where capacity is costly reversible, such as costly labor lay-offs, they give conditions under which the optimal capacity plan is a *target interval policy*. With the target interval policy it is optimal not to change the capacity as long as it is in the region defined by an interval. On the other hand, if the capacity is outside the interval, it is optimal to adjust the capacity to the nearest point on the boundary of the interval. In other words, if the initial capacity is below the lower target limit, then the aim is to bring the capacity up to that limit. And if the initial capacity is above the upper target limit, then the aim is to bring the capacity down to that limit. Otherwise, no capacity changes are made. (Eberly and Van Mieghem, 1997) generalize this result to multiple resources with linear or convex adjustment cost functions and concave operating profit functions for both finite and infinite planning horizons. (Angelus and Porteus, 2002) study optimal capacity and production planning for a make-to-stock system, where capacity can be reduced, as well as added. They consider both short life cycle products and products that can be carried over to future periods, incurring holding costs for the latter. In both cases a target interval policy is shown to be optimal. They characterize the target intervals for short life cycle products, assuming that demand first increases stochastically, then decreases.

In these studies capacity expansion and reduction are strategic decisions leading to the ownership of the capacity; there is either a salvage value or price for capacity reduction at the end of each planning period for unsold units/unused capacity. In our setting, we do not consider the ownership of the capacity, but consider the usage of the capacity temporarily. The reader is referred to (Van Mieghem, 2003) for a review of the literature on strategic capacity management under uncertainty.

In a contract setting environment (Kamien and Li, 1990) present conditions under which tactical outsourcing mechanisms should be carried out, and they show that such mechanisms have the effect of production smoothing. (Van Mieghem, 1999) uses a game-theoretical model to analyze outsourcing conditions for different types of contracts between a firm and its subcontractor. (Tan, 2004) analyzes an environment with a capacitated producer and a capacitated subcontractor. The availability of the subcontractor is subject to uncertainty; however, a level of availability is guaranteed by a contract. The producer decides how much to produce and how much to subcontract at a given time using a threshold-type policy that depends on the state of the inventory. Our work is different from this line of work in that we do not consider contracting issues, but we assume that there is a given contract (that sets the price of outsourcing option) with an exogenous supplier.

(Bradley, 2004) considers a continuous time inventory/production model to minimize the average cost. The inventory can be replenished through two possible resources, in-house production and a subcontractor, both with finite capacity. Assuming that the manufacturer uses a base-stock policy to control replenishment from the two sources, the authors propose to use a Brownian approximation of the optimal control problem for determining the fixed capacity level and optimal production quantities. They show through numerical studies the value of the outsourcing option. (Tan and Gershwin, 2004) study a similar continuous-time model with several subcontractors having different unit costs and capacities. We differ from these papers in two respects: (1) we consider a periodic review setting, and (2) we explicitly model capacity uncertainty.

(Yang *et al.*, 2005) consider a Markovian in-house production capacity, and the outsourcing option with setup cost. They show that the firm's optimal outsourcing policy is a capacity-dependent (s, S) policy. For the case of deterministic capacity they show that the optimal production policy is a modified base-stock policy. In their study they make the assumption that the outsourcing decision is made before the production decision and the outsourced amount is fully used, whereas in our setting the outsourcing and production decisions are simultaneous, enabling more flexible use of the outsourcing option.

(Tan and Alp, 2009) consider the usage of contingent capacity in terms of temporary workers. Assuming limited contingent and in-house capacity they show that the optimal operational policy, for any given fixed permanent capacity level, is of state-dependent order-up-to type. In a similar setting (Alp and Tan, 2008) also consider the permanent workforce size to be utilized through the planning horizon as a decision variable. Including fixed costs for both initiating production and for using contingent capacity, they provide the optimal solution to the single-period problem. For the multi-period problem they characterize the optimal policy for some special cases, and argue that even for a system where the regular capacity is deterministic and stationary, the form of the optimal policy is complicated. In (Pac *et al.*, 2009) this model is extended to include uncertainty in the contingent capacity received from external resources, where a certain number of workers can be guaranteed through contracts at a reservation cost. The decisions are the number of contracted contingent workers, the optimal level of permanent capacity, the number of workers to be hired, and the quantity of production in each period. (Pinker and Larson, 2003) consider a setting where holding inventory is not allowed and the absenteeism of regular workers is expected. The number of regular and contingent workers are fixed for the entire planning horizon, but the capacity is adjusted by using the labor force overtime (both permanent and contingent workers). (Mincsovics *et al.*, 2009) extend the model in (Alp and Tan, 2008) to include constant lead time associated with the acquisition of contingent capacity. They characterize the optimal policy for the operational decisions and the optimal permanent capacity level. They prove that the inventory, the pipeline contingent capacity, the contingent capacity to be ordered, and the permanent capacity are economic substitutes. The value of flexibility remains considerable even when the capacity acquisition lead time is relatively long.

3. A RATIONING MODEL IN THE EXISTENCE OF ADVANCE CAPACITY INFORMATION

In this chapter we study the rationing problem of a production/inventory system that serves customers from two different classes, which are distinguished by the penalty costs charged for unsatisfied demand. When rationing policy is used for allocation, the available inventory is allocated according to the customer type. Demand from a given customer class with lower penalty cost can be rejected even though the on hand inventory level is positive, with the expectation of future demand from a customer class with higher penalty cost. We assume that the capacity of the supplier is uncertain, but the supplier has agreed to provide advance capacity information for a certain number of future periods.

We give the details of our model and present the structural results in Section 3.1. The results of our numerical study and the managerial insights are given in Section 3.2. The proofs of our structural results are presented in Appendix A.

3.1. Model Formulation

We study a single product, periodic review model with uncertain capacity under the existence of multiple customer classes and Advance Capacity Information (ACI). We give details of our ACI framework in Section 3.1.1, then we present the dynamic programming formulation in Section 3.1.2, and provide structural results of the model in Section 3.1.3. Important notation is summarized in Table 3.1, we introduce the notation used as need arises.

3.1.1. Modeling Advance Capacity Information

Let K be the length of the capacity information horizon, that is, for a given period n, the capacity for the current period and the capacity realizations for the next $M = K - 1$ periods, are assumed to be known. In this paper we restrict K to be greater than or equal to one. The case $K = 0$, where the current period's capacity is also uncertain, can also be handled, but it changes the formulation of the problem and is omitted here. We assume that capacities in different periods are independent and identically distributed. Let $\underline{\mathfrak{C}}_n$ be the capacity information vector available at the beginning of period n:

$$\underline{\mathfrak{C}}_n = (C_n, C_{n+1}, \cdots, C_{n+M}),$$

where C_{n+i} is the capacity for period $n+i$, $i = 0, 1, .., M$. We assume that C_{n+i} is a realization of a random variable with distribution function $F_C(.)$. The ACI vector at the beginning of period $n+1$ is obtained from $\underline{\mathfrak{C}}_n$ as $\underline{\mathfrak{C}}_{n+1} = (C_{n+1}, C_{n+2}, \cdots, C_{n+M+1})$, where the last M entries of $\underline{\mathfrak{C}}_n$ constitute the first M entries of $\underline{\mathfrak{C}}_{n+1}$, and C_{n+M+1} is the new capacity availability information with distribution $F_C(.)$. In order to distinguish the capacity information of the current period from the ACI of future periods we write:

$$\underline{\mathfrak{C}}_n = (C_n, \mathfrak{C}_\mathfrak{n}),$$

where $\mathfrak{C}_\mathfrak{n} = (C_{n+1}, C_{n+2}, \cdots, C_{n+M})$. Then $\underline{\mathfrak{C}}_{n+1} = (\mathfrak{C}_n, C)$, where C is a random variable with distribution function $F_C(.)$.

We should note that for the sake of tractability, we assume that $F_C(.)$ is a stationary distribution and the capacity information that already became available does not change over time, that is, we assume perfect capacity information. We believe both of these variations can be handled with a similar approach presented here, at the expense of a more complicated notation and more invested derivations.

3.1.2. Development of Dynamic Programming Formulation

At the beginning of period n the decision maker observes the on hand inventory, I_n, and the capacity information, $\underline{\mathfrak{C}}_n$. Demand from customer class i in period n, D_n^i, is a stationary random variable, and $\mathfrak{D}_\mathfrak{n} = (D_n^1, D_n^2)$ is the demand vector. We drop the time subscript from notation whenever time period is clear from the context.

The following order of events take place in every period:

- After observing I_n and $\underline{\mathfrak{C}}_n$ an order of amount u_n is given (which is constrained by the available capacity), and the on hand inventory level is brought up to $y_n = I_n + u_n$. Lead time is assumed to be zero.
- Demand from both classes is observed.
- An amount of v_n^i is allocated to the customer from class i, with $i = 1, 2$.
- Unsatisfied demand is lost. Holding and penalty costs for lost sales are incurred at the end of the period.

Table 3.1. Model Parameters and Decision Variables.

C_n	capacity available at the beginning of period n.
D_n^i	random demand of customer class i, $i \in (1, 2)$.
d_n^i	realization of D_n^i.
$f_C(\cdot), F_C(\cdot)$	density and cumulative distribution function of capacity C.
$f_{D^i}(\cdot), F_{D^i}(\cdot)$	density and cumulative distribution function of demand D^i.
I_n	on hand inventory level at the beginning of period n.
y_n	inventory level after the order is received.
v_n^i	amount allocated to customer class i in period n.
p_i	penalty cost for not satisfying unit demand from customer class i.
h	holding cost.

We model the problem described above as a dynamic programming model. The objective is to minimize penalty and holding costs over a finite horizon of length N. We first model the allocation problem, given the available stock level and the demand realizations for the first and second class customers.

Let $L(y, \mathfrak{d}, \mathfrak{v})$ be the single period holding and backorder cost incurred at the end of a period when an allocation of v^1 and v^2 is made to customer classes 1 and 2 respectively (with $\mathfrak{v} = (v^1, v^2)$), after the realization of demand $\mathfrak{D}$, $\mathfrak{d} = (d^1, d^2)$, given the inventory level y:

$$L(y, \mathfrak{d}, \mathfrak{v}) = h(y - \sum_{i=1}^{2} v^i) + \sum_{i=1}^{2} p_i(d^i - v^i).$$

Let $J_n(I_n, \underline{\mathfrak{C}}_n)$ be the minimum cost of operating the system in periods $n, n+1, \cdots, N$, when the beginning inventory level is I_n and the ACI vector is $\underline{\mathfrak{C}}_n$. Then, the following model solves the allocation problem given y_n:

$$
\begin{aligned}
V_n(y_n, \mathfrak{d}_\mathfrak{n}, \underline{\mathfrak{C}}_n) = \quad & \min_{\mathbf{v_n^1, v_n^2}} \quad \{L(y_n, \mathfrak{d}_\mathfrak{n}, \mathfrak{v}_\mathfrak{n}) + E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]\} \\
& \text{s.t.} \quad 0 \le \sum_{i=1}^{2} v_n^i \le y_n \qquad (3.1) \\
& \qquad\;\; 0 \le v_n^i \le d_n^i \qquad \forall i \in \{1, 2\} \qquad (3.2) \\
& \qquad\;\; I_{n+1} = y_n - \sum_{i=1}^{2} v_n^i. \qquad (3.3)
\end{aligned}
$$

Equation 3.1 is the availability constraint. COnstraint (3.2) restricts the amount of allocation to a given class so that it does not exceed the demand of that class, and the constraint (3.3) specifies the stock level after the allocation. The subscripts on E show according to which random variables the expectations are taken.

Now, at the beginning of period n, the problem is to find the best order quantity given the available ACI. Then the minimum expected cost through periods $n, n+1, \cdots, N$, $J_n(I_n, \underline{\mathfrak{C}}_n)$ is defined as follows:

$$
J_n(I_n, \underline{\mathfrak{C}}_n) = \min_{\mathbf{I_n \le y_n \le I_n + C_n}} G_n(y_n, \mathfrak{C}_n),
$$

where

$$
G_n(y_n, \mathfrak{C}_n) = E_{\mathfrak{D}}[V_n(y_n, \mathfrak{D}_\mathfrak{n}, \underline{\mathfrak{C}}_n)].
$$

with $J_N(I_N, \underline{\mathfrak{C}}_N) = 0$.

3.1.3. Characterization of the Optimal Ordering Policy

In this section we first rewrite the allocation problem, then we present single-period and multiple-period problems and show that the optimal policy is a modified base-stock policy, where the base-stock level is dependent on the available capacity information. Proofs of our results are presented in Appendix A.

With the rationing policy, the demand of the first class customer is satisfied first as much as possible, but second class customer demand may not be satisfied, with the anticipation of future first class customer demand. We also assume that $p_1 \geq p_2 + h$, that is, not satisfying the second class customer and holding one unit of product for one period is less costly than not being able to satisfy the first class customer, and also $p_2 \geq h$.

3.1.3.1. A Reformulation of the Allocation Problem. In this section we reformulate the allocation policy described in Section 3.1.2. We show that when rationing is used, the allocation policy is actually equal to a policy, where we decide on the quantity left at the end of the current period (which is the initial inventory for the next period), which we denote as R_n. With the rationing policy, demand from customer class 1 is satisfied as much as possible, hence $v_n^1 = min(y_n, d_n^1)$. And $v_n^2 = y_n - min(y_n, d_n^1) - R_n$, where $y_n - min(y_n, d_n^1)$ is the quantity left after satisfying the first class customer. Here v_n^2 is expressed in terms of y_n, d_n^1, and R_n. With this approach the allocation problem is solved with only one decision variable, R_n, and only one constraint:

$$max(0, y_n - min(y_n, d_n^1) - d_n^2) \leq R_n \leq y_n - min(y_n, d_n^1). \tag{3.4}$$

The availability constraint (3.1) and the allocation constraint (3.2), are included in (3.4) in terms of R_n. The upper and lower bounds for R_n can be denoted as $R_n^L = max(0, y_n - min(y_n, d_n^1) - d_n^2)$ and $R_n^U = y_n - min(y_n, d_n^1)$.

We can now write the allocation problem as follows:

$$\begin{aligned} V_n(y_n, \mathfrak{d}_\mathfrak{n}, \underline{\mathfrak{C}}_n) = \quad & \min \quad \mathfrak{L}(y_n, \mathfrak{d}_\mathfrak{n}, R_n, \underline{\mathfrak{C}}_n) = hR_n + p_2(d_n^2 - (y_n - min(y_n, d_n^1) - R_n)) \\ & +p_1(d_n^1 - min(y_n, d_n^1)) + E_C[J_{n+1}(R_n, \underline{\mathfrak{C}}_{n+1})] \\ & \text{s.t.} \quad R_n^L \leq R_n \leq R_n^U. \end{aligned}$$

3.1.3.2. Single-period Problem. Since the cost associated with the end of the planning horizon are assumed to be zero there is no motivation to leave any inventory at the end of period N. The optimal decision is to satisfy all demand as long as possible. The single period objective function is:

$$\begin{aligned} G_N(y_N, \mathfrak{C}_N) = \quad & \int_{y_N}^{\infty} \int_0^{\infty} (p_1(d^1 - y_N) + p_2 d^2) dF_{D^2}(d^2) dF_{D^1}(d^1) \\ & + \int_0^{y_N} \int_0^{y_N - d^1} h(y_N - d^1 - d^2) dF_{D^2}(d^2) dF_{D^1}(d^1) \\ & + \int_0^{y_N} \int_{y_N - d^1}^{\infty} p_2(d^2 - (y_N - d^1)) dF_{D^2}(d^2) dF_{D^1}(d^1). \end{aligned} \tag{3.5}$$

The first term in Equation 3.5 is the penalty cost that occurs in case none of the customer classes can be satisfied fully, that is, when $y_N \leq d_N^1$. Second term is the holding cost which incurs when $d_N^1 + d_N^2 \leq y_N$, that is, when the sum of the demands for the two classes does not exceed the available amount. Third term is the penalty cost incurred when the inventory is enough to fully satisfy the first class customers but not the second class customers, that is, when $y_N \geq d_N^1$ and $d_N^2 \geq y_N - d_N^1$. In what follows, we state the optimal policy for the last period in the planning horizon, N. Since this follows from the convexity of the function $G_N(y_N, \mathfrak{C}_N)$, we state the result without proof.

Proposition 3.1. The optimal policy is a modified base-stock policy for the single-period problem. The optimal expected cost, $J_N(y_N, \underline{\mathfrak{C}}_N)$ is described as follows, where y_N^{opt} is the minimizer of $G_N(y_N, \mathfrak{C}_N)$:

$$J_N(I_N, \underline{\mathfrak{C}}_N) = \begin{cases} G_N(y_N^{opt}, \mathfrak{C}_N) & \text{if } y_N^{opt} - C_N \leq I_N \leq y_N^{opt} \\ G_N(I_N + C_N, \mathfrak{C}_N) & \text{if } I_N \leq y_N^{opt} - C_N \\ G_N(I_N, \mathfrak{C}_N) & \text{if } I_N \geq y_N^{opt}. \end{cases}$$

And the optimal inventory level for the one-period problem, y_N^*, is as follows:

$$y_N^*(\underline{\mathfrak{C}}_N) = \begin{cases} y_N^{opt} & \text{if } y_N^{opt} - C_N \leq I_N \leq y_N^{opt} \\ I_N + C_N & \text{if } I_N \leq y_N^{opt} - C_N \\ I_N & \text{if } I_N \geq y_N^{opt}. \end{cases}$$

<u>3.1.3.3. Multi-period Problem.</u> In this section we give the structural results for the multi-period problem.

Theorem 3.1. For the multi-period problem the following is true for period n:

- $J_n(I_n, \underline{\mathfrak{C}}_n)$ is convex in I_n for all $\underline{\mathfrak{C}}_n$, and $\partial E_C[J_n(I_n, \underline{\mathfrak{C}}_n)]/\partial I_n \geq -p_1$.
- Given y_n and $\mathfrak{d}_\mathfrak{n}$, $\mathfrak{L}(y_n, \mathfrak{d}_\mathfrak{n}, R_n, \underline{\mathfrak{C}}_n)$ is convex in R_n.
- $G_n(y_n, \mathfrak{C}_\mathfrak{n})$ is convex in y_n, hence the optimal ordering policy is a modified base-stock policy with $y_n^{opt}(\mathfrak{C}_n)$, minimizer of $G_n(y_n, \mathfrak{C}_\mathfrak{n})$. And the optimal expected cost at period n, $J_n(y_n, \underline{\mathfrak{C}}_n)$, is described as:

$$J_n(I_n, \underline{\mathfrak{C}}_n) = \begin{cases} G_n(y_n^{opt}(\mathfrak{C}_n), \mathfrak{C}_\mathfrak{n}) & \text{if } y_n^{opt}(\mathfrak{C}_n) - C_n \leq I_n \leq y_n^{opt}(\mathfrak{C}_n) \\ G_n(I_n + C_n, \mathfrak{C}_\mathfrak{n}) & \text{if } I_n \leq y_n^{opt}(\mathfrak{C}_n) - C_n \\ G_n(I_n, \mathfrak{C}_\mathfrak{n}) & \text{if } I_n \geq y_n^{opt}(\mathfrak{C}_n). \end{cases}$$

- The optimal inventory level for the multi-period problem, y_n^*, is described as:

$$y_n^* = \begin{cases} y_n^{opt}(\mathfrak{C}_n) & \text{if } y_n^{opt}(\mathfrak{C}_n) - C_n \leq I_n \leq y_n^{opt}(\mathfrak{C}_n) \\ I_n + C_n & \text{if } I_n \leq y_n^{opt}(\mathfrak{C}_n) - C_n \\ I_n & \text{if } I_n \geq y_n^{opt}(\mathfrak{C}_n). \end{cases}$$

Note that y_n^{opt} is a function of $\mathfrak{C}_n$, but not $\underline{\mathfrak{C}}_n$, since it is independent of C_n.

Remark 3.1. We have shown that in any period $\partial E_C[J_n(I_n, \underline{\mathfrak{C}}_n)]/\partial I_n \geq -p_1$. This can be interpreted as follows: by acquiring one more unit of inventory, the maximum possible decrease in cost can be achieved by satisfying one more demand from class 1, since $p_2 + h \leq p_1$, and the maximum possible decrease will be equal to $-p_1$.

Remark 3.2. We have shown the convexity of the allocation problem, given y_n and $\mathfrak{d}_{\mathfrak{n}}$, hence the optimal allocation variable, R_n^{opt}, is given by the first order condition

$$h + p_2 + \partial E_C[J_{n+1}(I_{n+1} = R_n, \underline{\mathfrak{C}}_{n+1})]/\partial R_n = 0,$$

which gives

$$\partial E_C[J_{n+1}(I_{n+1} = R_n, \underline{\mathfrak{C}}_{n+1})]/\partial R_n = -(h + p_2).$$

We have also shown that $\partial E_C[J_{n+1}(I_{n+1} = R_n, \underline{\mathfrak{C}}_{n+1})]/\partial R_n \geq -p_1$. Hence to be able to achieve the optimal value for R_n, the condition $-(h + p_2) \geq -p_1$ must be satisfied. This is equal to the condition that makes rationing meaningful, $h + p_2 \leq p_1$.

We next show that if capacity becomes larger in at least one of the periods, for which ACI is available, then y_n^{opt} does not increase. We also show that R_n^{opt} is also decreasing in the available information.

Proposition 3.2. Given two capacity vectors, $\mathfrak{C}_n$ and $\tilde{\mathfrak{C}}_n$ with $\tilde{\mathfrak{C}}_n \geq \mathfrak{C}_n$, that is, $\tilde{C}_{n+i} \geq C_{n+i}$ for $i = 1 \ldots k$, $\partial G_n(y_n, \mathfrak{C}_n)/\partial y_n \leq \partial G_n(y_n, \tilde{\mathfrak{C}}_n)/\partial y_n$ and hence $y_n^{opt}(\tilde{\mathfrak{C}}_n) \leq y_n^{opt}(\mathfrak{C}_n)$. Moreover R_n^{opt} is decreasing in $\mathfrak{C}_{n+1}$.

3.2. Numerical Analysis

We have conducted a numerical study to observe the effects of jointly using ACI and rationing, and also to identify important characteristics of the optimal ordering and allocation decisions that we are not able to obtain analytically. We first describe how the value of using ACI and rationing is evaluated in Section 3.2.1, then in Section 3.2.2 we present the experimental setting used in the numerical study. We present and discuss our numerical findings in Section 3.2.3.

3.2.1. Computational Environment: Measures to Evaluate

In this section we explain how the possible effects of jointly using ACI and rationing are evaluated. For the numerical computations we define four models. The properties of these models associated with capacity and rationing decisions are as follows:

- Model1: ACI is used in the ordering decision, but not in the rationing decision.
- Model2: No ACI is used in the ordering or rationing decisions.
- Model3: Allocation is done without rationing. ACI is used only when making the ordering decision.
- Model4: Allocation is done without rationing. No ACI is used.

We use these models to define measures to assess the benefits of using ACI and rationing in terms of inventory cost improvement, and to examine the effects of ACI and rationing on the fill rates for the first and second class customers, and also the rationed amount for the second class customers. These performance measures are defined in detail in Section 3.2.1.1 and Section 3.2.1.2.

We depict these models and the performance measures related to cost performance in Figure 3.1. In this figure the horizontal axis represents the degree of usage of ACI in ordering and rationing decisions. The vertical axis depict whether rationing is employed or not.

3.2.1.1. Value of using ACI and Rationing. We now define how we use the described models to evaluate the effects of using ACI and rationing.

The value of rationing, VOR, is defined as the percent cost reduction obtained (in the existence of ACI) by using rationing over the model with no rationing, Model3:

$$VOR = \frac{J_1^{Model3}(I_1, \underline{\mathfrak{C}}_1) - J_1^{opt}(I_1, \underline{\mathfrak{C}}_1)}{J_1^{Model3}(I_1, \underline{\mathfrak{C}}_1)},$$

where $J_1^{opt}(I_1, \underline{\mathfrak{C}}_1)$ is the optimal expected cost for the planning horizon, which was defined in Section 3.1.2.

The value of ACI, $VACI$, is defined as the percent cost reduction obtained by using ACI, which includes both value of using ACI in ordering decision and also in rationing decision:

$$VACI = \frac{J_1^{Model2}(I_1, \underline{\mathfrak{C}}_1) - J_1^{opt}(I_1, \underline{\mathfrak{C}}_1)}{J_1^{Model2}(I_1, \underline{\mathfrak{C}}_1)}.$$

In Model2 both order and allocation decisions are given without ACI. Hence, the difference between the optimal model and Model2 gives the value of ACI.

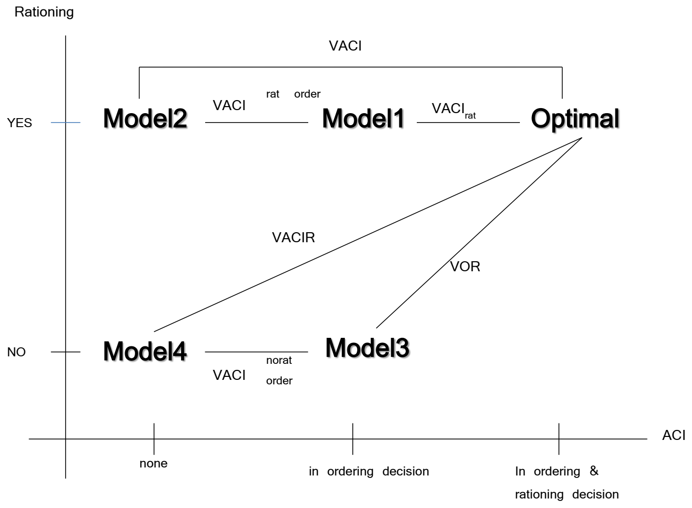

Figure 3.1. Measures to evaluate.

To distinguish between the value of ACI in the ordering and rationing decisions we define three other terms. First is the value of ACI in rationing:

$$VACI_{rat} = \frac{J_1^{Model1}(I_1, \underline{\mathfrak{C}_1}) - J_1^{opt}(I_1, \underline{\mathfrak{C}_1})}{J_1^{Model1}(I_1, \underline{\mathfrak{C}_1})}.$$

Here the optimal cost is compared to Model1 where rationing is done without using ACI.

Second is the value of using ACI in the ordering decision when there is rationing:

$$VACI_{order}^{rat} = \frac{J_1^{Model2}(I_1, \underline{\mathfrak{C}_1}) - J_1^{Model1}(I_1, \underline{\mathfrak{C}_1})}{J_1^{Model2}(I_1, \underline{\mathfrak{C}_1})}.$$

Here both models use rationing. $VACI_{order}^{rat}$ gives the percent cost reduction obtained by using Model1, where ACI is used only in the ordering decision, over Model2, where no ACI is used.

Third measure is similar to $VACI_{order}^{rat}$, except there is no rationing. It is the value of using ACI in the ordering decision when there is no rationing:

$$VACI_{order}^{norat} = \frac{J_1^{Model4}(I_1, \underline{\mathfrak{C}_1}) - J_1^{Model3}(I_1, \underline{\mathfrak{C}_1})}{J_1^{Model4}(I_1, \underline{\mathfrak{C}_1})}.$$

The combined value of using ACI and rationing at the same time can be captured, by comparing the optimal cost and the cost of Model4 where no ACI and no rationing is used:

$$VACIR = \frac{J_1^{Model4}(I_1, \underline{\mathfrak{C}_1}) - J_1^{opt}(I_1, \underline{\mathfrak{C}_1})}{J_1^{Model4}(I_1, \underline{\mathfrak{C}_1})}.$$

3.2.1.2. Fill Rate and Rationed Amount. In this section we present a fill rate measure and a rationing measure to see the success of the models in satisfying the first and second class customers. We use Monte Carlo simulation to calculate the rationing and fill rate measures.

With Monte Carlo simulation multiple trial runs are obtained using randomly generated capacity and demand. The fill rate for customer class i, β_i, is a measure of how much of the demand is satisfied, and it is defined as follows:

$$\beta_i \;=\; \frac{1}{N}\sum_{n=1}^{N}(v_n^i/d_n^i),$$

where N is the length of the planing horizon.

And the rationing measure, κ, can be defined when the on hand inventory is positive after satisfying the demand of class 1 customers (that is, when $y_n - v_n^1$ is positive) in any period n as:

$$\kappa \;=\; \frac{y_n - v_n^1 - v_n^2}{y_n - v_n^1}.$$

In reporting κ we choose $n = 1$, the first period in the planning horizon since the largest rationing is expected to happen at the beginning of the planning period with the aim of avoiding the penalty costs for not satisfying first class customer demand throughout the whole planning horizon.

3.2.2. Experimental Setting

For the numerical study we consider the combination of a set of input parameters for demands, (D^1, D^2), capacity for a given period, C, the ratio of the penalty costs for the two customer classes, p_1/p_2, and the holding cost, h, to see their effects on the value of ACI and rationing. The holding cost is fixed to one, the penalty cost for the second class customers, p_2, is fixed to five, and three different levels are considered for p_1 to vary p_1/p_2, $p_1 \in (15, 35, 45)$. And we consider three different levels for the following parameters: mean demand for customer class i, $\mu_i \in (3, 5, 7)$, mean capacity, $\mu_C \in (5, 7, 10)$, standard deviation of D^i, $\sigma_i \in (1, 2, 3)$, and standard deviation of C, $\sigma_C \in (3, 4.5, 9)$.

We choose discrete values for D^1, D^2 and C. Given the expected value and the variance, the demand and capacity distributions are generated by discretizing the normal distribution with the given expected value and variance, to obtain a 7-point distribution for demand, and 3-point distribution for capacity.

We generated 243 data sets using different parameter values. We compute the objective function value for a planning horizon of $N = 10$ periods, and an information horizon of $M = 2$ periods. Here we present only a subset of the numerical results we have obtained, other results are available upon request.

3.2.3. Analysis of the Effects of ACI and Rationing: Source of Improvement

We define the value of using ACI and/or rationing as the cost reduction obtained by implementing them. To compute the measures defining the value of ACI and rationing, the costs for a given model was calculated using the average of costs that result from different capacity realizations. We observe that the use of ACI can lead up to 36% cost reduction when allocation is done with rationing. On the average a reduction of 15% is obtained in 243 experiment settings. And by using rationing policy a reduction of 30% is possible, while the average reduction is 17%. The relative importance of the customer classes in terms of penalty costs also affects the results significantly, increasing the value of using ACI and rationing. A subset of the results for $VACI$ are presented in Table 3.2 for $p_1/p_2 = 35/5$. And Table 3.3 represents the results for VOR. In these tables the columns represents the changes as the capacity parameters change given a demand structure, and the rows reflect the change as demand parameters change.

We first discuss the relation of the different cost related measures defined to express the value of using rationing policy and ACI in Section 3.2.3.1. The analysis of observations regarding the behavior of the cost related measures, fill rates and rationed inventory amount as parameters change, is given in detail in Section 3.2.3.2, Section 3.2.3.3, and Section 3.2.3.4. We present some numerical results for the optimal solution in Section 3.2.3.5, and discuss the extent of ACI in Section 3.2.3.6.

<u>3.2.3.1. Comparing Cost Related Performance Measures.</u> In the numerical tests we observe some relations between the different measures defined in Section 3.2.1.1. We observe that the value of using advance information, $VACI$, can be roughly decomposed into two parts, using ACI in the rationing decision, $VACI_{rat}$, and using ACI in the ordering decision, $VACI^{rat}_{order}$:

$$VACI_{rat} + VACI^{rat}_{order} \cong VACI.$$

Table 3.2. VACI for $p_1/p_2 = 7$.

	(μ_1, σ_1)	(3, 1)	(3, 1)	(3, 1)	(3, 2)	(3, 2)	(5, 1)	(5, 1)	(5, 2)	(5, 2)
	(μ_2, σ_2)	(3, 1)	(5, 1)	(5, 2)	(5, 1)	(5, 2)	(3, 1)	(3, 2)	(3, 1)	(3, 2)
$\mu_C = 5$	$\sigma_C = 3$	24.68	16.37	15.36	14.62	13.54	15.57	15.42	12.43	12.13
	4.5	26.67	16.02	15.79	14.68	14.20	14.17	14.24	12.45	12.04
	9	22.39	14.90	14.90	14.20	14.11	12.52	12.65	10.73	10.78
$\mu_C = 7$	$\sigma_C = 3$	25.13	14.91	11.50	13.80	11.92	23.70	20.31	19.81	17.88
	4.5	30.71	21.50	19.32	18.88	17.03	24.89	23.37	19.88	18.78
	9	29.51	18.30	17.96	17.60	16.76	17.07	16.92	15.22	15.15
$\mu_C = 10$	$\sigma_C = 3$	8.48	8.49	2.22	4.22	3.89	18.15	9.18	11.41	9.27
	4.5	29.44	18.79	12.03	14.00	11.85	29.46	22.45	21.62	18.64
	9	34.61	26.92	24.81	23.85	21.91	27.32	26.08	23.02	21.88

Table 3.3. VOR for $p_1/p_2 = 7$.

	(μ_1, σ_1)	(3, 1)	(3, 1)	(3, 1)	(3, 2)	(3, 2)	(5, 1)	(5, 1)	(5, 2)	(5, 2)
	(μ_2, σ_2)	(3, 1)	(5, 1)	(5, 2)	(5, 1)	(5, 2)	(3, 1)	(3, 2)	(3, 1)	(3, 2)
$\mu_C = 5$	$\sigma_C = 3$	16.95	17.89	16.44	16.93	15.72	23.74	22.92	20.34	19.81
	4.5	21.76	22.80	21.67	20.62	19.57	24.19	23.70	20.87	20.59
	9	24.14	25.75	24.81	22.83	22.17	23.68	23.45	20.64	20.54
$\mu_C = 7$	$\sigma_C = 3$	5.81	9.51	6.95	9.39	8.20	22.28	19.23	17.83	16.06
	4.5	14.05	19.01	16.58	17.21	15.50	26.65	24.98	21.83	20.58
	9	22.61	25.54	24.26	22.74	21.59	25.88	25.29	22.08	21.79
$\mu_C = 10$	$\sigma_C = 3$	0.17	0.96	0.49	1.36	1.26	7.18	4.28	4.73	4.26
	4.5	3.40	9.19	5.91	7.48	6.54	21.50	15.33	14.42	13.08
	9	16.76	23.54	21.30	20.80	18.87	28.45	27.02	23.50	22.45

In Figure 3.2, $VACI$ is shown as the sum of $VACI_{rat}$ and $VACI_{order}^{rat}$ for a subset of the experiment settings. The sum is in some cases a little greater or smaller than one.

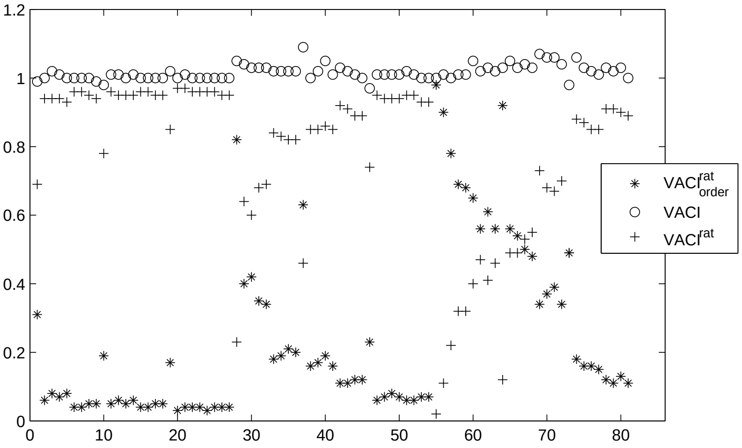

Figure 3.2. Composition of $VACI$.

A similar observation holds for $VACIR$. $VACIR$ can be roughly decomposed into the value of using rationing, VOR, and the value of using ACI in the ordering decision, $VACI_{order}^{rat}$:

$$VOR + VACI_{order}^{rat} \cong VACIR.$$

We also conclude that VOR is a combined measure of the value of using rationing policy and also the value of using ACI when rationing. Actually we observe that the maximum value $VACI_{rat}$ can take is equal to VOR, in a case where rationing is not done at all, which is a faulty allocation decision made in the absence of ACI.

Another observation is that the value of using ACI in ordering when there is no rationing is greater than when there is rationing:

$$VACI_{order}^{norat} > VACI_{order}^{rat}.$$

This can be explained as follows. When there is rationing there is a correcting action after the ordering decision, that is, if the inventory decision was incorrect for some reason (lack of information, for example), by a better allocation a better objective function can be obtained. Since in the case of rationing, a better objective function is obtained already (when compared to no rationing case), there is room for larger improvement by the use of ACI in the no rationing case.

3.2.3.2. Sensitivity Analysis. We first give some general observations with respect to the measures and then explain in detail the behavior of these measures as the model parameters change. Because of the decomposability properties defined in Section 3.2.3.1, we only give in detail the observations with respect to $VACI_{rat}$ and $VACI^{rat}_{order}$. Here are the main observations from the numerical experiments:

- The *tightness* plays an important role in determining the measures that define the value of ACI and rationing. We define *tightness* as a measure of the system's ability to satisfy demand, which is mainly a function of the utilization of the capacity and the variation of capacity. Tightness is of course also affected by the variation of demand. It is affected mainly by the first class demand since it is priority. We explain the behavior of value of ACI and rationing by tightness.
- The measures VOR, $VACI_{rat}$, $VACIR$ and $VACI$ all increase as p_1/p_2 increases. The increase is most significant in the values VOR, $VACI_{rat}$ and $VACIR$. But for $VACI^{rat}_{order}$ and $VACI^{norat}_{order}$ as tightness increases by the pressure of penalty cost, the order decision is less flexible and the value of having advance information in ordering decreases.
- We observe that when the tightness is too high (for example, when variation of capacity and μ_1/μ_C are high) ACI is not useful in terms of ordering, that is, $VACI^{rat}_{order}$ is low, since the plausible decision is to order as much as possible in that case, the ordering decision is not flexible. Whereas given the ordering decision, a correcting action can be taken by rationing inventory, and ACI can be more useful in the allocation decision (see Figure 3.2 for different cases where $VACI^{rat}_{order} \leq VACI_{rat}$ and vice versa).

  As the flexibility increases, by using ACI improvement in costs is possible. In that case there is room for improving the ordering decision and ACI can be more valuable. But the further decrease of tightness might not be useful, since then the ordering decision would be given myopically in that case. After a certain level as the tightness increases, $VACI_{rat}$ may also decrease since the action to be taken is not very flexible anymore, but it is to reserve inventory for the first class customers. Whenever tightness is very low it is plausible to satisfy all demand, and hence $VACI_{rat}$ is again low.

The effect of $p1/p2$ can be explained in a similar way, too. Whenever the ordering decision is not flexible (this time by the pressure of penalty cost) the allocation decision is more affected by ACI in terms of improving the expected costs, hence $VACI_{rat}$ becomes higher as $p1/p2$ increases.

Table 3.4. $VACI_{rat}$.

	(μ_1, σ_1)	(3, 1)	(3, 1)	(3, 1)	(3, 2)	(3, 2)	(5, 1)	(5, 1)	(5, 2)	(5, 2)
	(μ_2, σ_2)	(3, 1)	(5, 1)	(5, 2)	(5, 1)	(5, 2)	(3, 1)	(3, 2)	(3, 1)	(3, 2)
$\mu_C = 5$	$\sigma_C = 3$	18.57	15.34	14.20	13.59	12.43	14.74	14.52	11.46	11.20
	4.5	22.54	15.08	14.72	13.72	13.22	13.32	13.29	11.42	11.02
	9	18.55	14.24	14.06	13.40	13.29	11.76	11.82	9.92	9.92
$\mu_C = 7$	$\sigma_C = 3$	5.81	9.51	6.95	9.39	8.20	19.87	16.63	15.99	14.62
	4.5	14.05	18.66	16.47	15.77	14.49	22.34	20.96	17.42	16.51
	9	24.09	16.88	16.37	16.03	15.40	15.65	15.58	13.84	13.82
$\mu_C = 10$	$\sigma_C = 3$	0.17	0.96	0.49	1.36	1.26	7.18	4.28	4.73	4.26
	4.5	3.40	9.19	5.91	7.48	6.54	21.50	15.33	14.42	13.08
	9	21.50	23.14	21.79	20.74	19.17	24.47	23.66	20.20	19.63

We now present in detail a subset of the numerical results for $VACI_{rat}$ and $VACI^{rat}_{order}$.

Numerical results for $VACI_{rat}$: Table 3.4 represents a subset of the numerical results we obtained for $VACI_{rat}$ for $p_1/p_2 = 35/5$. We explain with numerical examples how the value of using information changes as model parameters change. We first give examples to show the effect of utilization on $VACI_{rat}$. As μ_C increases for a low value of variance, $\sigma_C = 3$, a decrease in $VACI_{rat}$ is observed for the demand scenarios corresponding to the low values of $\mu_1 = 3$. For these cases the capacity is not tight, μ_1/μ_C is low, and hence ACI does not add much value to the decision. For higher value of $\mu_1 = 5$, $VACI_{rat}$ first increases as μ_1/μ_C moves from 5/5 to 5/7, but when this further decreases from 5/7 to 5/10, a decrease in $VACI_{rat}$ is observed, which can be interpreted as follows: first increase of μ_C leads to better decision due to using ACI in terms of improving the costs. But the further increase of μ_C cannot result in a higher value of $VACI_{rat}$, since with $\mu_C = 10$ capacity is not tight anymore both type of customer demand can be satisfied.

The effect of μ_C also depends on σ_C. For a higher value of $\sigma_C = 9$, $VACI_{rat}$ increases with the increase of μ_C for all but $(\mu_1, \mu_2) = (3, 3)$ case. The increase is due to the increase of tightness which is caused by the high value of σ_C. And for the case of $(\mu_1, \mu_2) = (3, 3)$, tightness first increases as μ_C moves from 5 to 7, but it decreases as it further increases to 10, since then capacity is not as tight anymore.

Similar observations hold for the increase of σ_C. The change in $VACI_{rat}$ as σ_C alters can be summarized as follows: for $\mu_C = 5$, capacity is already tight, and the increase of σ_C (making capacity even tighter) leads to a decrease in $VACI_{rat}$. For $\mu_C = 7$, $VACI_{rat}$ increases as σ_C increases for the demand scenarios $(\mu_1, \mu_2, \sigma_1, \sigma_2) = (3, 3, 1, 1)$, $(\mu_1, \mu_2, \sigma_1, \sigma_2) = (3, 5, 2, 1)$ and $(\mu_1, \mu_2, \sigma_1, \sigma_2) = (3, 5, 2, 2)$. It first increases then decreases for the others. For a higher value of $\mu_C = 10$ we observe an increase for all values of mean demand, since capacity is not too tight, the increase of σ_C makes the available information more valuable.

We observe a decrease as σ_2 or σ_1 increases, in general. The decrease for σ_2 is smaller than for σ_1. Since the priority is to satisfy first class demand, tightness is firstly affected by the increase of σ_1.

Numerical results for $VACI^{rat}_{order}$: Table 3.5 represents a subset of the results obtained for $p_1/p_2 = 35/5$. As tightness decreases $VACI^{rat}_{order}$ increases up to a level, and as it decreases further, there is no more room for more improvement in costs, and $VACI^{rat}_{order}$ decreases. This was also the case for $VACI_{rat}$, but we can conclude that tightness affects $VACI^{rat}_{order}$ more in a way that for similar demand and capacity parameters the ordering decision is less flexible, and that a certain level of flexibility in the capacity decision is required to acquire cost improvement in the ordering decision. We first give some numerical examples to show the effect of utilization on $VACI^{rat}_{order}$. As μ_C increases, for $\sigma_C = 3$, $VACI^{rat}_{order}$ first increases as μ_C changes from 5 to 7, for the demand scenarios with low value of $\mu_1 = 3$ and then a decrease is observed as μ_C moves from 7 to 10. The first increase of μ_C increases $VACI^{rat}_{order}$, since capacity becomes less tight, but the further increase does not add any more value, since a certain level of flexibility is reached, demand is expected to be small and σ_C is also small. For the demand scenarios where $\mu_1 = 5$, and also for higher values of σ_C, $VACI^{rat}_{order}$ increases as μ_C increases, with the increase of tightness. As μ_1 increases, a decrease is observed for all capacity scenarios, but $(\mu_C, \sigma_C) = (10, 3)$, which is the most flexible case, with high μ_C and low cv_C. As μ_2 increases a decrease is observed for almost all capacity parameters, since the flexibility decreases as the utilization decreases further.

As σ_C increases, for $\mu_C = 7$ a decrease is observed for all demand scenarios, as tightness increases. And for $\mu_C = 10$, a decrease is observed for $(\mu_1, \mu_2, \sigma_1, \sigma_2) = (5, 3, 1, 1)$, while for other demand cases $VACI^{rat}_{order}$ first increases with the increase of σ_C from 3 to 4.5, then it decreases as tightness increases further.

3.2.3.3. Numerical Results for the Fill rates. We compare the fill rates for customer class 1 and 2, β_1 and β_2, for different models, to see the effects of using ACI and rationing. For customer class 1, we observe that, in general, the fill rate for the optimal model, β_1^{opt}, is greater than the fill rates for Model1 and Model2, β_1^{Model1} and β_1^{Model2}, and these are greater

Table 3.5. $VACI^{rat}_{order}$.

	(μ_1, σ_1)	(3, 1)	(3, 1)	(3, 1)	(3, 2)	(3, 2)	(5, 1)	(5, 1)	(5, 2)	(5, 2)
	(μ_2, σ_2)	(3, 1)	(5, 1)	(5, 2)	(5, 1)	(5, 2)	(3, 1)	(3, 2)	(3, 1)	(3, 2)
$\mu_C = 5$	$\sigma_C = 3$	7.51	1.21	1.36	1.18	1.27	0.97	1.06	1.09	1.04
	4.5	5.32	1.11	1.26	1.11	1.13	0.97	1.10	1.17	1.14
	9	4.71	0.78	0.98	0.92	0.95	0.86	0.94	0.91	0.96
$\mu_C = 7$	$\sigma_C = 3$	20.51	5.97	4.89	4.86	4.05	4.78	4.42	4.55	3.82
	4.5	19.38	3.50	3.41	3.69	2.97	3.29	3.04	2.97	2.71
	9	7.14	1.70	1.91	1.87	1.61	1.69	1.58	1.60	1.55
$\mu_C = 10$	$\sigma_C = 3$	8.32	7.60	1.74	2.90	2.66	11.83	5.12	7.02	5.23
	4.5	26.96	10.57	6.51	7.05	5.68	10.14	8.41	8.41	6.40
	9	16.71	4.92	3.86	3.92	3.39	3.78	3.17	3.54	2.80

than the fill rates for Model3 and Model4, β_1^{Model3} and β_1^{Model4}. β_1 for different models are closest to each other when cv_C is small. This is also true for β_2.

For β_2, in general, β_2^{Model1} and β_2^{Model2} are smaller than β_2^{Model3} and β_2^{Model4}. This is expected since in models β_2^{Model3} and β_2^{Model4} rationing is not used. And also β_2^{opt} is always smaller than when there is no rationing.

Table 3.6. Fill rate values obtained with different models: β_1.

	(μ_1, σ_1)	(3, 1)	(3, 1)	(3, 1)	(3, 2)	(3, 2)	(5, 1)	(5, 1)	(5, 2)	(5, 2)
	(μ_2, σ_2)	(3, 1)	(5, 1)	(5, 2)	(5, 1)	(5, 2)	(3, 1)	(3, 2)	(3, 1)	(3, 2)
$\mu_C = 5$	β_1^{opt}	0.95	0.95	0.95	0.85	0.85	0.89	0.89	0.88	0.878
$\sigma_C = 9$	β_1^{Model1}	0.94	0.94	0.94	0.84	0.84	0.89	0.89	0.88	0.88
	β_1^{Model2}	0.94	0.94	0.94	0.84	0.84	0.89	0.89	0.88	0.88
	β_1^{Model3}	0.9	0.84	0.84	0.75	0.75	0.78	0.79	0.78	0.78
	β_1^{Model4}	0.89	0.84	0.84	0.75	0.75	0.78	0.78	0.78	0.78
$\mu_C = 10$	β_1^{opt}	0.99	0.99	0.99	0.89	0.89	1	1	0.99	0.99
$\sigma_C = 3$	β_1^{Model1}	0.99	0.99	0.99	0.89	0.89	1	1	0.98	0.98
	β_1^{Model2}	0.99	0.99	0.99	0.89	0.89	1	1	0.98	0.98
	β_1^{Model3}	0.99	0.99	0.99	0.89	0.89	1	1	0.98	0.98
	β_1^{Model4}	0.99	0.99	0.99	0.89	0.89	1	1	0.98	0.98

The following observations are true for all models unless otherwise mentioned.

- As μ_C increases β_1 and β_2 both increase, the increase for β_2 is larger for all models other than Model3 and Model4. And as σ_C increases both β_1 and β_2 decrease. Again the decrease is larger for β_2 for all models other than Model3 and Model4. We can conclude that when there is a change in the capacity tightness, significant change can only be seen in β_2 when rationing is used. This is expected since it is more costly not to satisfy first class demand, and if there is any change in the availability the difference in β_2 is larger. A similar observation holds for the change in the penalty costs.
- As μ_1 increases there is a slight decrease for β_1, while the decrease can be very large for β_2, with the expectation of higher class 1 demand. And as μ_2 increases there is a slight

decrease for β_1, while the decrease can be very large for β_2. As σ_1 increases β_1 decreases and there is very small difference for β_2. Similarly as σ_2 increases β_1 alters slightly, while we can observe larger decrease for β_2.

These observations can be seen in Tables 4.3 and 3.7 for $p_1/p_2 = 35/5$.

Table 3.7. Fill rate values obtained with different models: β_2.

	(μ_1, σ_1)	(3, 1)	(3, 1)	(3, 1)	(3, 2)	(3, 2)	(5, 1)	(5, 1)	(5, 2)	(5, 2)
	(μ_2, σ_2)	(3, 1)	(5, 1)	(5, 2)	(5, 1)	(5, 2)	(3, 1)	(3, 2)	(3, 1)	(3, 2)
$\mu_C = 5$	β_2^{opt}	0.76	0.61	0.61	0.60	0.60	0.42	0.38	0.41	0.36
$\sigma_C = 3$	β_2^{Model1}	0.77	0.62	0.62	0.59	0.60	0.36	0.33	0.36	0.32
	β_2^{Model2}	0.77	0.62	0.62	0.59	0.60	0.36	0.33	0.36	0.32
	β_2^{Model3}	0.81	0.65	0.65	0.65	0.65	0.56	0.50	0.57	0.51
	β_2^{Model4}	0.80	0.65	0.65	0.65	0.65	0.56	0.50	0.58	0.51
$\mu_C = 10$	β_2^{opt}	0.94	0.91	0.90	0.90	0.88	0.86	0.78	0.85	0.74
$\sigma_C = 4.5$	β_2^{Model1}	0.94	0.91	0.91	0.91	0.89	0.89	0.79	0.88	0.77
	β_2^{Model2}	0.95	0.92	0.90	0.90	0.89	0.90	0.80	0.89	0.78
	β_2^{Model3}	0.95	0.92	0.90	0.91	0.89	0.90	0.80	0.89	0.78
	β_2^{Model4}	0.94	0.91	0.91	0.91	0.89	0.89	0.79	0.88	0.77

3.2.3.4. Numerical Results for the Rationed Amount. In the experiments we have conducted to see the effect of using ACI on the rationed amount, we observe that when tightness is low, for example, for $\mu_C = 10$ and $\sigma_C = 3$ or $\sigma_C = 4.5$, there is either no rationing or very small proportion of the available capacity is rationed (see Table 3.8). But when capacity is tight, for example, for $\mu_C = 5$ and $\sigma_C = 4.5$ or $\sigma_C = 9$, the rationed amount when ACI is not available, κ^{Model2}, is larger than the rationed amount for the optimal model where ACI is available, κ^{opt}. This shows that if ACI is available the allocation is better, such that, rationed amount (for the second class customer) is less, while keeping the fill rate for the first class customer high.

3.2.3.5. Behavior of y_n^{opt} and R_n. Some observations on the optimal allocation and ordering decisions are as follows: We have shown in Section 3.1.3.3, for a given period n, y_n^{opt}, decreases as $\mathfrak{C}_n$ increases, that is, if capacity is known to be larger in at least one of the periods for which capacity information is available, y_n^{opt} decreases. We have also observed that the decrease also depends on how much the capacity is increased, that is, the decrease in y_n^{opt} is limited by the total increase in the known future capacity. Moreover, the increase of the near future period's capacity is always more effective.

In all the experiments carried out we observe that the increase of the expected value of the capacity, μ_C, results in a decrease in y_n^{opt}, while y_n^{opt} increases as one of the following parameters increases, σ_C, σ_1, σ_2, μ_1, μ_2.

As the ratio p_1/p_2 increases y_n^{opt} increases, and y_n^{opt} is always higher when there is no rationing, as expected. R_n also increases as p_1/p_2 increases.

As μ_C increases R_n decreases, since capacity is less tight there is no need to hold excess

Table 3.8. Rationed amount for the optimal model and Model2.

	(μ_1, σ_1)	(3, 1)	(3, 1)	(3, 1)	(3, 2)	(3, 2)	(5, 1)	(5, 1)	(5, 2)	(5, 2)
	(μ_2, σ_2)	(3, 1)	(5, 1)	(5, 2)	(5, 1)	(5, 2)	(3, 1)	(3, 2)	(3, 1)	(3, 2)
$\mu_C = 5$										
$\sigma_C = 3$	κ^{Model2}	0.39	0.39	0.37	0.63	0.54	0.90	0.87	0.90	0.88
	κ^{opt}	0.39	0.46	0.44	0.54	0.49	0.80	0.76	0.83	0.81
4.5	κ^{Model2}	0.73	0.69	0.67	0.68	0.67	0.94	0.91	0.94	0.91
	κ^{opt}	0.50	0.57	0.56	0.66	0.64	0.86	0.83	0.86	0.84
9	κ^{Model2}	0.74	0.76	0.73	0.80	0.79	0.96	0.95	0.96	0.95
	κ^{opt}	0.62	0.68	0.67	0.71	0.70	0.89	0.87	0.91	0.89
$\mu_C = 10$										
$\sigma_C = 3$	κ^{Model2}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	κ^{opt}	0.00	0.00	0.00	0.03	0.03	0.04	0.04	0.14	0.13
4.5	κ^{Model2}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	κ^{opt}	0.02	0.06	0.05	0.15	0.14	0.20	0.17	0.35	0.34
9	κ^{Model2}	0.31	0.56	0.53	0.55	0.48	0.84	0.80	0.84	0.80
	κ^{opt}	0.26	0.39	0.35	0.51	0.49	0.64	0.61	0.73	0.69

inventory. And R_n increases as μ_1 or μ_2 increases, with a higher increase for μ_1. Actually we seldom observe positive increase in R_n as μ_2 increases for our parameter settings. This can be interpreted as follows: the first class customer demand has priority over the second class customers and whenever there is an increase in the first class demand y_n and R_n are increased at the same time not to take the risk of not being able to satisfy the first class demand. But, when only μ_2 is increased, only y_n might be increased to meet the larger demand, but R_n is not necessarily increased, since second class demand is not priority.

3.2.3.6. Extent of ACI. We have conducted another set of experiments to evaluate M, the number of future periods that information is available. We tested a set where $M = 1$, that is, only C_n and C_{n+1} are observed, and compared it with the case where $M = 2$, that is, C_n, C_{n+1} and C_{n+2} are observed. We have seen that when $M = 1$, the maximum cost reduction is 28% by using ACI, while it could reach a reduction of 36% for $M = 2$. And on the average 14% is observed for $M = 1$, while it is 18% for $M = 2$.

Having the extra information for period $n+2$ over the information for period $n+1$, has the least value when both the variation of capacity, cv_C, and also the utilization of capacity are low. In these cases, given that C_{n+1} is known, obtaining ACI for another period does not add much value in terms of reducing the expected costs. And it has the highest value when cv_C is not too low or too high, and the utilization is not too high. In that case there is some flexibility in both ordering and allocation decisions.

4. HEURISTIC POLICIES FOR ORDERING IN THE EXISTENCE OF ADVANCE CAPACITY INFORMATION

In this chapter we consider the ordering policies for an inventory system that faces stochastic demand under uncertain capacity. Under average cost criterion we construct heuristic methods to find ordering policies in the existence of ACI.

For an inventory/production system, ACI-dependent ordering policies was first introduced by (Jaksic *et al.*, 2011). In (Jaksic *et al.*, 2011), the authors show that for a finite horizon problem the optimal ordering policy is a state-dependent modified base-stock policy. They consider an ACI structure where the ACI for a number of future periods is fixed after it has been observed, as we assume in our model. They show that most benefits of using ACI can be reached with only limited future visibility, but the computation of the base-stock levels is very complicated.

The general dynamic programming model for the ordering problem for a single product under the existence of ACI can be written as follows:

$$J_n(I_n, \underline{\mathfrak{C}}_n) = \min_{\mathbf{I_n \leq y_n \leq I_n + C_n}} E_D\left[L(y_n) + E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]\right],$$

where $J_n(I_n, \underline{\mathfrak{C}}_n)$ is the minimum cost of operating the system in periods $n, n+1, \cdots, N$, whenever the beginning inventory level is I_n, the observed capacity is given by $\underline{\mathfrak{C}}_n$, and the inventory level is raised to y_n after the order arrives. The unsatisfied demand is backlogged. $L(y_n)$ is the cost for one period, in terms of backorder and holding costs, $L(y_n) = h(y_n - D_n)^+ + b(D_n - y_n)^+$, where $(a)^+ = \max(a, 0)$. Here the capacity information vector available at the beginning of period n consists of the observed realized capacity for periods $n, \cdots, n+M$, when ACI is available for M future periods:

$$\underline{\mathfrak{C}}_n = (C_n, \cdots, C_{n+M}).$$

In (Altug and Muharremoglu, 2011) advance supply information is available in terms of capacity forecasts provided by the supplier. They model the evolution of the capacity availability forecasts via the Martingale Method of Forecast Evolution, which was introduced by (Heath and Jackson, 1994), and show that state-dependent base-stock policies are optimal under the average cost criterion. The average cost can be defined as follows:

$$\lim_{\mathbf{N}\to\infty} \quad \frac{1}{N} J_1(I_1, \underline{\mathfrak{C}}_1).$$

Our purpose here is to introduce heuristic approaches to find the ordering decisions for an inventory system that observes ACI for a number of future periods. Specifically we propose to use ACI-dependent order-up-to levels to minimize the expected average cost. We evaluate the heuristic methods we propose via simulation and discuss their efficiency in reducing costs. The heuristics that we consider are presented in Section 4.1. Numerical observations are given in Section 4.2.

4.1. Heuristics

4.1.1. Heuristics Based on Asymptotic Behavior

In this section we propose to use a base-stock level for a given period n, dependent on the available capacity information, in the following form,

$$BS_n = s - \sum_{i=1}^{M} \beta_i^s C_{n+i}, \tag{4.1}$$

for some $0 \leq \beta_i^s \leq 1, \quad i \in (1, \cdots, M)$. Here M is the information horizon length and s is the constant term. For this dynamic policy where the information on capacity is updated every period, β_i^s is chosen to be a decreasing sequence, giving more weight to the nearer future. Here we choose $\beta_i^s = \beta^i$.

In this section we present two heuristics based on asymptotic results. First one is based on the logarithmic asymptotic for the steady-state waiting-time distribution for a single-server queue with unlimited waiting space and first-in first-out service discipline (without any explicit independence conditions on the interarrival and service times). The second one is based on a heavy traffic asymptotic approximation for the tail distribution of a reflected random walk with independent step sizes.

With the given base-stock policy, ignoring the possible overshoots (that is, the cases where on hand inventory exceeds the target order-up-to level for the given period), we now define the shortfall of the inventory level, Z_n. Z_n is the difference between the inventory level at the beginning of a period after the order for that period is received, y_n, and the base stock level, BS_n:

$$Z_n = BS_n - y_n$$

with $y_n = min(I_n + C_n, BS_n)$. Hence $Z_n = BS_n - min(I_n + C_n, BS_n)$. Using the fact that the inventory level at the beginning of period n is $I_n = y_{n-1} - D_{n-1}$, Z_n is rewritten as:

$$Z_n = max(0, BS_n - y_{n-1} + D_{n-1} - C_n)$$

By the definition of BS_n and BS_{n-1} we can write $BS_n = BS_{n-1} + \sum_{i=1}^{M} \beta^i C_{n-1+i} - \sum_{i=1}^{M} \beta^i C_{n+i}$. Replacing this in the last equation we obtain:

$$Z_n = max(0, Z_{n-1} + D_n + \sum_{i=1}^{M} \beta^i C_{n-1+i} - \sum_{i=1}^{M} \beta^i C_{n+i} - C_n)$$

From this equality we can conclude that Z_n is a reflected random walk with step size $X_n = D_n + \sum_{i=1}^{M} \beta^i C_{n-1+i} - \sum_{i=1}^{M} \beta^i C_{n+i} - C_n$. Note that the step sizes are dependent if $\beta \neq 1$.

Some inventory related examples for the model we have defined are as follows:
Example 1: Constant known capacity model, $C_{n+i} = c$, and $M = \infty$. Then $X_n = D_n - c$.

This constant capacity case is discussed in (Glasserman, 1997).

Example 2: $\beta = 1$, which results in i.i.d. step size, which is discussed in (Altug and Muharremoglu, 2011) for the case where evolution of the capacity availability is modeled via Martingale Method of Forecast Evolution.

Example 3: No ACI is available, $\beta = 0$, and $M = 0$. Then $X_n = D_n - C_n$, which is also discussed in (Glasserman, 1997).

Our objective is to minimize the average inventory related cost of the system that works under the proposed ACI-dependent ordering policy.

From the definition of the shortfall, $Z_n = BS_n - y_n$, we can write y_n as $y_n = BS_n - Z_n$. Hence, $y_n - D_n = BS_n - Z_n - D_n = s - \sum_{i=1}^{M} \beta^i C_{n+i} - Z_n - D_n$, where the last equation follows from the definition of the proposed base-stock level, BS_n. Now, since $y_n - D_n = I_{n+1}$ we can conclude that:

$$I_{n+1} = s - \sum_{i=1}^{M} \beta^i C_{n+i} - Z_n - D_n.$$

Let $A_n = \sum_{i=1}^{M} \beta^i C_{n-1+i} + Z_{n-1} + D_{n-1}$, and A_∞ be the steady-state random variable that A_n converges to, that is, $A_\infty = \lim_{n\to\infty} A_n$, provided that the limit exists.

Proposition 4.1. Under the ACI-dependent base-stock policy, $BS_n = s - \sum_{i=1}^{M} \beta^i C_{n+i}$, the optimal level of the constant term is given by the equation $s^* = F_{A_\infty}^{-1}(b/(b+h))$, where $F_{A_\infty}^{-1}$ is the inverse cumulative distribution function of the random variable A_∞.

Proof. We can write the average cost of this system as follows:

$$CO_\infty = hE[(y_\infty - D)^+] + bE[(D - y_\infty)^+],$$

where first term is the average holding cost and the second term is the average back-order cost. Let $I_\infty = y_\infty - D$. This cost term is actually equal to $CO_\infty = hE[I_\infty^+] + bE[-I_\infty^+]$. From the definition of A_∞, we can write I_∞ as follows:

$$I_\infty = s - \sum_{i=1}^{M} \beta^i C_i - Z_\infty - D = s - A_\infty.$$

Then the average cost function is $CO_\infty = h \int_0^s (s-x) dF_{A_\infty}(x) + b \int_s^\infty (x-s) dF_{A_\infty}(x)$. Taking the derivative with respect to s and equating it to zero we obtain $s^* = F_{A_\infty}^{-1}(b/(b+h))$. $\square$

To find s^* we first need the distribution of Z_∞. For this we further analyze the recursion $Z_n = max(0, Z_{n-1} + X_n)$. $Z_n = max(0, Z_{n-1} + X_n)$ is also the equation for the waiting time of customers of successive customers in a G/G/1 queue. The notation G/G/1 denotes a single-server queue with unlimited waiting space and the first-in first-out service discipline, with a general distribution of the sequences of inter-arrival and service times. Let W_n be the waiting time of the n^{th} customer. Then, for $n \geq 0$, the sequence W_n satisfies Lindley's equation $W_n = max(0, W_{n-1} + ST_{n-1} - IA_{n-1})$. Here IA_n is the random variable denoting the time between the arrivals of n^{th} and $n + 1^{st}$ customers, and ST_n is the random variable denoting the service time of the n^{th} customer.

Let W_∞ be the stationary distribution that W_n converges to. If the condition $E[ST_n - IA_n] < 0$ is satisfied then there exists a unique stationary distribution of the waiting time (Loynes, 1962). Note that the difference $ST_n - IA_n$ is the equivalent of the step size X_n. For the base-stock structure proposed in (4.1), the step-size X_n is restated here:

$$X_n = D_n + \sum_{i=1}^{M} \beta^i C_{n-1+i} - \sum_{i=1}^{M} \beta^i C_{n+i} - C_n.$$

This is equal to $D_n - \beta^M C_{n+M} + \sum_{i=1}^{M} (\beta^i - \beta^{i-1}) C_{n+i-1}$, and its expectation is equal to $\mu_D - \mu_C$ which is negative, since we assume $\mu_D < \mu_C$. Hence we can conclude that there exists a unique random variable Z_∞.

Computing the steady-state distribution for a G/G/1 queue is a computationally challenging problem. To find the distribution of Z_∞ we propose two different approaches based on asymptotics for tail distributions.

4.1.1.1. Logarithmic Small-tail Asymptotic. (Glynn and Whitt, 1994) find conditions for the steady-state waiting-time distribution, for a G/G/1 queue, to have small-tail asymptotics of the form $x^{-1} log P(W_\infty > x) \to -\eta*$ as $x \to \infty$ for some positive constant η^*. They show that under stationarity of the sequence of service times minus interarrival times, $ST_n - IA_n$ (which is the equivalent of the step size X_n), the necessary conditions are as follows:

- there exists a function ψ and positive constant η^* such that $n^{-1} log E[e^{\eta K_n}] \to \psi(\eta)$ as $n \to \infty$, for a neighborhood of η^*,
- $E[e^{K_n}] < \infty$ for $n \geq 1$,
- ψ is finite in a neighborhood η^* and differentiable at η^* with $\psi(\eta^*) = 0$ and $\psi'(\eta^*) > 0$.

Here K_n is the sum of step sizes, $K_n = \sum_{i=0}^{n} X_i$. These are also sufficient conditions for the existence of a stationary distribution for the waiting times.

In the special case of normally distributed K_n with mean μ_{K_n} and variance $\sigma_{K_n}^2$, the moment generating function is given as follows:

$$E[e^{\eta K_n}] = e^{\eta \mu_{K_n} + (1/2)\eta^2 \sigma_{K_n}^2}.$$

(Glynn and Whitt, 1994) show that, for the special case of normally distributed K_n, if K_n has negative mean μ_{K_n} and finite variance $\sigma_{K_n}^2$ for all $n \geq 1$, and $\mu_{K_n}/n \to \mu$ and $\sigma_{K_n}^2/n \to \sigma$ as $n \to \infty$ where $\mu < 0 < \sigma$, then $\psi(\eta) = \eta\mu + \eta^2\sigma^2/2$. The asymptotic decay rate, η^*, is the solution to the equation $\psi(\eta) = 0$, hence $\eta^* = -2\mu/\sigma^2$.

Let μ_D and σ_D^2 be the mean and variance of demand respectively, and μ_C and σ_C^2 be the mean and variance of capacity for any period.

Proposition 4.2. For the ACI-dependent base-stock policy with $BS_n = s - \sum_{i=1}^{M} \beta^i C_{n+i}$, for normally distributed demand and capacity, $\eta^* = -2(\mu_D - \mu_C)/(\sigma_D^2 + \sigma_C^2)$.

Proof. $K_n = \sum_{i=0}^{n} X_i$ can be written explicitly for $M = 2$ as follows:

$$K_n = \sum_{i=0}^{n} X_i = \sum_{i=1}^{n} D_i - \sum_{i=3}^{n} C_i + (\beta - 1)C_1 + (\beta^2 - 1)C_2 - \beta C_{n+1} - \beta^2 C_{n+2}.$$

K_n is normally distributed if D_n and C_n are normally distributed, and the mean and variance of K_n are given as follows,

$$\begin{aligned} \mu_{K_n} &= n\mu_D - (n-2)\mu_C + (\beta - 1)\mu_C + (\beta^2 - 1)\mu_C - \beta\mu_C - \beta^2\mu_C = n\mu_D - n\mu_C \\ \sigma^2_{K_n} &= n\sigma^2_D + (n-2)\sigma^2_C + (\beta - 1)^2\sigma^2_C + (\beta^2 - 1)^2\sigma^2_C + \beta^2\sigma^2_C + \beta^4\sigma^2_C. \end{aligned}$$

$\mu_{K_n}/n \to \mu = \mu_D - \mu_C$, and $\sigma^2_{K_n}/n \to \sigma^2 = \sigma^2_D + \sigma^2_C$ since as $n \to \infty$:

$$\sum_{i=0}^{n} X_i \to \sum_{i=1}^{\infty} D_i - \sum_{i=3}^{\infty} C_i + (\beta - 1)C_1 + (\beta^2 - 1)C_2.$$

We have defined K_n for $M = 2$ for normally distributed demand and capacity. Similar result holds for $M \geq 2$. □

From Proposition 4.2., we know for the distribution of Z_∞ that the following is satisfied, $z^{-1} log P(Z_\infty > z) \to -\eta*$ as $z \to \infty$ with $\eta^* = -2(\mu_D - \mu_C)/(\sigma^2_D + \sigma^2_C)$. Using this we generate values for Z_∞ and then we simulate the behavior of A_∞ using the generated values for Z_∞. Let s_g be the constant term in the base-stock level for this policy. Then s_g is found from the equation $s_g = F^{-1}_{A_\infty}(b/(b+h))$.

4.1.1.2. Heavy Traffic Asymptotic Approximation. The approximation considered here is based on the heavy traffic approximation for the tail of a reflected random walk with normally distributed step size introduced by (Siegmund, 1979). It has been used in the inventory literature in the works by (Glasserman, 1997), (Toktay and Wein, 2001) and (Altug and Muharremoglu, 2011).

For a reflected random walk Z_n with independent normally distributed step size X_n, (Siegmund, 1979) provides the following approximation as $z \to \infty$ and $\mu_D \to \mu_C$:

$$Pr\{Z_\infty > z\} = e^{-\vartheta(z+\lambda)},$$

with $\lambda = 0.583\sqrt{\sigma_D^2 + \sigma_C^2}$ and $\vartheta = 2(\mu_C - \mu_D)/(\sigma_D^2 + \sigma_C^2)$.

Using this approximation, and also the fact that in the sum of an exponential and normal random variables the exponential term dominates as $z \to \infty$, (Altug and Muharremoglu, 2011) gives an approximation for the limiting behavior of A_∞ as follows:

$$lim_{z\to\infty} Pr\{A_\infty > z\} = e^{-\vartheta(z+\lambda-\varphi)},$$

where $\varphi = M\mu_D + (1/2)\vartheta(M\sigma_D^2 + \sigma_C^2)$.

Assuming normally distributed demand and capacity, the step size X_n is normally distributed for our model. We propose to use the following approximation for the constant term, s, in the base-stock level:

$$s_a = (1/\vartheta)ln(1 + b/h) + \varphi - \lambda \tag{4.2}$$

with

$$\begin{aligned} \lambda &= 0.583\sqrt{\sigma_D^2 + \sigma_{eff}^2} \\ \vartheta &= 2(\mu_C - \mu_D)/(\sigma_D^2 + \sigma_{eff}^2) \\ \varphi &= M\mu_D + (1/2)\vartheta(M\sigma_D^2 + \sigma_{eff}^2). \end{aligned}$$

We use σ_{eff}^2 instead of σ_C^2 in the approximation defined in (Altug and Muharremoglu, 2011). σ_{eff}^2 is the variance of $\sum_{i=1}^{M} \beta^i C_{n-1+i} - \sum_{i=1}^{M} \beta^i C_{n+i} - C_n$, which is the part of the step size, X_n, associated with the capacity:

$$\sigma_{eff}^2 = \sigma_C^2\Bigg((\beta - 1)^2 + \sum_{i=1}^{M-1}(\beta^{i+1} - \beta^i)^2 + \beta^{2M}\Bigg).$$

The above mentioned tail approximation is developed for i.i.d. distributed step size for a reflected random walk. However, in our case X_n is not an i.i.d. sequence, hence s_a in 4.2 is a further approximation.

4.1.2. Weighted Cost Heuristic

In this section we present a heuristic that considers weighing the costs of the planning periods for which the capacity information is available plus a cost term associated with the periods for which the capacity information is not available. For period n, an ACI-dependent base-stock level, S_w, is found by minimizing the following expected cost:

$$\begin{aligned} WC(S) = \quad & \omega_0 E_{D_n}[L(S)] + \omega_{M+1} CA(S) \\ & + \sum_{j=1}^{M} \omega_j E_{D(n,n+j)}\Bigg[L\Bigg(S - D(n, n+j-1) + \rho C(n+1, n+j)\Bigg)\Bigg], \end{aligned}$$

where

- $\omega_0, \omega_1, \cdots, \omega_M$ are the weights associated with the current period and the next j periods respectively, for $j \in (1, \cdots M)$,
- ω_{M+1} is the weight associated with the cost term for which ACI is not observed,
- $D(n, n+j)$ denotes the sum of random demands over periods $n, \cdots, n+j$, with $j \in (1, \cdots, M)$,
- $C(n, n+j)$ is the sum of known capacities for periods $n, \cdots, n+j$, with $j \in (1, \cdots, M)$,
- ρ is the utilization of capacity, $\rho = \mu_D/\mu_C$,
- $L(y) = h(y - X)^+ + b(X - y)^+$ for a given random variable X and inventory level y,
- $CA(S) = E_{D+Z}[L(S)]$ is the expected average cost associated with the base-stock level S, for an inventory system where no information on capacity is available (with indepen-

dently and identically distributed demand and capacity):

$$CA(S) = \int_0^S h(S-u)d(F_D * F_Z)(u) + \int_S^\infty b(u-S)d(F_D * F_Z)(u),$$

where Z is the steady-state shortfall, and $F_D * F_Z$ is the convolution of demand and shortfall distributions for the aforementioned inventory system (Gullu, 1998).

$WC(S)$ is a weighted cost when the starting inventory level is S, and an amount of ρC_{n+j} is ordered for the periods for which ACI is available, $j \in (1, \cdots, M)$. The ordering level for these periods is proportional to the utilization level, ρ. The first term in $WC(S)$ is the cost of period n weighted with ω_0, second term is the sum costs for periods $n+1, \cdots, n+M$, weighted with $\omega_1, \cdots, \omega_M$, respectively. The last term is the average cost of a system for which ACI is not available, weighted with ω_{M+1}.

The weights are chosen such that as the utilization, ρ, decreases the current period's weight, ω_0, becomes larger, and ω_{M+1} increases as M decreases, to take into account the uncertainty as the ACI horizon is smaller. The following set of weights are used for this heuristic:

$$\begin{aligned}
\omega_0 &= \frac{M}{1+M}\frac{1-\rho}{1-\rho^{M+1}} \\
\omega_j &= \omega_0\rho^j \qquad j \in (1, \cdots, M) \\
\omega_{M+1} &= \frac{1}{1+M}.
\end{aligned}$$

4.2. Numerical Analysis

We run simulation tests to see the efficiency of the methods defined in Section 4.1.1 and Section 4.1.2 in reducing the inventory related costs. Average costs are calculated for different values of M, the number of future periods for which ACI is available. To evaluate the value of ACI or gain obtained by using different methods we define δ, the percent change in costs when comparing two different methods.

When $BS_n = s - \sum_{i=1}^{M} \beta^i C_{n+i}$ is used as a base-stock policy, for a given value of s, we compute the resulting cost as follows, we run simulation tests for different choices of β values with $0 \leq \beta \leq 1$, and choose the β value that gives the minimum average cost. We call this general structure β-adjusted base-stock policy. We compare different policies with each other again via simulation.

To compare the cost of the proposed heuristics that uses ACI with the cost of a system for which no ACI is available we use the fixed base-stock level, S_m, that minimizes the cost function $CA(S)$, defined in Section 4.1.2. We call this this inventory policy no-ACI policy.

4.2.1. Experimental Setting

To see the value of using ACI for the proposed heuristics we run simulation tests for a set of input parameters. We set the mean capacity, μ_C, to 20 and vary the demand mean, μ_D, to test for different utilization levels, $\rho = \mu_D/\mu_C$. Same way we set the holding cost, h, to 1 and choose different levels of penalty cost, b, to vary b/h. Six different levels of coefficient of variation are chosen for both demand and capacity, cv_D and cv_C, respectively, to test the effects of demand and capacity variation. Normally distributed demand and capacity are used in the tests. The parameter setting used in the numerical experiments are summarized in Table 4.1.

Table 4.1. Values for input parameters.

h	1
b	$10, 15, 20, 40$
μ_D	$16, 17, 18, 19$
cv_D	$0.1, 0.3, 0.5, 0.8, 1, 1.2$
μ_C	20
cv_C	$0.1, 0.2, 0.5, 0.75, 1, 1.5$
β	$0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$

4.2.2. Numerical Results

Here we present our numerical findings. We compare the average costs resulting from the no-ACI policy, the weighted cost heuristic that uses S_w, β-adjusted base-stock policy with s_a and s_g, and also the case where β is set to 1.

Our observations from the simulation runs are given below:

- From our observations we conclude that, for $M = 2$ periods of ACI, there is no significant difference between the aforementioned methods for the cases where the utilization is high, especially when variance of demand and capacity are also high, such as the cases where $\rho \in (0.9, 0.95)$, $cv_C \in (0.75, 1, 1.5)$ or $cv_D \in (0.8, 1, 1.2)$. We run tests to see if the different models considered resulted in different average costs. A subset of the obtained P-values can be seen in Table 4.2. From this table we observe that for the small parameters, the P-value is small. But the increase of one of the parameters may increase the P-value, and for the high P-values (for $P \geq \alpha$) we cannot reject the hypothesis that different methods give similar results, for example, at a level of significance of $\alpha = 0.5$.

Table 4.2. P-values for $M = 2$.

	$(\mu_D, \sigma_D = 18, 5.4)$			$(\mu_D, \sigma_D = 19, 5.7)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	0.00	0.00	0.00	0.00	0.48	0.94
15	0.00	0.00	0.02	0.00	0.56	0.92
20	0.00	0.00	0.08	0.03	0.90	1.00
40	0.00	0.00	0.20	0.18	0.78	0.99
	$(\mu_D, \sigma_D = 18, 9.5)$			$(\mu_D, \sigma_D = 19, 9.5)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	0.00	0.00	0.18	0.79	0.76	0.92
15	0.00	0.00	0.42	0.75	0.99	1.00
20	0.00	0.07	0.73	0.69	0.99	0.99
40	0.30	0.13	0.79	0.84	0.99	0.98

- We compare the average cost associated with the β-adjusted base-stock policy with s_a, defined in Section 4.1.1, to the average cost associated with the policy where $\beta = 1$. We observe significant cost reductions for the parameter settings, where statistically significant differences were observed. For the cases where the utilization and variance of demand and capacity are high, there is not significant difference between the two methods.

We define the percent change in the cost when β-adjusted base-stock policy is used instead of $\beta = 1$ as follows:

$$\delta = 100\frac{\widetilde{J}(s, \mathfrak{C}) - \widetilde{J}(s_a, \mathfrak{C})}{\widetilde{J}(s, \mathfrak{C})},$$

where $\widetilde{J}(s_a, \mathfrak{C})$ is the average cost for the β-adjusted base-stock policy that uses s_a and $\widetilde{J}(s, \mathfrak{C})$ is the average cost for $\beta = 1$. A subset of the results for δ are presented in Table 4.3 for $\rho = 0.8$ and $\rho = 0.85$, $cv_C \in (0.1, 0.2, 0.5)$, and $cv_D = 0.5$ for different values of b for $M = 2$. We also give the best β values that resulted in the minimum cost for the β-adjusted base-stock policy in Table 4.4.

Table 4.3. Percent change in costs: comparing $\widetilde{J}(s_a, \mathfrak{C})$ and $\widetilde{J}(s, \mathfrak{C})$ for $M = 2$.

	$(\mu_D, \sigma_D = 16, 8)$			$(\mu_D, \sigma_D = 17, 8.5)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
b=10	55.65	62.97	59.96	37.66	43.13	27.20
15	60.47	66.47	59.47	40.51	45.14	18.53
20	62.05	68.01	56.75	43.91	41.94	11.67
40	67.36	70.38	46.23	41.78	37.89	2.52

Table 4.4. Best β values for the β-adjusted base-stock policy.

	$(\mu_D, \sigma_D = 16, 8)$			$(\mu_D, \sigma_D = 17, 8.5)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
b=10	0.6	0.4	0.5	0.5	0.6	0.7
15	0.5	0.6	0.5	0.6	0.6	0.6
20	0.4	0.5	0.6	0.6	0.5	0.4
40	0.5	0.5	0.4	0.4	0.6	0.5

- We compare the average cost obtained by using the β-adjusted base-stock policy that uses s_a with the no-ACI policy. We observe that β-adjusted base-stock policy with s_a results in lower costs for the small capacity utilization and variation scenarios. A subset of the results for δ values are presented in Table 4.5 for $\rho = 0.8$ and $\rho = 0.85$, $cv_C \in (0.1, 0.2, 0.5)$, and $cv_D \in (0.1, 0.3)$, for different values of b for $M = 2$. Here δ is calculated in a similar manner as defined before, in terms of $\widetilde{J}(s_a, \mathfrak{C})$ and the average cost of the no-ACI policy, $\widetilde{J}(S_m)$, $\delta = 100\frac{\widetilde{J}(S_m) - \widetilde{J}(s_a, \mathfrak{C})}{\widetilde{J}(S_m)}$.

Table 4.5. Percent change in costs: comparing $\widetilde{J}(s_a, \mathfrak{C})$ and $\widetilde{J}(S_m)$ for $M = 2$.

	$(\mu_D, \sigma_D = 16, 1.6)$			$(\mu_D, \sigma_D = 17, 1.7)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	7.59	18.83	18.88	9.13	14.18	13.57
15	10.18	20.72	13.40	13.19	13.13	7.93
20	9.18	19.81	12.60	12.93	13.56	6.53
40	12.69	18.77	8.60	14.52	13.30	2.43
	$(\mu_D, \sigma_D = 16, 4.8)$			$(\mu_D, \sigma_D = 17, 5.1)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	12.89	10.05	4.82	13.53	13.47	9.25
15	8.83	14.29	7.48	19.32	16.18	6.81
20	12.44	9.61	7.77	16.92	15.95	6.20
40	9.96	10.39	7.06	17.86	17.40	5.09

- We compare the average cost obtained by using the β-adjusted base-stock policy that uses s_g with the no-ACI policy. A subset of the results for δ are presented in Table 4.6 for $\rho = 0.8$ and $\rho = 0.85$, $cv_C \in (0.1, 0.2, 0.5)$, and $cv_D \in (0.1, 0.3)$, for different values of b for $M = 2$. Here δ is defined in terms of $\widetilde{J}(s_g, \mathfrak{C})$, the average cost for the β-adjusted base-stock policy that uses s_g, and the average cost of the no-ACI policy, $\widetilde{J}(S_m)$, $\delta = 100\frac{\widetilde{J}(S_m) - \widetilde{J}(s_g, \mathfrak{C})}{\widetilde{J}(S_m)}$.
- We compare the weighted cost heuristic with the no-ACI policy to observe the value of ACI. We define the value of ACI, the percent change in the cost when weighted cost heuristic is used as $\delta = 100\frac{\widetilde{J}(S_m) - \widetilde{J}(S_w, \mathfrak{C})}{\widetilde{J}(S_m)}$, where $\widetilde{J}(S_w, \mathfrak{C})$ is the average cost for the weighted cost heuristic and $\widetilde{J}(S_m)$ is the average cost for the no-ACI policy. A subset of the results for δ are presented in Table 4.7 for $\rho = 0.8$ and $\rho = 0.85$, $cv_C \in (0.1, 0.2, 0.5)$,

Table 4.6. Percent change in costs: comparing $\widetilde{J}(s_g, \mathfrak{C})$ and $\widetilde{J}(S_m)$ for $M = 2$.

	$(\mu_D, \sigma_D = 16, 1.6)$			$(\mu_D, \sigma_D = 17, 1.7)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	5.83	17.38	8.40	10.26	15.20	7.46
15	2.42	17.83	6.10	4.11	14.14	0.75
20	5.20	14.57	1.93	10.32	10.66	1.54
40	3.33	20.61	4.24	4.25	18.61	3.54
	$(\mu_D, \sigma_D = 16, 4.8)$			$(\mu_D, \sigma_D = 17, 5.1)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	9.94	15.34	9.85	4.38	13.74	6.10
15	3.91	18.19	5.15	15.26	12.46	1.31
20	5.32	14.73	6.65	11.36	14.28	8.91
40	2.80	18.26	6.29	9.94	20.27	7.17

and $cv_D \in (0.1, 0.3)$, for different values of b and for $M = 2$.

Table 4.7. Percent change in costs: comparing $\widetilde{J}(S_w, \mathfrak{C})$ and $\widetilde{J}(S_m)$ for $M = 2$.

	$(\mu_D, \sigma_D = 16, 1.6)$			$(\mu_D, \sigma_D = 17, 1.7)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	1.26	7.64	28.63	4.59	9.77	18.65
15	2.84	9.72	29.53	5.51	13.69	15.22
20	4.55	6.68	27.28	8.34	8.18	21.29
40	7.82	12.98	29.63	10.12	12.52	12.05
	$(\mu_D, \sigma_D = 16, 4.8)$			$(\mu_D, \sigma_D = 17, 5.1)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	2.84	8.74	20.75	5.41	8.52	9.30
15	6.47	9.33	20.80	5.62	9.39	5.78
20	7.75	11.87	19.68	7.66	10.88	5.64
40	10.71	12.54	14.17	10.06	10.97	2.52

We observe that the weighted cost heuristic that uses S_w gives good results as M increases, even for the higher values of capacity variation and utilization. In Table 4.8

percent cost decrease over the no-ACI policy is seen for $M = 10$ for $cv_C \in (0.75, 1, 1.5)$.

4.2.3. Simulating the Shortfall

From the results of Section 4.2.2 we conclude that, for higher variance and utilization values, the only method to result in better costs compared to the no-ACI policy is the weighted cost heuristic as we increase M. For high parameters the other methods we do not obtain better results even if we increase M.

To overcome the poor results we propose to simulate the exact distribution of Z_∞ and A_∞ to obtain the constant term s in the base-stock policy, and obtain $s_{sim} = F^{-1}_{A_\infty}(b/(b+h))$. For this purpose we use the random walk definition of the form (Serfozo, 2009):

$$Z_n = \max\left\{ Z_{n-M} + \sum_{i=n-M+1}^{n} X_i, \sum_{i=n-M+2}^{n} X_i, \cdots, X_n, 0 \right\}.$$

From Table 4.9 we observe small values of δ for $M = 2$, that is, the cost reductions obtained by using s_{sim} are not large. Especially as the variation and utilization increases, we observe that s_{sim} gives similar results to the no-ACI policy and δ is close to zero. As M increases better results are obtained, even for higher variation and utilization. See Table 4.10 for high values of capacity variance ($cv_C \in (0.75, 1, 1.5)$), and Table 4.11 for high values of both capacity and demand variance ($cv_D \in (0.8, 1, 1.2)$) for $M = 10$.

Table 4.8. Percent change in costs: comparing $\widetilde{J}(S_w, \mathfrak{C})$ and $\widetilde{J}(S_m)$ for $M = 10$ for high capacity variance.

	$(\mu_D, \sigma_D = 16, 1.6)$			$(\mu_D, \sigma_D = 17, 1.7)$			$(\mu_D, \sigma_D = 18, 1.8)$		
	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$
$b = 10$	47.12	42.35	53.14	23.29	28.93	48.71	30.87	23.17	40.31
15	48.75	41.18	40.70	27.75	32.23	44.85	10.18	40.56	27.73
20	45.77	38.37	32.08	13.56	11.29	37.31	13.08	13.79	19.80
	$(\mu_D, \sigma_D = 16, 4.8)$			$(\mu_D, \sigma_D = 17, 5.1)$			$(\mu_D, \sigma_D = 18, 5.4)$		
	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$
$b = 10$	32.21	41.44	43.84	24.12	27.47	41.22	14.65	5.37	28.57
15	38.59	39.57	44.90	12.16	26.31	45.44	9.06	6.64	23.12
20	35.77	37.28	37.82	23.81	10.55	34.77	4.68	6.85	17.76
	$(\mu_D, \sigma_D = 16, 8)$			$(\mu_D, \sigma_D = 17, 8.5)$			$(\mu_D, \sigma_D = 18, 9)$		
	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$
$b = 10$	22.44	26.33	27.10	7.44	8.94	27.35	1.12	2.67	17.48
15	21.37	24.50	27.98	4.28	8.63	23.64	1.79	1.53	10.69
20	16.44	22.75	30.01	3.35	6.78	22.54	0.81	1.22	14.64

Table 4.9. Percent change in costs: comparing $\widetilde{J}(s_{sim}, \mathfrak{C})$ and $\widetilde{J}(S_m)$ for $M = 2$.

	$(\mu_D, \sigma_D = 16, 1.6)$			$(\mu_D, \sigma_D = 17, 1.7)$			$(\mu_D, \sigma_D = 18, 1.8)$			$(\mu_D, \sigma_D = 19, 1.9)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	0.68	18.49	17.98	7.66	22.52	15.23	12.20	18.70	6.57	9.02	3.23	0.47
15	2.88	19.89	16.68	6.83	24.00	12.93	15.13	17.36	1.96	6.70	4.90	1.35
20	5.16	21.51	18.42	12.44	24.28	9.90	14.28	17.44	5.33	2.74	5.04	2.25
40	6.80	25.88	15.88	17.37	24.83	6.51	14.21	16.23	6.17	4.31	1.44	1.32
	$(\mu_D, \sigma_D = 16, 4.8)$			$(\mu_D, \sigma_D = 17, 5.1)$			$(\mu_D, \sigma_D = 18, 5.4)$			$(\mu_D, \sigma_D = 19, 5.7)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	4.27	9.48	15.93	4.01	7.86	7.12	1.04	4.85	4.34	0.03	0.74	1.41
15	5.40	6.11	13.55	3.14	7.70	6.88	1.73	7.96	2.86	0.75	0.02	1.88
20	6.54	9.30	13.97	3.83	9.51	7.98	1.33	5.39	3.40	0.01	0.82	0.02
40	8.06	11.21	5.73	-0.12	1.53	2.19	0.60	1.15	0.77	0.23	0.41	0.20
	$(\mu_D, \sigma_D = 16, 8)$			$(\mu_D, \sigma_D = 17, 8.5)$			$(\mu_D, \sigma_D = 18, 9)$			$(\mu_D, \sigma_D = 19, 9.5)$		
	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$	$\sigma_C = 2$	$\sigma_C = 4$	$\sigma_C = 10$
$b = 10$	1.38	2.82	7.99	0.74	1.42	4.94	0.15	0.76	2.52	0.05	-0.12	-0.12
15	0.78	2.10	8.01	0.63	1.26	5.95	0.30	0.68	0.20	0.21	0.06	-0.48
20	1.53	2.66	8.35	0.67	1.79	2.07	0.29	0.24	1.67	0.17	0.27	-0.01
40	2.09	0.97	5.69	1.62	2.22	0.25	0.12	0.26	0.11	0.43	0.30	0.40

Table 4.10. Percent change in costs: comparing $\widetilde{J}(s_{sim}, \mathfrak{C})$ and $\widetilde{J}(S_m)$ for $M = 10$ for high capacity variation.

	$(\mu_D, \sigma_D = 16, 1.6)$			$(\mu_D, \sigma_D = 17, 1.7)$			$(\mu_D, \sigma_D = 18, 1.8)$			$(\mu_D, \sigma_D = 19, 1.9)$		
	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$
$b = 10$	26.99	24.97	22.87	24.90	24.32	23.14	20.69	20.58	22.93	8.67	15.18	21.01
15	26.09	24.68	23.16	24.57	24.09	23.03	20.66	21.92	23.25	7.93	11.96	20.59
20	27.07	25.08	23.76	21.90	22.71	22.60	16.66	21.91	22.54	7.33	7.67	19.82
40	26.90	25.92	22.60	22.20	21.18	23.79	13.40	19.29	21.06	1.78	5.97	20.70
	$(\mu_D, \sigma_D = 16, 4.8)$			$(\mu_D, \sigma_D = 17, 5.1)$			$(\mu_D, \sigma_D = 18, 5.4)$			$(\mu_D, \sigma_D = 19, 5.7)$		
	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$
$b = 10$	24.66	22.92	22.65	20.40	22.81	21.94	15.82	19.84	21.96	7.54	13.53	19.60
15	25.23	23.48	22.09	21.33	20.77	21.39	15.72	18.79	21.33	7.72	11.03	18.25
20	24.75	24.45	22.64	20.93	19.63	23.53	13.04	17.32	17.42	5.39	13.13	18.98
40	22.98	23.98	20.67	19.93	22.01	22.98	2.24	14.85	18.76	5.18	10.06	18.89
	$(\mu_D, \sigma_D = 16, 8)$			$(\mu_D, \sigma_D = 17, 8.5)$			$(\mu_D, \sigma_D = 18, 9)$			$(\mu_D, \sigma_D = 19, 9.5)$		
	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$
$b = 10$	21.65	22.11	21.30	19.15	19.75	21.35	9.92	15.28	20.04	2.27	9.55	18.19
15	20.70	22.18	21.29	17.66	19.81	21.81	3.56	14.96	19.12	4.64	7.56	17.71
20	19.59	20.49	20.80	15.85	19.70	21.75	8.37	12.22	18.56	4.29	9.79	17.34
40	20.55	18.14	22.57	13.70	11.36	20.33	8.40	10.81	19.46	2.49	6.46	15.03

Table 4.11. Percent change in costs: comparing $\widetilde{J}(s_{sim}, \mathfrak{C})$ and $\widetilde{J}(S_m)$ for $M = 10$ for high capacity and demand variation.

	$(\mu_D, \sigma_D = 16, 12.8)$			$(\mu_D, \sigma_D = 17, 13.6)$			$(\mu_D, \sigma_D = 18, 14.4)$			$(\mu_D, \sigma_D = 19, 15.2)$		
	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$
$b = 10$	13.92	18.06	19.52	7.62	14.20	17.37	2.17	8.22	12.72	11.32	1.12	3.67
15	13.66	17.90	18.80	7.52	12.42	17.87	1.54	3.72	16.89	9.99	1.92	4.41
20	11.67	17.33	18.04	2.63	10.54	18.96	1.83	5.48	15.46	7.94	4.60	2.78
40	11.08	14.43	20.03	5.34	7.33	18.00	1.02	5.80	14.87	12.16	2.40	2.71
	$(\mu_D, \sigma_D = 16, 16)$			$(\mu_D, \sigma_D = 17, 17)$			$(\mu_D, \sigma_D = 18, 18)$			$(\mu_D, \sigma_D = 19, 19)$		
	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$
$b = 10$	6.66	12.69	13.36	3.27	1.15	16.97	8.03	3.29	11.83	17.97	5.02	5.01
15	4.58	11.29	15.47	3.29	6.45	13.79	8.66	5.06	11.97	12.60	0.30	1.35
20	4.05	10.35	12.66	2.38	3.19	13.92	12.39	4.64	10.29	12.72	4.38	5.65
40	2.66	8.16	10.25	1.63	2.61	12.02	16.20	3.47	8.55	18.38	3.82	3.16
	$(\mu_D, \sigma_D = 19.2)$			$(\mu_D, \sigma_D = 17, 20.4)$			$(\mu_D, \sigma_D = 18, 21.6)$			$(\mu_D, \sigma_D = 19, 22.8)$		
	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$	$\sigma_C = 15$	$\sigma_C = 20$	$\sigma_C = 30$
$b = 10$	1.89	1.12	14.06	8.78	1.00	10.41	17.99	9.90	6.16	9.17	4.88	1.73
15	3.62	3.83	15.26	15.09	3.12	7.07	14.29	7.75	4.65	8.74	4.93	3.96
20	0.07	5.62	14.21	10.23	3.14	13.99	17.14	12.37	4.15	11.58	4.71	2.31
40	1.50	5.08	10.50	8.98	3.17	4.92	11.38	10.66	3.73	10.98	4.54	1.25

5. A CAPACITY FLEXIBILITY MODEL IN THE EXISTENCE OF ADVANCE CAPACITY INFORMATION

In this chapter we consider an inventory/production system under Advance Capacity Information with flexible capacity, where whenever the target base-stock level cannot be reached using the currently available capacity at a given planning period, the short amount can be bought from an exogenous supplier. Capacity of the inventory system we consider is the maximum amount that can be produced in a period, which we call the *regular capacity* of the system. We assume that the regular replenishment capacity for the inventory process is uncertain and non-stationary, but the manufacturer observes a signal on its realization in advance. Since obtaining the optimal policy for a system facing volatile demand in the existence of outsourcing option is difficult we propose an ACI-dependent order-up-to level policy for this system. We model the uncertainty in regular capacity explicitly, and we present an ACI process that tracks and updates the information on the regular capacity.

We introduce our model in Section 5.1. The solution procedure to find the optimal ordering levels is presented in Section 5.2. The results of our numerical study and the managerial insights are given in Section 5.3. The proofs of our structural results are presented in Appendix B.

5.1. Description of the Model

In this section we present and analyze a model where the maximum amount that can be produced in a period (the regular capacity of the system) is uncertain, but an information is available one period in advance. We first model the uncertainty in regular capacity and present an ACI process that tracks and updates the information on the of regular capacity. Then, we discuss the operating characteristics of the inventory system and propose an ACI dependent order-up-to level policy.

5.1.1. Modeling Capacity Uncertainty and the Evolution of the ACI Process

For the system that we consider the regular capacity for the inventory process is uncertain. Moreover, the distribution of capacity is non-stationary, and in any period n it follows the distribution of the random variable $\mathbb{C}_1$ with probability p and $\mathbb{C}_2$ with probability $q = 1 - p$. independent of any other period's capacity realization. It is natural to impose some order on these random variables. That is, it is natural to assume that the realizations of regular capacity in some periods are expected to be larger than the capacity realizations in other periods. For instance, in weeks where maintenance activities are carried out, or in cases where there are workforce disruptions, the regular capacity realizations are expected to be lower. Without loss of generality, we let $\mathbb{C}_1$ represent a "high" regular capacity variable, whereas $\mathbb{C}_2$ represents a "low" capacity variable. One may impose a stochastic ordering on $\mathbb{C}_1$ and $\mathbb{C}_2$ by assuming that $\mathbb{C}_1 \geq_{st} \mathbb{C}_2$ ($\geq_{st}$ means stochastically larger). Let $F_{\mathbb{C}_j}(.)$ be the distribution function for $\mathbb{C}_j$, $j = 1, 2$. Then, we require $F_{\mathbb{C}_1}(x) \leq F_{\mathbb{C}_2}(x)$ for all x. On the other hand, one can also consider a situation where $\mathbb{C}_j$ is not a random variable, but a constant, that is, $\mathbb{C}_j \equiv c_j$, $j = 1, 2$. In this case, we require $c_1 \geq c_2$. We note that by letting $c_1 = \infty$, and $c_2 = 0$ one can model a system where the capacity is either unrestricted, or completely unavailable. We should also note that our setup implies that the number of consecutive periods for which the regular capacity is observed from the "high" distribution is geometrically distributed with parameter p.

In general, at the beginning of a period, before making a decision on how much to order, the system manager does not know the capacity distribution for any of the subsequent periods. In this thesis we propose and model a situation where the system has knowledge on the distribution of the capacity for the current and the next period. We note that the particular realization of the capacity is still not known at the time of making a decision, but a window of information availability (the knowledge of the exact distribution) exists, for the current and the next period. Specifically, at the beginning of a period n, the system observes a signal on the availability of the regular capacity in the form of $ACI_n = (j_0, j_1)$ where $j_0 \in \{1, 2\}$, $j_1 \in \{1, 2\}$, and this advance information reveals that the regular capacity in period n will be realized from the random variable $\mathbb{C}_{j_0}$, whereas the regular capacity in period $n + 1$ will be realized from the random variable C_{j_1}. As time rolls one period, this information is updated, and ACI at the beginning of period $n + 1$ becomes $ACI_{n+1} = (j_1, J)$ where J is a random variable. $J = 1$ with probability p, and $J = 2$ with probability $q = 1 - p$. Therefore, the evolution of the advance information, $\{ACI_n, n = 0, 1, \ldots\}$ can be described by a simple Markov chain.

5.1.2. Description of the Replenishment Policy

As noted before, the amount of inventory that can be replenished by using the regular capacity is restricted. Any shortfall (in excess of the regular capacity) can be purchased from an outside supplier by paying a higher unit cost. We assume that the following order of events takes place in a typical period n.

(i) The inventory position at the beginning of the period, I_n, is observed.
(ii) ACI information on the regular capacity, $ACI_n = (j_0, j_1)$, is observed.
(iii) The replenishment order is placed. The current period's regular capacity, $\mathbb{C}_{j_0}$ is realized. If the current period's regular capacity is not sufficient to fulfill the order, then the shortfall is acquired from an exogenous supplier at a higher cost.
(iv) The demand for period n, D_n, is observed, and inventory holding/backorder costs are incurred.

We assume that the period length is long enough to justify that the production lead-time is zero. Our aim is to find a tractable and easily implementable policy which takes into account the availability of the ACI. To this end we propose an ACI-dependent order-up-to level policy. We should note that obtaining the true optimal policy for the system described above is difficult. Even for a system where the regular capacity is deterministic and stationary (that is, $\mathbb{C}_1 \equiv \mathbb{C}_2 \equiv c$, a constant) the form of the optimal policy is complicated (Alp and Tan, 2008). In this thesis we confine ourselves to time-stationary order-up-to level policies that partially depend on the observed ACI: if an ACI of $ACI_n = (j_0, j_1)$ is observed at the beginning of period n, then the inventory position of that period is raised to the level S_{j_1} (as long as there is no excess inventory from the previous period).

In order to see that an ACI-dependent order-up-to level policy is suitable, consider the following extreme situation. Suppose that in a period the regular capacity is either unrestricted or completely unavailable ($\mathbb{C}_1 \equiv \infty$ and $\mathbb{C}_2 \equiv 0$). If the next period's capacity is ∞, then this period's optimum inventory policy is to place an order to raise the inventory level up to the newsvendor solution, which is described in (Porteus, 2002) (since we already know that any amount that we order in the next period will be available). On the other hand, if the next period's capacity is 0, then the order amount should not only cover this period's demand, but also should take into account the (certain) unavailability of the capacity in the next and possibly in the future periods. Therefore, any reasonable inventory policy should depend on the observed ACI.

Note that the order-up-to level policy described above does not explicitly depend on the realization of the current capacity, $\mathbb{C}_{j_0}$. If the capacity constraint is a hard constraint (that is, if acquiring the shortfall from an exogenous supplier is not possible), then this form of the order-up-to level would be exact, as the current period's capacity will not have an influence on the determination of the order-up-to level (the current period's capacity determines whether this order-up-to level is achievable or not). However, as any shortfall can be fulfilled at the expense of paying a higher cost, the current period's capacity may become important in determining the order-up-to level. In that regard our analysis leads to approximately optimal order-up-to levels. On the other hand, including the full knowledge of capacity information would lead to a model which is quite untractable.

If one imposes a stochastic ordering on $\mathbb{C}_1$ and $\mathbb{C}_2$ as $\mathbb{C}_1 \geq_{st} \mathbb{C}_2$, then in an optimal solution $S_1 \leq S_2$ is expected. In the light of the foregoing discussion, we revise the third item in the order of events that takes place in a period as follows,

(iii) An order of size $u_n = \max\{S_{j_1} - I_n, 0\}$ is placed. The current period's regular capacity, $\mathbb{C}_{j_0}$ is realized. If $u_n > \mathbb{C}_{j_0}$, then the shortfall $u_n - \mathbb{C}_{j_0}$ is acquired from an exogenous supplier at a higher cost.

In our model the order-up-to level takes two values. This is due to the fact that the regular capacity in a period is realized from one of the two possible distributions, and in deciding the order-up-to level we only consider the information on the next period's capacity. The assumption of two possible capacity distributions is suitable in situations where the regular capacity is either severely disrupted or distributed around its usual average respective random durations. A typical example would be $\mathbb{C}_2 \equiv 0$ (complete capacity disruption) and $\mathbb{C}_1$ represents the capacity random variable when it is not disrupted. In cases where the capacity in a period can be realized from more than two possible distributions, our model can be used as an approximation by grouping possible capacity realizations into two groups.

5.2. Analysis of the Model

In this section we develop an expression for the average cost of the inventory system that we described in Section 5.1, from which one can obtain first order optimality equations for the order-up-to levels. Then, we discuss the properties of the optimal solution and describe the solution procedure. Proofs of our results are presented in Appendix B.

5.2.1. Derivation of the Average Cost Function

To obtain the average cost function we will consider regenerative cycles where without loss of generality the periods of a cycle is indexed as $1, 2, \ldots$. Consider the beginning of a cycle in period 1 with $I_1 = S_2$, and $ACI_1 = (1,2)$. A regenerative cycle is defined as the evolution of the process until the next time we observe $ACI_n = (1,2)$ and raise the inventory level to $I_n = S_2$. The length of the renewal interval is defined as:

$$T = \min\{n > 1 : ACI_{n+1} = (1,2)\}.$$

Note that because of the nature of the process, $T \geq 2$ ($T = 2$ occurs if $ACI_2 = (2,1)$ and $ACI_3 = (1,2)$). Also let

$$T_1 = \min\{n \geq 1 : ACI_{n+1} = (2,1)\},$$

as the length of a subinterval for which the order-up-to level stays at S_2 and switches from S_2

to S_1 at the beginning of period $T+1$. Also let

$$T_2 = \min\{n \geq 1 : ACI_{T_1+n+1} = (1,2)\}.$$

Note that T_1 and T_2 are geometrically distributed independent random variables with parameters p and q, respectively. Moreover, $T = T_1 + T_2$ and clearly $E[T] = (1/p) + (1/q)$ is the expected length of a renewal cycle. If $G(S_1, S_2)$ is the expected cost of a renewal cycle (to be derived subsequently), then the average cost per unit time is given by $G(S_1, S_2)/E[T]$, and the aim is to find S_1 and S_2 that minimize this average cost per period.

Let $D_1, D_2, \ldots$ be random demands for successive periods in a cycle, and define $D(k,m) = \sum_{i=k}^{m} D_i$ for $m \geq k$. Let $F_{D_i}(.)$ and $F_{D_{(k,m)}}(.)$ be the cumulative distribution function for D_i, and $D(k,m)$, respectively. Although per period demands are stationary random variables, we would like to keep the time related subscripts on the distribution functions for clarity of exposition. But for the sake of conciseness we write $F_i(.)$ and $F_{(k,m)}(.)$, respectively, instead of $F_{D_i}(.)$ and $F_{D_{(k,m)}}(x)$. And whenever there is no danger of confusion, we will simply write $F(.)$ for the distribution function of a single period's demand. We also write $B_i(.)$ instead of $F_{\mathbb{C}_i}(.)$, the distribution function of $\mathbb{C}_i$, for ease of read.

Let $\{\mathbb{C}_{j,i}, i = 1, 2, 3, \ldots\}$ be independent and identically distributed copies of $\mathbb{C}_j$ for $j = 1, 2$. $\mathbb{C}_{j,i}$ denotes the capacity random variable associated with the ith period in the cycle, whenever it is of type j. Define $\Delta = S_2 - S_1 \geq 0$.

Whenever the order-up-to level switches from S_2 to S_1 (at the beginning of period T_1+1), the inventory level may be above S_1 (recall that $S_1 \leq S_2$). Therefore, it is important to characterize the time at which it is feasible to bring the inventory position up to level S_1. For a given value of T_1 and T define

$$\tau = \min\{n \in \{T_1, T_1+1, \ldots, T\} : D(T_1, n) \geq \Delta\}, \tag{5.1}$$

as the first period on or after T_1 that the cumulative demand exceeds $\Delta = S_2 - S_1$. If no such

τ exists (if $D(T_1, T) < \Delta$) then we set $\tau = T$. Then, τ determines the first period in a cycle that the inventory level can be raised to the order-up-to level S_1. Figure 5.1 illustrates a typical sample path of the inventory process for $\mathbb{C}_i = c_i$, $i = 1, 2$ (constant capacity levels). The cycle starts with $ACI_1 = (1, 2)$, where c_1 is the currently available capacity and c_2 is the capacity observation for the next period. Hence the inventory level is raised to the base stock level of S_2. Then, for a number of periods we observe c_2's until the first time a c_1 is observed (which leads to T_1). Then, c_1's are observed until we observe the next $ACI = (1, 2)$ and the cycle ends. In Figure 5.1 the circles denote the target base-stock levels. As the cycle progresses, the inventory level is increased to its target level in a given period, either using the available regular capacity or by acquiring the shortfall from an outside supplier. Whenever the inventory level is higher than S_1, while S_1 is the target inventory level, no action is taken. The first time the target inventory shifts from S_2 to S_1 is when the first time $ACI = (2, 1)$ is observed.

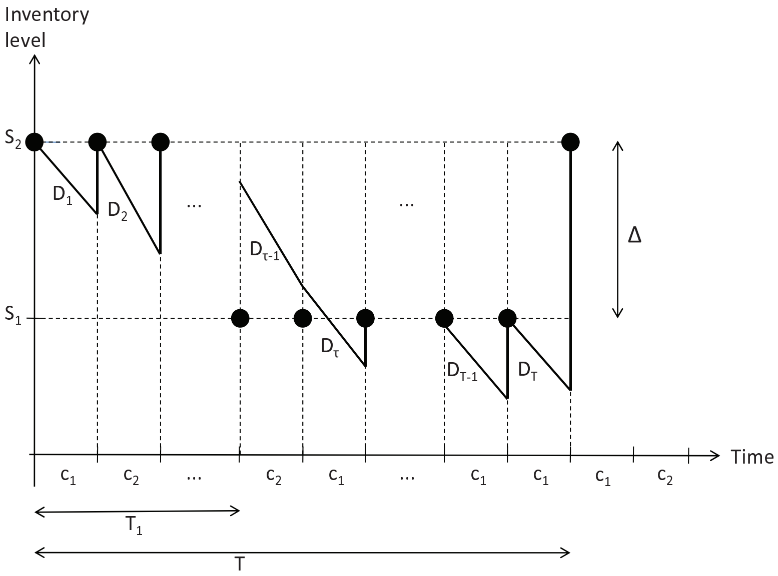

Figure 5.1. A renewal cycle.

Our next goal is to characterize the total expected cost of a renewal cycle. Let $L_i(y)$ be inventory related costs associated with period i:

$$L_i(y) = h(y - D_i)^+ + b(D_i - y)^+,$$

where h and b are the unit inventory holding and backorder costs per unit time, respectively, and $(a)^+ = \max(a, 0)$. The unit cost of acquiring items from an outside supplier is denoted by γ. We do not explicitly consider the unit cost of the regular capacity, but it can easily be included in the analysis. Suppose that $D_1, D_2, \ldots$ is a given demand sequence, and we are also given $\{\mathbb{C}_{j,i}, i = 1, 2, 3, \ldots\}$ for $j = 1, 2$ and the values of $T_1 \geq 1$ and $T_2 \geq 1$ (recall that $T = T_1 + T_2$). A specific demand sequence, along with T_1 determines the value of τ (see equation (5.1)).

Let $\widetilde{G}(S_1, S_2, T_1, T, \tau)$ be the random cycle cost given T_1, T, τ. Obviously, this cost also depends on the demand and capacity sequences. We suppress these dependencies in our notation for convenience. Then,

$$\begin{aligned}\widetilde{G}(S_1, S_2, T_1, T, \tau) =& \sum_{i=1}^{T_1-1} \{L_i(S_2) + \gamma(D_i - \mathbb{C}_{2,i+1})^+\} + L_{T_1}(S_2) + \gamma \widetilde{g}_{T_1} 1_{\{\tau=T_1\}} \\ &+ \sum_{i=T_1+1}^{\tau} L_i(S_2 - D(T_1, i-1)) + \gamma \sum_{i=T_1+1}^{T} \widetilde{g}_i 1_{\{\tau=i\}} \qquad (5.2) \\ &+ \sum_{i=\tau+1}^{T} \{L_i(S_1) + \gamma \widetilde{v}_i\},\end{aligned}$$

where $1_{(A)}$ is equal to 1 if A is true, and 0 otherwise, and

$$\widetilde{g}_i = \begin{cases} (D_{T_1} - \mathbb{C}_{2,i+1} - \Delta)^+ & i = T_1 \\ (D(T_1, i) - \mathbb{C}_{1,i+1} - \Delta)^+ & i = T_1 + 1, \cdots, T - 1 \\ (D(T_1, T) - \mathbb{C}_{1,i+1})^+ & i = T, \end{cases}$$

$$\widetilde{v}_i = \begin{cases} (D_i - \mathbb{C}_{1,i+1})^+ & i = \tau + 1, \ldots, T - 1 \\ (D_T + \Delta - \mathbb{C}_{1,i+1})^+ & i = T, \end{cases}$$

with the convention that $\sum_{i=a}^{b} = 0$ whenever $b < a$.

We explain 5.2 term by term. The renewal cycle starts with the inventory level S_2. In the first T_1-1 periods of the renewal cycle, the ACI for the next period is of type 2 (low capacity). Therefore the inventory is raised to level S_2 and at the end of period $i \in \{1, 2, \ldots, T_1-1\}$ an inventory/backorder cost of $L_i(S_2)$ is incurred. Moreover, at the end of each such period (at the beginning of period $i+1$) the capacity available is $\mathbb{C}_{2,i+1}$, and hence any demand in excess of this capacity level incurs a unit cost of γ. This explains the first term in the first line of 5.2. The second term in the first line of 5.2 is the inventory/backorder cost of period T_1. At the end of period T_1 the order-up-to level switches to S_1 and if $\tau = T_1$ then the starting inventory level at the beginning of period $T+1$ is S_1 and a unit cost of γ is paid if $S_1 - (S_2 - D_{T_1}) = D_{T_1} - \Delta > \mathbb{C}_{2,T_1+1}$, which explains the last term in the first line of 5.2. Next, we explain the second line of 5.2. If $\tau \geq T_1 + 1$ inventory related costs of a period $i = \{T_1+1, \ldots, \tau\}$ is $L_i(S_2 - D(T_1, i-1))$. Moreover, if $\tau \in \{T_1+1, \ldots, T\}$, then the cost of acquiring necessary shortfall from the outside supplier is given by $\gamma\widetilde{g}_i$ (regular capacity realizations for these periods are drawn from the random variable $\mathbb{C}_1$). Note that $\widetilde{g}_T$ is defined differently, as the order-up-to level switches back to S_2 at the end of period T. The third line of 5.2 vanishes if $\tau = T$. Otherwise, for $i = \tau+1, \ldots, T$, inventory/backorder costs are given by $L_i(S_1)$, and for $i = \tau+1, \ldots, T-1$ the cost for that period's shortfall is $\gamma\widetilde{v}_i$. Since the order-up-to level switches to S_2 at the end of period T, and the ACI becomes $ACI_{T+1} = (1, 2)$, a cost of $\gamma((S_2 - (S_1 - D_T) - \mathbb{C}_{1,T+1}))^+ = \gamma(\Delta + D_T - \mathbb{C}_{1,T+1})^+ = \gamma\widetilde{v}_T$ is incurred. This brings the inventory level at the beginning of period $T+1$ (the beginning of the next renewal cycle) to level S_2 and the current cycle ends.

By algebraic manipulation, which basically involves rewriting some of the summations in Equation 5.2, we obtain:

$$\begin{aligned}
\widetilde{G}(S_1, S_2, T_1, T, \tau) =& \sum_{i=1}^{T_1-1} \{L_i(S_2) + \gamma(D_i - \mathbb{C}_{2,i+1})^+\} + L_{T_1}(S_2) \\
&+ \sum_{i=T_1+1}^{T} L_i(S_2 - D(T_1, i-1))1_{\{\tau \geq i\}} - \sum_{i=T_1+1}^{T} L_i(S_1)1_{\{\tau \geq i\}} \\
&+ \sum_{i=T_1+1}^{T} L_i(S_1) + \sum_{i=T_1+1}^{T} \gamma\widetilde{v}_i - \sum_{i=T_1+1}^{T} \gamma\widetilde{v}_i 1_{\{\tau \geq i\}} + \sum_{i=T_1}^{T} \widetilde{g}_i 1_{\{\tau = i\}}.
\end{aligned} \tag{5.3}$$

Next, we observe that the event $\{\tau \geq i\}$ is equal to the event that $\{D(T_1, i-1) < \Delta\}$. Therefore, D_i is independent of $1_{\{\tau \geq i\}}$. Moreover,

$$E[1_{\{\tau \geq i\}}] = \Pr\{\tau \geq i\} = F_{(T_1, i-1)}(\Delta).$$

Define $\mathcal{L}(y) = E[L_i(y)]$, as the expectation of $L_i(y)$ over demand. Also let

$$\begin{aligned} g_i &= E[\widetilde{g}_i 1_{\{\tau = i\}}] & (5.4) \\ v_i &= E[\widetilde{v}_i], & (5.5) \end{aligned}$$

and

$$G_0(S_1, S_2, T_1, T) = E[\widetilde{G}(S_1, S_2, T_1, T, \tau)], \qquad (5.6)$$

where the expectations in 5.4, 5.5, and 5.6 are taken over the capacity and demand related random variables (hence τ does not appear anymore). Then,

$$\begin{aligned} & G_0(S_1, S_2, T_1, T) = \sum_{i=1}^{T_1 - 1} \{\mathcal{L}(S_2) + \gamma E[(D_i - \mathbb{C}_{2,i+1})^+]\} + \mathcal{L}(S_2) \\ & + \sum_{i=T_1+1}^{T} \int_0^{\Delta} \mathcal{L}(S_2 - x) dF_{(T_1, i-1)}(x) + \mathcal{L}(S_1) \sum_{i=T_1+1}^{T} (1 - F_{(T_1, i-1)}(\Delta)) \qquad (5.7) \\ & + \gamma \sum_{i=T_1+1}^{T} v_i (1 - F_{(T_1, i-1)}(\Delta)) + \gamma \sum_{i=T_1}^{T} g_i. \end{aligned}$$

Note that in 5.7

$$\begin{aligned}
&E[L_i(S_2 - D(T_1, i-1))1_{\{\tau \geq i\}}] = \int_0^{\Delta} \mathcal{L}(S_2 - x)dF_{(T_1,i-1)}(x),\\
&g_{T_1} = E[(D_{T_1} - \mathbb{C}_{2,T_1+1} - \Delta)^+ 1_{(\tau=T_1)}] = \int_{y=0}^{\infty}\int_{x=\Delta+y}^{\infty} (x - \Delta - y)dF_{T_1}(x)dB_2(y),\\
&g_i = E[(D(T_1, i) - \mathbb{C}_{1,i+1} - \Delta)^+ 1_{(\tau=i)}]\\
&= \int_{y=0}^{\infty}\int_{x_1=0}^{\Delta}\int_{x_2=\Delta+y-x_1}^{\infty} (x_1 + x_2 - y - \Delta)dF_i(x_2)dF_{(T_1,i-1)}(x_1)dB_1(y), \text{ and}\\
&v_i = E[(D_i - \mathbb{C}_{1,i+1})^+] = \int_{y=0}^{\infty}\int_{x=y}^{\infty}(x - y)dF_i(x)dB_1(y),
\end{aligned} \tag{5.8}$$

for $i = T_1 + 1, \ldots, T - 1$, and

$$\begin{aligned}
g_T &= E[(D(T_1, T) - \mathbb{C}_{1,T+1})^+ 1_{(\tau=T)}]\\
&= \int_{y=0}^{\infty}\int_{x_1=0}^{\Delta}\int_{x_2=y-x_1}^{\infty} (x_1 + x_2 - y)dF_T(x_2)dF_{(T_1,T-1)}(x_1)dB_1(y),\\
v_T &= E[(D_T + \Delta - \mathbb{C}_{1,T+1})^+\\
&= \int_{y=0}^{\infty}\int_{x=y-\Delta}^{\infty} (x + \Delta - y)dF_T(x)dB_1(y).
\end{aligned}$$

Finally, let $G(S_1, S_2) = E[G_0(S_1, S_2, T_1, T)]$ be the expected cost of a renewal cycle, where the last expectation is taken over the random variables T_1 and T:

$$G(S_1, S_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^m p^n G_0(S_1, S_2, m, n + m).$$

Then, the optimization problem becomes finding the values of S_1 and S_2 that minimize the average cost per period:

$$\min_{S_1,S_2} ACPP(S_1, S_2) = \frac{G(S_1, S_2)}{E[T]}. \tag{5.9}$$

5.2.2. Properties of the Optimal Solution

In this section our objective is to derive various properties of the optimal solution for the problem (5.9). Unfortunately, the average cost function, $ACPP(S_1, S_2)$ is not necessarily jointly convex in S_1 and S_2. Nevertheless, useful characteristics of an optimal solution can still be derived. We note that the denominator of $ACPP(S_1, S_2)$, $E[T]$, does not depend on (S_1, S_2). We also note that any behavior of $G_0(S_1, S_2, T_1, T)$ with respect to S_1 or S_2 for a fixed T_1 and T is preserved in $G(S_1, S_2)$, as $G(S_1, S_2)$ is simply a convex combination of countably many G_0 terms. We make a simple notational transformation and replace S_2 with Δ in the definition of the costs functions. For example, $G_0(S_1, S_2, T_1, T)$ becomes $G_0(S_1, \Delta, T_1, T)$. It is more convenient to work with (S_1, Δ) as decision variable pair, and S_2 can easily be recovered using $S_2 = \Delta + S_1$.

Proposition 5.1. *$ACPP(S_1, \Delta)$ is convex in S_1 for a given fixed value of Δ.*

Based on the first derivative of $G_0(S_1, \Delta, T_1, T)$ with respect to S_1 (defined below), we make two observations.

$$
\begin{aligned}
\partial G_0(S_1, \Delta, T_1, T)/\partial S_1 = \quad & \sum_{i=1}^{T_1} \mathcal{L}^{'}(S_1 + \Delta) + \sum_{i=T_1+1}^{T} \int_0^{\Delta} \mathcal{L}^{'}(S_1 + \Delta - x) dF_{(T_1, i-1)}(x) \\
& + \sum_{i=T_1+1}^{T} \left(1 - F_{(T_1, i-1)}(\Delta)\right) \mathcal{L}^{'}(S_1).
\end{aligned} \tag{5.10}
$$

Our first observation is that, for a given value of Δ the corresponding optimal value of S_1 does not depend on the capacity random variables $\mathbb{C}_1$ or $\mathbb{C}_2$. Note that we penalize the shortfall from the order-up-to level S_1 (by paying an expensive unit ordering cost) whenever the capacity is not sufficient, irrespective of the absolute value of S_1. Our second observation involves the behavior of the optimal S_1 with respect to Δ. By differentiating 5.10 with respect to Δ we obtain:

$$
\partial^2 G_0(S_1, \Delta, T_1, T)/\partial S_1 \partial \Delta = \sum_{i=1}^{T_1} \mathcal{L}^{''}(S_1 + \Delta) + \sum_{i=T_1+1}^{T} \int_0^{\Delta} \mathcal{L}^{''}(S_1 + \Delta - x) dF_{(T_1, i-1)}(x) \geq 0.
$$

Therefore, optimal S_1 decreases as Δ increases. The impact of the increase in Δ is to incur increased holding costs. The system decreases S_1 in order to balance this effect. The convexity result of Proposition 5.1. enables us to easily compute optimal value of S_1 for a given value of Δ.

We infer more properties of the optimal solution by checking the first order optimality condition with respect to Δ:

$$\begin{aligned}
\partial G_0(S_1,\Delta,T_1,T)/\partial\Delta &= \sum_{i=1}^{T_1}\mathcal{L}^{'}(S_1+\Delta)+\sum_{i=T_1+1}^{T}\int_0^{\Delta}\mathcal{L}^{'}(S_1+\Delta-x)dF_{(T_1,i-1)}(x)\\
&- \int_{y=0}^{\infty}\gamma(1-F(\Delta+y))dB_2(y) \qquad (5.11)\\
&- \gamma\sum_{i=T_1+1}^{T-1}\int_{y=0}^{\infty}\int_{x_1=0}^{\Delta}\int_{x_2=\Delta+y-x_1}^{\infty}dF(x_2)dF_{(T_1,i-1)}(x_1)dB_1(y)\\
&+ \gamma\int_{y=0}^{\infty}(1-F(y-\Delta))(1-F_{(T_1,T-1)}(\Delta))dB_1(y).
\end{aligned}$$

One immediate consequence of 5.11 is the following result on the effect of the “low” capacity random variable.

Proposition 5.2. Let $\mathbb{C}_2^a$ and $\mathbb{C}_2^b$ be two random variables, representing two different “low” capacity random variables, where $\mathbb{C}_2^a \geq_{st} \mathbb{C}_2^b$. Let Δ^a and Δ^b be optimal values for Δ under capacity random variables $\mathbb{C}_2^a$ and $\mathbb{C}_2^b$, respectively. Then, $\Delta^a \leq \Delta^b$.

The significance of Proposition 5.2. is that it enables us to narrow the search space for the optimal Δ as the distribution for $\mathbb{C}_2$ changes from one data set problem instance to another. If we find the optimal Δ value for a given random capacity $\mathbb{C}_2$, we know that the optimal Δ value for a stochastically larger $\mathbb{C}_2$ cannot be more than the one obtained previously.

By differentiating 5.11 with respect to S_1 we obtain:

$$\sum_{i=1}^{T_1}\mathcal{L}^{''}(S_1+\Delta)\sum_{i=T_1+1}^{T}\int_0^{\Delta}\mathcal{L}^{''}(S_1+\Delta-x)dF_{(T_1,i-1)}(x)\geq 0,$$

and therefore we conclude that optimal Δ decreases as S_1 increases.

5.2.3. A Characterization of the Optimal Solution

We obtain a characterization of the optimal solution to the model (5.9) whenever the solution satisfies the first order optimality equations:

$$\begin{aligned} \partial E[G_0(S_1, \Delta, T_1, T)]/\partial S_1 &= 0 \qquad (5.12) \\ \partial E[G_0(S_1, \Delta, T_1, T)]/\partial \Delta &= 0, \qquad (5.13) \end{aligned}$$

where the expected values are taken over T_1 and T. Let S_1^* and $S_2^* = \Delta^* + S_1^*$ be such a solution. First note that

$$\begin{aligned} E[\tau|T_1, T] &= \sum_{i=0}^{\infty} \Pr\{\tau > i|T_1, T\} \\ &= \sum_{i=0}^{T_1-1} \Pr\{\tau > i|T_1, T\} + \sum_{i=T_1}^{T-1} \Pr\{\tau > i|T_1, T\} \\ &= T_1 + \sum_{i=T_1+1}^{T} \Pr\{\tau \geq i|T_1, T\} \\ &= T_1 + \sum_{i=T_1+1}^{T} \Pr\{D(T_1, i-1) < \Delta\} = T_1 + \sum_{i=T_1+1}^{T} F_{(T_1,i-1)}(\Delta), \end{aligned}$$

where the second and the third lines follow by $\Pr\{\tau > T|T_1, T\} = 0$ and $\Pr\{\tau > i|T_1, T\} = 1$ for $i = 0, 1, \ldots, T_1 - 1$. Therefore,

$$E[\tau] = E[T_1] + E[\sum_{i=T_1+1}^{T} F_{(T_1,i-1)}(\Delta)].$$

Then, $E[\sum_{i=T_1+1}^{T}\left(1-F_{(T_1,i-1)}(\Delta)\right)] = E[T]-E[\tau]$. Define

$$\begin{aligned} H(\Delta) &= \int_{y=0}^{\infty}(1-F(\Delta+y))dB_2(y) \\ &+ E\left[\sum_{i=T_1+1}^{T-1}\int_{y=0}^{\infty}P\{D(T_1,i-1)<\Delta, D(T_1,i)>y+\Delta\}dB_1(y)\right. \\ &- \left.\int_{y=0}^{\infty}(1-F(y-\Delta))(1-F_{(T_1,T-1)}(\Delta))dB_1(y)\right], \end{aligned}$$

so that 5.11 and 5.13 imply that

$$E\left[\sum_{i=1}^{T_1}\mathcal{L}^{'}(S_1^*+\Delta^*)+\sum_{i=T_1+1}^{T}\int_0^{\Delta^*}\mathcal{L}^{'}(S_1^*+\Delta^*-x)dF_{(T_1,i-1)}(x)\right] = \gamma H(\Delta^*). \tag{5.14}$$

Similarly, by using 5.10, 5.12 and 5.14 we obtain that

$$\mathcal{L}^{'}(S_1^*)E\left[\sum_{i=T_1+1}^{T}(1-F_{(T_1,i-1)}(\Delta^*))\right] = -\gamma H(\Delta^*)$$

Noting that $\mathcal{L}^{'}(S_1^*) = -b+(h+b)F(S_1^*)$ and $E\left[\sum_{i=T_1+1}^{T}(1-F_{(T_1,i-1)}(\Delta^*))\right] = E[T]-E[\tau]$ we get

$$F(S_1^*) = \frac{b-\gamma R(\Delta^*)}{h+b} \tag{5.15}$$

where $R(\Delta^*) = H(\Delta^*)/(E[T]-E[\tau])$ in 5.15. It can be seen that $R(\Delta^*)\le 1$:

$$\begin{aligned} (1-F(\Delta+y)) &\le 1-F(\Delta), \\ P\{D(T_1,i-1)<\Delta, D(T_1,i)>y+\Delta\} &\le 1-F_{(T_1,i)}(\Delta), \\ (1-F(y-\Delta))(1-F_{(T_1,T-1)}(\Delta)) &\ge 0. \end{aligned}$$

Therefore, $H(\Delta)\le E[\sum_{i=T_1}^{T_1-1}(1-F_{(T_1,i)}(\Delta))] = E[\sum_{i=T_1+1}^{T_1}(1-F_{(T_1,i-1)}(\Delta))] = E[T]-$

$E[\tau]$. We should stress that $E[\tau]$ is also a function of Δ^*.

Equation 5.15 provides us a criterion for testing the quality of any numerical procedure for obtaining a first order condition based optimal solution. If the right hand side of (5.15) is not close to the left hand side, then we should further seek to improve the current solution in a numerical search method.

Note that for $\gamma = 0$, 5.15 reduces to $F(S_1^*) = \frac{b}{h+b}$, the newsvendor solution. An important special case is given by $C_1 \equiv \infty$ and $\mathbb{C}_2 \equiv 0$: full supply availability and supply disruption durations are respective geometrically distributed random variables. In this case $H(\Delta)$ simplifies to $H(\Delta) = 1 - F(\Delta)$.

We now present upper bounds on the optimal values of Δ and S_1. Let S_1^{UB} be the upper bound for S_1, and Δ_{UB} be the upper bound for Δ.

Proposition 5.3. There exists at least one Δ solving Equation 5.16. Let Δ_{UB} be the smallest solution of 5.16. Then, $\Delta^* \leq \Delta_{UB}$.

$$E[T_1]F(\Delta) + E\left[\sum_{i=T_1+1}^{T} F_{(T,i)}(\Delta)\right] = \frac{\gamma(E[T] - E[\tau]) + bE[\tau]}{h+b}. \tag{5.16}$$

Since we do not have a convexity result with respect to Δ, Proposition 5.3. can be used to limit the search space for the optimum Δ value.

Proposition 5.4. There exists an upper bound for S_1, which is equal to the newsvendor solution.

We have shown the existence of upper bounds. And we have shown that, given Δ, it is easy to search for S_1 since the inventory cost is convex in S_1 and also the first order condition does not depend on any of the possible capacity distributions. We have also characterized the first order conditions for the optimal solution.

We conclude this section by presenting the solution procedure to find Δ^* and S_1^*. This solution procedure is given in Figure 5.2.

```
**Initialize**: Set $\Delta = \Delta_{UB}$.
**Step 1**. Given $\Delta$, find the corresponding optimal $S_1$ by using the golden
    section search method on $[0, S_1^{UB}]$.
**Step 2.** STOP if equation 5.15 is satisfied. Otherwise decrease $\Delta$ and
    GO TO **Step 1**.
```

Figure 5.2. Solution procedure.

5.3. Numerical Analysis

In this section we present the results of our computational study that we have conducted to gain insights into the effects of using ACI and outsourcing on the costs. We first present the experimental setting used in the numerical study, then we describe how the value of using outsourcing and ACI are evaluated, present and discuss our numerical findings.

5.3.1. Experimental Setting

For the numerical study we use Gamma distributed random demand with varying mean, $\mu_D \in (3, 4, 5, 7)$, and standard deviation, $\sigma_D \in (1, 3, 5)$. Gamma distribution allows us to control the mean and variation of demand. We fix the backorder cost to $b = 10$ and the holding cost to $h = 1$, and vary the unit outsourcing cost, γ, with $\gamma \in (0, 5, 10, 15)$, to see the relative effect of these costs. We choose the cost parameters b and h so as to obtain a high service level, b being considerably higher than h. And in another case we fix b and γ, and vary h to see its effect on the percent cost reductions, with $h \in (2, 3, 4, 5)$.

For the capacities $\mathbb{C}_1$ and $\mathbb{C}_2$ we consider deterministic values with $\mathbb{C}_1 \equiv c_1$ and $\mathbb{C}_2 \equiv c_2$. We use five different levels for the probability of having high capacity, $p \in (0.1, 0.3, 0.5, 0.7, 0.9)$, and one of the following values of c_1 and c_2, $c_1 \in (6, 12, 20)$, and $c_2 \in (0, 4)$. We choose c_1 and c_2 to alter the capacity utilization and to observe how the utilization level affects the value of ACI and outsourcing. With these parameters we cover a wide range of capacity utilization. For example, for $p = 0.1$, $c_1 = 6$ and $c_2 = 4$, the expected capacity 4.2. And if $\mu_D = 5$, then the expected demand is approximately 20% higher than the expected regular capacity of a period. We choose fixed values for $\mathbb{C}_1$ and $\mathbb{C}_2$ to be able to eliminate the effect of variability of the capacity in a period, given that it is realized from high or low distribution. The case where $\mathbb{C}_1 \equiv 20$ and $\mathbb{C}_2 \equiv 0$, for example, is representative of a case where the capacity is either unrestricted or completely unavailable for the demand parameters that we use.

In our experiments we use a combination of a set of input parameters to see their effects on the value of ACI and value of outsourcing. We provide explanations to our results. We only present a subset of our numerical observations, but our findings are verified through several experiments.

We use Monte Carlo simulation to calculate the average costs of the systems we consider in our experiments. With Monte Carlo simulation multiple trial runs are obtained using randomly generated capacity and demand values in each period.

5.3.2. Value of Outsourcing

We study the benefit of using the outsourcing option under different parameters. The value of outsourcing is defined as the percent cost decrease caused by using the outsourcing opportunity. To evaluate the value of outsourcing we compare the costs of the ACI-dependent base-stock policy we propose in this thesis to a policy where ACI is observed, but there is no outsourcing option. For the latter policy we exhaustively search for the base-stock levels S_1 and S_2 since no optimality conditions can be defined for them. The value of outsourcing, VO, is defined as follows:

$$VO = 100\frac{ACPP^I - ACPP^F}{ACPP^I},$$

where $ACPP^F$ is the average cost of the flexible system that has an outsourcing option, and $ACPP^I$ is the average cost of the inflexible system with no outsourcing option.

The percent cost savings due to the outsourcing opportunity under various parameters are given in Table 5.1. From this table one can observe that very high cost reductions are

Table 5.1. VO for different levels of γ and cost parameters.

$c_1 = 12$ $c_2 = 0$ $p = 0.5$				$c_1 = 12$ $c_2 = 0$ $p = 0.7$			
(μ_D,σ_D)=	(5,1)	(5,3)	(5,5)	(μ_D,σ_D)=	(5,1)	(5,3)	(5,5)
$\gamma = 0$	95.58	86.90	84.67	$\gamma = 0$	84.60	68.97	50.86
5	73.90	72.20	68.79	5	53.54	43.64	23.66
10	58.49	51.16	56.48	10	35.61	24.56	2.70
15	47.49	41.95	45.52	15	16.48	5.42	0.24
$c_1 = 12$ $c_2 = 0$ $p = 0.5$				$c_1 = 12$ $c_2 = 4$ $p = 0.3$			
(μ_D,σ_D)=	(3,3)	(4,3)	(5,3)	(μ_D,σ_D)=	(5,1)	(5,3)	(5,5)
γ=0	56.71	74.71	86.90	γ=0	75.42	65.47	69.44
5	25.76	45.93	72.20	5	36.29	38.19	45.74
10	5.93	20.61	51.16	10	4.97	2.02	25.01
15	2.76	1.77	41.95	15	2.17	1.28	0.70

possible due to outsourcing. When outsourcing is more expensive VO decreases, having the highest value at $\gamma = 0$. Another important factor that determines the value of outsourcing is the capacity utilization, μ_D/μ_C. When the capacity utilization is significantly less than 1 (when the availability of the regular capacity is high), the outsourcing option is not very needed and it is used less, and the difference between the costs of the two systems becomes smaller. The difference is insignificant in some cases, especially when the variance of capacity is not high, as well. This can be observed from Table 5.1, for example, for $c_1 = 12$, $c_2 = 4$, $p = 0.3$ and $(\mu_D, \sigma_D) = (5, 5)$. We also observe that the value of the outsourcing option increases quickly as the demand level increases, for the same capacity parameters, for example, for capacity parameters $c_1 = 12$, $c_2 = 0$ and $p = 0.5$.

For similar utilization values we observe from our experimental results that as the variation of capacity increases the value of outsourcing also increases. For example, for the same demand parameters and similar utilization values of $\mu_D/\mu_C = 5/6.4$ and $\mu_D/\mu_C = 5/6$, as the coefficient of variation of capacity, $cv_C = \sigma_C/\mu_C$, increases from 0.57 (for $c_1 = 12$, $c_2 = 4$, $p = 0.3$) to 1 (for $c_1 = 12$, $c_2 = 0$, $p = 0.5$) VO also increases. The explanation is that when the variation of capacity is high the inventory level of the system with no outsourcing option becomes very high, and this system incurs high inventory costs. On the other hand, when availability of the capacity is low and its variance is high, this system incurs both high inventory holding and penalty costs.

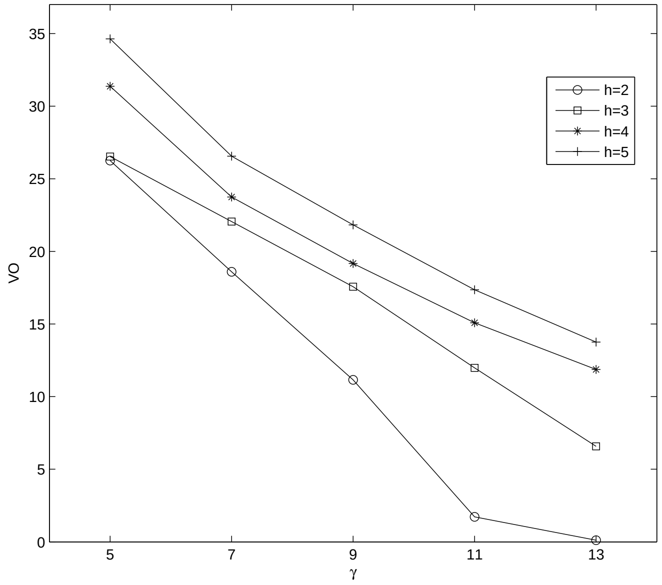

Figure 5.3. VO as a function of outsourcing and holding costs.

The outsourcing option is also more valuable when inventory costs are higher, again since the inflexible system carries too much inventory. This is illustrated in Figure 5.3, where VO is plotted for varying cost parameters with $h \in (2, 3, 4, 5)$ and $\gamma \in (5, 7, 9, 11, 13)$, for the following demand and capacity parameters, $(\mu_D = 5, \sigma_D = 5)$ and $(c_1 = 12, c_2 = 4, p = 0.5)$.

5.3.3. Value of ACI

We also check the possible percentage cost improvements achieved by using one-period ahead ACI, for a flexible system with the outsourcing option. We compare the costs of the ACI-dependent base-stock policy we propose in this thesis to a policy where inventory is managed by an optimal interval policy, also using the outsourcing option. With the interval policy (where capacity reduction is not allowed) the inventory level, I, is brought to be between the extremities of the interval S_1 and S_2, with $S_2 > S_1$, whenever possible. The short amount is outsourced if S_1 cannot be reached, and no action is taken if $I \geq S_2$. This can also be seen as a modified base-stock policy with a target level equal to S_2, where the inventory is not allowed to be lower than S_1. Since there is no optimality conditions for the optimal S_1 and S_2, we search for them exhaustively. Since the capacity information is not observed, we take c, the average expected capacity, as the capacity level. The target inventory level for this policy, y, is given as follows,

$$y = \begin{cases} S_1 & I_n \leq S_1 - c \\ I + c & S_1 - c \leq I \leq S_2 - c \\ S_2 & S_2 - c \leq I \leq S_2 \\ I & S_2 \leq I \end{cases}$$

We define the value of using ACI, $VACI$, as the percent cost decrease caused by using the ACI, which we calculate as follows:

$$VACI = 100 \frac{ACPP^{noACI} - ACPP^{ACI}}{ACPP^{noACI}},$$

where $ACPP^{ACI}$ is the average cost of the system that works under ACI, and $ACPP^{noACI}$ is the average costs of the system that does not observe ACI. Note that since both systems considered in this section use outsourcing option, we can consider cases where the utilization of capacity, μ_D/μ_C, is higher than 1.

Some numerical results for the value of ACI are given in Tables 5.2, 5.3 and 5.4. Having the capacity information even for one-period ahead can result in significant cost reductions, especially when the cost of outsourcing, γ, is high. The value of observing capacity information increases as γ increases.

For a given capacity, as the mean demand increases $VACI$ changes according to the utilization level. For example, when the utilization is increased from $\mu_D/\mu_C = 3/10.8$ to $5/10.8$ $VACI$ increases. But then further increasing μ_D/μ_C to $7/10.8$ we observe a decrease for $VACI$, since there is not as much capacity flexibility to use the obtained information by

Table 5.2. $VACI$ for different levels of γ, p and μ_D for $c_1 = 12$, $c_2 = 0$ and $\sigma_D = 3$.

$c_1 = 12$ $c_2 = 0$				
	(μ_D, σ_D)	(3, 3)	(5, 3)	(7, 3)
$p = 0.5$	$\gamma = 0$	0.51	0.03	3.50
$\mu_C = 6$	5	15.73	20.76	30.50
	10	23.99	26.07	52.79
	15	26.94	29.42	56.29
$p = 0.9$	0	0.03	0.27	0.08
$\mu_C = 10.8$	5	6.57	11.25	7.55
	10	13.00	20.23	12.76
	15	17.32	25.67	19.33

changing the ordering decisions. In a similar way, when capacity is flexible enough to alter the ordering decision, for example, when $\mu_D/\mu_C = 3/6$, increasing it to 5/6, and further to 7/6, results in an increase in $VACI$. These results can be observed from Table 5.2.

Table 5.3. $VACI$ for different levels of γ, p and cost parameters for $\mu_D = 5$ and $\sigma_D = 3$.

$\mu_D = 5$ $\sigma_D = 3$				
	(c_1, c_2)	(12, 4)	(20 ,4)	(12, 0)
$p = 0.5$	$\gamma = 0$	0.72	0.66	0.03
	5	6.27	8.85	20.76
	10	11.25	16.32	26.07
	15	13.62	22.27	29.42
$p = 0.9$	0	0.68	0.81	0.27
	5	2.15	5.03	11.25
	10	3.41	11.24	20.23
	15	4.09	16.66	25.67

We also analyze the relation between $VACI$ and capacity variability. This relation also depends on the capacity utilization level. Given the demand parameters, as the capacity variation increases $VACI$ increases as long as the utilization is not too high. This is due to the fact that there is not enough flexibility to react to the observed information. And given the expected value of the capacity, the increase in the capacity variance always results in an increase in $VACI$. These can be observed from Table 5.3: $p = 0.5$ yields a system where capacity variability is higher as compared to $p = 0.9$, and also given γ and p, the capacity variability increases as (c_1, c_2) moves from (12, 4) to (20, 4) and further to (12, 0). Although, the optimal interval policy does not take ACI into account, it is expected to perform well when the capacity is not too variable. And in fact it can perform better than the ACI-dependent base-stock policy.

As the variation of demand increases, both systems react by increasing the associated inventory levels that characterize the optimal solutions. But since our solution tries to achieve base-stock levels while the interval policy only tries to stay in the interval range (and probably incurring less outsourcing cost, since it does not always try to reach the higher boundary of the interval), the benefit of ACI can be smaller when the demand variability is high. Our results showing how $VACI$ changes as demand variability increases can be seen in Table 5.4 for $c_1 = 12$ and $c_2 = 0$.

Table 5.4. $VACI$ for different levels of γ, p and σ_D for $c_1 = 12$, $c_2 = 0$ and $\mu_D = 5$.

$c_1 = 12$ $c_2 = 0$				
	(μ_D, σ_D)	(5, 1)	(5, 3)	(5, 5)
$p = 0.1$	$\gamma = 0$	6.54	0.87	0.41
	5	2.38	3.51	3.15
	10	3.73	4.98	4.02
	15	5.34	5.38	4.56
$p = 0.3$	0	5.86	0.03	0.33
	5	26.62	20.76	12.57
	10	35.70	26.07	16.32
	15	38.86	29.42	19.99

6. CONCLUSIONS

In this thesis we analyze the effects of integrating available advance information on capacity into the inventory replenishment decisions, and identify the types of operating environments under which ACI is most valuable. We consider a production/inventory system that faces stochastic demand and a supplier with limited and uncertain capacity who has agreed to share its capacity information for a certain number of future periods. We consider three different inventory systems under these conditions.

In Chapter 3, we model the rationing problem of a production/inventory system that serves customers from two different classes. We incorporate the capacity information for a number of future periods into our model, and show that ACI-dependent ordering policy is optimal. We characterize the optimal allocation of available inventory. We analyze the effects of using ACI and rationing policy both in terms of cost and fill rates and provide managerial insights. We also investigate the conditions under which ACI or rationing policy is more valuable. A straightforward extension of this model would be an inventory system facing demand from multiple customer classes. We assume that the capacity information provided by the supplier does not change once it is announced, that is, we consider perfect ACI. It would be interesting to study a model with imperfect capacity information. In this thesis we characterize the optimal ACI-dependent base-stock and rationing levels, but they are difficult to compute analytically. Approximations for the ACI-dependent order-up-to levels and also allocation amounts can be considered. The rationing problem can also be extended to consider service level as a constraint. Here we do not model the consequences of possible demand rejection with rationing. A demand structure depending on the rationed amount could be considered.

In Chapter 4, we introduce heuristic approaches to find the ordering decisions to minimize the expected average cost for a system with uncertain demand and capacity. We first propose to use an ACI-dependent base-stock policy in a specific form where the available information is used giving more weight to the nearer future. For this form of base-stock policy, we propose two heuristics that are based on asymptotic results. First one is based on the logarithmic asymptotic for the steady-state waiting-time distribution for a single-server queue with unlimited waiting space and first-in first-out service discipline. The second one is based on a heavy traffic asymptotic approximation for the tail distribution of a reflected random walk. Then, we present a second type of heuristic that considers weighing the expected costs of the planning periods for which the capacity information is available plus a cost term associated with the periods for which the capacity is not observed. Minimizing the obtained cost function a base-stock level is found for a given period. We evaluate these methods via simulation and discuss their efficiency in reducing costs.

In Chapter 5, we study an inventory/production system with flexible capacity, where flexibility is obtained with an outsourcing option to replenish the inventory to reach the target inventory level in any planning period. For this problem we model the ACI as a signal on the regular capacity distribution. Since obtaining the optimal policy for the system with outsourcing option and ACI is difficult, we propose an ACI-dependent order-up-to level policy. We develop an expression for the resulting average cost. Then, using the first order conditions and the properties of the optimal solution we describe a solution procedure. Through a numerical analysis we show conditions under which the outsourcing option and ACI are more valuable. For this model we consider one-period ahead ACI, specifying the distribution of the regular capacity from two alternatives. A possible extension to this study could be a case where ACI is available for a general number of future periods, or a case where capacity can be realized from a general number of possible distributions. Modeling these extensions is not too difficult, but the derivation of the average cost per period and the computation of the order-up-to policies are considerably harder. Specifically, our approach, which is based on obtaining the average cost expression, cannot be easily extended to a case where information is available for more than two periods. Intuitively, the information on the capacity of the most immediate future period should have the greatest impact on the inventory decision. Therefore, as a future research direction one can think of heuristic approaches that combine our model with a simpler treatment of capacity information on the further away periods.

APPENDIX A: PROOFS OF PROPOSITIONS AND THEOREMS OF CHAPTER 3

Lemma 1. *$E_C[J_N(I_N, \underline{\mathfrak{C}}_N)]$ is differentiable everywhere and its first order derivative is greater than or equal to $-p_1$.*

Proof of Lemma 1. For $I_N \le y_N^{opt} - C_N$ and $I_N \ge y_N^{opt}$, $\partial E_C[J_N(I_N, \underline{\mathfrak{C}}_N)]/\partial I_N$ $= \partial J_N(I_N, \underline{\mathfrak{C}}_N)/\partial I_N$, since $J_N(I_N, \underline{\mathfrak{C}}_N)$ is not actually a function of C for the special case of N:

$$\begin{aligned}
\partial J_N(I_N, \underline{\mathfrak{C}}_N)/\partial I_N = \quad & \int_{I_N+C_N}^{\infty} -p_1 dF_{D^1} + \int_0^{I_N+C_N} \int_0^{I_N+C_N-d^1} h dF_{D^2}(d^2) dF_{D^1}(d^1) \\
& + \int_0^{I_N+C_N} \int_{I_N+C_N-d^1}^{\infty} -p_2 dF_{D^2}(d^2) dF_{D^1}(d^1) \\
\ge \quad & -p_1 + (p_1 - p_2) F_{D^1}(I_N + C_N) \\
& + \int_0^{I_N+C_N} \int_0^{I_N+C_N-d^1} h dF_{D^2}(d^2) dF_{D^1}(d^1) \\
& + \int_0^{I_N+C_N} p_2 F_{D_2}(I_N + C_N - d^1) dF_{D^1}(d^1) \ge -p_1.
\end{aligned}$$

For the case $I_N \le y_N^{opt} - C_N \le y_N^{opt}$, y_N^{opt} is achieved and the derivative is zero. And there is no discontinuity in the derivative. □

Proof of Theorem 3.1. We use backward induction to prove this theorem. The induction assumption is as follows: for a given period $n+1$, $J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})$ is convex in I_{n+1}, and $\partial E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]/\partial I_{n+1}$ is greater than or equal to $-p_1$.

The convexity result for the single-period problem, for period $n = N$, can be established by the convexity of the problem in y_N. In Lemma 1 the result for $\partial E_C/\partial I_N$ is shown.

We now give the structure of the ordering decision and conclude the proof by showing that the induction assumption also holds for period n. For this we first characterize the allocation decision.

When $y_n \le d_n^1$ the only possible allocation is to give y_n to the first class customers and hence $R_n = 0$ in this case. In case $y_n \geq d_n^1$ the allocation problem is equal to:

$$\begin{aligned}
\min \quad & \mathfrak{L}(y_n, \mathfrak{d}_\mathfrak{n}, R_n, \underline{\mathfrak{C}}_n) = hR_n + p_2(d_n^2 - (y_n - min(y_n, d_n^1) - R_n)) \\
& \quad + p_1(d_n^1 - min(y_n, d_n^1)) + E_C[J_{n+1}(R_n, \underline{\mathfrak{C}}_{n+1})] \\
\text{s.t.} \quad & R_n^L \leq R_n \leq R_n^U.
\end{aligned}$$

The first two terms in $\mathfrak{L}(y_n, \mathfrak{d}_\mathfrak{n}, R_n, \underline{\mathfrak{C}}_n)$ are linear in R_n and we know (by the induction assumption) that $E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]$ is convex in $I_{n+1} = R_n$, hence the allocation problem is convex in R_n. The following first order condition gives R_n^{opt}:

$$h + p_2 + \partial E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]/\partial R_n = 0.$$

By the convexity of $\mathfrak{L}(y_n, \mathfrak{d}_\mathfrak{n}, R_n, \underline{\mathfrak{C}}_n)$, whenever $max(0, y_n - min(y_n, d_n^1) - d_n^2) \leq R_n \leq y_n - min(y_n, d_n^1)$, the optimum can be reached and $R_n = R_n^{opt}$. If $R_n^{opt} \geq y_n - d_n^1$, then R_n^{opt} cannot be reached and by convexity best possible cost is achieved at $R_n^* = y_n - d_n^1$. And whenever $y_n - d_n^1 - d_n^2 \geq R_n^{opt}$, then again R_n^{opt} cannot be reached and by convexity the best possible cost is reached at $R_n^* = y_n - d_n^1 - d_n^2$.

Using the defined properties of the allocation decision, $G_n(y_n, \mathfrak{C}_n)$ is written as follows:

$$\begin{aligned}
G_n(y_n, \mathfrak{C}_n) &= \int_{y_n}^{\infty} \int_0^{\infty} \Big(p_1(d^1 - y_n) + p_2 d^2 + E_C[J_{n+1}(0, \underline{\mathfrak{C}}_{n+1}] \Big) dF_{D^2}(d^2) dF_{D^1}(d^1) \\
&+ \int_{y_n - R_n^{opt}}^{y_n} \int_0^{\infty} \Big(p_2 d^2 + h(y_n - d^1) + E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})] \Big) dF_{D^2}(d^2) dF_{D^1}(d^1) \\
&+ \int_0^{y_n - R_n^{opt}} \int_{y_n - d^1 - R_n^{opt}}^{\infty} \Big(p_2(d^2 - (y_n - d^1 - R_n^{opt})) + hR_n^{opt} \\
&\qquad + E_C[J_{n+1}(R_n^{opt}, \underline{\mathfrak{C}}_{n+1})] \Big) dF_{D^2}(d^2) dF_{D^1}(d^1) \\
&+ \int_0^{y_n - R_n^{opt}} \int_0^{y_n - d^1 - R_n^{opt}} \Big(h(y_n - d^1 - d^2) + E_C[J_{n+1}(y_n - d^1 - d^2, \underline{\mathfrak{C}}_{n+1})] \Big) dF_{D^2}(d^2) dF_{D^1}(d^1).
\end{aligned}$$

The first and second order partial derivative of $G_n(y_n, \mathfrak{C}_n)$ with respect to y_n are:

$$\begin{aligned}
\partial G_n/\partial y_n &= \int_{y_n}^{\infty} -p_1 dF_{D^1}(d^1) + \int_0^{y_n - R_n^{opt}} \int_{y_n - d^1 - R_n^{opt}}^{\infty} -p_2 dF_{D^2}(d^2) dF_{D^1}(d^1) \\
&+ \int_{y_n - R_n^{opt}}^{y_n} \int_0^{\infty} \Big(h + \partial E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big) dF_{D^2}(d^2) dF_{D^1}(d^1) \\
&+ \int_0^{y_n - R_n^{opt}} \int_0^{y_n - d^1 - R_n^{opt}} \Big(h + \partial E_C[J_{n+1}(y_n - d^1 - d^2, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big) dF_{D^2}(d^2) dF_{D^1}(d^1)
\end{aligned}$$

$$\begin{aligned}
&\partial^2 G_n/\partial y_n^2 = \\
&p_1 f_{D^1}(y_n) + \int_0^{\infty} \Big(h + \partial E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big)_{d^1 = y_n} dF_{D^2}(d^2) f_{D^1}(y_n) \\
&- \int_0^{\infty} \Big(h + \partial E_C[J_{n+1}(y_n - d^1, C_{n+1})]/\partial y_n\Big)_{d^1 = y_n - R_n^{opt}} dF_{D^2}(d^2) f_{D^1}(y_n - R_n^{opt}) \\
&+ \int_0^{y_n - R_n^{opt}} p_2 f_{D^2}(y_n - d^1 - R_n^{opt}) dF_{D^1}(d^1) - p_2 f_{D^1}(y_n - R_n^{opt}) \\
&+ \int_0^{y_n - R_n^{opt}} \Big(h + \partial E_C[J_{n+1}(y_n - d^1 - d^2, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big)_{d^2 = y_n - d^1 - R_n^{opt}} \\
&\quad f_{D^2}(y_n - d^1 - R_n^{opt}) \quad dF_{D^1}(d^1) \\
&+ \int_0^{y_n - R_n^{opt}} \int_0^{y_n - d^1 - R_n^{opt}} \partial^2 E_C[J_{n+1}(y_n - d^1 - d^2, \underline{\mathfrak{C}}_{n+1})]/\partial y_n^2 \quad dF_{D^2}(d^2) dF_{D^1}(d^1) \\
&+ \int_{y_n - R_n^{opt}}^{y_n} \int_0^{\infty} \partial^2 E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial y_n^2 \quad dF_{D^2}(d^2) dF_{D^1}(d^1).
\end{aligned}$$

The second derivative can be rewritten as follows:

$$\begin{aligned}
&\partial^2 G_n/\partial y_n^2 = (p_1 + h) f_{D^1}(y_n) + \Big(h + p_2 + \partial E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big)_{R_n^{opt}} \\
&\int_0^{y_n - R_n^{opt}} f_{D^2}(y_n - d^1 - R_n^{opt}) dF_{D^1}(d^1) \\
&- f_{D^1}(y_n - R_n^{opt}) \Big(h + p_2 - \partial E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big)_{R_n^{opt}} \\
&+ \int_0^{y_n - R_n^{opt}} \int_0^{y_n - d^1 - R_n^{opt}} \partial^2 E_C[J_{n+1}(y_n - d^1 - d^2, \underline{\mathfrak{C}}_{n+1})]/\partial y_n^2 \quad dF_{D^2}(d^2) dF_{D^1}(d^1) \\
&+ \int_{y_n - R_n^{opt}}^{y_n} \int_0^{\infty} \partial^2 E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial y_n^2 \quad dF_{D^2}(d^2) dF_{D^1}(d^1) \\
&+ f_{D^1}(y_n) \Big(\partial E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big)_{d^1 = y_n}.
\end{aligned}$$

The coefficient $h + p_2 - \partial E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]/\partial y_n)_{R_n^{opt}}$ is equal to zero, from the first order condition for the allocation problem. Hence $\partial G_n/\partial y_n$ can be rewritten as follows:

$$
\begin{aligned}
&\partial^2 G_n/\partial y_n^2 = (p_1+h) f_{D^1}(y_n)\\
&+\int_0^{y_n - R_n^{opt}} \int_0^{y_n - d^1 - R_n^{opt}} \partial^2 E_C[J_{n+1}(y_n - d^1 - d^2, \underline{\mathfrak{C}}_{n+1})]/\partial y_n^2 \quad dF_{D^2}(d^2) dF_{D^1}(d^1)\\
&+\int_{y_n - R_n^{opt}}^{y_n} \int_0^{\infty} \partial^2 E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial y_n^2 \quad dF_{D^2}(d^2) dF_{D^1}(d^1)\\
&+f_{D^1}(y_n)\Big(\partial E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big)_{d^1 = y_n}.
\end{aligned}
$$

From the induction assumption, $\partial E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial y_n)_{d^1=y_n}$ is larger than $-p_1$, and $E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]$ is convex. Therefore the second partial derivative is nonnegative. Thus there exists an optimal solution $y_n^{opt}(\mathfrak{C}_n)$ that minimizes G_n. Moreover the optimal policy is a modified base-stock policy, and $J_n(I_n, \underline{\mathfrak{C}}_n)$ is convex in I_n.

To show that $\partial E_C[J_n(I_n, \underline{\mathfrak{C}}_n)]/\partial I_n \geq -p_1$ we first show that $\partial G_n(y_n, \mathfrak{C}_n)/\partial y_n \geq -p_1$. For this we use the definition of $G_n(y_n, \mathfrak{C}_n)$ and the induction assumption that $\partial E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]/\partial I_{n+1} \geq -p_1$, which gives

$$
\begin{aligned}
\partial G_n(y_n, \mathfrak{C}_n)/\partial y_n \quad &\geq -p_1 + (p_1 - p_2) F_{D^1}(y_n - R_n^{opt})\\
&+(h - p_1 + p_2)\int_0^{y_n - R_n^{opt}} F_{D^2}(y_n - d^1 - R_n^{opt}) dF_{D^1}(d^1)\\
&\geq -p_1.
\end{aligned}
$$

Hence, by its definition $\partial J_n(I_n, \underline{\mathfrak{C}}_n)/\partial I_n \geq -p_1$. Now, by the Leibniz integral rule we can write $\partial E_C[J_n(I_n, \underline{\mathfrak{C}}_n)]/\partial I_n = E_C[\frac{\partial}{\partial I_n} J_n(I_n, \underline{\mathfrak{C}}_n)]$, Since $\partial J_n(I_n, \underline{\mathfrak{C}}_n)/\partial I_n \geq -p_1$, its expectation is also greater than $-p_1$, which completes the proof. □

Proof of Proposition 3.2. We prove this by induction: for $n = N$, $G_N(y_N, \mathfrak{C}_N)$ and y_N^{opt} satisfies $\partial G_N(y_N, \mathfrak{C}_N)/\partial y_N \leq \partial G_N(y_N, \tilde{\mathfrak{C}}_N)/\partial y_N$ and $\tilde{y}_N^{opt} \leq y_N^{opt}$, because the last period problem is actually independent of the capacity information. We assume this also holds for $n+1$, that is, $\partial G_{n+1}(y_n, \mathfrak{C}_{n+1})/\partial y_{n+1} \leq \partial G_{n+1}(y_{n+1}, \tilde{\mathfrak{C}}_{n+1})/\partial y_{n+1}$. y_{n+1}^{opt} is the solution to the first order condition, $\partial G_{n+1}(y_n, \mathfrak{C}_{n+1})/\partial y_{n+1} = 0$ and this gives $\tilde{y}_{n+1}^{opt} \leq y_{n+1}^{opt}$.

Next we check if $\partial J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})/\partial I_{n+1} \leq \partial J_{n+1}(I_{n+1}, \tilde{\underline{\mathfrak{C}}}_{n+1})/\partial I_{n+1}$. By the induction hypothesis $\tilde{y}_{n+1}^{opt} \leq y_{n+1}^{opt}$. There are two possible cases:

$$\begin{aligned} CaseA: &\quad \tilde{y}_{n+1}^{opt} - \tilde{C}_n \leq \tilde{y}_{n+1}^{opt} \leq y_{n+1}^{opt} - C_n \leq y_{n+1}^{opt} \\ CaseB: &\quad \tilde{y}_{n+1}^{opt} - \tilde{C}_n \leq y_{n+1}^{opt} - C_n \leq \tilde{y}_{n+1}^{opt} \leq y_{n+1}^{opt}. \end{aligned}$$

With $\tilde{\mathfrak{C}}_n \geq \mathfrak{C}_n$, for both cases, there are five possible intervals that I_{n+1} can fall into. These are shown in Figure A.1: The relation between $\partial J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})/\partial I_{n+1}$ and $\partial J_{n+1}(I_{n+1}, \tilde{\underline{\mathfrak{C}}}_{n+1})/\partial I_{n+1}$

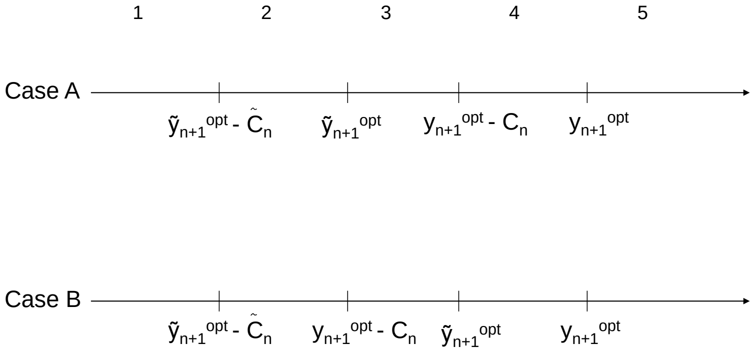

Figure A.1. Possible intervals for the inventory level I_{n+1}.

will be examined next:

- $I_{n+1} \in 1$: For both cases $\partial J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})/\partial I_{n+1} = \partial G_{n+1}(y_{n+1}, \mathfrak{C}_{n+1})/\partial y_{n+1}$ and $\partial J_{n+1}(I_{n+1}, \tilde{\underline{\mathfrak{C}}}_{n+1})/\partial I_{n+1} = \partial G_{n+1}(I_{n+1}, \tilde{\mathfrak{C}}_{n+1})/\partial y_{n+1}$. Since G_{n+1} is increasing in $\mathfrak{C}_{n+1}$, J_{n+1} is also increasing.
- $I_{n+1} \in 2$: For both cases $\partial J_{n+1}(I_{n+1}, \tilde{\underline{\mathfrak{C}}}_{n+1})/\partial I_{n+1} = \partial G_{n+1}(I_{n+1}, \tilde{\mathfrak{C}}_{n+1})/\partial y_{n+1} \geq 0$ and $\partial J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})/\partial I_{n+1} = \partial G_{n+1}(y_{n+1}^{opt}, \mathfrak{C}_{n+1})/\partial y_{n+1} = 0$, hence $\partial J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})/\partial I_{n+1} \leq \partial J_{n+1}(I_{n+1}, \tilde{\underline{\mathfrak{C}}}_{n+1})/\partial I_{n+1}$.
- $I_{n+1} \in 3$: Here for Case A, $\partial J_{n+1}(I_{n+1}, \tilde{\underline{\mathfrak{C}}}_{n+1})/\partial I_{n+1} = \partial G_{n+1}(I_{n+1}, \tilde{\mathfrak{C}}_{n+1})/\partial y_{n+1} \geq 0$ and $\partial J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})/\partial I_{n+1} = \partial G_{n+1}(I_{n+1}, \mathfrak{C}_{n+1})/\partial y_{n+1} \leq 0$. And for Case B, both derivatives are equal to zero since optimal values for G_{n+1} are achieved.
- $I_{n+1} \in 4$: For both cases $\partial J_{n+1}(I_{n+1}, \tilde{\underline{\mathfrak{C}}}_{n+1})/\partial I_{n+1} = 0$ and $\partial J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})/\partial I_{n+1} \leq 0$.
- $I_{n+1} \in 5$: For both cases $\partial J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})/\partial I_{n+1} = \partial G_{n+1}(y_{n+1} + C_{n+1}, \mathfrak{C}_{n+1})/\partial y_{n+1}$

and $\partial J_{n+1}(I_{n+1}, \underline{\tilde{\mathfrak{C}}}_{n+1})/\partial I_{n+1} = \partial G_{n+1}(I_{n+1} + \tilde{C}_{n+1}, \tilde{\mathfrak{C}}_{n+1})/\partial y_{n+1}$. Since $\partial G_{n+1}/\partial y_{n+1}$ is increasing in $\mathfrak{C}_{n+1}$ and $I_{n+1} + \tilde{C}_{n+1} \geq I_{n+1} + C_{n+1}$, we have $\partial J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})/\partial I_{n+1} \leq \partial J_{n+1}(I_{n+1}, \underline{\tilde{\mathfrak{C}}}_{n+1})/\partial I_{n+1}$.

We can conclude that $\partial J_{n+1}/\partial I_{n+1}$ is increasing in the capacity information. Next we prove that this also holds for $\partial E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]/\partial I_{n+1}$.

$$\begin{aligned}
\partial E_C[J_{n+1}(I_{n+1}, \underline{C}_{n+1})]/\partial I_{n+1} = &\quad \frac{\partial}{\partial I_{n+1}} \int_{C=0}^{\infty} J_{n+1}(I_{n+1}, \underline{\tilde{\mathfrak{C}}}_{n+1}) dF_C(C_{n+k+1}) \\
= &\quad \int_{C=0}^{\infty} \frac{\partial}{\partial I_{n+1}} J_{n+1}(I_{n+1}, \underline{\tilde{\mathfrak{C}}}_{n+1}) dF_C(C_{n+k+1}) \\
\geq &\quad \int_{C=0}^{\infty} \frac{\partial}{\partial I_{n+1}} J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1}) dF_C(C_{n+k+1}) \\
= &\quad \partial E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]/\partial I_{n+1}.
\end{aligned}$$

From this result $\partial E_C[J_{n+1}(I_{n+1} = R_n, \underline{\mathfrak{C}}_{n+1})]/\partial R_n$ is increasing in $\mathfrak{C}_{n+1}$, hence $\tilde{R}_n^{opt} \leq R_n^{opt}$.

To show that $\partial G_n(y_n, \mathfrak{C}_n)/\partial y_n$ is increasing in $\mathfrak{C}_n$ we show that $\partial G_n(y_n, \mathfrak{C}_n)/\partial y_n$ is increasing in C_{n+i} for $i \in (1, \cdots, k)$. Let's drop the time index for C_{n+i} for this proof. We now explicitly write R_n^{opt} as a function of C:

$$
\begin{aligned}
&\partial^2 G_n/\partial C\partial y_n \\
&= R_n^{opt\prime}(C)\Big(h + \partial E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big)_{d^1 = y_n - R_n^{opt\prime}(C)} f_{D^1}(y_n - R_n^{opt\prime}(C)) \\
&\int_{y_n - R_n^{opt}(C)}^{y_n} \partial^2 E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial C\partial y_n \;\; dF_{D^1}(d^1) \\
&+ p_2 f_{D^1}(y_n - R_n^{opt\prime}(C))R_n^{opt\prime}(C) - p_2 \int_0^{y_n - R_n^{opt}(C)} R_n^{opt\prime}(C) f_{D^2}(y_n - d^1 - R_n^{opt}(C))dF_{D^1}(d^1) \\
&- R_n^{opt\prime}(C)\int_0^{y_n - R_n^{opt}(C)}\Big(h + \partial E_C[J_{n+1}(y_n - d^1 - d^2, \underline{\mathfrak{C}}_{n+1})]/\partial y_n\Big)_{d^2 = y_n - d^1 - R_n^{opt\prime}(C)} \\
&f_{D^2}(y_n - d^1 - R_n^{opt\prime}(C))dF_{D^1}(d^1) \\
&+ \int_0^{y_n - R_n^{opt}(C)} \int_0^{y_n - d^1 - R_n^{opt}(C)} \partial^2 E_C[J_{n+1}(y_n - d^1 - d^2, \underline{\mathfrak{C}}_{n+1})]/\partial C\partial y_n \;\; dF_{D^2}(d^2)dF_{D^1}(d^1) \\
&= R_n^{opt\prime}(C) f_{D^1}(y_n - R_n^{opt}(C))\Big[h + p_2 + \partial E_C[J_{n+1}(R_n^{opt}(C), \underline{\mathfrak{C}}_{n+1})]/\partial I_{n+1}\Big] \\
&- R_n^{opt\prime}(C)\int_0^{y_n - R_n^{opt}(C)} f_{D^2}(y_n - d^1 - R_n^{opt}(C))dF_{D^1}(d^1) \\
&\Big[h + p_2 + \partial E_C[J_{n+1}(R_n^{opt}(C), \underline{\mathfrak{C}}_{n+1})]/\partial I_{n+1}\Big] \\
&+ \int_{y_n - R_n^{opt}(C)}^{y_n} \partial^2 E_C[J_{n+1}(y_n - d^1, \underline{\mathfrak{C}}_{n+1})]/\partial C\partial y_n \;\; dF_{D^1}(d^1) \\
&+ \int_0^{y_n - R_n^{opt}(C)} \int_0^{y_n - d^1 - R_n^{opt}(C)} \partial^2 E_C[J_{n+1}(y_n - d^1 - d^2, \underline{\mathfrak{C}}_{n+1})]/\partial C\partial y_n \;\; dF_{D^2}(d^2)dF_{D^1}(d^1).
\end{aligned}
$$

The first two terms are zero from the first order condition for the allocation problem. The last two terms are positive since $\partial E_C[J_{n+1}(I_{n+1}, \underline{\mathfrak{C}}_{n+1})]/\partial I_{n+1}$ is increasing in capacity.

We have shown that $\partial G_n/\partial y_n$ is increasing in $\mathfrak{C}_n$. As a result y_n^{opt} is decreasing in $\mathfrak{C}_n$. Note that y_n^{opt} is independent of C_n. □

APPENDIX B: PROOFS OF PROPOSITIONS AND THEOREMS OF CHAPTER 5

Proof of Proposition 5.1. It is sufficient to show the convexity of $G_0(S_1, \Delta, T_1, T)]$, as $E[T]$ does not depend on the decision variables, and the sum of convex functions is convex. Then,

$$\begin{aligned}\partial G_0(S_1, \Delta, T_1, T)/\partial S_1 = \quad & \sum_{i=1}^{T_1} \mathcal{L}^{'}(S_1 + \Delta) + \sum_{i=T_1+1}^{T} \int_0^{\Delta} \mathcal{L}^{'}(S_1 + \Delta - x) dF_{(T_1,i-1)}(x) \\ & + \sum_{i=T_1+1}^{T} \left(1 - F_{(T_1,i-1)}(\Delta)\right) \mathcal{L}^{'}(S_1),\end{aligned} \tag{B.1}$$

where $\mathcal{L}^{'}(y) = -b + (b+h)F(y)$. Taking one more derivative:

$$\begin{aligned}\partial^2 G_0(S_1, \Delta, T_1, T)/\partial S_1^2 \quad = \quad & \sum_{i=1}^{T_1} \mathcal{L}^{''}(S_1 + \Delta) + \sum_{i=T_1+1}^{T} \int_0^{\Delta} \mathcal{L}^{''}(S_1 + \Delta - x) dF_{(T_1,i-1)}(x) \\ & + \sum_{i=T_1+1}^{T} \left(1 - F_{(T_1,i-1)}(\Delta)\right) \mathcal{L}^{''}(S_1) \geq 0,\end{aligned}$$

since $\mathcal{L}^{''}(y) = (b+h)dF(y) \geq 0$, and this establishes the convexity of $APPC$ in S_1 for a fixed value of Δ. □

Proof of Proposition 5.2. We first note that $\mathbb{C}_2$ affects $\partial G_0(S_1, \Delta, T_1, T)/\partial \Delta$ through the expectation on the second line of Equation 5.11. Let $B_2^a(x)$ and $B_2^b(x)$ be distribution functions for $\mathbb{C}_2^a$ and $\mathbb{C}_2^b$, respectively. Since $(1 - F(\Delta + y))$ is a decreasing function of y, and $\mathbb{C}_2^a \geq_{st} \mathbb{C}_2^b$ we have

$$-\int_{y=0}^{\infty} \gamma(1 - F(\Delta + y)) dB_2^a(y) \geq -\int_{y=0}^{\infty} \gamma(1 - F(\Delta + y)) dB_2^b(y),$$

for all $\Delta \geq 0$, and the claim follows. Intuitively, if the realizations of capacity get smaller (stochastically), then the corresponding optimal Δ gets larger in order to avoid incurring excessive shortfall costs. □

Proof of Proposition 5.3. We consider terms of Equation $\partial E[G_0(S_1, \Delta, T_1, T)]/\partial \Delta$ (see equation 5.11) one by one and bound them: (i) Since $\mathcal{L}^{'}(y)$ is an increasing function

$$E\left[\sum_{i=1}^{T_1} \mathcal{L}^{'}(S_1 + \Delta)\right] \geq E[T_1]\mathcal{L}^{'}(\Delta).$$

(ii) The second term:

$$\begin{aligned}
&E\left[\sum_{i=T_1+1}^{T} \int_0^{\Delta} \mathcal{L}^{'}(S_1 + \Delta - x) dF_{(T_1,i-1)}(x)\right] \qquad \text{(B.2)}\\
&\geq E\left[\sum_{i=T_1+1}^{T} \int_0^{\Delta} \mathcal{L}^{'}(\Delta - x) dF_{(T_1,i-1)}(x)\right]\\
&= -bE\left[\sum_{i=T_1+1}^{T} F_{(T_1,i-1)}(\Delta)\right] + (h+b)E\left[\sum_{i=T_1+1}^{T} F_{(T_1,1)}(\Delta)\right]\\
&= -b(E[\tau] - E[T_1]) + (h+b)E\left[\sum_{i=T_1+1}^{T} F_{(T_1,i)}(\Delta)\right].
\end{aligned}$$

(iii) The third term:

$$\int_{y=0}^{\infty} \gamma(1 - F(\Delta + y)) dB_2(y) \leq \gamma(1 - F(\Delta)).$$

(iv) The fourth term:

$$\begin{aligned}
&\gamma E\left[\sum_{i=T_1+1}^{T-1} \int_{y=0}^{\infty} \int_{x_1=0}^{\Delta} \int_{x_2=\Delta+y-x_1}^{\infty} dF(x_2) dF_{(T_1,i-1)}(x_1) dB_1(y)\right]\\
= \quad &\gamma E\left[\sum_{i=T_1+1}^{T-1} \int_{y=0}^{\infty} \Pr\{D(T_1, i-1) \leq \Delta, D(T_1, i) \geq y + \Delta\} dB_1(y)\right]\\
\leq \quad &\gamma E\left[\sum_{i=T_1+1}^{T-1} \Pr\{\Delta, D(T_1, i) \geq \Delta\}\right] = \gamma E\left[\sum_{i=T_1+1}^{T-1} (1 - F_{(T_1,i)}(\Delta))\right].
\end{aligned}$$

(v) For the last term:

$$\gamma \int_{y=0}^{\infty} (1 - F(y - \Delta))(1 - F_{(T_1,T-1)}(\Delta)) dB_1(y) \geq 0.$$

Combining the right hand sides and the left hand sides of the inequalities and noting that

$$E[\sum_{i=T_1+1}^{T-1}(1-F_{(T_1,i)}(\Delta))]+(1-F(\Delta))=E[\sum_{i=T_1}^{T-1}(1-F_{(T_1,i)}(\Delta))]=E[T]-E[\tau]$$

yields

$$\partial E[G_0(S_1,\Delta,T_1,T)]/\partial\Delta \geq G'_{LB},$$

where

$$G'_{LB}(\Delta)=E[T_1]\mathcal{L}^{'}(\Delta)-b(E[\tau]-E[T_1])+(h+b)E\left[\sum_{i=T_1+1}^{T}F_{(T_1,i)}(\Delta)\right]-\gamma(E[T]-E[\tau]).$$

$G'_{LB}(\Delta)$ is continuous in Δ, $G'_{LB}(\Delta)\to -bE[T_1]-\gamma(E[T]-E[T_1])<0$ as $\Delta\to 0$ (since $E[\tau]\to E[T_1]$), and $G'_{LB}(\Delta)\to hE[T]>0$ as $\Delta\to\infty$ (since $E[\tau]\to E[T]$). Therefore, there is a solution to $G'_{LB}(\Delta)=0$. Let Δ_{UB} be the smallest solution. The claim and Equation 5.16 follow by rearranging $G'_{LB}(\Delta)$. □

Proof of Proposition 5.4. Since S_1 is decreasing in Δ it can take its highest value at $\Delta=0$. We have shown that given Δ the inventory cost is convex in S_1, hence S_1 can be found from the following first order condition,

$$\begin{aligned}\partial E[G_0(S_1,\Delta,T_1,T)]/\partial S_1 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^m p^n \partial G_0(S_1,\Delta,m,n)/\partial S_1\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^m p^n \sum_{i=1}^{m}\mathcal{L}^{'}(S_1+\Delta)\\ &\quad+\sum_{i=m+1}^{n}\int_0^{\Delta}\mathcal{L}^{'}(S_1+\Delta-x)dF_{(m,i-1)}(x)\\ &\quad+\sum_{i=m+1}^{n}\left(1-F_{(m,i-1)}(\Delta)\right)\mathcal{L}^{'}(S_1)=0.\end{aligned}$$

For $\Delta = 0$ the first order condition becomes,

$$\partial E[G_0(S_1, \Delta, T_1, T)]/\partial S_1 \;=\; \sum_{n=1}^{\infty} p^n \sum_{i=1}^{n} \mathcal{L}^{'}(S_1) = \mathcal{L}^{'}(S_1) \sum_{n=1}^{\infty} p^n n = 0.$$

From this equation we can conclude that the upper bound S_1^{UB} is equal to the newsvendor solution. $\square$

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