GEODESICS OF THE SCHWARZSCHILD BLACK HOLE

by

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ABSTRACT

GEODESICS OF THE SCHWARZSCHILD BLACK HOLE

Black holes are best known with their strong gravitational field from which even light can not escape. They are sometimes called *spacetime singularities* [1, 2]. There are observational evidence of existence of black holes [3]. The simplest and the best known one is the *Schwarzschild black hole* which has a static spherically symmetric gravitational field, uncharged and not rotating. However, there are still some ambiguities inherited in the solution of the Schwarzschild black hole. To get a better understanding of them one can look at how particles and light behave around the black hole, i.e. their geodesics. In this thesis we studied geodesics of the Schwarzschild black hole inside and outside of the event horizon.

First, derivation of the *Schwarzschild metric* is done using *Cartan's structural equations*. Then, it is verified that the geodesic of a particle in free spacetime is a straight line.

Afterwards, we have looked at geodesics of the Schwarzschild black hole. By calculating the equations for the geodesics of both - the particle and light - in two cases (angular momentum is zero and nonzero) we defined a potential and drew its graph and identified the orbits around the black hole. Then, we examined consistency of the graphs with our equations.

Lastly, by comparing the defined potential with the Newtonian potential, we designated how particles and light behave differently around the black hole.

ÖZET

SCHWARZSCHİLD KARA DELİĞİNİN JEODEZİKLERİ

Kara delikler, ışığın bile kaçamadığı çok güçlü çekim alanları ile tanınırlar. Uzay-zaman tekillikleri [1, 2] olarak da nitelenen kara deliklerin gözlemsel kanıtları günümüzde mevcuttur [3]. En basit ve en iyi bilinen türü küresel simetrik çekim alanına sahip, yüksüz ve açısal momentumu sıfır olan Schwarzschild kara deliğidir. Fakat hâlâ bu kara delik çözümüyle ilgili anlaşılamamış noktalar vardır. Kara delikleri anlamak için kütleli parçacıklarn ve ışığın kara delik etrafında nasıl hareket ettiğine yani jeodeziklerine bakabiliriz. Bu tezde de Schwarzschild kara deliğinin olay ufkunun içindeki ve dışındaki jeodezikleri inceledik.

İlk olarak Schwarzschild metriğini Cartan yapı denklemlerini kullanarak yeniden türettik. Sonrasında boş uzay-zamanda parçacıkların jeodeziklerinin düz bir çizgi olduğunu gösterdik.

Takiben, Schwarzschild kara deliği etrafında parçacıkların ve ışığın jeodeziklerini veren denklemleri açısal momentumun sıfır ve sıfırdan farklı olduğu durumlar için hesapladık. Elde edilen denklemlerden bir potansiyel tanımlayarak grafiğini çizdik. Bu grafikten kara delik etrafında parçacıklar ve ışık için hangi yörüngeler olduğunu belirledik.

Son olarak tanımladığımız potansiyeli Newton potansiyeli ile karşılaştırarak kara deliklerin hangi yönlerden farklı olduğunu gösterdik.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

$g_{\mu\nu}$	The second rank metric tensor
$G_{\mu\nu}$	Einstein tensor
$R_{\mu\nu}$	Ricci tensor
$T_{\mu\nu}$	Energy momentum tensor
ϕ	Scalar Field
K	Kretschman scalar
R	Ricci scalar

1. INTRODUCTION

Before Einstein, we understood *gravity* in a flat spacetime through the *Newton's Law of Universal Gravitation* which states that the force, $\vec{F}$, exerted on a body with mass m, by another mass M (where M and m are assumed to be spherical or point masses) located at the origin is given by

$$\vec{F} = -\frac{GmM}{r^2}\hat{r}, \tag{1.1}$$

where G is the universal gravitation constant and r is the distance from origin to m. Thus, the gravitational potential, ϕ, generated by M satisfies *Poisson's Equation*

$$\nabla^2\phi = 4\pi GM\delta(\vec{r}), \tag{1.2}$$

where $M\delta(\vec{r})$ is the mass density obtained by taking M as a point particle at the origin. The spherically symmetric solution of above equation gives the gravitational force as

$$\begin{aligned} \vec{F} &= -m\vec{\nabla}\phi \\ \vec{g} &\equiv -\vec{\nabla}\phi. \end{aligned} \tag{1.3}$$

where $\vec{g}$ is the gravitational field. Einstein on the other hand, described gravity on a curved spacetime with his *Einstein Field Equations*

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.4}$$

where $G_{\mu\nu}$ is the *Einstein Tensor* and $T_{\mu\nu}$ is the *Stress-Energy Tensor*. Einstein equations describe the geometry of spacetime based on the matter content in that spacetime and motion of matter is determined by this geometry [4]. All the metrics that describe the geometry of physical systems should obey the field equations.

2. THE SCHWARZSCHILD BLACK HOLE

Black hole concept was first demonstrated by Rev John Mitchell in 1783, [5]. He imagined a massive body, M, such that the escape velocity, v, of a mass m is equal to the speed of light, c. We know that the escape velocity of m can be found when its gravitational energy is equal to its kinetic energy

$$\frac{GmM}{r} = \frac{1}{2}mv^2, \tag{2.1}$$

$$v = \sqrt{\frac{2GM}{r}}. \tag{2.2}$$

Now take v=c, and see what is the mass to radius ratio

$$\frac{M}{r} \sim 10^{27} kg/m, \tag{2.3}$$

which shows that M is a very dense object such that 10^{27}kg is concentrated into a 1m-radius sphere. Today, a singularity in spacetime caused by such a dense object, M, is defined as *black hole*. For the mass M to be a black hole, it must be concentrated into its *Schwarzschild radius*, that is $r_s = \frac{2GM}{c^2}$. Spacetime geometry is summarized by a line element, ds, giving the spacetime distance between any two nearby points. Square of this line element, i.e. ds^2, is called *metric*. A Schwarzschild black hole describes an uncharged and non-rotating black hole and is summarized by the Schwarzschild metric given below,

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.4}$$

The metric is the unique solution of the vacuum Einstein equations under spherical symmetry assumption [6]. Notice that it has coordinate singularities at $r = r_s$ and $r = 0$. The first one stems from the choice of coordinate system, so that one can get rid off by changing the coordinate system. However, the latter is an essential curvature singularity [6,7]. This is shown by calculating *Kretschmann scalar*, K, that is [8]

$$K \equiv R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} = \frac{12 r_s^2}{r^6}, \tag{2.5}$$

where $R_{\mu\nu\alpha\beta}$ is the *Riemann curvature tensor*. Since K is coordinate independent the singularity at $r = 0$ can not be removed. Another point we want to emphasize is that, the metric is invariant under time-reversal, i.e. $t \to -t$. This means that a particle traveling forward in time is equivalent to the one traveling backward in time. However, physically particles can cross the event horizon in only one direction that is from exterior to interior region [7,9].

2.1. Derivation of the Schwarzschild Metric

Schwarzschild metric describes the geometry of spacetime around a non-rotating, uncharged and spherically symmetric body. A general static and spherically symmetric metric can be written as follows:

$$ds^2 = (f(r)dt)^2 - (g(r)dr)^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.6}$$

Our aim is to find $f(r)$ and $g(r)$ by solving vacuum Einstein equations. The derivation will be done using *Cartan's Structural Equations*. 3 steps of the derivation is in the following way:

2.1.1. Calculation of the Connection 1-forms Using Cartan's First Structural Equation

For the metric given below

$$ds^2 = (f(r)dt)^2 - (g(r)dr)^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.7}$$

we can take our orthonormal *basis 1-forms*, e^i, as

$$e^0 = f(r)dt, \quad e^1 = g(r)dr, \quad e^2 = rd\theta, \quad e^3 = r\sin\theta d\phi \tag{2.8}$$

Cartan's first structural equation is given by

$$de^i + w^i_j e^j = de^i + w^i_{jk} e^k \wedge e^j = 0, \tag{2.9}$$

where w^i_j and w^i_{jk} are called connection 1-forms and connection coefficients respectively and $i, j, k = 0, 1, 2, 3$. In the calculations we will omit the *wedge operator*, $\wedge$, it being understood that the product between all the forms is the wedge product. We would like to solve Equation 2.9 for each i. Our aim is to calculate $i = 0$ explicitly then, give the results for the remaining ($i = 1, 2, 3$). The first structure equation for $i = 0$ is

$$\begin{aligned} & de^0 + w^0_{10}e^0e^1 + w^0_{12}e^2e^1 + w^0_{13}e^3e^1 \\ & + w^0_{20}e^0e^2 + w^0_{21}e^1e^2 + w^0_{23}e^3e^2 \\ & + w^0_{30}e^0e^3 + w^0_{31}e^1e^3 + w^0_{32}e^2e^3 = 0. \end{aligned} \tag{2.10}$$

de^0 can be calculated easily as follows:

$$de^0 = d(f(r)dt) = f^{'}(r)drdt = f^{'}(r)\frac{e^1}{g(r)}\frac{e^0}{f(r)}, \tag{2.11}$$

where prime means derivative. Then, inserting the Equation 2.11 into Equation 2.10 and gathering e^0e^1 terms reveals

$$w^0_{10} = \frac{f^{'}(r)}{g(r)f(r)}. \tag{2.12}$$

Similarly from the other terms we have

$$w^0_{20} = w^0_{30} = 0 \quad w^0_{12} = w^0_{21} \quad w^0_{13} = w^0_{31}. \tag{2.13}$$

Now, let us construct w^i_j matrix. We will calculate w^0_1 explicitly, then give the results for the non-zero remaining ones

$$\begin{aligned} w^0_1 =& w^0_{10}e^0 + w^0_{11}e^1 + w^0_{12}e^2 + w^0_{13}e^3 = \frac{f'(r)}{f(r)g(r)}e^0 \\ w^1_2 =& \frac{-1}{g(r)r}e^2 \quad w^1_3 = \frac{-1}{g(r)r}e^3 \quad w^2_3 = \frac{-cot\theta}{r}e^3. \end{aligned} \tag{2.14}$$

So, w^i_j matrix is found to be

$$w^i_j = \begin{bmatrix} 0 & \frac{f'(r)}{f(r)g(r)}e^0 & 0 & 0 \\ & 0 & \frac{-1}{g(r)r}e^2 & \frac{-1}{g(r)r}e^3 \\ & & 0 & \frac{-cot\theta}{r}e^3 \\ & & & 0 \end{bmatrix}$$

2.1.2. Calculation of the Curvature 2-forms Using Cartan's Second Structural Equation

Cartan's second structural equation is given by

$$\Omega^i_j = dw^i_j + w^i_k w^k_j. \tag{2.15}$$

where Ω^i_j are called *curvature 2-forms* and defined (for the later use) in terms of Riemann tensor in the following way

$$\Omega^i_j = \frac{1}{2} R^i_{jkl} e^k e^l. \tag{2.16}$$

Again, we will explicitly calculate Ω^0_1 term, then give the results for the others.

$$\Omega^0_1 = dw^0_1 + w^0_2 w^2_1 + w^0_3 w^3_1$$

$$= d\left[\frac{f^{'}(r)}{f(r)g(r)} e^0\right] \tag{2.17}$$

$$= \frac{ff'g' - f''fg}{f^2 g^3} e^0 e^1$$

The remaining nonzero terms are found as

$$\Omega^0_2 = \frac{-f'}{fg^2 r} e^0 e^2 \quad \Omega^0_3 = \frac{-f'}{fg^2 r} e^0 e^2 \quad \Omega^1_2 = \frac{-g'}{g^3 r} e^1 e^2$$

$$\Omega^1_3 = \frac{-g'}{g^3 r} e^1 e^3 \quad \Omega^2_3 = (\frac{1}{r^2} - \frac{1}{g^2 r^2}) e^2 e^3. \tag{2.18}$$

So, Ω^i_j matrix is constructed

$$\Omega^i_j = \begin{bmatrix} 0 & \frac{ff'g'-f''fg}{f^2g^3} e^0 e^1 & \frac{-f'}{fg^2 r} e^0 e^2 & \frac{-f'}{fg^2 r} e^0 e^2 \\ & 0 & \frac{-g'}{g^3 r} e^1 e^2 & \frac{-g'}{g^3 r} e^1 e^3 \\ & & 0 & (\frac{1}{r^2} - \frac{1}{g^2 r^2}) e^2 e^3 \\ & & & 0 \end{bmatrix}$$

2.1.3. Solving $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} = 0$ to Find the Schwarzschild Metric

In this part we will solve Einstein's field equation in vacuum which is

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} = 0 \tag{2.19}$$

where the Einstein tensor is written in terms of the Ricci tensor, $R_{\mu\nu}$, and the Ricci scalar, R, as follows:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.20}$$

Before solving the above equation, first let us find the components of the Ricci tensor. To do this we should solve the equation below:

$$\Omega^{\mu}_{\nu} = \frac{1}{2} R^{\mu}_{\nu\lambda\kappa} e^{\lambda} e^{\kappa}. \tag{2.21}$$

We write Ω^0_1 explicitly, then give the results for the remaining ones:

$$\begin{aligned}
\Omega^0_1 &= \frac{1}{2} R^0_{1\lambda\kappa} e^{\lambda} e^{\kappa} \\
&= \frac{1}{2} [R^0_{101} e^0 e^1 + R^0_{102} e^0 e^2 + R^0_{103} e^0 e^3 \\
&= +R^0_{110} e^1 e^0 + R^0_{112} e^1 e^2 + R^0_{113} e^1 e^3 \\
&= +R^0_{120} e^2 e^0 + R^0_{121} e^2 e^1 + R^0_{123} e^2 e^3 \\
&= +R^0_{130} e^3 e^0 + R^0_{131} e^3 e^1 + R^0_{132} e^3 e^2] \\
&= \frac{f f' g' - f'' f g}{f^2 g^3} e^0 e^1.
\end{aligned} \tag{2.22}$$

So, from e^0e^1 terms we have

$$\frac{1}{2}(R^0_{101}e^0e^1 + R^0_{110}e^1e^0) = \frac{ff'g' - f''fg}{f^2g^3}e^0e^1. \tag{2.23}$$

Using the identities $R_{ijkl} = -R_{ijlk} = R_{jikl}$ and $e^ie^j = -e^je^i$, Equation 2.23 becomes

$$R^0_{101}e^0e^1 = \frac{ff'g' - f''fg}{f^2g^3}e^0e^1. \tag{2.24}$$

Then, R^0_{101} is found as

$$R^0_{101} = \frac{ff'g' - f''fg}{f^2g^3}. \tag{2.25}$$

Other components of the Riemann Tensor are calculated that is given below

$$\begin{gathered} R^0_{202} = \frac{-f^{'}}{fg^2r} \quad R^0_{303} = \frac{-f^{'}}{fg^2r} \quad R^1_{212} = \frac{g^{'}}{g^3r} \\ R^1_{313} = \frac{g^{'}}{g^3r} \quad R^2_{323} = \frac{g^2-1}{g^2r^2}. \end{gathered} \tag{2.26}$$

Notice that Equation 2.20 includes $R_{\mu\nu}$ and R which we have not calculated yet. From the contraction rule of the Riemann Tensor, one can find Ricci Tensor in the following way

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{2.27}$$

and, from the contraction of Ricci tensor with the metric tensor, we can find Ricci scalar, R, as follows

$$g^{\mu\nu}R_{\mu\nu} = R. \tag{2.28}$$

If we multiply Equation 2.20 by $g^{\mu\nu} = diag(1, -1, -1, -1)$ we see Ricci scalar is zero

$$\begin{aligned} g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) &= 0 \\ R + \frac{1}{2}4R &= 0 \\ R &= 0. \end{aligned} \tag{2.29}$$

So that, Equation 2.20 is reduced to

$$G_{\mu\nu} = \frac{8\pi G}{c^4}R_{\mu\nu}. \tag{2.30}$$

Remembering that we are trying to solve Equation 2.19, hence it is enough to solve the following equation

$$R_{\mu\nu} = 0. \tag{2.31}$$

We would like to show the explicit calculation for R_{00}, then give the results for the remaining ones.

$$\begin{aligned} R_{00} &= R^1_{010} + R^2_{020} + R^3_{030} \\ R_{00} &= \frac{f'g'}{fg^3} - \frac{f''}{fg^2} - \frac{2f'}{fg^2r} = 0. \end{aligned} \tag{2.32}$$

The remaining ones

$$R_{11} = \frac{f'g'}{fg^3} - \frac{f''}{fg^2} + \frac{2g'}{g^3r} = 0 \tag{2.33}$$

$$R_{22} = \frac{-f'}{fg^2r} + \frac{g'}{g^3r} - \frac{1}{g^2r^2} + \frac{1}{r^2} = 0 \tag{2.34}$$

$$R_{33} = \frac{-f'}{fg^2r} + \frac{g'}{g^3r} - \frac{1}{g^2r^2} + \frac{1}{r^2} = 0. \tag{2.35}$$

Let us subtract Equation 2.32 from Equation 2.33

$$R_{11} - R_{00} = \frac{2g'}{g^3r} + \frac{2f'}{fg^2r} = 0$$

$$\frac{f'}{fg^2r} = -\frac{g'}{g^3r} \tag{2.36}$$

$$\int \frac{f'}{f}dr = \int -\frac{g'}{g}dr.$$

By taking the above integral, $f(r)$ can be found

$$f(r) = \frac{C_1}{g(r)}, \tag{2.37}$$

where, C_1 is the integration constant. Now, if we substitute $f(r)$ into $R_{22} = 0$, Equation 2.34 becomes

$$\frac{2g'}{g^3r} - \frac{1}{g^2r^2} + \frac{1}{r^2} = 0. \tag{2.38}$$

Taking the integral Equation 2.38 leads to

$$\int \frac{g'}{g(1-g^2)}dg = \int \frac{1}{2r}dr. \tag{2.39}$$

Then, solving Equation 2.39, reveals $g(r)$

$$g(r) = \left(1 + \frac{C_2}{r}\right)^{-1/2}, \tag{2.40}$$

where C_2 is the integration constant. From Equation 2.37, $f(r)$ is found as

$$f(r) = C_1\left(1+\frac{C_2}{r}\right)^{1/2}. \tag{2.41}$$

What we have so far is

$$f(r) = \frac{C_1}{g(r)} = C_1\left(1+\frac{C_2}{r}\right)^{1/2}. \tag{2.42}$$

By substituting Equation 2.42 into $R_{22}=0$ equation, one can easily find $C_1 = 1$. To C_2 we look at the Newtonian limit of the Einstein's action, S_E, then compare with the Newtonian action, S_N.

$$S_N = \int\left(-mc^2 + \frac{1}{2}mv^2 + \frac{GmM}{r}\right)dt \tag{2.43}$$

$$S_E = D\int c\left[\left(1+\frac{C_2}{r}\right) - \frac{1}{c^2}\left(1+\frac{C_2}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 - \frac{r^2}{c^2}\left(\frac{d\Omega}{dt}\right)^2\right]^{1/2} dt, \tag{2.44}$$

where D is a constant to be determined. Let us look at the Newtonian limit, that is $r \gg C_2$. In this limit, the second term in the square bracket in S_E becomes

$$\frac{1}{c^2}\left(1+\frac{C_2}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 \approx \frac{1}{c^2}\left(1-\frac{C_2}{r}\right)\left(\frac{dr}{dt}\right)^2. \tag{2.45}$$

We can safely remove $\frac{C_2}{c^2 r}$ term in Equation 2.45 since c^2 multiplies r which is a greater limit than the Newtonian limit. So that we have

$$\frac{1}{c^2}\left(1+\frac{C_2}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 \approx \frac{1}{c^2}\left(1-\frac{C_2}{r}\right)\left(\frac{dr}{dt}\right)^2 \approx \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2. \tag{2.46}$$

Then Equation 2.44 becomes

$$S_E \approx D\int c\left[\left(1+\frac{C_2}{r}\right) - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 - \frac{r^2}{c^2}\left(\frac{d\Omega}{dt}\right)^2\right]^{1/2} dt. \tag{2.47}$$

Furthermore, we can treat the above square bracket as $(1-x)^{1/2}$ in the case $x \ll 1$. So that 1/2 power of the above the square bracket becomes multiplicative term in the Newtonian limit and we reach

$$S_E \cong D \int c \left[1 + \frac{C_2}{2r} - \frac{1}{2c^2}\left(\frac{dr}{dt}\right)^2 - \frac{r^2}{2c^2}\left(\frac{d\Omega}{dt}\right)^2\right] dt. \tag{2.48}$$

We can take $\theta = \frac{\pi}{2}$ to simplify the calculations so that the differential solid angle becomes $d\Omega = d\phi$. Writing Equation 2.48 with this simplification reveals

$$S_E \cong D \int c \left[1 + \frac{C_2}{2r} - \frac{1}{2c^2}\left(\frac{dr}{dt}\right)^2 - \frac{r^2}{2c^2}\left(\frac{d\phi}{dt}\right)^2\right] dt. \tag{2.49}$$

Notice that in polar coordinates velocity, v, is written as

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2. \tag{2.50}$$

So, Equation 2.49 becomes

$$S_E \cong D \int c \left[1 + \frac{C_2}{2r} - \frac{v^2}{2c^2}\right] dt. \tag{2.51}$$

We equate S_N and S_E to find C_1 and C_2

$$\begin{gathered} S_N \cong S_E \\ \int \left(-mc^2 + \frac{1}{2}mv^2 + \frac{GmM}{r}\right) dt \cong D \int c \left[1 + \frac{C_2}{2r} - \frac{v^2}{2c^2}\right] dt \end{gathered} \tag{2.52}$$

So it is clear from the first terms

$$D = -mc, \tag{2.53}$$

and, from the $1/r$ terms we have

$$C_2 = -\frac{2GM}{c^2}. \tag{2.54}$$

Finally, the Schwarzschild metric is derived

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.55}$$

Notice that the metric is independent of the temporal coordinate, t. It only depends on the mass of the black hole, M, which is the source of the gravitational field.

3. GEODESIC OF A FREE PARTICLE IN MINKOWSKI SPACETIME

A *curve* is a geodesic if it extremizes the distance between two fixed point [10]. In this chapter we would like to find what is the geodesic of a massive particle, m, in the absence of a gravitational potential, i.e. in the Minkowski spacetime. Let us write the relativistic energy for a free particle of mass m

$$E^2 = m^2 + \overrightarrow{P}^2, \tag{3.1}$$

and, angular momentum $\vec{L}$

$$\vec{L} = \vec{r} \times \vec{P} = m\vec{r} \times \frac{d\vec{r}}{d\tau} \tag{3.2}$$

where τ is the proper time and $\vec{P}$ is the momentum of the particle. Let us square the above equation for calculation purposes

$$\vec{L}^2 = m^2 \left[\vec{r}^2 (\frac{d\vec{r}}{d\tau})^2 - r^2 (\frac{dr}{d\tau})^2 \right]. \tag{3.3}$$

We want to find how $\vec{r}$ changes with respect to τ, so from Equation 3.3 we get

$$(\frac{d\vec{r}}{d\tau})^2 = \frac{L^2}{m^2 r^2} + (\frac{dr}{d\tau})^2. \tag{3.4}$$

If we insert Equation 3.4 into 3.1 we have

$$E^2 = m^2 \left[1 + \frac{L^2}{m^2 r^2} + (\frac{dr}{d\tau})^2 \right], \tag{3.5}$$

then, we get the below equation for the geodesic of a free particle in Minkowski spacetime

$$(\frac{dr}{d\tau})^2 = \frac{E^2}{m^2} - \frac{L^2}{m^2 r^2} - 1. \tag{3.6}$$

We can use the spherical coordinate system and set $\theta = \pi/2$ without loss of generality. So, the vector $\vec{r}$ in this coordinate system is written as

$$\vec{r} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \\ 0 \end{bmatrix}.$$

Taking the derivative of $\vec{r}$ with respect to τ

$$\frac{d\vec{r}}{d\tau} = \begin{bmatrix} \frac{dr}{d\tau}\cos\phi - r\sin\phi\frac{d\phi}{d\tau} \\ \frac{dr}{d\tau}\sin\phi + r\cos\phi\frac{d\phi}{d\tau} \\ 0 \end{bmatrix},$$

and squaring it results in

$$\left(\frac{d\vec{r}}{d\tau}\right)^2 = \left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2. \tag{3.7}$$

Then inserting above equation into Equation 3.4 reveals

$$\frac{d\phi}{d\tau} = \pm\frac{L}{mr^2}. \tag{3.8}$$

From the Equation 3.8 we see that

$$d\tau = \frac{mr^2}{L}d\phi, \tag{3.9}$$

now, we insert Equation 3.9 into Equation 3.5

$$E^2 = m^2 \left[1 + \frac{L^2}{m^2 r^2} + (\frac{L}{mr^2}\frac{dr}{d\phi})^2 \right], \tag{3.10}$$

then, we have

$$\begin{aligned} \left(\frac{dr}{d\phi}\right)^2 &= \left(\frac{E^2 - m^2}{L^2}\right) r^4 - r^2, \\ \int d\phi &= \int \frac{dr}{\sqrt{\left(\frac{E^2-m^2}{L^2}\right) r^4 - r^2}}. \end{aligned} \tag{3.11}$$

The result of the above integral which when solved for r gives a line equation which is,

$$r = \pm\sqrt{\frac{L^2}{E^2 - m^2}}\sqrt{1 + \frac{\cos^2(\phi - \phi_0)}{\sin^2(\phi - \phi_0)}} = \frac{C}{\sin\phi}, \tag{3.12}$$

where $C \equiv \pm\sqrt{\frac{L^2}{E^2 - m^2}}$. So that the particle with mass m moves on a straight line in the lack of potential. In other words, geodesic of a particle with mass m is a straight line in free space.

4. GEODESICS OF THE SCHWARZSCHILD BLACK HOLE

In this chapter, we look at the geodesics of a massive particle with mass m and of light, inside and outside of the Schwarzschild black hole horizon.

4.1. Null Geodesics with $l = 0$

Geodesic of light is called *null geodesic* and it is always $ds^2 = 0$ for light. In this section we look at the light with zero angular momentum, l, directed towards the center of the black hole i.e. $l = 0$ and $d\theta$=$d\phi$=0. Under these circumstances the Schwarzschild metric reduces to

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 = 0. \tag{4.1}$$

If we arrange above equation we get

$$\frac{dr}{cdt} = \pm \left(1 - \frac{2GM}{c^2 r}\right). \tag{4.2}$$

Let us solve the above equation

$$\int cdt = \pm \int \frac{dr}{1 - \frac{2GM}{c^2 r}}$$

$$= \pm \int \frac{rdr}{r - \frac{2GM}{c^2}} \tag{4.3}$$

$$= \pm \int \left(1 + \frac{\frac{2GM}{c^2}}{r - \frac{2GM}{c^2}}\right) dr$$

Then, we find

$$t = \pm\frac{1}{c}\left(r + \frac{2GM}{c^2}\ln|r - \frac{2GM}{c^2}|\right) + const. \tag{4.4}$$

Notice that, at the Schwarzschild radius $r = \frac{2GM}{c^2}$, time becomes infinite, i.e. $t = \pm\infty$. This means that, for an observer who sends a flash of light at r from the black hole, it takes infinite amount of time, t, for light to reach the horizon. Similarly, it takes infinite amount of time for light emitted at the horizon to reach the observer.

4.2. Non-Null Geodesics with $l = 0$

Geodesics of massive particles are called *non-null geodesic.* In this section, we look at what the geodesic of a particle with mass m and zero angular momentum per mass, $l = 0$, is around the Schwarzschild black hole. For such a particle we have the metric

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2. \tag{4.5}$$

Action, S, is written as

$$S = \int \sqrt{ds^2} = \int \mathcal{L} d\sigma. \tag{4.6}$$

For non-null geodesics we can choose $d\sigma = ds$ which makes $\mathcal{L} = 1$

$$\mathcal{L} = \sqrt{c^2\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{dt}{d\sigma}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left(\frac{dr}{d\sigma}\right)^2} = 1. \tag{4.7}$$

Using $\mathcal{L}$ let us solve Euler-Lagrange equation for the variable t

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t} &= \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) \\ 0 &= \frac{d}{ds}\left[c^2\left(1 - \frac{2GM}{c^2 r}\right)\dot{t}\right] \end{aligned} \tag{4.8}$$

where $\dot{t} = \frac{dt}{ds}$. From Equation 4.8 we see that inside of the square bracket is constant, i.e. independent of s. This constant is actually related to energy

$$\left(1 - \frac{2GM}{c^2 r}\right) c \frac{dt}{ds} = \frac{\epsilon}{c^2}, \tag{4.9}$$

where, ϵ is defined as energy per unit mass. We have found how t changes with respect to s. From now on, we will find how r changes with respect to s. To do this, we divide Equation 4.5 by ds^2

$$1 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{ds}\right)^2 \tag{4.10}$$

so that we have

$$\left(\frac{dr}{ds}\right)^2 = \left(1 - \frac{2GM}{c^2 r}\right)^2 c^2 \left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right). \tag{4.11}$$

Using the Equation 4.9, Equation 4.11 becomes

$$\left(\frac{dr}{ds}\right)^2 = \frac{2GM}{c^2 r} + \frac{\epsilon^2}{c^4} - 1. \tag{4.12}$$

Solving Equation 4.12 reveals

$$s = \sqrt{\frac{c^2}{2GM}} \int \frac{dr}{\sqrt{\frac{1}{r} + \frac{\epsilon^2}{2GMc^2} - \frac{c^2}{2GM}}}. \tag{4.13}$$

Further by substituting $\frac{2GM}{c^2} = r_s$, Equation 4.13 becomes

$$\begin{aligned} s &= \sqrt{\frac{1}{r_s}} \int \frac{dr}{\sqrt{\frac{1}{r} + \frac{1}{r_s}\left(\frac{\epsilon^2}{c^4} - 1\right)}} \\ &= \sqrt{\frac{1}{r_s}} \int \frac{dr}{\sqrt{\frac{1}{r} + A}} \end{aligned} \tag{4.14}$$

where we defined $A \equiv \frac{1}{r_s}\left(\frac{\epsilon^2}{c^4} - 1\right)$. The result of the integration in Equation 4.14 is found to be

$$s = \frac{r}{A}\sqrt{A + \frac{1}{r}} - \frac{\log\left[2r\left(A + \sqrt{A^2 + \frac{A}{r}}\right) + 1\right]}{2A^{3/2}} + const. \tag{4.15}$$

Notice that at $r = r_s$, s is finite in Equation 4.15. Since the relation between s and proper time, τ, of m is given by $ds^2 = c^2 d\tau^2$, this means that the particle m reaches to the horizon in a finite amount of proper time. Now, let us find how much time is needed from the point of view of an observer at ∞, for whom $\tau = t$. To calculate this, using Equation 4.12 we can write ds in terms of dr

$$ds = \frac{dr}{\sqrt{\frac{r_s}{r} + B}}, \tag{4.16}$$

where we defined $B \equiv \frac{\epsilon^2}{c^4} - 1$. Then, inserting the above equation into Equation 4.9 we have

$$t = \sqrt{\frac{1+B}{c^2}} \int \frac{dr}{(1 - \frac{r_s}{r})\sqrt{\frac{r_s}{r} + B}}. \tag{4.17}$$

The result of the above Equation 4.17 is found to be

$$\begin{aligned} t = {} & \frac{r_s}{\sqrt{B+1}} \ln\left(\frac{\left|\sqrt{\frac{Br+r_s}{r}} - \sqrt{B+1}\right|}{\sqrt{\frac{Br+r_s}{x}}} + \sqrt{B+1}\right) \\ & - \frac{r_s}{2B^{3/2}}(2B-1)\ln\left(\frac{\left|\sqrt{\frac{Br+r_s}{r}} - \sqrt{B}\right|}{\sqrt{\frac{Br+r_s}{r}} + \sqrt{B}}\right) \\ & + \frac{\sqrt{\frac{Br+r_s}{r}}}{\frac{B(Br+r_s)}{r_s r} - \frac{B^2}{r_s}} \end{aligned} \tag{4.18}$$

where it is assumed that $B > 0$. According to the Equation 4.18 at $r = r_s$, time t is infinite. This means that for an observer whose coordinate is (t, r), time needed for a massive particle to reach the black hole horizon is infinite.

4.3. Null Geodesics with $l \neq 0$

In this section, we look at light that is coming towards the black hole with an angle different than zero. The metric for such a beam of light is:

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) = 0. \quad (4.19)$$

To simplify the calculations we can set $\theta = \frac{\pi}{2}$ without loss of generality. Then, the metric becomes

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\phi^2 = 0. \quad (4.20)$$

Action is written as

$$S = \int \sqrt{ds^2} = \int \mathcal{L} d\sigma, \quad (4.21)$$

where Lagrangian is

$$\mathcal{L} = \sqrt{c^2 \left(1 - \frac{2GM}{c^2 r}\right) \left(\frac{dt}{d\sigma}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{d\sigma}\right)^2 - r^2 \left(\frac{d\phi}{d\sigma}\right)^2}. \quad (4.22)$$

Then, Euler-Lagrange equation for the variable t can be calculated as

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t} &= \frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) \\ 0 &= \frac{d}{d\sigma}\left[\left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}\right]. \end{aligned} \quad (4.23)$$

So that, inside of the square bracket should be equal to a constant which is actually related to energy

$$\left(1-\frac{2GM}{c^2r}\right)c\frac{dt}{d\sigma}=\frac{\epsilon}{c^2}, \tag{4.24}$$

where ϵ is the energy of light. Next, let us calculate Euler-Lagrange equation for the variable ϕ

$$\begin{aligned}\frac{d}{d\sigma}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right)&=\frac{\partial\mathcal{L}}{\partial\phi}\\ -\frac{d}{d\sigma}\left(r^2\dot{\phi}\right)&=0\end{aligned} \tag{4.25}$$

From the above equation it is clear that inside of the parentheses should be equal to a constant, which we call this constant, l, that is the angular momentum of light. So, we have

$$\frac{d\phi}{d\sigma}=\frac{l}{r^2c} \tag{4.26}$$

Now we would like to find how r changes with respect to ϕ i.e. $\frac{dr}{d\phi}$. To do this we use the Hamilton-Jacobi method. Before that, let us find the generalized momenta, P_q, using the formula $P_q=\frac{\partial\mathcal{L}}{\partial\dot{t}}$. For the three variables t, r, ϕ we have

$$\begin{aligned}P_t&=\frac{1}{\mathcal{L}}\left(1-\frac{r_s}{r}\right)\frac{c^2dt}{d\sigma},\\ P_\phi&=-\frac{r^2}{\mathcal{L}}\frac{d\phi}{d\sigma},\\ P_r&=-\frac{1}{\mathcal{L}}\left(1-\frac{r_s}{r}\right)^{-1}\frac{dr}{d\sigma}.\end{aligned} \tag{4.27}$$

Now, divide Equation 4.20 by ds^2 and use the formula given below

$$P_q = \frac{\partial S}{\partial q}, \tag{4.28}$$

where S is the Hamilton's principal function. So, we have

$$\left(\frac{ds}{d\sigma}\right)^2 = \frac{\mathcal{L}^2}{c^2}\left(1-\frac{r_s}{r}\right)^{-1}\left(\frac{dS}{dt}\right)^2 - \mathcal{L}^2\left(1-\frac{r_s}{r}\right)\left(\frac{dS}{dr}\right)^2 - \frac{\mathcal{L}^2}{r^2}\left(\frac{dS}{d\phi}\right)^2 = 0. \tag{4.29}$$

Notice that, the time t and the angle ϕ are cyclic coordinates. In addition, the total energy is conserved so that we can write the solution for Hamilton's principal function S, using additive seperation of variables (Staeckel conditions [11]).

$$S = -P_t t + P_\phi \phi + S_r(r). \tag{4.30}$$

Inserting Equation 4.30 into Equation 4.29, what we obtain:

$$\left(\frac{ds}{d\sigma}\right)^2 = \left(1-\frac{r_s}{r}\right)^{-1}\frac{\mathcal{L}^2 P_t^2}{c^2} - \left(1-\frac{r_s}{r}\right)\mathcal{L}^2\frac{\partial S_r(r)}{\partial r} - \frac{\mathcal{L}^2 P_\phi^2}{r^2} = 0. \tag{4.31}$$

So, $S_r(r)$ can be found as:

$$S_r(r) = \int \left(1-\frac{r_s}{r}\right)^{-1}\sqrt{\frac{P_t^2}{c^2} - \left(1-\frac{r_s}{r}\right)\frac{P_\phi^2}{r^2}}dr. \tag{4.32}$$

Derivative of Equation 4.30 with respect to P_ϕ is

$$\phi + \frac{\partial S}{\partial P_\phi} = \phi - \int \frac{P_\phi dr}{r^2\sqrt{\frac{P_t^2}{c^2} - \left(1-\frac{r_s}{r}\right)\frac{P_\phi^2}{r^2}}} = 0, \tag{4.33}$$

and, variation of above equation leads to

$$d\phi - \frac{P_\phi dr}{r^2\sqrt{\frac{P_t^2}{c^2} - \left(1-\frac{r_s}{r}\right)\frac{P_\phi^2}{r^2}}} = 0, \tag{4.34}$$

so that we have

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{P_\phi^2}{r^4\left[\frac{P_t^2}{c^2} - \left(1 - \frac{r_s}{r}\right)\frac{P_\phi^2}{r^2}\right]}, \tag{4.35}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{P_t}{P_\phi}\right)^2 \frac{r^4}{c^2} - r^2 + r_s r. \tag{4.36}$$

It can be easily found from Equation 4.27 that $\left(\frac{P_t}{P_\phi}\right)^2 = \epsilon^2/l^2$ is a constant. Then we write the above equation in the following way:

$$\left(\frac{dr}{d\phi}\right)^2 - \left(\frac{\epsilon}{lc}\right)^2 r^4 + r^2 - r_s r = 0. \tag{4.37}$$

We can define an analogue potential [12] such that,

$$V(r) = -\left(\frac{\epsilon}{lc}\right)^2 r^4 + r^2 - r_s r, \tag{4.38}$$

so that Equation 4.37 becomes

$$\left(\frac{dr}{d\phi}\right)^2 + V(r) = 0 \tag{4.39}$$

where in the above equation ϕ plays the role of time. We would like to make a graph to see how $V(r)$ changes with r. To to this let us write Equation 4.38 in a dimensionless manner

$$\frac{V(r)}{r_s^2} = -\left(\frac{r_s}{r_l}\right)^2 x^4 + x^2 - x \tag{4.40}$$

where $x \equiv r/r_s$ and $r_l \equiv lc/\epsilon$. Now, let us graph Equation 4.40 for a few representative $\left(\frac{r_s}{r_l}\right)^2$ values.

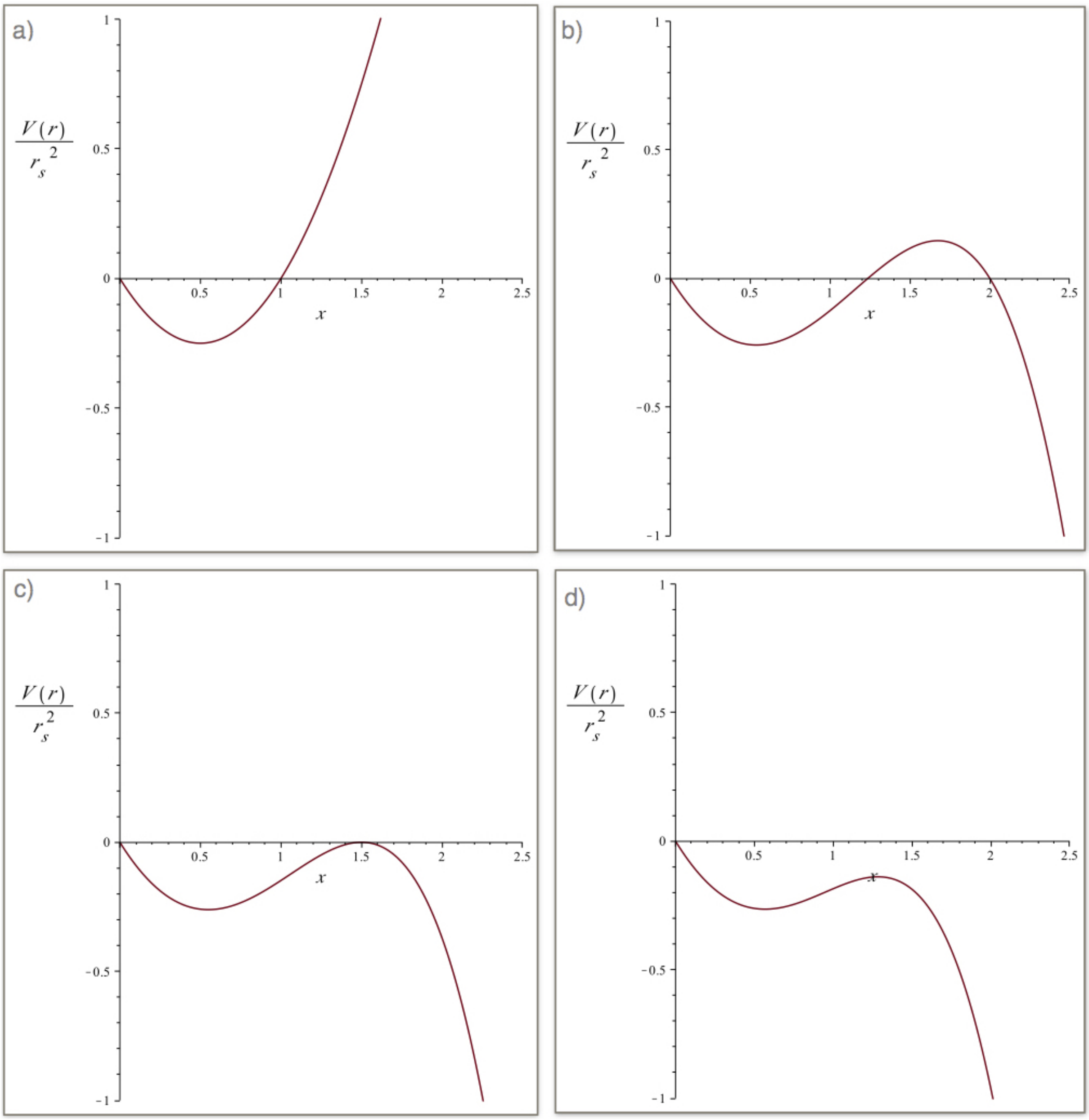

Figure 4.1: Normalized anologue potentail, $V(r)/r_s$, as a function of $x \equiv r/r_s$ for a few representative values of $\left(\frac{r_s}{r_l}\right)^2$, where $(a), (b), (c), (d)$ corresponds to $\left(\frac{r_s}{r_l}\right)^2 : (0, \frac{1}{8}, \frac{4}{27}, \frac{5}{27})$ respectively.

First of all, notice that $x = 1$ corresponds $r = r_s$ and the null geodesics are at $V(r) = 0$ by definition. The graph (a) in Figure 4.1 shows that for $\left(\frac{r_s}{r_l}\right)^2 \to 0$ limit, light can not go out of the Schwarzschild radius. According to the graph (b) for $\left(\frac{r_s}{r_l}\right)^2 = \frac{1}{8}$, light coming from infinity can only come close up to $2r_s$ to the black hole, then turns back to the infinity again. On the other hand, the light inside the black hole seems to go out of the Schwarzschild radius and then turns back to the inside.

But there is a problem in the region $1 < x < 1.3$. To be clearer we write the metric in terms of $V(r)$ using Equation 4.39

$$\left(1 - \frac{r_s}{r}\right)^2 c^2 dt^2 = \left[-V(r) + r^2\left(1 - \frac{r_s}{r}\right)\right] d\phi^2. \tag{4.41}$$

When we substitute $r = r_s$ on above equation we have

$$-V(r_s)d\phi^2 = 0. \tag{4.42}$$

Since at $r = r_s$, $V(r)$ is not zero, which can be seen on the graph, Equation 4.42 implies that

$$d\phi = 0, \tag{4.43}$$

which means that

$$\frac{d\phi}{dr} = 0. \tag{4.44}$$

Above equation leads to

$$\frac{dr}{d\phi} = \infty \tag{4.45}$$

which does not satisfies Equation 4.39. So that, there is an inconsistency in the region $r_s < r < 1.3r_s$. In the third graph, (c), where $r_s^2/r_l^2 = 4/27$ we see that light inside the horizon $(x < 1)$ seems to reach to $r = 1.5r_s$ which is the radius of unstable circular orbit. Notice also that, a geodesic coming from inside the black hole (for which time runs backwards) does meet with the geodesic outside the black hole (for which time runs forwards) at$r = 1.5r_s$. To see this better, look at the Equation 4.24. For $r < r_s$we should take $\frac{dt}{d\sigma} < 0$ to have $\epsilon > 0$. Actually, $\frac{dt}{d\sigma} < 0$ implies that time runs backwards. Even though light seems to go out of the horizon, the same inconsistency mention in the previous case exist in this case also. In the fourth graph (d), for $r_s^2/r_l^2 = 4/27$, light seems to go out of the black hole horizon without a potential barrier but the inconsistency stated before is also present here.

4.4. Non-Null Geodesics with $l \neq 0$

In this case, we will look at a massive particle with mass m approaches to the black hole with an angle different than zero. Again, set $\theta = \frac{\pi}{2}$ for the calculation purpose. So, the metric is

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\phi^2. \tag{4.46}$$

Action for this metric is written as

$$S = \int \sqrt{ds^2} = \int \mathcal{L} d\sigma, \tag{4.47}$$

for non-null geodesics we can choose $d\sigma = ds$ so that Lagrangian becomes one

$$\mathcal{L} = \sqrt{c^2 \left(1 - \frac{2GM}{c^2 r}\right) \left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{ds}\right)^2 - r^2 \left(\frac{d\phi}{ds}\right)^2} = 1. \tag{4.48}$$

Euler-Lagrange equation for the variable t

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t} &= \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) \\ 0 &= \frac{d}{ds}\left[\left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}\right]. \end{aligned} \tag{4.49}$$

So, inside of the square bracket is equal to a constant which is actually energy per mass, ϵ,

$$\left(1 - \frac{2GM}{c^2 r}\right) c^2 \frac{dt}{ds} = \frac{\epsilon}{c}. \tag{4.50}$$

Euler-Lagrange equation for the variable ϕ

$$\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = \frac{\partial\mathcal{L}}{\partial\phi}$$

$$-\frac{d}{ds}\left(r^2\dot{\phi}\right) = 0, \tag{4.51}$$

which means that inside of the parentheses is equal to a constant i.e. angular momentum per mass, l,

$$\frac{d\phi}{ds} = \frac{l}{r^2 c}. \tag{4.52}$$

Divide Equation 4.46 by ds^2

$$1 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{ds}\right)^2 - r^2 \left(\frac{d\phi}{ds}\right)^2 \tag{4.53}$$

so that, using Equation 4.50 and Equation 4.52 we can find $\left(\frac{dr}{ds}\right)^2$,

$$\left(\frac{dr}{ds}\right)^2 = \frac{\epsilon^2}{c^4} + \left(\frac{r_s - r}{r^3}\right)\frac{l^2}{c^2} + \left(\frac{r_s - r}{r}\right) \tag{4.54}$$

Write Equation 4.54 in this way

$$\left(\frac{dr}{ds}\right)^2 + V_e(r) = E, \tag{4.55}$$

where $V_e(r)$ and E are defined as follows

$$V_e(r) = \left(1 - \frac{r_s}{r}\right)\left[1 + \frac{l^2}{c^2 r^2}\right] - 1, \tag{4.56}$$

$$E = \frac{\epsilon^2}{c^4} - 1. \tag{4.57}$$

We would like to write $V_e(r)$ in the following way:

$$V_e(r) = -\frac{1}{x} + \frac{\lambda^2}{x^2} - \frac{\lambda^2}{x^3} \tag{4.58}$$

where $\lambda \equiv l/r_s c$ and $x \equiv r/r_s$. For a few representative λ^2 values, the graph of $V_e(r)$ to x can be seen below:

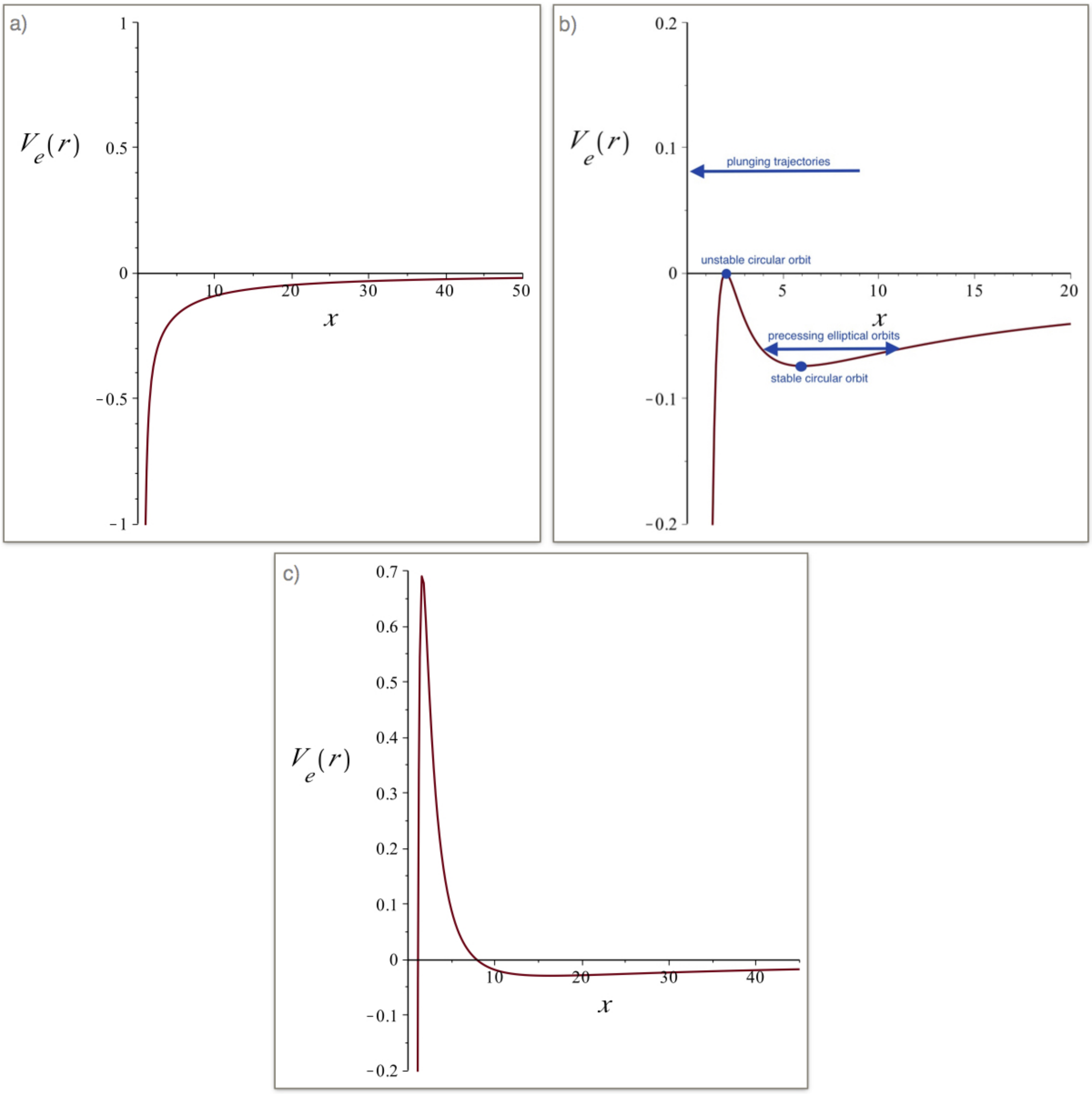

Figure 4.2: Effective potential, $V_e(r)$, as a function of $x \equiv r/r_s$ for a few representative values of $\lambda^2 \equiv \left(\frac{l}{r_s c}\right)^2$, where $(a), (b), (c)$corresponds to $\lambda^2 : (1, 4, 9)$ respectively.

According to Figure 4.2, the graph (a) shows that for $\lambda^2 = 1$ the particle with $E \geqslant 0$ coming from infinity falls into the center of the potential. According to the graph (b) that is for $\lambda^2 = 4$, for the particle with $E = 0$ coming from infinity, there are unstable circular orbits at $r = 2r_s$ so that it may fall in or escape back to infinity other than orbiting circularly. In addition to that, there is a stable circular orbit at the minimum of the $V_e(r)$, that is $r = 6r_s$. And between these two there are elliptical orbits. Similar to the previous case, in the graph (c) for $\lambda^2 = 9$, there are one unstable orbit, one stable orbit and precessing orbits for the particle. A particle coming from infinity with energy greater than the maximum value of $V_e(r)$, falls into the center of the potential.

5. COMPARISON BETWEEN NEWTONIAN PHYSICS AND GENERAL RELATIVITY

Let us explore what is common and not between Newtonian physics and general relativity. First of all, we will check if we reach the correct Newtonian limit out of the relativistic equations. To do this by using Equation 4.52, we write Equation 4.54 as follows

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4c^2}{l^2}\left(\frac{\epsilon^2}{c^4}-1\right) - r^2\left(1-\frac{r_s}{r}-\frac{c^2r_s}{l^2}r\right). \tag{5.1}$$

In Newtonian physics, non-relativistic energy per mass, ϵ_{nr}, is written as

$$\epsilon_{nr} = \frac{1}{2}\left(\frac{dr}{ds}\right)^2_{nr} + \frac{1}{2}r^2\left(\frac{d\phi}{ds}\right)^2_{nr} - \frac{GM}{r}, \tag{5.2}$$

using the chain rule

$$\frac{dr}{ds} = \frac{dr}{d\phi}\frac{d\phi}{ds} = \frac{dr}{d\phi}\dot{\phi}, \tag{5.3}$$

Equation 5.2 becomes

$$\epsilon_{nr} = \frac{1}{2}\left(\frac{dr}{d\phi}\right)^2_{nr}\dot{\phi}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r}. \tag{5.4}$$

Remembering that in Newtonian physics

$$\dot{\phi} = \frac{l}{r^2}, \tag{5.5}$$

where l is angular momentum per mass. So that inserting Equation 5.5 into Equation 5.4 we get,

$$\left(\frac{dr}{d\phi}\right)^2_{nr} = \frac{2\epsilon_{nr}r^4}{l^2} + \frac{c^2r_sr^3}{l^2} - r^2. \tag{5.6}$$

Since, the rest mass energy is not included in Newtonian physics, the relation between ϵ_{nr} and ϵ is given by

$$\epsilon = c^2 + \epsilon_{nr}. \tag{5.7}$$

Inserting Equation 5.7 into Equation 5.1 we have

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4c^2}{l^2}\left(\frac{c^4+\epsilon_{nr}^2+2\epsilon_{nr}c^2}{c^4}-1\right) - r^2\left(1-\frac{r_s}{r}-\frac{c^2r_s}{l^2}r\right). \tag{5.8}$$

In Newtonian limit ϵ_{nr}^2 and r_s/r terms can be neglected, so that we reach the correct Newtonian limit,

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{2\epsilon_{nr}r^4}{l^2} + \frac{c^2r_sr^3}{l^2} - r^2 = \left(\frac{dr}{d\phi}\right)^2_{nr}. \tag{5.9}$$

Now let us look at the terms of $V_e(r)$

$$V_e(r) = -\frac{r_s}{r} + \frac{l^2}{c^2r^2} - \frac{r_sl^2}{c^2r^3}. \tag{5.10}$$

The first term,$-1/r$, stems from the attractive gravitational potential and the second term is related to the centrifugal potential. These two correspond to the effective potential in Newtonian physics. The last term however, that is proportional to $-1/r^3$, is the relativistic correction to the Newtonian potential. For large r, first two terms are important and we can neglect the third one but for small r, the last term is not negligible. Also notice that it is with the minus sign, which shows that it contributes to the potential in an attractive way. Furthermore, as it can be seen on the Figure 4.7, there is a finite centrifugal potential barrier whereas in Newtonian physics it is infinite. Actually, addition of the correction term,$-1/r^3$, turns the infinite barrier into a finite barrier. This can seen in Figure 5.1, where the Newtonian effective potential is compared to $V_e(r)$.

$$V_e(r) = \underbrace{-\frac{1}{x} + \frac{\lambda^2}{x^2}}_{\text{Newtonian}} - \frac{\lambda^2}{x^3} \tag{5.11}$$

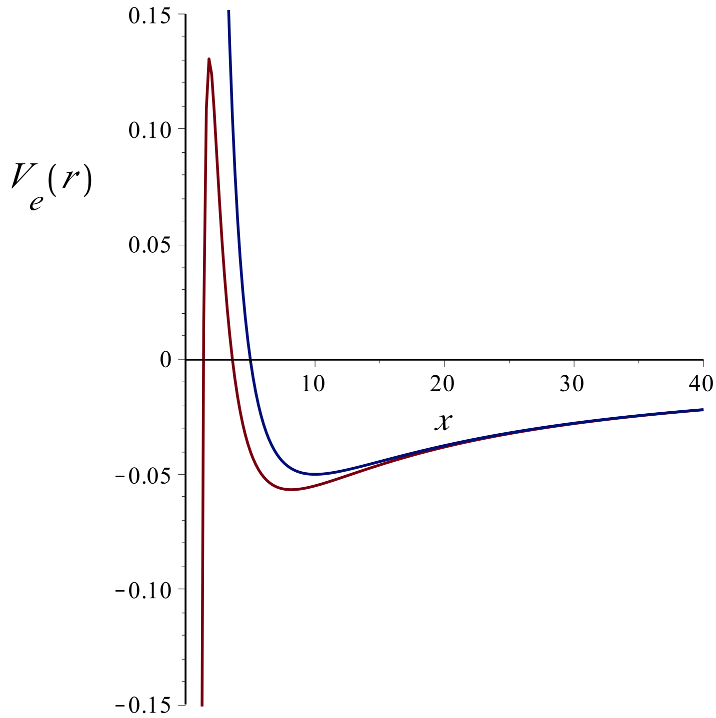

Figure 5.1: Relativistic correction to the Newtonian effective potential. The blue and red curves show the Newtonian effective potential and $V_e(r)$ respectively.

It is clear from Figure 5.1 that, a particle with energy E can never pass the Newtonian potential hence never plunge into the center of the potential. Whereas maximum point of the red line shows that, there is a finite potential barrier for the particle near the Schwarzschild black hole. So, whenever it has a greater energy than the maximum of the $V_e(r)$, the particle plunges into the black hole.

6. CONCLUSION

In this thesis we solved for the geodesics of both the particle with mass m and the light inside and outside of the Schwarzschild black hole horizon. The geodesics have been studied in the cases where the angular momentum l (of m and light) is zero and nonzero. Starting from the metric of each case we first calculated integral equations for the geodesics. Then, out of our equations we defined an effective potential and plotted it.

For null geodesics with $l = 0$ case, at the Schwarzschild radius, time t measured by an outside observer becomes infinite. So that, for the observer it takes infinite amount of time for light to reach the horizon. In the same manner, it takes infinite amount of time for light emitted at the horizon to reach the observer. Similarly, for non-null geodesics with $l = 0$, the particle m in the exterior region, reaches to the horizon in an infinite amount time, t, from the point of an outside observer. However, in the frame of mass m, it reaches the horizon in a finite amount of proper time, τ [13]. Notice that these geodesics are stopped at the $r = 0$ singularity because it is assumed that spacetime ends at that point. This shows that the theory is incomplete since it cannot be answered what happens as proper time continues to tick.

In the literature, the interior geometry of the Schwarzschild black hole is considered as a continuum of the exterior geometry. Therefore, infalling particles are considered to be ended at the central singularity at $r = 0$. However, in the interior region, *space* and *time* coordinates interchange to conserve casuality [8]. Therefore, r is treated as t, so that in the horizon, r is a temporal coordinate. This means that the particle in the horizon is not approaching through a static point but temporal. Thinking the interior region as a continuum of static exterior region leads to these conceptual difficulties.

For null geodesics with $l \neq 0$, we first defined an analogue potential with a dimensionless parameter, $\frac{r_s}{r_l}$, where $r_l = lc/\epsilon$. Then, the potential is plotted for different

l values. We realized that only the case, $l \to \infty$ limit, the geodesics stays in the Schwarzschild horizon. Whereas, in the cases where $0 < l < \infty$, null geodesics seem to reach out beyond the Schwarzschild radius. But, when the geodesics are re-written in terms of the analogue potential, we could not satisfy our equations. In other words, there is an inconsistency with these type of geodesics. New theories are accomplishing to get a better understanding of the black hole horizon [14].

For non-null geodesics with $l \neq 0$, we again defined an effective potential with the dimensionless parameter, $\lambda = l/r_s c$ and studied it. When $l = r_s c$, the particle with mass m falls into the center of the effective potential. For $l > r_s c$ cases there are stable and unstable circular orbits, precessing elliptical orbits and scattering orbits. But, the inconsistency mentioned in the previous case exists in here also. Lastly, we compared Newtonian physics and general relativity from the point of view of angular momentum. In Newtonian physics, angular momentum term in the expression of effective potential leads to infinite potential barrier. So that the particle, m, never falls into the center of the potential. On the other hand, there is a correction term coming from general relativity to the Newtonian effective potential. With this correction, the infinite potential barrier becomes finite. Hence, falling into the center of potential of massive particles and light becomes possible.

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