A RAMOND-NEVU-SCHWARZ STRING WITH ONE END FIXED

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ABSTRACT

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We study an open string with one end free and the other fixed on a D0-brane as a qualitative guide to the spectrum of hadrons containing one very heavy quark. We first consider the bosonic degrees of freedom, then introduce the fermionic degrees of freedom through the world sheet supersymmetry. The mixed boundary conditions break half of the world sheet supersymmetry and allow only odd-moded α and even-moded d oscillators in the Ramond sector, while the Neveu-Schwarz oscillators b's become odd-integer moded. Boson-fermion masses can still be matched if space-time is 9 dimensional; thus $SO(8)$ triality still plays a role in the spectrum, although full space-time supersymmetry does not survive. We quantize the system in a temporal-like gauge where $X^0 \sim \tau$. Although the gauge choice eliminates negative-norm states at the outset, there are still even-moded Virasoro and even(odd) moded super-Virasoro constraints to be imposed in the NS(R) sectors. The Casimir energy is now positive in both sectors; there are no tachyons. States for $\alpha' M^2 \leq 13/4$ are explicitly constructed and found to be organized into $SO(8)$ irreps by (super)constraints, which include a novel "$\sqrt{L_0}$" operator in the NS and $\Gamma^0 \pm I$ in the R sectors. GSO projections are not allowed. The pre-constraint states above the ground state have matching multiplicities, indicating spacetime supersymmetry is broken by the (super)constraints. A distinctive physical feature of the system is a slope twice that of the open RNS string. When both ends are fixed, all leading and subleading trajectories are eliminated, resulting in a spectrum qualitatively similar to the J/ψ and Υ particles.

ÖZET

BİR UCU BAĞLI RAMOND-NEVEU-SCHWARZ SİCİMİ

Bir ağır quark içeren hadron spektrumunu incelemek ve spektrum hakkında bazı kalitatif ipuçları elde edebilmek amacıyla bir ucu $D0$-bran üzerinde sabit duran diğer ucu ise serbestçe hareket edebilen bir Ramond-Neveu-Schwarz (RNS) sicimi çalışıldı. Uygulanan sınır koşulları nedeniyle süpersimetrinin yarısının elenmesine rağmen eğer uzay-zamanın boyutu $D = 9$ alınırsa, bozon-fermiyon kütlelerinin eşitlenebildiği görülebilir. $D = 9$ alındığı zaman meydana gelen bu durum $SO(8)$ gurubunun 'triality' özelliğinin spektrumda rol oynadığını gösterir. Sistem $X^0 \sim \tau$ şeklinde seçilen bir ayarda kuantize edildi. Sınır koşulları ve ayar şartı uygulandıktan sonra Ramond sektöründe sadece tek-tamsayı α ve çift-tamsayı d modları kalırken Neveu-Schwarz b modları tek-tamsayı modlara dönüştü. Ayar seçimi negatif-norm durumları elese de, fiziksel olmayan başka durumları yok etmek için çift-tamsayı modlu Virasoro ve çift(tek)-tamsayı modlu süper-Virasoro şartları NS(R) sektörleri üzerinde uygulandı. Casimir enerjisinin her iki sektör içinde pozitif olduğu ve her iki sektörde de takyon olmadığı gösterildi. $\alpha' M^2 \leq 13/4$ durumları açıkca yazıldı ve bunların (süper)Virasoro şartları uygulandığı takdirde $SO(8)$ gurubunun temsillerini oluşturacak şekilde organize oldukları gösterildi. Virasoro şartları NS sektöründe orjinal "$\sqrt{L_0}$" operatörü ve R sektöründe de $\Gamma^0 \pm I$ operatörünü içermektedir. GSO projeksiyonu gibi herhangi bir ilave şarta gerek kalmamıştır. Sistemin ayırt edici fiziksel özelliği Regge eğiminin standart açık siciminkinin iki katı olmasıdır. Her iki ucun da sabit tutulması durumunda ise bütün trajektoriler kaybolur ve spektrum kalitatif olarak J/ψ ve Υ parçcıklarının spektrumuna benzer.

TABLE OF CONTENTS

LIST OF SYMBOLS/ABBREVIATIONS

$a^{\alpha,b,d}$	Normal ordering constants
$(b,d)^{\mu}_{-n}$	Creation operators for world sheet fermions
F_r	Modes of the supercurrent in the Ramond sector
G_m	Modes of supercurrent in the Neveu-Schwarz sector
$h_{\alpha\beta}$	World sheet metric
J_{α}	World sheet supercurrent
ℓ	String length
L_m	Virasoro operators
$\mathbb{N}$	Number operator
S	World sheet action
T	String tension
$T_{\alpha\beta}$	World sheet energy-momentum tensor
X^{μ}	Space-time coordinates
α'	Regge slope
α^{μ}_{-n}	Creation operator of world sheet bosonic modes
Γ^{μ}	Gamma matrices in D-dimensional Minkowski space
$\eta_{\mu\nu}$	Space-time metric
τ, σ	World sheet coordinates
ψ^{μ}	World sheet fermions
GSO	Gliozzi-Scherk-Olive
DD	Dirichlet-Dirichlet
DN	Dirichlet-Neumann
ND	Neumann-Dirichlet
NN	Neumann-Neumann
NS	Neveu-Schwarz
R	Ramond
RNS	Ramond-Neveu-Schwarz

SO(D-1)	Special orthogonal group in (D-1)-dimension

1. INTRODUCTION

Before its reincarnation as a candidate for a fundamental Theory of Everything, String Theory was originally discovered [1, 2, 3] while trying to account for observed properties of hadron dynamics, and it was indeed qualitatively successful in reproducing features of hadron physics such as linear Regge trajectories for mesons, amplitudes with Dolen-Horn-Schmid duality [4], and desired high s and t behaviors [5]. The predictive power of the model was further increased by combining strings with the Quark model. In this picture, the nearly-massless up and down quarks and antiquarks were incorporated into the stringy description by being placed at open string endpoints, where Neumann boundary conditions ensured their moving at the speed of light. Furthermore, Harari-Rosner diagrams [6, 7] were useful for keeping track of internal quantum numbers and visualizing duality between Regge pole exchanges and resonances in the s-channel.

There was an obvious need to find an extension of the string model that could encompass baryons, which had masses similar to those of the mesons (except for the very light pion), and also lay on Regge trajectories with approximately the same slope. These similarities suggested a new kind of symmetry (albeit partially broken, in view of the non-vanishing mass differences) between bosons and fermions, and string model featuring such a symmetry were constructed by Ramond [8] and Neveu-Schwarz [9].

This line of research was abandoned upon the almost simultaneous realization that the microscopic theory of strong interactions had to be based on an unbroken Yang-Mills theory and that the string theory had a number of serious drawbacks: The spectrum had negative norm states and these decoupled from the theory if the space-time was 26-dimensional for bosonic string theory and 10-dimensional for Ramond-Neveu-Schwarz (RNS) model. Furthermore, the dual resonance models failed to explain the observed parton-like structure of hadrons; namely, the scaling property of high energy amplitudes for fixed s/t as $s, t \to \infty$. In 1974, Scherk and Schwarz [10] proposed a radical reinterpretation of string theory by suggesting that the superstring theory should be elevated to the status of a promising candidate for describing quantum gravity and other fundamental physical phenomena. The basic motivation behind this proposal was the existence of a massless spin-two state in the closed string spectrum, with couplings expected of a graviton. With this radical change, the length scale of the theory was changed from 10^{-13} cm, which is the typical length of the string combining two quarks, to 10^{-33} cm, which is the characteristic length scale of any quantum gravity theory. The partial success of strings in modelling hadrons was attributed to gluons forming string-like flux tubes between quarks.

It may be instructive to return to aforementioned oldest use of string theory as a phenomenological guide for the study of new generations of baryons and mesons containing at least one heavy (of the top, bottom or charmed variety) and one light quark. In an earlier work [11] such mesons were modelled as excitations of an open string with one end fixed on a $D0$-brane and the other end free. This gave rise to testable predictions such as a restriction to odd oscillator modes and a doubled Regge slope. Whether the description is of any merit will become apparent when higher spin excitations of heavy quark bearing mesons are found and studied. Strings with mixed boundary conditions have also been considered in [12, 13, 14, 15].

In this thesis, we examine the bosonic string with one end fixed [11] and the Ramond-Neveu-Schwarz version of it [16]. Our main motivation and hope is that this may provide qualitative hints about the spectrum of baryon and mesons with a single c, b, or a t quark, although we, of course, do not expect it to provide a heavy-hadron model that is realistic in all its details. Apart from its possible phenomenological usefulness, the system is also of some intrinsic string-theoretic interest in a number of respects. One of these is that the system we are studying can be thought of as an open string viewed from a frame where its mid-point is at rest if one disregards one half of the string. Exactly the same "half a Neumann-Neumann string" picture with even modes in the Ramond and odd modes in the Neveu-Schwarz sector is encountered in String Field Theory [17], where the interaction of two strings is affected by joining them up to mid-point and leaving the other halves free. Another point of interest is the use of a gauge halfway between the light-cone and the old covariant treatments. This gauge suggests itself in the mixed Neumann-Dirichlet string. The old covariant method of quantization of the even the standard bosonic string and its world sheet supersymmetry added version, or RNS version shortly, is described only very sketchily in standard reference works such as [18, 19, 20]; the detailed implementation of the constraints peculiar to our problem and the subsequent emergence of the physical spectrum turns out to be surprisingly intricate. Another interesting feature is the partial breaking of world sheet supersymmetry through the mixed boundary conditions and the way this is manifested in the space-time picture.

The plan and some of the main results of the thesis are as follows: In Chapter 2, an open bosonic string is introduced and the Neumann-Dirichlet (ND) boundary conditions are applied in a natural adaptation of the light-cone gauge of the Neumann-Neumann (NN) string. This leads to what might be called a "rest-frame gauge". The ND conditions allow only odd oscillator modes, which are then canonically quantized. The world sheet energy-momentum tensor is given; the Virasoro generators and their algebra are worked out. The normal ordering constant of L_0 is calculated. For the bosonic ND string, there is no tachyon in the spectrum and no critical space-time dimension. The asymptotic density of states are calculated using a modified version of the Hardy-Ramanujan analysis for the open string. In Chapter 3, the locally supersymmetric generalization of the Polyakov action is introduced and it is shown that how this more general action is simplified in the superconformal gauge and leads to super-constraints. Mode expansions are introduced and the result of reconciliation of the rest-frame gauge with supersymmetry transformations and the effect of boundary conditions on such expansions are worked out. Virasoro and super-Virasoro constraints in the Neveu-Schwarz (NS) and Ramond (R) sectors are considered. Some of the new features that emerge include the limitation of the Virasoro algebra to even modes, while the modes of the R- and NS-super- constraints can only be odd and even respectively. Then low-lying spectra of both sectors are listed. The organization and projection of physical states into specific $SO(D-1)$ irreducible representations via the constraints involves novelties such as a NS-constraint that may be regarded as a "bosonic sector square-root of L_0" and a R-constraint which is a truncated form of the usual Ramond-Dirac operator F_0. The spectrum does not exhibit full space-time supersymmetry, but it is at least possible to match the masses of bosons and fermions if $D - 1 = 8$, which means aspects of $SO(8)$ triality are still reflected in the states. Because of changes in the modes and their Casimir energies both the Neveu-Schwarz and Ramond ground states become massive. Both the absence of purely transverse states and a space-time dimension less than 10 are features of consistent string theories in subcritical dimensions. For higher states, we present a level multiplicity formula which counts the number of states before the constraints are imposed. Remarkably, the number of unconstrained bosonic and fermionic states (except for the bosonic ground state) are equal up to all the levels we have calculated. This indicates that the partial breaking of world sheet supersymmetry through mixed boundary conditions is reflected in the space-time picture in a surprisingly indirect way through the superconstraints. The thesis ends with concluding remarks in Chapter 4.

2. AN OPEN BOSONIC STRING WITH ONE END FIXED

2.1. A Brief Review of Open Bosonic String

A string is a one-dimensional object and correspondingly there are two possible topologies for a string: open and closed. In this thesis, only open strings will be considered. As the string moves in D-dimensional space-time, it sweeps out a two-dimensional surface called "world sheet" which is the two-dimensional analog of the one-dimensional world line of a point particle. This world sheet is described mathematically by specifying the position vector of the string in space-time denoted by $X^{\mu}(\sigma, \tau)$, $(\mu = 0, 1, \ldots, D-1)$, where σ and τ are two coordinates parameterizing the world sheet.

In analogy to the point particle case, where the action is proportional to the length of the world line traced out by point particle, the natural generalization for the string action would be a quantity which is proportional to the area of the world sheet

$$S = -\frac{T}{2} \int d\sigma \, d\tau \, \sqrt{-deth_{\alpha\beta}} \, h^{\alpha\beta} \, \eta_{\mu\nu} \, \partial_{\alpha} X^{\mu} \, \partial_{\beta} X^{\nu} \tag{2.1}$$

where T is the string tension, $h^{\alpha\beta}$ is the inverse of the world sheet metric and $\eta_{\mu\nu}$ is the metric of D-dimensional Minkowski space-time with $-\eta_{00} = \eta_{ii} = +1$, and indices are $\alpha, \beta = 1, 2$ and $i = 1, 2, \ldots, D-1$.

This action has some local symmetries regardless of the choice of background. These are the two-dimensional coordinate invariance, often called diffeomorphism invariance,

$$\begin{aligned} \delta X^{\mu} &= \xi^{\alpha} \, \partial_{\alpha} X^{\mu}, \\ \delta h^{\alpha\beta} &= \xi^{\gamma} \, \partial_{\gamma} h^{\alpha\beta} - \partial_{\gamma} \xi^{\alpha} \, h^{\gamma\beta} - \partial_{\gamma} \xi^{\beta} \, h^{\alpha\gamma}, \\ \delta(\sqrt{h}) &= \partial_{\alpha} \, (\xi^{\alpha} \, \sqrt{h}), \end{aligned} \tag{2.2}$$

and Weyl scaling

$$\delta h^{\alpha\beta} = \Lambda\, h^{\alpha\beta}, \tag{2.3}$$

where ξ^α and Λ are arbitrary infinitesimal functions of the world sheet coordinates σ^α.

Furthermore, there are also global symmetries from the point of view of the two-dimensional theory. They reflect the symmetries of the background. For flat Minkowski space, this global symmetry is the D-dimensional Poinçaré invariance, and described by

$$\begin{aligned} \delta X^\mu &= \omega^\mu{}_\nu X^\nu + a^\mu, \\ \delta h^{\alpha\beta} &= 0, \end{aligned} \tag{2.4}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. It should be emphasized that the parameters a^μ and $\omega_{\mu\nu}$ are not functions of the world sheet coordinates σ^α; that is why these are called the global symmetries from the viewpoint of the world sheet theory. There are also conserved currents associated with these symmetries and using the well-known Noether procedure, one can derive these currents. Invariance under translations gives the energy-momentum current

$$P^\mu_\alpha = T\, \partial_\alpha X^\mu, \tag{2.5}$$

while the invariance under the Lorentz transformations gives the angular momentum current

$$J^{\mu\nu}_\alpha = T\, (X^\mu\, \partial_\alpha X^\nu - X^\nu\, \partial_\alpha X^\mu). \tag{2.6}$$

The currents derived in this way are always conserved provided that the equations of motion are obeyed:

$$\partial_\alpha P^{\alpha\mu} = \partial_\alpha J^{\alpha\mu\nu} = 0. \tag{2.7}$$

Using the local symmetries of the action (two diffeomorphisms and one Weyl scaling), three independent components of $h_{\alpha\beta}$ can be fixed such that $h_{\alpha\beta} = \eta_{\alpha\beta}$. This is called the 'conformal gauge' and in this gauge the action, Equation (2.1), reduces to a simple free field action

$$S = -\frac{T}{2}\int d\sigma d\tau \; \eta^{\alpha\beta}\,\eta_{\mu\nu}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu. \tag{2.8}$$

Varying this action with respect to X^μ, taking $\delta X^\mu = 0$ at $\tau = \pm\infty$ as usual and applying integration by parts, one gets

$$\delta S = T\int_0^{\sigma_{max}} d\sigma \int_{-\infty}^{\infty} d\tau \; (\partial_\sigma^2 - \partial_\tau^2)X^\mu\,\delta X_\mu - T\int_{-\infty}^{\infty} d\tau \; (\partial_\sigma X^\mu)\,\delta X_\mu|_0^{\sigma_{max}}, \tag{2.9}$$

where the range of the σ coordinate has taken to be $[0, \sigma_{max}]$ and σ_{max} has not been specified yet! This variation of the action must be equal to zero in accordance with the principle of stationary action. As a result, the bulk term which is the first term in the above variation gives the equation of motion for X^μ

$$\nabla^2 X^\mu \equiv (\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^\mu = 0, \tag{2.10}$$

which is simply the two-dimensional free wave equation. For open strings, Equation (2.10) is necessary but not sufficient to ensure that $\delta S = 0$ under the general variation

$$X^\mu \to X^\mu + \delta X^\mu.$$

The surface term in Equation (2.9) must also be zero and this leads to the possible open string boundary conditions which will be considered in detail in the next section.

The wave equation, Equation (2.10), is derived from the gauge fixed form of a more general action. Thus, it must be supplemented by certain subsidiary conditions called "constraints" resulting from the gauge fixed equation of motion for the world sheet metric. This is basically due to the fact that these constraints make sure that the gauge fixing is valid at all times. Before fixing the gauge, the variation of the action, Equation (2.1), with respect to $h^{\alpha\beta}$ gives the energy-momentum tensor of the two-dimensional world sheet theory

$$T_{\alpha\beta}(\sigma,\tau) \equiv -\frac{2\pi}{\sqrt{-deth_{\alpha\beta}}}\frac{\delta S}{\delta h^{\alpha\beta}} = \frac{T}{2}\left(\partial_\alpha X^\mu\,\partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\,\partial_\gamma X^\mu\,\partial^\gamma X_\mu\right) = 0. \quad (2.11)$$

The field equation $\frac{\delta S}{\delta h^{\alpha\beta}}$ corresponding to the requirement $T_{\alpha\beta} = 0$ must now be imposed as a constraint on the solutions of Equation (2.10).

Since $T_{\alpha\beta}$ is a symmetric tensor, there are three constraints corresponding to the three gauge degrees of freedom that have been fixed. However, at the classical level $T_{\alpha\beta}$ is traceless associated with the Weyl invariance which means that one of the constraints is trivially satisfied and there are only two nontrivial constraints. It is conventional to go to the world sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$ in which the constraints become

$$T_{++} = \frac{T}{8}\,\partial_+ X^\mu\,\partial_+ X_\mu = 0, \quad (2.12)$$

$$T_{--} = \frac{T}{8}\,\partial_- X^\mu\,\partial_- X_\mu = 0, \quad (2.13)$$

$$T_{-+} = T_{+-} = 0 \quad (2.14)$$

2.2. Boundary Conditions and the Mode Expansions

To see the possible boundary conditions for open strings, one must consider the surface term in Equation (2.9) in detail:

$$(\partial_\sigma X^\mu)\,\delta X_\mu \,|_{\sigma=0}^{\sigma=\sigma_{max}} = 0. \quad (2.15)$$

This term is not automatically zero but can be made to vanish by proper choices of boundary conditions along σ direction. There are two types of boundary conditions:

The first type is

$$\partial_\sigma X^\mu = 0 \tag{2.16}$$

separately at both $\sigma = 0$ and $\sigma = \sigma_{max}$. These are called the "Neumann boundary conditions" and they prevent momentum from flowing off the ends of the string.

The second type is

$$\delta X_\mu = 0 \tag{2.17}$$

at both $\sigma = 0$ and $\sigma = \sigma_{max}$. These are the "Dirichlet boundary conditions".

Obviously, for open bosonic strings the same/different boundary conditions may be imposed on two ends and this leads to four different choices: Neumann-Neumann (NN), Neumann- Dirichlet (ND), Dirichlet-Neumann (DN), Dirichlet-Dirichlet (DD) bundary conditions. Of the four choices, the only consistent one with the D-dimensional Poincaré invariance is the NN boundary condition and in this case the end points of the open string move freely in space-time with the speed of light. On the other hand, if we relax the condition of Poincaré invariance, then the Dirichlet boundary condition can also be used. This type of boundary condition specifies the positions of the end points of the string in space-time and the only way this makes sense is if the open string ends on a physical object: a D-brane[1] .

Remembering the original motivation for the string theories in retrospect, one

[1]D stands for Dirichlet. D-branes are dynamical objects on which the open string endpoints are restricted to lie on. Their existence can be seen, in compactification procedure, by studying the $R \to 0$ limit of open string theory, where the different behaviours of open and closed strings lead to a seeming paradox. Furthermore, from the study of this limit one can argue that, on perturbative grounds, the usual Type I, IIA, IIB superstring theories are different states in a single theory, which also contains states with arbitrary configurations of D-branes.

can ask the following question: Is it possible to describe, in string theory language, a qualitative model for the excited states of mesons consisting of one very heavy quark/antiquark (of the Charmed, Bottom) and one very light antiquark/quark (Up or Down) in terms of a string with one end fixed and one end free? This question can be answered in the affirmative. Indeed, this model may be approximately described by using the Dirichlet boundary condition (placing the infinitely heavy quark at the origin of space coordinate system, for example) at one end and Neumann boundary condition at the other end. The resulting system will be referred to as the Neumann-Dirichlet, or ND string [11]. This model has, apart from being a guide to some expected properties of mesons with one very heavy quark or antiquark, also some novel string theoretic aspects and our emphasis will be more on this aspect because the data on Regge recurrences of such mesons is not available at this stage.

The general solution of the two-dimensional wave equation, Equation (2.10), can be Fourier expanded as

$$X^\mu(\sigma,\tau) = x^\mu + \ell^2 p^\mu \tau + i\frac{\ell}{2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu\, e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu\, e^{-in(\tau+\sigma)}\right) \tag{2.18}$$

where ℓ is a fundamental length introduced to ensure the correct dimension for X^μ and α_n^μ describe the Fourier components of the right-moving solutions of the wave equation and $\tilde{\alpha}_n^\mu$ those of the left-moving solutions. They are interpreted as oscillator coordinates. Furthermore, the relation between the fundamental length ℓ and the string tension T is

$$\ell = \sqrt{\frac{1}{\sigma_{max} T}}.$$

The infinitely massive quark can be located at the $\sigma = 0$ end and this point can be identified with the origin of space coordinates through the Dirichlet boundary condition

$$X^i(0,\tau) = 0. \tag{2.19}$$

This condition, when substituted into Equation (2.18), gives $x^i = p^i = 0$ and $\alpha_n^i = -\tilde{\alpha}_n^i$, $(i = 1, 2, \ldots, D-1)$. In modern parlance, this end of the string is confined to a $D0$-brane. Leaving the question of how the X^0 coordinate can be chosen to the next section, the Neumann boundary condition

$$\partial_\sigma X^\mu(\sigma_{max}, \tau) = 0 \tag{2.20}$$

can be adopted for the massless end. This leads to

$$\cos(n\sigma_{max}) = 0, \tag{2.21}$$

and this is satisfied either by taking $\sigma_{max} = \frac{\pi}{2}$ and $\alpha_n^\mu = 0$ for even α-modes or by taking $\sigma_{max} = \pi$ and $\alpha_n^\mu = 0$ for half-integer α-modes. One may adopt the usual range $[0, \pi]$ for the σ coordinate and work with the half-integrally moded oscillator coordinates. Another reason for preferring the range $[0, \frac{\pi}{2}]$ over the usual $[0, \pi]$ is that in the former, one works with the odd-numbered subset of the original integral modes rather than introducing an entirely new set of oscillators. Taking this range is also particularly advantageous in order to compare any result obtained for the bosonic ND string with that of the NN string. One should also note that for this range the relation between the fundamental length ℓ and the string tension T becomes $\ell_{ND} = \sqrt{\frac{2}{\pi T}}$ whereas the same relation for the NN string is $\ell_{NN} = \sqrt{\frac{1}{\pi T}}$ (in the rest of this thesis, ℓ_{ND} is meant wherever ℓ is used).

For the $[0, \frac{\pi}{2}]$ range, the disappearance of the even α-modes is quite similar to the same phenomenon in the elementary quantum mechanics problem of a "half-oscillator potential", i.e. a well with an infinite barrier at $x = 0$ and a harmonic oscillator potential $V = \frac{1}{2}kx^2$ for $x \geq 0$.

Thus, imposing the ND boundary conditions on the space components of the string coordinates results in the expansion

$$X^i(\sigma,\tau) = -\ell \sum_{r:odd} \frac{\alpha_r^i}{r} e^{-ir\tau} \sin(r\sigma), \tag{2.22}$$

and the reality condition, $X^{i\dagger} = X^i$, yields

$$\alpha_r^{i\dagger} = \alpha_{-r}^i$$

as usual.

2.3. Quantization

For ordinary NN strings, there is still a residual gauge symmetry that remains after setting $h_{\alpha\beta} = \eta_{\alpha\beta}$. Using this residual symmetry, one can make further gauge choices which can be noncovariant and solve the Virasoro constraints. For the NN string, the quantization method based on the noncovariant choice

$$X^+(\sigma,\tau) = x^+ + \ell^2 p^+ \tau \tag{2.23}$$

is called the "light-cone gauge" quantization [21] where $X^+ = (X^0 + X^{D-1})/\sqrt{2}$. What is done in this quantization method is to identify the τ coordinate of the world sheet with one of the light-cone target space coordinates. This amounts to setting α_n^+ operators to zero. Then using the constraint equations which are implemented at the operatorial level, α_n^- oscillators can be solved in terms of the $D-2$ transverse oscillator modes. This formalism is manifestly free of ghosts, though not manifestly covariant. The advantage of this method is that it is the shortest way of understanding the necessity of $D = 26$ for the bosonic NN string.

There are two types of manifestly covariant quantization methods. The first, called "old covariant quantization", is based on a description only in terms of X^μ. In this method, all components of X^μ are quantized and this leads to the propagation of negative norm states called 'ghosts'[2] . They lead to conflict with the probabilistic interpretation of quantum mechanics and thus must be eliminated from the full Hilbert space of states. This is achieved by imposing the Virasoro constraints on the states. These conditions are analogous to the Gupta-Bleuler condition in electrodynamics, in which the classical constraint $\partial_\mu A^\mu = 0$ is replaced by the requirement that the positive frequency components of the corresponding quantum operator should annihilate physical photon states. On the other hand, the modern covariant quantization approach involves "the Faddeev-Popov ghosts"[3] as well as X^μ and identification of BRST symmetries and currents.

In the case of ND string, only one end of the string describes a lightlike world line, while the other is timelike; hence, the full light-cone gauge treatment is inappropriate. But an analogy with the light-cone treatment of the NN string suggests itself here: The natural analog of Equation (2.23) is to identify the target space and world sheet 'times' via

$$X^0(\sigma,\tau) = l^2\, p^0\, \tau, \tag{2.24}$$

which may be called the "rest-frame gauge". This sets $x^0 = 0$ and $\alpha_n^0 = 0$; hence, states with negative norm are discarded at the outset. However, unlike the situation in the light-cone gauge of the NN string, absence of negative norm states does not mean that all the states obtained by hitting the ground state with the spacelike oscillators are automatically physical. The light-cone gauge of the NN string brings the possibility of solving α_n^- oscillators in terms of transverse oscillators but the rest frame gauge, which does not involve an off-diagonal metric, does not offer a similar possibility. There are still constraints which must be imposed on all possible states, and only the states annihilated by these constraints are the final physical ones although some of the states have been partially pruned by Equation (2.24). Therefore, in a sense, the quantization

[2]These ghosts should not be confused with the Faddeev-Popov ghosts.

[3]The term 'ghost' here is not used in the sense that they are states with negative norm and so does not threaten the unitarity of the theory. The term ghost in the context of Faddeev-Popov method is used to describe states which are only present in loop diagrams and the kinetic term appearing in their Lagrangian corresponds to that of bosons.

method adopted for the ND string is an intermediate approach between the light-cone and old covariant quantization.

Before proceeding to the quantization, it is essential to determine the Poisson brackets of the oscillator coordinates. This can be done by noting from Equation (2.8) that the Poisson brackets of the coordinates X^i and the canonical momenta $P^i_\tau = T\dot{X}^i$ at equal τ are

$$[P^i_\tau(\sigma), X^j(\sigma')]_{P.B} = -\delta^{ij}\,\delta(\sigma - \sigma'). \tag{2.25}$$

Inserting Equation (2.18) into this and using the formula

$$\pi\delta_{2\pi}(\sigma - \sigma') = \sum_{r:odd} e^{-ir(\sigma-\sigma')} \tag{2.26}$$

which may be obtained from the Poisson summation formula (a factor of $\cos\frac{(\sigma-\sigma')}{2}$ in the denominator of the left hand side is omitted since it becomes unity when the argument of the delta function vanishes) yield the expected Poisson brackets for the Fourier components:

$$[\alpha^i_r, \alpha^j_s]_{P.B} = i\,r\,\delta_{r+s}\,\delta^{ij}, \tag{2.27}$$

with r, s odd integers.

In the conventional canonical quantization procedure, the string space coordinates X^i are interpreted as quantum operators and the first quantization of the string is achieved by replacing the equal τ Poisson brackets by commutators, i.e.

$$[P^i_\tau(\sigma,\tau), X^j(\sigma',\tau)] = -i\,\delta^{ij}\,\delta(\sigma - \sigma'), \tag{2.28}$$

and this results in

$$[\alpha_r^i, \alpha_s^j] = r\, \delta_{r+s}\, \delta^{ij}, \tag{2.29}$$

where r, s are odd integers. This is a subalgebra of the bosonic NN string Heisenberg algebra.

It should be noted that the first quantized string theory results in a two-dimensional second quantized point particle quantum field theory on the world sheet. This is because the string position and momentum depend on two variables and upon quantization they may be viewed as a quantized field of some ordinary point particle theory.

The Hamiltonian of this two-dimensional world sheet theory, with the gauge choice Equation (2.24), becomes

$$\begin{aligned} H &= \int_0^{\frac{\pi}{2}} d\sigma\, (: X^\mu\, P_{\tau\mu} - L :) \\ &= \frac{T}{2} \int_0^{\frac{\pi}{2}} d\sigma\, (: \partial_\tau X^\mu\, \partial_\tau X_\mu : + : \partial_\sigma X^\mu\, \partial_\sigma X_\mu :) \\ &= \frac{T}{2} \int_0^{\frac{\pi}{2}} d\sigma\, \{: \partial_\tau X^i\, \partial_\tau X^i : + : \partial_\sigma X^i\, \partial_\sigma X^i : - l^4\, (p_0)^2\} \end{aligned} \tag{2.30}$$

where the normal ordering have been implicitly applied in order to have a vanishing vacuum expectation value for the Hamiltonian, i.e. $\langle 0|H|0\rangle = 0$. The infinite ground state energy caused by this procedure will be calculated and regularized later. Similarly, the components of the energy-momentum tensor

$$T_{\alpha\beta} = \frac{T}{2}\, (: \partial_\alpha X^\mu\, \partial_\beta X_\mu - \frac{1}{2}\, \eta_{\alpha\beta}\, \partial_\gamma X^\mu\, \partial^\gamma X_\mu :) \tag{2.31}$$

will become, in the gauge, Equation (2.24),

$$T_{00} = T_{11} = \frac{T}{4}\, \{: \partial_\tau X^i\, \partial_\tau X^i : + : \partial_\sigma X^i\, \partial_\sigma X^i : - l^4\, (p_0)^2\}, \tag{2.32}$$

$$T_{01} = T_{10} = \frac{T}{2}\, (: \partial_\tau X^i\, \partial_\sigma X^i :). \tag{2.33}$$

2.4. An Even Virasoro Algebra

The Poincaré invariance of the bosonic NN string is broken down to $SO(D-1)$ rotational symmetry about the origin of space since one end of the string is fixed in the target space. Thus, it may be anticipated that this symmetry breaking will lead to a Virasoro algebra with special features. In this system, although energy is conserved, momentum conservation can be no longer expected to hold just as in a mechanical problem with an infinite mass. This leads us to expect that the modes of the energy-momentum tensor involving the momentum operator will be absent. In classical terms, since the infinitesimal generators of the conformal algebra in two dimensions are in the form $L_n = -z^{n+1}\partial_z$, the generator of translations, the momentum operator, is $L_{-1} = -\partial_z$ and it is an odd-moded operator. Therefore, we not only identify the left- and right-moving oscillators but also exclude the even α-modes and this results in the elimination of odd Virasoro modes which cannot close upon commutation. It will be shown below that this actually happens.

To compute the nonzero modes of the world sheet energy-momentum tensor, first one should write the components in the light-cone coordinates:

$$T_{\pm\pm} = \frac{1}{2}(T_{00} \pm T_{01}) = \frac{T}{8}\{:(\partial_\tau X^i \pm \partial_\sigma X^i)^2: -(l^2 p_0)^2\}, \tag{2.34}$$

where

$$\partial_\tau X^i \pm \partial_\sigma X^i = \mp l \sum_{m=odd} \alpha^i_m \exp(-im(\tau \pm \sigma)). \tag{2.35}$$

This implies for $\tau = 0$

$$:(\partial_\tau X^i \pm \partial_\sigma X^i)^2 := l^2 \sum_{m:even} \sum_{s:odd} \exp(-im\sigma) : \alpha^i_s \, \alpha^i_{m-s} : . \tag{2.36}$$

It should be noted that, being the sum of two odd integers, m is necessarily even. Now we can calculate the mode expansions of the constraints $T_{\alpha\beta} = 0$ for the ND string

$$L_n = \int_0^{\frac{\pi}{2}} d\sigma \; (e^{in\sigma} T_{++} + e^{-in\sigma} T_{--}). \tag{2.37}$$

Obviously $T_{++}(\sigma) = T_{--}(-\sigma)$. Using this property and Equation (2.36), one gets

$$L_n = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\sigma \; e^{in\sigma} T_{++}. \tag{2.38}$$

The constraints $T_{\alpha\beta} = 0$ of the classical theory must now be imposed on states by demanding L_n annihilate physical states for $n > 0$. One must also further note that the only normal ordering ambiguity arises for L_0.

Defining $\alpha_0^\mu \equiv \ell p^\mu$, one gets

$$\begin{aligned} L_0 &= -\frac{1}{2}(\alpha_0^0)^2 + \frac{1}{2}\sum_{r:odd} : \alpha_r^i \, \alpha_{-r}^i : + a^\alpha, && (2.39) \\ L_n &= \frac{1}{2}\sum_{r:odd} : \alpha_r^i \, \alpha_{n-r}^i : \qquad \text{(n: even and non-zero)} \quad , && (2.40) \\ L_n &= 0 \qquad \text{(n: odd)}. && (2.41) \end{aligned}$$

where a^α is the previously mentioned normal ordering constant.

Obviously the Virasoro operators have their usual form in terms of odd oscillator modes, hence we may expect that they satisfy the familiar Virasoro algebra, maybe with a change in the central extension term of the algebra. This term can be most easily calculated using the following vacuum expectation value

$$\begin{aligned} A_m &= \langle 0 | [L_m, L_{-m}] | 0 \rangle = \langle 0 | L_m L_{-m} | 0 \rangle \\ &= \frac{1}{4} \langle 0 | \sum_{r,s:odd} \alpha_{m-r}^i \, \alpha_{r,i} \, \alpha_{-m-s}^j \, \alpha_{s,j} | 0 \rangle \\ &= \frac{1}{4} \langle 0 | \sum_{r,s:odd} \alpha_r^i \, \alpha_{m-r,i} \, \alpha_s^j \, \alpha_{-m-s,j} | 0 \rangle, \end{aligned}$$

where m is of course positive and even integer. The expectation value in the second line vanishes for $s > 0$ and for $r > m$, whereas the expectation value in the third line vanishes for $r < 0$ and for $s < -m$, yielding

$$A_m = \frac{1}{4} \langle 0| \sum_{r=1}^{m-1} \sum_{s=-m+1}^{-1} \alpha^i_{m-r}\, \alpha_{r,i}\, \alpha^j_{-m-s}\, \alpha_{s,j} |0\rangle$$

and this gives

$$A_m = \frac{D-1}{2} \sum_{r=1}^{m-1} r(m-r) \qquad (2.42)$$

where all the summations are over odd integers. In obtaining this result, Equation (2.29) is used successively and the dimension of space $D-1$ comes from the contraction of the space components of the metric, i.e. $D-1 = \eta_{ij}\eta^{ij}$. The odd integer summation in Equation (2.42) yields a modified central extension term and our even Virasoro algebra becomes

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{1}{2}(n-m)\,(\alpha^0_0)^2\,\delta_{n+m} + \frac{D-1}{24}\, n(n^2+2)\,\delta_{n+m}. \qquad (2.43)$$

If one could have calculated the sum in Equation (2.42) over all integers, one could find the well known Virasoro central extension term

$$A_m = \frac{D}{12}(m^3 - m). \qquad (2.44)$$

Changing L_0 to $L_0 - \frac{D-1}{16}$ in Equation (2.43), one can recover this well known form of the central extension term and the Virasoro algebra, Equation (2.43), becomes

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{1}{2}(n-m)\,(\alpha^0_0)^2\,\delta_{n+m} + \frac{D-1}{24}(n^3 - n)\,\delta_{n+m}, \qquad (2.45)$$

where only even integers are now allowed for n and m. Comparing the anomaly term here with that of the Virasoro algebra of NN string it should be noted that the coefficient D in the usual NN string Virasoro algebra has been changed to $(D-1)/2$; hence, the elimination of the odd Virasoro modes is seen to manifest itself as an apparent halving of the number of bosonic coordinates. It should be further recognized that there was an anomaly free subalgebra of NN string consisting of L_1, L_0, L_{-1} which is isomorphic to $SU(1,1)$ or $SL(2,R)$; but, for the ND string, this subalgebra reduces to $U(1)$.

2.5. The Spectrum

The Fock space is built up with the odd integer moded α^i_{-r} oscillators. Although the negative norm states are eliminated at the beginning by the chosen gauge, the physical states correspond to the subspace of states satisfying the Virasoro conditions. But one should be careful in implementing the constraints because even though in the classical theory $L_m = 0$ for all m, this can not be imposed on physical states since this is inconsistent with the commutators Equation (2.45):

$$\langle phys|[L_m, L_{-m}]|phys\rangle = \langle phys|\, 2mL_0\, |phys\rangle + \frac{D-1}{24}\,(m^3 - m)\,\langle phys|phys\rangle,$$

i.e. the left hand side is zero but the right hand side is not. Thus, it can not be required that $L_m|phys\rangle = 0$ for all m. Just as in the Gupta-Bleuler treatment of electrodynamics, the vanishing of L_m in the classical theory is replaced at the quantum level by the weaker requirement that the positive frequency components annihilate a physical state, i.e.

$$L_n\,|phys\rangle = 0, \quad (n \geq 0 \text{ and even}). \tag{2.46}$$

The zero mode of the Virasoro constraints gives a formula for the mass of a string state via the equation

$$L_0\,|phys\rangle = 0 \tag{2.47}$$

where L_0 is defined in Equation (2.39). Since one end of the string is fixed, all excitations can be analyzed in their common rest frame, i.e., one can take $p^i = 0$. This implies that $\alpha_0^0 = \ell p^0$ and $\alpha_0^i = \ell p^i = 0$. Thus,

$$\begin{aligned} -M^2 &= p^\mu p_\mu = p^0 p_0 + p^i p_i \\ \Rightarrow (\alpha_0^0)^2 &= \ell^2 M^2. \end{aligned}$$

Combining this result with the mass-shell condition Equation (2.47), one gets

$$(-\frac{1}{2}\ell^2 M^2 + N^{(\alpha)} + a^\alpha)\,|phys\rangle = 0, \tag{2.48}$$

where

$$N^{(\alpha)} = \sum_{i=1}^{D-1} \sum_{r=1,odd}^{\infty} \alpha_{-r}^i \, \alpha_r^i \tag{2.49}$$

is the number operator for α-modes, i.e., it counts the number of excitations of α-modes[4] . Finally, one can arrive at an expression for the mass of a state in terms of its internal states of oscillation using the zero frequency constraint as follow

$$M^2\,|phys\rangle = \frac{2}{\ell^2}\,(N^{(\alpha)} + a^\alpha)\,|phys\rangle. \tag{2.50}$$

To calculate the normal ordering constant, we will use the Polchinski's 'heuristic recipe' [19]. This involves two steps:

1. One adds the oscillator zero point energies, regularizing the divergent result via an analytic continuation of the ζ-function. This gives the normal ordering constant in (τ, σ) coordinates.

2. The L_n, and in particular L_0 are by contrast the modes in the expansion of $T(z)$ in powers of the z coordinate (on the Euclideanized world sheet $z = e^{-i\sigma+\tau}$); thus, one must add the non-tensor shift coming from the Schwarzian derivative of the transformation from (τ, σ) to z. For L_0, the shift is given by $c/24$ where c is the central

[4]The superscript α is introduced for later convenience

charge.

The zero point energy of the bosonic α oscillators is $(D-1)/2$ times a sum over positive odd integers, which, using $\zeta(-1) = -1/12$, is simply $\zeta(-1) - 2\zeta(-1) = 1/12$. This means

$$a^{\alpha} = \frac{D-1}{2}\frac{1}{12} + \frac{c}{24}$$

where c is $(D-1)/2$, where $(D-1)$ obviously comes from the number of space coordinates while the 2 in the denominator can be interpreted as the result of keeping of only odd modes. Then,

$$a^{\alpha} = \frac{D-1}{2}\frac{1}{12} + \frac{D-1}{2}\frac{1}{24} = \frac{D-1}{16}. \qquad (2.51)$$

To check whether the ND string exhibits linear Regge trajectories[5] and, if so, to find the Regge slope, the mass-shell condition should be written in the form $\alpha' M^2 = N + constant$. From Equation (2.50), one can easily see that it really behaves in this way, albeit with a slope

$$\alpha'_{ND} = \frac{1}{\pi T},$$

which is twice as large as the usual α'_{NN}. It would be interesting whether this slope doubling on the meson excited states with one light and one extremely heavy quark/antiquark is experimentally confirmed.

The aforementioned conclusions about the Regge trajectories are supported at the classical level by examining a ND string undergoing rigid rotation in the X^1X^2-plane with coordinates

$$X^0 = \ell^2 p^0 \tau,$$

[5]Regge trajectory is a curve obtained by plotting the energy squared on the x-axis and the angular momentum on the y-axis.

$$
\begin{aligned}
X^1 &= \ell^2 p^0 \sin\sigma \cos\tau, \\
X^2 &= \ell^2 p^0 \sin\sigma \sin\tau, \\
\vdots &= \vdots \\
X^k &= 0 \qquad (k = 3, \ldots, D-1).
\end{aligned}
$$

It can be easily verified that these satisfy both the equations of motion and the constraints of vanishing world sheet energy-momentum tensor. The energy of the configuration is, using Equation (2.5)

$$E = M = T \int_0^{\frac{\pi}{2}} d\sigma \, \partial_0 X^0 = \frac{\ell^2 \, p^0 \, \pi \, T}{2}, \tag{2.52}$$

while its angular momentum turns out to be, from Equation (2.6)

$$J = T \int_0^{\frac{\pi}{2}} d\sigma \, (X^1 \, \partial_0 X^2 - X^2 \, \partial_0 X^1) = \frac{\pi \, T \, (p^0 \, l^2)^2}{4}. \tag{2.53}$$

The rotation being rigid, angular momentum per mass square is maximized. The ratio of the former to the latter is the Regge slope, which again comes out to be $1/(\pi T)$.

Turning to the spectrum, the first few examples satisfying the constraints are as follows:

$N = 0$ and $\alpha' M^2 = (D-1)/16$:

$$|0\rangle$$

and this state is a massive scalar.

$N = 1$ and $\alpha' M^2 = (D+15)/16$:

$$|1\rangle \equiv \alpha^i_{-1} |0\rangle$$

and this state is a massive vector of $SO(D-1)$.

$N=2$ and $\alpha' M^2 = (D+31)/16$:

$$|2\rangle \equiv \{\alpha^i_{-1}\alpha^j_{-1} - \frac{\delta^{ij}}{D-1}\alpha^k_{-1}\alpha^k_{-1}\}|0\rangle$$

is a symmetric traceless second rank tensor of $SO(D-1)$ and this form is dictated by the L_2 action. The scalar $\alpha^k_{-1}\alpha^k_{-1}|0\rangle$ does not satisfy the Virasoro conditions $L_2, L_4, \ldots$. Furthermore, since the even α-modes are eliminated at the beginning, the states of the form $\alpha^i_{-2}|0\rangle$ will not appear in the spectrum.

$N=3$ and $\alpha' M^2 = (D+47)/16$:

$$|3\rangle_I \equiv \{\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1} - \frac{\delta^{ij}}{D+1}\alpha^n_{-1}\alpha^n_{-1}\alpha^k_{-1} - \frac{\delta^{ik}}{D+1}\alpha^n_{-1}\alpha^n_{-1}\alpha^j_{-1} - \frac{\delta^{jk}}{D+1}\alpha^n_{-1}\alpha^n_{-1}\alpha^i_{-1}\}|0\rangle,$$

and this tensorial form of the state is singled out by the Virasoro conditions. They also rule out $\alpha^i_{-3}|0\rangle$ and $\alpha^i_{-1}\alpha^k_{-1}\alpha^k_{-1} \mid 0\rangle$ type excitations to appear alone but allows the combination

$$|3\rangle_{II} \equiv \{\alpha^i_{-3} - \frac{3}{2(D+1)}\,\alpha^k_{-1}\alpha^k_{-1}\alpha^i_{-1}\}|0\rangle.$$

There are some important conclusions which can be drawn from the spectrum. First conclusion is that the vacuum state has a positive mass square $M^2 = (D-1)/(16\,\ell^2)$; hence, the ND string has a stable vacuum. This is different from the situation in the bosonic NN string where the ground state is a tachyon and thus, one is perturbing around an unstable vacuum. The second conclusion is related to the critical dimension of the ND string. At first it seems plausible that the ND string can exist in any dimension $D>1$. To check this, one should recall what is done in the bosonic NN string to derive the critical dimension. In the light-cone gauge quantization of such a system, the critical dimension can be obtained either by requiring the closure of the operator algebra or by verifying Lorentz covariance in representation space. The latter is easier to follow and will be taken over here. Specifically, in this method what has been done is to seek the consistency between the Lorentz transformation properties of the states created by the oscillator modes and the mass formula. Consequently, such a consistency requires $D = 26$. On the other hand, for the ND string since the fixed end condition breaks the D-dimensional Poincaré invariance down to $SO(D-1)$, all one has to check is whether this reduced rotational symmetry holds in the theory independently of the value of D. All of the states come in irreducible representations of $SO(D-1)$ since they are created by the $D-1$ space components of the oscillators and they are all massive as can be seen from the mass formula. Therefore the theory appears to be unitary and $SO(D-1)$ invariant for any D. However, the presence of an anomaly term proportional to $D-1$ in the even Virasoro algebra renders this conclusion suspect. On the other hand, there are general reasons to foresee that the conformal anomaly can not be canceled: Our Dirichlet boundary condition reduces the usual full open string disc to a half-a-disc with a diameter that is not allowed to change under conformal mappings (the appearance of only even Virasoro modes is related to this constraint; in classical terms, we only have the operators $L_{2n} \sim z^{2n+1}\partial_z$, which can only affect conformal transformations invariant under $z \longrightarrow -z$). Furthermore, world sheet and space-time properties are always intimately linked: the conformal invariance of the world sheet theory is transformed into the general coordinate invariance in space-time and, for the bosonic NN string consistency requires $D = 26$; similarly, modular invariance at one-loop level strongly constrains the spectrum and leads to absence of UV divergences and consistency allows only few possible spectra. Reasoning in the other direction, breaking of Poincaré invariance down to $SO(D-1)$ by choosing a special fixed point suggests a breakdown in conformal symmetry on the world sheet and it may be expected that the total anomaly can not be canceled.

2.6. Asymptotic Level Density and Hagedorn Temperature

The Virasoro operators select specific irreducible representations of $SO(D-1)$: One should note that there is always the highest rank tensor, at a given level N, built out of N α^i_{-1} oscillators, of course with appropriate subtractions of other lower irreducible representations that are created along the way. There is also a vector state created by α^i_{-N}. Although only odd-moded creation operators are allowed, the states with even N can still be obtained; for example, one can write 4=1+1+1+1 or 4=1+3, but is not allowed to use 4=4, 4=2+2, 4=2+ 1+1. This means that asymptotic level density for very highly excited states and the Hagedorn temperature [22] of the ND string will not be identical with those of the NN string.

Total number of states, denoted by d_n, with $\alpha' M^2 = n + a^\alpha$ is obtained from the coefficient of ω^n in the partition function $tr\omega^{\mathbb{N}}$, where $\mathbb{N}$ is the number operator, Equation (2.49). To calculate d_n, one first constructs the generating function

$$G(\omega) = \sum_{n=0}^{\infty} d_n \, \omega^n = tr\omega^{\mathbb{N}}.$$

Using the elementary methods of quantum statistical mechanics,

$$tr\omega^{\mathbb{N}} = \prod_{r=1,3,...}^{\infty} tr\omega^{\alpha_{-r}\cdot\alpha_r} = [g(\omega)]^{-(D-1)}$$

where

$$g(\omega) = \prod_{r=1,3,...}^{\infty} (1-\omega^r).$$

There is actually a subtle point here. In the NN string version of this calculation, the spectrum is entirely physical built out of $D-2=24$ transverse oscillators in the light-cone gauge; thus, d_n represents the true number of physical states at a given n. On the other hand, for the ND string, the $tr\omega^{\mathbb{N}}$ calculation yields all of the states generated by combinations of the α^i_{-r} before the Virasoro conditions eliminate the unphysical ones. Therefore this is an overestimation but the relative error decreases as $1/(D-1)^2$ with increasing D since the unwanted states come from the contraction of the space indices of two oscillators to give a scalar.

Obviously,

$$g(\omega) = \frac{f(\omega)}{f(\omega^2)}$$

where

$$f(\omega) = \prod_{n=1,2,\ldots}^{\infty} (1-\omega^n).$$

In order to investigate the asymptotic density of states, one must look at the behavior of the function $g(\omega)$ as $\omega \rightarrow 1$. A precise estimate in this limit can be obtained by noting that upon replacing ω by $e^{2\pi i\tau}$ in $g(\omega)$, $f(\omega)$ becomes closely related to the Dedekind eta function

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1,2,\ldots}^{\infty} (1-e^{2\pi i n\tau})$$

by

$$\eta(\tau) = e^{i\pi\tau/12} f(e^{2\pi i\tau}).$$

Using the modular transformation formula (or the S-transformation property of the eta function)

$$\eta(-1/\tau) = (-i\tau)^{1/2}\,\eta(\tau),$$

one can get

$$g(\omega) = \frac{1}{\sqrt{2}}\,\omega^{1/24}\,q^{1/24}\,\frac{f(q^2)}{f(q)},$$

where

$$q = e^{2\pi^2/ln\omega}.$$

As $\omega \longrightarrow 1$,

$$g(\omega) \longrightarrow A\,e^{\pi^2/12ln\omega}$$

or

$$tr\omega^{\mathbb{N}} \rightarrow A\, e^{-\pi^2 (D-1)/12\, ln\omega} \tag{2.54}$$

where A is constant.

One can project out d_n from the generating function via a contour integral on a small circle about the origin, i.e.

$$\begin{aligned} d_n &= \frac{1}{2\pi i} \oint d\omega\, \frac{G(\omega)}{\omega^{n+1}} \\ &= \frac{1}{2\pi i} \oint d\omega\, \frac{tr\omega^{\mathbb{N}}}{\omega^{n+1}} \end{aligned}$$

Substituting Equation (2.54) into this equation, we have

$$d_n \approx \frac{1}{2\pi i} \oint d\omega\, \exp\left[-\frac{\pi^2(D-1)}{12\, ln\omega} - (n+1) ln\omega\right].$$

Performing the saddle point approximation for large n around the extremum point

$$ln\omega^* = -\pi\, \sqrt{(D-1)/12(n+1)}$$

one arrives at the asymptotic expression for the level density

$$d_n \approx \exp\left[\pi\, \sqrt{\frac{n(D-2)}{3}}\right]\, (D-1)^{1/4}\, n^{-3/4} \tag{2.55}$$

where only the dependance of d_n on the dimension of space-time and the level n are displayed omitting some irrelevant multiplicative constants.

The asymptotic expression for the level density of NN string in D dimensions is

$$d_n \approx \exp\left[\pi\sqrt{\frac{2n(D-2)}{3}}\right] n^{-3/4} n^{-(D-2)/4}. \tag{2.56}$$

Comparing Equation (2.55) with Equation (2.56), one observes that the argument of the exponential changes from $\sqrt{n}$ to $\sqrt{2n}$. This stems from the fact that the ND spectrum is built using only odd-moded oscillators, whereas in the NN string both odd and even-moded oscillators contribute to create a state at level n. Nevertheless, one can see that $d_n(ND) > d_n(NN)$ for $D = 2$ and $D = 3$, which is due to the relatively larger contribution of the longitudinal oscillator modes to the ND string for these low dimensions. For higher D, the exponential dominates as n goes to infinity and the NN string has the greater multiplicity at a given large n.

An interesting limit with stringy behavior is for high temperatures: One can write the density of levels as a function of mass asymptotically as

$$d_n \approx e^{m/m_0}.$$

Note that the level density grows so rapidly with mass that the partition function $tre^{-\beta H}$ of the free theory can not be defined beyond a maximum temperature $T = m_0$ - called the Hagedorn temperature. The level density formula, Equation (2.55), shows that the Hagedorn temperature of the ND string is

$$T_{ND} = \frac{1}{\pi}\sqrt{\frac{3}{2\,\alpha'_{NN}\,(D-1)}},$$

where we have used $\alpha'_{NN} = 2\,\alpha'_{ND}$. The Hagedorn temperature for the NN string in D dimensions follows from the same formula when $D-1$ is replaced by $D-2$.

2.7. The Dimension of Space-time

Before introducing the world sheet supersymmetry in the next chapter, there are three important questions that need to be answered. The first is the unaddressed expectation that the problem should involve an infinite mass associated with the fixed end. The second and the third involve an apparent conflict between two facts: On the one hand, the theory appears to be unitary and $SO(D-1)$ invariant for any D, while on the other hand, the presence of a non-zero anomaly term in the Virasoro algebra casts doubt on the possibility of imposing the constraints $T_{\alpha\beta} = 0$ in a consistent way independently of the value of D. The answer to the first two question is that although the ND string, unlike the bosonic NN string, does not sharply require a specific dimension such as 26, it nevertheless prefers D to be as high as possible.

In relation to the first question, we note that the mass of the scalar ground state is proportional to $D-1$; hence an infinite D is consistent with one end of the string being immovable. Another consequence of this infinite D limit is that the group $SO(D-1)$ with $(D-1)(D-2)/2$ generators and the full inhomogeneous Lorentz group with only $2D-1$ additional generators 'merge up to order $1/D$'; thus, in a sense, the ND string recovers its Poincaré invariance as D goes to infinity. This is evidenced also by a fact we observed earlier: the overestimation in the density of states vanishes like $1/D$. Interestingly, in the rather different approach to the problem taken in [13], D has to be taken to infinity to make a saddle point calculation possible.

The third question, namely the presence of the anomaly term in the Virasoro algebra, gets affected by this choice of D; the coefficient of the anomaly is now infinite. Since conformal invariance is violated by the immobile end, a conformal anomaly has to appear somewhere in the model; its becoming infinite is perhaps an indication that like most other infinite quantities in physics, it is to be disregarded! Indeed, the problems normally associated with the conformal anomaly have already been solved: there are no negative norm states, and the Virasoro conditions have organized and selected states into $SO(D-1)$ irreducible representations.

The critical dimension, $D = 26$, of the bosonic NN string is reduced to $D = 10$ by the consistency requirements when the world sheet supersymmetry is introduced. Furthermore, only in this critical dimension the spectrum is space-time supersymmetric. One can expect a similar situation for the ND string, namely having introduced the world sheet supersymmetry to the system, requiring a spectrum which is supersymmetric in the space-time sense may lead to a specific critical dimension. This possibility will be examined in detail in the next chapter.

3. A RAMOND-NEVEU-SCHWARZ STRING WITH ONE END FIXED

3.1. The Classical RNS String

In the previous chapter the open bosonic string with one end fixed was introduced. Although this system has very interesting features, it conspicuously lacks one important ingredient: fermions. One of the motivations of this work is, apart from shedding light on the novel string theoretic properties of the open string with one end fixed, to construct a model providing some qualitative features of hadrons with a single c, b quark. Thus, in this sense, the model should be extended in such a way to include fermions also. Encompassing baryons requires a new kind of symmetry between the bosons and fermions, and string models of this type were constructed by Ramond [8] and Neveu-Schwarz [9]. This new symmetry is called as "supersymmetry" which unites bosons and fermions into a single multiplet. But at this point one should note that this symmetry must be partially broken for the hadron physics domain in view of the non-vanishing mass differences.

In fact supersymmetry was first discovered in the string theory by Gervais and Sakita [23] who showed that an extension of the usual bosonic action Equation (2.1) has a symmetry converting bosons into fermions and vice versa. One important point about supersymmetry should be clarified here: The supersymmetry of the early spinning string of Ramond and Neveu-Schwarz was a two-dimensional world sheet supersymmetry relating the space-time coordinates $X^{\mu}(\sigma, \tau)$ to fermionic partners $\psi^{\mu}(\sigma, \tau)$ which are two-component world sheet spinors. This world sheet supersymmetry does not directly mean that the theory is also space-time supersymmetric; but what we want are space-time fermions. A string spectrum with space-time supersymmetry is obtained after a suitable truncation found by Gliozzi, Scherk, and Olive (GSO) [24] and this truncation of spectrum not only leads to space-time supersymmetric spectrum but also eliminates the tachyonic ground state from the spectrum. Although our system does not have a tachyon, the possibility of GSO truncation for our spectrum will be investigated when we construct the spectrum of our system.

There is also the so called Green-Schwarz [25] formalism in which the space-time supersymmetry is manifest at the cost of manifest world sheet supersymmetry. It uses in a crucial way the triality property of $SO(8)$, but this formalism will not be considered in this work.

In the rest of this section, the local generalization of the Equation (2.1) will be given and the constraints of this new theory including world sheet fermions will be obtained. The key to the construction of this locally supersymmetric action is to add more fields to the theory. In addition to the supersymmetric pair

$$(X^\mu(\sigma,\tau), \psi^\mu(\sigma,\tau))$$

one should incorporate a 'zweibein' $e^a_\alpha(\sigma,\tau)$ and a Rarita-Schwinger field $\chi_{A\alpha}(\sigma,\tau)$ which is a two-component Majorana spinor and a world sheet vector. In this notation, A is the world sheet spinor index, the a index of zweibein is a Lorentz index taking part in local Lorentz transformations whereas α is called an Einstein index taking part in coordinate transformations of the world sheet. Einstein indices are raised and lowered with the world sheet metric $h_{\alpha\beta}$ and Lorentz indices with the Lorentz metric η_{ab}. The zweibein allows to transform Lorentz into Einstein indices and vice versa. The introduction of zweibein is necessary if one wants to describe spinors on a curved manifold since the group $GL(n,R)$ does not have a spinor representation whereas the the tangent space group $SO(1,D-1)$ does.

The complete action, first written by Brink, Di Vecchia, Howe [26] and Deser and Zumino [27], is

$$S = -\frac{1}{2\pi}\int d^2\sigma\, e\,(h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X_\mu \;-\; i\,\overline{\psi}^\mu\,\rho^\alpha\,\nabla_\alpha\psi_\mu \;+\; 2\,\overline{\chi}_\alpha\,\rho^\beta\,\rho^\alpha\,\psi^\mu\,\partial_\beta X_\mu$$
$$+\;\frac{1}{2}\,\overline{\psi}^\mu\,\psi_\mu\,\overline{\chi}_\alpha\,\rho^\beta\,\rho^\alpha\,\chi_\beta) \qquad (3.1)$$

where ρ^α are the curved space gamma matrices, ∇_α is the covariant derivative on the two-dimensional manifold, $e \equiv |det e^a_\alpha| = \sqrt{h}$ and $\overline{\psi}^\mu$ indicates $\psi^{\mu\dagger}\rho^0$ as usual.

This action is invariant under two-dimensional supersymmetry transformations

$$\begin{aligned}
\delta X^\mu &= \overline{\epsilon}\,\psi^\mu, \\
\delta\psi^\mu &= -i\,\rho^\alpha\,\epsilon\,(\partial_\alpha X^\mu - \overline{\psi}^\mu\chi_\alpha), \\
\delta e^a_\alpha &= -2i\,\overline{\epsilon}\,\rho^a\,\chi_\alpha, \\
\delta\chi_\alpha &= \nabla_\alpha\epsilon,
\end{aligned} \tag{3.2}$$

and Weyl transformations

$$\begin{aligned}
\delta X^\mu &= 0, \\
\delta\psi^\mu &= -\frac{1}{2}\Lambda\,\psi^\mu, \\
\delta e^a_\alpha &= \Lambda\,e^a_\alpha, \\
\delta\chi_\alpha &= \frac{1}{2}\Lambda\,\chi_\alpha.
\end{aligned} \tag{3.3}$$

There is also another fermionic local symmetry due to certain identities in two dimensions[6] given by

$$\begin{aligned}
\delta e^a_\alpha &= \delta\psi^\mu = \delta X^\mu = 0, \\
\delta\chi_\alpha &= i\,\rho_\alpha\,\eta.
\end{aligned} \tag{3.4}$$

where η is an arbitrary Majorana spinor. Furthermore, by construction, this action is also manifestly invariant under local two-dimensional Lorentz transformations and reparameterizations. An action with all these symmetries describes a 'superconformal' theory. In the purely bosonic model the conformal symmetry of the action, Equation (2.1), resulted in the Virasoro constraints after covariant gauge fixing. Following the same path for the superconformal theory here will lead to new constraints which are the supersymmetric extension of the bosonic Virasoro constraints and they can do for

[6] $\rho^\alpha\rho_\beta\rho_\alpha = 0$ in two dimensions.

fermions what the Virasoro conditions do for bosons.

Having fixed the action and its symmetries, one can make a covariant gauge choice which leads to simple field equations and constraint conditions as in the bosonic theory. The number of components of the zweibein is four. It is possible to gauge these components of the zweibein into the form $e^a_\alpha = \delta^a_\alpha$ by using four local bosonic symmetries of the theory which are two world sheet reparametrizations, one local Lorentz, and one Weyl scaling. There are also four local fermionic symmetries: the two supersymmetries (ϵ) and two superconformal symmetries (η). These can be used locally to set the four components of χ_α to zero. This gauge choice, namely

$$e^a_\alpha = \delta^a_\alpha, \tag{3.5}$$

$$\chi_\alpha = 0, \tag{3.6}$$

is called the 'superconformal gauge' which is the analog of the gauge choice $h_{\alpha\beta} = \eta_{\alpha\beta}$ for the open bosonic string theory. In this gauge the action, Equation (3.1), becomes

$$S = -\frac{1}{2\pi} \int d^2\sigma \, \{\partial_\alpha X^\mu \, \partial^\alpha X_\mu - i\,\overline{\psi}^\mu \, \rho^\alpha \, \partial_\alpha \psi_\mu\} \tag{3.7}$$

where ρ^α represents the two-dimensional Dirac matrices and a convenient basis satisfying

$$\{\rho^\alpha, \rho^\beta\} = -2\,\eta^{\alpha\beta} \tag{3.8}$$

is

$$\rho^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \rho^1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

and the analog of the 'γ^5' is

$$\rho^3 = \rho^0 \rho^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It should be noted that the constant in front of the Equation (2.8) is changed to $1/(2\pi)$ in Equation (3.7) to make use of the notation of [16] and compare the results easily with some well-known references such as [18, 19, 20, 28].

The upper and lower components of ψ will be denoted by ψ_- and ψ_+[7], respectively. One should note that the Dirac operator, $i\rho^\alpha \partial_\alpha$, is real in this representation; thus the components of the world sheet spinor should be taken as real[8].

Furthermore, the world sheet supersymmetry transformations under which this new gauge fixed action is invariant become

$$\delta X^\mu = \bar{\epsilon}\, \psi^\mu, \tag{3.9}$$

and

$$\delta \psi^\mu = -i\, \rho^\alpha\, \partial_\alpha X^\mu\, \epsilon, \tag{3.10}$$

with ϵ a constant anticommuting spinor.

The first term of the Equation (3.7) was considered in the previous chapter; thus, from now on, only the second term will be considered throughout this chapter.

The fermionic part of the action in the light-cone coordinates, $\sigma^\pm = \tau \pm \sigma$, becomes

$$S_F = \frac{i}{\pi} \int d^2\sigma\, (\psi_+^\mu\, \partial_- \psi_{+\mu} + \psi_-^\mu\, \partial_+ \psi_{-\mu}). \tag{3.11}$$

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[7] We will see that $\partial_\pm \psi_\mp = 0$ which justifies the $\pm$ label on ψ.

[8] Such a two-component real spinor is known as a Majorana spinor.

The variation of this action with respect to ψ^μ is

$$\begin{aligned}\delta S_F &= \frac{i}{\pi}\int d^2\sigma\,[\delta\psi_+\,\partial_-\psi_+ + \psi_+\,\partial_-(\delta\psi_+) + (+\longleftrightarrow -)],\\ &= \frac{i}{\pi}\int d^2\sigma\,[2\,\delta\psi_+\,(\partial_-\psi_+) + 2\,\delta\psi_-\,(\partial_+\psi_-) + \partial_-(\psi_+\,\delta\psi_+) + \partial_+(\psi_-\,\delta\psi_-)],\\ &= \frac{2i}{\pi}\int d^2\sigma\,[\delta\psi_+\,(\partial_-\psi_+) + (+\longleftrightarrow -)] + \frac{i}{\pi}\int_0^{\sigma_{max}} d\sigma\,\partial_\sigma[\psi_-\,\delta\psi_- - \psi_+\,\delta\psi_+],\end{aligned}$$

where

$$\delta\psi_\pm|_{\tau\to\pm\infty} = 0$$

as usual. Demanding $\delta S_F = 0$ in accordance with the principle of stationary action, the bulk term, which is the first term in the last line of above variation, leads to the equations of motion for the fermi coordinates

$$\begin{aligned}\partial_+\psi_- &= 0,\\ \partial_-\psi_+ &= 0.\end{aligned} \tag{3.12}$$

But at the same time the surface term must be also set to zero necessarily to make sure that $\delta S_F = 0$. This surface term will be considered in the next section in detail.

These equations of motion must be supplemented by the equations of motion of the fields e^a_α and χ_α evaluated in the superconformal gauge Equation (3.5). The constraints are

$$\begin{aligned}T_{\alpha\beta} &= \frac{1}{2}\left(\partial_\alpha X^\mu\,\partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\,\partial_\gamma X^\mu\,\partial^\gamma X_\mu\right)\\ &+ \frac{i}{4}\left(\overline{\psi}^\mu\,\rho_\alpha\,\partial_\beta\psi_\mu + \overline{\psi}^\mu\,\rho_\beta\,\partial_\alpha\psi_\mu - \eta_{\alpha\beta}\,\overline{\psi}^\mu\,\rho^\gamma\,\partial_\gamma\psi_\mu\right) = 0\end{aligned} \tag{3.13}$$

and

$$J_\alpha \equiv -\frac{\pi}{2e}\frac{\delta S}{\delta\chi^\alpha} = \frac{1}{2}\rho^\beta\,\rho_\alpha\,\psi^\mu\,\partial_\beta X_\mu = 0. \tag{3.14}$$

Thus the resulting equations, which will be called as 'super-constraints', are the vanishing of the energy-momentum tensor and the supercurrent on the world sheet.

The energy-momentum tensor is traceless again as in the bosonic theory. The analogous restriction for the supercurrent is[9]

$$\rho^\alpha J_\alpha = 0. \tag{3.15}$$

Actually the supercurrent could have been obtained from Equation (3.9) and Equation (3.10) by using the Noether method for an ϵ which is not constant. In this case, the variation of the action could be of the general form

$$\delta S \sim \int d^2\sigma \, (\partial_\alpha \bar{\epsilon}) \, J^\alpha,$$

and this J^α would be indeed the same as the response of the action Equation (3.1) to variation of the Rarita-Schwinger field, i.e. Equation (3.15); but in this way the constraint $J^\alpha = 0$ would be just a postulated condition. On the other hand, deriving it by gauge fixing of a suitable two-dimensional supergravity Lagrangian, which is the way followed here, one can easily see that why this condition must be there instead of postulating it.

3.2. Boundary Conditions and Mode Expansions

The bosonic space-time coordinates X^μ satisfies the wave equation, Equation (2.10), again and the boundary conditions for them are as in Chapter 2: Locate the infinitely massive 'quark' at the $\sigma = 0$ end and identify this point with the origin of target space coordinates through the Dirichlet boundary condition and adopt the Neumann boundary condition for the massless end. The latter leads to $\sigma_{max} = \pi/2$.

For the fermi coordinates ψ^μ, from the variation of S_F, vanishing of the surface

[9]This is again a consequence of $\rho^\alpha \rho^\beta \rho_\alpha = 0$ in two dimensions.

term requires

$$\psi_+ \, \delta\psi_+ - \psi_- \, \delta\psi_- = 0 \tag{3.16}$$

at each end of the open string and this necessitates $\psi_+^\mu = \pm\psi_-^\mu$, and hence $\delta\psi_+^\mu = \pm\delta\psi_-^\mu$, at $\sigma = 0$ and $\sigma = \pi/2$. In the usual open RNS string, the convention is to set, without loss of generality,

$$\psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau). \tag{3.17}$$

But this condition, unlike the usual RNS string, requires a reconciliation with the world sheet supersymmetry at the $\sigma = 0$ end due to our Dirichlet boundary condition. However, this question will be left to the next section and the convention for the $\sigma = 0$ end will be taken here. On the other hand, the relative sign at the other end now becomes meaningful. Leaving the choice of ψ^0 to the Section 3.4, which is intimately related to the gauge choice, the two cases arising from the sign choice at $\sigma = \pi/2$ end leads to the following mode expansions:

The Ramond (R) boundary condition is

$$\psi_+^i(\frac{\pi}{2},\tau) = \psi_-^i(\frac{\pi}{2},\tau) \tag{3.18}$$

and this leads to the exclusively even-moded expansions,

$$\psi_-^i(\sigma,\tau) = \sum_{n:even} d_n^i e^{-in(\tau-\sigma)}, \tag{3.19}$$

$$\psi_+^i(\sigma,\tau) = \sum_{n:even} d_n^i e^{-in(\tau+\sigma)}. \tag{3.20}$$

For the usual RNS string these sums run over all integers. This means that in the R-sector we will work with the even-numbered subset of the original integral modes.

The Neveu-Schwarz (NS) boundary condition is

$$\psi^i_+(\frac{\pi}{2},\tau) = -\psi^i_-(\frac{\pi}{2},\tau) \tag{3.21}$$

and this results in the odd-integer modes

$$\psi^i_-(\sigma,\tau) = \sum_{r:odd} b^i_r e^{-ir(\tau-\sigma)}, \tag{3.22}$$

$$\psi^i_+(\sigma,\tau) = \sum_{r:odd} b^i_r e^{-ir(\tau+\sigma)}. \tag{3.23}$$

It should be noted that while the boundary conditions merely restrict the Ramond oscillators d^i_n to even and the bosonic α^i_r to odd modes, the Neveu-Schwarz oscillators actually changes from half-integral to odd integral oscillators. This will lead to a massive NS and a massive R ground state. Furthermore, the even modes are appropriate for the description of 'space-time fermions'. For the usual RNS string, the half-integer modes of the NS-sector was appropriate for the description of bosonic states which are different from the states of bosonic theory of the previous chapter. Although the half-integer modes are changed to odd integers, this sector is still appropriate for the description of bosons. This will be explained in Section 3.4.2 .

3.3. Broken Global World Sheet Supersymmetry

For the open string with one end fixed, the supersymmetry transformations Equation (3.9) and Equation (3.10) require special care to be reconciled with the Dirichlet condition at the $\sigma = 0$ end. As a result of this reconciliation, the supersymmetry is broken at this end where the Dirichlet boundary conditions are imposed for the space components of the bosonic coordinates. Actually this is not an unexpected situation since it is known that the half of the supersymmetry is broken by D-branes and, in D-brane language, $\sigma = 0$ end of the open string is restricted to lie on a $D0$-brane in our case. Thus, one can expect the breaking of half of the supersymmetry at the beginning because of the presence of a D-brane in the system. This will be shown in the following.

The infinitesimal form of the world sheet supersymmetry transformation of the bosonic coordinates is $\delta X^\mu = \bar{\epsilon}\,\psi^\mu$ where ϵ is a constant anticommuting spinor. At the $\sigma = 0$ end, the space part of this condition becomes, when the Dirichlet condition is imposed

$$\begin{aligned} \delta X^i(\sigma = 0, \tau) = 0 &= \bar{\epsilon}\,\psi^i, \\ &= \begin{bmatrix} \epsilon_-^\dagger & \epsilon_+^\dagger \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \psi_-^i \\ \psi_+^i \end{bmatrix}, \\ &= i\,(\epsilon_+^\dagger\,\psi_-^i - \epsilon_-^\dagger\,\psi_+^i)|_{\sigma=0} \end{aligned} \tag{3.24}$$

This would normally be satisfied either by taking

$$\psi_+^i(0,\tau) = \psi_-^i(0,\tau), \tag{3.25}$$

and

$$\epsilon_+ = \epsilon_-, \tag{3.26}$$

or by taking

$$\psi_+^i(0,\tau) = -\psi_-^i(0,\tau), \tag{3.27}$$

and

$$\epsilon_+ = -\epsilon_-. \tag{3.28}$$

But it should be noted that the second alternative is ruled out by the choice of boundary conditions at the $\sigma = 0$ end and one is restricted to work with the first alternative, which eliminates half of the world sheet supersymmetry. It would also be possible that the fermion boundary conditions at the two ends could have been reversed and the second alternative could have been chosen; but, even in this case, the obvious conclusion will be again that our mixed Dirichlet-Neumann conditions are compatible with only half of the usual supersymmetry transformations. One can wonder whether this breaking of world sheet supersymmetry will manifest itself in the spectrum. Indeed this happens and will be shown in Section 3.6 .

3.4. Quantization

The quantization method followed for the bosonic ND string was between two different but equivalent quantization methods of the bosonic NN string. For the bosonic string with one end fixed at the origin of space, it was natural to choose the gauge condition $X^0 \sim \tau$. This is the analog of the light-cone gauge of the bosonic NN string, namely $X^+ \sim \tau$ which is suited to a string with both ends moving at the speed of light. Indeed, the first excited state of such a free open string is a massless vector boson with $D-2$ polarization, which is the number of transverse oscillators used in the light-cone gauge. Our problem, in contrast, leads to a massive vector with $D-1$ polarization states, which is in accord with Equation (2.24). In this sense the quantization method is similar to the light-cone gauge quantization. However, requiring physical states satisfy the Virasoro conditions makes the quantization method similar to the old covariant quantization scheme. The similar path will be followed for the RNS string of one end fixed, i.e. a noncovariant gauge will be chosen and the Hilbert space of states will be built up; but, at the end, the super-Virasoro constraints will be imposed on these states to get the physical states.

3.4.1. Gauge Choice

In the light-cone gauge quantization of the usual RNS string, having chosen the gauge $X^+ \sim \tau$, one can apply local supersymmetry transformations which preserve this gauge choice. These transformations turn out to be sufficient to gauge away ψ^+ completely. Thus, one can make the gauge choice $\psi^+ = 0$. For the RNS string with one end fixed, remembering the gauge choice, Equation (2.24), one can gauge away ψ^0 completely in a similar way by applying local supersymmetry transformations, i.e. one can choose the gauge

$$\psi^0 = 0. \tag{3.29}$$

It should also be noted that, as a consistency check, under a global supersymmetry transformation

$$\delta X^0 = \bar{\epsilon}\,\psi^0 = 0,$$

so that the X^0 choice is not altered by this gauge choice.

This is similar to setting all the ψ^+ and X^+ oscillators to zero in the light-cone gauge treatment of the usual RNS string. The gauge choice Equation (3.29), combining with the fact that half of the supersymmetry is broken, leads to

$$0 = \bar{\epsilon}\,\psi^0 = i\,(\epsilon^\dagger\,\psi^0_- - \epsilon^\dagger\,\psi^0_+). \tag{3.30}$$

This condition kills all the b^0_r modes of the NS-sector. On the other hand, for the R-sector, all d^0_n modes are zero except d^0_0. To see this explicitly one can write the mode expansion of ψ^0 as

$$\psi^0_\mp(\sigma,\tau) = \sum d^0_n e^{-in(\tau\mp\sigma)}, \tag{3.31}$$

and substitute this into the previous equation. Then this equation is only compatible with d^0_0 being nonzero. In Section 3.4.2, it will also be shown that d^i_0's satisfy an algebra which is the algebra of space components of gamma matrices of the Clifford algebra. Thus, in this sense, this is just as well, since we would not want to lose an element of the Clifford algebra in $(1, D-1)$ dimensions.

With the rest-frame gauge chosen above, the α^0_r, b^0_r, and d^0_n modes and their associated negative norm states are discarded at the outset. However, unlike the situation in the light-cone gauge, this does not mean that all the states obtained by hitting the ground state with the spacelike oscillators are automatically physical. There are still constraints which must be imposed on all possible states, and the only states annihilated by these constraints are the final physical ones. This is similar to noncovariant treatments of gauge theories where one first sets $A_0 = 0$, but then also imposes $\vec{\nabla}.\vec{A} = 0$ to isolate the physical degrees of freedom.

3.4.2. Canonical Quantization and Emergence of Space-Time Fermions

It was shown that the dynamics of the coordinates $X^{\mu}(\sigma,\tau)$ and $\psi^{\mu}(\sigma,\tau)$ are given by a free two-dimensional massless Klein-Gordon equation and a Dirac equation supplemented by some certain constraints. The quantization of these coordinates is just that of free two-dimensional field theory and this was shown for the bosonic coordinates in the previous chapter where the analysis leads to the commutation relation for the α-modes Equation (2.29), namely

$$\left[\alpha_r^i, \alpha_s^j\right] = r\,\delta_{r+s}\,\delta^{ij}$$

with r, s odd integers.

The quantization of the fermionic coordinates is achieved in a similar manner. The canonical anticommutation relations of the fermionic coordinates are

$$\left\{\psi_A^i(\sigma,\tau), \psi_B^j(\sigma',\tau)\right\} = \pi\,\delta(\sigma-\sigma')\,\delta^{ij}\,\delta_{AB}. \tag{3.32}$$

Substituting the known mode expansions, one obtains the anticommutation relations of the modes. These are

$$\left\{b_r^i, b_s^j\right\} = \delta^{ij}\,\delta_{r+s} \qquad \text{(r, s: odd)} \tag{3.33}$$

for the NS-sector, and

$$\left\{d_m^i, d_n^j\right\} = \delta^{ij}\,\delta_{m+n} \qquad \text{(m, n: even)} \tag{3.34}$$

for the R-sector. Finally, the d_0^0 mode obeys

$$\{d_0^0, d_0^0\} = -I. \tag{3.35}$$

Combining this with the previous anticommutation relation, one can write

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\, \delta_{m+n}, \tag{3.36}$$

with m, n even integers.

α_{-r}^0, b_{-r}^0, and d_{-n}^0 with $r > 0$ and $n > 0$ are creation operators. When, in the next section, the analogues of the bosonic number operator Equation (2.49) will be derived from the zero-frequency part of the super-Virasoro constraints, it will be shown that they increase the eigenvalue of $\mathbb{N}^{(\alpha)}$ and $\mathbb{N}^{(b)}$ by r units and $\mathbb{N}^{(d)}$ by n units, respectively.

The vacuum of the Fock space in the NS-sector is defined by

$$\alpha_r^i\,|0\rangle_{NS} = b_r^i\,|0\rangle_{NS} = 0, \tag{3.37}$$

for $r > 0$, and a generic state of this sector is

$$|NS\rangle = \prod_j (\alpha_{-r_j}^i)^{q_j} \prod_i (b_{-r_i}^i)^{p_i} |0\rangle_{NS}. \tag{3.38}$$

Similarly the vacuum of the R-sector is defined by

$$\alpha_r^i\,|0\rangle_R = d_m^i\,|0\rangle_R = 0, \tag{3.39}$$

for $m, r > 0$, and a generic state of this sector is similar to that of the NS-sector Equation (3.38) with $b_{-r_i}^i$ is replaced by $d_{-m_i}^i$.

There is actually an important issue concerning the d_0^μ: When the number operators will be derived, one will see that $\mathbb{N}^{(d)}$ will not contain d_0^μ. So what is the effect of the action of d_0^μ on the Ramond ground state?

d_0^μ are neither creation nor annihilation operators; thus, acting on any state, they generate a new state with the same $\mathbb{N}^{(d)}$ eigenvalue. In particular, $|0\rangle_R$ is a family of states, all satisfying $d_m^i|0\rangle_R = 0$ with $m > 0$, and related to each other by d_0^μ. The d_0^μ satisfy

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}, \tag{3.40}$$

which is essentially the Clifford algebra in D dimensions. This can be made explicit by defining the gamma matrices with $\Gamma^\mu \equiv i\sqrt{2}\, d_0^\mu$ in $(1, D-1)$-dimensional Minkowski space. Then

$$\{\Gamma^\mu, \Gamma^\nu\} = -2\,\eta^{\mu\nu}. \tag{3.41}$$

Although the explicit form of $\mathbb{N}^{(d)}$ has not been derived yet, we said that d_0^μ do not appear in it. Thus Γ^μ commutes with $\mathbb{N}^{(d)}$ and this means that every state in the representation of the algebra for d_m^μ $(m \neq 0)$ at each mass level should also form a representation of the algebra for Γ^μ and therefore correspond to a space-time spinor of $SO(1, D-1)$. In particular, this is the case with the Ramond vacuum, $|0\rangle_R$. To make this explicit, it is better use the notation $|0\rangle_R^a$ where a is the spinor index and the range of it has not been specified yet. This is a Majorana spinor since $\psi_\mp^\mu$ are real. Explicit construction of $|0\rangle_R^a$ can be made as follows: (For simplicity, it is better consider $SO(D)$ rather than $SO(1, D-1)$ and restrict ourselves to case with D:even; but the result for odd D will be given at the end also.)

One can define operators a_k and $a_k^\dagger$ for $k = 1, \ldots, D/2$ such that

$$\Gamma_k = a_k + a_k^\dagger, \tag{3.42}$$

$$\Gamma_{D/2+k} = \frac{1}{i}(a_k - a_k^\dagger). \tag{3.43}$$

Then the Clifford algebra takes the form

$$\{a_k, a_{k'}\} = 0 = \left\{a_k^\dagger, a_{k'}^\dagger\right\}, \tag{3.44}$$

and

$$\left\{a_k, a_{k'}^\dagger\right\} = \delta_{kk'}. \tag{3.45}$$

This is the usual algebra for $D/2$ pairs of fermionic creation and annihilation operators. A ground state, $|0\rangle$, can be defined such that it is annihilated by all a_k: $a_k|0\rangle = 0$. Then the full representation can be easily constructed by applying the creation operators on the ground state and the representation content with their multiplicities when this is done will be as follows: The state $|0\rangle$ with multiplicity 1, the state $a_i^\dagger|0\rangle$ with multiplicity $D/2$, the state $a_i^\dagger a_j^\dagger|0\rangle$ with multiplicity $\frac{D/2(D/2-1)}{2}$, ..., the state $a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_{D/2}}^\dagger|0\rangle$ with multiplicity 1.

The dimension of the representation is just the total number of such states which is also equal to their multiplicities. The sum of the multiplicities gives

$$1 + D/2 + \frac{D/2(D/2-1)}{2} + \ldots + 1 = 2^{D/2} \tag{3.46}$$

which is the sum of the binomial coefficients of $(1+1)^{D/2}$. This means that there are $2^{D/2}$ components of $|0\rangle_R^a$, i.e. the (space-time) spinor index a runs from 1 to $2^{D/2}$. If D were an odd number, the number of components will be $2^{(D-1)/2}$.

Therefore, one can conclude that the Ramond ground state is a spinor of $SO(1, D-1)$ while in the NS-sector it is possible to choose a unique nondegenerate ground state which may be identified as a spin-0 state. This means that the boundary conditions that give even moded d oscillators must give fermionic states, that is the sector with the even moded world sheet spinors is fermionic. On the other hand, odd moded world sheet spinors lead to bosonic states. The bosonic sector of the RNS string should not be confused with the bosonic string of Chapter 2. The oscillators, all being space-time vectors, cannot change bosons into fermions or vice versa. Whether a state is fermionic or bosonic depends on the ground state it is built on.

3.5. Super-Virasoro Constraints

Constraint equations arise from gauge fixing of a gauge-invariant Lagrangian. For the RNS string, such a gauge fixing leads to the constraints $T_{\alpha\beta} = 0$ and $J_\alpha = 0$ at the classical level. When the system is quantized, they are implemented on the states:

$$T^+_{\alpha\beta}\,|phys\rangle = 0, \tag{3.47}$$

$$J^+_\alpha\,|phys\rangle = 0, \tag{3.48}$$

where $T^+_{\alpha\beta}$ and J^+_α are the positive frequency components of the energy-momentum tensor and super-current, respectively. It should be recalled from Section 2.5 that the reason of imposing this weaker requirement is to prevent the inconsistency with the (anti)commutator of the super-Virasoro modes.

The energy-momentum tensor of the fermionic part is

$$T_{\alpha\beta} = \frac{i}{4}\,(\overline{\psi}^\mu\,\rho_\alpha\,\partial_\beta\psi_\mu + \overline{\psi}^\mu\,\rho_\beta\,\partial_\alpha\psi_\mu - \eta_{\alpha\beta}\,\overline{\psi}^\mu\,\rho^\gamma\,\partial_\gamma\psi_\mu)$$

The components of the energy-momentum tensor in the light-cone coordinates, imposing also the rest frame gauge, are

$$\begin{aligned} T_{++} &= \frac{1}{2}\,(T_{00} + T_{01}) = \frac{i}{2} : \Psi^i_+\,\partial_+\Psi_{+i} :, && (3.49)\\ T_{--} &= \frac{1}{2}\,(T_{00} - T_{01}) = \frac{i}{2} : \Psi^i_-\,\partial_-\Psi_{-i} :, && (3.50)\\ T_{-+} &= T_{+-} = 0. && (3.51) \end{aligned}$$

Furthermore, $\partial^\alpha T_{\alpha\beta} = 0$ leads to $\partial_+ T_{--} = 0$ and $\partial_- T_{++} = 0$ provided that equations of motion are satisfied.

The super-current is

$$J_\alpha = \frac{1}{2} \rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu .$$

The nonzero components of the super-current, again in the light-cone coordinates, are

$$J_{+A} = \begin{bmatrix} 0 \\ \psi_+ . \partial_+ X \end{bmatrix}, \qquad J_{-A} = \begin{bmatrix} \psi_- . \partial_- X \\ 0 \end{bmatrix}.$$

where A is the spinor index with $A = 1, 2$. Conveniently only the nonzero components of J_{+A} and J_{-A} are referred as J_+ and J_-; and these nonzero components are simply

$$J_+ = \psi_+^i \, \partial_+ X_i - \psi_+^0 \, \partial_+ X_0, \tag{3.52}$$

$$J_- = \psi_-^i \, \partial_- X_i - \psi_-^0 \, \partial_- X_0. \tag{3.53}$$

The conservation law of the super-current yields $\partial_- J_+ = \partial_+ J_- = 0$, again provided that the equations of motion for X^μ and ψ^μ are satisfied.

Since the equations of motion for X^μ and ψ^μ imply that $\partial_\pm X$ and $\psi_\pm$ are functions of $\sigma^\pm$, the conservation laws of the constraints yield that $T_{\pm\pm}$ and $J_\pm$ are functions of $\sigma^\pm$ only, too.

In the rest of this section, the Fourier modes of these constraints will be constructed in terms of the harmonic oscillator modes and their algebras will be examined

3.5.1. Neveu-Schwarz Constraints

The super-Virasoro operators are the Fourier coefficients of the energy-momentum tensor and the super-current. Furthermore, for open strings there is one independent set of these coefficients since right- and left-movers are identified by the boundary conditions. The coefficients of the energy-momentum tensor are

$$L_m^{(b)} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\sigma\ (e^{im\sigma}\, T_{++}(\sigma) + e^{-im\sigma}\, T_{--}(\sigma)). \tag{3.54}$$

Using $T_{++}(\sigma) = T_{--}(-\sigma)$ and extending the region of integration to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, one gets

$$L_m^{(b)} = \frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\sigma\ e^{im\sigma}\, T_{++}(\sigma). \tag{3.55}$$

For $m = 0$ the above definition disagrees by an additive constant with what should properly be called L_0. Substituting the mode expansions, this becomes

$$L_m^{(b)} = \frac{1}{2}\sum_{r:odd}(r + \frac{m}{2}) : b^i_{-r}\, b^i_{m+r} : + a^b\, \delta_{m,0} \tag{3.56}$$

where m is seen to be necessarily even. It is normal ordering for $m = 0$ which brings the constant a^b. The regularization recipe for calculating such constant will be given later. Thus the $L_m^{(b)}$ complement the $L_m^{(\alpha)}$ which are also even. Actually this is an indication of the consistency of the mode elimination caused by our mixed boundary conditions.

The fermionic generators of this sector are obtained similarly using $J_+(\sigma) = J_-(-\sigma)$, which results in

$$G_m = \frac{\sqrt{2}}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\sigma\ e^{im\sigma}\, J_+(\sigma). \tag{3.57}$$

In terms of the mode expansions one finds

$$G_m = \sum_{r:odd} \alpha^i_{-r}\, b^i_{m+r} \tag{3.58}$$

where m is again forced to be even. The analog of this operator in the usual RNS string is half-integrally moded.

The super-Virasoro algebra for this sector is

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{2}(m-n)(\alpha_0^0)^2\,\delta_{m+n} + A_m\,\delta_{m+n}, && (3.59)\\
[L_m, G_n] &= (\frac{m}{2}-n)G_{m+n}, && (3.60)\\
\{G_m, G_n\} &= 2L_{m+n} - 2(a^\alpha + a^b)\delta_{m+n} + (\alpha_0^0)^2\delta_{m+n} + B_m\delta_{m+n} && (3.61)
\end{aligned}$$

where $L_m = L_m^{(\alpha)} + L_m^{(b)}$.

Obviously all super-Virasoro operators have their usual forms, say as in [18], except for the fact that their mode characters are changed. This naturally leads us to expect that the central extension terms must also change. The α-mode anomaly $A(m)$ was calculated in Equation (2.42) with the result

$$A_m^{(\alpha)} = \frac{(D-1)}{24}\,(m^3 + 2m).$$

With a similar procedure the anomaly due to b-modes can also be calculated, and this can be easily done via

$$\begin{aligned}
A_m^{(b)} &= \langle 0 \mid [L_m^{(b)}, L_{-m}^{(b)}] \mid 0\rangle \\
&= \langle 0 \mid L_m^{(b)} L_{-m}^{(b)} \mid 0\rangle \\
&= \frac{1}{4}\langle 0 \mid \sum_{r,s:odd} (r+\frac{m}{2})(s-\frac{m}{2}) b_{-r}^i\, b_{m+r}^i\, b_{-s}^j\, b_{-m+s}^j \mid 0\rangle \\
&= \frac{1}{4}\langle 0 \mid \sum_{r,s:odd} (r+\frac{m}{2})(s-\frac{m}{2}) b_{m+r}^i\, b_{-r}^i\, b_{-m+s}^j\, b_{-s}^j \mid 0\rangle, && (3.62)
\end{aligned}$$

where m is, of course, even and positive. The expectation value in the third line vanishes for $r > 0$ and $s > m$, whereas the expectation value in the fourth line vanishes for $s < 0$ and $m < -r$. This yields

$$\begin{aligned}
A_m^{(b)} &= \frac{1}{4}\langle 0 \mid \sum_{r=-m+1}^{-1} \sum_{s=1}^{m-1} (r+\frac{m}{2})(s-\frac{m}{2}) b_{-r}^i\, b_{m+r}^i\, b_{-s}^j\, b_{-m+s}^j \mid 0\rangle, && (3.63)\\
&= 2\,(D-1)\sum_{s=1}^{m-1}(s-\frac{m}{2})^2 && (3.64)
\end{aligned}$$

where all the summations above are over odd integers and $\eta_{ij}\eta^{ij} = D - 1$ is the number of space coordinates. The summation in the second line, when taken over all integers, produces the well-known NS-sector central extension term. On the other hand, our odd integer summation gives

$$A_m^{(b)} = \frac{(D-1)}{48}(m^3 - 4m). \tag{3.65}$$

Hence the total central extension term of the first commutator in the super-Virasoro algebra of NS-sector is

$$A_m = \frac{(D-1)}{16}m^3. \tag{3.66}$$

The analog of this term in the usual RNS string is [18]

$$A_m = \frac{D}{8}(m^3 - m). \tag{3.67}$$

It should be noted again as in the purely bosonic case of Chapter 2 that the coefficient D has been changed to $(D-1)/2$ and this approximate halving of the anomaly is the manifestation of the elimination of the odd Virasoro modes.

The central extension term appearing in the anticommutator of the G_m can be similarly calculated with the result

$$B_m = \langle 0 \mid G_m G_{-m} \mid 0 \rangle, \tag{3.68}$$

$$= \frac{(D-1)}{4}m^2. \tag{3.69}$$

The central extension term of the same anticommutator in the usual RNS string is [18]

$$B_r = \frac{D}{2}(r^2 - 1/2), \tag{3.70}$$

and again D of this term has been replaced by $(D-1)/2$ in our case.

At this point, by taking into account these new central extension terms, it should be noted that the anomaly-free $OSp(1|2)$ closed subalgebra of the NS-sector of the usual RNS string formed by L_1, L_0, L_{-1}, $G_{1/2}$, $G_{-1/2}$ is now reduced to the Abelian super-subalgebra of L_0 and G_0.

3.5.2. Ramond Constraints

The method followed in this section is very similar to that of the previous section. The super-Virasoro operators of this sector are

$$L_m^{(d)} = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\sigma \, e^{im\sigma} \, T_{++}(\sigma) \tag{3.71}$$

and

$$F_m = \frac{\sqrt{2}}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\sigma \, e^{im\sigma} \, J_+(\sigma). \tag{3.72}$$

Substituting the known mode expansions of T_{++} and J_+, we find

$$L_m^{(d)} = \frac{1}{2} \sum_{n:even} (n + \frac{m}{2}) : d_{-n}^i \, d_{m+n}^i : + a^d \delta_{m,0} \tag{3.73}$$

where m must be even, and

$$F_s = \sum_{r:odd} \alpha_{-r}^i \, d_{r+s}^i \tag{3.74}$$

where s is necessarily odd. An exception is the operator

$$f_0 = \alpha_0^0 \, d_0^0 \tag{3.75}$$

which is all that survives here from the generalized Dirac operator F_0 of the standard RNS string. The super-Virasoro algebra of this sector is as follows:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{2}(m-n)(\alpha_0^0)^2\,\delta_{m+n} + A_m\delta_{m+n}, && (3.76)\\
[L_m, F_r] &= (\frac{m}{2} - r)F_{m+r}, && (3.77)\\
\{F_r, F_s\} &= 2L_{r+s} - 2(a^\alpha + a^d)\delta_{m+n} + (\alpha_0^0)^2\delta_{r+s} + B_r\delta_{r+s}, && (3.78)\\
[L_m, f_0] &= 0, && (3.79)\\
\{f_0, f_0\} &= -(\alpha_0^0)^2 && (3.80)
\end{aligned}
$$

where $L_m = L_m^{(\alpha)} + L_m^{(d)}$ and the commutators (or anticommutators, as appropriate) of $(\alpha_0^0)^2$ or (f_0) with other operators vanish.

The central extension terms of the first commutator of this algebra can be calculated as in the previous section, giving

$$A_m^{(d)} = \frac{(D-1)}{24}(\frac{m^3}{2} - 3m^2 + 4m). \qquad (3.81)$$

Then the total central extension term of the first commutator, including the contribution of the α modes also, will be

$$A_m = \frac{(D-1)}{16}(m^3 + 4m). \qquad (3.82)$$

The analog of this term for the standard RNS string is, [18],

$$A_m = \frac{D}{8}m^3, \qquad (3.83)$$

and one can easily notice that D is again replaced by $(D-1)/2$ in our case.

A similar treatment for the central extension term of anticommutators of fermionic generators yields

$$B_r = \frac{(D-1)}{4}(r^2+1). \tag{3.84}$$

The analogous term of the standard RNS string is, [18],

$$B_m = \frac{D}{2}m^2, \tag{3.85}$$

and as usual D of this term becomes $(D-1)/2$ in our case.

3.6. The Spectrum

3.6.1. The Normal Ordering Constants

In Chapter 2, the prescription for the calculation of normal ordering constant of bosonic α-oscillators was given. The same method will be applied to Neveu-Schwarz b-oscillators and Ramond d-oscillators with two changes due to the fermionic nature of these modes: The first change is that in adding zero point energies of the oscillators we have an extra minus sign. The second change is that c, the central charge, has an extra 1/2 factor for world sheet fermions.

In taking into account these modifications, first we will calculate the normal ordering constant of the Neveu-Schwarz sector. The zero point energy of the odd integral Neveu-Schwarz world sheet b-oscillators is

$$-\frac{D-1}{12}\frac{1}{12},$$

and the shift is

$$\frac{D-1}{2}\frac{1}{2}\frac{1}{24},$$

giving

$$a^b = -\frac{D-1}{32}. \tag{3.86}$$

For the d-oscillators, the zero point energy is

$$-\frac{D-1}{2}\left(-\frac{1}{6}\right),$$

The first minus being due to the fermionic nature of the d's and the $-1/6$ coming from a sum of even positive integers. Adding the same shift as in the case of Neveu-Schwarz sector, we get

$$a^d = 3\,\frac{D-1}{32}. \tag{3.87}$$

With these results, one can now find the total normal ordering constants of both sectors. For the NS sector,

$$\begin{aligned} L_0(NS) &= a^\alpha + a^b \\ &= \frac{D-1}{16} - \frac{D-1}{32} \\ &= \frac{D-1}{32}. \end{aligned} \tag{3.88}$$

Similarly for the Ramond sector

$$\begin{aligned} L_0(R) &= a^\alpha + a^d \\ &= \frac{D-1}{16} + 3\,\frac{D-1}{32} \\ &= 5\,\frac{D-1}{32}. \end{aligned} \tag{3.89}$$

It should be noted that the normal ordering constant here is nonzero, unlike that of the R-sector of the standard RNS string. To understand the reason of the appearance of such a normal ordering constant in our case, it had better remember the situation in the standard RNS string briefly: There is a relation between the of the fermionic super-Virasoro constraint F_0 and that of the other super-Virasoro constraint. This relation can be succinctly written[10] as $F_0^2 = L_0$. Actually this is the analog of the fact that Dirac equation is the 'square-root' of the Klein-Gordon equation, F_0 being the generalized Dirac operator and L_0 Klein-Gordon operator. Introducing a normal ordering constant into the constraint equation of L_0, i.e. $(L_0 - \mu^2) \mid phys\rangle = 0$, necessitates a similar term in the constraint equation of F_0 due to the relation between them. This means that the form of the constraint must be $(F_0 - \mu) \mid phys\rangle = 0$. However, F_0 has no normal ordering ambiguity in passing to the quantum theory. Furthermore, F_0 is an anticommuting operator so it will be quite unnatural to add to it a (commuting) c-number. Thus the usual procedure in the RNS string is to put $\mu = 0$ at the beginning; but later this option is justified when one considers the spectrum. Since our system in the R-sector does not have a relation like $F_0^2 = L_0$, a normal ordering constant can appear in L_0 constraint without causing an inconsistency of the aforementioned type.

3.6.2. Mass Formulas in the NS and R-Sectors

With the calculated normal ordering constants, the mass-shell conditions can be written as

$$(-\alpha'_{ND} M^2 + \mathbb{N}^{(a)} + \mathbb{N}^{(b)} + \frac{D-1}{32}) \mid phys\rangle = 0, \tag{3.90}$$

for the NS-sector and

$$(-\alpha'_{ND} M^2 + \mathbb{N}^{(a)} + \mathbb{N}^{(d)} + 5\,\frac{(D-1)}{32}) \mid phys\rangle = 0, \tag{3.91}$$

for the R-sector where $\alpha'_{ND} = \ell^2/2$ is twice as large as the standard α'_{NN}. This doubling of the slope relative to α'_{NN} of the usual open string is one of the principal distinguishing features of our system. A quick way of understanding this result is to consider a classical string with one end fixed in its highest angular momentum (leading

[10] Since the exact form of F_0 and L_0 are irrelevant to our subsequent considerations and arguments, they are not given here. The only thing about them to recall is that their forms are different from ours.

Regge trajectory) mode, where it rotates rigidly. This can also be viewed as an ordinary open string in the same mode with its center of mass at rest and then throwing away one half. In order to preserve the relation $J \sim \alpha' M^2$ with J and M both being halved, the slope has to be doubled.

Since half of the world sheet supersymmetry has been broken at the beginning by the restriction $\epsilon_+ = \epsilon_-$ due to the familiar phenomenon of halving of supersymmetry via D-branes [20], it is not expected that the space-time supersymmetry appears the way it does in the RNS string subjected to the GSO projection [24]. In our case, the fixed end is nothing but a $D0$-brane. Thus, while the spectrum is not expected to be fully supersymmetric, there are two further options in the degree of supersymmetry breaking: The first option is to take $D-1 = 8n$. In this case the masses of the states in the NS-sector have the same values as those in the R-sector for $\alpha'_{ND} M^2 \geq n$, partially preserving supersymmetry in the mass spectrum. The second option is $D-1 \neq 8n$ and in this case mass values in the two sectors are completely different and supersymmetry is completely broken.

In the subsequent arguments, the minimal supersymmetry breaking option $n = 1$, $D = 9$ will be considered. All states being massive, it is expected that all states are irreducible representations of $SO(8)$, which is clearly a remnant of the space-time supersymmetry enjoyed by the ten-dimensional superstring. We now examine the spectra of two sectors separately.

3.6.3. Neveu-Schwarz Spectrum

The Fock space of the states is constructed by applying α^i_{-r} and b^i_{-r}, r:odd, oscillators on the scalar Neveu-Schwarz ground state $\mid 0\rangle$. Although the negative norm states are eliminated by the gauge choice, our suggested quantization procedure requires that the physical states constitutes a subspace of the full Hilbert space satisfying the following super-Virasoro constraints:

$$G_{2m} \mid phys\rangle \;=\; 0, \qquad m > 0 \tag{3.92}$$

$$L_{2n} \mid phys\rangle = 0, \qquad n > 0 \tag{3.93}$$

$$L_0 \mid phys\rangle = 0, \tag{3.94}$$

and the last constraint leads to the mass-shell condition

$$\alpha'_{ND} M^2 = N^{(\alpha)} + N^{(b)} + \frac{D-1}{32}. \tag{3.95}$$

There is also a G_0 constraint that has to be handled separately. An examination of the NS-constraint algebra of Section 3.5.1 shows that all the physical state conditions follow as soon as one adjoins G_2 to G_0 and L_0, i.e., it is enough to check only these constraints on the states whether they are physical. If one considers the anticommutator of G_0's, one gets from the algebra

$$\begin{aligned} 2G_0^2 &= 2L_0 + (\alpha_0^0)^2 - 2(a^\alpha + a^b) \\ &= 2(N^\alpha + N^b) \end{aligned} \tag{3.96}$$

and if $L_0 \mid phys\rangle = 0$ is imposed on the states, it necessarily follows from this that the G_0 condition must be

$$G_0 \mid phys\rangle = \sqrt{N^\alpha + N^b} \mid phys\rangle. \tag{3.97}$$

This means that G_0 does not in fact annihilate physical states; it instead requires them be eigenstates with the mass as the eigenvalue up to some constant.

It should be noted that this amounts to taking the square root of the Klein-Gordon-like operator L_0. The novelty here is that this happens in the R-sector of the RNS string but in our case it happens in the bosonic sector of the theory! This condition also seems to be in conflict with the fact that G_0 does not have a normal ordering ambiguity. On the other hand, this condition follows from the super-Virasoro algebra of this sector; thus we can not throw it away. Perhaps a reasonable comment on this issue is that this is one of the unusual things introduced due to the mixed

Dirichlet-Neumann boundary conditions, such as the existence of infinite conformal anomaly but disappearance of all problems associated with this anomaly at the end.

The low-lying physical states of this sector are obtained as follows: Actually, since negative-metric states have been barred from the beginning, it is not immediately obvious what role is left for the above constraints to play. If we use the L_n directly, the answer in the bosonic NS sector turns out to be that all α^i_{-n} and b^i_{-n} oscillators for $n > 1$ are ruled out, and the surviving states are automaticaly organized into $SO(8)$ irreps. Hence the daughter trajectories are eliminated from the spectrum. The G_n constraints prune the remaining states even further; for example, a potential $N = 3$ state of the form $\alpha^i_{-1}\alpha^j_{-1}b^k_{-1} \mid 0\rangle$ is prohibited by the G_2 constraint. Finally, the G_0 constraint allows only specific linear combinations of the states that have survived that far. This is obviously different from what happens in the light-cone gauge in the usual RNS string, where all combinations of $\alpha^i_{-n}, b^i_{-n} (i = 1, ..., 8)$ oscillators on the vacuum are guaranteed to produce physical states, which then combine with the others at the same mass to give $SO(9)$ irreps. There being no obvious pattern to the allowed states beyond what we have just mentioned, we limit ourselves to displaying below the contents of the first four levels.

$N = 0$ and $\alpha'_{ND}M^2 = 1/4$:

$$\mid 0\rangle$$

and this state is a massive scalar, providing a stable vacuum for this sector. This is unlike the situation in the NS-sector of RNS string.

$N = 1$ and $\alpha'_{ND}M^2 = 5/4$:

We start with the two massive vector states

$$\mid \alpha\rangle \equiv \alpha^i_{-1} \mid 0\rangle, \qquad \mathbf{8_v}$$

$$\mid \beta\rangle \equiv b^i_{-1} \mid 0\rangle, \qquad \mathbf{8_v}$$

which are allowed by the G_2 constraint; but, when G_0 acts on them it leads to

$$\begin{aligned} G_0[\alpha^i_{-1} \mid 0\rangle] &= b^i_{-1} \mid 0\rangle, \\ G_0[b^i_{-1} \mid 0\rangle] &= \alpha^i_{-1} \mid 0\rangle, \end{aligned}$$

which are not equal to zero; thus, G_0 does not allow them in this form. However, the combination

$$\frac{1}{\sqrt{2}}(\mid \alpha\rangle + \mid \beta\rangle)$$

is permitted by G_0. It is better to see this explicitly:

$$\begin{aligned} G_0[\frac{1}{\sqrt{2}}(\mid \alpha\rangle + \mid \beta\rangle)] &= \frac{1}{\sqrt{2}}(\mid \beta\rangle + \mid \alpha\rangle) \\ &= \frac{\sqrt{N^\alpha + N^b}}{\sqrt{2}}(\mid \alpha\rangle + \mid \beta\rangle), \end{aligned}$$

since $N^\alpha + N^b = 1$ at this level; and that is why the given combination above is the only allowed state of this level.

$N = 2$ and $\alpha'_{ND} M^2 = 9/4$:

Under G_2, the massive tensor states

$$\begin{aligned} \mid 1\rangle &\equiv \left\{\alpha^i_{-1}\alpha^j_{-1} - \frac{\delta^{ij}}{D-1}\alpha^k_{-1}\alpha^k_{-1}\right\} \mid 0\rangle, \qquad \mathbf{35_v} \\ \mid 2\rangle &\equiv b^i_{-1}b^j_{-1} \mid 0\rangle, \qquad \mathbf{28} \\ \mid 3\rangle &\equiv \left\{\alpha^i_{-1}b^j_{-1} + \alpha^j_{-1}b^i_{-1} - \frac{2\delta^{ij}}{D-1}\alpha^k_{-1}b^k_{-1}\right\} \mid 0\rangle, \qquad \mathbf{35_v} \\ \mid 4\rangle &\equiv \{\alpha^i_{-1}b^j_{-1} - \alpha^j_{-1}b^i_{-1}\} \mid 0\rangle, \qquad \mathbf{28} \end{aligned}$$

are the allowed combinations. One should recall that since the even α and b modes have been eliminated at the beginning, there cannot be excitations of the form $\alpha^i_{-2} \mid 0\rangle$ and $b^i_{-2} \mid 0\rangle$. Furthermore, out of these four, the condition $G_0 \mid phys\rangle = \mid phys\rangle$ of this sector allows only

$$\mid 1\rangle + \frac{1}{\sqrt{2}} \mid 3\rangle, \qquad \mathbf{35_v}$$

and

$$\mid 2\rangle + \frac{1}{\sqrt{2}} \mid 4\rangle, \qquad \mathbf{28_v}.$$

$N = 3$ and $\alpha'_{ND} M^2 = 13/4$:

$$\begin{aligned}
\mid 1\rangle &\equiv \{\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1} - \frac{\delta^{ij}}{D+1}\alpha^n_{-1}\alpha^n_{-1}\alpha^k_{-1} - \frac{\delta^{ik}}{D+1}\alpha^n_{-1}\alpha^n_{-1}\alpha^j_{-1} \\
&\qquad - \frac{\delta^{jk}}{D+1}\alpha^n_{-1}\alpha^n_{-1}\alpha^i_{-1}\} \mid 0\rangle, \mathbf{112_v} \\
\mid 2\rangle &\equiv \{b^i_{-1}b^j_{-1}\alpha^k_{-1} + b^k_{-1}b^j_{-1}\alpha^i_{-1} + b^i_{-1}b^k_{-1}\alpha^j_{-1} + \frac{2\delta^{jk}}{2-D}b^i_{-1}b^n_{-1}\alpha^n_{-1}\} \\
&\qquad - \frac{2\delta^{ik}}{2-D}b^j_{-1}b^n_{-1}\alpha^n_{-1}\} \mid 0\rangle, \mathbf{160_v} \\
\mid 3\rangle &\equiv \{b^i_{-1}b^j_{-1}b^k_{-1}\} \mid 0\rangle, \qquad \mathbf{56_v} \\
\mid 4\rangle &\equiv \{b^i_{-1}b^j_{-1}\alpha^k_{-1} - b^i_{-1}b^k_{-1}\alpha^j_{-1} - b^k_{-1}b^j_{-1}\alpha^i_{-1}\} \mid 0\rangle, \qquad \mathbf{56_v}
\end{aligned}$$

The tensorial form of the first state is dictated by L_2, which is obtained from the anticommutator of G_2 and G_0. As mentioned earlier, the same L_2 prohibits states with higher oscillators of the form $\alpha^i_{-3} \mid 0\rangle$, The forms of the second and fourth states are determined by the action of G_2. Finally, only the combination

$$\frac{1}{\sqrt{2}}(\sqrt{3} \mid 3\rangle + \mid 4\rangle), \quad \mathbf{56_v}$$

obeys $G_0 \mid phys\rangle = \frac{\sqrt{3}}{\sqrt{2}} \mid phys\rangle$ constraint of this sector.

3.6.4. Ramond Spectrum

The physical states of this sector must satisfy

$$F_{2m+1} \mid \psi\rangle = 0, \quad m > 0 \tag{3.98}$$

$$L_{2n} \mid \psi\rangle = 0, \quad n > 0 \tag{3.99}$$

$$L_0 \mid \psi\rangle = 0. \tag{3.100}$$

However, examining the super-Virasoro algebra of this sector, one can see that these infinite set of conditions can be reduced to just the F_1, F_3 and L_0 constraints. Furthermore, taking the square root of the anticommutator of f_0, one can obtain the f_0 constraint which simplifies to

$$(\Gamma^0 + I) \mid \psi\rangle = 0 \tag{3.101}$$

where $\mid \psi\rangle$ is the Ramond ground state. This is what remains of the Dirac equation.

$\mid \psi\rangle$ is a spinor of $SO(1, D-1)$ and since in our case $D = 9$ option has been chosen to have the minimal supersymmetry breaking option, $\mid \psi\rangle$ must be a Majorana spinor in our $(8,1)$-dimensional Minkowski space-time. A spinor in D dimensions has $2^{D/2}$, or $2^{(D-1)/2}$, complex components depending on whether D is even or odd. Since $D = 9$ in our case, $\mid \psi\rangle$ has $2^4 = 16$ copmlex components at the beginning; but imposing Majorana condition, it cuts half of them and we are left with 16 real components. These 16 components consist of linear combinations of the two independent $SO(8)$ spinors, which we will denote by 8_s and 8_c. These are projected out of a 16-component spinor by the operators $\Gamma^0 + I$ and $\Gamma^0 - I$, respectively. Thus Equation (3.101) indeed serves as a Dirac equation in halving the number of independent components.

The situation here is strikingly different from that of the RNS string but the results in some critical respects are very similar in the following sense: For the standard RNS string when one writes the spectrum, one sees that there is a tachyon in the NS-sector and there are two different chirality spinors in the R-sector [28]. The spectrum is not supersymmetric in the space-time sense and even the vacuum of the NS-sector is not stable. Then one applies the GSO projection and this eliminates the tachyon of the NS-sector and chooses one of the two chiralities of the R-sector. The resulting spectrum is, at the end, supersymmetric. The GSO projection at the tree level is an ad hoc constraint, i.e., it is applied to make the spectrum space-time supersymmetric. Its necessity is seen, on the other hand, when one takes into account the one-loop corrections to tree level amplitudes. At one-loop level the GSO projection is required to satisfy the modular invariance[11] of the amplitudes. In our case, on the other hand, there is no tachyon in the spectra of both sectors and one of the two chiralities in the R-sector is singled out by one of the super-Virasoro constraints, i.e. without applying a GSO projection-like additional requirement. Although both our model and the standard RNS string, at the end, do not contain a tachyon and both use a single chirality, the ways of implementing these two common properties are obviously different. However, this is a reason to believe that our model is not in conflict with the usual RNS string in these critical respects.

The states are constructed by applying the combination of creation operators with $N = 0, 1, 2$ on the ground state $\mid \psi\rangle$. The physically allowed combinations are obtained by imposing the F_1, F_3, L_0 and f_0 constraints:

$N = 0, \alpha'_{ND} M^2 = 5/4$:

$$\mid 0\rangle \equiv \mid \psi_\beta\rangle, \quad \psi_\beta \sim \mathbf{8}_\mathrm{s}$$

It should be noted that the f_0 constraint Equation (3.101) has eliminated $\mathbf{8}_\mathrm{c}$ and kept $\mathbf{8}_\mathrm{s}$.

[11] Invariance of the partition function and amplitudes under the global diffeomorphisms of the torus which are represented by the group $PSL(2, Z)$. These transformations change the modular parameter but leave the torus invariant. This 'global consistency' condition has no analog in point particle QFT's and is responsible for many stringy features and improvements on point particle theories such as the absence of UV divergence even in the case of divergent bosonic one-loop partition function

$N=1, \alpha'_{ND}M^2 = 9/4$:

$$|\,1\rangle \equiv |\,\psi^{ij}_{\beta'}\rangle \equiv (\alpha^i_{-1}d^j_0 + \alpha^j_{-1}d^i_0 - \frac{\delta^{ij}}{D-1}\alpha^k_{-1}d^k_0)\,|\,\psi_{\beta'}\rangle, \quad \mathbf{35_v} \otimes \mathbf{8_c}$$

is the only permissible combination. Imposing $(\Gamma^0 + I)\,|\,1\rangle = 0$ forces $|\,\psi_{\beta'}\rangle$ to be $\mathbf{8_c}$ since Γ^0 anticommutes with d^i_0.

$N=2, \alpha'_{ND}M^2 = 13/4$:

$$\begin{aligned}|\,\psi^{ij}_{\beta}(\alpha\alpha)\rangle + |\,\psi^{ij}_{\beta}(dd)\rangle &\equiv (\alpha^i_{-1}\alpha^j_{-1} - \frac{\delta^{ij}}{D-1}\alpha^k_{-1}\alpha^k_{-1} \\ &- d^i_{-2}d^j_0 - d^j_{-2}d^i_0 + \frac{2\delta^{ij}}{D-1}d^k_{-2}d^k_0)\,|\,\psi_{\beta}\rangle, \quad \mathbf{35_v} \otimes \mathbf{8_s}\end{aligned}$$

The $\alpha\alpha$ and dd parts of this separately satisfy the Virasoro constraints Equation (3.103) and Equation (3.104), but the superconstraints Equation (3.102) force them into this particular combination. Because of the even number of d modes, the basic spinor is $\mathbf{8_s}$ since each jump of Γ^0 through each d it gets a minus sign but for two d's it is $+1$ at the end. The $SO(8)$ transformation properties of these states have been indicated above, but one must be careful in distinguishing between these boldface numbers and the actual physical degrees of freedom. The NS spectrum, where the boldface numbers are identical with the number of physical states, is free of this complication. The R-sector $SO(8)$ generators are built out of α^i_n's and d^i_n's, where r is odd and n is even, and thus transform the R-states exactly as indicated as in boldface. However, a look at the $N=1$ state shows that we cannot be dealing with 35x8 physical states: the d^i_0 merely shuffle the 8 components of the ground state spinor. Thus we have at most 8x8 $=$ 64 states, but since the tensor is traceless, 8 spinor components corresponding to $\psi^{ii}_{\beta'}$ are absent, and the true physical content is $\mathbf{56_s}$. This is not as unfamiliar a situation as it might first appear: consider a scalar field θ and its gradient $\partial_\mu\theta$. The latter transforms as a D-vector, but the physical degree of freedom is still just θ, which is a scalar.

Examining the physical content of the $N=2$ states, we see that $|\,\psi^{ij}_{\beta}(\alpha\alpha)\rangle$ contains 35x8 $=$ 280 states, but these do not belong to a single irrep. Among the 280 states there is a $\mathbf{56_c}$ of the form $d_0^i \mid \psi_\beta^{ij}(\alpha\alpha)\rangle$ and the rest is $\mathbf{224_c}$. The part $\mid \psi_\beta^{ij}(dd)\rangle$ represents another $\mathbf{56_c}$, because the d_0^i does not increase the number of degrees of freedom. This brings the number down from 280 to 64, and tracelessness in ij takes off another 8. Thus the representation content of $N = 2$ state is $\mathbf{224_c} + \mathbf{56_c} + \mathbf{56_c}$. Counting the α and d modes separately and adding the numbers may seem surprising, but it is again not new: in the Higgs phenomenon, the massless photon field A_μ and $\partial_\mu\theta$ are added to form the massive vector field B_μ. Although all three formally transform as 4-vectors, the final B_μ has 2+1 = 3 degrees of freedom.

Having written down a few levels of the spectra in both sectors, one may wonder naturally that whether there are any remnants of space-time supersymmetry in the spectrum. Actually the only remnant is seen at the mass level $\alpha'_{ND}M^2 = 5/4$ where the state of the NS-sector is $\mathbf{8_v}$ of the $SO(8)$ and that of the R-sector is $\mathbf{8_s}$ of the $SO(8)$. The remaining states do not form a supermultiplet since their degrees of freedom do not match as far as the other listed levels are concerned.

3.6.5. Density of States

Upper bounds for the number of bosonic and fermionic excitations at a given level can be calculated by slightly modifying standard techniques. The most important difference between the standard light-cone calculation and the one here is that our calculation gives the number of states *before* Virasoro and super-Virasoro constraints are applied. Thus the total number of pre-constraint string states at each level is given by $tr\omega^{\mathbb{N}}$. Now let us calculate this trace for the two sectors separately.

Neveu-Schwarz Sector: Counting only odd a and b modes leads to the result

$$\begin{aligned} g_{NS}(\omega) &= tr\omega^{\mathbb{N}} \\ &= tr\omega^{\mathbb{N}_a+\mathbb{N}_b} \\ &= \{\prod_{r=1,3,5\ldots} \frac{1+\omega^r}{1-\omega^r}\}^{(D-1)}. \end{aligned} \tag{3.102}$$

Note that the GSO projection operator found in the usual light-cone formula is absent here.

Ramond Sector: Even moded d and odd moded α oscillators give

$$\begin{aligned} g_R(\omega) &= \lambda \times tr\omega^{\mathbb{N}} \\ &= \lambda \times tr\omega^{\mathbb{N}_\alpha + \mathbb{N}_d} \\ &= \lambda \times \omega^{\frac{D-1}{8}} \{ \prod_{r=1,3,5\ldots} \frac{1+\omega^{r+1}}{1-\omega^r} \}^{(D-1)}. \end{aligned} \tag{3.103}$$

The factor λ represents the degeneracy of the spinor ground state. Prior to the f_0 constraint, $\lambda = 16$. $\omega^{\frac{D-1}{8}}$ is due to the fact that the ground state of this sector has $\frac{D-1}{8}$ unit higher mass than that of the Neveu-Schwarz sector.

For $D = 9$ the expressions have the following expansions:

$$\begin{aligned} g_{NS}(\omega) &= 1 + 16\omega + 128\omega^2 + 704\omega^3 + 3072\omega^4 + 11488\omega^5 + 38400\omega^6 + \ldots, \\ g_R(\omega) &= 16\omega + 128\omega^2 + 704\omega^3 + 3072\omega^4 + 11488\omega^5 + 38400\omega^6 + \ldots. \end{aligned}$$

We have unfortunately not been able to find an analog of Jacobi's "abstruse identity" showing $g_{NS}(\omega) - 1 = g_R(\omega)$ to all orders. Assuming the multiplicities continue to be equal, the surprising conclusion seems to be that the halving of world sheet supersymmetry shows its effects on the space-time spectrum not directly, but through the (super)constraints.

3.6.6. Density of States and Microscopic Origin of Black Hole Entropy

In Sections (2.6) and (3.6.5) we have calculated the asymptotic level density for open bosonic strings and density of states for RNS version of it with one end on a D0-brane. This kind of state counting is a very important ingredient in finding black hole entropy microscopically and in this section this aspect of density of states will be considered.

First we should clarify why the microscopic origin of black hole entropy is important. In the mid 1970's, Hawking and Bekenstein proposed that black holes are thermal systems that obey the laws of thermodynamics [29, 30, 31, 32, 33, 34, 35]. In fact black holes have an entropy proportional to the area of the event horizon. Hawking wrote down a formula for what this entropy should be. It is a famous formula that says the entropy is one-quarter the area of the event horizon of the black hole. To reach this conclusion, he used a macroscopic thermodynamic argument. In physical systems the thermodynamic entropy has a statistical interpretation in terms of counting microscopic configurations with the same macroscopic properties and if what he was saying is correct, there should also be some microscopic explanation for this entropy. For black holes this has been a long-standing puzzle and in the subsequent 20 years nobody could figure out what are the degrees of freedom that the Bekenstein-Hawking entropy is counting. Now D-branes provide the necessary machinery to solve this problem.

Actually, before going to the state counting associated with D-branes, we should say that these D-branes are identified with some specific solutions, called p-branes, of corresponding supergravity theories. In fact, they are complementary descriptions of the same object: p-brane solutions are non-perturbative configurations of supergravity theories and D-branes are their description in string theory. One can also construct some specific configurations of p-branes called intersections and these intersecting configurations are important in constructing black hole solutions. Although charged black hole solutions can easily be constructed in (super)gravity theories, the important point for getting them from intersecting brane solutions in ten/eleven dimensions is that one can automatically have a string/M-theory interpretation. In particular, the interpretation of a black hole as a particular configuration of branes allows one to calculate the entropy of the black hole by considering the number of massless degrees of freedom in string theory. This can then be compared to the area of the lower dimensional black hole horizon to provide a microscopic derivation of the Bekenstein-Hawking entropy formula [36]. This is a large subject which has been reviewed in detail in [37, 38, 39, 40]. Here only one of the simplest cases will be considered shortly and we will not mention the form of such p-brane solutions of supergravity theories in this thesis.

One of the first interesting calculations to be done for a black hole derived from ten-dimensional p-brane solution is the entropy calculation carried out by counting the number of string states that are present for D1-D5 system and then comparing it to the geometric entropy of the corresponding soliton solution. D1-D5 system can be summarized as follows: N_5 D5-branes wrap along the coordinates x_1, x_2, x_3, x_4, x_5 and N_1 D1-branes wrap along x_1 direction. The intersecting brane configuration solution can be written for this system but since we are just interested in state counting issue, we have not written it here. This configuration is first compactified on four-dimensional torus T^4 along the directions x_2, x_3, x_4, x_5 which are transverse to D1-brane and then along x_1 direction on a circle of radius R. The quantized momentum along this direction is then in the form $P = N_w/R$ where N_w is another integer like N_1 and N_5. The compactified configuration is an extremal charged black hole in $D = 5$ with the entropy

$$S = 2\pi \sqrt{N_1 \, N_5 \, N_w}. \tag{3.104}$$

The momentum in the brane configuration, on the other hand, can be viewed as the momentum carried by fundamental open strings moving around the x_1 circle. The entropy of the system is determined by the number of ways this momentum can be partitioned among an arbitrary number of fundamental strings. It turns out that the important strings are those ending on both a D1-brane and D5-brane; and the microscopic counting of the degeneracy of D-brane states in string theory also yields the same entropy value Equation (3.104).

It is possible to construct other black hole solutions from intersecting branes. The method is simply to construct an intersecting brane configuration and toroidally compactify on the relative transverse space. It is really an interesting attempt, for example, to compactify some recently found localized solutions for D2-D6 system [41], and D3-D5, D4-D4 systems [42] to lower dimensional black holes and to find their geometric entropies and to compare the results with the microscopic state counting of the mentioned type in the previous paragraph. Of course, in compactifying the intersecting brane configuration, one must be careful in getting the solutions with nonzero horizon area.

4. CONCLUSIONS

Our main concern in this thesis has been twofold: The first was to see whether an RNS string with one fixed end would provide qualitative phenomenological hints and distinctive signatures about the spectrum of hadrons with one very heavy quark. The second concern was the more formal one of working out the novel consequences of the unusual mixed Dirichlet-Neumann boundary conditions and seeing whether the resulting system can be quantized in a consistent way. Just as in the original attempts to apply string theory to hadrons, these two aims are partially in conflict, most notably in the dimension of space-time.

Ignoring this conflict as was customarily done in the old string-based hadrodynamics, we can summarize the predictions of our model as follows: (i) There is no space-time supersymmetry except for the fact that meson and baryon Regge trajectories have the same slope, leading to equal meson and baryon masses for the higher states. (ii) The Regge slope is twice that of the one observed for lighter hadrons.

An interesting point here is that if we consider an interaction where the free ends of two of our strings join while the heavy quark ends are kept fixed, we are led to a model in which $X^{\mu} \sim \Sigma \alpha_n^{\mu} e^{-in\tau} sin(n\sigma)$ with σ in the range $[0, \pi]$. For such strings with both ends fixed on $D0$-branes, all α_n modes and, consequently, all L_n are allowed. In this sector, there will be "daughter excitations", with the same spin but equally spaced masses but not in the sense there are any leading linear Regge trajectories above them (the latter cannot be present because the balancing of centrifugal force against tension in a rigid rotation mode that defines leading trajectories is imposible when both ends are fixed). This is in qualitative, and one hopes, not entirely fortuitous accord with the multiplicity of $b\bar{b}$ and $c\bar{c}$ states of spin one. In any case, the Coulomb part of the QCD potential is known to play an important role in the dynamics of heavy quark-antiquark systems, ensuring deviations from a mass spectrum based on the purely string-based picture.

Turning to string-theoretic issues, we start with the question of whether using the more conventional σ range $[0, \pi]$ would have made a physical difference. One may anticipate that the simultaneous doubling of the σ range and the halving of the mode index n will result in an equivalent physical system, and this is indeed what happens. The mixed boundary conditions now produce half-integral a and b modes and integral Ramond d modes, but the allowed physical spectrum is exactly as the one above except for an overall scaling. In particular, the vanishing of the NS vacuum energy and the preference for a $(8,1)$ space-time and $SO(8)$ symmetry remain. We prefer working with odd a and b and even d modes because this leads to the disappearence of odd L_n's. We have broken Poincaré invariance by fixing one end of the string at a special point in space; this is consistent with discarding the operators L_1 and L_{-1} which involve the generator of space translations.

The GSO projection turns out to be inapplicable to the mixed boundary condition superstring. This is already suggested by the fact that low-lying states do not exhibit space-time supersymmetry beyond matching masses and Regge slopes in the two sectors, but there are more basic manifestations of the incompatibility. For example, our final physical states are not homogeneous in the number of space-time fermions. Furthermore, in the R-sector we already have the remnant Dirac equation operators $\Gamma^0 \pm I$, but not in the role they play in the GSO projection. Since the mode structure here is different from the standard RNS string (for example, our b operators add an integer rather than half an integer to the squared mass), one could not in any case have expected the GSO procedure to work in the usual way.

A final conclusion is on the dimension of space-time: A natural way of obtaining the value of space-time dimension in bosonic string theory or in superstring theory is to get a condition for cancellation of conformal anomaly. For the bosonic ND string, this method seems to fail because of the immobile end of the string; but problems associated with the conformal anomaly vanish. On the other hand, in Chapter 3 we have taken $D = 9$; but one should note that this choice was not been dictated by the anomaly cancellation requirement but the minimal supersymmetry breaking option. Of course, this this does not mean that D must be 9. The good point about this choice is that it leads to $SO(8)$ group and this group plays a crucial role in obtaining the space-time spectrum.

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