ON THE GEOMETRIC OBJECTS OF SAME TYPE AS CHRISTOFFEL SYMBOLS OF EHRESMANN ε–CONNECTIONS

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ABSTRACT

In this thesis the higher order Ehresmann ε–connections are studied as geometric objects. We prove that, any two Lie subgroups of the r^{th} order jet group G_n^r which are isomorphic to G_n^1 are conjugate. It follows from this result that the geometric objects defined by such subgroups are equivalent to the Ehresmann ε–connections.

KISA ÖZET

Bu tezde, yüksek mertebe Ehresmann ε–konneksiyonları geometrik nesneler olarak ele alınmıştır. Mertebesi r olan jet grubunun, birinci mertebe jet grubuna izomorfik olan altgruplarının eşlenik olduğu ispatlanmıştır. Bunun sonucu olarak, bu tür altgruplarla tanımlanan geometrik nesnelerin, Ehresmann ε–konneksiyonlarına eşdeğer olduğu gösterilmiştir.

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LIST OF SYMBOLS

B_n^k	Space of k-linear mappings
$\mathbb{C}$	Complex numbers
$F^r(M)$	r^{th} order frame bundle on M
G_n^r	r^{th} order jet group on an n-dimensional manifold
$j_x^r(f)$	r-jet of the function f at the point x.
$J^r(M,N)$	r^{th} order jet bundle with source M and target N
$\Re$	Real numbers
$\times$	Cartesian product
$\Gamma^i_{i_1 \dots i_k}$	Christoffel symbols of order k

1. INTRODUCTION

Since the beginning of the 20^{th} century, attempts have been made to define a concept general enough to include all objects that appear in differential geometry. A differential geometric object of order r on an n-dimensional manifold M is originally defined in [1]. A quite complete treatment of the theory appeared in [2] and a geometric object at a point p of M is defined as a set of m numbers, given in each coordinate system defined at p, with a consistent transformation rule which uses the derivatives of the coordinate transformations up to order r. Then, the geometric objects are classified according to the law of transformation of the set of m numbers. However, it is also possible that two geometric objects with different laws of transformation may define same entity. For example, consider a non-zero covariant vector $v=(v_1,v_2)$ on a two-dimensional manifold. It can also be given as $u=\left(\frac{v_1}{v_1^2+v_2^2},\frac{v_2}{v_1^2+v_2^2}\right)$, since there is a one-to-one correspondence between u and v. However, the rules of transformation of these entities are quite different. Later attempts to formulate the theory of geometric objects were made in [3] and [4] with the fiber bundle approach. In [3] not the components of the geometric object are primary but the space of the object and group of transformations operating on it. A. Nijenhuis first introduced the concept of natural bundles that made the problem independent of coordinates in his talk at the 1958 International Congress of Mathematicians. A rather complete historical survey can be found in his paper [5]. For the most recent developments in this field, see [6].

There are also a great number of papers concerning classification of geometric objects of order r with m components on an n-dimensional manifold, for several combinations of the parameters m,n,r. For the most general results about homogeneous geometric objects, see [7]

The purpose of this paper is to study the equivalence problem of special homogeneous objects of order r with codimension $m_r = n^3(n^{r-2}-1)/(n-1)$, namely the number of Christoffel symbols of Ehresmann ε-connections order r for r≥2, n>1, as defined in . We will shortly call them by Christoffel Symbols throughout the text. We adopt the fibre bundle approach developed in [3].

Let $G_n^r = G_n^1 \times B_n^2 \times ... \times B_n^r$ be the jet group of order r at the origin of $\Re^n$. For a subgroup H of G_n^r this problem is equivalent to find all conjugacy classes of subgroups of G_n^r isomorphic to H [3]. In order to study objects of same type as Christoffel symbols of order r, we choose H=G_n^1, that is, the general linear group. If the left coset space G_n^r / G_n^1 is identified with the subgroup $\{e\} \times B_n^2 \times ... \times B_n^r$ by the map $[(e,f_2,\dots,f_r)] \rightarrow (e,f_2,\dots,f_r)$, the standard action of G_n^r on G_n^r / G_n^1 becomes,

$$(a,f_2,\dots,f_r)(\mathbf{0})(e,h_2,\dots,h_r) = (a,f_2,\dots,f_r)(e,h_2,\dots,h_r)(a,0,\dots,0)^{-1}$$

Let $F^r(M) \rightarrow M$ be the r^th order frame bundle of M where dimM=n and $E^r(M,(\mathbf{0})) \rightarrow M$ be the associated bundle with fiber $\{e\} \times B_n^2 \times ... \times B_n^r$. Then, there is a one-to-one correspondence between

(i) The sections of $E^r(M,(\mathbf{0})) \rightarrow M$,

(ii) G_n^1-invariant sections of $F^k(M) \rightarrow F^1(M)$, which are called Ehresmann ε-connections [8],

(iii) Functions $\Gamma^i_{j_1 j_2}(x),\dots,\Gamma^i_{j_1 \dots j_r}(x)$ subject to the transformation rule

$$\left(\frac{\partial y^i}{\partial x^j},\dots,\frac{\partial^r y^i}{\partial x^{j_1}\dots\partial x^{j_r}}\right)\left(e,\Gamma^i_{j_1 j_2}(x),\dots,\Gamma^i_{j_1 \dots j_r}(x) \right)\left(\frac{\partial y^i}{\partial x^j},0\dots,0\right)^{-1}$$

which are called the Christoffel symbols of Ehresmann ε-connections [13].

Suppose T(θ) is a subgroup of G_n^r of the form $\{(a,\theta_2(a),\dots,\theta_r(a)), a \in G_n^1\}$, where $\theta_k : G_n^1 \rightarrow B_n^k$ are differentiable and hence analytic functions. Then, another associate bundle $E^r(M,(\theta))$ can be defined by the action

$$(a,f_2,\dots,f_r)(\theta)(e,h_2,\dots,h_r) = (a,f_2,\dots,f_r)(e,h_2,\dots,h_r)(a,\theta_2(a),\dots,\theta_r(a))^{-1}.$$

Thus, $E^r(M,(\theta))$ is a seemingly new geometric object bundle. Now, the problem is to obtain whether $E^r(M,(\mathbf{0}))$ and $E^r(M,(\theta))$ are equivalent in the sense of [3]. We will prove that the stabilizer subgroups of these two actions are conjugate and hence, the geometric object bundles defined by such subgroups are equivalent.

2. NOTATION AND PRELIMINARIES

2.1. Principal Fibre Bundles

In order to define geometric object bundles, we first define principal bundles. They can be defined either as the equivalence class of coordinate bundles [9] or in a coordinate free manner as follows [10]:

DEFINITION 2.1.1. Let M be a differentiable manifold and G a Lie group. A differentiable principal bundle over M with group G consists of a manifold P and an action of G on P satisfying the following conditions;

(i) G acts freely on P on the right, $(u,a)\in P\times G\to ua=R_a u\in P$

(ii) M is the quotient space of P by the equivalence relation induced by G, $M=P/G$, and the canonical projection $\pi:P\to M$ is differentiable.

(iii) P is locally trivial, that is, every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U\times G$ in the sense that there is a diffeomorphism

$\Psi:\pi^{-1}(U)\to U\times G$ with $\Psi(u)=(\pi(u),\varphi(u))$ where φ is a mapping of $\pi^{-1}(U)$ into G satisfying $\varphi(ua)=\varphi(u)a$ for all $a\in G_n^1$.

Then $P(M,G,\pi)$ is called the principal bundle with bundle space P, base space M, structure group G and projection π. For each $x\in M$, $\pi^{-1}(x)$ is a closed submanifold of P called the fibre.

Using the third condition one can choose an open covering $\{U_\alpha\}$ of M, each $\pi^{-1}(U_\alpha)$ provided with a diffeomorphism $u\to(\pi(u), \varphi_\alpha(u))$ of $\pi^{-1}(U_\alpha)$ onto $U_\alpha\times G$ such that $\varphi_\alpha(ua)=\varphi_\alpha(u)a$.

If $u\in U_\alpha\cap U_\beta$ then $\varphi_\beta(ua)\varphi_\alpha^{-1}(ua)=\varphi_\beta(u)\varphi_\alpha^{-1}(u)$ that implies $\varphi_\beta(u)\varphi_\alpha^{-1}(u)$ depends only on $\pi(u)$. Let $\psi_{\beta\alpha}:U_\alpha\cap U_\beta\to G$ where $\psi_{\beta\alpha}(\pi(u))=\varphi_\beta(u)\varphi_\alpha^{-1}(u)$. The mappings are called transition functions of the bundle corresponding to the open covering $\{U_\alpha\}$ of M satisfying

$$\psi_{\beta\alpha}(x)=\psi_{\beta\gamma}(x)\psi_{\gamma\alpha}(x) \text{ for all } x\in U_\alpha\cap U_\beta\cap U_\gamma \tag{1}$$

THEOREM 2.1.2. Let M be a differentiable manifold, $\{U_\alpha\}$ be an open covering of M, G be a Lie group and $\psi_{\beta\alpha}:U_\alpha\cap U_\beta\to G$ for all nonempty $U_\alpha\cap U_\beta$ in such a way that (1) is satisfied. Then one can construct a differentiable principal fibre bundle with base manifold M, structure group G.

For the proof see [11].

Given a principal fibre bundle P(M,G) and a manifold F on which G acts on the left as $(a,x)\in G\times F\to ax\in F$, one can construct a new bundle in the following way. On the product manifold P×F, a right action of G can be defined as,

$(u,x)\in P\times F\to(ua,a^{-1}x)\in P\times F$

The quotient set of P×F by this action is denoted $E=P\times_G F$. The mapping $\pi_E:P\times_G F\to M$ where $\pi_E([u,x])=\pi(u)\in M$, called the projection of E onto M. For each $x\in M$, $\pi_E^{-1}(x)$ is called the fibre of E over x. The differentiable structure of P×F induces a differentiable structure on E [11].

DEFINITION 2.1.3. The above defined fibre bundle with base space M, structure group G and fibre F is called the associate bundle of P(M,G) and denoted by $E(M,G,F,\pi_E)$ [11].

2.2. Jet Groups and Frame Bundles

Let M and N be differentiable manifolds of dimension m and n respectively. Let U and V be neighborhoods of a point x in M and let $f:U\to N$ and $g:V\to N$. Then, f and g give rise to same r-jet at x if they have the same partial derivatives up to order r at x with respect to some local coordinate systems in M and N. Using chain rule of partial derivatives of order r, one can show that this definition is independent of choice of coordinate system. The equivalence class of f, thus defined, is denoted by $j^r_x(f)$. The set of r-jets $j^r_x(f)$ at x is denoted by $J^r_x(M,N)$ and let $J^r(M,N)=\bigcup_{x\in M} J^r_x(M,N)$.

Let $\alpha(j^r_x(f))=x$ and $\beta(j^r_x(f))=f(x)$, where x and $f(x)$ are called the source and the target of $j^r_x(f)$ respectively. Then, $J^r(M,N)$ is a fibre bundle over M, N and M×N with projections α,β and $\alpha\times\beta$ respectively.

If r>q a natural projection is defined by $\mu_r^q : J^r(M,N) \to J^q(M,N)$, where $\mu_r^q(j_x^r(f)) = j_x^q(f)$. If r>q>p, then $\mu_q^p \circ \mu_r^q = \mu_r^p$. Let $j_x^r(f) \in J^r(M,N)$ and $j_x^r(g) \in J^r(N,P)$, where $f(x)=y$, then $j_x^r(g \circ f) = j_y^r(g) \circ j_x^r(f)$

An r-jet $j_x^r(f) \in J^r(M,N)$ is said to be invertible, if there is an r-jet $j_x^r(g) \in J^r(N,M)$ such that $j_x^r(f) \circ j_y^r(g) = j_y^r(\mathrm{id}_y)$, $j_y^r(g) \circ j_x^r(f) = j_x^r(\mathrm{id}_x)$, where id_x is the identity map of M onto itself defined in a neighborhood of x. It follows from the implicit function theorem that $j_x^r(f)$ is invertible if and only if $j_x^1(f)$ is.

Let $J_0^r(\mathfrak{R}^n, M)$ denotes the space of r-jets of real n-dimensional vector space $\mathfrak{R}^n$ into an n-dimensional manifold M with source at the origin 0 of $\mathfrak{R}^n$. An invertible r-jet $j_0^r(f) \in J_0^r(\mathfrak{R}^n, M)$ is called an r-frame of M at $x=f(0)$. The set of r-frames of M will be denoted by $F^r(M)$. It is a fibre bundle over M with natural projection π, $\pi(j_0^r(f)) = f(0)$. The structure group of $F^r(M)$ is defined as follows: Let G_n^r be the set of invertible r-jets $j_0^r(g) \in J_0^r(\mathfrak{R}^n, \mathfrak{R}^n)$ with source 0 and target 0. The G_n^r is a group with multiplication defined by the composition of jets. Let $u \in F^r(M)$ and $s \in G_n^r$. Then, $u = j_0^r(f) \in J_0^r(\mathfrak{R}^n, M)$ and $s = j_0^r(g) \in J_0^r(\mathfrak{R}^n, \mathfrak{R}^n)$. Define $us = j_0^r(g \circ f) \in J_0^r(\mathfrak{R}^n, M)$. Then, G_n^r acts transitively on each fibre of $F^r(M)$.

DEFINITION 2.2.1. $F^r(M)$ is a principal fibre bundle over M with group G_n^r and is called the bundle of r-frames on M [10].

Choosing standard coordinates in $\mathfrak{R}^n$, $G_n^r = GL(n,\mathfrak{R}) \times B_n^2 \times \ldots \times B_n^r$ where each B_n^k is the space of k-linear mappings of $\mathfrak{R}^n \times \ldots \times \mathfrak{R}^n \to \mathfrak{R}^n$, the elements of G_n^r can be identified with pairs $\left(a_{j_1}^i, f_{j_1 j_2}^i, \ldots, f_{j_1 \cdots j_r}^i\right)$, $1 \leq i, j_1, \ldots, j_r \leq n$. Then, group operation can be written as

$$\left(a_{j_1}^i, f_{j_1 j_2}^i, \ldots, f_{j_1 \cdots j_r}^i\right)\left(b_{j_1}^i, g_{j_1 j_2}^i, \ldots, g_{j_1 \cdots j_r}^i\right)$$

$$= \left(a_{s_1}^i b_{j_1}^{s_1}, a_{s_1}^i g_{j_1 j_2}^{s_1} + f_{s_1 s_2}^i b_{j_1}^{s_1} b_{j_2}^{s_2}, \ldots, a_{s_1}^i g_{j_1 \cdots j_r}^{s_1} + f_{s_1 \cdots s_r}^i b_{j_1}^{s_1} \ldots b_{j_r}^{s_r} + \Phi\left(f_{j_1 j_2}^i, \ldots, f_{j_1 \cdots j_{r-1}}^i, b_{j_1}^i, \ldots, g_{j_1 \cdots j_{r-1}}^i\right)\right)$$

and also by

$$\left(a, f_2, \ldots, f_r\right)\left(b, g_2, \ldots, g_r\right) = \left(ab, ag_2 + f_2 b, \ldots, ag_r + f_r b + \Phi\left(f_2, \ldots, f_{r-1}, b, g_2, \ldots, g_{r-1}\right)\right)$$

where $(af)(x_1, \ldots, x_r) = a(f(x_1, \ldots, x_r))$, $(fa)(x_1, \ldots, x_r) = f(ax_1, \ldots, ax_r)$ and Φ is a polynomial in its entries. For the details on jet groups and frame bundles we refer [10].

2.3. Bundle of Geometric Objects and Equivalence of Geometric Objects

Let M be a differentiable manifold with an atlas (U_i,φ_i) and G be an effective topological transformation group of a Hausdorff space Y. If U is a subspace of G, then $(U\bullet y)$ is a subspace of Y for all y in Y. Furthermore it is assumed that,

(i) The subspace $(U\bullet y)$ is open in $(G\bullet y)$ for each U open in G and $y\in Y$.

(ii) Let $h_x: G^r_n\to G$ for each $x\in M$ be a homomorphism of G^r_n onto G that depends continuously on x.

For $x\in U_i\cap U_j$, define $g_{ij}(x)=h_x(j^r_0(\varphi_j^{-1}\varphi_i))$ then, $g_{kj}(x)=g_{ki}(x)g_{ij}(x)$

It follows from Theorem 2.1.2 that there exists a fibre bundle with bundle space B, base space M, fibre Y, group G, and transition functions $g_{ij}(x)$. Another allowable coordinate system leads to another bundle equivalent to B.

DEFINITION 2.3.2. The fibre bundle (B,M,Y,G,h) defined above is called a bundle of geometric objects of order $\leq$ k over M of type h. A point of this bundle is called a geometric object. A cross-section of this bundle is called a geometric object field [3].

REMARK: The natural projection $\mu^k_{k+1}:G^{k+1}_n\to G^k_n$ is a homomorphism. Then, a geometric object bundle of order $\leq$ k is also of order $\leq$ k+1. The bundle of geometric objects is said to be of order k if it is of order $\leq$ k but not of order $\leq$ k-1.

Let B and B′ be two bundles of geometric objects over the same base space M, with fibres Y and Y′ groups G and G′ respectively. Let $h: G^k_n\to G$, $h': G^k_n\to G'$.

DEFINITION 2.3.3. Two geometric object bundles, B and B′ are equivalent if there exists a continuous map f:B→B′, satisfying the following commutative diagrams,

$$\text{(i)}\quad \begin{array}{ccc} B & \xrightarrow{f} & B' \\ \downarrow p & & \downarrow p' \\ M & \xrightarrow{id_M} & M \end{array} \qquad \text{(ii)}\quad \begin{array}{ccccccc} U_i\cap U_j\times Y & \xrightarrow{\varphi_i} & p^{-1}(U_i\cap U_j) & \xrightarrow{f} & p'^{-1}(U_i\cap U_j) & \xrightarrow{\varphi_i'^{-1}} & U_i\cap U_j\times Y' \\ \downarrow g_{ij} & & & & & & \downarrow g_{ij} \\ U_i\cap U_j\times Y & \xrightarrow{\varphi_i} & p^{-1}(U_i\cap U_j) & \xrightarrow{f} & p'^{-1}(U_i\cap U_j) & \xrightarrow{\varphi_i'^{-1}} & U_i\cap U_j\times Y' \end{array}$$

Let $\lambda_i: U_i\cap U_j\times Y\to U_i\cap U_j\times Y'$ where $\lambda_i(x)=\varphi_i'^{-1}\circ f\circ\varphi_i$ be the map from Y onto Y′. Then $\lambda_i(x)=\lambda_j(x)$ for each $x\in U_i\cap U_j$ hence, λ_i is well-defined.

The first condition asserts that f carries each fibre Y_x to Y'_x homeomorphically. The second is the condition of coordinate independence of the mappings between local trivializations.

THEOREM 2.3.4. Two bundles of geometric objects are equivalent if and only if the structure groups G and G′ acting on the fibres Y and Y′ respectively are related as $g'=\lambda(x)\circ g\circ\lambda^{-1}(x)$ [3].

Proof:

$\Rightarrow$: Assume the geometric object bundles are equivalent. Then,
$\lambda_i(x)=\varphi_i'^{-1}\circ f\circ\varphi_i=\varphi_i'^{-1}\varphi_j'\circ(\varphi_j'^{-1}\circ f\circ\varphi_j)\circ\varphi_j^{-1}\varphi_i=g'_{ij}(x)\circ\lambda_j(x)\circ g_{ji}(x)$ for all $x\in U_i\cap U_j$.
$\Rightarrow\lambda(x)=g'_{ij}(x)\circ\lambda(x)\circ g_{ji}(x)$ since $\lambda_i(x)=\lambda_j(x)$ for all $x\in U_i\cap U_j$.
$\Rightarrow g'_{ij}(x)=\lambda(x)\circ g_{ji}(x)\circ\lambda^{-1}(x)$ for all $x\in U_i\cap U_j$.
$\Leftarrow$: Assume $g'_{ij}(x)=\lambda(x)\circ g_{ji}(x)\circ\lambda^{-1}(x)$ for all $x\in U_i\cap U_j$, where $\lambda(x)$ is a homeomorphism of Y onto Y′ depending continuously on x. Then, $\lambda(x):M\times Y\to Y'$ is continuous. Define f of B onto B′ as $\varphi'(x,\lambda(x)y)=f\circ\varphi(x,y)$. Then f satisfies the conditions for equivalence,
(i) $p\circ\varphi'(x,\lambda(x)y)=p'\circ f\circ\varphi(x,y)\Rightarrow x=p'\circ f\circ\varphi(x,y)\Rightarrow p\circ\varphi(x,y)=p'\circ f\circ\varphi(x,y)\Rightarrow p=p'\circ f$
(ii) $g'_{ij}(x)=\lambda(x)\circ g_{ji}(x)\circ\lambda^{-1}(x)$ is our assumption.

Note that, the sufficient condition for the equivalence of fibre bundles is that the function f defined above is an arbitrary homeomorphism [9]. However, in the case of geometric object bundles, f must be an inner automorphism on each fibre.

2.4. Transitive Geometric Object Bundles

DEFINITION 2.4.1. A geometric object bundle is said to be transitive if its structure group is a transitive topological transformation group of the fibre Y [3].

Let $H_y=\{g\in G, gy=y\}$ be the stabilizer group of $y\in Y$. Note that H_y is a closed subgroup. Since G is transitive there exists a one-to-one correspondence between G/H and Y which is actually the orbit of y. Consider the natural left action of G on G/H. There is a geometric object bundle of type h over M with group G and fibre G/H, so define $\lambda:Y\to G/H$ where $\lambda(gy)=gH$ and $g'=\lambda(y)g\,\lambda^{-1}(y)$.

THEOREM 2.4.2. Any transitive geometric object bundle with group G is equivalent to a bundle of same type and with the coset space G/H as fibre [3].

Proof: Construct λ as stated above and use Theorem 2.3.4 [3].

Let K be the closed subgroup of G_n^r such that $h(K)=H$. Let K_0 be the intersection of all subgroups conjugate to K. Then K_0 is the largest normal subgroup of K that is normal in G_n^r. If the natural left action of G_n^r over G_n^r/K is effective, then $K_0=\{e\}$. If it is not, then the action of G_n^r/K_0 on G_n^r/K is effective. In order to show that G_n^r/K_0 is a topological transformation group of G_n^r/K, assume F is open in G_n^r. Then, FK_0 is open in G_n^r/K_0 with the quotient topology. If $aK \in G/K$, $FK_0 \bullet aK = F \bullet aK$ which is open in G_n^r/K with quotient topology. Then, G_n^r/K_0 is a topological transformation group acting on G_n^r/K. Hence, $(M, G_n^r/K_0, G_n^r/K, h')$ where h' is the natural homomorphism of G_n^r onto G_n^r/K_0, is a geometric object bundle.

THEOREM 2.4.3. The bundles of geometric objects $(M, G_n^r/K_0, G_n^r/K, h')$ and $(M, G, G/H, h)$ are equivalent [3].

Proof:

Let $\tau: G_n^r/K \to G/H$ with $\tau(aK)=h(a)H$. Then τ is onto since h is. $h(a)H=h(b)H \Rightarrow h(a^{-1}b) \in H \Rightarrow a^{-1}b \in K \Rightarrow aK=bK$. Hence τ is a bijection. Now, the diagram below is commutative,

$$\begin{array}{ccc} G_n^r & \xrightarrow{\mu} & G_n^r/K \\ \downarrow h & & \downarrow \tau \\ G & \xrightarrow{\nu} & G/H \end{array}$$

where ν, μ, h are open mappings and hence τ and τ^{-1} are continuous. Hence, τ is a homeomorphism and each fibre G_n^r/K is homeomorphic to G/H.

Let $g \in G$, $g' \in G_n^r/K_0$ with $g=h(a)$, $g'=h'(a)=aK_0$ for some $a \in G_n^r$. If $bK \in G_n^r/K$ then, $h'(a)bK=abK$,

$\Rightarrow \tau(abK)=h(ab)H=h(a)h(b)H=h(a)\tau(bK)=g \circ \tau(bK)$

$\Rightarrow \tau(abK)=\tau(h'(a)bK)=\tau g'(bK)=\tau \circ g'(bK) \Rightarrow g'=\tau^{-1} \circ g \circ \tau$ [3].

THEOREM 2.4.4. Two transitive geometric object bundles of same order with fibres Y and Y′ are equivalent if and only if the closed groups K and K′ of G^r_n are conjugate where K and K′ are the preimages of the stabilizers of arbitrarily chosen points of $y \in Y$ and $y' \in Y'$ respectively [3].

Proof:

$\Rightarrow$:Let (M,Y,G,h) and (M,Y',G',h') be two equivalent geometric object bundles. Then there are corresponding geometric object bundles $(M, G^r_n/K_0, G^r_n/K, \sigma)$ and $(M, G^r_n/K'_0, G^r_n/K', \sigma')$ Since these two bundles are equivalent, there is a homeomorphism τ of G^r_n/K onto G^r_n/K' with $\tau(\sigma(g)y)=\sigma'(g)\tau(y)$ $g \in G^r_n$, $y \in G^r_n/K$. Let $\sigma(a)$ $a \in K$ fixes a point $bK \in G^r_n/K$. Note that, $\sigma(a)bK=abK=bK \Rightarrow b^{-1}abK=K \Rightarrow \sigma(b^{-1}ab)K=K$ for $a \in K$ and for some $b \in G^r_n$. Hence $b^{-1}Kb$ is the preimage of the stabilizer of the point $K \in G^r_n/K$ and it is the complete set of points with this property. Same argument holds for the set $c^{-1}K'c$ for some $c \in G^r_n$. Then $\tau(\sigma(b^{-1}ab)K)=\sigma'(b^{-1}ab)\tau(K)=\sigma'(b^{-1}ab)K'=\tau(K)=K'$.It follows that $b^{-1}ab \in c^{-1}K'c$ for all $a \in K \Rightarrow b^{-1}Kb \subset c^{-1}K'c$. Hence K and K′ are conjugate.
$\Leftarrow$:Assume K and K′ are conjugate, i.e., $K'=gKg^{-1} \Rightarrow K'_0=K_0 \Rightarrow G^r_n/K_0 = G^r_n/K'_0$. Define q: $G^r_n \to G^r_n/K'_0$ where $q(a)=gag^{-1} \Rightarrow q(K)=K'$. By Theorem 2.3.4 the geometric object bundles are equivalent [3].

3. CHRISTOFFEL SYMBOLS OF EHRESMANN ε-CONNECTIONS AS GEOMETRIC OBJECTS

3.1. Definition of Christoffel Symbols of Ehresmann ε–connections

Let $F^r(M)\to M$ be the r^{th} order frame bundle of M of order r. Let Γ be a section of $\mu^1_r : F^r(M)\to F^1(M)$. For details on higher order frame bundles we refer [10] and the references therein. If $(U;x^i)$ is a coordinate neighborhood in M and $V=(\mu^1_r)^{-1}(U)\subseteq F^1(M)$, then Γ is given on $(V;x^i,\mathrm{x}^i_j)$ by

$$\Gamma(x,\mathbf{x})=\left(x,\mathbf{x},\Gamma^i_{j_1j_2}(x,\mathbf{x}),\ldots,\Gamma^i_{j_1\cdots j_r}(x,\mathbf{x})\right)$$

DEFINITION 3.1.1. G^1_n–invariant sections of $\mu^1_r:F^r(M)\to F^1(M)$ are called Ehresmann ε–connections.

THEOREM 3.1.2. There is a one-to-one correspondence between

(i) G^1_n–invariant sections of $F^r(M)\to F^1(M)$, called Ehresmann ε-connections [8].

(ii) Functions $\Gamma^i_{j_1j_2}(x),\ldots,\Gamma^i_{j_1\cdots j_r}(x)$ subject to the transformation rule

$$\left(\frac{\partial y^i}{\partial x^j},\ldots,\frac{\partial^r y^i}{\partial x^{j_1}\ldots\partial x^{j_r}}\right)\left(e,\Gamma^i_{j_1j_2}(x),\ldots,\Gamma^i_{j_1\cdots j_r}(x)\right)\left(\frac{\partial y^i}{\partial x^j},0,\ldots,0\right)^{-1}=\left(e,\Gamma^i_{j_1j_2}(y),\ldots,\Gamma^i_{j_1\cdots j_r}(y)\right)$$

which are called the Christoffel Symbols of Ehresmann ε-connections [13].

Proof:

(i)⇒(ii): Suppose Γ be a G^1_n–invariant section of $F^r(M)\to F^1(M)$ then,

$$\Gamma^i_{j_1\cdots j_k}\left(x^i,\mathrm{x}^i_m a^m_j\right)=\Gamma^i_{m_1\cdots m_k}\left(x^i,\mathrm{x}^i_j\right)a^{m_1}_{j_1}\ldots a^{m_k}_{j_k}\quad\text{for all }\left(a^i_j\right)\in G^1_n,\ 2\le k\le r.$$

Now, let $\mathrm{x}^i_j=\delta^i_j$, replace x^i_j by a^i_j and write $\Gamma^i_{j_1\cdots j_k}\left(x^i,\delta^i_j\right)=\Gamma^i_{j_1\cdots j_k}\left(x^i\right)$ then,

$$\Gamma^i_{j_1\cdots j_k}(x,\mathbf{x})=\Gamma^i_{m_1\cdots m_k}(x)\,\mathrm{x}^{m_1}_{j_1}\ldots\mathrm{x}^{m_k}_{j_k}.$$

Let $\Gamma^{i}_{j_1\cdots j_k}(x,\mathbf{x})=X^{i}_{j_1\cdots j_k}$ since,

$$\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\left(x^i_{i_1},\ldots,x^i_{j_1\cdots j_k}\right)=\left(y^i_{i_1},\ldots,y^i_{j_1\cdots j_k}\right)$$

then,

$$\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\left(x^i_{i_1},\ldots,x^i_{j_1\cdots j_k}\right)\left(y^i_{i_1},0,\ldots,0\right)^{-1}=\left(y^i_{i_1},\ldots,y^i_{j_1\cdots j_k}\right)\left(y^i_{i_1},0,\ldots,0\right)^{-1}$$

$$\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\left(x^i_{i_1},\ldots,x^i_{j_1\cdots j_k}\right)\mu^1_k\left(\left(y^i_{i_1},\ldots,y^i_{j_1\cdots j_k}\right)^{-1}\right)=\left(y^i_{i_1},\ldots,y^i_{j_1\cdots j_k}\right)\left(y^i_{i_1},0,\ldots,0\right)^{-1}$$

$$=\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\left(x^i_{i_1},\ldots,x^i_{j_1\cdots j_k}\right)\mu^1_k\left(\left(\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\left(x^i_{i_1},\ldots,x^i_{j_1\cdots j_k}\right)\right)^{-1}\right)$$

$$=\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\left(x^i_{i_1},\ldots,x^i_{j_1\cdots j_k}\right)\mu^1_k\left(\left(x^i_{i_1},\ldots,x^i_{j_1\cdots j_k}\right)\right)^{-1}\mu^1_k\left(\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\right)^{-1}$$

$$=\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\left(x^i_{i_1},\ldots,x^i_{j_1\cdots j_k}\right)\left(x^i_{i_1},0,\ldots,0\right)^{-1}\mu^1_k\left(\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\right)^{-1}$$

$$=\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\left(e,\Gamma^i_{i_1 i}(x),\ldots,\Gamma^i_{j_1\cdots j_k}(x)\right)\mu^1_k\left(\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\right)^{-1}$$

$$\Rightarrow\left(\frac{\partial y^i}{\partial x^{i_1}},\ldots,\frac{\partial^k y^i}{\partial x^{i_1}\ldots\partial x^{i_k}}\right)\left(e,\Gamma^i_{i_1 i}(x),\ldots,\Gamma^i_{j_1\cdots j_k}(x)\right)\left(\frac{\partial y^i}{\partial x^{i_1}},0\ldots,0\right)^{-1}=\left(e,\Gamma^i_{i_1 i}(y),\ldots,\Gamma^i_{j_1\cdots j_k}(y)\right)$$

(ii)$\Rightarrow$(i):Let $\Gamma^i_{j_1 j_2}(x),\ldots,\Gamma^i_{j_1\cdots j_r}(x)$ subject to the transformation rule,

$$\left(\frac{\partial y^i}{\partial x^{j}},\ldots,\frac{\partial^r y^i}{\partial x^{j_1}\ldots\partial x^{j_r}}\right)\left(e,\Gamma^i_{j_1 j}(x),\ldots,\Gamma^i_{j_1\cdots j\,r}(x)\right)\left(\frac{\partial y^i}{\partial x^{j}},0,\ldots,0\right)^{-1}=\left(e,\Gamma^i_{j_1 j}(y),\ldots,\Gamma^i_{j_1\cdots j\,r}(y)\right).$$

Then, let $\Gamma^i_{j_1\cdots j_k}\left(x^i,x^i_m\right)=\Gamma^i_{m_1\cdots m_k}\left(x^i\right)x^{m_1}_{j_1}\ldots x^{m_k}_{j_k}$ for all $\left(a^i_j\right)\in G^1_n$. Clearly $\Gamma(x,\mathbf{x})=\left(x,\mathbf{x},\Gamma^i_{j_1 j_2}(x,\mathbf{x}),\ldots,\Gamma^i_{j_1\cdots j_r}(x,\mathbf{x})\right)$ is a G^1_n–invariant section of $F^r(M)\rightarrow F^1(M)$.

For r=2 the Christoffel Symbols of Ehresmann ε–connections transforms as,

$$\left(\frac{\partial y^i}{\partial x^{j}},\frac{\partial^2 y^i}{\partial x^{j_1}\partial x^{j_2}}\right)\left(e,\Gamma^i_{j_1 j_2}(x)\right)\left(\frac{\partial y^i}{\partial x^{j}},0\right)^{-1}=\left(e,\Gamma^i_{j_1 j_2}(y)\right)$$

$$=\left(\frac{\partial y^i}{\partial x^{j}},\frac{\partial^2 y^i}{\partial x^{j_1}\partial x^{j_2}}+\frac{\partial y^i}{\partial x^{r_1}}\Gamma^{r_1}_{j_1 j_2}(x)\right)\left(\frac{\partial x^j}{\partial y^i},0\right)=\left(e,\frac{\partial^2 y^i}{\partial x^{r_1}\partial x^{r_2}}\frac{\partial x^{r_1}}{\partial y^{j_1}}\frac{\partial x^{r_2}}{\partial y^{j_2}}+\frac{\partial y^i}{\partial x^{r_1}}\Gamma^{r_1}_{s_1 s_2}(x)\frac{\partial x^{s_1}}{\partial y^{j_1}}\frac{\partial x^{s_2}}{\partial y^{j_2}}\right)$$

$$\Rightarrow\Gamma^i_{j_1 j_2}(y)=\frac{\partial y^i}{\partial x^{r_1}}\Gamma^{r_1}_{s_1 s_2}(x)\frac{\partial x^{s_1}}{\partial y^{j_1}}\frac{\partial x^{s_2}}{\partial y^{j_2}}+\frac{\partial^2 y^i}{\partial x^{r_1}\partial x^{r_2}}\frac{\partial x^{r_1}}{\partial y^{j_1}}\frac{\partial x^{r_2}}{\partial y^{j_2}}$$

In order to construct the bundle of Christoffel symbols over a manifold M, let $E^r(M,(0))\to M$ be the associate bundle with base space M, fibre $\{e\}\times B^2_n\times...\times B^r_n$ and group G^r_n where the action (0) of G^r_n on $\{e\}\times B^2_n\times...\times B^r_n$ is defined as

$$(a,f_2,\ldots,f_r)(0)(e,h_2,\ldots,h_r)=(a,f_2,\ldots,f_r)(e,h_2,\ldots,h_r)(a,0,\ldots,0)^{-1}$$

for $(e,h_1,\ldots,h_r)\in\{e\}\times B^2_n\times...\times B^r_n$ and $(a,f_1,\ldots,f_r)$, $(a,0,\ldots,0)\in G^r_n$.

The fibres of this bundle are in one-to-one correspondence with $G^r_n/T(0)$, where $T(0)=\{(a,0,\ldots,0), a\in G^1_n\}$ and $G^r_n/T(0)$ is the left coset space.

PROPOSITION 3.1.3. The mapping $\rho:\{e\}\times B^2_n\times...\times B^r_n\to G^r_n/T(0)$ with $\rho(e,f_2,\ldots,f_r)=[(e,f_2,\ldots,f_r)]$ is a bijection.

Proof:

Let $[(a,f_2,\ldots,f_r)]\in G^r_n/T(0)$. Surjectivity of ρ is obvious from definition. Since $(a,f_2,\ldots,f_r)=(a,0,\ldots,0)(e,a^{-1}f_2,\ldots,a^{-1}f_r)\Rightarrow[(a,f_2,\ldots,f_r)]=[(e,a^{-1}f_2,\ldots,a^{-1}f_r)]$.

Let $[(e,h_2,\ldots,h_r)],[(e,g_2,\ldots,g_r)]\in G^r_n/T(0)$ with $[(e,h_1,\ldots,h_r)]=[(e,g_2,\ldots,g_r)]$. Then, $(e,h_2,\ldots,h_r)=(a,0,\ldots,0)(e,g_2,\ldots,g_r)$ $\exists(a,0,\ldots,0)\in T(0)$ $\Rightarrow(a,h_2,\ldots,h_r)=(a,ag_2,\ldots,ag_r)$ $\Rightarrow$ $a=e$, $h_m=g_m$. Hence ρ is injective.

Since $T(0)$ is isomorphic to G^1_n, and $G^r_n/T(0)$ is identified with the subgroup $(e,B^2_n\times...\times B^r_n)$ through ρ, the standart action of G^r_n on G^r_n/G^1_n becomes

$$(a,f_2,\ldots,f_r)(0)(e,h_2,\ldots,h_r)=(a,f_2,\ldots,f_r)(e,h_2,\ldots,h_r)(a,0,\ldots,0)^{-1}$$

Let Γ be a section of $E^r(M,(0))\to M$, then there exists unique functions $\Gamma^i_{j_1j_2}(x),\ldots,\Gamma^i_{j_1\ldots j_r}(x)$, given in each coordinate neighborhood $(U;x^1,\ldots,x^r)$ such that Γ on U is given by,

$\Gamma(x)=[(x,e,0,\ldots,0),(e,\Gamma^i_{j_1j_2}(x),\ldots,\Gamma^i_{j_1\ldots j_r}(x))]$. Hence, on $(U;x^1,\ldots,x^r)\cap(V;y^1,\ldots,y^r)$,

$$\Gamma(y)=[(y,\frac{\partial y^i}{\partial x^{j_1}},\ldots,\frac{\partial^r y^i}{\partial x^{j_1}\ldots\partial x^{j_r}}),(e,\Gamma^i_{j_1j_2}(y),\ldots,\Gamma^i_{j_1\ldots j_r}(y))]$$

$$=[(x,e,0,\ldots,0)(\frac{\partial y^i}{\partial x^{j_1}},\ldots,\frac{\partial^r y^i}{\partial x^{j_1}\ldots\partial x^{j_r}}),(e,\Gamma^i_{j_1j_2}(y),\ldots,\Gamma^i_{j_1\ldots j_r}(y))]$$

$$=[(x,e,0,\ldots,0),\ (\frac{\partial y^i}{\partial x^{j_1}},\ldots,\frac{\partial^r y^i}{\partial x^{j_1}\ldots\partial x^{j_r}})^{-1}(0)(e,\Gamma^i_{j_1j_2}(y),\ldots,\Gamma^i_{j_1\ldots j_r}(y))]$$

$$\Rightarrow(e,\Gamma^i_{j_1j_2}(y),\ldots,\Gamma^i_{j_1\ldots j_r}(y))=(\frac{\partial y^i}{\partial x^{j_1}},\ldots,\frac{\partial^r y^i}{\partial x^{j_1}\ldots\partial x^{j_r}})(0)(e,\Gamma^i_{j_1j_2}(x),\ldots,\Gamma^i_{j_1\ldots j_r}(x)).$$

3.2. Geometric Objects Of Same Type As Christoffel Symbols

In order to construct geometric objects of same type as Christoffel symbols, one may search for different associated bundles of $F^r(M)\to M$ defined by different actions of G^1_n. In fact as stated in [3], the problem reduces to determination of all conjugacy classes of subgroups isomorphic to G^1_n. Now, we will consider the subgroups of the form $T(\theta)=\{(a,\theta_2(a),\ldots,\theta_r(a)),a\in G^1_n\}$

By this subgroup, one can define another associate bundle $E^r(M,(\theta))\to M$ of $F^r(M)\to M$ with base space M, fibre $\{e\}\times B^2_n\times\ldots\times B^r_n$ and group G^r_n acting on $\{e\}\times B^2_n\times\ldots\times B^r_n$ as,

$$(a,f_2,\ldots,f_r)(\theta)(e,h_2,\ldots,h_r) = (a,f_2,\ldots,f_r)\,(e,h_2,\ldots,h_r)\,(a,\theta_2(a),\ldots,\theta_r(a))^{-1}$$

It is easy to check that $E^r(M,(\theta))\to M$ is a transitive geometric object bundle.

THEOREM 3.2.1. The stabilizer subgroup of the action $(\mathbf{0})$ of G^r_n on $\{e\}\times B^2_n\times\ldots\times B^r_n$, $St(e,h_2,\ldots,h_r)$, is a subgroup of the form $T(\phi)$ where $\phi=(\phi_2,\ldots,\phi_r)$ is a polynomial in the entries $a,h_2,\ldots,h_r$. Furthermore the correspondence between $(e,h_2,\ldots,h_r)$ and $(\phi_2,\ldots,\phi_r)$ is injective [12].

Proof:

Consider the stabilizer subgroup $St(e,0,\ldots,0,h_k)$ of the action $(\mathbf{0})$ of G^k_n on $\{e\}\times B^2_n\times\ldots\times B^k_n$, i.e.

$(a,f_2,\ldots,f_k)(\mathbf{0})\,(e,0,\ldots,0,h_k)=(e,0,\ldots,0,h_k)$

$(a,f_2,\ldots,f_k)\,(e,0,\ldots,0,h_k)(a,0,\ldots,0)^{-1}=(e,0,\ldots,0,h_k)$

$(a,f_2,\ldots,f_k)\,(e,0,\ldots,0,h_k)= (e,0,\ldots,0,h_k)(a,0,\ldots,0)$

$(a,f_2,\ldots,f_k+ah_k)= (e,0,\ldots,0,h_ka)\Rightarrow f_k=h_ka-ah_k$.

Hence $St(e,0,\ldots,0,h_k)=\{(a,0,\ldots,0,\ h_ka-ah_k)\mid a\in G^1_n\}$.

If $h_ka-ah_k= d_ka-ad_k$ for all $a\in G^1_n$ then,

$(h_k-d_k)a-a(h_k-d_k)=0$

$\Rightarrow$ Choosing $a=\lambda e$ where $\lambda\neq 0,1$ gives $h_k=d_k$ which proves the injectivity.

For k=2,

$St(e,h_2)=\{(a,h_2a-ah_2)\mid a\in G^1_n\}$ and the correspondence between (e,h_2) and (a,h_2a-ah_2) is injective.

Assuming the truth of the statement for $k-1$, let

$(a,f_1,\ldots,f_k)(0)\ (e,h_2,\ldots,h_k)=(e,h_2,\ldots,h_k)$.

Projecting this equation to G_n^{k-1} and using induction hypothesis,

$f_m=\theta_m(a,h_2,\ldots,h_m)$, $2\leq m\leq k-1$ where f_m is of order m in a.

Since, $(a,f_2,\ldots,f_k)(e,h_2,\ldots h_k)(a,0,\ldots,0)^{-1}=(e,h_2,\ldots h_k)$

$(a,f_2,\ldots,f_k)(e,h_2,\ldots h_k)=(e,h_2,\ldots h_k)\ (a,0,\ldots,0)=(a,h_2a,\ldots,h_ka)$

$\Rightarrow f_k+ah_k+P(f_2,\ldots,f_{k-1},h_2,\ldots h_{k-1})=h_ka.$

$\Rightarrow f_k=h_ka-ah_k-P(f_2,\ldots,f_{k-1},h_2,\ldots h_{k-1}).$

Let $f_k=d_ka-ad_k-P(f_2,\ldots,f_{k-1},d_2,\ldots d_{k-1})$.

By the induction hypothesis $d_n=h_n$ for $n\leq k-1$ then,

$h_ka-ah_k=d_ka-ad_k\Rightarrow h_k=d_k$ [12].

There is also a converse problem: Whether any subgroup $T(\phi_2,\ldots,\phi_r)$ is of the form $St(e,h_2,\ldots,h_r)$ for some $h_2,\ldots,h_r$? The following proposition reduces the problem to the case of $T(0,\ldots,0,\phi_r)$

PROPOSITION 3.2.2. For $r>1$, the subgroups of the form $T(\phi_2,\ldots,\phi_r)$ are stabilizers iff the subgroups of the form $T(\upsilon)$ where $\upsilon=(0,\ldots,0,\upsilon_r)$, are stabilizers [12].

Proof:

$\Leftarrow$: For $r=2$, the proposition is trivial. Assume the proposition is true for $r-1$ and suppose that $T(\phi_2,\ldots,\phi_r)$ is given with $r\geq3$. Then there exist $h_2,\ldots,h_{r-1}$ such that $T(\phi_2,\ldots,\phi_{r-1})=St(h_2,\ldots,h_{r-1})$ i.e.,

$(a,\phi_2(a),\ldots,\phi_{r-1}(a))(e,h_2,\ldots,h_{r-1})(a,0,\ldots,0)^{-1}=(e,h_2,\ldots,h_{r-1})$ for all $a\in G_n^1$. Define a subgroup H of G_n^r by

$H=(e,h_2,\ldots,h_{r-1},0)^{-1}T(\phi_2,\ldots,\phi_r)(e,h_2,\ldots,h_{r-1},0)$. If $\mu_r^{r-1}:G_n^r\rightarrow G_n^{r-1}$ is the projection homomorphism, then it can be shown that

$\mu(H)=\{(a,0,\ldots,0),a\in G_n^1$. In fact,

$\mu[(e,h_2,\ldots,h_{r-1},0)^{-1}(a,\phi_2(a),\ldots,\phi_r(a))(e,h_2,\ldots,h_{r-1},0)]$

$=(e,h_2,\ldots,h_{r-1})^{-1}(a,\phi_2(a),\ldots,\phi_{r-1}(a))(e,h_2,\ldots,h_{r-1})$

$=(e,h_2,\ldots,h_{r-1})^{-1}(e,h_2,\ldots,h_{r-1})(a,0,\ldots,0)=(a,0,\ldots,0).$

It follows that H is a subgroup of G_n^r of the form $T(0,\ldots,0,\upsilon_r)$.

Thus, $T(0,\ldots,0,\upsilon_r)=St(e,\ldots,0,t_r)$ for some t_r, or equivalently

$(e,h_2,\ldots,h_{r-1},0)^{-1}(a,\phi_2(a),\ldots,\phi_r(a))(e,h_2,\ldots,h_{r-1},0)\ (e,0,\ldots,0,t_r)\ (a,0,\ldots,0)^{-1}$

$=(e,0,\ldots,0,t_r)$ [11].

Let us define h_r by $(e,h_2,\ldots,h_{r-1},0)\ (e,\ldots,0,t_r)=(e,h_2,\ldots,h_{r-1},h_r)$, which implies that $T(\phi_2,\ldots,\phi_r)=St(e,h_2,\ldots,h_r)$.

Note that $\{(a,0,\ldots,0,\upsilon_r(a)),\ a\in G_n^1\}$ is a subgroup $T(\upsilon)$ of G_n^r iff

$\upsilon_r(ab)=a\upsilon_r(b)+\upsilon_r(a)b$ for all $a,b\in G_n^1$

PROPOSITION 3.2.3. Given a $T(\upsilon)$, there exists an $h_r\in B_n^r$ such that $T(\upsilon)=St((e,0,\ldots,0,h_r)$ [12].

Proof:

$\upsilon_r(ab)=a\upsilon_r(b)+\upsilon_r(a)b$ for all $a,b\in G_n^1$. Let $a=\lambda e$,

Since $\upsilon_r(\lambda b)=\upsilon_r(b\lambda)$,

$\lambda\upsilon_r(b)+\upsilon_r(\lambda e)b=b\upsilon_r(\lambda e)+\lambda^r\ \upsilon_r(b)$ for all $a,b\in G_n^1$

$(\lambda-\lambda^r)\ \upsilon_r(b)=b\upsilon_r(\lambda e)-\upsilon_r(\lambda e)b$ for all $b\in G_n^1$

$\upsilon_r(b)=bh_r-h_rb$ for some $h_r\in B_n^r$ [12].

So far we have proven that every stabilizer subgroup of the action **(0)** is a subgroup of the form $T(\phi)$ and vice versa. Now it remains to show that the stabilizer subgroup of the action (θ) is of the form $T(\zeta)$ and hence the stabilizer subgroups defined by two actions are conjugate.

THEOREM 3.2.4. The stabilizer subgroup of the action (θ) of G_n^r on $\{e\}\times B_n^2\times\ldots\times B_n^r$, $St_\theta(e,h_2,\ldots,h_r)$, is a subgroup of the form $T(\zeta)$ where $\zeta=(\zeta_2,\ldots,\zeta_r)$ and each ζ_k is a polynomial in the entries $h_2,\ldots,h_k$. Furthermore, The stabilizer subgroups of the actions **(0)** and (θ) of G_n^r on $\{e\}\times B_n^2\times\ldots\times B_n^r$, $St(e,h_2,\ldots,h_r)$ and $St_\theta(e,h_2,\ldots,h_r)$ respectively, are conjugate.

Proof:

Let $(a,t_2,\ldots,t_r)\in St_\theta(e,h_2,\ldots,h_r)$ and $(a,f_2,\ldots,f_r)\in St(e,h_2,\ldots,h_r)$, then,

$(a,t_2,\ldots,t_r)(\theta)(e,h_2,\ldots,h_r)=(a,t_2,\ldots,t_r)(e,h_2,\ldots,h_r)\ (a,\theta_2(a),\ldots,\theta_r(a))^{-1}$

$\Rightarrow (a,t_2,\ldots,t_r)=(e,h_2,\ldots,h_r)(a,\theta_2(a),\ldots,\theta_r(a))\ (e,h_2,\ldots,h_r)^{-1}=(a,\zeta_2(a),\ldots,\zeta_r(a))$.

$(a,f_2,\ldots,f_r)=(e,h_2,\ldots,h_r)(a,0,\ldots,0)(e,h_2,\ldots,h_r)^{-1}$

It follows from Proposition3.2.2 and Proposition3.2.3 that

$(a,\theta_2(a),\ldots,\theta_r(a))=(e,d_2,\ldots,d_r)(a,0,\ldots,0)(e,d_2,\ldots,d_r)^{-1}$

for some $(e,d_2,\ldots,d_r)\in\{e\}\times B_n^2\times\ldots\times B_n^r$.

$\Rightarrow(a,t_2,\ldots,t_r)=(e,h_2,\ldots,h_r)\,(e,d_2,\ldots,d_r)(a,0,\ldots,0)(e,d_2,\ldots,d_r)^{-1}(e,h_2,\ldots,h_r)^{-1}$

$=(e,q_2,\ldots,q_r)(a,f_2,\ldots,f_r)(e,q_2,\ldots,q_r)^{-1}$ where,

$(e,q_2,\ldots,q_r)=(e,h_2,\ldots,h_r)(e,d_2,\ldots,d_r)\,(e,h_2,\ldots,h_r)^{-1}$.

It follows from Theorem 2.4.4 that the geometric object bundles defined by these two actions are equivalent.

4. CONCLUSION

In this thesis the higher order Christoffel symbols of Ehresmann ε–connections are studied as geometric objects. It is proven that the geometric object bundle induced by the subgroup of G_n^r of the form $T(\theta)$ is equivalent to the bundle of Christoffel symbols of Ehresmann ε–connections of order r. In [12] it is proven that, the subgoups of G_n^r isomorphic to G_n^1 must be of the form $T(\theta)$. Hence, the Christoffel symbols of Ehresmann ε–connections forms the unique geometric object bundle with base space M, fibre G_n^r/G_n^1, structure group G_n^1.

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