# ALGEBRO-GEOMETRIC SOLUTIONS OF THE KADOMTSEV–PETVIASHVILI EQUATION

by

Fatma Çiçek

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# ABSTRACT


## ALGEBRO-GEOMETRIC SOLUTIONS OF THE KADOMTSEV–PETVIASHVILI EQUATION

I. M. Krichever suggested a method to solve nonlinear partial differential equations in the form of the Zaharov-Shabat Equation $[L - \partial_y, A - \partial_t] = 0$ where $L$ and $A$ are differential operators including derivatives only with respect to the $x$ variable in 1976 [1]. The method uses so called Baker-Akhiezer functions on Riemann surfaces and provides periodic and conditionally periodic solutions to such nonlinear equations that can be expressed in terms of the so called Riemann $\theta$-function, a $\theta$-function defined on some $n$ dimensional complex space where the Riemann matrix of the function corresponds to a Riemann surface. In this thesis, we will mainly consider the Kadomtsev-Petviashvili equation (or KP equation) $\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left[u_t - \frac{1}{4}(6uu_x + u_{xxx})\right]$ which is an example of the Zaharov-Shabat equation. Following the expository paper of B. A. Dubrovin [2], we will present the construction of such solutions to the KP equation given as $u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + yV + tW + z_0) + c$. It was observed that this construction allows one to investigate the sufficient conditions on arbitrary vectors $U$, $V$, and $W$ that make the above function $u(x, y, t)$ a solution to the KP equation. We explain the answer to this question for Riemann surfaces of small genera, and mention the result for more general Riemann surfaces which are both given in [2].

# ÖZET

## KADOMTSEV-PETVIASHVILI DENKLEMİNİN CEBİRSEL-GEOMETRİK ÇÖZÜMLERİ

Zaharov-Shabat denklemi, $L$ ve $A$ sadece $x$ değişkenine göre türev içeren diferansiyel operatörler olmak üzere $[L-\partial_y, A-\partial_t]=0$ şeklindeki nonlineer kısmi diferansiyel denklemlerdir. 1976 yılında, Krichever bu formdaki nonlineer kısmi diferansiyel denklemlerin çözülmesi için yeni bir metot önerdi [1]. Bu metot Riemann yüzeylerinde tanımlanan Baker-Akhiezer fonksiyonlarını kullanıyordu ve bu tür denklemlere periyodik veya şartlı periyodik çözümler sağlıyordu. Bu çözümler, Riemann teta fonksiyonu denilen $n$ boyutlu kompleks uzayda tanımlanmış ve $n$ cinsli Riemann yüzeylerine karşılık gelen teta fonksiyonları cinsinden ifade edilebilen fonksiyonlardı. Bu tezde asıl olarak, Zaharov-Shabat denkleminin bir örneği olan Kadomtsev-Petviashvili denklemi (veya KP denklemi) $\frac{3}{4}u_{yy}=\frac{\partial}{\partial x}\left[u_t-\frac{1}{4}(6uu_x+u_{xxx})\right]$ incelenecektir. KP denkleminin $u(x,y,t)=2\frac{\partial^2}{\partial x^2}\log\theta(xU+yV+tW+z_0)+c$ formundaki çözümlerinin inşası verilecektir. Bu kurgu, KP denkleminin bu şekilde ifade edilebilen diğer tüm çözümlerini bulmak için $U$, $V$ ve $W$ vektörlerinin sağlaması gereken yeterli koşulların incelenmesi sorusunu getirmiştir. Bu sorunun, cinsi 1, 2 veya 3 olan Riemann yüzeyleri için cevabını açıklayacak ve genel Riemann yüzeyleri için elde edilen sonuçtan [2] bahsedeceğiz. Çözümlerin inşasını ve bundan elde edilebilen sonuçları B. A. Dubrovin'in makalesinden [2] çalışacağız.

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# LIST OF SYMBOLS

| | |
|---|---|
| $A(P)$ | The Abel map at a point $P$ on a Riemann surface |
| $\det M$ | The determinant of a matrix $M$ |
| exp | The exponential function |
| $\mathrm{Im}\ z$ | The imaginary part of a complex number $z$ |
| $\mathrm{Im}\ M$ | The imaginary part of a complex matrix $M$ |
| $\mathrm{rank}\, M$ | The rank of a matrix $M$ |
| $\mathrm{Re}\ z$ | The real part of a complex number $z$ |
| $\mathrm{Re}\ M$ | The real part of a complex matrix $M$ |
| $\mathrm{supp}\, f$ | The support of a function $f$ |
| $\langle x, y\rangle$ | The Euclidean scalar product of complex vectors $x$ and $y$ |
| $u_t$ | Partial derivative of $u(t, x)$ with respect to $t$ |
| $u_{x_j}$ | Partial derivative of $u(t, x)$ with respect to ith spatial variable $x_i$ |
| $\mathbb{Z}^+$ | The set of positive integers |
| $\mathbb{C}$ | The set of complex numbers |
| $\mathbb{C}^n$ | The set of $n$ dimensional complex vectors |
| $\mathbb{P}^n$ | The real projective space of dimension $n$ |
| $\mathbb{CP}^n$ | The complex projective space of dimension $n$ |
| $\frac{1}{2}(\mathbb{Z}_2)^n$ | The set of $n$ dimensional vectors with all components 0 or $\frac{1}{2}$ |
| $\delta_{kl}$ | Kronecker delta function |
| $\theta(z)$ | Riemann theta function |
| $\theta[\alpha, \beta](z)$ | Riemann theta function with characteristics $\alpha$ and $\beta$ |
| $\Gamma'$ | The canonical curve of a Riemann surface $\Gamma$ |
| $H_1(\Gamma)$ | First homology group of a Riemann surface $\Gamma$ |
| $J(\Gamma)$ | The Jacobi variety of a Riemann surface $\Gamma$ |
| $\mathcal{M}(\Gamma)$ | The space of meromorphic functions on a Riemann surface $\Gamma$ |
| $\mathcal{M}^1(\Gamma)$ | The space of meromorphic differentials on a Riemann surface $\Gamma$ |

# LIST OF ACRONYMS/ABBREVIATIONS

| | |
|---|---|
| KdV Equation | Korteweg-de Vries Equation |
| KP Equation | Kadomtsev-Petviashvili Equation |

# 1. INTRODUCTION

The KdV (Korteweg-de Vries) equation is the most celebrated nonlinear equation exhibiting a remarkable behaviour. This is, it describes nonlinear water waves that retain their shapes for a noticably long time. The KdV equation was later solved by the inverse scattering transform in 1967 [3]. It is the foremost, by far the most well-known (completely) integrable equation in $1+1$ dimensions (1 space, 1 time).

The KP equation is an integrable generalization of the KdV equation to $2+1$ dimensions (2 space, 1 time). It is an important equation because it is one of the possible generalizations of the KdV equation and it describes many physical phenomena such as evolution of water waves, and propagation of waves in ferromagnetic media. As the KdV equation, the KP equation was also solved by inverse scattering method [4] in 1974, and then, in 1976, Krichever proposed a new method to find a family of solutions to the Zaharov-Shabat equation ( [1] and [5] ). Inverse scattering method provided rapidly decreasing (in some sense) solutions to the KP equation whereas Krichever's method found conditionally periodic solutions to the KP equation.

The aim of this thesis is to present the algebraic and geometric method developed by Krichever, establishing a connection between the theory of nonlinear partial differential equations and algebraic geometry. The method is eligible to develop such a relation because the constructed solutions are functions of Riemann theta functions corresponding to Riemann surfaces, hence they contain information about the surface. Some basic knowledge on Riemann surfaces and on differential equations is the necessary background.

The second chapter is on fundamental concepts, definitions and theorems about theta functions, Jacobi varieties of Riemann surfaces, and divisor functions defined on Riemann surfaces. The results presented are classical and many of the proofs are skipped. In the third chapter, we explain the construction of solutions to the KP equation based on an article of Dubrovin [2], and Baker-Akhiezer functions that are meromorphic functions on Riemann surfaces with essential singularity of the form $\exp(q(\omega))$ for some polynomial $q(\omega)$ will work as a basic instrument with the nice properties they satisfy. The constructed family of solutions will be of the form

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + yV + tW + z_0) + c$$

where $c$ is an arbitrary complex number and $z_0$ is a vector carrying the information of parameters which are nonspecial divisors on the surface.

In the last chapter, we follow the discussion by B.A. Dubrovin in [2] for the effectivization of the formulae of the constructed solutions to the KP equation. That is, a function $u(x,y,t)$ in the form above is substituted into the KP equation and the necessary conditions on the vectors $U$, $V$ and $W$ so that $u(x,y,t)$ will be a solution to the KP equation are studied. Only the cases of Riemann surfaces having small genera are explained in detail, and we mention the result by Dubrovin [2] proven for almost all Riemann surfaces.

# 2. THETA FUNCTIONS AND JACOBI VARIETIES OF RIEMANN SURFACES

In our discussion, the geometric setting will be that of Riemann surfaces which are defined as below.

**Definition 2.1.** *A* Riemann surface *is a second countable, connected, Hausdorff topological space which is a one dimensional complex manifold.*

Some well-known examples of Riemann surfaces are the complex plane $\mathbb{C}$, the complex projective line $\mathbb{CP}^1$, i.e. the set of one dimensional subspaces of $\mathbb{C}^2$ and any projective smooth plane curve, i.e. a curve that can be given by the zero locus in $\mathbb{P}^2$ of a nonsingular homogeneous polynomial of two variables.

We will denote a Riemann surface by $\Gamma$ and we assume that $\Gamma$ is compact.

## 2.1. Theta Functions

**Definition 2.2.** *A symmetric $g \times g$ matrix $B = [B_{jk}]$ with negative definite real part* $\mathrm{Re}B = [\mathrm{Re}B_{jk}]$ *is called a* Riemann matrix.

**Definition 2.3.** *A* Riemann theta function *is a function $\theta : \mathbb{C}^g \to \mathbb{C}$ defined by the series*

$$\theta(z \mid B) = \sum_{N \in \mathbb{Z}^g} \exp\left(\frac{1}{2}\langle BN, N\rangle + \langle N, z\rangle\right) \tag{2.1}$$

*where $B$ is a $g \times g$ Riemann matrix and $\langle x, y\rangle = \sum_{j=1}^{g} x_j y_j$ is the Euclidean scalar product.*

The Riemann theta function corresponding to a Riemann matrix $B$ is denoted as

$\theta(z \mid B)$. When there is no confusion about the Riemann matrix $B$, we will write $\theta(z)$ to denote the Riemann theta function $\theta(z \mid B)$.

**Proposition 2.1.** *The function $\theta(z) = \theta(z \mid B)$ corresponding to a Riemann matrix $B$ is analytic on $\mathbb{C}^g$.*

*Proof.* First we write the general term of the series as

$$\exp\left(\frac{1}{2}\langle BN, N\rangle + \langle N, z\rangle\right) = \exp\left(\frac{1}{2}\mathrm{Re}\langle BN, N\rangle + \mathrm{i}\frac{1}{2}\mathrm{Im}\langle BN, N\rangle + \langle N, z\rangle\right).$$

We ignore the term with the imaginary part to check the absolute convergence. If $\lambda < 0$ is the maximum eigenvalue of $\mathrm{Re}B$, then $\mathrm{Re}\langle BN, N\rangle \leq \lambda\langle N, N\rangle$ and this implies the uniform and absolute convergence of the series on compact subsets of $\mathbb{C}^g$, hence the analyticity on $\mathbb{C}^g$.

**Example 2.1.** *Let $g = 1$. Then for $B \in \mathbb{C}$ with $\mathrm{Re}B < 0$*

$$\theta(z) = \theta(z \mid B) = \sum_{N\in\mathbb{Z}} \exp\left(\frac{1}{2}BN^2 + Nz\right) = \sum_{N\in\mathbb{Z}} \exp\left(\pi \mathrm{i} N^2 b + 2\pi \mathrm{i} N z\right)$$

*where the last equation follows from a simple change of variable $B \mapsto \dfrac{B}{2\pi\mathrm{i}} = b$, $z \mapsto \dfrac{z}{2\pi\mathrm{i}}$. Note that this is the Jacobi theta function.*

It can be easily seen that a Riemann theta function $\theta(z) = \theta(z \mid B)$ satisfies the following translation properties:

$$\theta(z + 2\pi\mathrm{i}e_j) = \theta(z)$$
$$\theta(z + f_j) = \exp\left(\frac{-1}{2}B_{jj} - z_j\right)\theta(z)$$

where $(e_j)_k = \delta_{jk}$, $f_j = Be_j$ . We deduce that $\theta(z \mid B)$ satisfies the transformation law

$$\theta(z + 2\pi\mathrm{i}N + BM) = \exp\left(\frac{-1}{2}\langle BM, M\rangle - \langle M, z\rangle\right)\theta(z) \tag{2.2}$$

for $M, N \in \mathbb{Z}^g$.

### 2.2. Theta Functions with Characteristics

Let $\alpha, \beta \in \mathbb{R}^g$ with $0 < \alpha_j, \beta_k < 1$ for each $j$ and $k$. The theta function with characteristics $\alpha$ and $\beta$, $\theta[\alpha, \beta]$, is defined as

$$\theta[\alpha, \beta](z) = \exp\left(\frac{1}{2}\langle B\alpha, \alpha\rangle + \langle z + 2\pi \mathrm{i}\beta, \alpha\rangle\right)\theta(z + 2\pi \mathrm{i}\beta + B\alpha).$$

**Definition 2.4.** *A characteristics $[\alpha, \beta]$ with all $\alpha_j, \beta_k = 0$ or $\frac{1}{2}$ is said to be* even *if $4\langle\alpha, \beta\rangle = 0 \pmod 2$. Otherwise, it is said to be* odd.

The theta function defined in (2.1) is easily seen to be even. For theta functions with characteristics, we will see that theta functions with odd (resp. even) characteristics are odd (resp. even).

**Proposition 2.2.** *If the characteristics $[\alpha, \beta]$ is even (resp. odd), then the function $\theta[\alpha, \beta]$ is even (resp. odd).*

*Proof.* See Assertion 1.1.2 of [2].

We introduce a notation which will be useful to express the result of the following theorem. For a $g \times g$ Riemann matrix $B$, let $\hat{\theta}[\alpha, \beta](z) = \theta[\alpha, \beta](z|2B)$ and $\frac{1}{2}(\mathbb{Z}_2)^g = \{n = (n_1, \ldots, n_g) : n_j = 0, \frac{1}{2}\}$

**Theorem 2.1** (Addition Theorem of $\theta$-functions)**.** *Let $\alpha$, $\beta$, $\gamma$ and $\delta$ be vectors in $\mathbb{R}^g$. Then*

$$\begin{aligned}&\theta[\alpha, \gamma](z^1 + z^2)\theta[\beta, \delta](z^1 - z^2)\\ &\qquad = \sum_{\epsilon \in \frac{1}{2}(\mathbb{Z}_2)^g} \hat{\theta}\left[\frac{\alpha + \beta}{2} + \epsilon, \gamma + \delta\right](2z^1)\hat{\theta}\left[\frac{\alpha - \beta}{2} + \epsilon, \gamma - \delta\right](2z^2).\end{aligned}$$

*Proof.* See Theorem 1.4.1. in [2].

We will present more results about theta functions, e.g. about zeros of theta functions defined on Riemann surfaces, in the later sections.

### 2.3. Jacobi Variety of a Riemann Surface

Let $\Gamma$ be a compact Riemann surface of genus $g$.

**Definition 2.5.** *A differential $\omega$ on $\Gamma$ is said to be* holomorphic *if it can be written in the form*

$$\omega = f(z)\mathrm{d}z$$

*in a neighbourhood of any point where $f(z)$ is an analytic function in the corresponding neighbourhood and $z$ a local coordinate function on $\Gamma$.*

For the first homology group $H_1(\Gamma)$, there is a basis of cycles $a_1, a_2, \ldots, a_g; b_1, b_2, \ldots, b_g$ with the intersection indices (see page 16 of [6])

$$a_j \circ a_k = b_j \circ b_k = 0, \quad a_j \circ b_k = \delta_{jk} \quad j, k = 1, 2, \ldots, g. \tag{2.3}$$

$\Gamma$ becomes a $4g$-gon $\widetilde{\Gamma}$ if it is cut along these cycles. Whenever we take a basis for $H_1(\Gamma)$, we mean a basis satisfying the conditions (2.3). Choose such a basis of cycles $a_1, a_2, \ldots, a_g; b_1, b_2, \ldots, b_g$ for $H_1(\Gamma)$. Let $\omega$ be a closed differential on $\Gamma$, i.e $\mathrm{d}\omega = 0$, and let

$$A_j = \oint_{a_j} \omega, \quad B_j = \oint_{b_j} \omega$$

be its $a$-periods and $b$-periods, respectively, over these cycles. Fix a point $P_0 \in \Gamma$ that does not lie on the basis cycles $a_i, b_j$, and define a function $f : \widetilde{\Gamma} \to \mathbb{C}$ as

$$f(P) = \int_{P_0}^{P} \omega \quad \text{for} \quad P \in \widetilde{\Gamma}.$$

To see that this map is well-defined, note that any two paths from $P_0$ to a point $P$ differ by a closed path which can be written as a linear combination of the basis cycles in $H_1(\Gamma)$. On $\widetilde{\Gamma}$, such a curve is homologically zero.

**Lemma 2.1.** *Let $\omega$ and $\omega'$ be two closed differentials on $\Gamma$ with periods $A_1, A_2, \ldots, A_g$, $B_1, B_2, \ldots, B_g$; $A_1', A_2', \ldots, A_g'$, $B_1', B_2', \ldots, B_g'$, respectively. Then*

$$\iint_{\Gamma} \omega \wedge \omega' = \oint_{\partial\widetilde{\Gamma}} f\omega' = \sum_{j=1}^{g} (A_j B_j{}' - A_j{}' B_j). \tag{2.4}$$

*Proof.* See Lemma 2.1.1 in [2].

**Corollary 2.1.** *Let $\omega$ be a nonzero holomorphic differential on $\Gamma$. Then the periods $A_1, A_2, \ldots, A_g$; $B_1, B_2, \ldots, B_g$ of $\omega$ satisfies*

$$\mathrm{Im} \sum_{j=1}^{g} A_j B_j < 0. \tag{2.5}$$

*Proof.* Apply (2.4) to $\omega$ and its conjugate $\overline{\omega}$. Note that if $\omega$ is holomorphic, then $\omega$ is closed and this implies its conjugate $\overline{\omega}$ is also closed.

**Corollary 2.2.** *A holomorphic differential $\omega$ is completely determined by its $a$-periods $A_1, \ldots, A_g$ or $b$-periods $B_1, \ldots, B_g$.*

*Proof.* Suppose that two holomorphic differentials $\omega$ and $\tilde{\omega}$ have the same $a$-periods (respectively, $b$-periods). Then the differential $\omega - \tilde{\omega}$ is holomorphic and all its $a$-periods (resp. $b$-periods) are 0. Equation (2.5) implies $\omega - \tilde{\omega}$ is the zero differential, hence $\omega$ and $\tilde{\omega}$ are the same.

For any Riemann surface $\Gamma$ of genus $g$, one can construct $g$ many linearly independent holomorphic differentials (see Section 10.1 in [7]). Corollary 2.2 implies that the space of holomorphic differentials on $\Gamma$ has dimension $g$. Hence, we can choose a basis $\omega_1, \dots, \omega_g$ for this space satisfying

$$\oint_{a_k} \omega_j = 2\pi \mathrm{i} \delta_{jk} \quad \text{for} \quad j, k = 1, \dots, g. \tag{2.6}$$

Define a $g \times g$ matrix $B$

$$B_{jk} = \oint_{b_k} \omega_j \quad \text{for} \quad j, k = 1, \dots, g. \tag{2.7}$$

**Theorem 2.2.** *$B_{jk}$ is a Riemann matrix.*

*Proof.* The equation (2.4) applied to $\omega_j$ and $\omega_k$ gives

$$0 = \sum_{l=1}^{g} A_l B_l' - A_l' B_l = \sum_{l=1}^{g} 2\pi \mathrm{i} \delta_{jl} B_{kl} - 2\pi \mathrm{i} \delta_{kl} B_{jl} = B_{kj} - B_{jk}$$

and this proves $B$ is symmetric. Consider a nonzero holomorphic differential $\omega = \sum_{i=1}^{g} c_i \omega_i$ where $c_1, \dots, c_g \in \mathbb{R}$. Then by (2.6) and (2.7)

$$A_j = 2\pi \mathrm{i} c_j, \quad B_j = \sum_{k=1}^{g} c_k B_{jk} \quad \text{for each} \quad j = 1, \dots, g.$$

By Corollary 2.1,

$$2\pi \sum_{j,k=1}^{g} c_j c_k \mathrm{Re} B_{jk} = \mathrm{Im} \sum_{j=1}^{g} A_j \overline{B_j} < 0.$$

and so $\mathrm{Re}B$ is a negative definite matrix.

**Proposition 2.3.** *Let $B = [B_{jk}]$ denote the $g \times g$ Riemann matrix associated with a Riemann surface $\Gamma$ of genus $g$. Let $e_1, \dots, e_g$ be the standard basis for $\mathbb{C}^g$ over $\mathbb{C}$ given* *by* $(e_j)_k = \delta_{jk}$*, and correspondingly define the vectors* $f_1, \ldots, f_g$ *as* $f_j = B(e_j)$ *for each* $j = 1, \ldots, g$*. Then the vectors* $e_1, \ldots, e_g, f_1, \ldots, f_g$ *are linearly independent over* $\mathbb{R}$*.*

*Proof.* For some real numbers $c_1, \ldots, c_{2g}$, assume

$$2\pi \mathrm{i} \sum_{j=1}^{g} c_j e_j + \sum_{k=1}^{g} c_{g+k} f_k = 0. \tag{2.8}$$

Taking the real part of both sides of this equation, we get $\mathrm{Re}B\left(\sum_{k=1}^{g} c_{g+k} e_k\right) = 0$. But $\mathrm{Re}B$ is negative definite, so nonsingular. Hence the vector $\sum_{k=1}^{g} c_{g+k} e_k$ is the zero vector and this implies all the coefficients $c_{g+1}, \ldots, c_{2g}$ are zero by the linear independence of the vectors $e_1, \ldots, e_g$ over $\mathbb{R}$. This implies $c_1 = c_2 = \cdots = c_g = 0$ again by linear independence of the vectors $e_1, \ldots, e_g$. Hence, the vectors $e_1, \ldots, e_g, f_1, \ldots, f_g$ are linearly independent over the reals.

**Definition 2.6.** *The quotient space of* $\mathbb{C}^g$ *with the lattice* $\Lambda$ *generated by* $2\pi \mathrm{i} N$ *and* $BM$ *with* $M, N \in \mathbb{Z}^g$ *or the* $2g$ *dimensional torus constructed from the matrix of periods of holomorphic differentials on a Riemann surface* $\Gamma$ *is called* the Jacobi variety of $\Gamma$*. It is denoted as* $J(\Gamma) = T^{2g} = \mathbb{C}^g/\Lambda$*.*

The definition of the Jacobi variety of a Riemann surface does not depend on the choice of the basis for the first homology group of the Riemann surface (See page 28 in [2]).

### 2.4. Abel Map and Abel's Theorem

Given a Riemann surface $\Gamma$ of genus $g$, let $J(\Gamma)$ be its associated Jacobi variety. For a fixed point $P_0$ on $\Gamma$, the Abel mapping $A : \Gamma \to J(\Gamma)$ is defined as follows

$$\begin{aligned} &A : P \mapsto (A_1(P), \ldots, A_g(P)) \\ &A_j(P) = \oint_{P_0}^{P} \omega_j \quad j = 1, \ldots, g. \end{aligned} \tag{2.9}$$

We now demonstrate that the integrals in (2.9) are path independent and hence the Abel map is well-defined. Take $P \in \Gamma$ and assume $\gamma_1$ and $\gamma_2$ are two curves from $P_0$ to $P$, and let $A(P)$ and $\tilde{A}(P)$ denote the corresponding values of the Abel map, respectively. These curves differ by a closed curve, say it is $\gamma$. Let $a_1, \ldots, a_g; b_1, \ldots, b_g$ be the basis for $H_1(\Gamma)$. We can write

$$\gamma = \sum_{k=1}^{g} m_k a_k + \sum_{l=1}^{g} n_l b_l$$

Using the linearity of the integral

$$A_j(P) - \tilde{A}_j(P) = \sum_{k=1}^{g} m_k \oint_{a_k} \omega_j + \sum_{l=1}^{g} n_l \oint_{b_l} \omega_j \equiv 0 \quad \text{in} \quad J(\Gamma).$$

**Remark 2.1.** *The Abel map is defined from any Riemann surface of genus $g$ to its Jacobi variety, i.e. a quotient of $\mathbb{C}^g$. The theta function in* (2.1) *has $\mathbb{C}^g$ as its domain. By composition with the Abel map, we can define a theta function on any Riemann surface.*

Now we note the following basic result about functions on Riemann surfaces which will be used in the statement of the so called Abel's Theorem.

**Theorem 2.3.** *Let $f$ be a nonconstant meromorphic function on a compact Riemann surface $\Gamma$. Then number of zeros of $f$ is equal to the number of poles of $f$.*

*Proof.* See Proposition 4.12 in [8].

Abel's Theorem is an important result which gives a necessary and sufficient condition on a set of points to be the set of zeros and poles of a meromorphic function on a Riemann surface.

**Theorem 2.4** (**Abel's Theorem**). *Let* $\Gamma$ *be a compact Riemann surface, and let* $\Lambda$ *be the lattice defining the Jacobi variety of* $\Gamma$*. Suppose that* $P_1, \ldots, P_n; Q_1, \ldots, Q_n$ *are points on* $\Gamma$*. Those points are zeros and poles of a meromorphic function on* $\Gamma$ *if and only if*

$$\sum_{j=1}^{n} [A(P_j) - A(Q_j)] \equiv 0 \pmod{\Lambda}.$$

*Proof.* See page 30 of [2].

## 2.5. Divisors on a Riemann Surface

**Definition 2.7.** *Let* $f$ *be a complex-valued function on a Riemann surface* $\Gamma$*. Define the support of* $f$*, denoted by* $\operatorname{supp} f$*, as the set* $\operatorname{supp} f = \{p \in \Gamma : f(p) \neq 0\}$*.*

**Definition 2.8.** *A* divisor $D$ *on a Riemann surface* $\Gamma$ *is a function* $D : \Gamma \to \mathbb{C}$ *such that* $\operatorname{supp} D$ *is a discrete subset of* $\Gamma$*.*

We assume that the Riemann surface $\Gamma$ is compact, in that case any open cover of $\Gamma$ has a finite open subcover. Since $\operatorname{supp} D$ is a discrete subset of $\Gamma$, it is locally finite, and hence finite when $\Gamma$ is compact. We denote a divisor $D$ with a finite sum

$$D = \sum_{j=1}^{n} D(p_j)p_j = \sum_{j=1}^{n} m_j p_j.$$

**Definition 2.9.** *The* degree of a divisor $D = \sum_{j=1}^{n} m_j p_j$ *is defined as*

$$\deg D = \sum_{j=1}^{n} m_j.$$

**Definition 2.10.** *A divisor $D$ is said to be* principal *if there exists a meromorphic function $f : \Gamma \to \mathbb{C}$ with zeros at $P_1, \ldots, P_n$ of multiplicities $p_1, \ldots, p_n$ and with poles at $Q_1, \ldots, Q_n$ of multiplicities $q_1, \ldots, q_n$ such that*

$$D = p_1 P_1 + \cdots + p_n P_n - q_1 Q_1 - \cdots - q_n Q_n$$

*and such a divisor $D$ is denoted by $(f)$.*

Let $\mathfrak{D}$ be the set of all divisors on a compact Riemann surface $\Gamma$. Let $D$ and $\tilde{D}$ be elements of $\mathfrak{D}$ with

$$D = \sum_{j=1}^{n} m_j P_j \quad \text{and} \quad \tilde{D} = \sum_{j=1}^{\tilde{n}} n_j \tilde{P}_j.$$

The sum divisor of $D$ and $\tilde{D}$ is denoted by $D + \tilde{D}$ which is defined as

$$D + \tilde{D} = \sum_{j=1}^{n} m_j P_j + \sum_{j=1}^{\tilde{n}} n_j \tilde{P}_j.$$

$\mathfrak{D}$ has an abelian group structure with this addition operation. Also, this addition operation can be used to define an equivalence relation on $\mathfrak{D}$.

**Definition 2.11.** *Two divisors $D$ and $\tilde{D}$ are said to be* equivalent *if the divisor $D - \tilde{D}$ is principal.*

**Definition 2.12.** *Let $\Gamma$ be a Riemann surface. A differential $\Omega$ on $\Gamma$ is said to be* meromorphic *if for any point $P \in \Gamma$, in some neighbourhood of $P$, it can be written in* *the form*

$$\Omega = f(z)\mathrm{d}z$$

*where $f(z)$ is a meromorphic function at $P$.*

We denote the space of meromorphic differentials on $\Gamma$ by $\mathcal{M}^1(\Gamma)$.

**Definition 2.13.** *A divisor corresponding to a meromorphic differential $\Omega$ is said to be a* canonical divisor *and it is denoted by $(\Omega)$.*

**Example 2.2.** *All canonical divisors are equivalent with this equivalence relation defined on $\mathfrak{D}$. Let $\mathcal{C}$ denote the class of canonical divisors.*

We can extend the Abel mapping to $A : \mathfrak{D} \to \mathbb{C}$ linearly and using this extension, we rewrite Abel's Theorem.

$$D = \sum_{j=1}^{n} m_j P_j \mapsto A(D) = \sum_{j=1}^{n} m_j A(P_j)$$

**Theorem 2.5** (Abel's Theorem)**.** *Two divisors $D$ and $\tilde{D}$ are equivalent if and only if*

$$\deg D = \deg \tilde{D} \quad \textit{and} \quad A(D) \equiv A(\tilde{D}) \quad \textit{in} \quad J(\Gamma).$$

**Definition 2.14.** *Let $D$ and $\tilde{D}$ be two divisors on $\Gamma$. We say $D \succeq \tilde{D}$ if $D(p) \geq \tilde{D}(p)$ for all $p \in \Gamma$.*

Clearly, this relation is a partial ordering on $\mathfrak{D}$. Let $D$ be a divisor on $\Gamma$. Let $\mathcal{M}(\Gamma)$ denote the space of meromorphic functions on a Riemann surface $\Gamma$ and define

$$L(D) = \{f \in \mathcal{M}(\Gamma) : (f) \succeq -D\}.$$

$L(D)$ is a vector space over $\mathbb{C}$ and we define the number $\mathrm{l}(D)$ to be its dimension over $\mathbb{C}$. It can be shown easily that the spaces $L(D)$ and $L(\tilde{D})$ are isomorphic if $D$ and $\tilde{D}$ are equivalent. For example, if $D$ and $\tilde{D}$ are canonical divisors, then $\mathrm{l}(D) = \mathrm{l}(\tilde{D})$.

Similarly we define the space $I(D)$ as

$$I(D) = \{\Omega \in \mathcal{M}^{(1)}(\Gamma) : (\Omega) \succeq -D\} \quad \text{and} \quad i(D) = \dim_{\mathbb{C}} I(D).$$

**Theorem 2.6** (Riemann-Roch Theorem)**.** *Let $D$ be a divisor on a compact Riemann surface of genus $g$. Then*

$$\mathrm{l}(D) = \deg D - g + 1 + i(D).$$

For a proof of Riemann-Roch Theorem, see Theorem 3.11 in [8].

**Definition 2.15.** *A divisor $D$ is called* nonspecial *if $i(D) = 0$. Otherwise it is called* special.

Finally, we will state a theorem that will be needed for proving the well-definedness of Baker-Akhiezer functions in the next chapter.

Let $\Gamma$ be a Riemann surface of genus $g$, and let $D = P_1 + \cdots + P_g$ be a nonspecial divisor on $\Gamma$. Let $P_0$ denote the initial point of the Abel map, let $a_1, \ldots, a_g$ denote the $a$-cycles of $H_1(\Gamma)$. Let $\omega_1, \ldots, \omega_g$ denote the basis holomorphic differentials of $\Gamma$ and let $B$ denote the Riemann matrix associated with $\Gamma$.

**Theorem 2.7.** *With the above notation, the function $F : \Gamma \to \mathbb{C}^g$ defined as $F(P) = \theta(A(P) - A(D) - K)$ has exactly $g$ zeros $P = P_1, \ldots, P = P_g$ where the vector $K$ is the Riemann constant vector given by*

$$K_j = \frac{2\pi \mathrm{i} + B_{jj}}{2} - \frac{1}{2\pi \mathrm{i}} \sum_{k \neq j} \oint_{a_k} \omega_k(P) \int_{P_0}^{P} \omega_j \qquad j = 1, \ldots, g. \tag{2.10}$$

*Proof.* See page 38 of [2].

# 3. BAKER-AKHIEZER FUNCTIONS AND NONLINEAR EQUATIONS

## 3.1. Baker-Akhiezer Functions

In 1860's, Clebsch and Gordan worked on generalizations of exponential functions on a Riemann surface. It was already known that a holomorphic function on a compact Riemann surface must be constant. Hence, a function on a compact Riemann surface having exponential growth should possess an essential singularity of the exponential type. Baker-Akhiezer functions were defined to be functions with essential singularity of exponential type and in addition to this condition, they were defined to have possible poles at points which are in the support of a nonspecial divisor. In 1920's, Burchnall and Chaundy [9] and then Baker [10] investigated the relations of the Baker-Akhiezer type functions with commuting differential operators. Akhiezer considered an example of such functions for the hyperelliptic case $k^2 = P(z)$ where $P$ is a polynomial of odd degree [11], and he also considered their relations with the spectral theory. In our case, Baker-Akhiezer functions will be our basic tool to find a family of solutions to the KP equation, and the method used can be generalized to apply to some other nonlinear differential equations of higher degree.

**Theorem 3.1.** *Let $\Gamma$ be a Riemann surface of genus $g$, $D = \sum_{i=1}^{g} P_i$ a nonspecial divisor on $\Gamma$, $Q$ an arbitrary fixed point on $\Gamma$ not in the support of $D$. Suppose $z = z(P)$ is a local parameter centered at $Q$, and put $\omega(P) = \dfrac{1}{z(P)}$. Let $q(\omega)$ be a nonconstant polynomial. Then, up to a constant, there exists a unique function $\Phi : \Gamma \to \mathbb{C}$ satisfying the following properties:*

*(i) $\Phi$ is meromorphic everywhere on $\Gamma$ except $Q$ and has poles possibly and only at $P_1, \ldots, P_g$.*

*(ii) The function $\Phi(P)\exp(-q(\omega(P))$ is analytic in a neighbourhood of $Q$.*

*Proof.* First, we will prove the existence. Let $\Omega$ be a meromorphic differential on $\Gamma$

with principal part of the form $\mathrm{d}q(\omega)$ at $Q$. We assume poles of $\Omega$ do not lie on the basis cycles $a_1, \ldots, a_g; b_1, \ldots, b_g$ of $H_1(\Gamma)$. By (2.6), we can assume that the $a$-periods of $\Omega$ vanish by adding a holomorphic differential to $\Omega$ if necessary. Define a vector $U$ as the following:

$$U_j = \oint_{b_j} \Omega \quad j = 1, \ldots, g. \tag{3.1}$$

Let $P_0$ be a point of $\Gamma$ other than $Q$ and define a function $\Phi$ on $\Gamma$ using the Abel map $A(P) = \left(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_g\right)$ with $P_0$ as the initial point of integration,

$$\Phi(P) = \exp\left(\oint_{P_0}^{P} \Omega\right) \frac{\theta(A(P) - A(D) + U - K)}{\theta(A(P) - A(D) - K)} \tag{3.2}$$

where the path of integration in the integral $\oint_{P_0}^{P} \Omega$ and in the Abel mapping are the same. Note that the $\theta$-function used is the Riemann $\theta$-function corresponding to the Riemann matrix $B$ that corresponds to the surface $\Gamma$. The vector $K$ is the vector of Riemann constants given in the equation (2.10) and it depends only on the initial point $P_0$ of the Abel map $A$.

We need to check that $\Phi$ is well-defined on $\Gamma$ and for this, we will show that the value of $\Phi(P)$ at any $P \in \Gamma$ is independent of the chosen path between $P_0$ and $P$. Any path of integration from $P_0$ to P differs from another by a closed path, say it is $\gamma$. We can express $\gamma$ as a sum of the basis cycles as

$$\gamma = \sum_{j=1}^{g} m_j a_j + \sum_{k=1}^{g} n_k b_k \quad \text{for some integers} \quad m_j, n_k.$$

Define two vectors $M$ and $N$ in $\mathbb{Z}^g$

$$M = (m_1, \ldots, m_g), \quad N = (n_1, \ldots, n_g).$$

By (2.2), the quotient of $\theta$-functions in (3.2) is to be multiplied by

$$\frac{\exp\left(-\frac{\langle BM,M\rangle}{2} - \langle M, A(P) - A(D) + U - K\rangle\right)}{\exp\left(-\frac{\langle BM,M\rangle}{2} - \langle M, A(P) - A(D) - K\rangle\right)} = \exp(-\langle M, U\rangle).$$

The exponential term in (3.2) is to be multiplied by $\exp(\langle M, U\rangle)$. Hence the value of $\Phi(P)$ remains unchanged.

By Theorem 2.7, $\Phi$ has poles at $P_1, \dots, P_g$. Finally, $\Phi$ has an essential singularity of the form $\exp(q(\omega))$ at $Q$.

It remains to prove the uniqueness up to a constant. Let $\Phi$ and $\tilde{\Phi}$ be two Baker-Akhiezer functions corresponding to the divisor $D$ and to the polynomial $q(\omega)$. By Theorem 2.3, $\tilde{\Phi}$ has exactly $g$ zeros on $\Gamma$, say $Q_1, \dots, Q_g$. Let $\tilde{D}$ be the divisor $\tilde{D} = Q_1 + \cdots + Q_g$. Then the function $\dfrac{\Phi(P)}{\tilde{\Phi}(P)}$ is meromorphic with all its poles at $Q_1, \dots, Q_g$. Since $D$ is a nonspecial divisor, $i(D) = 0$ and by Riemann-Roch Theorem (Theorem 2.6) $\mathrm{l}(D) = 1$. Since $D$ and $\tilde{D}$ are equivalent, $\mathrm{l}(D) = \mathrm{l}(\tilde{D}) = 1$. Hence the function $\dfrac{\Phi(P)}{\tilde{\Phi}(P)}$ is constant.

**Definition 3.1.** *A function whose existence is proved in the above theorem is called a* Baker-Akhiezer function *corresponding to the point $Q$, to the local parameter $z(P)$, to the polynomial $q(\omega)$, and to the nonspecial divisor $D = P_1 + \cdots + P_g$. This function is unique up to a constant multiple.*

### 3.2. Construction of a Family of Solutions to the KP Equation and the KdV Equation

The KP equation (or Kadomtsev-Petviashvili equation) is the following third-order nonlinear differential equation

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left[u_t - \frac{1}{4}(6uu_x + u_{xxx})\right]. \tag{3.3}$$

It was introduced in a paper of Boris Kadomtsev and Vladimir Petviashvili in 1970 [12] investigating the stability of soliton solutions to the Korteweg-de Vries equation (so called the KdV equation) with respect to transverse perturbations. The KdV equation is the following third order nonlinear equation in one spatial and one temporal variable

$$u_t = \frac{1}{4}(6uu_x + u_{xxx}), \tag{3.4}$$

while the KP equation includes two spatial variables. Both the KP and the KdV equations have been used extensively as a model for the evolution of water waves. Also, they can be both solved by inverse scattering method ( [4] and [3]) for rapidly decreasing functions.

We will describe an algebro-geometric method, using Baker-Akhiezer functions on Riemann surfaces to find some solutions to the KP equation [5]. These solutions will be given as a function of the theta function associated with the surface.

Let $\Gamma$ be a compact Riemann surface of genus $g$ and let $D = P_1 + \cdots + P_g$ be a nonspecial divisor on $\Gamma$. We choose a point $Q$ in $\Gamma$ and a local parameter $z = z(P)$ with $z(Q) = 0$. Put $\omega(P) = \dfrac{1}{z(P)}$. For our purposes, we will work with the polynomial $q(\omega) = t\omega^3 + y\omega^2 + x\omega$ only, where $x$, $y$, and $t$ are complex parameters. An extension of this discussion will be given in Theorem 3.4. From Theorem 3.1 in section 3.1, we know there exists a unique Baker-Akhiezer function $\Phi(x, y, t; P)$ which has the following expansion in a neighbourhood of $Q$:

$$\begin{aligned}\Phi(x, y, t; P) &= \exp(q(\omega))\left(1 + \xi_1 z + \xi_2 z^2 + \dots\right)\\ &= \exp(t\omega^3 + y\omega^2 + x\omega)\left(1 + \frac{\xi_1}{\omega} + \frac{\xi_2}{\omega^2} + \dots\right)\end{aligned} \tag{3.5}$$

where $\xi_j = \xi_j(x, y, t)$ and the series is a series expansion around $Q$ of an analytic function. Now we will define two operators. First define two functions

$$u = -2\frac{\partial \xi_1}{\partial x}, \tag{3.6}$$

$$v = 3\xi_1\frac{\partial \xi_1}{\partial x} + 3\frac{\partial^2 \xi_1}{\partial x^2} - 3\frac{\partial \xi_2}{\partial x}. \tag{3.7}$$

Let the operators $L_1$ and $A_1$ be defined as

$$L_1 = \frac{\partial^2}{\partial x^2} + u, \quad A_1 = \frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} + v. \tag{3.8}$$

We investigate the action of these differential operators on $\Phi$.

$$\begin{aligned}\exp(-q(\omega))L_1\Phi = &\left(\frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial x}\frac{1}{\omega} + \cdots\right) + \left(\omega^2 + \xi_1\omega + \xi_2 + \cdots\right)\\ &+ \left(\frac{\partial^2 \xi_1}{\partial x^2}\frac{1}{\omega} + \frac{\partial^2 \xi_2}{\partial x^2}\frac{1}{\omega^2} + \cdots\right) + \left(\frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial x}\omega + \cdots\right)\\ &+ u\left(1 + \frac{\xi_1}{\omega} + \frac{\xi_2}{\omega^2} + \cdots\right),\end{aligned}$$

$$\begin{aligned}\exp(-q(\omega))A_1\Phi = &\left(\frac{\partial^2 \xi_1}{\partial x^2} + \frac{\partial^2 \xi_2}{\partial x^2}\frac{1}{\omega} + \cdots\right) + \left(\frac{\partial \xi_1}{\partial x}\omega + \frac{\partial \xi_2}{\partial x} + \cdots\right)\\ &+ \left(\frac{\partial \xi_1}{\partial x}\omega + \frac{\partial \xi_2}{\partial x} + \cdots\right) + \left(\omega^3 + \xi_1\omega^2 + \xi_2\omega + \cdots\right)\\ &+ \left(\frac{\partial^3 \xi_1}{\partial x^3}\frac{1}{\omega} + \frac{\partial^3 \xi_2}{\partial x^3}\frac{1}{\omega^2} + \cdots\right) + \left(\frac{\partial^2 \xi_1}{\partial x^2} + \frac{\partial^2 \xi_2}{\partial x^2}\frac{1}{\omega} + \cdots\right)\\ &+ \left(\frac{\partial^3 \xi_1}{\partial x^3}\frac{1}{\omega} + \frac{\partial^3 \xi_2}{\partial x^3}\frac{1}{\omega^2} + \cdots\right) + \left(\frac{\partial \xi_1}{\partial x}\omega + \frac{\partial \xi_2}{\partial x}\omega^2 + \cdots\right)\\ &+ \frac{3}{2}u\left(\frac{\partial \xi_1}{\partial x}\frac{1}{\omega} + \frac{\partial \xi_2}{\partial x}\frac{1}{\omega^2} + \cdots\right) + \frac{3}{2}u\left(\omega + \xi_1 + \frac{\xi_2}{\omega} + \cdots\right) + v\left(1 + \frac{\xi_1}{\omega} + \frac{\xi_2}{\omega^2} + \cdots\right).\end{aligned}$$

On the other hand, we observe that the first order $t$ derivative of $\Phi$ and the third order $x$ derivative of $\Phi$ share the same main term. Similarly, the first order $y$ derivative of $\Phi$ and the second order $x$ derivative of $\Phi$ have the same main term.

$$\frac{\partial \Phi}{\partial y} = \omega^2 \exp q(\omega)\left(1+\frac{\xi_1}{\omega}+\frac{\xi_2}{\omega^2}+\dots\right)+\exp q(\omega)\left(\frac{\partial \xi_1}{\partial y}\frac{1}{\omega}+\frac{\partial \xi_2}{\partial y}\frac{1}{\omega^2}+\dots\right),$$
$$\frac{\partial \Phi}{\partial t} = \omega^3 \exp q(\omega)\left(1+\frac{\xi_1}{\omega}+\frac{\xi_2}{\omega^2}+\dots\right)+\exp q(\omega)\left(\frac{\partial \xi_1}{\partial t}\frac{1}{\omega}+\frac{\partial \xi_2}{\partial t}\frac{1}{\omega^2}+\dots\right).$$

From these equations, we see that the difference between $L_1\Phi$ and $\frac{\partial \Phi}{\partial y}$ is of the order of $\frac{1}{\omega}$, and so is the difference between $A_1\Phi$ and $\frac{\partial \Phi}{\partial t}$.

**Lemma 3.1.** *The function $\Phi$ above satisfies*

$$\left(-\frac{\partial}{\partial y}+L_1\right)\Phi = \mathcal{O}\left(\frac{1}{\omega}\right)\exp(t\omega^3+y\omega^2+x\omega), \tag{3.9}$$
$$\left(-\frac{\partial}{\partial t}+A_1\right)\Phi = \mathcal{O}\left(\frac{1}{\omega}\right)\exp(t\omega^3+y\omega^2+x\omega). \tag{3.10}$$

*where $u$ and $v$ are as defined in* (3.6) *and* (3.7).

**Theorem 3.2.** *Let $\Phi = \Phi(x, y, t; P)$ be a Baker-Akhiezer function corresponding to the polynomial $q(\omega) = t\omega^3 + y\omega^2 + x\omega$ and to a nonspecial divisor $D$ of degree $g$. Then $\Phi$ is a solution of the system*

$$\frac{\partial \Phi}{\partial y} = L_1\Phi, \quad \frac{\partial \Phi}{\partial t} = A_1\Phi. \tag{3.11}$$

*where $L_1$ and $A_1$ are as in* (3.8).

*Proof.* Consider the functions

$$\phi_1 = \left(-\frac{\partial}{\partial y}+L_1\right)\Phi \quad \text{and} \quad \phi_2 = \left(-\frac{\partial}{\partial t}+A_1\right)\Phi.$$

From equations (3.9) and (3.10) in Lemma 3.1, we conclude that both $\phi_1$ and $\phi_2$ are Baker-Akhiezer functions corresponding to the polynomial $q(\omega) = t\omega^3 + y\omega^2 + x\omega$. But $\phi_1$ and $\phi_2$ are not constant multiples of each other and the same equations imply that

$\phi_1$ and $\phi_2$ vanish at $Q$. Hence we conclude that both $\phi_1$ and $\phi_2$ are identically 0 on $\Gamma$.

**Remark 3.1.** *In the last result, we used the uniqueness property (up to a constant) of the Baker-Akhiezer function. This property serves as the key point to find a family of solutions to the KP equation.*

**Corollary 3.1.** *The functions $u$ in* (3.6) *and $v$ in* (3.7) *are solutions of the system of the equations*

$$\frac{3}{2}u_y + \frac{3}{2}u_{xx} - 2v_x = 0, \tag{3.12}$$

$$v_y - u_t + u_{xxx} + \frac{3}{2}uu_x - v_{xx} = 0. \tag{3.13}$$

*Proof.* Assume that the commutator operator $X = \left[L_1 - \frac{\partial}{\partial y}, A_1 - \frac{\partial}{\partial t}\right]$ is nonzero. From the system of equations (3.11), we know that the kernel of the operator $X$ includes an infinite family of functions $\{\Phi(x, y, t; P)\}$ parametrized by nonspecial divisors on the underlying Riemann surface. Computation of this commutator gives

$$\begin{aligned} X &= \left[L_1 - \frac{\partial}{\partial y}, A_1 - \frac{\partial}{\partial t}\right] \\ &= \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y} + u\right)\left(\frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} - \frac{\partial}{\partial t} + v\right) \\ &\quad - \left(\frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} - \frac{\partial}{\partial t} + v\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y} + u\right) \\ &= \left(\frac{3}{2}u_{xx} - \frac{3}{2}u_y + 2v_x\right)\frac{\partial}{\partial x} + u_t - v_y - u_{xxx} - \frac{3}{2}uu_x + v_{xx} \end{aligned}$$

$X$ is a differential operator containing differentiation only with respect to $x$, so its kernel must be finite dimensional because we already know that the kernel is not just zero, it contains an infinite dimensional space. We deduce that the commutator operator $X$ is identically zero.

**Corollary 3.2.** *The function $u$ in* (3.6) *is a solution of the KP equation*

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left[u_t - \frac{1}{4}(6uu_x + u_{xxx})\right].$$

*Proof.* We take the partial derivative of (3.12) with respect to $y$, and take the partial derivative of (3.13) with respect to $x$ and multiply the latter by 2. Adding these two equations gives the KP equation given above.

### 3.3. An Explicit Expression of the Solutions to the KP Equation

Using the expansion of the Baker-Akhiezer function in (3.5), we may write down an explicit formula for the constructed solutions of the KP equation.

**Theorem 3.3.** *The constructed solutions of the KP equation in Section 3.2 are of the form*

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + yV + tW + z_0) + c \tag{3.14}$$

*where* $\theta = \theta(z(P))$ *is the theta function corresponding to* $\Gamma$, $z_0 = -A(D) - K$ *is a vector determined by a nonspecial divisor* $D$ *of degree* $g$, $K$ *is the Riemann constant vector given in* (2.10), $c$ *is a constant in* $\mathbb{C}$, *and the vectors* $U$, $V$ *and* $W$ *are given by*

$$U_j = \oint_{b_j}\Omega_1, \quad V_j = \oint_{b_j}\Omega_2, \quad W_j = \oint_{b_j}\Omega_3 \qquad j = 1,\dots,g, \tag{3.15}$$

*where* $\Omega_1$, $\Omega_2$, *and* $\Omega_3$ *are differentials with zero a-periods and having principal parts* $\mathrm{d}\omega$, $\mathrm{d}(\omega^2)$, *and* $\mathrm{d}(\omega^3)$ *at* $Q$, *respectively.*

*Proof.* By (3.6), it is enough to find $\xi_1$ which is the coefficient of $\frac{1}{\omega}$ in the expansion of the Baker-Akhiezer function $\Phi$ around $Q$ in (3.5). From (3.5), we know

$$\log\Phi(x,y,t;P) = t\omega^3 + y\omega^2 + x\omega + \eta_0 + \frac{\eta_1}{\omega} + \frac{\eta_2}{\omega^2} + \cdots$$

where $\eta_1 = \xi_1$ and each $\eta_i$ is a function of $x, y, t$, and $P$. Using (3.2), we deduce that the coefficient of $\frac{1}{\omega}$ in the expansion of the function

$$\Psi(P) = \log \frac{\theta(A(P) + xU + yV + tW + z_0)}{\theta(A(P) + z_0)}$$

around $Q$ is $\eta_1 - cx - by - at$. The coefficient $\eta_1$ does not depend on the initial point of integration in the definition of the Abel map because of the uniqueness of the Baker-Akhiezer function. We choose $Q$ as that point so that $A(Q) = 0$. We will use this fact and the following result (this is lemma 2.1.2 in [2]).

**Lemma 3.2.** *Let $\omega$ be a local coordinate on $\Gamma$ with $\omega(Q) = 0$ for some $Q \in \Gamma$, and let $\Omega_Q^{(n)}$ be a meromorphic differential with a single pole at $Q$ of multiplicity $n+1$ for $n \geq 1$, i.e. a meromorphic differential with principal part*

$$\Omega_Q^{(n)} = \frac{\mathrm{d}\omega}{\omega^{n+1}}$$

*around $Q$. If $\omega_j = f_j(z)\mathrm{d}z$ is the local expression for each of the basis holomorphic differentials $\omega_j$ around $Q$, then $\oint_{b_j} \Omega_Q^{(n)} = \frac{1}{n!}\frac{\mathrm{d}^{n-1} f_j(z)}{\mathrm{d}\omega^{n-1}}\big|_{z=0}$ for $j = 1, \ldots, g$ and $n \in \mathbb{Z}^+$.*

Now we will expand the Abel map $A(P)$ near $Q$ in the form

$$A(P) = -\frac{1}{\omega}U + \mathcal{O}(\frac{1}{\omega^2}). \tag{3.16}$$

Then we rewrite $\Psi(P)$ using Taylor expansion and (3.16):

$$\begin{aligned}
\Psi(P) =& \log \frac{\theta(A(P) - A(D) - K + xU + yV + tW)}{\theta(A(P) - A(D) - K)} \\
=& \log \theta((x - \frac{1}{\omega})U + yV + tW - A(D) - K + \mathcal{O}(\omega^{-2})) \\
& - \log \theta(-\frac{1}{\omega}U - A(D) - K + \mathcal{O}(\omega^{-2})) \\
=& \log \theta(xU + yV + tW - A(D) - K) - \frac{1}{\omega}\partial_x \log \theta(xU + yV + tW - A(D) - K) \\
& - \log \theta(-\frac{1}{\omega}U - A(D) - K + \mathcal{O}(\omega^{-2})) + \mathcal{O}(\omega^{-2}).
\end{aligned} \tag{3.17}$$

Note that the term $-\log\theta(-\frac{1}{\omega}U - A(D) - K + \mathcal{O}(\omega^{-2}))$ in (3.17) is independent of $x$, call it $f(\omega)$. We conclude

$$\eta_1 - cx - by - at = -\partial_x \log\theta(xU + yV + tW + z_0) + a(\omega).$$

where $a(\omega)$ denotes the coefficient of $\dfrac{1}{\omega}$ in $f(\omega)$. We also have $u = -2\dfrac{\partial\eta_1}{\partial x}$ by (3.6) because $\eta_1 = \xi_1$. Therefore, with the above constant $c$, $u$ has the form

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + yV + tW + z_0) + c$$

where we let $z_0 = -A(D) - K$.

We deduce the existence of infinitely many solutions to the KP equation indexed by nonspecial divisors of degree $g$ on the surface $\Gamma$.

## 3.4. The KdV Equation as a Special Case

We recall that the Korteweg-de Vries equation, or shortly the KdV equation, has the following form

$$u_t = \frac{1}{4}(6uu_x + u_{xxx}).$$

This equation describes the evolution of a solitary wave, a wave that preserves its shape while moving. In 1834, John Scott Russell observed the water waves that were preserving their shapes while moving in a canal and then implemented experiments on water waves. The mathematical model showing that the motion of shape-preserving water waves is governed by the KdV equation was given by Diederik Korteweg and Gustav de Vries in 1895 [13]. As we already mentioned, the KP equation is one of the possible generalizations of the KdV equation in the sense that the KP equation includes two spatial variables whereas the KdV equation includes one spatial variable. Hence, we will manipulate the method we used to find solutions to the KP equation and will obtain solutions to the KdV equation.

Let $\Gamma$ be a Riemann surface of genus $g$ and let $D$ be a nonspecial divisor of degree $g$ on $\Gamma$. Suppose $Q$ is a point on $\Gamma$ such that there exists a meromorphic function $\lambda : \Gamma \to \mathbb{C}$ with a unique double pole at $Q$. We will demonstrate the existence of such a function for hyperelliptic surfaces in Section 3.4.1. Define a local parameter $\omega^{-1}(P) = \dfrac{1}{\sqrt{\lambda(P)}}$ in a neighbourhood of $Q$. Let $q(\omega) = t\omega^3 + y\omega^2 + x\omega$, as before. We write

$$\Phi(x, y, t; P) = \exp(y\lambda(P))\varphi(x, t; P)$$

so that $\varphi$ is the Baker-Akhiezer function having the same poles as $\Phi$ and $\varphi \exp(-t\omega^3 - x\omega)$ is analytic at $Q$.

**Corollary 3.3.** (Corollary to Theorem 3.2) $\varphi$ *is a solution of the system of equations*

$$L_1\varphi = \lambda\varphi \quad \textit{and} \quad A_1\varphi = \frac{\partial \varphi}{\partial t}.$$

*where the operators* $L_1$ *and* $A_1$ *are as given in* (3.8).

We see that in this case the functions $u$ in (3.6) and $v$ in (3.7) are independent of $y$ because the function $\varphi$ is a function of $x$, $t$ and $P$ only. Hence the KP equation

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left[u_t - \frac{1}{4}(6uu_x + u_{xxx})\right]$$

simplifies and the function

$$u(x, t) = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + tW + z_0) + c \tag{3.18}$$

becomes a solution to the KdV equation for any constant $c \in \mathbb{C}$.

### 3.4.1. An Example

A hyperelliptic surface $\Gamma$ of genus $g$ is an algebraic curve given by

$$k^2 = P(z) = \prod_{j=1}^{\deg P} (z - z_j)$$

where $P(z)$ is a polynomial of degree $2g+1$ or $2g+2$. We write $\Gamma = \{(z,k) \in \mathbb{C}^2 : k^2 = P(z)\}$.

Assume $\deg P = 2g+1$. Then the branch points of $\Gamma$ are $z_1, \ldots, z_{2g+1}$ and the point at infinity. The following lemma demonstrates the existence of a meromorphic function with a single double pole that is defined on hyperelliptic curves $k^2 = P(z) = \prod_{j=1}^{2g+1} (z - z_j)$ where $P$ is a polynomial of degree $2g+1$. This result was used in Section 3.4. Hence, in the hyperelliptic case, we can apply the method in Section 3.4 to find a family of solutions in terms of theta functions to the KdV equation.

**Lemma 3.3.** *Let $\Gamma$ be a hyperelliptic curve of genus $g$ that is given by $k^2 = P(z) = \prod_{j=1}^{2g+1} (z - z_j)$ for a polynomial $P$ of degree $2g+1$. Let $D$ be a nonspecial divisor on $\Gamma$. Suppose that the point $Q$ is the point at infinity, and $\omega = \sqrt{z}$. Then the vector $V$ in* (3.15) *is the zero vector and the function $u(x,y,t)$ in* (3.14) *simplifies to $u(x,t)$ in* (3.18).

The proof requires a few remarks. To prove the meromorphicity of a function $\lambda$ on a Riemann surface, we need to check the meromorphicity of the function $\lambda \circ \phi^{-1}$ , $\phi$ being a local parameter. First, we note that the function $\lambda(P) = z$ for $P = (z,k) \in \Gamma$ has a single second order pole at $Q$ because the local parameter $\phi$ around $Q$ (the point at infinity), is $\phi(P) = \frac{1}{\sqrt{z}}$. Around points of $\Gamma$ other than the branch points, we can choose $z = z(P)$ as the local parameter. The vector $V$ is the zero vector because $V_j = \oint_{b_j} \Omega_2$ for each $j = 1, 2, \ldots, g$ and the differential $\Omega_2$ is exact, $\Omega_2 = \mathrm{d}z$. Hence the function

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + tW + z_0) + c$$

is a solution to the KdV equation.

### 3.5. Other Nonlinear Equations

The KP equation is an example of a more general class of equations called the Zaharov-Shabat equation $L_t - A_y = [A, L]$ or in the equivalent form we used for the KP equation $[L - \partial_y, A - \partial_t] = 0$ where $L$ and $A$ are operators of arbitrary degree including derivatives only with respect to the $x$ variable. The algebro-geometric construction we presented above was proposed by I. M. Krichever in 1976 for a class of periodic solutions of the more general Zaharov-Shabat equation, and these solutions can be expressed in terms of the Riemann theta function [1]. Moreover, his method provides a classification of commutative rings of differential operators [14]. The following theorem is the result of this method applied to nonconstant polynomials $q(\omega)$ of arbitrary degree.

**Theorem 3.4.** *Let $\Gamma$ be a Riemann surface of genus $g$, $D$ a nonspecial divisor of degree $g$ on $\Gamma$, $Q$ a point in $\Gamma$ not in the support of $D$, and $\omega^{-1}(P)$ a local coordinate map centered at $Q$. Let $q(\omega)$ be a nonconstant polynomial of degree $n$.*

*(i) Suppose that $\Phi_q = \Phi_q(x, y; P)$ is the Baker-Akhiezer function on $\Gamma$ corresponding to the divisor $D$ with the essential singularity $\exp(x\omega + yq(\omega))$ at $Q$. There is an ordinary differential operator $L_q$ which is uniquely determined by the equation*

$$\frac{\partial \Phi_q}{\partial y} = L_q \Phi_q.$$

*Moreover, the coefficients of $L_q$ can be expressed recursively in terms of the coefficients of the reciprocal powers of $\omega$ in the expansion of the function $\Phi_q \exp(-x\omega - yq(\omega))$.*

*(ii) For any pair of polynomials $q_1(\omega)$ and $q_2(\omega)$, there corresponds a solution of a*

*nonlinear equation given by*

$$\left[-\frac{\partial}{\partial y}+L_{q_1},-\frac{\partial}{\partial t}+L_{q_2}\right]=0.$$

*This solution can be expressed in terms of the $\theta$-function of $\Gamma$.*

(iii) *Let $\lambda(P)$ be a function on $\Gamma$ with a single pole at $Q$. There is an operator $L_\lambda$ which has its eigenfunctions corresponding to the eigenvalue $\lambda(P)$ as the Baker-Akhiezer functions corresponding to the divisor $D$ with essential singularity $\exp(x\omega)$ at $Q$. In this case, for $q(\omega)$ being the principal part of the expansion of $\lambda(P)$ around $Q$, $L_\lambda = L_q$.*

(iv) *The divisor $D$ gives a homomorphism from the ring of the meromorphic functions with a single pole at $Q$ into a commutative algebra of ordinary differential operators with the assignment $\lambda \mapsto L_\lambda$.*

*Proof.* We content with a short explanation. For the proof, see Theorem 1.1 and Theorem 2.3 in [5]. Part (*i*) and part (*ii*) can be understood from the construction for the KP equation by observing that in that discussion $L_1$ had degree 2 coinciding with the power of $y$ in the polynomial $q(\omega) = t\omega^3 + y\omega^2 + x\omega$. The statements of (*iii*) and (*iv*) can be intuitively comprehended by considering the construction of solutions to the KdV equation in Section 3.4 as a special case of these general statements.

# 4. EFFECTIVIZATION OF THE FORMULAE FOR THE SOLUTIONS OF THE KdV EQUATION AND THE KP EQUATION

In this chapter, we will use the explicit formulae for constructed solutions of the KdV equation and the KP equation given in equations (3.18) and (3.14), respectively to obtain a system of algebraic equations in $U$, $V$, and $W$. We recall that the vectors $U$, $V$, and $W$ are $b$-periods of some meromorphic differentials. By the effectivization of the formula for the solution $u$, we will find conditions on arbitrary vectors $U$, $V$, and $W$ so that the corresponding function $u$ in the form (3.18) (or (3.14)) will be a solution to the KdV equation (or KP equation). First, we will consider the cases genera 1 and 2 for the KdV equation, and then we will use a similar reasoning while considering the KP equation case for genera $1, 2$, and 3.

## 4.1. KdV Equation

Assume that the KdV equation

$$u_t = \frac{1}{4}(6uu_x + u_{xxx})$$

has a solution of the form

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + tW + z_0).$$

Our aim is to find the necessary conditions on $U$ and $W$. These solutions correspond to the case $c = 0$ in Section (3.18) which can be achieved by a linear change in

$W$ in terms of $U$ and $W$. By direct substitution of $u(x,t)$ into the KdV equation,

$$\frac{\partial}{\partial x}\left(-2\frac{\partial^2 \log\theta}{\partial x\partial t}+3\Big(\frac{\partial^2 \log\theta}{\partial x^2}\Big)^2+\frac{1}{2}\frac{\partial^4 \log\theta}{\partial x^4}\right)=0.$$

where we wrote $\theta = \theta(xU + tW + z_0)$. Assume the Riemann matrix $B = [B_{ij}]$ is indecomposable. Then

$$-2\frac{\partial^2 \log\theta}{\partial x\partial t}+3\Big(\frac{\partial^2 \log\theta}{\partial x^2}\Big)^2+\frac{1}{2}\frac{\partial^4 \log\theta}{\partial x^4}=4C$$

for some constant $C$. A rearrangement of this equation gives

$$\frac{\partial^4\theta}{\partial x^4}\theta-4\frac{\partial^3\theta}{\partial x^3}\frac{\partial\theta}{\partial x}+3\left(\frac{\partial^2\theta}{\partial x^2}\right)^2-4\frac{\partial^2\theta}{\partial x\partial t}\theta+4\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial t}+8C\theta^2=0. \tag{4.1}$$

Now we will apply Theorem 2.1 (the Addition Theorem). For $z^1 + z^2 = w^1$, $z^1 - z^2 = w^2$, it gives

$$\theta(z^1)\theta(z^2)=\sum_{n\in\frac{1}{2}(\mathbb{Z}_2)^g}\hat{\theta}[n](w^1)\hat{\theta}[n](w^2). \tag{4.2}$$

where $[n]$ stands for the characteristics $[n, 0]$ and $\hat{\theta}[n](w) = \theta[n, 0](w \mid 2B)$.

**Lemma 4.1.** *The equation* (4.1) *is equivalent to the following system of* $2^g$ *equations indexed with* $n \in (\mathbb{Z}_2)^g$ *in the unknowns* $U_1,\dots,U_g;W_1,\dots,W_g$*:*

$$\partial_U^4\hat{\theta}[n]-\partial_U\partial_W\hat{\theta}[n]+C\hat{\theta}[n]=0. \tag{4.3}$$

*where we used the notations*

$$
\begin{aligned}
\hat{\theta}_{jk\ldots}[n] &= \hat{\theta}_{jk\ldots}[n](0). \\
\hat{\theta}_{jk\ldots}[n](w) &= \frac{\partial}{\partial w_j}\frac{\partial}{\partial w_k}\ldots\hat{\theta}[n](w), \\
\partial_U^4\hat{\theta}[n] &= \sum_{j,k,l,m=1}^{g} U_jU_kU_lU_m\hat{\theta}_{jklm}[n], \\
\partial_U\partial_W\hat{\theta}[n] &= \sum_{j,k=1}^{g} U_jW_k\hat{\theta}_{jk}[n].
\end{aligned}
\tag{4.4}
$$

*Proof.* It is conventional to define the operators

$$
\begin{aligned}
X_{z^1} &= \sum_{j=1}^{g} U_j\frac{\partial}{\partial z_j^1}, \qquad & X_{z^2} &= \sum_{j=1}^{g} U_j\frac{\partial}{\partial z_j^2}, \\
T_{z^1} &= \sum_{j=1}^{g} W_j\frac{\partial}{\partial z_j^1}, \qquad & T_{z^2} &= \sum_{j=1}^{g} W_j\frac{\partial}{\partial z_j^2}.
\end{aligned}
\tag{4.5}
$$

Similarly the operators $X_{w^1}$, $X_{w^2}$, $T_{w^1}$ and $T_{w^2}$ are defined. These operators satisfy

$$
\begin{aligned}
X_{z^1} &= X_{w^1} + X_{w^2}, \qquad & X_{z^2} &= X_{w^1} - X_{w^2}, \\
T_{z^1} &= T_{w^1} + T_{w^2}, \qquad & T_{z^2} &= T_{w^1} - T_{w^2}.
\end{aligned}
\tag{4.6}
$$

Using (4.5), we can rewrite equation (4.1) as

$$
\left(X_{z^1}^4 - 4X_{z^1}^3X_{z^2} + 3X_{z^1}^2X_{z^2}^2 + 4X_{z^1}T_{z^2} - 4X_{z^1}T_{z^1} + 8C\right)\theta(z^1)\theta(z^2)\Big|_{z^1=z^2=z} = 0. \tag{4.7}
$$

Using (4.6), we write (4.7) in terms of $X_{w^1}$, $X_{w^2}$, $T_{w^1}$ and $T_{w^2}$. We will apply this operator to the right side of the equation (4.2) and evaluate it for $w^1 = 2z$, $w^2 = 0$. Note that the function $\hat{\theta}[n]$ is even by the Proposition 2.2 in chapter 2, so it suffices to consider the terms with even total degree. Doing the computation and applying this simplification, the result is

$$
8(X_{w^2}^4 - X_{w^2}T_{w^2} + C)\sum_{n\in\frac{1}{2}(\mathbb{Z}_2)^g}\hat{\theta}[n](w^1)\hat{\theta}[n](w^2)\Big|_{w^1=2z,w^2=0} = 0.
$$

The $\theta$-functions $\hat{\theta}[n](2z)$ are linearly independent. Then all the coefficients of the above equation are zero, hence the result.

#### 4.1.1. $g = 1$ case

Put $U = U_1$, $W = W_1$. The equation (4.3) gives

$$U^4\hat{\theta}^{(4)}[0] - UW\hat{\theta}^{(2)}[0] + C\hat{\theta}[0] = 0,$$
$$U^4\hat{\theta}^{(4)}[1/2] - UW\hat{\theta}^{(2)}[1/2] + C\hat{\theta}[1/2] = 0.$$

By scaling if necessary, we may assume that $U = 1$. Then

$$W = \frac{\hat{\theta}^{(4)}\hat{\theta}[1/2] - \hat{\theta}^{(4)}[1/2]\hat{\theta}[0]}{\hat{\theta}^{(2)}[0]\hat{\theta}[1/2] - \hat{\theta}^{(2)}[1/2]\hat{\theta}[0]}.$$

#### 4.1.2. $g = 2$ case

Let $A = [A_{ij}]$ be the following matrix $4 \times 4$ matrix

$$A = \{\hat{\theta}_{11}[n], \hat{\theta}_{12}[n], \hat{\theta}_{22}[n], \hat{\theta}[n]\}.$$

where $n \in \frac{1}{2}(\mathbb{Z}_2)^2$. We assume that $A$ has full rank. Suppose its inverse is given a $4 \times 4$ matrix $D$

$$D = \{d_n^{11}, d_n^{12}, d_n^{22}, d_n\}.$$

The equation (4.3) provides

$$\partial_U^4\hat{\theta}[n] - U_1W_1\hat{\theta}_{11}[n] - (U_1W_2 + U_2W_1)\hat{\theta}_{12}[n] - U_2W_2\hat{\theta}_{22} - C\hat{\theta}[n] = 0.$$

for each $n \in \frac{1}{2}(\mathbb{Z}_2)^2$. Using this equation, we define the polynomials

$$\begin{aligned} U_1W_1 &= \sum_{n\in\frac{1}{2}(\mathbb{Z}_2)^2} d_n^{11}\partial_U^4\hat{\theta}[n] = P_{11}(U_1,U_2) = P_{11}(U), \\ U_2W_2 &= \sum_{n\in\frac{1}{2}(\mathbb{Z}_2)^2} d_n^{22}\partial_U^4\hat{\theta}[n] = P_{22}(U_1,U_2) = P_{22}(U), \\ U_1W_2+U_2W_1 &= \sum_{n\in\frac{1}{2}(\mathbb{Z}_2)^2} d_n^{12}\partial_U^4\hat{\theta}[n] = P_{12}(U_1,U_2) = P_{12}(U). \end{aligned} \tag{4.8}$$

From these equations, we obtain a homogeneous equation of degree 6 in the following form

$$P(U) = P(U_1,U_2) = U_1^2P_{22}(U) - U_1U_2P_{12}(U) + U_2^2P_{11}(U) = 0. \tag{4.9}$$

$U_1$ and $U_2$ are solutions to (4.9) and then the vector $W$ can be found from (4.8).

## 4.2. KP Equation

### 4.2.1. $g = 1$ case

We suppose that the KP equation

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left(u_t - \frac{1}{4}(6uu_x + u_{xxx})\right)$$

has a solution $u$ of the form

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + yV + tW + z_0).$$

We proceed as in the previous section. Direct substitution gives an equation in the $\theta$-function

$$\theta_{xxxx}\theta - 4\theta_{xxx}\theta_x + 3\theta_{xx}^2 + 4\theta_x\theta_t - 4\theta_{xt}\theta + 3\theta_{yy}\theta - 3\theta_y^2 + 8C\theta^2 = 0. \tag{4.10}$$

where we denoted the integration constant by $4C$. Then we apply the Addition Theorem (Theorem 2.1) and make a similar reasoning with the operators. This provides the result below.

**Lemma 4.2.** *The equation* (4.10) *is equivalent to the system of equations*

$$\partial_U^4\hat{\theta}[n] - \partial_U\partial_W\hat{\theta}[n] + \frac{3}{4}\partial_V^2\hat{\theta}[n] + C\hat{\theta}[n] = 0, \quad n \in \frac{1}{2}(\mathbb{Z}_2)^g, \tag{4.11}$$

*where we use the notation in* (4.4) *and*

$$\partial_V^2\hat{\theta}[n] = \sum_{j,k=1}^{g} V_j V_k \hat{\theta}_{jk}[n].$$

**Remark 4.1.** *The system* (4.11) *is invariant under the following type of transformations*

$$\begin{gathered} U \mapsto \alpha U, \quad V \mapsto \pm(\alpha^2 V + 2\alpha\beta U), \\ W \mapsto \alpha^3 W + 3\alpha^2\beta V + 3\alpha\beta^2 U, \quad C \mapsto \alpha^4 C. \end{gathered} \tag{4.12}$$

*where $\alpha$ and $\beta$ are complex numbers with $\alpha \neq 0$.*

If $g = 1$, then $U$, $V$, and $W$ are just numbers, so with an invariance transformation of the equation (4.10), we can make $V = 0$ which is the case of the KdV equation discussed in the previous section. Now we consider the cases of larger genera.

### 4.2.2. $g = 2$ case

We assume that $U$ and $V$ are linearly independent, if not, again by using the invariance transformation of the equation (4.10), we can achieve $V = 0$ (so the KdV case). Since the dimension is 2, there are constants $a$ and $b$ satisfying

$$W = aU + bV.$$

If we define another function $w(x, y)$

$$w(x + at, y + at) = u(x, y, t)$$

then we have

$$w(x, y) = 2\frac{\partial^2}{\partial x^2} \log \theta(xU + yV + z_0)$$

and that $w$ satisfies the equation

$$3w_{yy} - 4aw_{xx} + 4bw_{xy} + (3w^2 + w_{xx})_{xx} = 0.$$

As in Section 4.1.2, we solve (4.11) and get

$$\begin{aligned} W_j &= \frac{3}{4}\frac{V_j^2}{U_j} + \frac{P_{jj}(U)}{U_j} \quad j = 1, 2 \\ (U_2V_1 - U_1V_2)^2 &= -\frac{4}{3}P(U_1, U_2). \end{aligned}$$

where the polynomials $P_{11}(U)$, $P_{12}(U)$, $P_{22}(U)$, $P(U)$ are determined from the equations (4.8) and (4.9).

Now, we set $U_1 = 1$ and $V_1 = 0$ using the invariance property in (4.12) and put $U_2 = z$ a complex parameter. Then

$$U = (1, z), \quad V = \left(\pm\frac{2\mathrm{i}\sqrt{P(1, z)}}{\sqrt{3}}, 0\right), \quad W = (-zP_{22}(1, z) + P_{12}(1, z), P_{22}(1, z)),$$
$$a = P_{22}(1, z), \quad b = \frac{\sqrt{3}(P_{12}(1, z) - zP_{22}(1, z))}{2\mathrm{i}\sqrt{P(1, z)}}.$$

where we assume $P(1, z) \neq 0$ which means $V_2 \neq 0$, i.e. $V$ is not the zero vector.

### 4.2.3. $g = 3$ case

The vectors $U$, $V$ and $W$ are linearly independent. Hence we do not have a reduction to the KdV equation. We consider the equation

$$\partial_U^4\hat{\theta}[n] - \partial_U\partial_W\hat{\theta}[n] + \frac{3}{4}\partial_V^2\hat{\theta}[n] + C\hat{\theta}[n] = 0 \quad n \in \frac{1}{2}(\mathbb{Z}_2)^g$$

as a linear equation in 7 unknowns

$$\begin{gathered} U_1W_1 - \frac{3}{4}V_1^2, \quad U_1W_2 + U_2W_1 - \frac{3}{2}V_1V_2, \quad U_1W_3 + U_3W_1 - \frac{3}{2}V_1V_3, \\ U_2W_2 - \frac{3}{4}V_2^2, \quad U_2W_3 + U_3W_2 - \frac{3}{2}V_2V_3, \quad U_3W_3 - \frac{3}{4}V_3^2, \quad -C; \end{gathered}$$

with the $8 \times 7$ matrix

$$A = \{\hat{\theta}_{11}[n], \hat{\theta}_{12}[n], \hat{\theta}_{33}[n], \hat{\theta}[n]\}, \quad n \in \frac{1}{2}(\mathbb{Z}_2)^3.$$

We assume that $A$ has rank $7 = \frac{3.(3+1)}{2} + 1$. Let $n_1$,..., $n_7$ in $\frac{1}{2}(\mathbb{Z}_2)^3$ be such that the $7 \times 7$ matrix

$$M = \{\hat{\theta}_{11}[n], \hat{\theta}_{12}[n], \hat{\theta}_{33}[n], \hat{\theta}[n]\} \quad n = n_1, \ldots, n_7.$$

has nonzero determinant. Let $D = \{d_n^{jk}, d_n\}$ where $n = n_1, \ldots, n_7$ denote the inverse matrix of $M$. Then by (4.11),

$$P_{jj}(U) = U_jW_j - \frac{3}{4}V_j^2 \quad j = 1, 2, 3; \tag{4.13}$$

$$U_jW_k + U_kW_j - \frac{3}{2}V_jV_k = P_{jk}(U) \quad j, k = 1, 2, 3; \quad j \neq k, \tag{4.14}$$

where the polynomials $P_{jk}(U)$ are given by

$$P_{jk}(U) = \sum_{l=1}^{7} d_{n_l}^{jk} \partial_U^4 \hat{\theta}[n_l].$$

Using equation (4.13) in (4.14), we get

$$Q_{jk}(U) := -\frac{3}{4}(U_j V_k - U_k V_j)^2 = U_j^2 P_{kk} - U_j U_k P_{jk} + U_k^2 P_{jj} \quad j,k = 1,2,3; \quad j < k,$$

which provides

$$U_j V_k - U_k V_j = \frac{2\mathrm{i}}{\sqrt{3}} \sqrt{Q_{jk}(U)} \quad j,k = 1,2,3; \quad j < k.$$

Then $V$ is determined from

$$\begin{aligned} V_1 &= -\lambda(U_3\sqrt{Q_{13}(U)} + U_2\sqrt{Q_{12}(U)}), \\ V_2 &= -\lambda(U_1\sqrt{Q_{12}(U)} + U_3\sqrt{Q_{23}(U)}), \quad \lambda = \frac{2\mathrm{i}}{\sqrt{3}(U_1^2 + U_2^2 + U_3^2)}, \\ V_3 &= -\lambda(U_2\sqrt{Q_{23}(U)} + U_1\sqrt{Q_{13}(U)}). \end{aligned}$$

For the compatibility of all the equations, the signs of the square-roots must be in accordance with the equation

$$U_1\sqrt{Q_{23}(U)} - U_2\sqrt{Q_{13}(U)} + U_3\sqrt{Q_{12}(U)} = 0.$$

Finally, $W$ can be determined from condition (4.13).

### 4.2.4. The KP Equation, larger genera

In this section, we will prove that the solution set $\{U, V, W, C\}$ to (4.11) depends only on the vector $U = (U_1, \ldots, U_g)$, up to some linear transformations.

**Definition 4.1.** *Let $\Gamma$ be a Riemann surface of genus $g \geq 2$, and let $\{\omega_1, \cdots, \omega_g\}$ be a basis for the holomorphic differentials on $\Gamma$. Define a mapping $\psi$*

$$\psi : \Gamma \to \mathbb{CP}^{g-1}$$
$$P \mapsto (\omega_1(P) : \cdots : \omega_g(P)).$$

*Note that for each $j = 1, 2, \ldots, g$, and for each $P \in \Gamma$, $\omega_j(P)$ denotes the value of $f_j(P)$ where $\omega_j = f_j(z)\mathrm{d}z$ around $P$. We define the canonical curve $\Gamma'$ to be the image of this map, $\Gamma' = \psi(\Gamma) \subset \mathbb{CP}^{g-1}$.*

**Remark 4.2.** *By Lemma 3.2, the vector $U = (U_1 : \cdots : U_g) \in \mathbb{CP}^{g-1}$ in our construction in Section 3.3 is*

$$(U_1(P) : \cdots : U_g(P)) = (\omega_1(P) : \cdots : \omega_g(P)).$$

*Thus the canonical curve $\Gamma'$ is included in the solution set for the vector $U$ in the system* (4.11). *We will give an outline of the proof that the vector $U$ satisfying* (4.11) *is on the canonical curve and there are no other solutions for $U$.*

**Lemma 4.3.** *Let $B = [B_{jk}]$ be the Riemann matrix of a Riemann surface $\Gamma$ of genus $g$, and let $\theta = \theta(z)$ denote the corresponding $\theta$-function. Then the following condition holds:*

$$\mathrm{rank}\{\hat{\theta}_{jk}[n], \hat{\theta}[n]\} = \frac{g(g+1)}{2} + 1. \tag{4.15}$$

*Proof.* See lemma 4.3.1 in [2].

**Lemma 4.4.** *For matrices $B = [B_{jk}]$ satisfying the condition* (4.15), *the vectors $V$, $W$*

*and the number $C$ are uniquely determined by the vector $U$, up to the transformations*

$$V \mapsto \pm(V + 2\alpha U), \quad W \mapsto W + 3\alpha V + 3\alpha^2 U,$$

*where $\alpha$ is a complex-valued parameter.*

*Proof.* Suppose that the triples $\{V^1, W^1, C^1\}$ and $\{V^2, W^2, C^2\}$ correspond to the same vector $U$. For both sets, we have the equality (4.11) and subtraction of these two equations gives

$$\sum_{j,k=1}^{g} \left[ \frac{3}{4}(V_j^1 V_k^1 - V_j^2 V_k^2) - U_j(W_k^1 - W_k^2) \right] \hat{\theta}_{jk}[n] + (C^1 - C^2)\hat{\theta}[n] = 0.$$

The condition (4.15) implies

$$\begin{gathered} \frac{3}{4}[(V_j^1)^2 - (V_j^2)^2] - U_j(W_j^1 - W_j^2) = 0, \\ \frac{3}{2}(V_j^1 V_k^1 - V_j^2 V_k^2) - U_j(W_k^1 - W_k^2) - U_k(W_j^1 - W_j^2) = 0, \\ C^1 = C^2. \end{gathered} \tag{4.16}$$

By compatibility of these equations

$$(U_j V_k^1 - U_k V_j^1)^2 = (U_j V_k^2 - U_k V_j^2)^2,$$

hence

$$\begin{gathered} U_j(V_k^1 \pm V_k^2) = U_k(V_j^1 \pm V_j^2) \\ \text{and} \quad V = \pm(V + 2\alpha U) \end{gathered} \tag{4.17}$$

$W$ can be found by substituting (4.17) into the first equation of (4.16).

**Theorem 4.1.** *Let $B = [B_{jk}]$ be a Riemann matrix corresponding to a Riemann surface $\Gamma$ of genus $g \geq 4$. Then for $\Gamma$ in general position, the image of the projection $(U, V, W, C) \mapsto U$ from the solution set of the system (4.11) into $\mathbb{CP}^{g-1}$ is the canonical curve of $\Gamma$.*

*Proof.* For $g = 2$ and $g = 3$, see Theorem 4.3.1 in [2]. For $g \geq 4$, the complete proof is given in Theorem 4.3.2 in [2]. The proof considers the cases below separately and uses the corresponding facts.

case.1 $\Gamma$ is a general curve (some special, but possible cases omitted) of genus $g \geq 5$.
case.2 $\Gamma$ is a hyperelliptic curve.
case.3 $\Gamma$ is a trigonal curve.
case.4 $\Gamma$ is a plane curve of degree 5.

Fact.1 Let $(\theta)$ be a $\theta$-divisor. The mapping $\rho$ from $J(\Gamma)$ into $\mathbb{CP}^{g-1}$

$$\rho : z \in \text{supp}((\theta)) \mapsto \nabla\theta(z) = (\theta_1(z) : \cdots : \theta_g(z))$$

has rank $g - 1$ almost everywhere.

Fact.2 Let $\Gamma$ be a Riemann surface of genus $g \geq 5$. Define a set

$$\mathcal{S} = \{z \in \Gamma : \theta(z) = 0, \nabla\theta(z) = 0\}.$$

Note that $\mathcal{S} \subset J(\Gamma)$. Let $z \in \mathcal{S}$ be substituted in the equation (4.10). Then we have

$$\sum_{j,k=1}^{g} U_j U_k \theta_{jk}(z) = 0 \quad \text{for} \quad \theta(z) = 0, \quad \theta_j(z) = 0 \quad \text{for} \quad j = 1, \ldots, g. \qquad (4.18)$$

With the exceptions (2a) and (2b) below, the canonical curve $\Gamma'$ is the intersection of the tangent cones with the points in $\mathcal{S}$

$$\sum_{j,k=1}^{g} x_j x_k \theta_{jk}(z) = 0 \quad \text{and} \quad z \in \mathcal{S} \tag{4.19}$$

with homogeneous coordinates $(x_1 : \cdots : x_g)$.

(2a) Suppose that $\Gamma$ is a trigonal curve, i.e. there exists a meromorphic function $f$ on $\Gamma$ with a single third-order pole at a point $Q$. Then the equation (4.19) and

$$\sum_{j,k,l=1}^{g} x_j x_k x_l \theta_{jkl}(z) = 0 \quad \text{for} \quad z \in \mathcal{S} \quad \text{and} \quad \theta_{jk}(z) = 0 \quad \text{for} \quad j, k = 1, \ldots, g. \tag{4.20}$$

(2b) Suppose that $\Gamma$ is a smooth plane curve of degree 5. Then it is of genus 6 by the Genus-Degree Formula that gives the genus $g$ of a nonsingular plane curve and its degree $d$ satisfy $g = \frac{1}{2}(d-1)(d-2)$. The equation (4.19) defines a Veronese variety of the form $(x^2 : xy : xz : y^2 : yz : z^2)$ in $\mathbb{CP}^5$.

# 5. CONCLUSION

In this thesis our goal was to study the algebro-geometric construction of a family of solutions in terms of theta functions to the KP equation. The technique is due Krichever [1]. A consequence of this result utilizing the algebraic nature of these solutions is provided by the effectivization of the formulae for the constructed solutions, this is given in [2]. We used a large theory of Riemann surfaces including knowledge of Riemann theta functions and Baker-Akhiezer functions, and the method used is applicable to nonlinear equations of Zaharov-Shabat type. In our way of explaining these, we followed a survey article by Dubrovin [2].

Finally, we want to mention two problems in algebraic geometry that are related to this construction. The constructed solutions were in terms of Riemann theta functions corresponding to Riemann surfaces and this situation brought about new approaches for two old problems, e.g. a new proof for the classical Torelli theorem and a partial result for the solution of (still open) Schottky problem.

The classical Torelli theorem [15] implies that two algebraic curves of the same genus are birationally equivalent, i.e. there is an invertible rational map between them, if and only if the Riemann matrices corresponding to them are the same. One consequence of this result is that a Riemann surface is determined by its Jacobi variety and hence in order to study a Riemann surface, one can study Jacobi variety of that Riemann surface. By Theorem 4.2.4 and using properties of the map $\rho$, Dubrovin [2] gave a new proof to the Torelli Theorem.

Novikov conjectured that the compatibility of the system (4.11) will provide necessary and sufficient conditions on an arbitrary Riemann matrix to be a Riemann matrix corresponding to a Riemann surface, or matrix of $b$-periods of holomorphic differentials. Finding these conditions is a famous open problem called the Schottky problem or as it is sometimes called the Riemann problem. When genus of the surface is $1, 2$ or $3$, the solution to this problem simple and can be found in [2]. Schottky found the necessary conditions on a Riemann matrix to be Riemann matrix of $b$-periods of holomorphic differentials on a Riemann surface for the case $g = 4$ [16]. Indeed, the fact that explicit solutions to the KP equation are given in terms of Riemann theta functions turns out to provide a partial solution [17] to the Schottky problem. This solution by Dubrovin is partial since he proved that the relations on a theta function that are implied by (4.11) are distinctive properties of theta functions of Riemann surfaces except some possible irreducible components of the variety of all theta functions.

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