ASPECTS OF STRING THERMODYNAMICS

by

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ABSTRACT

ASPECTS OF STRING THERMODYNAMICS

In this thesis, we study thermodynamics of closed strings and open strings on D-branes in the microcanonical ensemble. Both single string and multi-string density of states are obtained from energy spectrum. In the microcanonical description, the temperature always become smaller than the limiting Hagedorn temperature, reaching it in the infinite energy limit. The stability of both systems are verified in the microcanonical picture. We also determine the early time cosmic evolution sourced by open strings on D-branes. The energy-momentum tensor of this system is determined from the thermodynamical data and the field equations are determined from consistency using Bianchi identities. We observe that although pressures are large, they can be ignored in field equations compared to energy. This simplifies field equations a lot and an approximate but analytic solution can be obtained.

ÖZET

SİCİM TERMODİNAMİĞİNİN GENEL ÖZELLİKLERİ

Bu tezde, kapalı ve D-zarları üzerindeki açık sicimlerin termodinamik özelliklerini mikrokanonik toplulukta çalıştık. Her iki tek sicim ve çoklu-sicim durum yoğunlukları enerji spektrumundan elde edilmiştir. Mikrokanonik açıklamada, sıcaklık her zaman limit Hagedorn sıcaklığından küçüktür, o sıcaklığa sonsuz enerji limitinde ulaşır. Her iki sistemin kararlılığı mikrokanonik temsilde onaylanmıştır. Ayrıca D-zarları üzerindeki açık sicim kaynaklı erken zaman kozmik genişlemeyi tanımladık. Bu sistemin enerji-momentum tensörü termodinamik veriden ve alan denklemleri Bianchi benzerliği kullanılarak süreklilikten elde edilmiştir. Basınç, büyük olmasına rağmen alan denklemlerinde enerjinin yaninda ihmal edilebilmiştir. Bu, alan denklemlerini oldukça basitleştirmiş ve yaklaşık ama analitik bir çözüm elde edilebilmiştir.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS/ABBREVIATIONS

f	The ratio of volumes of the spatial Neumann and Dirichlet directions
$g_{\mu\nu}$	Metric tensor
g_s	String coupling constant
H	Hubble parameter
k_B	Boltzman constant
$\mathcal{L}_0$	Effective Lagrangian
m_i	Momentum number
n	Oscillator level
n_+	Level number for right movers
n_-	Level number for left movers
R_D	Dirichlet radius
R_N	Neumann radius
$R_{\mu\nu}$	Ricci curvature tensor
T_H	Hagedorn temperature
$T_{\mu\nu}$	Energy-momentum tensor
T_s	String tension
V_D	Volume of the spatial Dirichlet directions
V_N	Volume of the spatial Neumann directions
$\alpha^{'}$	Slope parameter
η	Conformal time
ϕ	Dilaton
w_i	Winding number

1. INTRODUCTION

String theory is a candidate for a unified theory of all interactions and quantum gravity (see e.g. [1], [2]), which is based on the assumption that elementary particles in the universe are not point like excitations, but they are 1-dimensional objects which are composed of loops of vibrating strings. One assumes that to each different vibrational mode of the string there corresponds to an elementary particle. String theory has very interesting properties. For instance, consistency of the theory requires existence of extra dimensions in addition to the observed space-time dimensions. In general, string theory includes both closed and open strings which have two end points. Assuming simple joining and splitting interactions one can see that open strings can always form closed strings. However, the opposite, i.e. the splitting of a closed string into two open strings, is avoided by a conservation rule. Therefore, one can consider theories with or without open strings. There is also a subdivision between bosonic and supersymmetric strings. A bosonic string lives in 26 space-time dimensions and being tachyonic its ground state is not well-defined. On the other hand, a supersymmetric string lives in a 10 dimensional space-time. There are five known consistent superstring theories: type-I (both closed and open strings), closed type-II A, closed type-II B, and two heterotic string theories $SO(32)$ and $E_8 \times E_8$. Recent studies also revealed the existence of a more fundamental 11-dimensional theory, called M-theory (for a review see e.g. [3]). These different string theories (and M-theory) are connected by different symmetries (or dualities). For instance, T-duality is known as the exchange of compactification radius R with $1/R$ that transforms type-II A theory to type-II B and vice versa. The existence of higher dimensional objects (branes) in string theory is crucial for duality transformations to work. As an important example, in closed string theories D-branes are hypersurfaces on which open strings can end with Dirichlet boundary conditions.

The issue of thermodynamics of strings has not been studied too much in the literature (see [4], [5], [6] for earlier work). However, thermodynamics is important for understanding the early time cosmic evolution (see e.g. [7]). In order to study thermodynamics in the presence of gravity, one should assume suitable conditions to avoid formation of black holes due to gravitational collapse. In string theory this can be achieved by taking a small string coupling constant and low energy density. As we will see, some aspects of string thermodynamics are very interesting. Since there is an exponential degeneracy in the number of states, thermal partition function does not converge above the so called Hagedorn temperature. Therefore, in such extreme cases the thermodynamics of strings should be studied in the microcanonical ensemble. Moreover, due to the existence of winding modes, some thermal properties depend on the topology or the size of the compactification radius.

In standard cosmology, big-bang singularity remains as an issue to be resolved from a physical point of view. For instance, the initial condiditons, which tell us in what state the universe emerged from big-bang, are lacking. Since string theory is a candidate of quantum gravity, it should explain or resolve this singularity problem and provide initial conditions. In string theory there are different scenarios about early time cosmology and big-bang. String gas cosmology [9] is one possibility that we focus on this thesis. In string gas cosmology, it is assumed that strings are in thermal equilibrium at extreme energies with all possible excitations like winding modes, which are expected to play important cosmological roles in both early and late time evolutions (see [10], [13] for recent reviews).

In this thesis, we first study thermal properties of closed and open strings and then consider an early time cosmology dominated by open strings attached to D-branes. Our motivation is to extend the results appeared in the literature on closed strings (see e.g. [11], [12]) to include open string excitations. The plan of the thesis is as follows: In chapter II, we review thermodynamics of closed strings and open strings attached to parallel D-branes which are uniformly distributed along some compact directions in a small radius regime. In chapter III, we discuss the cosmology of strings in thermal equilibrium in a totally compact space. We use basic thermodynamical properties which are obtained in the previous chapter in dilaton-gravity equations to determine the early time cosmological evolution. In chapter IV we discuss our results and conclude.

2. Thermodynamics of Strings

2.1. Microcanonical vs. Canonical Ensembles

We consider a system which is characterized by a set of quantum numbers r with corresponding energies E_r. The statistical description of this system at the most fundamental level can be studied in the microcanonical ensemble. Assuming that it is mechanically and adiabatically isolated, the equilibrium probability distribution for the microcanonical ensemble is given by

$$p_r = \frac{1}{\Omega(E)} \begin{cases} 1 & \text{for } E+\delta E > E_r > E, \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

where $\Omega(E)$ is the number of states accessible to the system with energy E

$$\Omega(E) = \sum_r \delta(E - E_r)\delta E. \tag{2.2}$$

The entropy of the system is given by

$$S(E) = k_B \ln \Omega(E), \tag{2.3}$$

where k_B is the Boltzmann constant and one can reach the fundamental result of thermodynamics as

$$\frac{\partial S}{\partial E} = \frac{1}{T}. \tag{2.4}$$

We will use suitable units for the temperature and set $k_B = 1$.

In contrast to microcanonical ensemble, in the canonical ensemble the system is maintained at a constant temperature in contact with a sufficiently large reservoir. The combined system can be analyzed in the microcanonical ensemble and the probability of finding the quantum state r can be determined as

$$p_r = \frac{e^{-\beta E_r}}{Z}, \tag{2.5}$$

where the normalization constant

$$Z(\beta) = \sum_r e^{-\beta E_r} \tag{2.6}$$

is known as the *partition function* and $\beta \equiv 1/T$. In the canonical ensemble the entropy is given by

$$S = k_B(\ln Z + \beta \bar{E}), \tag{2.7}$$

where $\bar{E}$ is the average energy, which can be found as

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta}. \tag{2.8}$$

There is a precise mathematical relationship between $Z(\beta)$ and $\Omega(E)$. From (2.2) it is easy to see that $Z(\beta)$ is the Laplace transform of $\Omega(E)$ (here we set $\delta E = 1$)

$$Z(\beta) = \int_0^{+\infty} dE e^{-\beta E} \Omega(E). \tag{2.9}$$

On the other hand, using the integral representation of the Dirac delta function

$$\delta(E - E_r) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\beta e^{i\beta(E-E_r)} \tag{2.10}$$

$\Omega(E)$ can be rewritten as

$$\Omega(E) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\beta e^{i\beta(E)} \sum_r e^{(-i\beta)E_r}. \tag{2.11}$$

Thus we have

$$\Omega(E) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\beta E} Z(i\beta) d\beta, \tag{2.12}$$

which can in principle be used to calculate $\Omega(E)$ from $Z(\beta)$ using complex analysis.

In ordinary systems $\Omega(E)$ typically grows with energy like E^f where f is the number of the degrees of freedom. Since $e^{-\beta E}$ is a rapidly decaying function, in the integral (2.9) there exists an energy E^* around which $e^{-\beta E}\Omega(E)$ is sharply peaked. Thus the integral can approximately be evaluated as

$$Z \cong e^{-\beta E^*}\Omega(E^*)\Delta E. \tag{2.13}$$

From (2.8) one sees that $E^* = \bar{E}$ and thus the most probable canonical energy is sharply peaked around the average energy. Using (2.13) in (2.7) one recovers the definition of entropy (2.3) in the microcanonical ensemble so that the canonical ensemble becomes indistinguishable from the microcanonical ensemble at the average energy.

To illustrate this point let us consider an ideal gas of N particles in a box of volume V. The number of states $\Omega(E)$ with energy E is given by the volume available in the phase space which can be found as

$$\Omega(E) \cong V^N E^{\frac{3N}{2}}, \tag{2.14}$$

where the first factor is the volume of the position space and the second factor is the volume of a $3N$-dimensional sphere in the momentum space determined from $E = \sum_i \frac{\vec{p}_i^2}{2m}$. From this we see that the entropy of the system is

$$S = k(N\ln V + \frac{3N}{2}\ln E). \tag{2.15}$$

On the other hand, the partition function is given by

$$Z(\beta) = \prod_i \int d^3\vec{r}_i \int d^3\vec{p}_i e^{-\beta\vec{p}_i^2/2m}. \qquad (2.16)$$

The integral in the position space again gives a factor of V^N and momentum integrals are simple Gaussians. A straightforward calculation then gives

$$\ln Z(\beta) \cong N(\ln V - \frac{3}{2}\ln\beta). \qquad (2.17)$$

The total mean energy of the gas is

$$\bar{E} = -\frac{\partial}{\partial\beta}\ln Z = \frac{3}{2}\frac{N}{\beta} \qquad (2.18)$$

and the entropy of the system is found as

$$S = k(N\ln V + \frac{3N}{2}\ln\bar{E}), \qquad (2.19)$$

which is same with the entropy calculated from microcanonical ensemble (2.15). Thus we see that microcanonical and canonical ensembles are equivalent at the average energy of the system. Let us point out that in the above calculation we treated particles as distinguishable. For identical particles one should divide Ω and Z by $N!$, namely $\Omega \to \frac{1}{N!}\Omega$ and $Z \to \frac{1}{N!}Z$ to reduce the overcounting of states. As is well known, in this way the extensivity of entropy is ensured.

As we will see in the next section, in string theory the equivalence of microcanonical and canonical ensembles breaks down at extreme energies since $\Omega(E)$ grows exponentially with energy as $\Omega(E) \approx e^{\beta_H E}$ where the constant parameter $\beta_H = \frac{1}{T_H}$ is the so called Hagedorn temperature. Due to this exponential degeneracy it is not possible to analyze strings using the canonical ensemble at temperatures higher than the Hagedorn temperature $T > T_H$ since the partition function does not converge. We need to use the microcanonical ensemble to explore the string thermodynamics in such extreme regimes.

2.2. Closed Strings

We focus on type-II superstring theories and consider a closed string propagating in a 10-dimensional flat space-time. We take d-dimensions to be non-compact and assume the remaining $(9-d)$ dimensions to be compact circles. A closed string in this background is labeled by the following quantum numbers; n_+, n_-, m_i, w_i, $m_{i'}$, where $i = 1, ..., 9-d$, $i' = 1, ..., d$. Here n_+ and n_- are the level numbers for right and left movers respectively and (m_i, w_i) are the momentum and winding numbers. For convenience, we compactify the non-compact dimensions on large circles of radius $R_{i'}$. Thus the momentum components along these directions are quantized and $m_{i'}$ is the corresponding momentum number. There is no winding for non-compact directions and we finally take $R_{i'} \to \infty$ limit at the end of the calculation. The numbers n_+ and n_- are related by the level matching condition as (see e.g. [8])

$$n_- = n_+ + \vec{m}.\vec{w}. \tag{2.20}$$

Let R_i to denote the radius of the i^{th} compact direction. Then the energy of the closed string is given by [8]

$$E^2 = \frac{2}{\alpha'}(n_+ + n_-) + \sum_{i'} \left(\frac{m_{i'}}{R_{i'}}\right)^2 + \sum_i \left(\frac{m_i}{R_i}\right)^2 + \sum_i \left(\frac{w_i R_i}{\alpha'}\right)^2, \tag{2.21}$$

where α' is the slope parameter which is related to string tension as $T_s = \frac{1}{2\pi\alpha'}$. Equation (2.21) can be viewed as the generalization of the relativistic energy formula $E^2 = \vec{p}^2 + m^2$ in the presence of winding modes. The first term on the right hand side of (2.21) is like the "rest mass", and the "momenta" corresponding to momentum and winding numbers are given by

$$p_i = \frac{m_i}{R_i} \quad , \quad p_i^{(w)} = \frac{w_i R_i}{\alpha'} = w_i (2\pi R_i) T_s. \tag{2.22}$$

The quantization of momentum p along a compact direction can be deduced from the quantum mechanical wave function e^{ipx}. To get a single valued wave function on the circle, i.e. for the wave function not to change under $x \to x + 2\pi R$, the momentum should take discrete values $p = n/R$ with n being an integer. On the other hand, the winding number w counts the number of times a string winding a compact direction and the corresponding energy is simply given by the tension times the length. Let us note that in the above formulas we are using units such that $c = 1$ and $\hbar = 1$; thus mass, length and time can be measured in terms of energy units (recall that we have already set $k_B = 1$ and the temperature can also be measured in energy units).

For a given level number there corresponds to different closed string states. Asymptotically the degeneracy for either left-moving or right-moving oscillator is given by [9]

$$d(n_+) \cong (2n_+)^{-\frac{11}{4}} \exp(\beta_H \sqrt{\frac{n_+}{2\alpha'}}) \quad , \quad d(n_-) \cong (2n_-)^{-\frac{11}{4}} \exp(\beta_H \sqrt{\frac{n_-}{2\alpha'}}), \tag{2.23}$$

where

$$\beta_H = 2\pi\sqrt{2\alpha'}. \tag{2.24}$$

As we will see below, this is the source of exponential degeneracy that destroys the equivalence of microcanonical and canonical ensembles mentioned in the previous section. The total degeneracy for closed strings is given by $d(n_+)d(n_-)$.

We first calculate the number of states $\omega(E)$ accessible with energy E for a *single* string in the microcanonical ensemble. The degeneracy function $\omega(E)$ can be found as

$$\begin{aligned}\omega(E) = &\sum_{n_+}\sum_{m_i}\sum_{w_i}\sum_{m_{i'}} d(n_+)d(n_-) \\ &\delta\left[E - \sqrt{\frac{2}{\alpha'}(n_+ + n_-) + \left(\frac{m_{i'}}{R_{i'}}\right)^2 + \left(\frac{m_i}{R_i}\right)^2 + \left(\frac{w_i R_i}{\alpha'}\right)^2}\right],\end{aligned} \tag{2.25}$$

where n_- is fixed from (2.20). Note that in (2.25) the delta-function only counts the states with the given energy E. We are going to analyze the regime with $E \gg 1/\sqrt{\alpha'}$ and thus $n_+ \gg 1$. Therefore, we first replace sum over n_+ by an integral which can be performed by the use of the delta function as

$$\omega(E) = \frac{\alpha^{'}}{2} \sum_{m_i} \sum_{w_i} \sum_{m_{i'}} d(n_+) d(n_+ + m_i w_i)\, E, \tag{2.26}$$

where $n_+ = n_+(E)$ is now seen as a function of E and other quantum numbers by (2.21). To get the leading order asymptotic behavior we use the degeneracy expression (2.23) in (2.26). Expanding $n_+(E)$ in a series for large E around $n_+(E) = \frac{\alpha^{'} E^2}{2}$, one can calculate $\omega(E)$ order by order in inverse powers of energy. The term that gives the greatest contribution can be found as

$$\begin{aligned}\omega(E) = \frac{\alpha^{'} E}{2} \sum_{m_i} \sum_{w_i} \sum_{m_{i'}} \left(\frac{\alpha^{'}}{2} E^2 \right)^{-\frac{11}{2}} \\ \exp \left[\beta_H E - \frac{\beta_H}{2E} \left(\frac{m_{i'}}{R_{i'}} \right)^2 - \frac{\beta_H}{2E} \left(\frac{m_i}{R_i} \right)^2 - \frac{\beta_H}{2E} \left(\frac{w_i R_i}{\alpha'} \right)^2 \right] + \cdots\end{aligned} \tag{2.27}$$

As an approximation let us replace sums over integers m_i, w_i, $m_{i'}$ by continuous integrals which become simple Gaussians. Carrying out the integrals one reaches a simple equation

$$\omega(E) = c \frac{e^{\beta_H E}}{E^{1+\frac{d}{2}}} V, \tag{2.28}$$

where

$$c = \left(\frac{2}{\alpha'^3} \right)^{d/4} \tag{2.29}$$

and $V = \prod_{i'} R_{i'}$ can be interpreted as the volume of the system along non-compact directions. It is important to note that the constant c is equal to 1 when all dimensions are compact ($d = 0$). As we will see, this is important to ensure the stability of the system.

Approximating sums by integrals in (2.27) requires that the numbers multiplying m_i^2, w_i^2, $m_{i'}^2$ in the exponentials to be much less than 1. Using (2.24), this implies

$$R_i^2 \gg \frac{\sqrt{\alpha'}}{E}, \qquad R_i^2 \ll E\sqrt{\alpha'^3}, \qquad R_{i'}^2 \gg \frac{\sqrt{\alpha'}}{E}. \tag{2.30}$$

Since we take $R_{i'} \to \infty$, the last condition is satisfied trivially. In string theory, it is always possible to take $R_i \geq \sqrt{\alpha'}$ by applying a T-duality transformation $R_i \to \alpha'/R_i$. Therefore, the first condition involving R_i is satisfied for $E \gg 1/\sqrt{\alpha'}$. The second condition on R_i gives an upper bound for the radius for a given energy E, showing that in using (2.31) one should restrict to small enough compactification radii. We call this to be the small radius regime. As discussed in [6] the form of (2.28) will change in a large radius regime.

Having obtained the single string density of states, we now proceed with the calculation of the total degeneracy $\Omega(E)$. As discussed in [6] this calculation depends on the number of non-compact dimensions d in a nontrivial way. Here we simply analyze the case with $d = 0$, i.e. all dimensions are taken to be compact and from (2.28) we have

$$\omega(E) = \frac{e^{\beta_H E}}{E}. \tag{2.31}$$

We assume that the single string density of states is given by (2.31) for $E > \Lambda$ and ignore lower energy states. As noted above (2.31) is valid in the small radius regime determined by (1.30), i.e. $R_i^2 \ll E\sqrt{\alpha'^3}$ (large radius corrections can be found in [6]). Let Ω_N to denote the multi-string density of states with N strings and total energy E. Since strings are identical, Ω_N is given by

$$\Omega_N(E) = \frac{1}{N!}\int_\Lambda^E \prod_{i=1}^N dE_i \omega(E_i)\delta\left[(\sum E_i) - E\right]. \tag{2.32}$$

By defining $x_i = E_i/E$ and using (2.31) we convert this integral to

$$\Omega_N(E) = \frac{1}{N!}\frac{\exp(\beta_H E)}{E}\int_{\Lambda/E}^{1}\prod_{i=1}^{N}\frac{dx_i}{x_i}\delta\left[(\sum x_i) - 1\right]. \tag{2.33}$$

Using the integral representation of the delta function, (2.33) can be transformed into

$$\Omega_N(E) = \frac{1}{2\pi N!}\frac{\exp(\beta_H E)}{E}\int_{-\infty}^{\infty} d\alpha \exp(-i\alpha)\left[\int_{\Lambda/E}^{1}\frac{dx}{x}\exp(i\alpha x)\right]^N, \tag{2.34}$$

The total density of states $\Omega(E)$ is given by

$$\Omega(E) = \sum_N \Omega_N(E). \tag{2.35}$$

Defining a function $F(\alpha, \Lambda/E)$

$$F(\alpha, \Lambda/E) = \int_{\alpha\Lambda/E}^{\alpha}\frac{dx}{x}\exp(ix).$$

the sum over N can be performed to get

$$\Omega(E) = \frac{1}{2\pi}\frac{\exp(\beta_H E)}{E}\int_{-\infty}^{\infty} d\alpha \exp(-i\alpha)\exp(F(\alpha, \Lambda/E)). \tag{2.36}$$

By noting that $F^*(\alpha) = F(-\alpha)$, we divide the integral into two parts.

$$\Omega(E) = \frac{1}{2\pi}\frac{\exp(\beta_H E)}{E}\int_{-\infty}^{\infty} d\alpha\frac{1}{2}\left[\exp(-i\alpha)F(\alpha) + \exp(i\alpha)F^*(\alpha)\right]. \tag{2.37}$$

Using now the Euler formula $e^{i\alpha} = \cos\alpha + i\sin\alpha$, one can expand the exponentials and use the angle addition formula for cosine to get

$$\begin{aligned}\Omega(E) = \frac{1}{2\pi}\frac{\exp(\beta_H E)}{E}\int_{-\infty}^{\infty} d\alpha \exp\left[(\int_0^{\alpha}\frac{\cos x - 1}{x}dx)\ln(\frac{E}{\Lambda})\right] \\ \left[\cos(\int_0^{\alpha}\frac{x-\sin x}{x}dx) - \sin(\int_0^{\alpha}\frac{x - \sin x}{x}dx)\frac{\alpha\Lambda}{E}\right].\end{aligned} \tag{2.38}$$

Defining the convergent numerical factors as

$$a = \frac{1}{\pi}\int_0^\infty d\alpha \cos(\int_0^\alpha \frac{x-\sin x}{x}dx)\exp(\int_0^\alpha \frac{\cos x - 1}{x}dx), \tag{2.39}$$

$$b = \frac{1}{\pi}\int_0^\infty \alpha d\alpha \sin(\int_0^\alpha \frac{x-\sin x}{x}dx)\exp(\int_0^\alpha \frac{\cos x - 1}{x}dx), \tag{2.40}$$

one finally reach the following result for total number density of states [9]

$$\Omega(E) = \frac{\exp(\beta_H E)}{E}(\frac{E}{\Lambda}a - b + \cdots). \tag{2.41}$$

Using the definition given by (2.3) we get entropy of the system

$$S = \ln\Omega(E) \approx \beta_H E - \frac{b\Lambda}{aE} + constant \quad . \tag{2.42}$$

The temperature is given by

$$\frac{1}{T} = \beta = \frac{dS}{dE} = \beta_H + \frac{b\Lambda}{aE^2} + \cdots , \tag{2.43}$$

thus it is always smaller than the Hagedorn temperature since $E > \Lambda$. As one increases the energy, the temperature approaches to the Hagedorn temperature from below. Therefore, in the microcanonical ensemble, the maximum physically allowed temperature is the Hagedorn temperature.

As mentioned previously, the numerical prefactor in (2.31) is exactly equal to 1. If the constant c in (2.31) were different from 1, the leading order contribution to entropy differs from (2.42):

$$S \approx \beta_H E + (c-1)\ln E. \tag{2.44}$$

The stability of a system in the microcanonical ensemble requires $d\beta/dE < 0$. Thus from (2.44) it can be seen that for $c > 1$, $d\beta/dE$ is positive and for $c < 1$ it is negative. In the case of $c = 1$, the logarithmic term vanishes and stability is ensured by the leading term in (2.42). It is remarkable to note that, getting both a critical value $c = 1$ and then a leading order correction with suitable sign is very non-trivial, which is dictated by the quantum dynamics of strings. To sum up, in a totally compact space the thermodynamics of closed strings is well defined in the microcanonical approach.

2.3. Open Strings

In closed string theories, D-branes are extended objects on which open strings can end. In this subsection we study thermodynamics of open strings attached to $\mathcal{N}$ parallel d_N dimensional D-branes in a *totally compact toroidal space* in the weak string coupling. The brane directions are called Neumann and the transverse directions are called Dirichlet, which are labeled by the indices $i = 1, ...d_N$ and $i^{'} = 1, ...d_D$, respectively, where $d_N + d_D = 9$ (see figure 2.1). An open string in this background is labeled by quantum numbers m_i, $w_{i^{'}}$, n, where m_i is the momentum and $w_{i^{'}}$ is the winding numbers and n is the oscillator level. In addition, two Chan-Paton factors should be specified indicating on which D-branes the two ends of the open string are attached. Note that there are d_N momentum numbers and d_D winding numbers in the system, namely there is no momentum mode along Dirichlet and no winding mode along Neumann directions. T-duality can be used to make all radii larger than the T-selfdual radius $R = \sqrt{\alpha'}$. In this subsection we use string units and set $\alpha^{'} = 1$, thus we take

$$R_N, R_D \geq 1. \tag{2.45}$$

The energy of the single open string is given by [14]

$$E^2 = \sum_i \frac{m_i^2}{R_i^2} + \sum_{i^{'}} w_{i^{'}}^2 R_{i^{'}}^2 + n, \tag{2.46}$$

where R_i and $R_{i^{'}}$ are the corresponding Neumann and Dirichlet radii.

For open strings attached to the parallel D-branes, the level-density with a par-

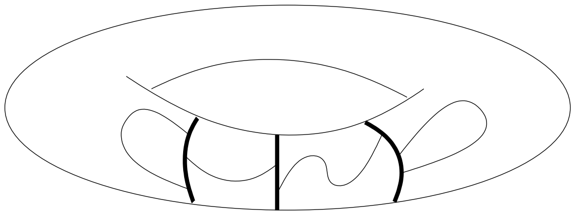

Figure 2.1. Parallel D-branes in a compact space and open strings attached to them.

ticular value of the oscillator level is given by [14]

$$d(n) \cong n^{-11/4} e^{\beta_H \sqrt{n}}, \tag{2.47}$$

where

$$\beta_H = 2\sqrt{2}\pi \tag{2.48}$$

is the Hagedorn temperature for open strings. The degeneracy function $\omega(E)$, which counts the number of states with energy E can be written as

$$\omega(E) = \sum_n \sum_{m_i} \sum_{w_{i'}} \delta\left[E - \sqrt{\sum_i \frac{m_i^2}{R_i^2} + \sum_{i'} w_{i'}^2 R_{i'}^2 + n}\right] d(n)\, \mathcal{N}^2. \tag{2.49}$$

The $\mathcal{N}^2$ term arises from the combinatorics of the Chan-Paton factors, i.e. the number of ways of attaching the ends of the strings to D-branes. We would like to evaluate (2.49) for $E \gg 1$ in an asymptotic series. As in closed string calculation, we first replace sum over n by an integral and use the delta function to obtain

$$\omega(E) = \sum_{m_i} \sum_{w_{i'}} 2\mathcal{N}^2\, E\, d(n), \tag{2.50}$$

where $n = n(E)$ is now seen as a function of E. Using the degeneracy expression (2.47) in (2.50) to get the asymptotic behavior, one obtains

$$\omega(E) dE \approx \sum_{m_i} \sum_{w_{i'}} \mathcal{N}^2 E^{(-9)/2} e^{\beta_H \sqrt{n(E)}} dE. \tag{2.51}$$

As before using (2.46) we expand $n(E)$ around $n(E) = E^2$ as

$$\sqrt{n(E)} = E - \sum_i \frac{m_i^2}{2ER_i^2} - \sum_{i'} \frac{w_{i'}^2 R_{i'}^2}{2E} + \cdots \quad . \tag{2.52}$$

and plug this into (2.51).

To proceed, we would like to approximate the summations over m_i and $w_{i'}$ by Gaussian integrals. This can be done if

$$R_i^2 \gg 1/E \quad , \quad R_{i'}^2 \ll E \quad . \tag{2.53}$$

The first condition is satisfied trivially since $E \gg 1$ and thus sum over m_i can be converted to an integral. If the second condition $R_{i'}^2 \ll E$ is also satisfied, $w_{i'}$ sum can also be converted to an integral. Otherwise, for $R_{i'}^2 \gg E$, one should keep only $w_{i'} = 0$ term. Therefore, given a radius R_0 along a Dirichlet direction, there is a threshold energy $E_0 = R_0^2$ such that above this energy thermal strings are energetic enough to wind around Dirichlet directions.

In the following we take Neumann and Dirichlet direction to have common respective values, $R_i = R_N$ and $R_{i'} = R_D$. In the light of the above comments, the single string density of states in different energy limits can be calculated as

$$\omega(E) = \begin{cases} f e^{\beta_H E} & E \gg R_D^2, \\ f(E/E_0)^{-d_D/2} e^{\beta_H E} & E \ll R_D^2, \end{cases} \tag{2.54}$$

where f is defined as

$$f = \frac{V_N}{V_D} \mathcal{N}^2, \tag{2.55}$$

$V_N = (R_N)^{d_N}$ and $V_D = (R_D)^{d_D}$ are the volumes of the spatial Neumann and Dirichlet directions. As discussed in [14], for $E \gg 1$, it is possible to rewrite $\omega(E)$ as a summation over a quantum number l as

$$\omega(E) = f e^{\beta_H E} \sum_l g_l e^{-\frac{l^2 E}{R_D^2}} \tag{2.56}$$

where g_l is a degeneracy factor given by $g_l \cong 2Vol(S^{d_D-1}) l^{d_D-1}$ for $l \gg 1$. One can see the equivalence of (2.54) and (2.56) by noting that the summation over l only the first term $l = 0$ contributes for $E \gg R_D^2$. However, for $E \ll R_D^2$ we can replace the sum with a Gaussian integral. As for closed strings, one can define the small and large radius regimes for $E \gg R_D^2$ and $E \ll R_D^2$ respectively.

In the following we consider the small radius regime and take

$$\omega(E) = f e^{\beta_H E}. \tag{2.57}$$

Let $\Omega_N(E)$ to denote the multi-string density of states with N open strings attached to parallel D-branes. As before $\Omega_N(E)$ can be written as

$$\Omega_N(E) = \frac{1}{N!} \prod_{i=1}^{N} \int_0^E \omega(E_i)\, dE_i\, \delta \left[(\sum_i E_i) - E \right]. \tag{2.58}$$

Using the integral representation of the delta function, (2.58) can be transformed into

$$\Omega_N(E) = \frac{1}{2\pi E N!} \int_{-\infty}^{\infty} d\alpha e^{-i\alpha} \left[\int_0^E d\epsilon\, \omega(\epsilon) e^{i\alpha} \right]^N. \tag{2.59}$$

To find the total density of states $\Omega(E) = \sum_N \Omega_N(E)$, we define a function $F(\alpha)$

$$F(\alpha) = \int_0^E d\epsilon\, \omega(\epsilon) e^{i\alpha\epsilon/E}. \tag{2.60}$$

and thus the sum over N can be performed to get

$$\Omega(E) = \frac{1}{2\pi E} \int_{-\infty}^{\infty} d\alpha\, e^{-i\alpha} e^{F(\alpha)}. \tag{2.61}$$

By putting $\omega(E) = fe^{\beta_H E}$ into (2.60) one obtains

$$F(\alpha) = fE\frac{e^{i\alpha + E\beta_H} - 1}{i\alpha + E\beta_H}. \tag{2.62}$$

$F(\alpha)$ is a regular function in the complex α-plane which vanishes as $\alpha \to \pm\infty$ and can be written as the sum of two singular functions $F(\alpha) = F_1(\alpha) + F_2(\alpha)$, where

$$F_1 = fE\frac{e^{i\alpha+\beta_H E}}{i\alpha + \beta_H E} \quad , \quad F_2 = fE\frac{-1}{i\alpha + \beta_H E}. \tag{2.63}$$

Since $F(\alpha)$ is an analytic function in the complex α-plane we can deform the integral contour near the origin by extending it through the imaginary axis to circle around the point $iE\beta_0$ (see figure 2.2). We see that in the expansion of $\exp(F_1)$ in powers of F_1 only first term contributes to the integral along the deformed contour. By excluding the poles F_1 and F_2, to evaluate the first term the contour can be closed from above and the integral along semi-circle at infinity vanishes. Therefore we have

$$\Omega(E) \cong \Omega_0(E) = \frac{1}{2\pi E}\oint d\alpha \exp\left(\frac{-fE}{i\alpha + E\beta_H}\right) e^{-i\alpha}. \tag{2.64}$$

Closing the contour below, since there is no contribution coming from lower semi-circle the integral equals to the residue at $\alpha = iE\beta_H$. We expand the first exponential and evaluate the residue at $\alpha = iE\beta_H$ term by term giving

$$\Omega(E) \cong e^{\beta_H E}\sum_k \frac{(fE)^k}{k!(k-1)!}\frac{1}{E} = \sqrt{\frac{f}{E}}\exp(\beta_H E)I_1(2\sqrt{fE}) \tag{2.65}$$

$$\cong \frac{f^{1/4}}{E^{3/4}}\exp\left(\beta_H E + 2\sqrt{fE}\right), \qquad E \gg 1, \tag{2.66}$$

where I_1 is the modified Bessel function of first kind. Using (2.65) we find the entropy of the system as

$$S \cong \beta_H E + 2\sqrt{fE}. \tag{2.67}$$

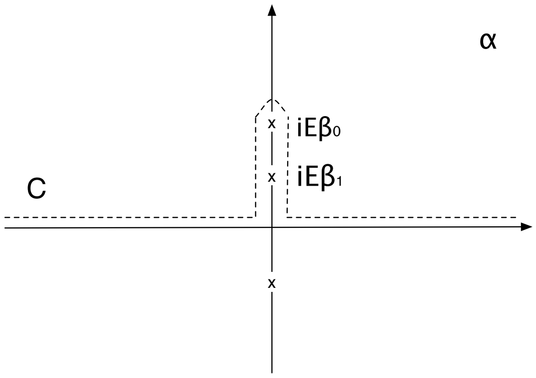

Figure 2.2. The deformed integration contour C.

The temperature is then given by

$$\frac{1}{T} = \beta = \frac{dS}{dE} = \beta_H + \sqrt{\frac{f}{E}}. \tag{2.68}$$

Since $d\beta/dE$ is negative, stability of the system is ensured. As for closed strings T is always smaller than the open string Hagedorn temperature T_H.

3. Dilaton Gravity Equations

In general relativity Einstein's equations are used to determine the curvature of spacetime in the presence of a source characterized by an energy-momentum tensor. They can be written as

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = T_{\mu\nu}, \tag{3.1}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature and $T_{\mu\nu}$ is the the energy-momentum tensor. The energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0 \tag{3.2}$$

is achieved by the contracted Bianchi identity

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R. \tag{3.3}$$

In string theory, there is a scalar field ϕ called dilaton that non-minimally couples to metric. Dilaton is a fundamental field in string theory. For instance, the closed string coupling constant g_s, which determines the strength of closed string interactions, is fixed by dilaton as $g_s = e^{2\phi}$. We first write down the source free dilaton-gravity equations which can be derived from the low-energy effective action

$$S = \int d^{10}x \sqrt{-g}\, e^{-2\phi} \left[R + 4(\nabla\phi)^2 \right]. \tag{3.4}$$

From varying this action, the field equations can be found as

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi = 0, \tag{3.5}$$

$$R + 4\nabla^2\phi - 4(\nabla\phi)^2 = 0. \tag{3.6}$$

We would like to couple a conserved energy-momentum tensor for a gas of strings to the dilaton gravity background. The equations of motion for a gas of winding D-branes coupled to dilaton are obtained in [15]. As in that discussion, our aim is to determine the dilaton-gravity equations, this time for open strings attached to D-branes (we ignore back-reaction of D branes on the geometry). The coupling of open strings to gravity is inversely proportional to the open string coupling constant e^{ϕ}. We therefore add a matter term to the action (3.4) in the following way

$$S = \int d^{10}x \sqrt{-g}\, \left[e^{-2\phi}(R + 4(\nabla\phi)^2) + e^{-\phi}\mathcal{L}_0\right], \tag{3.7}$$

where $\mathcal{L}_0$ is the effective Lagrangian for open strings. Varying this action with respect to $g_{\mu\nu}$ one obtains

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{2}\left[R + 4\nabla^2\phi - 4(\nabla\phi)^2\right] g_{\mu\nu} = e^{\phi}\, T_{\mu\nu} \tag{3.8}$$

where $T_{\mu\nu}$ is the energy momentum tensor

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}\left(\sqrt{-g}\,\mathcal{L}_0\right). \tag{3.9}$$

On the other hand, variation with respect to ϕ gives

$$R + 4\nabla^2\phi - 4(\nabla\phi)^2 + \frac{1}{2}\mathcal{L}_0 e^{\phi} = 0, \tag{3.10}$$

As discussed in [15], the form of $\mathcal{L}_0$ can be determined from the Bianchi identity and energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$. Taking the divergence of (3.8) with ∇^μ, we find

$$(\nabla_\nu\phi)\mathcal{L}_0 = T_{\nu\lambda}\,\nabla^\lambda\phi. \tag{3.11}$$

In a cosmological setting, one usually assumes

$$\phi = \phi(t), \quad g_{ti} = 0, \quad T_{ti} = 0, \tag{3.12}$$

where t is the time coordinate and i labels a spatial direction. Thus from (3.11) it can be seen that

$$\mathcal{L}_0 = T_{tt}\, g^{tt} = -\rho, \tag{3.13}$$

which $\mathcal{L}_0$ is precisely the Lagrangian for hydrodynamical matter [16]. Therefore, the field equations are given by (3.8) and (3.10) where $\mathcal{L}_0 = -\rho$.

We take the metric for this system as

$$ds^2 = -dt^2 + e^{2B(t)}(d\vec{x})^2 + e^{2C(t)}(d\vec{y})^2, \tag{3.14}$$

where the radii of the Neumann and Dirichlet directions are defined as

$$R_N = e^B, \quad R_D = e^C. \tag{3.15}$$

The energy-momentum tensor is assumed as $T_{\mu\nu} = diag(\rho, p_N, p_D)$. The equations (3.8) and (3.10) then imply (see the appendix)

$$\ddot{B} = -K\dot{B} + e^{\phi}p_N + \frac{1}{2}e^{\phi}\rho, \tag{3.16}$$

$$\ddot{C} = -K\dot{C} + e^{\phi}p_D + \frac{1}{2}e^{\phi}\rho, \tag{3.17}$$

$$\ddot{\phi} = -K\dot{\phi} - \left(\frac{3 - d_N - d_D}{4}\right) e^{\phi}\rho + \left(\frac{d_N p_N + d_D p_D}{2}\right) e^{\phi}, \tag{3.18}$$

$$K^2 = d_N\dot{B}^2 + d_D\dot{C}^2 + 2e^{\phi}\rho, \tag{3.19}$$

where

$$K = d_N\dot{B} + d_D\dot{C} - 2\dot{\phi}. \tag{3.20}$$

We derive the energy-momentum tensor from the entropy of this system obtained in the previous section. For a given $S(E, V_N, V_D)$ the temperature and the pressures are found as

$$\frac{1}{T} = \frac{\partial S}{\partial E}, \quad P_N = TV_N\frac{\partial S}{\partial V_N}, \qquad P_D = TV_D\frac{\partial S}{\partial V_D}. \tag{3.21}$$

The variables E, P_N and P_D are related to the densities in (3.16)-(3.19) as

$$\rho = \frac{E}{V}, \quad p_N = \frac{P_N}{V}, \quad p_D = \frac{P_D}{V} \tag{3.22}$$

where $V = V_N V_D$. Assuming adiabaticity of the cosmic evolution, i.e. $dS = 0$, implies

$$\dot{E} + d_N\dot{B}P_N + d_D\dot{C}P_D = 0, \tag{3.23}$$

which is equivalent to conversation of energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$. Using the thermodynamical relationships given by (3.22), (2.67) implies

$$\frac{1}{T} = \beta_H + \sqrt{\frac{f}{E}}, \quad P_N = T\sqrt{fE}, \quad P_D = -T\sqrt{fE}. \tag{3.24}$$

We can obtain an approximate solution of above equations as follows. In the small radius regime, f is of order unity $f \sim \mathcal{O}(1)$, so temperature is approximately equal to the Hagedorn temperature $1/T \sim \beta_H$ and can be taken constant. Also by looking at the right-hand side of (3.16)-(3.18) one can see that pressures can be ignored compared to energy. We assume that the universe starts out at the string radii $B(0) = C(0) = 0$ and $\phi(0) = \phi_0$. Ignoring the pressures we can then set $B = C$. Moreover adiabaticity implies $E = constant$. Defining a *conformal time* η, the system for B and ϕ can be rewritten as

$$B'' = \frac{1}{2}Ee^{9B-3\phi} \tag{3.25}$$

$$\phi'' = \frac{3}{2}Ee^{9B-3\phi} \tag{3.26}$$

$$72B'^2 + 4\phi'^2 - 36B'\phi' = 2Ee^{9B-2\phi}, \tag{3.27}$$

where

$$e^{9B-2\phi}d\eta = dt, \tag{3.28}$$

and prime denotes derivative with respect to η.

From (3.25) and (3.26) one sees

$$\phi = 3B + c\eta + \phi_0 \tag{3.29}$$

which can be used to get a single second order differential equation for B. Setting $B(0) = 0$ and choosing $\eta = 0$ we find

$$B = \lambda\left(e^{3c\eta} - 1\right) - \frac{c}{3}\eta, \tag{3.30}$$

$$\phi = 3\lambda\left(e^{3c\eta} - 1\right) - 2c\eta + \phi_0, \tag{3.31}$$

where

$$c^2 = \frac{E}{18\lambda}e^{-3\phi_0} \tag{3.32}$$

and λ is a free positive constant. This last condition follows from (3.27). The Hubble parameter which is the rate of expansion can be determined as

$$H = \dot{B} = e^{-9B+2\phi}B' = c\ \exp\left[-3\lambda(e^{3c\eta} - 1) - c\eta + 2\phi_0\right]\left[3\lambda e^{3c\eta} - 1/3\right]. \tag{3.33}$$

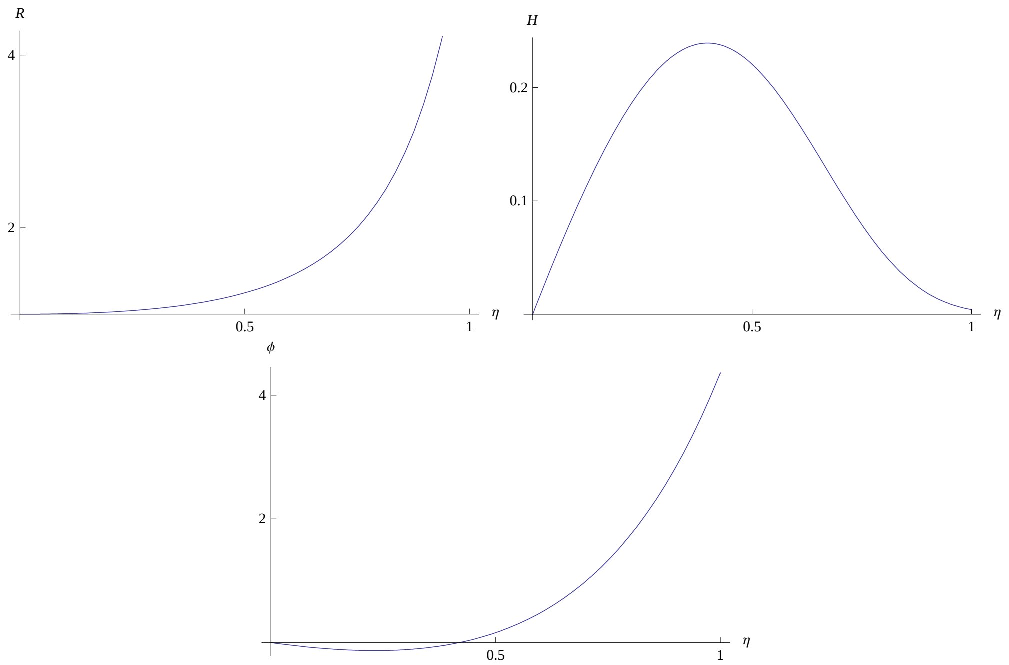

Figure 3.1. The plots of $R = e^B$, H and ϕ for $\lambda = 1/9$. We take $c = 1$ in the graphs.

Also note the initial Hubble rate

$$H_0 = c\, e^{2\phi_0} \left[3\lambda - 1/3\right]. \tag{3.34}$$

From (3.33) it can be seen that for $\lambda \geq 1/9$ the dimensions continuously expand where the expansion speed initially vanishes for $\lambda = 1/9$. For $\lambda < 1/9$, the initial speed is negative giving a period of contraction which later turns into an ongoing expansion. The plots of the functions $\exp(B), \phi$ and H for $\lambda = 1/9$ are given in figure 3.1.

In [17] we checked whether the assumption of thermal equilibrium can be justified in this background. This is a known problem for closed strings and it is crucial to check if the problem persists for open strings. For thermal equilibrium to be maintained one requires $\Gamma > H$, where Γ is the interaction rate. We found in [17] that even one can initially fine tune constants such that $\Gamma < H$ (recall that in the above solution $H = 0$ for $\lambda = 1/9$), in time H increases and becomes greater than Γ. Therefore, maintaining thermal equilibrium is not possible for open strings as in closed strings.

4. CONCLUSIONS

In this thesis, we review the thermodynamics of closed strings and open strings attached to D-branes and determine the corresponding dilaton-gravity solution for open strings. In the first section of chapter II, by considering a system in the microcanonical and canonical ensembles, we see that these descriptions are equivalent for ordinary systems like an ideal gas. However, for a system having exponential degeneracy, there is a limiting temperature above which the partition function diverges and canonical description breaks down. In the following two sections we show that both closed and open strings have this exponential degeneracy and thus microcanonical ensemble should be used to describe them. We first obtain the single string density of states and conclude the second section with the calculation of total degeneracy in a totally compact space. Moreover, by calculating the entropy we see that the maximum physically allowed temperature is the Hagedorn temperature and the stability of the system is ensured. In the last section, we obtain the single and total density of states for open strings attached to D-branes. As for closed strings, we see that temperature is always smaller than the Hagedorn temperatue and the system is stable.

In chapter III, we first obtain the dilaton-gravity equations sourced by open strings attached to D-branes. Then, we use the thermodynamical properties of open strings on D-brane backgrounds to calculate the energy-momentum tensor. It turns out that for large energies the temperature becomes nearly equal to the Hagedorn temperature and can be taken as a constant. Moreover, we see that pressures can be ignored in field equations compared to energy. We therefore rewrite and solve the field equations by ignoring the presures and defining a conformal time. These solutions tell us about the expansion of space. As discussed in [17] although the initial conditions are finely tuned to fulfill thermal equilibrium, in time the expansion rate becomes larger than the interaction rate. This was a known problem for closed strings [18], [19]. Although open strings have a different dilaton coupling to gravity and they have different thermal properties, we observe that the same problem persists for them. However, the existence of suitable initial conditions consistent with thermal equilibrium shows that the cosmology of open string gases may differ substantially from that of closed strings.

APPENDIX A: DERIVATION OF FIELD EQUATIONS

In this appendix we give an explicit derivation of the equations (3.16)-(3.19). Firstly, consider a metric of the form (in this appendix we are not using the Einstein summation convention)

$$ds^2 = -dt^2 + \sum_k e^{2B_k(t)} dx_k^2. \tag{A.1}$$

The non-zero Christoffel symbols for (A.1) are

$$\Gamma^t_{kl} = \dot{B}_k e^{2B_k} \delta_{kl}, \quad \Gamma^k_{lt} = \dot{B}_k \delta_{kl}. \tag{A.2}$$

where dot denotes derivative with respect to t. By computing all components of the Riemann tensor, one can obtain the Ricci tensor as

$$R_{tt} = -\sum_k \left(\ddot{B}_k - \dot{B}_k{}^2 \right), \tag{A.3}$$

$$R_{kl} = \ddot{B}_k e^{2B_k} \delta_{kl} + \dot{B}_k \left(\sum_m \dot{B}_m \right) e^{2B_k} \delta_{kl}. \tag{A.4}$$

The Ricci scalar can be found

$$R = \sum_k \left(\ddot{B}_k + \dot{B}_k{}^2 + \dot{B}_k \sum_l \dot{B}_l \right). \tag{A.5}$$

Assuming that $\phi = \phi(t)$ and $T_{\mu\nu} = diag(\rho, p_k)$, the field equations (3.8) and (3.10) can be rewritten as

$$R_{tt} = \frac{1}{2} e^{\phi} \rho - 2\ddot{\phi} \tag{A.6}$$

$$R_{kl} = \frac{1}{2} e^{2B_k} \delta_{kl} e^{\phi} \rho + e^{\phi} e^{2B_k} p_k \delta_{kl} + 2\dot{B}_k e^{2B_k} \dot{\phi} \delta_{kl}. \tag{A.7}$$

Using (A.3) and (A.4) in (A.6) and (A.7) and taking linear combinations to simplify the expressions we get

$$\ddot{B}_k = -\dot{B}_k \sum_k \dot{B}_k + 2\dot{B}_k\dot{\phi} + e^{\phi}\rho + \frac{1}{2}e^{\phi}\rho, \tag{A.8}$$

$$\ddot{\phi} = -\dot{\phi} \sum_k \dot{B}_k + 2\dot{\phi}^2 - \frac{3}{4}e^{\phi}\rho + \frac{1}{2}\left(\frac{\rho}{2} + p_k\right)e^{\phi}\delta_{kl}, \tag{A.9}$$

$$K^2 = \sum_k \dot{B}_k^{\ 2} + 2e^{\phi}\rho, \tag{A.10}$$

where

$$K = \sum_k \dot{B}_k - 2\dot{\phi}. \tag{A.11}$$

In our case, from (3.14) we have

$$B_k(t) = \begin{cases} B(t) & \text{for} \quad k = 1, ..., d_N, \\ C(t) & \text{for} \quad k = 1, ..., d_D. \end{cases} \tag{A.12}$$

Similarly momentum p_k is decomposed as

$$p_k = \begin{cases} \mathrm{p}_N & \text{for} \quad k = 1, ..., d_N, \\ \mathrm{p}_D & \text{for} \quad k = 1, ..., d_D. \end{cases} \tag{A.13}$$

Putting the corresponding values of B_k and p_k into the equations (A.8), (A.9), (A.10) and (A.11) we obtain

$$\ddot{B} = -K\dot{B} + e^{\phi}p_N + \frac{1}{2}e^{\phi}\rho, \tag{A.14}$$

$$\ddot{C} = -K\dot{C} + e^{\phi}p_D + \frac{1}{2}e^{\phi}\rho, \tag{A.15}$$

$$\ddot{\phi} = -K\dot{\phi} - \left(\frac{3 - d_N - d_D}{4}\right)e^{\phi}\rho + \left(\frac{d_N p_N + d_D p_D}{2}\right)e^{\phi} \tag{A.16}$$

$$K^2 = d_N\dot{B}^2 + d_D\dot{C}^2 + 2e^{\phi}\rho. \tag{A.17}$$

where

$$K = d_N\dot{B} + d_D\dot{C} - 2\dot{\phi}. \tag{A.18}$$

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