AGT AND ELLIPTIC HIGGSED NETWORKS

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ABSTRACT

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In this thesis we want to introduce a current work in progress where elliptically deformed networks of DIM algebra intertwiners are used to obtain the partition function of 4d quiver gauge theories. To achieve this we start by introducing the relevant background of the AGT correspondence whose higher dimensional extension has the DIM algebra as its symmetry. The first part then gives a review of the AGT relation as well as its higher dimensional extensions to 5 and 6d. Then part two begins with a brief of DIM algebra and how its itertwiners are used to build the Higgsed networks whose matrix elements give the partition function of 3d quiver gauge theories. Finally, we deform the algebra and use the deformed networks to obtain the 4d uplift of the 3d theories.

ÖZET

AGT VE ELİPTİK HİGGSLENMİŞ AĞLAR

Bu tezde, eliptik olarak deforme olmuş DIM cebri iç içe geçmiş ağlarının 4d titreme ölçer teorilerinin bölme fonksiyonunu elde etmek için kullanıldığı, devam etmekte olan güncel bir çalışmayı tanıtmak istiyoruz. Bunu başarmak için, yüksek boyutlu uzantısı simetrisi olarak DIM cebirine sahip olan AGT yazışmasının ilgili arka planını tanıtarak başlıyoruz. İlk bölüm daha sonra AGT ilişkisinin yanı sıra 5 ve 6d'ye yüksek boyutsal uzantılarının bir incelemesini verir. Daha sonra ikinci bölüm, kısa bir DIM cebiri ve onun yineleyicilerinin matris elemanları 3B titreme ölçer teorilerinin bölme fonksiyonunu veren Higgsed ağlarını oluşturmak için nasıl kullanıldığı ile başlar. Son olarak, cebiri deforme ettik ve deforme olmuş ağları 3 boyutlu teorilerin 4 boyutlu yükseltmesini elde etmek için kullanıyoruz.

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LIST OF SYMBOLS

A_{N-1}	Classical group of the A type for a degree N group
$\mathbb{C}^n$	n-dimensional complex space
$\mathfrak{g}$	Lie algebra
$\hat{\mathfrak{g}}$	Affine Lie algebra
$\hat{\hat{\mathfrak{g}}}$	Double affine Lie algebra
G	Gauge group
$\mathcal{N}$	Number of supersymmetries
$\mathcal{O}$	Operator
$\mathbb{R}^n$	n-dimensional real space
$\mathcal{R}$	Representation of a guage group
S	Action
$\mathcal{U}$	Universal envelopping algebra
Z	Partition function
$\mathbb{Z}$	The set of integers
$\mathbb{Z}_+$	The set of positive integers
α	Differential form
β	Gauge theory beta function
δ	Coproduct
ϵ_1, ϵ_2	Equivariant parameter
$\epsilon_{\alpha\beta}$	Spinors metric
$\Gamma_{q,p}$	Elliptic gamma function
θ	Fermionic degree of freedom in superspace
θ_q/θ_p	Jacobi q/p theta function
Θ	Instanton theta angle

LIST OF ACRONYMS/ABBREVIATIONS

2d	Two Dimensional
3d	Three Dimensional
4d	Four Dimensional
5d	Five Dimensional
6d	Six Dimensional
AGT	Alday-Gaiotto-Tachikawa
BPS	Bogomol'nyi–Prasad–Sommerfield
CFT	Conformal field theory
DF	Dotsenko-Fateev
DIM	Ding-Iohara-Miki
dim	Dimension
DOZZ	Dorn-Otto-Zamolodchikov-Zamolodchikov
Eu_G	G-equivariant Euler characteristic
iff	if and only if
im	Image of a map
ker	Kernel of a map
Pf	Pfaffian of a matrix
SCFT	Superconformal field theory
SUSY	Supersymmetry
$SU(N)$	Special unitary group of degree N
$SO(N)$	Special orthogonal group of degree N
SYM	Super Yangl-Mills
Sym	The symmetric group
tr	Trace of a matrix (or an operator)
$U(N)$	Unitary group of degree N
Vir	Virasoro

1. INTRODUCTION

Gauge theories are very important in contemporary theoretical physics. Indeed, the Standard Model of particle physics is built on gauge theories. But despite its success, the strong force is not fully understood and this is because it is modeled by QCD with many phenomena such as confinement being in the strongly coupled regime of theory. In that regime, perturbation theory fails to give answers as non-perturbative effects rise and this makes computations much harder. Introducing supersymmetry in our gauge theory provides a laboratory that help studying these strongly coupled theories at low energies analytically, and these supersymmetric gauge theories are extensively studied by theorists. Another symmetry which provides a powerful tool in studying QFTs is conformal symmetry. In particular, conformally invariant QFTs in 2d, or 2d CFTs for short, can be solved exactly.

In [1], Alday, Gaiotto and Tachikawa (AGT) conjectured a duality between two a priori different theories in different dimensions, the AGT correspondence. This correspondence relates the Nekrasov isntanton partition function in 4d $\mathcal{N}$=2 Super Yang-Mills (SYM) theory and the Virasoro conformal blocks of 2d Liouville theory. The importance of this 4d/2d correspondence has been growing ever since because some aspects that are difficult to understand on one side of the correspondence can be understood on the other side. For this purpose, there is an AGT "dictionary" relating parameters and mathematical structures between the two sides. This thesis will serve two purposes. The first is two give a review of the AGT and the underlying topics needed to understand the correspondence. In the second part we give a review of the "Higgsed Network Calculus" formalism and present original results (a current work in progress to be out soon).

The first part starts with the 4d side. In Chapter 2, a brief of pure $\mathcal{N}$=2 SYM with gauge group SU(2) is given and the low-energy Seiberg-Witten solution is explained. Next, in Chapter 3, we focus on the instanton part of the partition function of those class S-theories, which is the main mathematical object on the 4d side. We introduce Nekrasov's localization solution and instanton counting using Young diagrams. Chapter 4 is where we move to the 2d side of the correspondence. We give a brief of Conformal Field Theories in 2d, ending with the 4-point correlators and the conformal blocks, which is the main object on the 2d side and the corresponding object to the instanton part of the partition function in 4d. Chapter 5 is the final chapter of the first part and here we give a generalization to Seiberg-Witten's pure $SU(2)$ theory where we add matter to get a superconformal field theory. This generalization goes by the name "class-$\mathcal{S}$ theories" and it is the partition function of this class of theories that gives the 4d side of the correspondence. Then we introduce the 4d/2d AGT relation and the dictionary relating the two sides of the correspondence and then discuss extensions of the correspondence to 5d and 6d with q- and elliptically deformed Virasoro conformal blocks.

In the second part, the main topic is Zenkevich's Higgsed network calculus, which employs Ding-Iohara-Miki (DIM) algebra so we start by giving a review of this algebra in Chapter 6. It is a quantum toroidal algebra and thus we first give a brief on quantum algebras and quantum affine algebras before introducing quantum double affine (toroidal) algebras in general and DIM algebra in specific and its representation theory. Then, in Chapter 7, we introduce, briefly, the Higgsed network calculus formalism where a network of DIM intertwiners is used to get the holomorphic blocks of 3d quiver gauge theories [2]. Finally, Chapter 8 is where we give the original results of our current work. The full scope of our work is investigating two natural generalizations of those linear quiver 3d theories. One is to elliptically deform DIM algebra to get a 4d uplift of those 3d theories. This part is the work in progress and the part that will be included in the thesis. Two sources have been of great help through the first part of the thesis which are [3] and [4].

2. SEIBERG-WITTEN THEORY

Supersymmetry has been an interesting laboratory where gauge theories are studied. The analytical control over quantum field theories that is achievable in such supersymmetric theories makes them a very powerful tool in studying those quantum theories. Just as any kind of symmetry, supersymmetry constrains theories that enjoy such symmetry and the more supersymmetry a theory has the more constrained it is. To study field theories without taking into account gravitational effects, the allowed number of supersymmetries are $1, 2$ and 4. The case of $\mathcal{N} = 1$ susy is not constrained enough while that with $\mathcal{N} = 4$ is very constrained. The case of $\mathcal{N} = 2$ however is balanced between the two being constrained enough while having interesting phenomena to be studied. The Lagrangian of this theory is written in terms of a holomorphic function called the prepotential which encodes the properties of the theory and computing the explicit form of this function amounts to solving the theory. This is why the work by Seiberg and Witten is of great importance and marked a milestone towards the end of the previous millenium.

Many reviews have been written on the topic of this chapter. Out of those, I benefited the most while writing this chapter from [5–11].

2.1. Supersymmetry

Supersymmetry (SUSY) is a hypothetical (haven't been experimentally verified, yet!) space-time symmetry of particle physics which asserts that for every observed bosonic (fermionic) particle there exists a "superpartner" which is fermionic (bosonic). In a supersymmetric theory, bosons and fermions are treated equally and instead of having one particle states consisting of either one, we have supermultiplets (irreducible representations of the supersymmetry algebra) containing both a boson (or a fermion) and its superpartner. It was introduced as "loophole" in the Coleman and Mandula no-go theorem which states that a dynamically non-trivial quantum field theory (one with a non-trivial S-matrix) has a symmetry algebra that is the direct product of the Poincaré algebra and any internal algebra whose elements commute with each other and this direct product symmetry cannot be extended. This means that internal quantum numbers behave like scalars under the action of Poincare generators and no multiplet can have particles of different spin. The symmetry generators meant for that no-go theorem are bosonic generators that obey certain commutation relations. Supersymmetry evades this constraint because supersymmetry generators are fermionic ones that obey anticommutation relations. Then, since, as was mentioned, supersymmetry is a spacetime symmetry, the Poincare algebra is extended to the Super-Poincaré algebra and we have relativistic field theory with a symmetry algebra of Super-Poincaré $\otimes$ internal algebra. Supercharges commute with translations and thus particles and their superpartners belonging to the same multiplet have equal masses. Since this mass degeneracy has not been observed yet, if supersymmetry is a physical symmetry, it must be spontaneously broken.

2.1.1. SUSY algebra

The supersymmetry algebra (from here on we will call it superalgebra for brevity) is an extension of the Poincaré algebra with spinor charges Q called *supercharges*. The latter has the usual spacetime translation (from here on we will just say translation but what we mean is spacetime translations which, in addition to the obvious spatial translations, include time translations as well) generators P_μ, and Lorentz generators $L_{\mu\nu}$ where μ, ν are spacetime indices that take the value 0,1,2,3 since we're dealing with field theories in 4d in this chapter. The supercharges has the following properties:

(i) They transform as Weyl spinors under the action of the Lorentz group $SO(1,3)$.
(ii) They commute with the translation generators $[P_\mu, Q^I_\alpha] = 0$.
(iii) They have the following commutation relations among themselves

$$\{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^I_J \tag{2.1}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta} Z^{IJ} \tag{2.2}$$

where I and J take the values $1, \cdots, \mathcal{N}$ and $\mathcal{N}$ is the number of copies of supersymmetry with $\mathcal{N} = 1$ called minimal supersymmetry and $\mathcal{N} > 1$ called extended supersymmetry, and α is a Weyl spinor label of the spinorial representation of the Lorentz group, that is, Q^I_α transforms under the $(1/2, 0)$ representation; a left-handed Weyl spinor while $\bar{Q}_{\dot{\alpha}I}$ transforms under the $(0, 1/2)$ representation; a right-handed Weyl spinor. $\bar{Q}_{\dot{\alpha}I}$ is the conjugate of Q^I_α; $\bar{Q}_{\dot{\alpha}I} = (Q^I_\alpha)^\dagger$. We have $\sigma^\mu = (\sigma^0, \sigma^i)$ with $i = 1, 2, 3$ and $\bar{\sigma}^\mu = (\sigma^0, -\sigma^i)$ where σ^0 is the identity matrix and σ^i are the Pauli matrices. $\epsilon_{\alpha\beta}$ is the totally antisymmetric tensor in 2d and is equal to $i\sigma^2$. It is used to raise and lower spinorial indices,

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta, \quad \chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta \qquad\qquad \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}, \quad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}$$

with the convention

$$\epsilon^{12} = -\epsilon_{12} = 1, \qquad \epsilon^{\dot{1}\dot{2}} = -\epsilon_{\dot{1}\dot{2}} = 1. \tag{2.3}$$

Supersymmetry admits a central extension and Z^{IJ} are the *central charges* of this extension; they commute with all the generators of the superalgebra. They are antisymmetric in the indices I and J, and commute with all generators of the superalgebra. The supersalgebra has a global $U(1)$ symmetry acting on all the supercharges. In addition, when $\mathcal{N} > 1$, the supercharges of the different copies can be arranged in a multiplet so that an action of a unitary transformation belonging to $SU(\mathcal{N})$ rotate them into one another. These two symmetries, $U(1)$ and $SU(\mathcal{N})$, goes by the name $R-$ *symmetries* and denoted by $U(1)_R$ and $SU(\mathcal{N})_R$.

We are dealing with gauge theories without gravity so we must restrict our analysis to theories with spin ≤ 1. Higher spin particles are related to gravity; spin 2 is the graviton and spin 3/2 is its fermionic superpartner, the gravitino. Supersymmetric theories dealing with such higher spin particles have local supersymmetry and are called supergravity theories, for obvious reasons. We will thus be focusing on theories with global, or rigid, supersymmetry. Also, since our ultimate goal is the $\mathcal{N} = 2$ Yang-Mills, we will focus our analysis to $\mathcal{N} = 1, 2$. Thus, the supermultiplets of importance to our analysis will contain particles of spin 1; gauge fields, spin 1/2; fermion fields and spin 0; scalar fields. These are the "usual" quantum fields (strictly speakin, we are talking about the particles as states of the irreducible representation and not yet fields) but here there are two different aspects than what happens in any other quantum field theory:

(i) The fields belong to the same multiplets of the superalgebra as superpartners that differ by a spin of 1/2.

(ii) The fields of the same supermultiplet have the same mass due to the fact that supercharges commute with the translation charges.

The supermultiplets we will encounter, according to the preceding discussion, are thus multiplets of $\mathcal{N} = 1, 2$. For $\mathcal{N} = 1$, there are two types of supermultiplets, the gauge multiplet and the matter multiplet. The gauge (or vector) multiplet contains the force mediating particle; the gauge field A_μ and its superpartner, the gluino, which is a spin-3/2 Weyl fermion λ_α. As in ordinary quantum field theories, gauge fields are Lie algebra-valued and thus the $\mathcal{N} = 1$ gauge multiplet transforms in the adjoint representation of the gauge group G of our supersymmetric gauge theory. The matter (or chiral) multiplet has the matter content of the theory, a complex scalar field ϕ and its superpartner which is a spin-1/2 Weyl fermion ψ_α. Like matter fields of ordinary quantum field theories, the matter multiplet transforms in any representation $\mathcal{R}$ of the theory's gauge group G. We can write this as

- $\mathcal{N} = 1$ *Gauge Multiplet* is (A_μ, λ_α)
- $\mathcal{N} = 1$ *Chiral Multiplet* is (ψ_α, ϕ)

Now, for $\mathcal{N} = 2$, we have, again, two types of multiplets. These are the gauge multiplet and the hypermultiplet. The gauge multiplet contains the gauge field A_μ and its superpartner, the gaugino λ_α, which is again a left Weyl fermion in addition to another Weyl fermion, the one from the $\mathcal{N} = 1$ chiral multiplet, ψ_α and a complex scalar field

ϕ. So the $\mathcal{N}=2$ gauge multiplet can be seen as the direct sum of an $\mathcal{N}=1$ gauge multiplet and an $\mathcal{N}=1$ chiral multiplet. As we mentioned earlier, the gauge field is Lie algebra-valued, and hence the two $\mathcal{N}=1$ multiplets must be in the adjoint representation of the gauge group G. Thus, the $\mathcal{N}=2$ gauge multiplet is in the adjoint representation. The other multiplet of the extended supersymmetry is the multiplet with the matter content and it goes conventionally by the name *Hypermultiplet*. It contains two complex scalar fields $H_1, \bar{H}_2$ and two Weyl fermions $\psi_{1\alpha}$ and $\bar{\psi}_{2\dot{\alpha}}$, all in the same representation $\mathcal{R}$ of the gauge group. It can be composed as the direct sum an $\mathcal{N}=1$ chiral multiplet $(H_1, \psi_{1\alpha})$ and an $\mathcal{N}=1$ chiral multiplet $(\bar{H}_2, \bar{\psi}_{2\dot{\alpha}})$ (because $(H_2, \psi_{2\alpha})$ is an antichiral multiplet that transforms in the complex conjugate representation $\bar{\mathcal{R}}$ to that of $(H_1, \psi_{1\alpha})$). We can put it again like in the $\mathcal{N}=1$ case as:

- $\mathcal{N}=2$ *Gauge Multiplet is* $(A_\mu, \lambda_\alpha, \psi_\alpha, \phi)$
- $\mathcal{N}=2$ *Hypermultiplet is* $(H_1, \bar{H}_2, \psi_{1\alpha}, \bar{\psi}_{2\dot{\alpha}})$

And, in the $\mathcal{N}=1$ form, these two multiplets can be composed as:

- ($\mathcal{N}=2$ Gauge Multiplet) = ($\mathcal{N}=1$ Gauge Multiplet (A_μ, λ_α)) $\oplus$ ($\mathcal{N}=1$ Chiral Multiplet (ψ_α, ϕ) in the adjoint representation of G)
- ($\mathcal{N}=2$ Hypermultiplet) = ($\mathcal{N}=1$ Chiral Multiplet $(H_1, \psi_{1\alpha})$) $\oplus$ ($\mathcal{N}=1$ Chiral Multiplet $(\bar{H}_2, \bar{\psi}_{2\dot{\alpha}})$) both in the same representation $\mathcal{R}$ of G.

2.1.2. Superfields

All what we have studied so far is at the algebra level and our particles are, so far, states of the relevant supermultiplet. But our end goal is to study quantum field theories and thus we need to represent these multiplets and their corresponding states as fields. As we elaborated, supersymmetric theories combine particles of different spins into the same multiplet and this applies as well when we represent these multiplets as fields. Thus, a field in a supersymmetric theory, whether bosonic of fermionic, will contain fields of different spins. To accommodate those differing spin fields we introduce Grassmannian (anticommuting) coordinates, θ_α which are Weyl fermion and its complex conjugate $\bar{\theta}^{\dot{\alpha}} = (\theta_\alpha)^\dagger$. They commute with the ordinary bosonic Minkowski coordinates and anticommute with each other;

$$[x^\mu, \theta_\alpha] = [x^\mu, \bar{\theta}^{\dot{\alpha}}] = \{\theta_\alpha, \theta_\beta\} = \{\theta_\alpha, \bar{\theta}^{\dot{\beta}}\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0. \tag{2.4}$$

They combine with bosonic or fermionic fields to get the required spin-statistics of fields. These fermionic coordinates extend our Minkowski space into *superspace*. It's the arena where the supermultiplets are represented and the resulting fields are called *superfields*.

Superfields are functions of both bosnic and fermionic coordinates. By virtue of the anticommutativity of the Grassmannian coordinates, $\theta^{\alpha 2} = \bar{\theta}^{\dot{\alpha}2} = 0$, the Taylor expansion of a superfield yields a finite expression in powers of θ and $\bar{\theta}$. A general bosonic superfield has the expansion

$$\begin{aligned} S(x, \theta, \bar{\theta}) &= \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + \theta\theta F(x) + \bar{\theta}\bar{\theta} g^*(x) \\ &\quad + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\rho(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)\,. \end{aligned} \tag{2.5}$$

with the component fields ϕ, A_μ, F, g and D being bosonic, while the fields ψ, χ, λ and ρ are fermionic. Here, the spinor indices are not shown explicitly and we have the juxtaposition of two fermions means their inner product resulting in a Lorentz scalar quantity. This inner product contraction is denoted by:

$$\begin{aligned} \psi\chi &= \psi^\alpha \chi_\alpha, & \bar{\psi}\bar{\chi} &= \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \\ \chi\sigma^m\bar{\psi} &= \chi^\alpha \sigma^m_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}, & \bar{\chi}\bar{\sigma}^m\psi &= \bar{\chi}_{\dot{\alpha}}\bar{\sigma}^{m\dot{\alpha}\alpha}\psi_\alpha \end{aligned} \tag{2.6}$$

This generic form contains too many field components than a supermultiplet contains and is thus reducible. Applying constraints on this form of the superfield allows us to get a superfield with the correct number of field components; the one corresponding to the supermultiplet.

1- The *Chiral Superfield* Φ. It is the superfield corresponding to the chiral multiplet and thus it is the superfield giving the matter content of the theory. It is obtained by imposing a chirality condition on the generic superfield S. We have the chiral superfield and the antichiral superfield obtained by imposing the conditions

$$\bar{D}_{\dot{\alpha}}\Phi = 0, \qquad D_{\alpha}\bar{\Phi} = 0 \tag{2.7}$$

respectively, with the superderivatives D_α and $D_{\dot{\alpha}}$ defined by

$$D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \qquad \bar{D}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{2.8}$$

and the differentiation and integration of θ coordinates are defined by

$$\frac{\partial}{\partial\theta^\alpha}(1, \theta^\beta, \bar{\theta}^{\dot{\beta}}) \equiv \int d\theta^\alpha (1, \theta^\beta, \bar{\theta}^{\dot{\beta}}) \equiv (0, \delta_\alpha{}^\beta, 0)\,. \tag{2.9}$$

We also define the fermionic θ differentials by

$$d^2\theta \equiv \frac{1}{4}d\theta^\alpha d\theta_\alpha, \qquad d^2\bar{\theta} \equiv \frac{1}{4}d\bar{\theta}_{\dot{\alpha}}d\bar{\theta}^{\dot{\alpha}}, \qquad d^4\theta \equiv d^2\theta d^2\bar{\theta}\,, \tag{2.10}$$

where the factors of $\frac{1}{4}$ is because $\theta\theta = 2\theta^1\theta^2$ and thus, when integrating, we end up with the regular integration rules

$$\int d^2\theta\ \theta\theta = \int d^2\bar{\theta}\ \bar{\theta}\bar{\theta} = \int d^4\theta\ \theta\theta\bar{\theta}\bar{\theta} = 1. \tag{2.11}$$

Solving for the chiral and antichiral superfield we get

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) &= \phi(x_+) + \sqrt{2}\theta\psi(x_+) + \theta\theta F(x_+) \\ \Phi^\dagger(x,\theta,\bar{\theta}) &= \phi^*(x_-) + \sqrt{2}\bar{\theta}\bar{\psi}(x_-) + \bar{\theta}\bar{\theta}\bar{F}(x_-) \end{aligned} \tag{2.12}$$

where

$$x^\mu_\pm = x^\mu \pm i\theta\sigma^\mu\bar{\theta}, \tag{2.13}$$

and the spinor contraction is implicit. The field components of the expansion are ϕ a complex scalar field, ψ is a fermion field and F is an auxiliary scalar field, that is, a non-dynamical field that can be eliminated using the equations of motion. We see that the component fields of the Chiral (and the antichiral) superfield corresponds (apart from the auxiliary field F) exactly to the states of the $\mathcal{N} = 1$ chiral multiplet.

2- The *Vector Superfield* V. To obtain the vector superfield we impose reality on S; $V^\dagger = V$. This is not enough to reduce the number of field components of the resulting vector field to those corresponding to the components of the $\mathcal{N} = 1$ gauge multiplet. This is because V has a gauge symmetry

$$V \rightarrow V - i(\Lambda - \bar{\Lambda}) \tag{2.14}$$

where Λ is chiral superfield. To obtain the gauge superfield, which is the gauge invariant vector superfield, we impose a gauge known as the *Wess-Zumino* gauge. In this gauge, V takes the form:

$$V_{WZ}(x,\theta,\bar{\theta}) = \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{2.15}$$

We now have the gauge superfield and we see that the components of such field are the gauge field A_μ and the gaugino field λ, which matches that of the $\mathcal{N} = 1$ gauge multiplet. Like the case of the chiral superfield, the expansion of the gauge superfield has an auxiliary field; D, which can be eliminated by the equations of motion, just like F. In this gauge, $V^3 = 0$ and any expansion in powers of V truncates at the second power.

Just like in non-supersymmetric gauge theories, we want to construct a field strength, or in our case, a superfield strength. This superfield, however, is a spinor superfield unlike the bosonic ones we have encountered so far. We have a chiral and antichiral superfield strength and they are constructed from the gauge superfield, respectively, as:

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}D^2 \bar{D}_{\dot{\alpha}} V. \tag{2.16}$$

Just like V, W_α is gauge invariant. We see that, because the product of three operators D or $\bar{D}$ vanishes,

$$\bar{D}_\alpha W_\alpha = D_\alpha \bar{W}_{\dot{\alpha}} = 0 \tag{2.17}$$

which confirms that indeed they are chiral and antichiral superfields. They further satisfy the constraint

$$\bar{D}_\alpha \bar{W}^{\dot{\alpha}} = D^\alpha W_\alpha. \tag{2.18}$$

From this constraint we deduce that

$$\mathrm{Im}(\mathrm{D}^\alpha \mathrm{W}_\alpha) = 0. \tag{2.19}$$

Solving for W we get the expansion:

$$W_\alpha(x_+) = -i\lambda_\alpha(x_+) + \theta_\alpha D(x_+) - \frac{i}{2}\theta^\beta F_{\alpha\beta}(x_+) + \theta\theta\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\bar{\lambda}^{\dot{\beta}}(x_+) \tag{2.20}$$

with $F_{\alpha\beta} = F_{\mu\nu}\sigma^\mu_{\alpha\dot{\gamma}}\sigma^{\nu\ \dot{\gamma}}_{\beta}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We see that indeed the expansion components of the superfield strength contains $F_{\mu\nu}$, the Abelian field strength tensor.

This analysis for the gauge superfield and the superfield strength was for the Abelian case. Moving to the non-Abelian case, the gauge fields are $A_\mu = A^a_\mu T^a$

where T^a's are the non-Abelian Lie algebra generators with the commutation relations $[T^a, T^b] = if^{abc}T_c$. For this case, the gauge symmetry in Equation (2.14) becomes

$$e^{2V} \to e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda}, \qquad e^{-2V} \to e^{i\Lambda} e^{-2V} e^{-i\bar{\Lambda}} \tag{2.21}$$

where g is the gauge coupling and Λ is a Lie algebra-valued chiral superfield (we will see why we put it in an exponential form when we discuss the coupling of gauge and chiral multiplets). Hence, V and its expansion components in Equation (2.15) becomes Lie algebra-valued as well. Then, we have the non-Abelian superfield strength which becomes::

$$W_\alpha = \frac{1}{8}\bar{D}^2(e^{2V} D_\alpha e^{-2V}), \tag{2.22}$$

with the expansion

$$W^a = (-i\lambda + \theta D - \frac{i}{2}\theta F + \theta\theta\sigma^\mu \nabla_\mu \bar{\lambda})\, T^a, \tag{2.23}$$

and we see that here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the non-Abelian strength field and $\nabla_\mu \lambda = \partial_\mu \lambda - i[A_\mu, \lambda]$ is the gauge covariant derivative (for a field in the adjoint representation).

2.1.3. Supersymmetric Lagrangians

Now that we have our superfields, the next step is to construct lagrangians. We need the lagrangians to build the action of our supersymmetric gauge theory and be able to extract useful information and physical observables. Unbroken supersymmetry requires that these lagrangians be invariant under supersymmetry (or changing by a total derivative). We will see that our supersymmetric lagrangians are ordinary gauge and matter lagrangians but with constraints on the parameters of the theory.

A susy transformation of a superfield S has the form

$$\delta_\xi S = (\xi Q + \bar{\xi}\bar{Q})S, \tag{2.24}$$

where ξ is the infinitesimal parameter of susy transformation and is a left-handed Weyl spinor. Just as translations and Lorentz transformations in ordinary space are represented by a differential operator, a susy transformation in superspace is represented by a differential operator. The supercharge Q is the operator of such transformation and has the operator representation

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \qquad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{2.25}$$

We need our action to be invariant under the action of such symmetry or changing by a total derivative. So, we check the results of the susy transformation on the individual components of our chiral and gauge superfields. We find that the field components satisfying this condition are the auxiliary fields F and D. Since they are the components corresponding to $\theta\theta$ and $\theta\theta\bar{\theta}\bar{\theta}$, respectively, we could use the fermionic integrals in Equation (2.11) to extract these two terms. Thus, we have two types of supersymmetric invariant lagrangians;

$$\text{F}-\text{terms} \qquad \mathcal{L}_F = \quad F = \int d^2\theta\ \Phi \tag{2.26}$$

$$\text{and its complex conjugate} \qquad \bar{F} = \int d^2\bar{\theta}\ \bar{\Phi} \tag{2.27}$$

$$\text{D}-\text{terms} \qquad \mathcal{L}_D = \quad \frac{1}{2}D = \int d^4\theta\ V, \tag{2.28}$$

and we have our Lagrangians (the properly named lagrangian not the lagrangian density that we have been calling lagrangians by a conventional abuse of notation) $L = \int d^3x\mathcal{L}$.

1. F-terms lagrangians. We saw that we could construct a supersymmetric invariant lagrangian from chiral (and antichiral) superfields by extracting the coefficient of $\theta\theta$. But a holomorphic (a function of only chiral superfields) function of chiral superfields is itself a chiral superfield. Thus, we could generalize Equation (2.26) to any holomorphic (and antiholomorphic functions corresponding to Equation (2.27)) function $\mathcal{W}(\Phi^i)$. Such functions are known as *superpotentials*. Then Equation (2.26) and Equation (2.27) can be combined in a generalized expression written as:

$$\mathcal{L} = \int d^2\theta \; \mathcal{W}(\Phi^i) + \int d^2\bar{\theta} \; \bar{\mathcal{W}}(\bar{\Phi}^i) \tag{2.29}$$

As we mentioned before, the superfield strength is a chiral (or antichiral in case of $\bar{W}_{\dot{\alpha}}$) superfield. Thus, we could construct an F-term lagrangian from W_α. This is done by contracting two W's in order to get a Lorentz invariant expression. We also take the trace over the gauge index (this is the non-Abelian case we are discussing and remember that there, V and W_α are Lie algebra-valued) to get a gauge invariant expression. This is because, for a non-Abelian theory, $W_\alpha \rightarrow e^{i\Lambda} W_\alpha e^{-i\Lambda}$ under a gauge transformation. So we want something of the sort $\sim trW_\alpha W^\alpha$. The F-term in the expansion of $trW_\alpha W^\alpha$ is

$$trW_\alpha W^\alpha \mid_{\theta\theta} = -2i\lambda^a \sigma^\mu \nabla_\mu \bar{\lambda}^a + D^a D^a - \frac{1}{2} F^{\mu\nu,a} F^a_{\mu\nu} + \frac{i}{2} F^{\mu\nu,a} \tilde{F}^a_{\mu\nu} \tag{2.30}$$

where we used the normalization $trT^aT^b = \delta^{ab}$ and with $\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma,a}$ the dual field strength. We see that the expansion has the ordinary kinetic Yang-Mills term and imaginary term $\sim \tilde{F}F$. This term is the topological charge that gives non-perturbative (instanton) contribution to the action. Including the hermitian conjugate term, we can anticipate our F-term action now to be of the form:

$$\mathcal{L} \sim \int d^2\theta \; trW_\alpha W^\alpha + \int d^2\bar{\theta} \; tr\bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \tag{2.31}$$

But this expression is missing the topological term which is important in Yang-Mills theories. We thus introduce the *complexified gauge coupling* $\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$ with Θ being the instanton theta angle. Then, we can write the desired F-term lagrangian with the topological term present and the correct normalization of the Yang-Mills kinetic term as

$$\mathcal{L} = \frac{1}{8\pi}\,\mathrm{Im}\Big[\tau\int d^2\theta\; trW_\alpha W^\alpha\Big]$$
$$= \frac{1}{g^2}tr\big(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^2 - i\lambda\sigma^\mu\nabla_\mu\bar{\lambda} + \frac{\Theta g^2}{32\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu}\big) \quad (2.32)$$

2. D-terms lagrangians. One way to construct a vector superfield is to combine a chiral and an antichiral superfield. Indeed, $(\Phi\bar{\Phi})^\dagger = \Phi\bar{\Phi}$ which is the constraint that vector superfields are required to satisfy. We have that, under the gauge transformation in Equation (2.21), $\Phi^i \to (e^{i\Lambda})^i{}_j\ \Phi^j$ and $\bar{\Phi}_i \to \bar{\Phi}_j\ (e^{-i\bar{\Lambda}})_i{}^j$. Now, since Λ does not necessarily equal $\bar{\Lambda}$, $\Phi\bar{\Phi}$ is not gauge invariant. A gauge invariant term would be the term $\bar{\Phi}e^{2V}\Phi$ which, for the $\mathcal{N}=1$ case, resembles coupling the chiral multiplet to the gauge multiplet; the gauge-matter coupling. We then have our D-term lagrangian

$$\mathcal{L} = \int d^2\theta\ d^2\bar{\theta}\ \bar{\Phi}e^{2V}\Phi. \quad (2.33)$$

As we mentioned before, in the Wess-Zumino gauge, expanding a function of V, the expansion truncates at V^2. Thus, $e^{2V} = 1 + 2V + 2V^2$ which has the expansion

$$\mathcal{L} = |\nabla^\mu\phi|^2 - i\psi\sigma^\mu\nabla_\mu\bar{\psi} + \bar{F}F + i\sqrt{2}\bar{\phi}\lambda\psi - i\sqrt{2}\bar{\psi}\bar{\lambda}\psi + \bar{\phi}D\phi. \quad (2.34)$$

This is for the case of a renormalizable quantum field theory. If we relax this condition we could write this D-term as a more general form

$$\mathcal{L} = \int d^2\theta\ d^2\bar{\theta}\ K(\Phi, \bar{\Phi}e^{2V}) \quad (2.35)$$

where the function K is called a *Kaehler potential*. In this case, the scalar field components of the chiral superfield can be seen as the coordinates of a target Riemannian manifold known as a *Kaehler Manifold* which is a complex manifold with a positive definite, non-singular metric that can be defined as the second derivative of a scalar function (the Kahler potential);

$$g_{ij} = \frac{\partial^2 K}{\partial^i \phi \partial^j \bar{\phi}} \tag{2.36}$$

with g_{ij} being the Kaehler metric. This lagrangian has a Kaehler transformation symmetry of the form

$$K(\phi, \bar{\phi}) \rightarrow K(\phi, \bar{\phi}) + \Lambda(\phi) + \bar{\Lambda}(\bar{\phi}). \tag{2.37}$$

Now that we have gotten the expressions for both F- and D-lagrangians, let's put a different labeling on those terms. The F-term with the spinor superfield $\mathcal{W}_\alpha$ gave the component fields for a Yang-Mills theory and thus it can be labelled as the pure super Yang-Mills (SYM) term. The superpotential W and the Kaehler potential K (without the e^{2V} factor coupling the chiral and gauge multiplets) are both functions of the chiral multiplet and their expansion component fields are the scalar and spin-$\frac{1}{2}$fields which are the matter fields. The Kaehler term gives the kinetic terms for these matter fields (the 2^{nd} derivative of the scalar field and the 1^{st} derivative of the fermion field), while the superpotential gives higher-order non-derivative terms; potential terms. Thus, the lagrangian with these 3 terms is called the matter multiplet. Then, coupling the SYM term with matter we get (for a general non-renormalizable theory)

$$\begin{aligned}\mathcal{L} = \frac{1}{8\pi} \operatorname{Im}\Big[\tau \int d^2\theta \; trW_\alpha W^\alpha\Big] + \int d^2\theta \; d^2\bar{\theta} \; K(\Phi, \bar{\Phi} e^{2V}) \\ + \int d^2\theta \; \mathcal{W}(\Phi^i) + \int d^2\bar{\theta} \; \bar{\mathcal{W}}(\bar{\Phi}^i).\end{aligned} \tag{2.38}$$

2.1.4. $\mathcal{N} = 2$ Lagrangians

All what we have done for Lagrangians has been for an $\mathcal{N} = 1$ supersymmetric theory. Extending susy with another copy of supercharges impose more constraint on our Lagrangian and hence on its form. There are two methods to arrive at this desired form. The first is to start from the $\mathcal{N}=1$ form Equation (2.38) and impose the $\mathcal{N}=2$ constraints on that expression. The second is to extend our superspace with another copy of fermionic coordinates to correspond to the second susy copy of supercharges and work with an $\mathcal{N}=2$ chiral multiplet and impose the chirality conditions we did with the $\mathcal{N}=1$ chiral multiplet and then construct the Lagrangian. We will briefly exploit both methods and see they are equivalent and we end up with the same expression.

(a)The first thing to try is to start with Equation (2.38) and impose further constraints on it to get the $\mathcal{N}=2$ Lagrangian. As we elaborated before, the $\mathcal{N}=2$ gauge multiplet consists of an $\mathcal{N}=1$ gauge multiplet and an $\mathcal{N}=1$ chiral multiplet, both in the adjoint representation of the gauge group. Also, just as in the $\mathcal{N}=1$ case, the pure SYM will be described by the gauge multiplet. Hence, the pure $\mathcal{N}=2$ SYM Lagrangian takes the form

$$\mathcal{L}_{SYM}^{\mathcal{N}=2} = \frac{1}{8\pi}\,\mathrm{Im}\Big[\tau\int d^2\theta\; trW_\alpha W^\alpha\Big] + \int d^2\theta\; d^2\bar{\theta}\; \bar{\Phi}e^{2V}\Phi. \tag{2.39}$$

with the component expansion

$$\begin{aligned}\mathcal{L}_{SYM}^{\mathcal{N}=2} = \frac{1}{g^2}tr\big(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^2 - i\lambda\sigma^\mu\nabla_\mu\bar{\lambda} + \frac{\Theta g^2}{32\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu} + |\nabla^\mu\phi|^2 \\ - i\psi\sigma^\mu\nabla_\mu\bar{\psi} + \bar{F}F + i\sqrt{2}\bar{\phi}\{\lambda,\psi\} + i\sqrt{2}\phi\{\bar{\lambda},\bar{\psi}\} + \bar{\phi}[D,\phi]\big)\end{aligned} \tag{2.40}$$

where the commutator and anticommutators appear because now ϕ, ψ are both in the adjoint representation. We could go a step further and eliminate the auxiliary fields. Then, the three terms containing such fields

$$\frac{1}{g^2}tr\big(\frac{1}{2}D^2 + \bar{F}F + \bar{\phi}[D,\phi]\big) \tag{2.41}$$

becomes

$$\mathcal{V} = -\frac{1}{2g^2}tr\big([\bar{\phi},\phi]\big)^2 \tag{2.42}$$

which is identified as the scalar potential of the theory $\mathcal{V}$.

We discussed that for an extended susy $\mathcal{N}$, there is an $SU(\mathcal{N})$ R-symmetry that rotates the $\mathcal{N}$ copies of susy into each other other. Hence, and since the particle states of the supermultiplets are obtained by the action of these supercharges on some vacuum; acting as creation operators, these states, and their counterpart fields, will belong to mulltiplets under the action of this $SU(\mathcal{N})_R$ symmetry. For our case in $\mathcal{N} = 2$, the fermionic fields of the $\mathcal{N} = 2$ gauge multiplet (λ, ψ) belong to a doublet under the $SU(2)_R$ action, while A_μ and ϕ behave as singlets. From this discussion we infer that the interchange of these fermionic fields should be a symmetry of the Lagrangian. Hence, the fields should be treated on the same footing and the terms with these fields should have the same normalization. From this we find that an $\mathcal{N} = 2$ Lagrangian cannot have a superpotential term since it is a function of the $\mathcal{N} = 1$ chiral superfields and thus contains only one of these fermionic fields and the interactions they describe will not have the other one belonging to the $\mathcal{N} = 1$ gauge superfield. Another constraint on the Lagrangian is the relative normalization. Here, and unlike the $\mathcal{N} = 1$ case, the chiral and gauge multiplets are not separately susy invariant but rather belong to the same supermultiplet and thus the kinetic terms should have the same normalization. Thus, the scalar superfield Φ in Equation (2.39) is actually not the same as in the minimal susy case but rather rescaled as $\Phi \to \Phi/g$ (that would amount to adding $1/g^2 \times$ Equation (2.34) to Equation (2.32) to get the field content of an $\mathcal{N} = 2$ gauge multiplet).

As for the matter Lagrangian, which is described by the $\mathcal{N} = 2$ Hypermultiplet, it consists of two $\mathcal{N} = 1$ chiral superfields in conjugate representations $H_1 = (H_1, \psi_1)$ and $H_2 = (H_2, \psi_2)$. The matter Lagrangian will then be of the form

$$\mathcal{L}^{\mathcal{N}=2}_{matter} = \sum_{i=1}^{N_f} \int d^2\theta \; d^2\bar{\theta} \; (\bar{H}^i_1 e^{2V_{\mathcal{R}}} H^i_1 + H^i_2 e^{-2V_{\mathcal{R}}} \bar{H}^i_2) \\ + \int d^2\theta \; \Big[(\sqrt{2} H^i_1 \Phi H^i_2 + m_i H^i_1 H^i_2) + h.c \Big] \quad (2.43)$$

where m is a matrix for the mass of the hypermultiplet, $\mathcal{R}$ is a generic representation of the gauge group and N_f is the number of flavours. The $SU(2)_R$ symmetry argument applies here as well but for the hypermultiplet the scalar fields H_1, H_2 are the ones belonging to the doublet under the $SU(2)_R$ action while the fermionic fields behave as singlets. This is obvious in the symmetric appearance of H's in the matter Lagrangian.

(b) We could instead try a similar analysis to the $\mathcal{N} = 1$ case. In the $\mathcal{N} = 2$ case we have a second susy copy of supercharges and an $SU(2)_R$ symmetry rotating the charges among themselves. Similarly, we introduce a second copy of the fermionic coordinates in superspace and together with the first copy, they behave as a doublet under this $SU(2)_R$ symmetry. This set of coordiantes is denoted by $\tilde{\theta} = (\tilde{\theta}_\alpha, \bar{\tilde{\theta}}_{\dot{\alpha}})$. Then, our set of coordinates for the extended superspace is $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}, \tilde{\theta}_\alpha, \bar{\tilde{\theta}}_{\dot{\alpha}})$. A generic superfield in this superspace is defined as a function of these coordinates. Thus, an $\mathcal{N} = 2$ chiral superfield takes the form $\Psi = \Psi(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}, \tilde{\theta}_\alpha, \bar{\tilde{\theta}}_{\dot{\alpha}})$, and it obeys the equivalent of the chirality constraint in Equation (2.7) but the $\mathcal{N} = 2$ one,

$$\bar{\tilde{D}}_{\dot{\alpha}} \Psi(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}, \tilde{\theta}_\alpha, \bar{\tilde{\theta}}_{\dot{\alpha}}) = 0, \qquad \tilde{D}_\alpha \bar{\Psi}(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}, \tilde{\theta}_\alpha, \bar{\tilde{\theta}}_{\dot{\alpha}}) = 0 \tag{2.44}$$

and we have the set of superderivatives $\tilde{D} = (\tilde{D}_\alpha, \bar{\tilde{D}}_{\dot{\alpha}})$ defined similar to Equation (2.8) but with $\theta \to \tilde{\theta}$. Introducing shifted coordinates similar to (2.13),

$$\tilde{x}^\mu_+ = x^\mu_+ + i\tilde{\theta}\sigma^\mu \bar{\tilde{\theta}} = x^\mu + i\theta\sigma^\mu\bar{\theta} + i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}} \tag{2.45}$$

we could write the form of the superfield Ψ satisfying the condition in Equation (2.44) as

$$\Psi = \Phi(\tilde{x}^\mu_+, \theta) + \sqrt{2}\tilde{\theta}W(\tilde{x}^\mu_+, \theta) + \tilde{\theta}\tilde{\theta}G(\tilde{x}^\mu_+, \theta). \tag{2.46}$$

We see that this expansion gives the $\mathcal{N} = 2$ chiral superfield in terms of the $\mathcal{N} = 1$ chiral superfield Φ and spinor superfield strength W_α. The third function, G, is written in the form

$$G(\tilde{x}_+^\mu, \theta) = \int d^2\bar{\theta}\ \bar{\Phi}(\tilde{x}_+^\mu - i\theta\sigma^\mu\bar{\theta}, \theta, \bar{\theta})\, e^{-2V(\tilde{x}_+^\mu - i\theta\sigma^\mu\bar{\theta}, \theta, \bar{\theta})}. \tag{2.47}$$

We can put the $\mathcal{N} = 2$ SYM Lagrangian in the compact simple form

$$\mathcal{L}_{SYM}^{\mathcal{N}=2} = \frac{1}{4\pi}\mathrm{Im}\ \mathrm{tr} \int d^2\theta d^2\tilde{\theta}\ \frac{1}{2}\tau\Psi^2. \tag{2.48}$$

This form is for a renormalizable theory. More generally, we could write the previous Lagrangian in terms of a holomorphic function of the superfield Ψ called the prepotential $\mathcal{F}(\Psi)$. Then, writing the Lagrangian and expanding $\mathcal{F}(\Psi)$ in terms of the component fields in Equation (2.46),

$$\begin{aligned} \mathcal{L}_{SYM}^{\mathcal{N}=2} &= \frac{1}{4\pi}\mathrm{Im}\ \mathrm{tr} \int d^2\theta d^2\tilde{\theta}\ \mathcal{F}(\Psi) \\ &= \frac{1}{8\pi}\mathrm{Im}\left(\int d^2\theta\ \mathcal{F}_{ab}(\Phi) W^{a\alpha} W^b_\alpha + 2\int d^2\theta d^2\bar{\theta}\ (\bar{\Phi} e^{2V})^a \mathcal{F}_a(\Phi) \right) \end{aligned} \tag{2.49}$$

with $\mathcal{F}_{ab}(\Phi) = \partial\mathcal{F}/\partial\Phi^a\partial\Phi^b, \mathcal{F}_a(\Phi) = \partial\mathcal{F}/\partial\Phi^a$ and a, b are global gauge indices. Since we are interested in the work of Seiberg-Witten, these indices are $SU(2)$ indices and thus take the value $1, 2$. If we now compare Equation (2.49) to the non-renormalizable form of Equation (2.39)

$$\mathcal{L}_{SYM}^{\mathcal{N}=2} = \frac{1}{8\pi}\ \mathrm{Im}\left[\tau \int d^2\theta\ trW_\alpha W^\alpha\right] + \int d^2\theta\ d^2\bar{\theta}\ K(\Phi, \bar{\Phi}e^{2V}) \tag{2.50}$$

we conclude that $\mathcal{F}_{ab}(\Phi)$ acts as a generalized coupling and $\mathrm{Im}\big[(\Phi \bar{e}^{2V})^a \mathcal{F}_a(\Phi)\big]$ is the Kaehler potential.

2.2. Moduli Space

As we mentioned, susy has not been verified experimentally and no mass degeneracy has been detected where two superpartners share the same mass as dictated by susy. Thus we concluded that, even if susy is physical and not just purely academic, it must be broken. But now we are studying a supersymmetric theory and thus we require susy to be unbroken. For that to happen, a necessary condition is the vanishing of the scalar potential term of Equation (2.42) of our theory for the vacuum state. Then, we see that we must have $tr\big([\bar{\phi},\phi]\big)^2 = 0$ or $[\bar{\phi},\phi] = 0$. But for this to happen, we don't need the scalar fields to vanish but it is sufficient that they commute. Now, since these scalar fields belong to the gauge multiplet and thus are Lie algebra valued, they must belong to the Cartan subalgebra $\mathfrak{h}$ of the gauge algebra (the toral subalgebra),

$$\phi = \sum_i^r a^i h^i. \tag{2.51}$$

Here r is the rank of the gauge algebra and a^i are complex constants called *moduli*. In our case, $\mathfrak{g} = \mathfrak{su}(2)$ and thus only one generator belongs to the Cartan subalgebra and we have $\phi = \frac{1}{2}a^3\sigma^3$. This is still not a unique choice since, by virtue of the Weyl reflection symmetry, $a, -a$ are gauge equivalent quantities. Thus, we should have a gauge inequivalent description for these scalar fields and this expression is $\mathrm{tr}\phi^2 = \frac{1}{2}a^2$. This is classically speaking but on the quantum level what we are interested in is the vacumm expectation value VEV of the fields. Thus the quantity of interest is $u = \langle tr\phi^2\rangle$ with $\langle\phi\rangle = \frac{1}{2}a\sigma^3$. This value is not exact though and we will why see in a moment. u is a quantity parametrizing the space of gauge inequivalent vacua which is called the *moduli space*, $\mathcal{M}$. It is a complex manifold of dimension r whose coordinates are the values of a.

A consequence of this non-vanishing VEV is that ϕ breaks the gauge symmetry generating mass for the gauge field. For the case we are interested in, the symmetry is broken from $G = SU(2) \to H = U(1)$ with two *Goldstone* bosons that the gauge field absorbs acquiring mass and one component left massless. Hence, $A^a_\mu, a = 1, 2$ are massive gauge fields with mass $m = \sqrt{2}a$ while A_μ^3 remains massless. Now, because A_μ belong to the gauge multiplet as a superpartner for the gaugino λ, they both share the same mass and hence $\lambda_\alpha^a, a = 1, 2$ are massive fermionic fields of mass $m = \sqrt{2}a$ while λ_α^3 is massless. Not just that, but because we are in $\mathcal{N} = 2$ multiplet, λ and ψ are an $SU(2)$ doublet and hence they are related as well and we have $\psi_\alpha^a, a = 1, 2$ a massive field of mass $m = \sqrt{2}a$ while ψ_α^3 is massless. We then see that our $SU(2)$ theory is broken to a $U(1)$ theory and the gauge indices in Equation (2.49) take one value; 3, and can thus be dropped. Also, in the expansion e^{2V}, only 1 survives. Thus, we have an Wilsonian effective Lagrangian describing the $U(1)$ massless modes remaining taking the form

$$\mathcal{L}_{effective}^{\mathcal{N}=2} = \frac{1}{8\pi}\mathrm{Im}\Bigg(\int d^2\theta \ \mathcal{F}^{''}(\Phi)W^\alpha W_\alpha + 2\int d^2\theta d^2\bar{\theta} \ \bar{\Phi}\mathcal{F}^{'}(\Phi)\Bigg) \tag{2.52}$$

with $\mathcal{F}^{''}(\Phi) = \frac{\partial^2 \mathcal{F}}{\partial^i\phi\partial^i\phi}$. We see that we have the generalized holomorphic coupling as $\mathrm{Im}\mathcal{F}^{''}(\Phi)$ and the Kaehler potential $K = \mathrm{Im}\bar{\Phi}\mathcal{F}^{'}(\Phi)$. From Equation (2.36) we can get the metric on the Kaehler manifold and it is $g_{ij} = \frac{\partial^2}{\partial^i\phi\partial^j\bar{\phi}}\mathrm{Im}\bar{\Phi}\mathcal{F}^{'}(\Phi) = \mathrm{Im}\mathcal{F}^{''}(\Phi)$. This can also be seen if we expand the second term in Equation (2.52) to find the leading term $\sim \mathrm{Im}\mathcal{F}^{''}(\Phi)| \partial_\mu\phi |^2$. There it can be seen that $\mathrm{Im}\mathcal{F}^{''}(\Phi)$ acts as a metric in the space of fields. Hence, the Moduli space $\mathcal{M} = \frac{SU(2)}{U(1)\times Weyl}$ is a Kaehler manifold and the scalar fields are the coordinates on this manifold. Another note to make is that, by virtue of the $\mathcal{N} = 2$ susy, the holomorphic coupling $\tau(a) = \mathrm{Im}\mathcal{F}^{''}(a)$ is the same as the metric on the moduli space (to be more accurate, this is the case when ϕ is replaced by its VEV a). We can write a definition of a line element in the moduli space then as

$$\mathrm{d}s^2 = \mathrm{Im}\mathcal{F}^{"}(a)\mathrm{d}a\mathrm{d}\bar{a} = \mathrm{Im}\tau(a)\mathrm{d}a\mathrm{d}\bar{a} \tag{2.53}$$

with $\bar{a}$ the complex conjugate of a.

The moduli space of $\mathcal{N} = 2$ SYM is a Kaehler manifold $\mathcal{M}^V \otimes \mathcal{M}^H$ parametrized by the scalar fields of the theory. $\mathcal{M}^V$ is the subspace of the moduli space where the scalar fields that acquire VEVs are those belonging to the vector multiplet V. It is a special Kaehler manifold parametrized by the VEV of the scalar fields ϕ. This subspace is called the *Coulomb Branch* because these ϕ's Higgs the gauge group G to its maximal torus $U(1)^r$ with r being the rank of the group and ϕ takes the form of Equation (2.51). Thus, the remaining theory is an Abelian gauge theory of $r - U(1)$ gauge fields with Coulomb like-interactions; r-copies of electromagnetism. In our case in Equation (2.52), this is just one copy of $U(1)$.

$\mathcal{M}^H$ is the subspace where the scalar fields that acquire VEVs are the H's belonging to the hypermultiplet. It's a hyperKaehler manifold and goes by the name *Higgs Branch*. This is because the gauge group is now fully broken and all gauge fields acquire masses. There is also the case where both ϕ's and H's have non-vanishing VEVs and are called *mixed branches*. What we are after is a quantum supersymmetric gauge theory and what we will focus on is the Coulomb branch because it is the subspace of the moduli space receiving quantum corrections. That is why the expression $u = \langle \mathrm{tr}\phi^2 \rangle$ was mentioned to be not exact and it should be $u = \langle \mathrm{tr}\phi^2 \rangle +$ quantum corrections. The Higgs branch on the other hand, if it exists in a classical theory, remains exact in its quantum version.

Now comes the question on which relies the whole Seiberg-Witten analysis: do the coordinates $a, \bar{a}$ suffice to cover the whole moduli space? It turns out this is not the case and we need more than one patch to cover the whole moduli space. This can be seen if we remember that our moduli space metric is $\mathrm{Im}\tau(a) = \mathrm{Im}\frac{\partial^2 \mathcal{F}}{\partial^i \phi \partial^i \phi}$. For our theory to be unitary, $\tau(a)$ must be positive definite. But since $\mathcal{F}(a)$ is a holomorphic function, $\mathrm{Im}\frac{\partial^2 \mathcal{F}}{\partial^i \phi \partial^i \phi}$ is a harmonic function and cannot have a minimum anywhere in the moduli space. Thus, $\mathrm{Im}\tau(a)$ cannot be positive definite which is not the case. So we come to the conclusion that $\mathrm{Im}\tau(a)$ cannot be globally defined but rather *locally* holomorphic of a. The points where some coordinates fail to be defined is a *singular* point at which $\mathrm{Im}\tau(a) \to 0$. When a point of this sort is approached, a different coordinate system is needed at which $\mathrm{Im}\tau(a)$ is non-vanishing.

2.3. Electromagnetic Duality

We have seen that the moduli space possesses singularities and thus has the structure of an orbifold. To see why these singularities arise, let's look at the classical $\mathcal{N} = 2$ SYM. We have seen that a generic VEV for the scalar field breaks the gauge symmetry to its maximal torus and our theory is described by an effective action with all massive degrees of freedom integrated out. This is all good until we are at the point $a = 0$. There, the gauge symmetry is unbroken and the subgroup of the broken symmetry is enhanced back to the original symmetry group at this point and we have the massless particles that were integrated out appearing again. At such point, new coordinates are needed. Originally we have the coordinates $a, \bar{a}$ parametrizing the patch of the moduli space around $u = \infty$ where the theory is asymptotically free; weakly coupled region, and semiclassical analysis is valid. The Seiberg and Witten analysis [12] showed that the quantum moduli space of $\mathcal{N} = 2$ SYM has three singularities. The $u = \infty$ semiclassical singularity of the weakly coupled region and the $\pm\Lambda^2$ singularities around which the theory is strongly coupled, with Λ a dynamical quantum scale. The point $u = 0$ is no longer a singularity; it doesn't survive quantization. It was also shown in [12] that, by virtue of the electromagnetic duality, that one could study the strongly coupled region by means of *dual* variables in terms of which we would have a weakly coupled dual theory and the analysis becomes easier. Since our effective theory is a $U(1)$ theory, we attempt to find this dual theory following the dualization of Maxwell's electrodynamics.

Source free Maxwell's equations $\partial_\mu F^{\mu\nu} = \partial_\mu \tilde{F}^{\mu\nu} = 0$ have a symmetry exchanging electric and magnetic fields. This duality, $F^{\mu\nu} \leftrightarrow \tilde{F}^{\mu\nu}$, is obscured though at the Lagrangian level. This is because the Bianchi identity, $\partial_\mu \tilde{F}^{\mu\nu}$, is solved by writing $F^{\mu\nu}$ as an antisymmetric derivative of the gauge field A^μ and the equations of motion are just $\partial_\mu F^{\mu\nu}$. To restore this duality we must have both the field equations $\partial_\mu F^{\mu\nu} = 0$ and the Bianchi identities $\partial_\mu \tilde{F}^{\mu\nu} = 0$ emerging as equations of motion. This is done as follows: on the partition function level, we have the measure integrating over the gauge fields. We thus need to treat the field strength $F^{\mu\nu}$ as the independent field in the measure, and introduce the Bianchi identities as a constraint through a Lagrange multiplier field which we call $A_{D\mu}$. This could be written as

$$\begin{gathered} Z = \int \mathcal{D}A \; \exp\left[-\int d^4x \; \tfrac{1}{4e^2} F_{\mu\nu}F^{\mu\nu}\right] \\ \downarrow \\ Z = \int \mathcal{D}F\mathcal{D}A_D \; \exp\left[-\int d^4x \; \tfrac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\int d^4x \; \varepsilon_{\mu\nu\rho\sigma} A^{\mu}_{D} \partial^{\nu} F^{\rho\sigma}\right] \end{gathered} .$$

This constraint term could be integrated by parts and it then takes the form

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} A^{\mu}_{D} \partial^{\nu} F^{\rho\sigma} = \frac{1}{2}\tilde{F}^{\mu\nu}_{D} F_{\mu\nu} \tag{2.54}$$

We then complete the square and integrate out the unconstrained field strength F which leaves us with

$$Z = \int \mathcal{D}A_D \; \exp\left[-\int d^4x \; \frac{1}{4e_D^2} F_{D\mu\nu}F^{\mu\nu}_{D}\right] \tag{2.55}$$

where $e_D = \frac{1}{e}$ is the dual $U(1)$ coupling. To see the physical meaning of Equation (2.55), we take a look at Maxwell's equations in the presence of electric and magnetic sources,

$$\partial_{\mu} F^{\mu\nu} = J^{\nu}_{e} \tag{2.56}$$

$$\partial_{\mu} \tilde{F}^{\mu\nu} = J^{\nu}_{m} \tag{2.57}$$

Then, using Equation (2.57), we could write the constraint term as $\sim A^{\mu}_{D} J^{\mu}_{m}$. We see then that the Lagrange multiplier is a "magnetic" gauge field that couples to magnetic sources and our dual Lagrangian in Equation (2.55) describes the dynamics of this "magnetic photon". Another note to make is that, since the original $U(1)$ gauge coupling e is a weak coupling, the dual coupling is strong, as we have $e_D = \frac{1}{e}$. We see then that indeed the dual theory describes the dynamics of a strongly coupled theory.

The same procedure is followed for the supersymmetric theory, with the gauge field replaced by the vector superfield and the field strength by the spinor superfield strength. The first term in our effective action in Equation (2.52) is the supersymmetric version of the unconstrained gauge Lagrangian while the Bianchi identity is a component in the expansion of the constraint in Equation (2.19) satisfied by the supefield strength; Im $D_\alpha W^\alpha$. Hence, imposing this constraint via a Lagrange multiplier superfield V_D and making the superfield strength as the independent field of integration in the measure at the partition function level, we get

$$
\begin{gathered}
Z = \int \mathcal{D}V \ \exp\left[\tfrac{i}{8\pi}\mathrm{Im}\int d^4x d^2\theta \ \mathcal{F}''(\Phi) W^\alpha W_\alpha\right] \\
\downarrow \\
Z = \int \mathcal{D}W\mathcal{D}V_D \ \exp\left[\tfrac{i}{8\pi}\mathrm{Im}\int d^4x \left(d^2\theta \ \mathcal{F}''(\Phi) W^\alpha W_\alpha + \tfrac{1}{2}\int d^2\theta d^2\bar{\theta} \ V_D D_\alpha W^\alpha \right)\right].
\end{gathered}
$$

We then could rewrite the contsraint term as we did with the non-supersymmetric case to be able to integrate out the superfied W,

$$
\begin{aligned}
\int d^2\theta d^2\bar{\theta} \ V_D D_\alpha W^\alpha = -\int d^2\theta d^2\bar{\theta} \ D_\alpha V_D W^\alpha &= \int d^2\theta \ \bar{D}^2 \ (D_\alpha V_D W^\alpha) \\
&= \int d^2\theta \ \bar{D}^2 \ (D_\alpha V_D) W^\alpha = -4\int d^2\theta \ W_{D\alpha} W^\alpha
\end{aligned} \tag{2.58}
$$

In the first equality we simply integrated by parts. In the second, we used $\int d\bar{\theta} = \bar{D}$, up to a spacetime derivative which of course doesn't affect the Lagrangian. In the third we used the chirality condition in Equation (2.18); $\bar{D}_{\dot{\alpha}} W_\alpha = 0$ while in the fourth we used an analog of Equation (2.17) to define the dual superfield strength $W_{D\alpha} = -\frac{1}{4}\bar{D}^2 D_\alpha V_D$. We then have the partition function as

$$
Z = \int \mathcal{D}W\mathcal{D}V_D \ \exp\left[\frac{i}{8\pi}\mathrm{Im}\int d^4x \left(d^2\theta \ \mathcal{F}''(\Phi) W^\alpha W_\alpha - 2\int d^2\theta \ W_{D\alpha} W^\alpha \right)\right] \tag{2.59}
$$

which upon completing the square and integrating out W yields the dual action

$$Z = \int \mathcal{D}V_D \ \exp\left[\frac{i}{8\pi}\mathrm{Im}\int d^4x d^2\theta \left(-\frac{1}{\mathcal{F}''(\Phi)} W_{D\alpha} W_D^\alpha \right)\right] \tag{2.60}$$

We see then that the effective action has a dual description in terms of the dual superfield strength and the dual coupling $-\frac{1}{\mathcal{F}''(\Phi)}$.

This is just half the job done though because the Kaehler term in the effective action is still written in terms of the original chiral superfield and the dual coupling is a function of that chiral superfield. Then, a dual description for the chiral superfield is needed in order to write a fully dual effective action. To do that we define a dual chiral superfield

$$\Phi_D = \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} \tag{2.61}$$

and a dual prepotential $\mathcal{F}(\Phi_D)$ defined by the relationship

$$\frac{\partial \mathcal{F}_D(\Phi_D)}{\partial \Phi_D} = -\Phi. \tag{2.62}$$

We note that together, Equation (2.61) and Equation (2.62) constitute a Legendre transformation

$$\mathcal{F}_D(\Phi_D) = \mathcal{F}(\Phi) - \Phi \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} = \mathcal{F}(\Phi) - \Phi\Phi_D \tag{2.63}$$

Then, we can write the Kaehler term of the effective action as

$$\begin{aligned} &\mathrm{Im}\int d^2\theta d^2\bar{\theta} \ \bar{\Phi}\frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} \\ &\qquad = \mathrm{Im}\int d^2\theta d^2\bar{\theta} \left(-\frac{\partial \mathcal{F}(\Phi_D)}{\partial \Phi_D} \right)^\dagger \Phi_D = \mathrm{Im}\int d^2\theta d^2\bar{\theta} \ \bar{\Phi}_D \frac{\partial \mathcal{F}_D(\Phi_D)}{\partial \Phi_D} \end{aligned} \tag{2.64}$$

The last thing to write the dual action is to express the dual coupling as a function of Φ_D; $\tau_D(\Phi_D)$. This can be done by exploiting Equation (2.61) and Equation (2.62)

$$\mathcal{F}_D^{''}(\Phi_D) = -\frac{d\Phi}{d\Phi_D} = -\frac{1}{\mathcal{F}''(\Phi)} = -\frac{1}{\tau(\Phi)} \tag{2.65}$$

Then, defining $\tau_D(\Phi_D) \equiv \mathcal{F}_D^{''}(\Phi_D)$ we find that the relationship between the couplings is exactly like in the non-supersymmetric case;

$$\tau_D(\Phi_D) = -\frac{1}{\tau(\Phi)} \tag{2.66}$$

The full dual effective action then takes the form

$$\mathcal{L}_{effective}^{dual} = \frac{1}{8\pi}\mathrm{Im}\Bigg(\int d^2\theta\ \mathcal{F}_D^{''}(\Phi_D)W_{D\alpha}W_D^{\alpha} + 2\int d^2\theta d^2\bar{\theta}\ \bar{\Phi}_D\mathcal{F}_D^{'}(\Phi_D)\Bigg) \tag{2.67}$$

The similarity of constructing this dual Lagrangian to the one we got for the $U(1)$ electrodynamics case suggest that this dual action describes the dynamics of a magnetic (dual) photon A_D^{μ} and a magnetically charged dual Higgs field Φ_D. This magnetic Higgs field couple to magnetically charged particles; monopoles and dyons. These latter particles appear in the spectrum of systems with spontaneously broken $SU(2)$ symmetry, like our case, as states in the hypermultiplet and the coupling is similar to that of Equation (2.43). There's another form in which a duality group of the effective action is manifest. If we write Equation (2.52) as

$$\mathcal{L}_{effective} = \frac{1}{8\pi}\mathrm{Im}\int d^2\theta\ \frac{d\Phi_D}{d\Phi}W^{\alpha}W_{\alpha} - \frac{i}{16\pi}\int d^2\theta d^2\bar{\theta}\ (\bar{\Phi}\Phi_D - \bar{\Phi}_D\Phi), \tag{2.68}$$

we find two apparent symmetry transformations under which the action is invariant. The first is $\Phi_D \to \Phi, \Phi \to -\Phi_D$;

$$\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix} \to \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix}. \tag{2.69}$$

The second is

$$\begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \to \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \tag{2.70}$$

under which the first term of Equation (2.68) changes by the term

$$\frac{b}{8\pi}\mathrm{Im}\int d^4x d^2\theta \; W^\alpha W_\alpha = \frac{b}{16\pi}\int d^4x \; F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\pi b n \tag{2.71}$$

where $n \in \mathbb{Z}$ is the instanton number. Since the action appears in the path integral as e^{is}, the transformation in Equation (2.70) leaves the effective action invariant as long as $b \in \mathbb{Z}$. In particular, for $b = 1$, we have the symmetry transformation written as

$$\begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \tag{2.72}$$

It is now obvious what duality group the effective action has. The transformations

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.73}$$

are the two generators of the modular group $SL(2, \mathbb{Z})$ and thus the effective action is invariant under the action of this group. Another note to make before ending this section is related to the moduli space metric in Equation (2.53). Since $\mathcal{F}''(a) = \frac{d\Phi_D}{d\Phi} \mid_{\Phi=a} = \frac{da_D}{da}$, Equation (2.53) can be written as

$$ds^2 = \mathrm{Im}(da_D d\bar{a}) = \frac{i}{2}(da d\bar{a}_D - da_D d\bar{a}). \tag{2.74}$$

It is then apparent that the moduli space metric (also called Zamolodchikov metric) is $SL(2, \mathbb{Z})$-invariant. This metric is, unlike the non-dual invariant one, non-singular and can be used at the singular points (from the point of view of one of the descriptions) of the moduli space. This is because the duality we achieved can be used to go from the non-perturbative spectrum of a strong coupling theory to the perturbative spectrum of the dual weakly coupled theory, since the couplings are inversely proportional. Hence, just like the the original, or "electric", theory is defined in the weakly coupled patch and the VEV of Φ; a (to be more accurate, the VEV of the scalar field component ϕ of the chiral superfield Φ) is used as a coordinate in that region of the moduli space, the dual, or "magnetic", theory is the one defined around the two singularities in the strongly coupled patch and the VEV of Φ_D; a_D is used as a coordinate in that region of the moduli space. To put it quantitatively, the electric prepotential is

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_{class} + \mathcal{F}_{1-loop} + \mathcal{F}_{inst} && (2.75) \\ &= \frac{1}{2}\tau a^2 + \frac{i}{\pi}a^2 \ln\left(\frac{a^2}{\Lambda^2}\right) + \frac{1}{2\pi i}a^2 \sum_{\ell=1}^{\infty} c_\ell \left(\frac{\Lambda}{a}\right)^{4\ell} && (2.76) \end{aligned}$$

where c_ℓ are coefficients giving instanton contributions for each number of intsantons. The coordinate in the patch where this prepotential is defined is a with the dual VEV giving the preptential through the expression

$$a_D = \frac{\partial \mathcal{F}}{\partial a}. \tag{2.77}$$

On the other hand, the magnetic prepotential is

$$\mathcal{F}_D = \frac{1}{2}\tau_D a_D^2 - \frac{i}{4\pi}a_D^2 \ln\left(\frac{a_D}{\Lambda}\right) - \frac{1}{2\pi i}\Lambda^2 \sum_{\ell=1}^{\infty} c_\ell^D \left(\frac{i a_D}{\Lambda}\right)^{\ell} \tag{2.78}$$

with the coordinate of the patch where it is defined is a_D and the prepotential is given by the electirc VEV through

$$a = \frac{\partial \mathcal{F}_D}{\partial a_D} \tag{2.79}$$

Both descriptions gives the same physics but which one we use depends on where we are in the quantum moduli space.

2.4. Monodromies

From the previous discussions we conclude that the geometry of the moduli space is non-trivial and that the physics that the $\mathcal{N} = 2$ low energy effective Lagrangian describes is in direct relation to this geometry. The prepotential is related to the coordinates of this complex space via the relation in Equation (2.62) when $\langle\phi\rangle = a$ and a is defined in terms of the moduli space parameter u as

$$a_D(u) = \frac{\partial \mathcal{F}(a)}{\partial a} = \frac{ia}{\pi}\Big(\ln\frac{a^2}{\Lambda^2} + 1\Big) = \frac{i}{\pi}\sqrt{2u}\Big(\ln\frac{2u}{\Lambda^2} + 1\Big) \tag{2.80}$$

$$a(u) = \sqrt{2u} \tag{2.81}$$

where the form after the second equality in Equation (2.80) will be proven in a moment and the last equality is the parametrized form. Thus, determining the moduli space geometry or, equivalently, the form of the coordinates amounts to determining the prepotential and hence solving the theory. This is done by studying how the coordinates behave as the parameter u changes. From the forms of Equation (2.80) and Equation (2.81) we see that, since the moduli space is complex, they are multivalued functions with the branch points being the singularities of the moduli space. Hence, a change of the parameter u will best appropriately be through a contour in this space. If the contour encircles one of the singularities, the functions do not return to their original values and this is the concept of *monodromy*.

Let us first analyze the weakly coupled region. In the limit a (or, equivalently, u) $\to \infty$, the theory is asymptotically free and semiclassical analysis is justified. The one-loop contribution is given by

$$\mathcal{F}_{1-loop}(\Psi) = \frac{i}{2\pi}\Psi^2 \ln\frac{\Psi^2}{\Lambda^2} \tag{2.82}$$

Then, as $u \to \infty$, the above expression becomes

$$\mathcal{F}(a) = \frac{i}{2\pi}a^2 \ln\frac{a^2}{\Lambda^2} \tag{2.83}$$

and we have

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{ia}{\pi}\big(\ln\frac{a^2}{\Lambda^2}+1\big) \tag{2.84}$$

which is what we stated without proof in Equation (2.80). As we have $u = e^{i\theta}|u|$, taking a closed contour around $u=\infty$ amounts to $a \to e^{i\pi}a$ and $\ln a^2 \to \ln a^2 + 2\pi i$, and hence $a_D \to -a_D + 2a$. We can then write the monodromy matrix M_∞ acting on the column vector $\begin{pmatrix} a_D \\ a \end{pmatrix}$ as

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{2.85}$$

Now the spectrum of our original effective theory contains electric particles of charge n with n the integral electric charge ($q = ne$). These particles' dynamics is described by the original Lagrangian and are represented by hypermultiplets which are coupled to the vector multiplet via the term

$$\sqrt{2}n\Phi H_1 H_2. \tag{2.86}$$

which are massless in the original $SU(2)$ high energy microscopic Lagrangian and thus saturate the so-called BPS bound. When $\langle\phi\rangle = a \neq 0$, the chiral fields H_1 and H_2 are massive but their masses still satisfy the BPS constraint $m = \sqrt{2}|Z|$ where Z is the central charge of the susy algebra. For electric particles the expression for the central charge is $Z = |an|$. The same also happens with magnetic particles whose dynamics are given by the dual Lagrangian. Monopoles then are massive excitations satisfying the BPS bound with $Z = |a_D m|$ and m being the integral magnetic charge. Since the central charge is additive, a particle of mixed charge, called *dyon*, satisfy

$Z = |an + a_D m|$, or

$$\begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{2.87}$$

As we have seen, the coordinate vector $\begin{pmatrix} a_D \\ a \end{pmatrix}$ transforms with the monodromy matrix M_∞ according to Equation (2.85). Another way to look at the action of this monodromy is to see it as an action on the row charge vector from the right; $\begin{pmatrix} m & n \end{pmatrix}$ transforming as $\begin{pmatrix} m & n \end{pmatrix} M$. This transformation of the charge vector has two implications. Firstly, since the charge numbers n and m are integers, monodromy matrices must have integral entries. Secondly, since two dyons must satisfy the Dirac quantization condition $nm' - mn' = 2\pi\mathbb{Z}$, transformed charge vectors

$$\begin{pmatrix} m & n \end{pmatrix} M, \qquad \begin{pmatrix} m' & n' \end{pmatrix} M \tag{2.88}$$

must still obey that condition. Thus, if we define a charge matrix whose determinant is the quantization condition;

$$C = \begin{pmatrix} m' & n' \\ m & n \end{pmatrix}, \qquad det(C) = nm' - mn' \tag{2.89}$$

then the rows of this matrix are both transformed with M and we then have the transformed charge matrix $\tilde{C}$ with the determinant

$$det(\tilde{C}) = \tilde{n}\tilde{m}' - \tilde{m}\tilde{n}' = det(C)det(M) \tag{2.90}$$

Thus, M must be a matrix with a unit determinant. A 2×2 matrix with integral entries and a unit deteminant is an element of $SL(2, \mathbb{Z})$ and we see again that the modular group plays another role in our analysis.

Now we move on to the strongly coupled patch around the two singularities $\pm\Lambda^2$. The semiclassical analysis is not accurate now and we do not start off with the perturbative expression for the prepotential. Instead we use renormalization group analysis using the $U(1)$ β-function of the dual theory. Since our effective action is for a $U(1)$ theory, we put the instanton theta angle $\Theta = 0$. Hence, the dual coupling

$$\tau_D = \frac{4\pi i}{e_D^2(a_D)} \tag{2.91}$$

We also have the "magnetic" β-function as

$$\beta_D(e_D) \equiv \mu \frac{de_D}{d\mu} = \frac{e_D^3}{8\pi^2} \tag{2.92}$$

Then, and since the dual VEV is proportional to the renormalization scale μ; $a_D \sim \mu$, we could use Equation (2.91) and Equation (2.92) to write the expression

$$a_D \frac{d\tau_D}{da_D} = -\frac{i}{\pi} \tag{2.93}$$

or, after integration,

$$\tau_D = -\frac{i}{\pi} \ln a_D \tag{2.94}$$

and since $\tau_D = -\frac{da}{da_D}$, integrating once more leads to the dual coordinate

$$a = c_0 + \frac{i}{\pi} a_D \ln a_D + \frac{i}{\pi} a_D \tag{2.95}$$

Remember that we are in the vicinity of the strong coupling singularity Λ^2. There, a_D is the appropriate coordinate to use and hence it depends linearly on the parameter u; $a_D \approx c(u - \Lambda^2)$ and thuse we have

$$a = c_0 + \frac{i}{\pi} c(u - \Lambda^2) \ln c(u - \Lambda^2) + \frac{i}{\pi} c(u - \Lambda^2) \tag{2.96}$$

Then, taking a counterclockwise contour around Λ^2 amounts to $(u-\Lambda^2) \to e^{2\pi i}(u-\Lambda^2)$ and this results in $a \to a - 2a_D$, or

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} = M_{\Lambda^2} \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{2.97}$$

To obtain the monodromy around the third singularity we could apply a similar analysis but it is easier to use a topological argument. The contour around $u = \infty$ can be deformed into two paths, the first following the contour around $u = \Lambda^2$ and the second following $u = -\Lambda^2$, both contours counterclockwise. Then, this argument could be expressed quantitatively as a factorisation condition on the monodromy matrices as

$$M_\infty = M_{\Lambda^2} M_{-\Lambda^2} \tag{2.98}$$

It is easy now to solve for $M_{-\Lambda^2}$ and the result is

$$M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{2.99}$$

As we mentioned before, at these singularities, massive excitations that were integrated out to obtain the Wilsonian effective action become massless due to symmetry enhancement. So what are the massless particles for these two singularities? As we elaborated before, the monodromy action can be seen from the BPS bound as an action on the charge vector from the right and for particles saturating this bound, Z must be invariant. Hence, for such particles, the charge vector should be invariant under the monodromy action, that is, a left eigenvector of the monodromy matrix with a unit eigenvalue;

$$\begin{pmatrix} m & n \end{pmatrix} M = \begin{pmatrix} m & n \end{pmatrix} \tag{2.100}$$

Solving for the charge vector satisfying this condition for the strongly coupled singularities we find that at $u = \Lambda^2$ the massless excitation is a magnetic monopole $\begin{pmatrix} m & n \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, while for $u = -\Lambda^2$ the massless particle is a dyon of charge $\begin{pmatrix} m & n \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix}$.

2.5. Elliptic Curve Solution

Monodromy problems like the one we have with a, a_D are usually hard to solve. It was the genius of Seiberg and Witten in transforming the problem to a geometric one that can be solved through the theory of Riemann surfaces. The observation they made is due to the properties of the coupling τ, namely that it is complex and with a positive definite imaginary part. Such properties are the same that the moduli of Riemann surfaces have. In our case, the Riemann surface corresponding to the moduli space is an elliptic curve whose modulus is identified with the holomorphic coupling and which can be written as

$$y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u) \tag{2.101}$$

with x and y being the coordinates of this complex surface and u parametrizing the modulus τ. To see what type of surface this is we note that y has the value of a square root and thus, since we are in a complex surface, is multivalued. To be a single valued function it must be described by two branches, each branch is a complex $x-$plane, or sheet. The two sheets have branch points at $\Lambda^2, -\Lambda^2, u$ and ∞ and we choose the branch cuts from Λ^2 to $-\Lambda^2$ and the second from u to ∞. Compactifying both two sheets by adding the point at ∞ they become Riemann spheres and, when connected through the cuts, the surface we obtain is topologically a torus.

A torus has a basis of two independent closed cycles A and B. We choose the A-cycle to encircle the branch cut between Λ^2 and $-\Lambda^2$ while the $B-$cycle to encircle the two branch points Λ^2 and u going from the first to the second on the upper sheet (which is now a sphere after stereographic projection) and back from the second to the first on the lower sheet. This choice of basis is not unique but defined up to transformations:

$$\begin{pmatrix} A \\ B \end{pmatrix} \to M \begin{pmatrix} A \\ B \end{pmatrix} \qquad M \in SL(2,\mathbb{Z}) \tag{2.102}$$

It is obvious the similarity of this basis transformation and the monodromy of a and a_D.

The modulus of the torus, also called the period matrix, $\tau(u)$ is defined as the ratio of the two periods of the torus ω_D and ω;

$$\tau(u) = \frac{\omega_D(u)}{\omega(u)} \tag{2.103}$$

and these two periods are defined as

$$\omega_D(u) = \oint_B c\frac{dx}{y}, \qquad \omega(u) = \oint_A c\frac{dx}{y} \tag{2.104}$$

Since we identified this modulus with the holomorphic coupling, which has the from $\tau = \frac{\partial a_D}{\partial a}$, we see that the two periods should be of the form

$$\omega_D(u) = \frac{da_D(u)}{du}, \qquad \omega(u) = \frac{da(u)}{du} \tag{2.105}$$

Then, integrating Equation (2.105) with respect to u gives the Seiberg-Witten solution

$$a_D = \oint_B c'\, d\lambda, \qquad a = \oint_A c'\, d\lambda \tag{2.106}$$

where the 1-form $d\lambda$ is called the *Seiberg-Witten differential* and takes the form

$$d\lambda = \frac{(x-u)dx}{y} \tag{2.107}$$

We then have the solution written explicitly as

$$a_D(u) = c' \int_{\Lambda^2}^{u} dx \frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}, \qquad a = c' \int_{-\Lambda^2}^{\Lambda^2} dx \frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}} \tag{2.108}$$

where the factor of -2 arising from the u-integration of Equation (2.105) and the factor of 2 arising from expanding the cyclic integrals along the torus cycles have been absorbed into $c \to c'$. Then, performing these two integrals in the limit where $u \to$ one of the singularities, we see that, for $c' = \frac{\sqrt{2}}{\pi} \quad (c = -\frac{\sqrt{2}}{4\pi})$, the solution is what we obtained from the monodromy properties of the moduli space coordinates.

3. NEKRASOV PARTITION FUNCTION

We have seen that to solve the supersymmetric $\mathcal{N} = 2$ gauge theory amounts to obtaining the functional form of the prepotential. Just as in the non-supersymmetric case though, instanton contributions are very hard to compute. That is why it is the part of the prepotential that gets special attention and it is the part that is the most non-trivial to compute. The Seiberg-Witten approach is not efficient to compute instanton contributions beyond the first or second coefficients c_ℓ (or c^D_{ell}). It was Nekrasov [13] who made it possible to make progress when he computed the instanton contribution and got the prepotential using localization.

3.1. Yang-Mills Instantons

We start by giving a brief on instantons in the non-supersymmetric Yang-Mills theory. This section will follow [14].

Instantons are finite-energy field configurations that minimize the Euclidean Yang-Mills action

$$S = \frac{1}{4g^2} \int d^4x \ \mathrm{tr}(F_{mn}F^{mn}) \tag{3.1}$$

where $m, n = 1, 2, 3, 4$ are Euclidean indices and the measure $d^4x = d\tau d^3x$ with $\tau = it$ is the Euclidean time (the analytically continues Minkowskian time). We use the conventions for the gauge fields, fields strengths and group theoretic quantities as in the previous chapter. This can be written in the form

$$S = \frac{1}{4g^2} \int d^4x \Big[\pm \mathrm{tr}(F_{mn}\tilde{F}^{mn}) + \frac{1}{2}\mathrm{tr}(F_{mn} \mp \tilde{F}^{mn})^2 \Big] \tag{3.2}$$

The second term is positive definite and thus we have the action

$$S_{inst} \geq \frac{8\pi^2}{g^2}|k| \tag{3.3}$$

where k is the instanton number

$$k = \pm\frac{1}{32\pi^2}\int d^4x \mathrm{tr} F_{mn}\tilde{F}^{mn} \tag{3.4}$$

with the positive sign for instantons and the minus for anti-instantons. The equality in Equation (3.3) holds only when

$$F_{mn} = \tilde{F}_{mn} \tag{3.5}$$

$$F_{mn} = -\tilde{F}_{mn} \tag{3.6}$$

with the first being the self-duality condition for the case of instantons ($k > 0$) while the second is the anti-self dual condition for the case of anti-instantons ($k < 0$). This instanton number is a topological invariant or, in mathematical jargon, what is called a Pontryagin index. It is the winding number of maps from the boundary of our spacetime as a manifold to the gauge group manifold

$$\partial\mathcal{M} \mapsto G \tag{3.7}$$

For our case, the boundary of our 4d spacetime is a 3-sphere and the gauge group is $SU(2)$, which is, as a manifold, a 3-sphere too. Thus Equation (3.7) becomes

$$S^3_{\partial\mathcal{M}} \mapsto S^3_G \tag{3.8}$$

These maps are the homotopy groups $\pi_3(S^3) \in \mathbb{Z}$. Thus, the instanton number k is an integer. From the properties of homotopy groups we know that these homotopy maps cannot be deformed into one another; they are non-trivial, and thus theories with different instanton numbers have distinct field configurations. Since instantons and anti-instantons are related by a parity transformation, the analysis is usually done for instantons and the same procedure applies to anti-isntantons as well, and this is what we will follow.

As we mentioned, instantons are finite energy field configurations. This means that in the Yang-Mills action

$$S \sim \int dr\ r^3\ F_{mn}F^{mn} \tag{3.9}$$

F_{mn} must fall faster than $1/r^2$ as $r \to \infty$. This in turns means that the gauge field A_m must fall of faster than $1/r$ as $r \to \infty$. This requires A_m to be a pure gauge in that limit. Instanton (or anti-instanton) solution must obey this constraint and, for an $SU(2)$ gauge group, the self dual 1-instanton gauge field obeying the finite energy condition takes the form

$$A^a_m = 2\frac{\eta^a_{mn}(x-X)^n}{(x-X)^2+\rho^2} \tag{3.10}$$

with η^a_{mn} is the 't Hooft antisymmetric symbol defined by

$$\eta^a_{mn} = \epsilon^a_{mn} \quad m,n = 1,2,3 \quad \eta^a_{m4} = -\eta^a_{4m} = \delta^a_m \tag{3.11}$$

These symbols are self-dual and thus the field strength corresponding to this 1-isnaton solution

$$F^a_{mn} = -4\eta^a_{mn}\frac{\rho^2}{[(x-X)^2+\rho^2]^2} \tag{3.12}$$

is self-dual. This solution is seen to have eight parameters (also known as collective coordinates or moduli in solitonic language), four corresponding to the spacetime location of the instanton's center of mass X_m, a scale parameter indicating the size of the instanton ρ in addition to three global $SU(2)$ rotations that takes one solution to another one.

Now, to get an $SU(N)$ instanton, all we have to do is to embed the $SU(2)$ solution we got in $SU(N)$;

$$A_m^{SU(N)} = \begin{pmatrix} 0 & 0 \\ 0 & A_m^{SU(2)} \end{pmatrix} \tag{3.13}$$

where the $SU(2)$ one instanton solution is the 2×2 matrix in the lower right corner. As we discussed, global gauge transformations take one solution to another and thus, a generic $SU(N)$ instanton solution takes the form

$$A_m^{SU(N)} = U^\dagger \begin{pmatrix} 0 & 0 \\ 0 & A_m^{SU(2)} \end{pmatrix} U \tag{3.14}$$

This U, however, should not be an element of $SU(N-2)$ nor $U(1)$ since they obviously leave the $SU(2)$ solution invariant. Thus,

$$U \in \frac{SU(N)}{SU(N-2) \times U(1)} \tag{3.15}$$

Counting the collective coordinates of this $SU(N)$ instanton, we have five for the instanton's center of mass location and size while those for the global gauge transformations are $N^2 - (N-2)^2 - 1 = 4N - 5$. Thus, the instanton moduli space $\mathcal{M}$ has the dimension $4N$. For the case of k instantons, the $SU(N)$ moduli space dimension is then

$$\dim\, \mathcal{M}_k = 4kN. \tag{3.16}$$

3.2. Instanton Moduli Space

As we have seen, the moduli space for instantons is parametrized by $4kN$ collective coordinates with k being the number of instantons and N denoting the degree of the gauge group $SU(N)$. Out of those, 5 coordinates can describe a sort of non-compactness of the moduli space. For one, the instanton size described by ρ can become arbitrarily small. This type of non-compactness is called the *UV non-compactness*. The other coordinate with non-compactness is the instanton's center of mass' coordinate as it can go to infinity. This is known as the *IR non-compactness*.

First we deal with the UV problem. Our moduli space $\mathcal{M}_k$ has the dimension of $4N \times k$. So, if one of the instantons become of zero size, we can compactify the dimension corresponding to that instanton leaving us with an effective $k-1$ moduli space $\mathcal{M}_{k-1}$ and a point corresponding to the zero-size instanton; that is, $\mathcal{M}_k = \mathcal{M}_{k-1} \times \mathbb{R}^4$. This is called *Uhlenbeck compactification*. This process exchanges our moduli space with the hyperkaehler orbifold $\tilde{\mathcal{M}}_k$

$$\widetilde{\mathcal{M}}_k = \mathcal{M}_k \bigcup \mathcal{M}_{k-1} \times \mathbb{R}^4 \bigcup \mathcal{M}_{k-2} \times Sym^2(\mathbb{R}^4) \bigcup \cdots \bigcup Sym^k(\mathbb{R}^4) \tag{3.17}$$

where the use of the symmetrizer Sym is due to the indistinguishability of the point-like instantons.

Next we address the IR problem. This problem is related to the non-compactness (in the topological sense) of the $\mathbb{R}^4$ part of the moduli space. To see this we look at the instanton partition function definition

$$Z_{inst} = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_k} 1 \tag{3.18}$$

with q^k being the weight of the k-th instanton section. It equals Λ^β for the case of an asymptotically free theory and $\exp(2\pi i \tau_{UV})$ for a conformally invariant theory. Thus, the spacetime part of this integral is

$$\int_{\mathbb{R}^4} d^4x = \infty \tag{3.19}$$

To cure this problem, we put in a regularizing factor and the result is (working in $\mathbb{C}^2 \cong \mathbb{R}^4$)

$$\int_{\mathbb{C}^2} d^2z_1 d^2z_2 \to \int_{\mathbb{C}^2} d^2z_1 d^2z_2 \; e^{-(\epsilon_1|z_1|^2+\epsilon_2|z_2|^2)} \sim \frac{1}{\epsilon_1\epsilon_2} \tag{3.20}$$

3.2.1. Symplectic Geometry

Now we review some facts from symplectic geometry that will be of great utility. For this review we follow [15].

Definition 1. *A pair* (M^{2n}, ω) *where* M *is a manifold of even dimension and* ω *is a symplectic (non-degenerate closed) 2-form.*

A function H is called a $Hamiltonian$ function generated by a Hamiltonian vector field $V_H \in Vect(M)$ which is defined using the symplectic form by the relation

$$\omega_x(Y, V_H(x)) = (dH)_V(Y) \qquad \forall\; Y \in T_xM \tag{3.21}$$

or, in terms of the interior product map,

$$\imath_{V_H}\omega = -dH \tag{3.22}$$

Another definition for the Hamiltonian, which will be of more utility to us, is given with the so called *momentum map.* If our symplectic manifold has a Lie group action g which acts on M via symplectic transformations; $g : G \to Symp(M)$.

Definition 2. *The momentum map* $\mu : M \to \mathfrak{g}^*$ *defines a Hamiltonian function* μ^ξ

$$d\mu^\xi = -\imath_{V_\xi}\omega \tag{3.23}$$

for a vector field V_ξ provided three conditions

(i) $\mu^\xi : M \to \mathbb{R}$, $\mu^\xi(x) \equiv \langle \mu(x), \xi \rangle$*.*

(ii) V_ξ *is the induced vector field* $V^\#$ *generated by the one-parameter subgroup* $\{\exp(t\xi) | t \in \mathbb{R}\}$*.*

(iii) μ *satisfies the G-equivariance condition for maps (a map* $f : X \to Y$ *between two G-spaces is G-equivariant if it satisfies* $f(g.x) = g.f(x) \quad \forall x \in X, g \in G$*) with respect to the coadjoint represntation;* $\mu(g.x) = Ad_g^*.\mu(x) \quad \forall x \in M, g \in G$*.*

$\forall \xi \in \mathfrak{g}$. *The action g is then a Hamiltonian action and the quadruple* (M, ω, G, μ) *is a Hamiltonian G-space.*

Now comes the gist. Consider a Hamiltonian action of a torus $\mathbb{T}^r$ generated by $\xi \in \mathfrak{t} = Lie(\mathbb{T})$. The momentum map is then $\mu : M \to \mathfrak{t}^*$ associated to a Hamiltonian vector field V_ξ and we have the following expression

$$\int_M \frac{\omega^n}{n!} e^{-\langle \mu(x), \xi \rangle} = \sum_{\{f\}} \frac{e^{-\langle \mu(f), \xi \rangle}}{\prod_{i=1}^n w_{i,\xi}(f)} \tag{3.24}$$

with $\frac{\omega^n}{n!}$ being the *Liouville form*, $\{f\} = \{x \in M | \ V_\xi(x) = 0\}$ the set of fixed points with respect to the torus action and $w_{i,\xi}(f)$ are the weights of the torus action on the tangent space to M at the fixed point f. This is the Duistermaat-Heckman formula for equivariant localization [16] and it is the used computation technique to compute the prepotential.

Reflecting on the regularization in Equation (3.20), we have our manifold $M = \mathbb{C}^2$, the symplectic form $\omega = \frac{i}{2} \sum_{k=1}^2 dz_k \wedge d\bar{z}_k$, the momentum map $\mu(z_1, z_2) = \frac{1}{2}(\epsilon_1 |z_1|^2, \epsilon_2 |z_2|^2)$ and $r = 2$ so the torus is $\mathbb{T}^2 = S^1 \times S^1$ giving the Hamiltonian $H = \mu^\xi = \frac{1}{2}(\epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2)$ with $\vec{\epsilon} = (\epsilon_1, \epsilon_2)$ being the weight vector corresponding to the action of $\vec{\xi}$. Using Equation (3.22), we find $V_\xi = i \sum_{k=1}^2 \epsilon_k (z_k \partial_{z_k} - \bar{z}_k \partial_{\bar{z}_k})$ the vector field generating this torus action (if we go back to R^4 with the real symplectic form $\omega = \sum_{k=1}^2 dx_k \wedge dy_k$, this vector field would be $V_\xi = \sum_{k=1}^2 \epsilon_k (-y_k \partial_{x_k} + x_k \partial_{y_k})$ which is the familiar angular momentum generator). Our space has the origin as the only fied points and thus Equation (3.24) for our case gives the result in Equation (3.20).

3.3. Equivariant Localization

This section follows [17, 18]. In [13], Nekrasov used supersymmetric localization techniques to reduce the infinite dimensional integral over the instanton moduli space to a sum over a space of fixed instanton configurations.

To be able to do this, the $\mathcal{N} = 2$ SYM has to be transformed to its topological version; the topological Yang-Mills or, in mathematical literature, the Donaldson-Witten theory. This is a topological field theory of the Witten type. A quantum field theory is said to be topological if the operators of the theory $\mathcal{O}_i$, called *topological observables*, have metric independent-correlation functions. Namely,

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle = 0 \tag{3.25}$$

To achieve such a theory, a procedure, called *supersymmetric twisting*, needs to be done. This twisting is a coupling of the fields of the theory to gravity and it basically amounts to identifying the $SU(2)_{\mathcal{R}}$ index with the right chiral spinor $SU(2)_R$ (in this chapter, we use $\mathcal{R}$ to denote the R-symmetry while R is uded for the right handed spinorial representation);

$$\bar{Q}_{\dot{\alpha}I} \to \bar{Q}_{\dot{\alpha}\dot{\beta}} \tag{3.26}$$

with the *topological supercharge* defined as the trace of this new supercharge

$$\bar{\mathcal{Q}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{Q}_{\dot{\alpha}\dot{\beta}} \tag{3.27}$$

Under this twisting, the field content of the $\mathcal{N} = 2$ theory change their spin and the new multiplets are used to build the topological theory's action and localization techniques can be used. But before we look into supersymmetric localization we first review equivariant localization since it builds the needed foundation.

Eventually, what we are interested in is to compute integrals for theories with some symmetry group G; an integral over a G-manifold M (a manifold with a Lie group action $x \to g.x \quad \forall x \in M, g \in G$).

Definition 3. *A group action on a manifold is free iff the isotropy subgroups are trivial. More mathematically, if the stabilizer of G, $Stab(G) = \{g|g.x = x\}$, is just the identity element; $g.x = x \iff g = e$.*

When the group action is free, the orbit space M/G is a smooth manifold and it constitutes the base space of a principal G-bundle

$$\begin{tikzcd} M \arrow[d, "P"] & G \arrow[l] \\ M/G & \end{tikzcd} .$$

Then, ordinary de Rham cohomology can be applied to this orbit space.

Definition 4. *The k-th de Rham cohomology group $H^k(M)$ is the space of closed differential forms modulo exact differential forms.*

$$H^k(M) = \frac{ker\ d|_{\Lambda^k M}}{im\ d|_{\Lambda^{k-1} M}} \tag{3.28}$$

with

$$ker\ d = \{\alpha \in \Lambda M | d\alpha = 0\}, \quad im\ d = \{\alpha \in \Lambda^{k-1} M, \beta \in \Lambda^k M | d\alpha = \beta\} \tag{3.29}$$

with

$$\Lambda M = \bigoplus_{k=0}^{dimM} \Lambda^k M = \{\alpha = \sum_{k=0}^{dimM} \alpha_k | \alpha_k \in \Lambda^k M\} \tag{3.30}$$

is the exterior algebra of M with differential forms α. It is equipped with two maps: the exterior derivative

$$d = d^\mu x \frac{\partial}{\partial x^\mu} \qquad d : \Lambda^k M \to \Lambda^{k+1} M \tag{3.31}$$

and the contraction map or interior product

$$\imath_V = V^\mu \frac{\partial}{\partial x^\mu} \qquad \imath_V : \Lambda^k M \to \Lambda^{k-1} M \tag{3.32}$$

for some vector field V.

All of this builds on the G-action being free. But what if this action is not free? That is, if there is a set of fixed points under the translation action of the elements of G; $g.x = x$ with $g \neq e$. The orbit space then is not smooth but pathological with singularities at these fixed points; it is an orbifold. This, however, can be remedied by the fact that cohomology is homotopy invariant. That is, two cohomolgy groups are isomorphic if their underlying manifolds are homotopic. Thus, we search for a homotopic manifold M^* with two properties

(i) M^* is homotopic to M.
(ii) M^* has a free G-action.

This homotopic space is usually the *universal bundle EG*.

Definition 5. *A universal bundle is a contractible space with a free G-action.*

Now we can define equivariant cohomology (geometric definition).

Definition 6. *The equivariant cohomology of a space M on which G does not act freely is the de Rham cohomology of the quotient of the direct product of that space with a universal bundle and G.*

$$H^\bullet_G(M) = H^\bullet(M_G) \quad \textit{with } M_G \equiv (M \times EG)/G \tag{3.33}$$

This equivariant cohomology is a generalization of de Rham cohomology and they both coincide when the G-action is free on M. From here on we will be working with equivariant cohomology so I will drop the G subscript for the ease of notation.

Just like ordinary de Rham cohomology groups, an equivariant cohomology group has an algebraic definition as follows. The k-th equivariant de Rham cohomology group is defined as the space of equivariantly closed forms modulo equivariantly exact forms.

$$H^k(M) = \frac{\ker\, d_V|_{\Lambda^k_V M}}{\mathrm{im}\, d_V|_{\Lambda^{k-1}_V M}} \tag{3.34}$$

where $\Lambda_V M = \{\alpha \in \Lambda M | \mathcal{L}_V \alpha = 0\}$ is the space of equivariant forms which is also the subspace of the exterior algebra annihilated by the Lie derivative (we will see why in a moment) and d_V is the *equivariant differential*

$$d_V \equiv d + \imath_V \tag{3.35}$$

This differential squares to

$$d_V^2 = d^2 + \imath_V^2 + d\imath_V + \imath_V d = \mathcal{L}_V \tag{3.36}$$

where we used Cartan's definition of the Lie derivative with respect to a vector field V and the fact that d and $\imath_V$ are nilpotent by virtue of the antisymmetry of differential forms. Equivariant integration of polyforms is defined as the integration of their top form component

$$\int_M \alpha \equiv \int_M \alpha_{\mathrm{dim}M} \tag{3.37}$$

Using Stoke's theorem, we find that equivariant integration of polyforms on boundaryless spaces; closed manifolds, is independent of the representative chosen from the equivariant cohomology class. This can be seen as follows. For an equivariantly exact form, its top form is

$$(\alpha)_n = (d_V\beta)_n = d\beta_{n-1} + \imath_V\beta_{n+1} \tag{3.38}$$

with $n = \mathrm{dim}M$. But the differential form has vanishing components with degree $> \mathrm{dim}M$. Thus, the second term above vanishes and we have that equivariantly exact forms are exact in ordinary cohomological sense (the same applies for equivariantly closed forms that are exact; their top form component is closed in the ordinary sense). Applying Stoke's theorem we get

$$\int_M \alpha_n = \int_M d_V\beta = \int_M d\beta = \int_{\partial M} \beta = 0 \tag{3.39}$$

We can then define equivariant integration as integration of cohomology classes of polyforms

$$\int_M (\alpha + d_V\beta) = \int_M \alpha = \int_M [\alpha] \tag{3.40}$$

The equivariant integral in Equation (3.37) was first computed by Berline and Vergne [19] and Atiyah and Bott [20] and it takes the form

$$\int_M \alpha = (2\pi)^{dimM/2} \sum_{x_k \in M_V} \frac{\alpha_0(x_k)}{\prod_{i=1}^{dimM/2} \omega_i(x_k)} \tag{3.41}$$

with α_0 the zeroth component of the form α, the ω's are the weights of the torus action as in the Duistermaat-Heckamn formula and x_k's are the fixed points of the vector field V on M; $M_V = \{x \in M | V(x) = 0\}$.

3.4. Supersymmetric Localization

This section draws from [17, 21, 22]. Now that we have the mathematical tool to use, we should put these expressions in a more physical notation (supersymmetric field theoretic notation). In the twisted supersymmetric theory, the topological charge in Equation (3.27) is the equivalent of the equivariant exterior derivative d_V and we have a corresponding $\mathcal{Q}$-*cohomology* instead of the equivariant cohomology. Table 3.1 gives a sort of a "dictionary" between the mathematical and physical notations.

Table 3.1. Equivariant vs $\mathcal{Q}$-cohomology

Equivariant	**Supersymmetric**
d_V	$\mathcal{Q}$
$d_V^2 = \mathcal{L}_V$	$\mathcal{Q}^2 = B$
Even/Odd polyforms	Bosons/Fermions
$d_V\alpha = 0$	$\mathcal{Q}S = 0$
$\int_M \alpha = \int_M \alpha e^{td_V\beta}$ for $\mathcal{L}_V\beta = 0$	$Z = \int_{BPS} \mathcal{D}X\ e^{-S[X]} = \int_{BPS} \mathcal{D}X\ e^{-S[X]-t\mathcal{Q}\gamma}$ for $\mathcal{Q}\gamma = 0$

We see that the Lie derivative corresponds to a bosonic operator, since the supercharge, which is fermionic, squares to a Boson. The fourth row shows that operators that corresponds to an equivariant polyform are BPS-observables and the space of equivariant forms M_V corresponds in supersymmetric theory to the space of BPS field configurations.

The last entry needs more elaboration. As we discussed in the last section, equivariant integration over closed manifolds only depends on equivariant cohomology class and not on the particular representative. The same also holds in $\mathcal{Q}$-cohomology and integration of $\mathcal{Q}$-exact operators vanishes. We can use this to perform a trick that comes in handy when computing integrals. We can use any representative of cohomology class that makes it easier for us to compute it. Thus, we introduce a deformed action and the deformed partition function takes the following form

$$Z(t) = \int \mathcal{D}X \ e^{-S_t[X]} \equiv \int \mathcal{D}X \ e^{-S[X]-t\mathcal{Q}\gamma} \tag{3.42}$$

with $B\gamma = 0$ and we dropped the BPS subscript. To see that this deformed partition function is of the same class as the original undeformed one we show that it is t-independent

$$\frac{d}{dt}Z = -\int \mathcal{D}X \ \mathcal{Q}\gamma \ e^{-S[X]-t\mathcal{Q}\gamma} = \int \mathcal{D}X \ \mathcal{Q}(\gamma e^{-S[X]-t\mathcal{Q}\gamma}) = 0 \qquad \forall t \tag{3.43}$$

where we integrated by parts, dropping the boundary term which is absent and using the fact that the action is $\mathcal{Q}$-closed, $B\gamma = 0$ and the assumption that our quantum field theory is not anomalous; the measure is invariant under the topological supersymmetry. This t-independence enables us to use any value for t that makes our life easier when computing integrals. In particular, $t = 0$ gives the undeformed partition function and the t-independence shows that both the deformed and undeformed partition functions belong to the same $\mathcal{Q}$-cohomology class and in the limit $t \to \infty$, the integral localizes to BPS configurations as anticipated. The partition function is nothing but the VEV of 1

$$Z = \langle 1 \rangle = \int \mathcal{D}X \ e^{-S[X]} \tag{3.44}$$

Then the previous analysis extends to correlators of BPS observables; correlators of BPS observables ($\mathcal{Q}$-closed operators) belong to $\mathcal{Q}$-cohomology classes and the correlator is insensitive to the addition of $\mathcal{Q}$-exact operators $\mathcal{QO}$

$$\langle \mathcal{O}_{\mathcal{BPS}} \rangle = \langle \mathcal{O}_{\mathcal{BPS}} + \mathcal{QO} \rangle \tag{3.45}$$

A standard choice for γ is the form dual to the vector field V_ξ;

$$\gamma = g(V, \cdot) = g_{\mu\nu} V^\nu dx^\mu = V_\mu dx^\mu \tag{3.46}$$

where we dropped ξ for brevity. This dual form has the $\mathcal{Q}$-action (with $\mathcal{Q}$ interpreted as the equivariant derivative d_V)

$$\mathcal{Q}\gamma = V^2 + dV = V^2 + \partial_\mu V_\nu dx^\mu \wedge dx^\nu = V^2 + \frac{1}{2}(R_V)_{\mu\nu} dx^\mu \wedge dx^\nu \tag{3.47}$$

with $(R_V)_{\mu\nu}$ the curvature of the vector field V. Working in a more physics-friendly notation, we denote 1-forms as fermions; $dx^\mu \equiv \psi^\mu$. Accounting for the action having fermionic fields, the integral in Equation (3.42) takes the form

$$Z(t) = \int \mathcal{D}X\mathcal{D}\psi \; e^{-S[X,\psi] - tV^2 - \frac{t}{2}(R_V)_{\mu\nu}\psi^\mu\psi^\nu} \tag{3.48}$$

The path integral over the fermionic degrees can bde done to give

$$\int \mathcal{D}\psi \; e^{-\frac{1}{2}\psi^\mu M_{\mu\nu}\psi^\nu} = \sqrt{detM} = Pf(M) \tag{3.49}$$

where $Pf(M)$ is the Pfaffian of a matrix defined as $Pf(M)^2 \equiv det(M)$. Then, and in the limit $t \to \infty$, Equation (3.48) gives the result

$$Z = \int \mathcal{D}X \; e^{-S[X]} Eu_G(X) \tag{3.50}$$

with

$$Eu_G(X) = \frac{1}{(2\pi)^n} Pf(R) \tag{3.51}$$

the G-equivariant Euler characteristic of the tangent bundle to the space we are integrating on and which R is its curvature. When that space is the instanton moduli space and the BPS configurations are instantons, we find from Equation (3.18)

$$Z_{inst} = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_k} Eu_G(\mathcal{M}_k) \tag{3.52}$$

with

$$Z_k(a, m, \epsilon_1, \epsilon_2) = \int_{\mathcal{M}_k} Eu_G(\mathcal{M}_k) \tag{3.53}$$

the k-th instanton function. We then have the full Nekrasov partition function

$$\begin{aligned} Z(a, m, \Lambda, \epsilon_1, \epsilon_2) &= Z_{class}(a, \Lambda) \times Z_{pert}(a, m, \Lambda, \epsilon_1, \epsilon_2) \times \sum_{k=0}^{\infty} q^k Z_k(a, m, \epsilon_1, \epsilon_2) \\ &= \exp \frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}(a, m, \Lambda, \epsilon_1, \epsilon_2) \end{aligned} \tag{3.54}$$

and we will give the form of these partition functions in the next section. When we send the regularized spacetime volume back to infinity we get the prepotential;

$$\mathcal{F}(a, m, \Lambda) = \lim_{\epsilon_1, \epsilon_2 \to 0} \ln Z(a, m, \Lambda, \epsilon_1, \epsilon_2) \tag{3.55}$$

3.5. Instanton Combinatorics

As we have seen, the instanton partition function is computed by performing the equivariant integral. This localization process, as we have discussed for supersymmetric localization, reduces the integral to BPS field configuration, which are the instantons localized at the fixed points of spacetime, or, to be more accurate, the fixed points of the deformed spacetime $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ (which is also known as the omega background). For the torus action $\vec{\xi}$, the fixed points set is just the origin and the BPS particles are instantons at the origin. $U(N)$ instantons are labelled by an N-tuple of Young diagrams $\vec{Y} = (Y_1, \cdots, Y_N)$. The instanton number is given by the total number of boxes $|\vec{Y}| = \sum_{i=1}^{N} |Y_i|$ and the instantons contribution to the full partition function weighted by $q^{|\vec{Y}|}$. For our case, quivers have $U(2)$ gauge groups and thus we have the BPS instantons labelled by $\vec{Y} = (Y_1, Y_2)$ and instanon number by $|\vec{Y}| = |Y_1| + |Y_2|$. A brief of Young diagrams is given in Appendix A.

Now we give the results for the instanton partition function in terms of these Young diagrams. For the case of a linear quiver with N $U(2)$ gauge groups [1] , the partition function is then

$$\begin{aligned} Z_{\text{inst}} = \sum_{\vec{Y}_1, \vec{Y}_2, \ldots, \vec{Y}_N} & \left(\prod_{i=1}^{N} q_i^{|\vec{Y}_i|} z_{\text{vector}}(\vec{a}_i, \vec{Y}_i) \right) z_{\text{antifund}}(\vec{a}_1, \vec{Y}_1, \mu_1) z_{\text{antifund}}(\vec{a}_1, \vec{Y}_1, \mu_2) \\ & \times \left(\prod_{i=1}^{N-1} z_{\text{bifund}}(\vec{a}_i, \vec{Y}_i; \vec{a}_{i+1}, \vec{Y}_{i+1}; m_i) \right) z_{\text{fund}}(\vec{a}_N, \vec{Y}_N, \mu_3) z_{\text{fund}}(\vec{a}_N, \vec{Y}_N, \mu_4) \quad (3.56) \end{aligned}$$

where the subscripts (vector, fund, $\cdots$) denotes the contribution of the building representations of the quiver (and thus, the theory). A vector in this section will always denote pair; $\vec{a} = (a_1, a_2)$. Then, $\vec{a}_i = (a_{i,1}, a_{i,2})$ is the VEV of the vector multiplet's adjoint scalar, $\vec{Y}_i$ is the pair of Young diagrams specifying the fixed instanton, q_i is the exponentiated gauge coupling of the i-th $SU(2)$ gauge group, m_i is the mass of the bifundamental hypermultiplet charged under $SU(2)_i$ and $SU(2)_{i+1}$, $\mu_{1,2,3,4}$ are the masses of the hypermultiplets and we define $\epsilon_+ \equiv \epsilon_1 + \epsilon_2$

[1]Strictly speaking, this is the case for the instanton partition function and how the original computations were done [13]. Yet we follow [1] that we can decouple $U(1)$ gauge fields and that's why the AGT correspondence is for the $SU(2)$ gauge theory not $U(2)$.

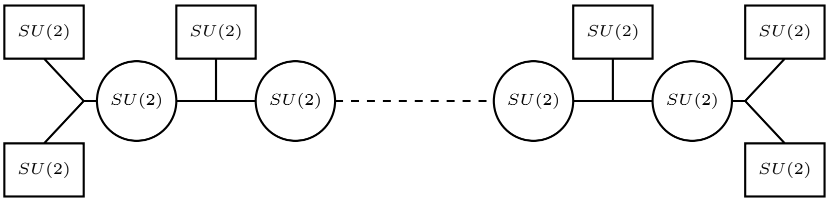


Figure 3.1. Linear quiver of N-SU(2) vector multiplets

For the case of a necklace quiver, the quiver closes on itself and each two vectors are joined with a bifundamental. Hence, the first and last vector $SU(2)_1, SU(2)_N$ are joined together with a bifundamental charged under these two gauge groups instead of the four fundamentals present in the linear quiver.

The fundamental and antifundamental contributions are defined as

$$z_{\text{fund}}(\vec{a}, \vec{Y}, m) = \prod_{i=1}^{2} \prod_{s \in Y_i} (\phi(a_i, s) - m + \epsilon_+), \tag{3.57}$$

$$z_{\text{antifund}}(\vec{a}, \vec{Y}, m) = z_{\text{fund}}(\vec{a}, \vec{Y}, \epsilon_+ - m) \tag{3.58}$$

where, for the box $s = (i, j)$, $\phi(a, s)$ is defined as

$$\phi(a, s) = a + \epsilon_1(i-1) + \epsilon_2(j-1). \tag{3.59}$$

The bifundamental on the other hand couples to two vector multiplets and thus, unlike fundamentals and antifundamentals, it is given with respect to two Young diagrams and two Coulomb branch parameters

$$z_{\text{bifund}}(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; m) = \prod_{i,j=1}^{2} \prod_{s \in Y_i} (E(a_i - b_j, Y_i, W_j, s) - m) \prod_{t \in W_j} (\epsilon_+ - E(b_j - a_i, W_j, Y_i, t) - m) \tag{3.60}$$

with the function $E(a, Y_1, Y_2, s)$ defined by

$$E(a, Y_1, Y_2, s) = a - \epsilon_1 L_{Y_2}(s) + \epsilon_2(A_{Y_1}(s) + 1). \tag{3.61}$$

In the case we have an adjoint hypermultiplet, Equation (3.56) should get a contribution Z_{adj}. Now, as was elaborated in the discussion following Equation (5.7), we could get a SCFT by the coupling of a vector multiplet to an adjoint hypermultiplet. This can be seen as a bifundamental charged under the same vector multiplet, twice. Hence, this adjoint contribution to the instanton partition function is

$$z_{\text{adj}}(\vec{a}, \vec{Y}, m) = z_{\text{bifund}}(\vec{a}, \vec{Y}, \vec{a}, \vec{Y}, m). \tag{3.62}$$

Then, setting the mass of this adjoint to zero, we only have a vector multiplet and its contribution is given by

$$z_{\text{vector}}(\vec{a}, \vec{Y}) = 1/z_{\text{adj}}(\vec{a}, \vec{Y}, 0). \tag{3.63}$$

Now that we have given the instanton part of the partition function, let us look at the perturbative part. As is known in supersymmetric theories, perturbative contributions higher than one loop is absent. Thus, the perturbative partition function will be solely of one-loop contribution. Our definition will be built from the logarithm of the Barnes double gamma function

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \log \Gamma_2(x + \epsilon_+ | \epsilon_1, \epsilon_2). \tag{3.64}$$

where the Barnes gamma function defined as

$$\Gamma_r(z|\vec{\omega}) = \prod_{\vec{n} \in \mathbb{Z}_{>0}} (z + \vec{\omega}.\vec{n})^{-1} \tag{3.65}$$

For the case of a linear quiver with N gauge groups, the one-loop part of Nekrasov's partition function is given by

$$Z_{\text{1-loop}} = \left(\prod_{i=1}^{N} z^{\text{1-loop}}_{\text{vector}}(\vec{a}_i)\right) z^{\text{1-loop}}_{\text{antifund}}(\vec{a}_1,\mu_1) z^{\text{1-loop}}_{\text{antifund}}(\vec{a}_1,\mu_2) \\ \times \left(\prod_{i=1}^{N-1} z^{\text{1-loop}}_{\text{bifund}}(\vec{a}_i,\vec{a}_{i+1},m_i)\right) z^{\text{1-loop}}_{\text{fund}}(\vec{a}_N,\mu_3) z^{\text{1-loop}}_{\text{fund}}(\vec{a}_N,\mu_4) \quad (3.66)$$

where the individual contributions are given by

$$z^{\text{1-loop}}_{\text{vector}}(\vec{a}) = \prod_{i<j} \exp\left[-\gamma_{\epsilon_1,\epsilon_2}(a_i - a_j - \epsilon_1) - \gamma_{\epsilon_1,\epsilon_2}(a_i - a_j - \epsilon_2)\right], \quad (3.67)$$

$$z^{\text{1-loop}}_{\text{fund}}(\vec{a},\mu) = \prod_{i} \exp\left[\gamma_{\epsilon_1,\epsilon_2}(a_i - \mu)\right], \quad (3.68)$$

$$z^{\text{1-loop}}_{\text{antifund}}(\vec{a},\mu) = \prod_{i} \exp\left[\gamma_{\epsilon_1,\epsilon_2}(-a_i + \mu - \epsilon_+)\right], \quad (3.69)$$

$$z^{\text{1-loop}}_{\text{bifund}}(\vec{a},\vec{b},m) = \prod_{i,j} \exp\left[\gamma_{\epsilon_1,\epsilon_2}(a_i - b_j - m)\right]. \quad (3.70)$$

Again, the contribution of an adjoint hypermultiplet is given by identifying the two Coulomb branch parameters under which a bifundamental is charged $z^{\text{1-loop}}_{\text{bifund}}(\vec{a},\vec{a},m)$.

Finally, the classical part of the partition function is given by

$$Z_{class} = e^{-2\pi i \tau a^2} \quad (3.71)$$

4. LIOUVILLE THEORY

In this section we leave the 4d side of the correspondence and start our review of the 2d side. As was the case with susy, and any type of symmetry actually, conformal symmetry provides an aid in solving quantum field theories because of the constraints it imposes on the theory. Nevertheless, conformal field theories are not generally easy to solve exactly. Two dimensional theories are different though because the symmetry algebra is infinite dimensional and can be solved exactly.

In the previous chapters we have discussed the partition function of $\mathcal{N} = 2$ SYM with gauge group $SU(2)$. Thus, in the 2d side of the correspondence we study Liouville field theory with Virasoro symmetry. Higher rank gauge groups of the A type; A_{N-1}, would amount to studying Toda field theories with $\mathcal{W}$-symmetry [23]. That is, 4d SYM with gauge group $SU(N)$ and A_{N-1} Toda field theory. We will not discuss these higher generalizations in this thesis though. This chapter then focuses on the A_1 Toda field theory; Liouville field theory.

4.1. 2d Conformal Algebra

We start by giving a brief of 2d Conformal Field Theory (CFT). The first two sections will mainly follow [24].

Conformal transformations are coordinate transformations that act on the metric by a Weyl transformation; they leave the metric invariant up to a scale factor;

$$g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{4.1}$$

It can be seen from this definition that for the case when $\Omega(x) = 1$, the Poincaré algebra is a subalgebra of the conformal algebra.

In 2d, transformations with this property are holomorphic (and antiholomorphic) coordinate transformations of the complex plane to itself

$$z \mapsto f(z) = z + \epsilon z^{n+1}, \qquad \bar{z} \mapsto \bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}\bar{z}^{n+1} \tag{4.2}$$

with $n \in \mathbb{Z}$ and ϵ being an infinitesimal parameter for the transformation. This gives the following conformal metric transformation

$$ds^2 = dzd\bar{z} \longmapsto ds^2 = \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} dzd\bar{z} \tag{4.3}$$

with the scale factor $\Omega = |\frac{\partial f}{\partial z}|^2$. These transformations (holomorphic and antiholomorphic) induce transformations on the fields via the generators ℓ and $\bar{\ell}$ which have the differential operator representation

$$\ell_n = -z^{n+1}\frac{\partial}{\partial z}, \qquad \bar{\ell}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}} \tag{4.4}$$

These operators generate two copies of the $Witt$ algebra, a holomorphic and an anti-holomorphic copy, with commutation relations

$$[\ell_n, \ell_m] = (n-m)\ell_{m+n}, \qquad [\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{m+n}, \qquad [\ell_n, \bar{\ell}_m] = 0 \tag{4.5}$$

This algebra is a classical algebra for conformally invariant classical field theory. We are interested though in a conformal quantum field theory and, in a quantum mechanical theory, symmetries are realized projectively on states (modulo phase multiplication of states in the Hilbert space). Projective representations of an algebra are equivalent to the central extension of the algebra. Then, the symmetry algebra of our conformal field theory (from here on this means the conformal quantum field theory) is the centrally extended Witt-algebra, the so called $Virasoro$ algebra. It has the generators $(L_n)_{n\in\mathbb{Z}}$ and $\mathbf{1}$, and obey the commutation relations (for the holomorphic copy)

$$[\mathbf{1}, L_n] = 0 \quad , \quad [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{4.6}$$

where the number c is called the central charge. There is of course an antiholomorphic copy of the Virasoro algebra. But since both are independent copies, the full symmetry algebra decomposes into $Vir \times \overline{Vir}$ acting on a decomposed state space $\mathcal{H} \times \bar{\mathcal{H}}$ and it will suffice to study just one copy and the same results will apply to the other one.

We then discuss representation theory of the Virasoro algebra. Consistent with the axioms of quantum field theory, we are interested in unitary highest weight representations. A *Verma Module* is a representation of the Virasoro algebra satisfying these two conditions.

Definition 7. *Verma Module $\mathcal{V}_\Delta$. It is the vector space carrying a highest weight representation of the Virasoro algebra. The highest weight state, also called a primary state, $|v\rangle$ has a conformal dimension Δ which is the eigenvalue of the zeroth Virasoro generator;*

$$L_0 \, |v\rangle = \Delta \, |v\rangle \tag{4.7}$$

and is annihilated by the positive generators

$$L_n \, |v\rangle = 0 \quad n > 0. \tag{4.8}$$

Lower states are created from this primary state by the negative generators $L_{-n}, n > 0$ and they form a basis $\boldsymbol{B}$ for the module

$$\boldsymbol{B} = \Big\{ \prod_{i=1}^{k} L_{-n_i|v\rangle} \Big\}_{0 \leq n_1 \leq \cdots \leq n_k} \tag{4.9}$$

A lower state, $\big\{ \prod_{i=1}^{k} L_{-n_i|v\rangle} \big\}$, has the conformal dimension $\Delta + N$ where $N = \sum_{i=1}^{k} n_i$ is called the level of the state and levels with $N \geq 1$ are called descendent states.

The level N can be seen as the number of boxes for a Young diagram and the set $\{n_i\}$ in Equation (4.9) of strictly increasing positive integers as forming partitions of that number (to be more accurate, it is the set of basis vectors labeled by the integers that form the partition). Verma modules are characterized by their conformal dimension and each Verma module constitutes a family of a primary field and its descendants. The families are distinct since, for every conformal dimension, only one primary state exists and no field descends from more than one primary state. Figure 4.1 shows a family of states up to level 3.

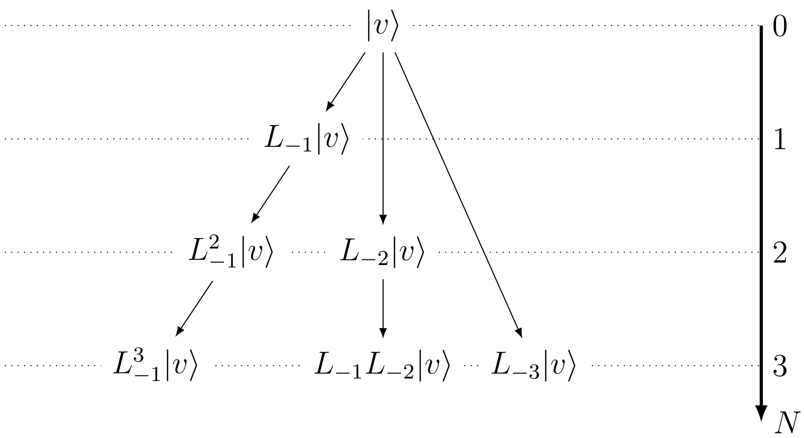


Figure 4.1. Conformal family

4.2. Conformal Fields

As we are studying a field theory, the next step after studying the symmetry on the algebraic level is to study the action of the generators of our symmetry on the fields. Just as we did with the algebra, we will limit our exposition to holomorphic fields, those depending only on z, and the exact same analysis will aply to antiholomorphic fields.

Knowing the states, it is easy to get the fields as a result of the *state-field correspondence* which assigns a field to every state in the spectrum. Thus, fields can be seen as functions indexed by the states (because of this one-one correspondence) and taking values at a certain point on the Riemann sphere

$$|w\rangle \mapsto V_{|w\rangle}(z) \tag{4.10}$$

This mapping is injective so no two fields come from the same state. The action of the Virasoro generators on the fields is then defined as

$$L_n V_{|w\rangle}(z) = V_{L_n|w\rangle}(z) = L_n V(z) \tag{4.11}$$

From this correspondence we see that, for the primary state $|v\rangle$ in our Verma module of conformal dimension Δ, there corresponds a primary field

$$|v\rangle \mapsto V_\alpha(z) \tag{4.12}$$

and, under the action of the algebra generators, has properties

$$L_0 V_\alpha(z) = \Delta_\alpha V_\alpha(z) \quad , \quad L_{n>0} V_\alpha(z) = 0 \tag{4.13}$$

where α is the label of the conformal family and Δ_α is the conformal dimension of this family. From Equation (4.4) we can see that the action of L_{-1} on fields could be finded as

$$L_{-1} V(z) = \frac{\partial}{\partial z} V(z) \tag{4.14}$$

Under a conformal transformation of the coordinates $z \mapsto f(z)$, fields transforming as

$$V(z) \mapsto V'(z') = \big(\frac{\partial f}{\partial z}\big)^{\Delta} V(f(z)) \tag{4.15}$$

are called primary fields of conformal dimension Δ.

Similarly, by the states-fields correspondence, descendant states map to descendant fields.

As we discussed, 2d conformal field theory has an infinite dimensional symmetry algebra and this strongly constrains the theory. It does so to an extent that, even though field theories are usually defined by an action that is used to compute observables and properties of our theory, we can do without a Lagrangian description of the theory. In this approach the only object needed is the energy-momentum tensor and the information it holds about the behaviour of fields under conformal transformations.

Definition 8. *We define the energy-momentum tensor via the Laurent series*

$$T(y) = \sum_{n\in\mathbb{Z}} \frac{L_n}{(y-z)^{n+2}} \tag{4.16}$$

From this definition we can see the action of the tensor on a generic field $V(z)$ as

$$T(y)V(z) = \sum_{n\in\mathbb{Z}} \frac{L_n V(z)}{(y-z)^{n+2}} \tag{4.17}$$

In the case of a primary field, this gives the result

$$T(y)V_\alpha(z) = \frac{\Delta_\alpha}{(y-z)^2} V_\alpha(z) + \frac{1}{(y-z)} \frac{\partial}{\partial z} V_\alpha(z) + \cdots \tag{4.18}$$

where the ellipses denote regular terms. This is an example of an *Operator Product Expansion*, or OPE, which we will discuss in a bit. We can also use Equation (4.17) to define the action of the Virasoro generators on fields by the complex integral

$$L_n V(z) = \frac{1}{2\pi i} \oint dy\ (y-z)^{n+1} T(y) V(z) \tag{4.19}$$

We now come to the important object in quantum field theories, correlation functions.

Definition 9. *A correlation function, or N-point function, is a number assigned to N-fields at N different points on the Riemann sphere*

$$\Big\langle V_1(z_1) \cdots V_N(z_N) \Big\rangle. \tag{4.20}$$

In a conformal field theory, an auxiliary $(N+1)$-point function $Z(y)$ is used to study the $N-point$ function Z, where we insert the energy momentum tensor in the latter to define the former;

$$Z = \left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \right\rangle \quad , \quad Z(y) = \left\langle T(y) \prod_{i=1}^{N} V_{\alpha_i}(z_i) \right\rangle \ . \tag{4.21}$$

When the fields are primaries, this gives

$$Z(y) = \sum_{i=1}^{N} \left(\frac{\Delta_i}{(y-z_i)^2} + \frac{1}{y-z_i} \frac{\partial}{\partial z_i} \right) Z \ . \tag{4.22}$$

In the limit as $y \to \infty$, we have $T(y) \sim O(\frac{1}{y^4})$. Hence, in the expansion of Equation (4.22), the coefficients of y^{-1}, y^{-2}, y^{-3} must vanish. Expanding, we find these vanishing coefficients to be

$$\sum_{i=1}^{N} \partial_{z_i} Z = \sum_{i=1}^{N} \left(z_i \partial_{z_i} + \Delta_i \right) Z = \sum_{i=1}^{N} \left(z_i^2 \partial_{z_i} + 2\Delta_i z_i \right) Z = 0 \ . \tag{4.23}$$

These three equations are the global conformal *Ward identities*. Solving these equations, we find the 2, 3 and 4-point functions to be

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \rangle = \frac{N(\Delta_1)\delta_{\Delta_1,\Delta_2}}{z_{12}^{2\Delta_1}} \tag{4.24}$$

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle = \frac{C_{\alpha_1,\alpha_2,\alpha_3}}{\prod_{i<j} z_{ij}^{\Delta_i+\Delta_j-\epsilon_{ijk}\Delta_k}} \tag{4.25}$$

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) \rangle = \frac{G(\mathcal{Z})}{\prod_{i<j} z_{ij}^{\Delta_i+\Delta_j-\sum_k \frac{\Delta_k}{3}}} \tag{4.26}$$

where $z_{ij} = z_i - z_j$, $N(\Delta)$ is a normalization constant which we can take to be 1, $C_{\alpha_i,\alpha_j,\alpha_k}$ are called product coefficients (they are like structure coefficients of an algebra) and $\mathcal{Z} = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ is the so-called cross ratio. From here on we will drop the α label on the product coefficients to not clutter the equations.

Now, as the number of fields in the correlator increases, the task of solving the Ward identities will certainly become tougher. To overcome this difficulty, *Operator Product Expansions (OPEs)* are introduced. It reduces N-point correlators to an N-1 point correlator and OPE coefficients in the limit where the arguments of the fields coincide. It can thus be used to reduce any correlator to a 2-point correlator and OPE coefficients

$$V_{|w_1\rangle}(z_1)V_{|w_2\rangle}(z_2) \underset{z_1\to z_2}{=} \sum_i C_{12}^i(z_1,z_2)V_{|w_i\rangle}(z_2) \tag{4.27}$$

where $|w_i\rangle$ are the basis states of the spectrum S. It can be further factorized to two sums, one over the primary states with the index labeling the different conformal families; different momenta, and one over the descendants of each conformal family. This expansion, for the OPE of primary fields, takes the form

$$V_{\alpha_i}(z_1)V_{\alpha_j}(z_2) \underset{z_1\to z_2}{=} \sum_\alpha \sum_k \beta_{ij}^{\alpha,(k)} C_{ij}^\alpha(z_1,z_2)V_\alpha^k(z_2) \ . \tag{4.28}$$

where k labels the sum over the descendants. For the 2-point expansion in Equation (4.24), we get

$$\langle V_{\alpha_i}(z_1)V_{\alpha_j}(z_2)\rangle \underset{z_1\to z_2}{=} \sum_\alpha C_{ij}^\alpha |z_{12}|^{2(\Delta_\alpha-\Delta_1-\Delta_2)} \left(V_{\Delta_\alpha}(z_2)+O(z_1-z_2)\right) \tag{4.29}$$

where the subleading terms are for the descendants' contribution. Doing the same for a 3-point function, the result we get is

$$\left\langle \prod_{i=1}^3 V_{\alpha_i}(z_i)\right\rangle = C_{12}^3|z_{12}|^{2(\Delta_3-\Delta_1-\Delta_2)}|z_{13}|^{2(\Delta_2-\Delta_1-\Delta_3)}|z_{23}|^{2(\Delta_1-\Delta_2-\Delta_3)}. \tag{4.30}$$

From this we see that the OPE expansion coefficients are the same as the product coefficients of 3-point functions in Equation (4.25). For the case of 4-point functions, we use conformal symmetry to map the 4 points that are the arguments of the fields to the special points $z, 0, \infty, 1$ on the Riemann sphere. We then use the OPE of the two fields $V_{\alpha_1}(z), V_{\alpha_2}(0)$ and that of the other two $V_{\alpha_3}(\infty), V_{\alpha_4}(1)$ to obtain the result

$$\begin{aligned}\Big\langle V_{\alpha_1}(z)V_{\alpha_2}(0)V_{\alpha_3}(\infty)V_{\alpha_4}(1)\Big\rangle &\underset{z\to 0}{=} \sum_{\alpha} C_{12}^{\alpha}|z|^{2(\Delta_\alpha-\Delta_1-\Delta_2)} \\ &\times \Big(\Big\langle V_{\Delta_\alpha}(0)V_{\alpha_3}(\infty)V_{\alpha_4}(1)\Big\rangle + O(z)\Big)\,, \\ &\underset{z\to 0}{=} \sum_{\alpha} C_{12}^{\alpha}C_{34}^{\alpha}|z|^{2(\Delta_\alpha-\Delta_1-\Delta_2)}\Big(1+O(z)\Big). \end{aligned} \tag{4.31}$$

Again, the descendants' contribution is factorized and we can define this contribution as

$$\mathcal{F}_{12}^{34}(\alpha|z) \underset{z\to 0}{\equiv} |z|^{2(\Delta_\alpha-\Delta_1-\Delta_2)}\Big(1+O(z)\Big). \tag{4.32}$$

This is the 4-point function *conformal block*. Now, inside the correlators, fields commute and we could have taken another combination of OPE of the fields. So, instead of *s-channel* conformal blocks; $\mathcal{F}_{12}^{(s)34}(\alpha|z)$ we got, we could have gotten the *t-channel* conformal blocks by using the combination $V_{\alpha_1}(z), V_{\alpha_4}(1)$ and $V_{\alpha_2}(0), V_{\alpha_3}(\infty)$ which yields the expansion

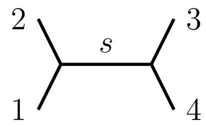
$$\Big\langle V_{\alpha_1}(z)V_{\alpha_2}(0)V_{\alpha_3}(\infty)V_{\alpha_4}(1)\Big\rangle = \sum_{\alpha} C_{14}^{\alpha}C_{23}^{\alpha}\mathcal{F}_{14}^{(t)23}(\alpha|z)\ , \tag{4.33}$$

where the t-channel conformal blocks are defined as

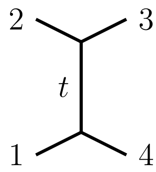
$$\mathcal{F}_{14}^{(t)23}(\alpha|z) \underset{z\to 1}{=} (z-1)^{\Delta_\alpha-\Delta_1-\Delta_4}\Big(1+O(z-1)\Big) \tag{4.34}$$

These two expressions (in addition to the third combination option, the *u-channel*) should be the same and this constraint on the 4-point functions is called *crossing* *symmetry* which, as a quantum field theory amplitude, looks like

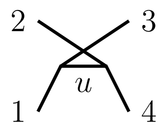
$$\sum_s C_{12s}C_{s34}$$



$$= \sum_t C_{14t}C_{t23}$$



$$= \sum_u C_{13u}C_{u24}$$



(4.35)

A conformal field theory should have its product coefficients obeying this crossing symmetry permutaion invariance.

4.3. Liouville Theory

The previous analysis is general and applies to any 2d conformal field theory. In particular, 3 and 4-point functions take the forms that we got but it is the value of the product coefficients that take different values depending on the the conformal field theory we are studying. The theory of interest for the AGT correspondence is Liouville theory. It is a theory of a scalar field subject to a potential term of the form $4\pi\mu e^{2b\phi}$ known as the Liouville potential and coupled to gravity via the term $QR\phi$. Thus, it has an action of the form

$$S[\phi] = \frac{1}{4\pi}\int d^2z\ \sqrt{g}(g^{\nu\rho}\partial_\nu\phi\partial_\rho\phi + QR\phi + 4\pi\mu e^{2b\phi}) \tag{4.36}$$

where R is the Ricci scalar, Q is the gravity coupling strength, or the background charge, and μ is the cosmological constant. The quantization of this theory has a central charge

$$c = \bar{c} = 1 + Q^2 \tag{4.37}$$

with

$$Q = b + \frac{1}{b} \tag{4.38}$$

For this theory, the primaries take the form $V_\alpha = e^{2\alpha\phi}$ with α taking a value in the range

$$\alpha = (b + \frac{1}{b})/2 + iP = Q/2 + iP, \qquad P \in \mathbb{R} \tag{4.39}$$

We see now that the momenta α are not discrete values anymore but belongs to a continuum. These kind of theories are called irrational conformal field theories and Liouville theory is one of these. The conformal dimension is now parametrized by the continuous value of α as

$$\Delta(\alpha) = \alpha(Q - \alpha) \tag{4.40}$$

This form indicates that a reflection of the form $\alpha \to Q - \alpha$ leaves the expression and, consequently, the theory invariant. Thus, primaries have a reflection identification defined by

$$V_\alpha = R(\alpha) V_{Q-\alpha} \tag{4.41}$$

where the reflection coefficient $R(\alpha)$ is defined as

$$R(\alpha) = -\lambda^{Q-2\alpha} \frac{\Gamma(b(2\alpha - Q))\Gamma(\frac{1}{b}(2\alpha - Q))}{\Gamma(b(Q - 2\alpha))\Gamma(\frac{1}{b}(Q - 2\alpha))} \tag{4.42}$$

with

$$\lambda = \left(\frac{\pi\Gamma(b^2)}{\Gamma(1 - b^2)} \mu \right)^{1/b} \tag{4.43}$$

From Equation (4.38), Equation (4.42) and Equation (4.43) we could see that there is a duality of the form $b \to 1/b$. The final task then is to find the form of the product coefficients and the 4-point correlator. The former is given be the DOZZ formula and takes the form

$$C_{\alpha_1\alpha_2\alpha_3} = \frac{(b^{2/b-2b}\lambda)^{Q-\sum_{i=1}^{3}\alpha_i}\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_1+\alpha_3-\alpha_2)} \tag{4.44}$$

where

$$\Upsilon_b(x) = \frac{1}{\Gamma_2(x|b,b^{-1})\Gamma_2(Q-x|b,b^{-1})} \tag{4.45}$$

with Γ_2 the Barnes gamma function defined in Equation (3.65). Finally, the 4-point function in Equation (4.33) takes the form

$$\left\langle V_{\alpha_1}(z,\bar{z})V_{\alpha_2}(0)V_{\alpha_3}(\infty)V_{\alpha_4}(1)\right\rangle = \frac{1}{2}\int_{Q/2+i\mathbb{R}} d\alpha\; C_{\alpha_1\alpha_2\alpha}C_{\alpha\alpha_3\alpha_4}||\mathcal{F}_{12}^{(s)34}(\alpha|z)||^2 \tag{4.46}$$

where the factor 1/2 is to avoid the double counting coming from the identification $\alpha \sim Q-\alpha$. Here we have explicitly written the holomorphic and antiholomorphic factors of the algebra for the conformal blocks in the form

$$||\mathcal{F}_{12}^{34}(\alpha|z)||^2 = \mathcal{F}_{12}^{34}(\alpha|z)\bar{\mathcal{F}}_{12}^{34}(\alpha|\bar{z}) \tag{4.47}$$

5. AGT

5.1. Class-$\mathcal{S}$ Theories

Our analysis through of the 4d side has been for a pure $SU(2)$ theory. We want to go a step further and start adding matter. This is achieved by adding hypermultiplets which couple to the vector multiplet. Then we add to our effective Lagrangian the matter term of Equation (2.43), which we write again for convenience

$$\mathcal{L}_{matter}^{\mathcal{N}=2} = \sum_{i=1}^{N_f} \int d^2\theta \, d^2\bar{\theta} \, (\bar{H}_1^i e^{2V_{\mathcal{R}}} H_1^i + H_2^i e^{-2V_{\mathcal{R}}} \bar{H}_2^i) + \int d^2\theta \left[(\sqrt{2} H_1^i \Phi H_2^i + m_i H_1^i H_2^i) \right]$$

with the gauge indices suppressed. Adding flavours affects the running coupling and we have the one-loop β-function

$$\beta \propto -(N_c - 2N_f) \tag{5.1}$$

for an $SU(2)$ vector multiplet and flavours in the fundamental representation. A consequence of this is that adding four hypermultiplets, or hypers from here on, results in a vanishing β-function. Thus, for such number of flavours, the coupling is exactly marginal and the gauge theory is scale invariant; a superconformal field theory (SCFT). This construction can be done geometrically and this SCFT can be associated to a 2d Riemann surface. This is already a step towards the AGT correspondence because it's this Riemann surface that connects the partition function of the gauge theory associated to the surface and the correlators of Liouville theory that lives on the surface. The following discussion mainly follows that of [25].

This geometric realization of the gauge theory can be "constructed" from two building blocks, a three punctured sphere and a cylinder. To each puncture of the sphere we associate an $SU(2)$ flavour, or a hyper. This construction is called the *trifundamental matter* and is denoted by Q_{aiu} where the indices a, i, u are $SU(2)$ flavour indices each taking the values $1, 2$. The cylinder on the other hand represents an $\mathcal{N} = 2$ $SU(2)$ vector multiplet with a complexified coupling τ as shown in Figure 5.1.

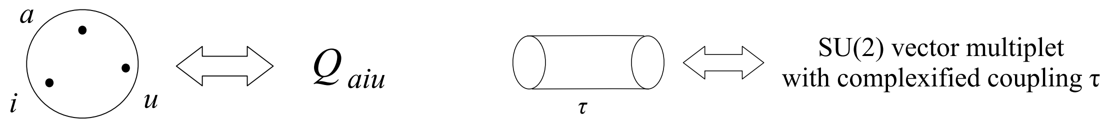


Figure 5.1. The Trifundamental and the vector multiplet as Riemann surfaces

The process of gluing punctures from the spheres with cylinders corresponds to building a gauge theory. We thus start building the SCFT we described earlier. Starting with two three punctured spheres, Q_{aiu}, Q'_{bsx} and a cylinder, we glue two punctures one from each sphere through the cylinder. This process amounts to identifying one puncture from each sphere with the other by gauging these two $SU(2)$ flavours, say u and b, by the dynamical $SU(2)$ gauge that the cylinder represents. The end result then is two gauged three punctured spheres, Q_{aiu}, Q'_{usx}, connected via a tube glued to both, or a four punctured sphere. The indices a, i, s, x are $SU(2)$ flavour symmetries while u is a dynamical $SU(2)$ gauge symmetry and all indices take the values $1, 2$.

This theory has four doublet (under the $SU(2)$ gauge) hypers or, in $\mathcal{N} = 1$ language, (H^i_{1a}, H^{ia}_2) with $i = 1, 2, 3, 4$ and $a = 1, 2$ is the $SU(2)$ gauge index. Another way to look at this theory is if we combine those 8 $\mathcal{N} = 1$ scalars into the scalar q_I with $I = 1, \cdots 8$. Then, the superpotential term in $\mathcal{L}^{\mathcal{N}=2}_{matter}$ can be written as

$$\mathcal{W} \propto q^a_I \Phi_{ab} q^b_J \delta^{IJ} + m^{IJ} q^a_I q^b_J \epsilon_{ab} \tag{5.2}$$

Then, since the indices a, b are symmetric, so is I, J. We then see that we have an $SO(8)$ symmetry and this is called flavour symmetry enhancement;

$$SU(2)_a \times SU(2)_i \times SU(2)_s \times SU(2)_x \subset SO(8) \tag{5.3}$$

A simpler schematic way of representing an $\mathcal{N} = 2$ $SU(2)$ SCFT is through a *quiver*. This is a directed graph with nodes and squares. The nodes give the gauge group of the theory and the squares give the flavour symmetry. So if we have a quiver with n-nodes and m-squares we have an $G = \prod_i^n SU(2)_i$ gauge theory with hypers transforming under the representation $\mathcal{R}_j \oplus \bar{\mathcal{R}}_j$ according to $\bigoplus_j^m n_j \mathcal{R}_j \oplus \bar{\mathcal{R}}_j$ with n_j the multiplicity of different representations under which the hypers transform, and an overall flavour symmetry group $\prod_j U(n_j)$. This flavour symmetry gets enhanced to $\prod_j SO(2n_j)$ for $\mathcal{R}$ being a pseudoreal representations, as we have seen before since the fundamental representation is pseudoreal (A pseudoreal representation is a representation whose generators $T_{\mathcal{R}}$ are not real; $-(T_{\mathcal{R}})^* \neq T_{\mathcal{R}}$, but are related to their complex conjugates via a unitary transformation J with $JJ^* = 1$; $-(T_{\mathcal{R}})^* = JT_{\mathcal{R}}J^*$).

The building block for the quivers are the so-called three legged pants which represents the matter trifundamental. Each leg represents a flavour and connecting two legs amounts to gauging two flavour symmetries, introducing a dynamical gauge theory.

In the previous example of four flavours, the $\mathcal{N} = 1$ chiral multiplets transform as

$$2_a \otimes 2_i \oplus 2_s \otimes 2_x = 8_v \tag{5.4}$$

where 8_v is the vector representation of the enhanced $SO(8)$ flavour symmetry (see Figure 5.2).

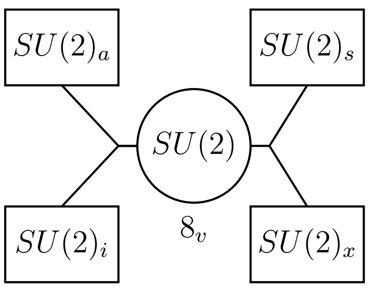


Figure 5.2. Vector representation of $SO(8)$

It is apparent that this four punctured sphere can be rotated so that the matter fields transform as

$$2_a \otimes 2_s \oplus 2_i \otimes 2_x = 8_s \tag{5.5}$$

with 8_s being the spinor representation of $SO(8)$ (Figure 5.3)

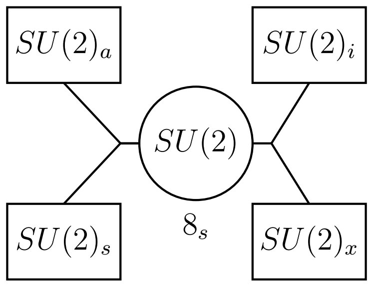


Figure 5.3. Spinor representation of $SO(8)$

or

$$2_a \otimes 2_x \oplus 2_s \otimes 2_i = 8_c \tag{5.6}$$

where 8_c is the cospinor representation of $SO(8)$ (Figure 5.4).

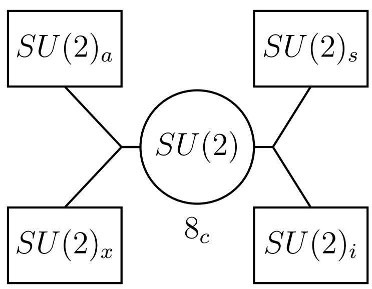


Figure 5.4. Cospinor representation of $SO(8)$

These three gauge theories have their matter content transform under different representations of the flavour symmetry group. They also have different coupling constants. However, they are related by the S-duality which as we have seen relates different couplings and, in our case here, also rotates flavour symmetries into each other. Thus they are considered to be a single theory but with different descriptions. This "triality" of descriptions is showed in Figure 5.5 in the Riemann surface representation as the

Gaiotto curve $\mathcal{C}_{0,4}$.

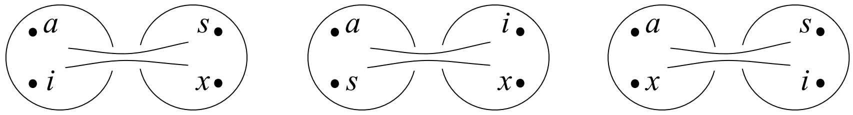


Figure 5.5. $SO(8)$ flavour triality

This is not the whole story. The $\beta-$function relation to N_c and N_f in Equation (5.1) was just for matter fields in the fundamental representation of $SU(2)$. But, more generally, (yet still for an $SU(2)$ vector),

$$\beta(g) \propto -(4 - 2\sum_i T(R_i)) \tag{5.7}$$

where the sum goes over the matter content and $T(R_i)$ is the Dynkin index of the representation of the ith matter field. For the fundamental ($fund$), bifundamental ($bifund$) and adjoint (Ad) representations we have

$$T(fund) = 1/2, \qquad T(bifund) = 1, \qquad T(Ad) = 2 \tag{5.8}$$

Thus, we see that we could have a conformal theory in different ways. The first, the one we discussed, is to couple 4 fundamental flavours, or fundamentals for short, to each gauge group. The second, is to couple 2 fundamentals and one bifundamental. But a flavour in the bifundamental representation must couple to another gauge group, transforming under the representation $SU(2)_1 \times SU(2)_2$. Thus, we would have two gauge groups, each coupled to 2 fundamentals and one bifundamental. This shown in Figure 5.6 (both the quiver representation and the Gaiotto curve).

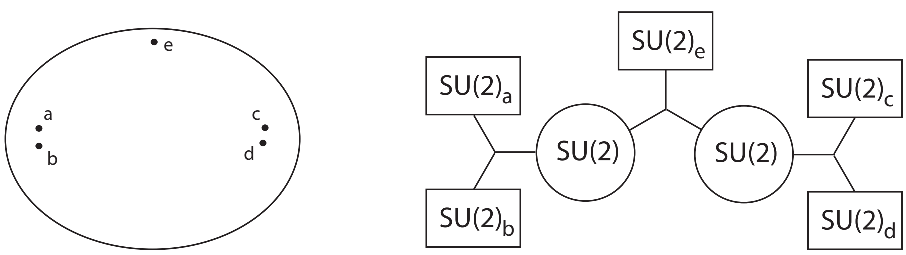


Figure 5.6. The 5-punctured sphere ($\mathcal{C}_{0,5}$) and its quiver representation

The last choice is to couple the gauge group to a flavour in the adjoint representation which is shown in Figure 5.7.

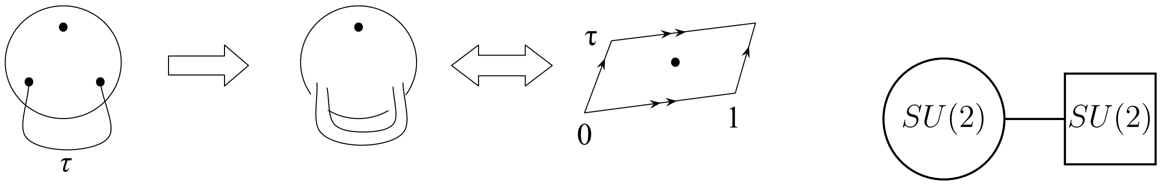


Figure 5.7. The torus with one puncture ($\mathcal{C}_{1,1}$) and its quiver representation

To recapitulate, for each theory we could associate a quiver or, geometrically, a Riemann surface $\mathcal{C}_{g,n}$ with n punctures and a genus g. The theory is denoted by $\mathcal{T}_{g,n}$ and the surface $\mathcal{C}_{g,n}$ is the Gaiotto curve of that theory. This class of theories was first shown by Gaiotto [26].

5.2. 4d/2d Correspondence

Now that we have studied both sides of the correspondence, the gauge theory side described by a genus-g Riemann surface with n-punctures; the Gaiotto curve $\mathcal{C}_{g,n}$ of the theory $\mathcal{T}_{g,n}$ and the 2d side corresponding to the conformal field theory living on the surface of that Riemann surface, we are in a place to give the statement of the correspondence.

$$\int da \; || \; Z_{S^4}^{Nek}[\mathcal{T}_{g,n}] \; ||^2 = \left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i, \bar{z}_i \right\rangle_{\mathcal{C}_{g,n}} \tag{5.9}$$

This states that the partition function on S^4 of the $\mathcal{T}_{g,n}$ corresponds to the n-point function of Liouville theory living on $\mathcal{C}_{g,n}$. To see which object on each side corresponds to which on the other, let us work an example, the case of a four punctured sphere $\mathcal{T}_{0,4}$, which is a vector multiplet coupled to four fundamentals and 4-point function of Liouville theory living on the Gaiotto curve of that theory; $\mathcal{C}_{0,4}$. The integral of the the gauge theory partition function takes the form

$$\int da \; || \; Z_{S^4}^{Nek}[\mathcal{T}_{0,4}] \; ||^2 = \int da \; Z_{1-loop}(a, \vec{m}) || Z_{class}(a, \tau) Z_{inst}(a, \vec{m}, \tau) ||^2 \tag{5.10}$$

while the Liouville theory correlator takes the form of Equation (4.46)

$$\left\langle V_{\alpha_1}(z)V_{\alpha_2}(0)V_{\alpha_3}(\infty)V_{\alpha_4}(1)\right\rangle \sim \int d\alpha\, C_{\alpha_1\alpha_2\alpha}C_{\alpha\alpha_3\alpha_4}||\mathcal{F}^{(s)34}_{12}(\alpha|z)||^2$$

Thus, comparing the two expressions we find an AGT dictionary between the gauge theory and conformal field theory which we give in Table 5.1.

Table 5.1. AGT dictionary

Gauge theory	**CFT**
$\int ||Z_{Nek}||^2$	4-point function
Z_{1-loop}	3-point coefficients (DOZZ product)
$Z_{class}Z_{inst}$	Conformal blocks
Coupling constant $q = e^{2\pi i\tau}$	cross ratio z
Equivariant parameters ϵ_1, ϵ_2	$b, \frac{1}{b}$
Coulomb branch parameter a	internal momenta α
hypermultimplet masses m_i	external momenta α_i

This can be seen graphically if we depict the quiver diagram of this four-punctured sphere theory and the s-channel amplitude for the 4-point function. where a is the VEV

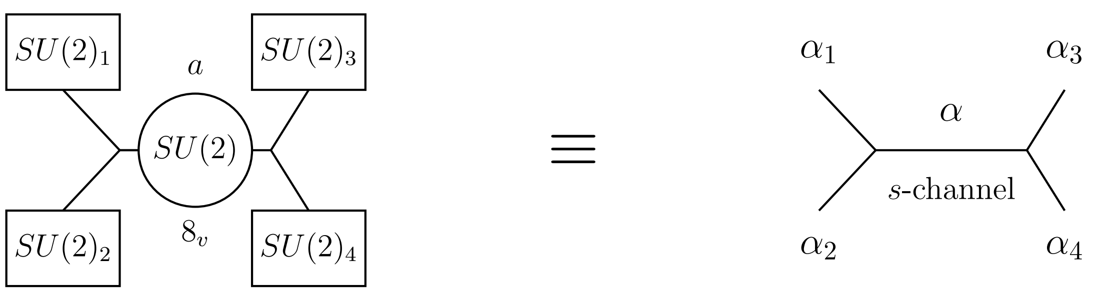


Figure 5.8. Correspondence between the vector representation of $SO(8)$ and the s-channel amplitude

of the scalar of the vector multiplet (the Coulomb branch parameter) corresponding to the node $SU(2)$. The same also applies for the other two representations of $SO(8)$; namely, the spinor and cospinor representations with the t- and u-channel amplitudes, respectively.

5.3. AGT Extensions

Supersymmetric gauge theories in 4d has natural generalizations to 5- and 6d by enlarging the non-compact 4d spacetime with compact manifolds. For 5d this compact dimension is a circle S^1 and for 6d it is a product of two circles; a torus $\mathbb{T}^2 \simeq S^1 \times S^1$. On the 2d side, Liouville theory (or Toda theory for higher rank gauge groups) has a natural extension through q-deformation, which is a 1-parameter deformation, as well as elliptic deformation, which is a 2-parameter deformation. Thus a natural question is in order; do higher dimensional correspondences like the AGT, called AGT-like correspondences, exist? The answer is yes!

In [27], a 5d/q-Virasoro correspondence was first proposed and in [28], an equivalence between the instanton partition function of $U(N)$ gauge theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1$ and the conformal blocks of q-Liouville theory on a sphere with $N+2$ punctures was shown.

For the q-deformed version of Liouville theory, another representation of the conformal blocks is more convenient. Previously we gave the conformal blocks as a sum over the descendants of every primary in the theory. Here it is more convenient to use the Dotsenko-Fateev (DF) representation (also known as the Coulomb-gas representation). In this representation, Liouville conformal blocks has the form

$$\mathcal{F}_{\{\alpha_0,\cdots,\alpha_{N+1}\}}(z_1,\cdots,z_N) = \langle V_{\alpha_0}(0)\cdots V_{\alpha_N}(z_N)V_{\alpha_{N+1}}(\infty)Q^a\rangle \tag{5.11}$$

where we have used global conformal transformations to send $z_0, z_{N+1} \to 0, \infty$ and Q is the so-called *screening charge* defined as the contour integral of *screening currents* S

$$Q = \oint dy\ S(y) \tag{5.12}$$

and we have used the notation Q^a to denote the insertion of a screening charges. We could then write this identification as

$$Z^{5d}_{inst}|_{a=a_r} \simeq \langle \prod_{i=0}^{N+1} V_{\alpha_i}(z_i) \prod_{j=1}^{r} \oint dy_j S(y_j) \rangle_{q-Vir} \tag{5.13}$$

with a_r are some tuned Coulomb branch parameters [2] and we have r screening charges with the integration over the Coulomb branch with the aforementioned parameters.

In [29, 30] a further 1-parameter deformation was studied and another extension of the AGT was given. Concisely, it states that the instanton partition function on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times \mathbb{T}^2$ identifies with the conformal blocks of the elliptically deformed Liouville theory and we can write it as

$$Z^{6d}_{inst}|_{a=a_r} \simeq \langle \prod_{i=0}^{N+1} V_{\alpha_i}(z_i) \prod_{j=1}^{r} \oint dy_j S(y_j) \rangle_{ell-Vir} \tag{5.14}$$

Figure 5.9 shows the gauge theories and their higher dimensional extension, the conformal theories and their 1-parmeter deformation and the corresponces between them.

$$\begin{tikzcd}
Z^{6d}_{inst} \arrow[rr, leftrightarrow, "\text{AGT-like}"] & & \mathcal{F}_{ell-Vir} \\
Z^{5d}_{inst} \arrow[u, "S^1"] \arrow[rr, leftrightarrow, "\text{AGT-like}"] & & \mathcal{F}_{q-Vir} \arrow[u, swap, "\text{elliptic-deformation}"] \\
Z^{4d}_{inst} \arrow[u, "S^1"] \arrow[rr, leftrightarrow, "\text{AGT}"] & & \mathcal{F}_{Vir} \arrow[u, swap, "q\text{-deformation}"]
\end{tikzcd}$$

Figure 5.9. AGT correspondence and its extensions

[2]These are the values of the parameters at which the Coulomb branch meets the Higgs branch of the moduli space and it can be reached by tuning the hypermultiplets' masses so that they are equal to the Coulomb branch parameters

6. DIM ALGEBRA

Ding-Iohara-Miki (DIM) algebra is an algebraic structure of growing interest both in mathematics and physics. It is the symmetry behind the 5d extension of the AGT relation as the Nekrasov partition function on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1$ is invariant under the action of the DIM action and we have the deformation parameters q, t of the Virasoro algebra identified with the DIM algebra parameters q_1, q_2 which we will see later. DIM algebra also plays the central role in refined topological strings, where the main object, the refined topological vertex, can be identified with the intertwining operator of Fock representations of DIM algebra.

The main importance of DIM algebra, for which we introduce it here, is because the intertwining maps of this algebra are used in building networks of operators. The matrix elements of these operators give the 3d gauge theory partition function. This will be the topic of the next two chapters.

Notations

$\mathbb{Z}$: The set of integers, $\quad \mathbb{N} := \{0, 1, 2, \cdots\}, \quad \mathbb{Z}_+ := \{1,\ 2,\ \cdots\}$,

$\mathbb{Q}$: The set of rational numbers,

$\mathbb{Q}(q, t)$: The field of rational functions of q, t over $\mathbb{Q}$, with $q, t \in \mathbb{C}^\times$

$\mathbb{C}$: The set of complex numbers, $\quad \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$,

$$\mathbb{F}[[z, z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} w_n z^n | w_n \in \mathbb{F} \right\},$$

$$\mathbb{F}[z, z^{-1}] = \left\{ \sum_{n\in\mathbb{Z}} w_n z^n | w_n \in \mathbb{F}, \text{ with all but finitely many } w_n = 0 \right\},$$

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n \quad \in \mathbb{C}[[z, z^{-1}]].$$

6.1. Algebraic preliminaries

DIM algebra is a quantum double affine algebra and to understand what this means we need to understand Drinfeld's quantum affinization of a quantum algebra (which is itself the quantization of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$) having a Hopf algebra structure. Thus, we start by giving the definition of Hopf algebras and see what a quantum algebra is.

6.1.1. Hopf algebras

The exposition of this subsection will follow that of [31] and [32]. To define a Hopf algebra we need the concepts of an algebra and a bialgebra and we give those.

Definition 10. *An algebra is a linear space A together with two linear maps*

$$\begin{aligned} \textit{product} \quad m \ &: \ A{\otimes}A \to A \\ \textit{unit} \quad \eta \ &: \ a \in \mathbb{C} \to a\mathbf{1} \in A \quad \textit{where } \mathbf{1} \textit{ is the unit element in } A \end{aligned}$$

with the composition of these maps

$$\begin{aligned} &\textit{(i)} \ \ m \circ (m{\otimes}\mathbf{1}) = m \circ (\mathbf{1}{\otimes}m) \ \ \textit{(associativity)} \\ &\textit{(ii)} \ \ m \circ (\mathbf{1}{\otimes}\eta) = m \circ (\eta{\otimes}\mathbf{1}) = \mathbf{1} \ \ \textit{(unit)} \end{aligned}$$

taking $A \otimes A \otimes A \to A$.

Definition 11. *A bialgebra B is an algebra A together with two linear maps, the coproduct*

$$\Delta : B \to B{\otimes}B$$

satisfying the following properties

(i) $\Delta(ab) = \Delta(a)\Delta(b)$ *(Δ is an algebra homomorphism)*

(ii) $(\Delta \otimes \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes \Delta) \circ \Delta$ *which takes $B \to B \otimes B \otimes B$ (coassociativity)*

and the counit $\varepsilon : a\mathbf{1} \in B \to a \in \mathbb{C}$, which obeys the identity, $(1 \otimes \varepsilon)\Delta = (\varepsilon \otimes 1)\Delta = \mathbf{1}$.

Definition 12. *A Hopf algebra is a bialgebra B together with a map a, called the antipode*

$$S : B \to B$$

which is antihomomorphism; S(ab) = S(b)S(a), with the following property:

$$m \circ (S \otimes \mathbf{1}) \circ \Delta = m \circ (\mathbf{1} \otimes S) \circ \Delta = \eta \circ \varepsilon$$

Now comes the object of interest, quantum algebras.

Definition 13. *For a complex simple Lie algebra $\mathfrak{g}$, the quantum universal enveloping algebra, or quantum algebra for short, is the q-deformation of the universal enveloping algebra of $\mathfrak{g}$, $\mathcal{U}_q(\mathfrak{g})$, which is a unital associative algebra over $\mathbb{C}$ with the following q-deformed commutation relations between the generators $X_i^{\pm}, H_i, i = 1, 2, ..., l =$ rank of the algebra,*

$$[H_i, H_j] = 0, \qquad [H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}, \qquad [X_i^+, X_j^-] = \delta_{ij} \frac{q_i^{H_i/2} - q_i^{-H_i/2}}{q_i^{1/2} - q_i^{-1/2}}$$

with $q_i = q^{(\alpha_i, \alpha_i)/2}$ and q-deformed Serre relations:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q_i} \left(X_i^{\pm}\right)^k X_j^{\pm} \left(X_i^{\pm}\right)^{n-k} = 0, \quad i \neq j$$

where q is the deformation parameter defined as the exponent of a complex number $h; q = e^h$, and we have the q-analog of a number m as $[m]_q = \dfrac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}$, *which gives the regular number in the limit* $q \to 1$, *or, equivalently,* $h \to 0$, $n = 1 - a_{ij}$ *where* $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ *is the Cartan matrix of* $\mathfrak{g}$, $(\cdot,\cdot)$ *is the scalar product of the roots* α *with the normalization* $(\alpha, \alpha) = 2$, *and the q-deformed binomial* $\binom{n}{k}_q = \dfrac{[n]_q!}{[k]_q![n-k]_q!}, ; [m]_q! = [m]_q[m-1]_q....[1]_q$.

This is Drinfeld's definition of a quantum algebra (it goes by the name quantum group as well) first introduced in [33]. There is also Jimbo's definition [34] but it's Drinfeld's that we mention as we use his realization of quantum affine algebras in the next subsection.

Drinfeld also observed in [33] that $\mathcal{U}_q(\mathfrak{g})$ has a Hopf algebra structure with the action of the comultiplication, counit and antipode on the quantum algebra's generators as follows:

$$\begin{gathered}
\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \\
\Delta(X_i^{\pm}) = X_i^{\pm} \otimes q_i^{H_i/4} + q_i^{-H_i/4} \otimes X_i^{\pm} \\
\varepsilon(H_i) = \varepsilon(X_i^{\pm}) = 0 \\
S(H_i) = -H_i, \quad S(X_i^{\pm}) = -q_i^{\pm 1/2} X_i^{\pm}
\end{gathered}$$

6.1.2. Quantum affine Lie algebras

Since what we are after is to understand DIM algebra, which is a quantum double affine algebra, we need to introduce the affinization of a quantum algebra, twice. So we start by giving the notion of a quantum affine algebra in this subsection, which is the quantum affinization of the quantum algebra given previously. The q-deformation of the simple Lie algebra introduced in the previous subsection is also valid if we replace the simple Lie algebra $\mathfrak{g}$ with the corresponding untwisted affine Lie algebra $\hat{\mathfrak{g}}$ and it was given by Drinfeld in [35]. Figure 6.1 is imported from [36].

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{Affinization}} & \hat{\mathfrak{g}} \\ \Big\downarrow{\scriptstyle \text{Quantization}} & & \Big\downarrow{\scriptstyle \text{Quantization}} \\ \mathcal{U}_q(\mathfrak{g}) & \xrightarrow{\text{Quantum affinization}} & \mathcal{U}_q(\hat{\mathfrak{g}}) \end{array}$$

Figure 6.1. Affinization and quantization of Lie algebras

First, let's recall the notion of an affine Kac-Moody algebra. Kac-Moody algebras are generalizations of finite dimensional simple Lie algebras to infinite dimensional Lie algebras $\mathfrak{g}$ with the Cartan matrix property of being positive-definite relaxed. The class of untwisted affine Kac-Moody algebras $\hat{\mathfrak{g}}$, which is what we are interested in, is that for which the determinant of the Cartan matrix is zero. They are the extension of the loop algebra $\mathcal{L}(\mathfrak{g})$ with a central element γ

$$\hat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\gamma \quad \text{where} \quad \mathcal{L}(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}.$$

Then, the quantum affine algebra, $\mathcal{U}_q(\hat{\mathfrak{g}})$, is the quantization of the untwisted affine Lie algebra, $\hat{\mathfrak{g}}$. It is an associative unital algebra with the set of generators,

$$\{x_i^+, x_i^-, k_i, k_i^{-1}, i = 0, 1, ..., l \text{ (the rank of the Lie algebra } \mathfrak{g})\} \tag{6.1}$$

and satisfying some relations that we don't need to mention here but can be found in [37]. This is the aforementioned first definition given by Drinfeld in [33].

In [35], Drinfeld realized another presentation of the quantum affine algebra (which is known as Drinfeld's second realization of quantum affine algebras), which is isomorphic to the previous associative unital algebra, with the set of generators,

$$\{x_{ik}^{\pm}, \phi_{im}, \psi_{in}, \gamma^{\pm 1/2}, i = 1, 2, ..., l, k \in \mathbb{Z}, m, n \in \mathbb{Z}_+\} \tag{6.2}$$

with γ being a central element defined as $\gamma \equiv k_i^{-1} = k_i$. These generators are the Fourier modes of the so-called *Drinfeld currents* of the algebra which obey the relations of the algebra and whose expansions are defined in terms of a formal indeterminnate as

$$x_i^{\pm}(z) = \sum_{k\in\mathbb{Z}} z^{-k} x_{ik}^{\pm}, \quad \psi_i(z) = \sum_{n\in\mathbb{Z}_+} z^{-n}\psi_{in}, \quad \phi_i(z) = \sum_{m\in\mathbb{Z}_+} z^{m}\phi_{im}.$$

Again, we don't need these relations and only refer to them in [37].

This isomorphism between the two presentations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ are as follows,

$$x_i^{\pm} \to x_{i0}^{\pm}, \qquad k_i \to \psi_{i0}, \qquad k_i^{-1} \to \phi_{i0}.$$

Note that many authors us the notation ψ^{-} instead of ϕ (this is the one we will use here), then the generators in the Drinfeld presentation are, $\{x_{ik}^{\pm}, \psi_{in}^{\pm}, \gamma^{\pm 1/2}\}$.

6.1.3. Quantum toroidal algebras

This subsection draws from [38]. As we explained in the previous section, affine Kac-Moody algebras are loop algebras supplemented with a central element and, possibly, a grading. If we apply a further affinization, we get a double loop,hence the name toroidal algebra with two central elements (and two gradings if we introduce gradings in the affinization process).This procedure is classical. On the quantum side, Drinfeld's affine quantization can be applied to any quantum Kac-Moody algebra. Thus, we can apply this procedure to the quantum affine algebra of the previous section, obtaining a quantum toroidal algebra. Figure 6.2 explains these classical and quantum transitions where $\hat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\gamma_1$, and $\hat{\hat{\mathfrak{g}}} = \mathcal{L}_{\text{double}}(\mathfrak{g}) \oplus \mathbb{C}\gamma_1 \oplus \mathbb{C}\gamma_2$,with $\mathcal{L}(\mathfrak{g}) = \mathbb{C}[t^{\pm}] \otimes \mathfrak{g}$, is the loop algebra and, $\mathcal{L}_{\text{double}}(\mathfrak{g}) = \mathbb{C}[t^{\pm}, q^{\pm}] \otimes \mathfrak{g}$ is the double loop algebra.

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\text{Affinization}"] \arrow[d, "\text{Quantization}"] & & \underset{\text{Affine algebra}}{\hat{\mathfrak{g}}} \arrow[rr, "\text{Affinization}"] \arrow[d, "\text{Quantization}"] & & \underset{\text{Double affine algebra}}{\hat{\hat{\mathfrak{g}}}} \arrow[d, "\text{Quantization}"] \\
\mathcal{U}_q(\mathfrak{g}) \arrow[rr, "\text{Quantum affinization}"] & & \underset{\text{Quantum affine algebra}}{\mathcal{U}_q(\hat{\mathfrak{g}})} \arrow[rr, "\text{Quantum affinization}"] & & \underset{\text{Quantum toroidal algebra}}{\mathcal{U}_{q,t}(\hat{\hat{\mathfrak{g}}})}
\end{tikzcd}$$

Figure 6.2. Double affinization and quantization of Lie algebras

Quantum toroidal algebras were introduced in [39]. The quantum toroidal algebra of $\mathfrak{gl}_n$ is formulated in terms of the Drinfeld currents

$$x_i^{\pm}(z) = \sum_{k\in\mathbb{Z}} z^{-k} x_{i,k}^{\pm}, \quad \psi_i^{\pm}(z) = \sum_{k\in\mathbb{N}} z^{\mp k} \psi_{i,\pm k}^{\pm}$$

with $i = 0, 1, ..., n-1$. Note that there is another notation for the generators of this algebra, which is

$$x_i^{+}(z) \to E_i(z), \quad x_i^{-}(z) \to F_i(z), \quad \psi_i^{\pm}(z) \to K_i^{\pm}(z),$$

but we will use the former one.

The operators $x_i^{\pm}(z)$ are associated to the simple roots α_i of $\hat{\mathfrak{g}}$; they are the root generators. On the other hand, the operators $\psi_i^{\pm}(z)$ describe the Cartan sector of the algebra, they are naturally associated to the coroots $\alpha_i^{\vee}$; the Cartan generators.

For the underlying simple Lie algebra of rank $n-1$, the second central element of the quantum toroidal algebra is defined in terms of the Cartan zeroth generators as

$$(\prod_{i=0}^{n-1} \psi_{i,0}^{+})^{-1} = \prod_{i=0}^{n-1} \psi_{i,0}^{-} \tag{6.3}$$

The defining relations for a quantum toroidal algebra can be found in [38], but we won't put them here since in the next section we will be providing the version of the relations for the quantum toroidal algebra of interest; DIM algebra.

6.2. Introducing DIM Algebra

We will follow [40] in this section. The algebra constructed by Ding-Iohara [41] and Miki [42] is the quantum toroidal deformation of $\mathfrak{gl}_1$. In Drinfeld's presentation, we have just one set of Drinfeld currents (that for which i=0), so we drop the i subscript and DIM algebra has the generators

$$\{x_k^{\pm}, \psi_l^{\pm}, \gamma \; | k, l \in \mathbb{Z}\} \tag{6.4}$$

with γ the first central element. The algebra depends on three parameters $q_1, q_2, q_3 \in \mathbb{C}^{\times}$ constrained by the relation $q_1 q_2 q_3 = 1$. These parameters are conventionally chosen as

$$q_1 = q, \quad q_2 = t^{-1}, \quad q_3 = t/q \tag{6.5}$$

so that we have the first two as the deformation parameters of the algebra and the third obtained by the constraint $q_1 q_2 q_3 = 1$. The algebra is symmetric under the action of the symmetry group $Sym(3)$ permuting these deformation parameters $(q, t^{-1}, t/q)$. In 5d gauge theories, these parameters are identified with the equivariant parameters of the Ω background ϵ_1, ϵ_2 through

$$q_1 = e^{-R\epsilon_1}, \quad q_2 = e^{-R\epsilon_2}. \tag{6.6}$$

with R being the radius of the compact dimension. The algebra has two central elements, the first is $\gamma = q_3 = \sqrt{t/q}$ and the second is $\psi_0^- = (\psi_0^+)^{-1} = \gamma_\perp = \sqrt{t/q}$. The second is usually defined according to Equation (6.3) as $\psi_0^-(\psi_0^+)^{-1}$ and together, these two elements define a central charge vector

$$\Gamma = (\gamma^{2c}, \gamma_\perp^{2\bar{c}}) \tag{6.7}$$

with central charges $(c, \bar{c}) \in (\mathbb{N}, \mathbb{N})$. This vector will be denoted $\Gamma^{(c,\bar{c})}$.

We now define our algebra.

Definition 14. *The Ding-Iohara-Miki (DIM) algebra, $\mathcal{U} := \mathcal{U}(q,t)$ is a unital associative algebra over $\mathbb{Q}(q,t)$ generated by the Drinfeld currents*

$$x^{\pm}(z) = \sum_{n\in\mathbb{Z}} x_n^{\pm} z^{-n}, \qquad \psi^{\pm}(z) = \sum_{n\in\mathbb{N}} \psi_n^{\pm} z^{-n}, \tag{6.8}$$

and the central element $\gamma^{\pm 1/2}$, satisfying the defining relations

$$\psi^{\pm}(z)\psi^{\pm}(w) = \psi^{\pm}(w)\psi^{\pm}(z), \tag{6.9}$$

$$\psi^{+}(z)\psi^{-}(w) = \frac{g(\gamma^{+1}w/z)}{g(\gamma^{-1}w/z)}\psi^{-}(w)\psi^{+}(z), \tag{6.10}$$

$$\psi^{+}(z)x^{\pm}(w) = g(\gamma^{\mp 1/2}w/z)^{\mp 1}x^{\pm}(w)\psi^{+}(z), \tag{6.11}$$

$$\psi^{-}(z)x^{\pm}(w) = g(\gamma^{\mp 1/2}z/w)^{\pm 1}x^{\pm}(w)\psi^{-}(z), \tag{6.12}$$

$$[x^{+}(z), x^{-}(w)] = \frac{(1-q)(1-1/t)}{1-q/t}\Big(\delta(\gamma^{-1}z/w)\psi^{+}(\gamma^{1/2}w) - \delta(\gamma z/w)\psi^{-}(\gamma^{-1/2}w)\Big), \tag{6.13}$$

$$x^{\pm}(z)x^{\pm}(w) = g(z/w)^{\pm 1}x^{\pm}(w)x^{\pm}(z). \tag{6.14}$$

The structure function $g(z)$ encodes the dependence on the parameters q_1, q_2, q_3 of the algebra:

$$g(z) = \prod_{a=1,2,3} \frac{1-q_a z}{1-q_a^{-1}z}. \tag{6.15}$$

Explicitly we have

$$g(z) := \frac{G^{+}(z)}{G^{-}(z)}, \qquad G^{\pm}(z) := (1-q^{\pm 1}z)(1-t^{\mp 1}z)(1-q^{\mp 1}t^{\pm 1}z). \tag{6.16}$$

where $G^{\pm}(z)$ are analytic polynomials in $z \in \mathbb{C}^{\times}$.

The algebra $\mathcal{U}$ has a Hopf algebra structure defined by the action of the Hopf algebra maps on elements of the algebra as follows:
Coproduct Δ (which is often called Drinfeld's coproduct):

$$\Delta(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2}, \tag{6.17}$$

$$\Delta(x^+(z)) = x^+(z) \otimes 1 + \psi^-(\gamma_{(1)}^{1/2} z) \otimes x^+(\gamma_{(1)} z), \tag{6.18}$$

$$\Delta(x^-(z)) = x^-(\gamma_{(2)} z) \otimes \psi^+(\gamma_{(2)}^{1/2} z) + 1 \otimes x^-(z), \tag{6.19}$$

$$\Delta(\psi^{\pm}(z)) = \psi^{\pm}(\gamma_{(2)}^{\pm 1/2} z) \otimes \psi^{\pm}(\gamma_{(1)}^{\mp 1/2} z), \tag{6.20}$$

where $\gamma_{(1)}^{\pm 1/2} = \gamma^{\pm 1/2} \otimes 1$ and $\gamma_{(2)}^{\pm 1/2} = 1 \otimes \gamma^{\pm 1/2}$.
Counit ε:

$$\varepsilon(\gamma^{\pm 1/2}) = 1, \qquad \varepsilon(\psi^{\pm}(z)) = 1, \qquad \varepsilon(x^{\pm}(z)) = 0. \tag{6.21}$$

Antipode S:

$$S(\gamma^{\pm 1/2}) = \gamma^{\mp 1/2}, \tag{6.22}$$

$$S(x^+(z)) = -\psi^-(\gamma^{-1/2} z)^{-1} x^+(\gamma^{-1} z), \tag{6.23}$$

$$S(x^-(z)) = -x^-(\gamma^{-1} z) \psi^+(\gamma^{-1/2} z)^{-1}, \tag{6.24}$$

$$S(\psi^{\pm}(z)) = \psi^{\pm}(z)^{-1}. \tag{6.25}$$

6.3. Representations of DIM algebra

A generic representation R of DIM algebra is a left $\mathcal{U}$-module M. It is of level $\ell = (c, \bar{c})$, where $c, \bar{c}$ are the central charges defined previously, if for any $\alpha \in M$ the central elements have the following action on the module

$$R(\gamma^2)\alpha = (t/q)^c \alpha, \quad R(\gamma_{\perp}^2)\alpha = (t/q)^{\bar{c}} \alpha \tag{6.26}$$

We are interested in two classes of these representations, the so-called vertical representations $(0, m)$ and the horizontal representations $(1, n)$.

The vertical representation $\pi_x(\cdot)$

We have a representation $\pi_x(\cdot)$ of $\mathcal{U}$ on $\mathcal{V}_x = \mathbb{Q}(q^{1/2}, t^{1/2})[x, x^{-1}]$ by

$$\pi_x(\gamma^{\pm 1/2}) = 1, \tag{6.27}$$

$$\begin{aligned}\pi_x(\psi^+(z)) &= \frac{(1 - q^{1/2}t^{-1}x/z)(1 - q^{-1/2}tx/z)}{(1 - q^{1/2}x/z)(1 - q^{-1/2}x/z)} \\ &= 1 + \sum_{n \geq 1}(1 - q/t)(1 - t)\frac{1 - q^n}{1 - q}q^{-n/2}(x/z)^n,\end{aligned} \tag{6.28}$$

$$\begin{aligned}\pi_x(\psi^-(z)) &= \frac{(1 - q^{1/2}t^{-1}z/x)(1 - q^{-1/2}tz/x)}{(1 - q^{1/2}z/x)(1 - q^{-1/2}z/x)} \\ &= 1 + \sum_{n \geq 1}(1 - q/t)(1 - t)\frac{1 - q^n}{1 - q}q^{-n/2}(z/x)^n,\end{aligned} \tag{6.29}$$

$$\pi_x(x^{\pm}(z)) = c^{\pm 1}(1 - t^{\mp 1})\delta(q^{\mp 1/2}x/z)T_{q^{\mp 1},x}, \tag{6.30}$$

where $c \in \mathbb{Q}(q^{1/2}, t^{1/2})^{\times}$. And $T_{q^{\pm 1},x}$ in the last equation is the q-shift operator defined by $T_{q^{\pm 1},x}f(x) = f(q^{\pm 1}x)$.

The horizontal representation $\rho(\cdot)$

We have a representation $\rho(\cdot)$ of $\mathcal{U}$ on $\mathcal{F}$ by $\rho(\gamma^{\pm 1/2}) = (t/q)^{\pm 1/4}$, $\rho(\psi^{\pm}(z)) = \varphi^{\pm}(z)$, $\rho(x^+(z)) = c\,\eta(z)$ and $\rho(x^-(z)) = c^{-1}\xi(z)$, where $c \in \mathbb{Q}(q^{1/2}, t^{1/2})^{\times}$, with $\mathcal{F}$ being the Fock space constructed by the action of the negative modes of the q-deformed Heisenberg algebra,

$$[a_m, a_{-n}] = m\frac{1 - q^{|m|}}{1 - t^{|m|}}\delta_{m,n} \tag{6.31}$$

on a Fock vacuum state. The vertex operators $\varphi^{\pm}(z), \eta(z), \xi(z)$ are defined as

$$\eta(z) := \exp\left(\sum_{n>0}\frac{1-t^{-n}}{n}a_{-n}z^{n}\right)\exp\left(-\sum_{n>0}\frac{1-t^{n}}{n}a_{n}z^{-n}\right), \tag{6.32}$$

$$\xi(z) := \exp\left(-\sum_{n>0}\frac{1-t^{-n}}{n}(t/q)^{n/2}a_{-n}z^{n}\right)\exp\left(\sum_{n>0}\frac{1-t^{n}}{n}(t/q)^{n/2}a_{n}z^{-n}\right), \tag{6.33}$$

$$\varphi^{+}(z) := \exp\left(-\sum_{n>0}\frac{1-t^{n}}{n}(1-t^{n}q^{-n})(t/q)^{-n/4}a_{n}z^{-n}\right), \tag{6.34}$$

$$\varphi^{-}(z) := \exp\left(+\sum_{n>0}\frac{1-t^{-n}}{n}(1-t^{n}q^{-n})(t/q)^{-n/4}a_{-n}z^{n}\right). \tag{6.35}$$

The last object we need to introduce before ending this chapter is the intertwining operator $\Phi^{\mathcal{F}}_{\mathcal{V}_x\otimes\mathcal{F}} : \mathcal{V}_x\otimes\mathcal{F} \to \mathcal{F}$ which should satisfy the condition $\Phi^{\mathcal{F}}_{\mathcal{V}_x\otimes\mathcal{F}}\Delta(a) = a\,\Phi^{\mathcal{F}}_{\mathcal{V}_x\otimes\mathcal{F}}$ for any $a \in \mathcal{U}$, and it has a free-field representation as

$$\Phi(y) = \exp\left(\sum_{n>0}\frac{1}{n}\frac{1-t^{n}}{1-q^{n}}q^{n/2}t^{-n}a_{-n}y^{n}\right)\exp\left(-\sum_{n>0}\frac{1}{n}\frac{1-t^{n}}{1-q^{n}}q^{n/2}a_{n}y^{-n}\right). \tag{6.36}$$

7. HIGGSED NETWORKS

In the last chapter we discussed DIM algebra and mentioned couple of its utility in physics. One of those was the use of DIM intertwining operators to obtain the partition function of 3d linear quiver gauge theories. More accurately, we will be getting the holomorphic blocks [3] of those 3d theories and these are identified with the 3d partition function. This has the advantage that it gives the 3d theory directly without the need for an intermediate step where an auxiliary 5d theory is first obtained and then tune its parameters to get the end result of the 3d theory.

7.1. Intertwiners and their Duals

The main object of these last two chapters will be the intertwining operator, or Higgsed vertex as it is called in [2]. We introduced it in Equation (6.36) but will instead use the form used in [2] in this chapter to have consistent results. The form

$$\Phi(w) = e^{-\epsilon_2 Q} w^{\frac{P}{\epsilon_1}} \exp\left[-\sum_{n>0} \frac{w^n}{n}\frac{1-t^{-n}}{1-q^n} a_{-n}\right] \exp\left[\sum_{n>0} \frac{w^{-n}}{n}\frac{1-t^n}{1-q^{-n}} a_n\right] \tag{7.1}$$

is different than Equation (6.36) in that the variable W is rescaled by $q^{-1/2}$ and the zero modes P and Q are introduced. These operators commute with the bosonic Heisenberg generators a_n and obey the commutation relation

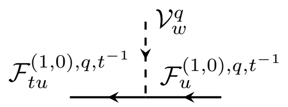
$$[P, Q] = 1. \tag{7.2}$$

Their action on the Fock vacuum $|\varnothing, u\rangle$ is

$$P\,|\varnothing, u\rangle = \ln u\,|\varnothing, u\rangle\,, \quad e^{\alpha Q}\,|\varnothing, u\rangle = |\varnothing, e^{\alpha} u\rangle \tag{7.3}$$

[3]These were introduced in [43] as the building blocks of 3d $\mathcal{N} = 2$ partition functions

where u is the vacuum charge. The parameters q, t are defined as $q = e^{\epsilon_1}$, $t = e^{\beta\epsilon_1} = e^{-\epsilon_2}$ with β a complex parameter and the intertwiner has the graphical representation to be used in the network

$\Phi(w) =$ 

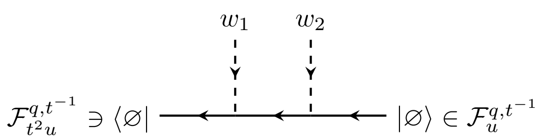
(7.4)

with the intertwiner $\Phi(w) : \mathcal{F}_u^{(1,0),q,t^{-1}} \otimes \mathcal{V}_w^q \to \mathcal{F}_{tu}^{(1,0),q,t^{-1}}$. The superscript on $\mathcal{F}$ gives information on the central charge vector; $(1,0)$ means that it's of level $\ell = (1,0)$, that is, a horizontal representation with $m = 0$ and q, t^{-1} are the parameters q_1, q_2. Then we have the central charge vector for this case as

$$\Gamma^{(1,0)} = (\gamma^2, \gamma_\perp^0) = (q_3^2, q_3^0) = (t/q, 1) \tag{7.5}$$

The dashed line in Equation (7.4) denotes the vertical representation part of the intertwining map while the solid line is the horizontal Fock representation.

We start with the simplest (non-trivial) network that we can build with this intertwiner. This is achieved by the matrix element of two intertwiners between two Fock vacuums



(7.6)

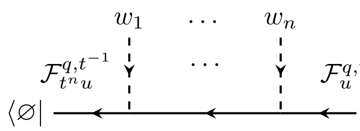
which, using Equation (7.1), the Heisenberg commutation relations in Equation (6.31) and the Baker-Campbell-Hausdorff formula to bring the Heisenberg modes inside the two intertwiners to a normal ordered form, gives the result

$$\begin{aligned}(7.6) = \langle t^2u, \varnothing|\Phi(w_1)\Phi(w_2)|u, \varnothing\rangle &= w_1^{\log_q u+\beta} w_2^{\log_q u} \exp\left[-\sum_{n>0}\frac{1}{n}\left(\frac{w_2}{w_1}\right)^n \frac{1-t^{-n}}{1-q^{-n}}\right] \\ &= w_1^{\log_q u+\beta} w_2^{\log_q u}\frac{\left(\frac{q}{t}\frac{w_2}{w_1}; q\right)_\infty}{\left(q\frac{w_2}{w_1}; q\right)_\infty}.\end{aligned} \tag{7.7}$$

where the exponent at the end of the first line is expanded as

$$\begin{aligned}\exp\left[-\sum_{n>0}\frac{1}{n}\left(\frac{w_2}{w_1}\right)^n\frac{1-t^{-n}}{1-q^{-n}}\right] &= \exp\left[\sum_{n>0}\sum_{m=0}\frac{1}{n}\left(\frac{w_2}{w_1}q^{m+1}\right)^n(1-t^{-n})\right]\\ &= \prod_{m=0}^{\infty}\left(1-q\frac{w_2}{w_1}q^m\right)^{-1}\prod_{m=0}^{\infty}\left(1-\frac{q}{t}\frac{w_2}{w_1}q^m\right)\end{aligned} \tag{7.8}$$

to give the final result in terms of the q-Pochhammer $(x;q)_\infty$, which is defined in Appendix B, Equation (B.1). Following the same procedure for n-insertions of intertwiners we get the matrix element of an n-intertwiner network



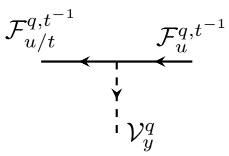
$$= A(\vec{w})\left(\prod_{i=1}^{n} w_i^{\log_q u+\beta(i-1)}\right)\prod_{i<j}\frac{\left(\frac{w_i}{w_j};q\right)_\infty}{\left(t\frac{w_i}{w_j};q\right)_\infty} \tag{7.9}$$

with the q-periodic prefactor

$$A(\vec{w}) = \prod_{i<j}\left[\left(\frac{w_i}{w_j}\right)^\beta\frac{\theta_q\left(t\frac{w_i}{w_j}\right)}{\theta_q\left(\frac{w_i}{w_j}\right)}\right] \tag{7.10}$$

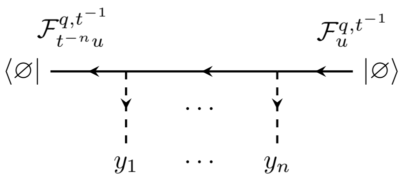
and the theta functions are defined in Equation (B.2). To get the form of Equation (7.9) we used Equation (B.4).

To build an actual network, we need another operator, the dual intertwiner $\Phi^*(y):\mathcal{F}^{(1,0)}_u \to \mathcal{F}^{(1,0)}_{u/t}\otimes\mathcal{V}^q_y$



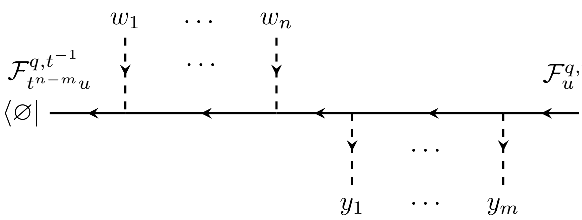
$$\begin{aligned}= \Phi^*(y) = e^{\epsilon_2 Q}y^{\beta-\frac{P}{\epsilon_1}}\exp\left[\sum_{n>0}\frac{y^n}{n}\left(\frac{t}{q}\right)^{\frac{n}{2}}\frac{1-t^{-n}}{1-q^n}a_{-n}\right]&\\ \times\exp\left[-\sum_{n>0}\frac{y^{-n}}{n}\left(\frac{t}{q}\right)^{\frac{n}{2}}\frac{1-t^n}{1-q^{-n}}a_n\right]&\end{aligned} \tag{7.11}$$

which, for n insertions, has the matrix element



$$= \left(\prod_{i=1}^{n} y_i^{-\log_q u+\beta(i-n+1)}\right)\prod_{i<j}\frac{\left(\frac{y_j}{y_i};q\right)_\infty}{\left(t\frac{y_j}{y_i};q\right)_\infty}. \tag{7.12}$$

Now we combine both operators Φ and Φ^* in a single matrix element with n intertwiners and m dual intertwiners to get



$$= A(\vec{w})B(\vec{w},\vec{y})\prod_{i=1}^{n} w_i^{\log_q u+\beta(i-2m-1)}$$

$$\times\prod_{i=1}^{m} y_i^{-\log_q u+\beta(i-2n+1)}\prod_{k<l}^{n}\frac{\left(\frac{w_k}{w_l};q\right)_\infty}{\left(t\frac{w_k}{w_l};q\right)_\infty}\prod_{i<j}^{m}\frac{\left(\frac{y_j}{y_i};q\right)_\infty}{\left(t\frac{y_j}{y_i};q\right)_\infty}\prod_{a=1}^{m}\prod_{b=1}^{n}\frac{\left(t\sqrt{\frac{q}{t}}\frac{w_b}{y_a};q\right)_\infty}{\left(\sqrt{\frac{q}{t}}\frac{w_b}{y_a};q\right)_\infty}, \tag{7.13}$$

where $B(\vec{w},\vec{y})$ is an additional q-periodic prefactor defined as

$$B(\vec{w},\vec{y}) = \prod_{a=1}^{m}\prod_{b=1}^{n}\left[\left(\frac{y_a}{w_b}\right)^{\beta}\frac{\theta_q\left(t\sqrt{\frac{q}{t}}\frac{y_a}{w_b}\right)}{\theta_q\left(\sqrt{\frac{q}{t}}\frac{y_a}{w_b}\right)}\right]. \tag{7.14}$$

The q-Pochhammer factors in Equation (7.13) are seen to resemble the holomorphic block integrand for $\mathcal{N}=2$ bifundamental hypermultiplets charged under the group $U(n)\times U(m)$. This can be seen if we make the identification

$$\bar{\mu}_i \equiv \sqrt{\frac{q}{t}}w_i \quad i=1,\cdots,n \qquad\qquad \mu_j = y_j \quad j=1,\cdots,m \tag{7.15}$$

We then have the bifundamental contribution to the $D^2\times S$ partition function where $D^2\simeq R^2_\epsilon$ is the so-called cigar geometry and it is the regularized version of R^2 with the equivariant parameter ϵ; the 3d Ω-background.

We can also interpret the resulting network in terms of the Hanany-Witten brane picture [44]. We have a single NS5-brane with n D3 branes on one side and m on the other. This has the interpretation as nm hypermultiplets in the bifundamental representation charged under $U(n) \times U(m)$. The D3-branes are semi-infinite which means the gauge coupling $\to 0$ and the gauge symmetry reduces to a global flavour symmetry. In the algebra picture, horizontal lines resembles the NS5-branes and N of these lines correspond to $N-1$ gauge groups. Thus, the network we have is for bifundamentals charged under $U(n) \times U(m)$ with the gauge fields frozen and the symmetry is global.

7.2. Vertical Gluing

The next step further, and the last one, needed in the process of building a network of operators is to glue vector representations. This is done by identifying the vector representation of an intertwiner acting on some Fock space with the dual intertwiner acting on another Fock space. The resulting network then, unlike the previous insertions, acts on a tensor product of spaces. The simplest network built this way is built from gluing an intertwiner and its dual, resulting in the operator $\mathcal{Q}_q^{q,t^{-1}} : \mathcal{F}_{u_1}^{q,t^{-1}} \otimes \mathcal{F}_{u_2}^{q,t^{-1}} \to \mathcal{F}_{t^{-1}u_1}^{q,t^{-1}} \otimes \mathcal{F}_{tu_2}^{q,t^{-1}}$:

$$
\begin{aligned}
\mathcal{Q}_q^{q,t^{-1}} &= \sum_{k\in\mathbb{Z}} \begin{array}{ccc} \mathcal{F}_{t^{-1}u_1}^{q,t^{-1}} & \longleftarrow & \mathcal{F}_{u_1}^{q,t^{-1}} \\ & \Big\downarrow \mathcal{V}_{q^k w}^{q} & \\ \mathcal{F}_{tu_2}^{q,t^{-1}} & \longleftarrow & \mathcal{F}_{u_2}^{q,t^{-1}} \end{array} = \sum_{k\in\mathbb{Z}} \begin{array}{c} \Phi^*(q^k w) \\ \otimes \\ \Phi(q^k w) \end{array} = \int_{-\infty}^{\infty} d_q w \begin{array}{c} \Phi^*(w) \\ \otimes \\ \Phi(w) \end{array} \\
&= \sum_{k\in\mathbb{Z}} e^{-\epsilon_2(Q_1-Q_2)} \left(q^k w\right)^{\beta+\frac{P_1-P_2}{\epsilon_1}} \exp\left[-\sum_{n>0} \frac{w^n}{n} q^{nk} \frac{1-t^{-n}}{1-q^n} \left(a_{-n}^{(2)} - \left(\frac{t}{q}\right)^{\frac{n}{2}} a_{-n}^{(1)}\right)\right] \\
&\qquad \times \exp\left[\sum_{n>0} \frac{w^{-n}}{n} q^{-nk} \frac{1-t^n}{1-q^{-n}} \left(a_n^{(2)} - \left(\frac{t}{q}\right)^{\frac{n}{2}} a_n^{(1)}\right)\right], \qquad (7.16)
\end{aligned}
$$

where the superscript for the Heisenberg modes (and the subscripts for the zero modes) indicates which Fock space (upper for 1 and lower for 2) these generators act on.

Let us work a simple network with $n = 3$ and $m = 1$ in Equation (7.13). The

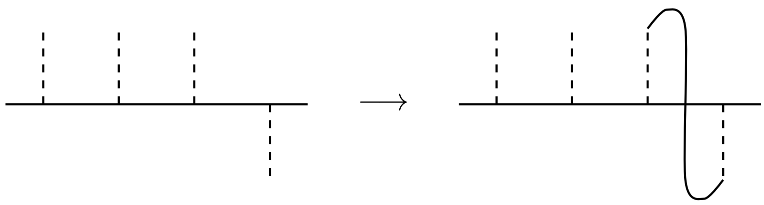
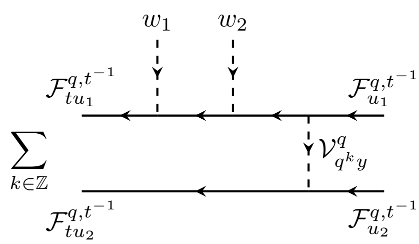


Figure 7.1. Gluing $D3$ branes

results is one glued operator and two intertwining insertions



(7.17)

The matrix element of this network is

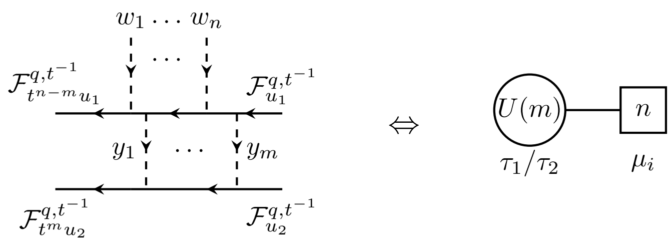
$$\begin{array}{ccc} \langle\varnothing| & & |\varnothing\rangle \\ \otimes & \left(7.17\right) & \otimes \\ \langle\varnothing| & & |\varnothing\rangle \end{array} = A(w_1, w_2)B(w_1, w_2, y)\left(\prod_{i=1}^{2} w_i^{\log_q u+\beta(i-3)}\right)$$

$$\times \frac{\left(\frac{w_1}{w_2}; q\right)_\infty}{\left(t\frac{w_1}{w_2}; q\right)_\infty} \int_{-\infty}^{\infty} d_q y \, y^{\log_q \frac{u_2}{u_1} - 2\beta} \prod_{i=1}^{2} \frac{\left(t\sqrt{\frac{q}{t}}\frac{w_i}{y}; q\right)_\infty}{\left(\sqrt{\frac{q}{t}}\frac{w_i}{y}; q\right)_\infty}. \quad (7.18)$$

The resulting theory from this network is a $U(1)$ gauge theory with 2 flavours in the fundamental representation constituting a $U(2)$ flavour group of this matter content. The Coulomb branch parameter of the gauge group is y and it results from the gauging of the two falvours as seen in Figure 7.1.

7.3. General Constructions

For more general networks, we will have different number of NS5-branes and stacks of D3-branes in between giving gauge groups as well as semi-infinte ones giving the matter content of the gauge theory. A general network of 2 NS5-branes



(7.19)

with the corresponding matrix element of this network as follows

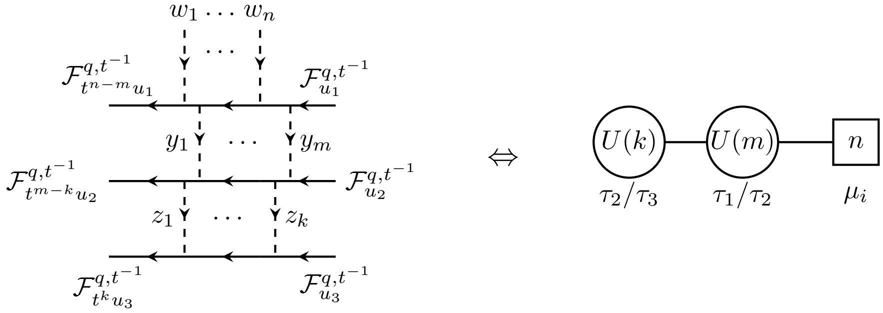
$$\begin{matrix} \langle\varnothing| & & |\varnothing\rangle \\ \otimes & \Big(7.19\Big) & \otimes \\ \langle\varnothing| & & |\varnothing\rangle \end{matrix} \sim \prod_{k<l}^{n} \frac{\left(\frac{w_k}{w_l};q\right)_\infty}{\left(t\frac{w_k}{w_l};q\right)_\infty} \oint_{\mathcal{C}} d^m y \prod_{i=1}^{m} y^{\log_q \frac{u_2}{u_1} - 2\beta - 1} \frac{\Delta_m^{(q,t)}(\vec{y})}{\bar{\Delta}_{m,n}^{(q,t)}(\vec{y},\vec{w})}, \tag{7.20}$$

where

$$\Delta_m^{(q,t)}(\vec{y}) = \prod_{k\neq l}^{m} \frac{\left(\frac{y_k}{y_l};q\right)_\infty}{\left(t\frac{y_k}{y_l};q\right)_\infty}, \qquad \bar{\Delta}_{m,n}^{(q,t)}(\vec{y},\vec{w}) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{\left(\sqrt{\frac{q}{t}}\frac{w_i}{y_j};q\right)_\infty}{\left(t\sqrt{\frac{q}{t}}\frac{w_i}{y_j};q\right)_\infty}. \tag{7.21}$$

and with $\mathcal{C}$ some contour whose poles are the singularities of the Pochhammers in the denominator according to their definition in Equation (B.1). This network has the interpretation as a $U(m)$ gauge theory with n hypers in the fundamental representation of $U(m)$.

Introducing more NS5-branes result in a larger number of stakcs of D3-branes and, thus, more gauge groups (or nodes in the quiver representations). The network



(7.22)

which has the interpretation as a gauge theory whose gauge group is $U(m)\times U(k)$ with n hypers in the fundamental of $U(m)$. It has the following matrix element

$$
\begin{array}{ccc}
\langle\varnothing| & & |\varnothing\rangle \\
\otimes & & \otimes \\
\langle\varnothing| & \left(7.22\right) & |\varnothing\rangle \\
\otimes & & \otimes \\
\langle\varnothing| & & |\varnothing\rangle
\end{array}
\sim \prod_{k<l}^{n} \frac{\left(\frac{w_k}{w_l};q\right)_\infty}{\left(t\frac{w_k}{w_l};q\right)_\infty}
$$

$$
\times \oint_{\mathcal{C}} d^m y \oint_{\mathcal{C}} d^k z \prod_{i=1}^{m} y^{\log_q \frac{u_2}{u_1}-2\beta-1} \prod_{i=1}^{k} z^{\log_q \frac{u_3}{u_2}-2\beta-1} \frac{\Delta_k^{(q,t)}(\vec{z})\Delta_m^{(q,t)}(\vec{y})}{\bar{\Delta}_{k,m}^{(q,t)}(\vec{z},\vec{y})\bar{\Delta}_{m,n}^{(q,t)}(\vec{y},\vec{w})}, \quad (7.23)
$$

As we mentioned before, a general network with N horizontal lines will have a quiver representation with $N-1$ nodes and the external semi-infinite D3-branes will correspond to hypermultiplets charged under the node corresponding to the stack these branes are attached to.

8. ELLIPTIC HIGGSED NETWORKS

8.1. Elliptic Deformation Algorithm

The first step in going from the 3d theory to the 4d theory is to elliptically deform the intertwiners of DIM algebra. This 1-parameter deformation, as we mentioned, will result in networks whose vacuum-vacuum matrix elements are the partition functions of the 4d gauge theory. The ellipticizing algorithm we follow in this thesis is after [45].

The deformation algorithm starts by having a vertex operator of the form

$$X(w) = \exp\left(\sum_{n>0} X^{-}_{-n} a_{-n} w^{n}\right) \exp\left(\sum_{n>0} X^{+}_{n} a_{n} w^{-n}\right) \tag{8.1}$$

with $X^{\pm}_{\pm n} \in \mathbb{C}$ and the operators $\{a_n\}_{n\in\mathbb{Z}/\{0\}}$ are bosonic operators satisfying the q-deformed Heisenberg algebra in Equation (6.31), which we recite again for convenience

$$[a_m, a_{-n}] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m,n}.$$

The deformation algorithm can then be achieved by following three steps.

(i) Introduce the elliptic deformation parameter p into the Heisenberg algebra through a factor as follows

$$[a_m, a_{-n}] = m(1-p^{|m|})\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m,n} \tag{8.2}$$

(ii) Define a second qp-deformed Heisenberg algebra whose modes satisfy the commutation relation

$$[b_m, b_{-n}] = m\frac{1-p^{|m|}}{(qt^{-1}p)^{|m|}}\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m,n}, \qquad [b_m, a_n] = 0 \tag{8.3}$$

(iii) Define the elliptically deformed version of the vertex operator in Equation (8.1) as

$$X^{(e)}(w) \equiv X_a(p;w)X_b(p;w) \tag{8.4}$$

where the factors $X_a(p;w), X_b(p;w)$ are defined as vertex operators whose generators are the Heisenberg modes a_n, b_n, respectively according to

$$X_a(p;w) \equiv \exp\left(\sum_{n>0}\frac{1}{1-p^{|n|}}X^-_{-n}a_{-n}w^n\right)\exp\left(\sum_{n>0}\frac{1}{1-p^{|n|}}X^+_n a_n w^{-n}\right) \tag{8.5}$$

$$\begin{aligned} X_b(p;w) \equiv \exp\Bigg(&-\sum_{n>0}\frac{p^{|n|}}{1-p^{|n|}}X^-_n b_{-n}w^{-n}\Bigg) \\ &\times \exp\left(-\sum_{n>0}\frac{p^{|n|}}{1-p^{|n|}}X^+_{-n}b_n w^n\right) \end{aligned} \tag{8.6}$$

Our vertex operators that we are interested in, the intertwiner Φ defined in Equation (7.1) and the dual intertwiner Φ^* defined in Equation (7.11) have the some form as Equation (8.1). Thus, we can deform these two operators to obtain their elliptic versions $\Phi^{(e)}$ and $\Phi^{*(e)}$, respectively. We can then use these deformed operators as vertices to build networks of intertwiners just like the previous chapter, and end up with the required partition function, but for a 4d theory this time.

8.2. Elliptic Intertwiners and their Duals

Now that we have the algorithmic procedure to follow, the next step is to perform it and get the elliptic version of the operators and use them to build the networks required to get the partition function in 4d.

As we mentioned, the intertwining operator in Equation (7.1) has the form of that in Equation (8.1) with

$$X^-_{-n} = -\frac{1}{n}\frac{1-t^{-n}}{1-q^n}, \qquad X^+_n = -\frac{q^n}{n}\frac{1-t^n}{1-q^n} \tag{8.7}$$

Then, introducing the second Heisenberg algebra, we define the elliptic intertwiner

$$\Phi^{(e)}(w) \equiv e^{-\epsilon_2 Q} w^{\frac{P}{\epsilon_1}} \; \Phi_a^{(e)}(w)\Phi_b^{(e)}(w) \tag{8.8}$$

with $\Phi_a^{(e)}(w)$ and $\Phi_b^{(e)}(w)$ defined as

$$\begin{aligned} \Phi_a^{(e)}(w) \equiv \exp\left(-\sum_{n>0} \frac{w^n}{n} \frac{1}{1-p^{|n|}} \frac{1-t^{-n}}{1-q^n} a_{-n} \right) \\ \times \exp\left(-\sum_{n>0} \frac{w^{-n}}{n} \frac{1}{1-p^{|n|}} q^n \frac{1-t^n}{1-q^n} a_n \right) \end{aligned} \tag{8.9}$$

and

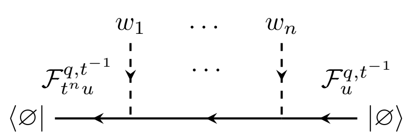
$$\begin{aligned} \Phi_b^{(e)}(w) \equiv \exp\left(\sum_{n>0} \frac{w^{-n}}{n} \frac{p^{|n|}}{1-p^{|n|}} q^n \frac{1-t^n}{1-q^n} b_{-n} \right) \\ \times \exp\left(\sum_{n>0} \frac{w^n}{n} \frac{p^{|n|}}{1-p^{|n|}} \frac{1-t^{-n}}{1-q^n} b_n w^n \right) \end{aligned} \tag{8.10}$$

Now we follow the same steps of the 3d constructions. First, the first non-trivial network of two elliptic intertwiners. Their vacuum–vacuum matrix element is

$$\langle t^2 u, \varnothing | \Phi^{(e)}(w_1)\Phi^{(e)}(w_2) | u, \varnothing \rangle = w_1^{\log_q u + \beta} w_2^{\log_q u} \frac{\Gamma_{q,p}(q w_2/w_1)}{\Gamma_{q,p}(q w_2/t w_1)} \tag{8.11}$$

where $\Gamma_{q,p}(x)$ is the elliptic gamma function which we define in Equation (B.6). Note that in the limit $p \to 0$ Equation (B.8), we obtain the undeformed result of Equation (7.7).

Next, we compute the matrix element of n-elliptic intertwiners and we get the result



$$= A(\vec{w}) \left(\prod_{i=1}^{n} w_i^{\log_q u + \beta(i-1)} \right) \prod_{i<j} \frac{\Gamma_{q,p}(t w_i/w_j)}{\Gamma_{q,p}(w_i/w_j)} \tag{8.12}$$

with the same q-periodic prefactor $A(\vec{w})$ defined in Equation (7.10). To get the form of Equation (8.12) we used the reflection property of the elliptic gamma function in Equation (B.7). Then we move to the elliptic version of the dual intertwiner in Equation (7.11). Comparing its form with Equation (8.1), we find

$$X^-_{-n} = \frac{1}{n}\left(\frac{t}{q}\right)^{\frac{n}{2}}\frac{1-t^{-n}}{1-q^n}, \qquad X^+_n = \frac{q^n}{n}\left(\frac{t}{q}\right)^{\frac{n}{2}}\frac{1-t^n}{1-q^n} \tag{8.13}$$

Then, following the deformation procedure, the elliptic dual interwiner takes the form

$$\Phi^{*(e)}(y) \equiv e^{\epsilon_2 Q} w^{\beta-\frac{P}{\epsilon_1}}\ \Phi^{*(e)}_a(y)\Phi^{*(e)}_b(y) \tag{8.14}$$

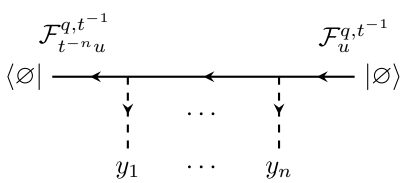
with $\Phi^{*(e)}_a(y)$ and $\Phi^{*(e)}_b(y)$ defined as

$$\begin{aligned}\Phi^{*(e)}_a(y) \equiv \exp&\left(\sum_{n>0}\frac{y^n}{n}\frac{1}{1-p^{|n|}}\left(\frac{t}{q}\right)^{\frac{n}{2}}\frac{1-t^{-n}}{1-q^n}a_{-n}\right)\\ &\times\exp\left(-\sum_{n>0}\frac{y^{-n}}{n}\frac{1}{1-p^{|n|}}q^n\left(\frac{t}{q}\right)^{\frac{n}{2}}\frac{1-t^n}{1-q^n}a_n\right)\end{aligned} \tag{8.15}$$

and

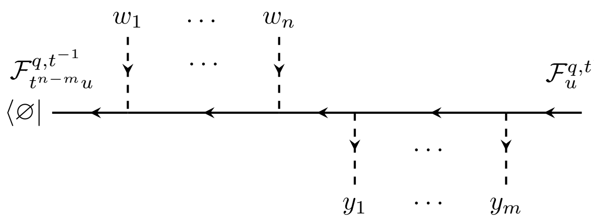
$$\begin{aligned}\Phi^{*(e)}_b(y) \equiv \exp&\left(-\sum_{n>0}\frac{y^{-n}}{n}\frac{p^{|n|}}{1-p^{|n|}}q^n\left(\frac{t}{q}\right)^{-\frac{n}{2}}\frac{1-t^n}{1-q^n}b_{-n}\right)\\ &\times\exp\left(-\sum_{n>0}\frac{y^n}{n}\frac{p^{|n|}}{1-p^{|n|}}\left(\frac{t}{q}\right)^{-\frac{n}{2}}\frac{1-t^{-n}}{1-q^n}b_n w^n\right)\end{aligned} \tag{8.16}$$

Again, we follow the steps of the previous chapter. First, the matrix element of n-dual elliptic intertwiners



$$= \left(\prod_{i=1}^{n} y_i^{-\log_q u+\beta(i-n+1)}\right)\prod_{i<j}\frac{\Gamma_{q,p}(ty_j/y_i)}{\Gamma_{q,p}(y_j/y_i)}. \tag{8.17}$$

Then, we combine both operators $\Phi^{(e)}$ and $\Phi^{*(e)}$ in a single matrix element with n intertwiners and m dual intertwiners to get



$$= A(\vec{w})B(\vec{w},\vec{y})$$

$$\times \prod_{i=1}^{n} w_i^{\log_q u+\beta(i-2m-1)} \prod_{i=1}^{m} y_i^{-\log_q u+\beta(i-2n+1)} \prod_{k<l}^{n} \frac{\Gamma_{q,p}(tw_k/w_l)}{\Gamma_{q,p}(w_k/w_l)} \prod_{i<j}^{m} \frac{\Gamma_{q,p}(ty_j/y_i)}{\Gamma_{q,p}(y_j/y_i)}$$
$$\times \prod_{a=1}^{m}\prod_{b=1}^{n} \frac{\Gamma_{q,p}(\sqrt{\frac{q}{t}}w_b/y_a)}{\Gamma_{q,p}(\sqrt{t\frac{q}{t}}w_b/y_a)}, \quad (8.18)$$

with the q-periodic factor $B(\vec{w},\vec{y})$ defined as in Equation (7.14). Just like its non-elliptic version, this network is for nm hypers in the bifundamental representation charged under the group $U(n)\times U(m)$ which is a global symmetry due the gauge fields being frozen. The only difference here is that this is a 4d theory and the elliptic gamma function gives the 4d holomorphic blocks as defined on the geometry $R^2\times T^2$.

8.3. Vertical Gluing

The last step in building networks, like the undeformed case, is gluing vector representations. The resulting network, again, is an operator acting on a tensor product of Fock spaces and we use $1,2$ as superscripts to indicate which Fock space the Heisenberg modes (and the subscripts for the zero modes) act on; upper for 1 and lower for 2. The network built built by gluing an intertwiner and its dual, results in the operator

$$\mathcal{Q}_q^{(e)\ q,t^{-1}} : \mathcal{F}^{q,t^{-1}}_{u_1} \otimes \mathcal{F}^{q,t^{-1}}_{u_2} \to \mathcal{F}^{q,t^{-1}}_{t^{-1}u_1} \otimes \mathcal{F}^{q,t^{-1}}_{tu_2} \quad (8.19)$$

which is defined as

$$
\mathcal{Q}_q^{(e)\ q,t^{-1}} = \sum_{k\in\mathbb{Z}} \begin{array}{ccc} \mathcal{F}^{q,t^{-1}}_{t^{-1}u_1} & & \mathcal{F}^{q,t^{-1}}_{u_1} \\ \longleftarrow & & \longleftarrow \\ & \mathcal{V}^q_{q^k w} & \\ \longleftarrow & & \longleftarrow \\ \mathcal{F}^{q,t^{-1}}_{tu_2} & & \mathcal{F}^{q,t^{-1}}_{u_2} \end{array} = \sum_{k\in\mathbb{Z}} \begin{array}{c} \Phi^{*(e)}(q^k w) \\ \otimes \\ \Phi^{(e)}(q^k w) \end{array} = \int_{-\infty}^{\infty} d_q w \begin{array}{c} \Phi^{*(e)}(w) \\ \otimes \\ \Phi^{(e)}(w) \end{array} =
$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} e^{-\epsilon_2(Q_1-Q_2)} \left(q^k w\right)^{\beta+\frac{P_1-P_2}{\epsilon_1}} \exp\left[-\sum_{n>0}\frac{w^n}{n}q^{nk}\frac{1}{1-p^n}\frac{1-t^{-n}}{1-q^n}\left(a^{(2)}_{-n}-\left(\frac{t}{q}\right)^{\frac{n}{2}}a^{(1)}_{-n}\right)\right] \\
&\qquad\times \exp\left[\sum_{n>0}\frac{w^{-n}}{n}q^{-nk}\frac{1}{1-p^n}\frac{1-t^n}{1-q^{-n}}\left(a^{(2)}_{n}-\left(\frac{t}{q}\right)^{\frac{n}{2}}a^{(1)}_{n}\right)\right] \\
&\qquad\times \exp\left[-\sum_{n>0}\frac{w^{-n}}{n}q^{-nk}\frac{p^n}{1-p^n}\frac{1-t^n}{1-q^{-n}}\left(b^{(2)}_{-n}-\left(\frac{t}{q}\right)^{-\frac{n}{2}}b^{(2)}_{-n}\right)\right] \\
&\qquad\qquad\times \exp\left[\sum_{n>0}\frac{w^{n}}{n}q^{nk}\frac{p^n}{1-p^n}\frac{1-t^{-n}}{1-q^{n}}\left(b^{(2)}_{n}-\left(\frac{t}{q}\right)^{-\frac{n}{2}}b^{(2)}_{n}\right)\right] \qquad (8.20)
\end{aligned}
$$

Following the same steps of the previous chapter we start with the simple network of one glued operators and two external insertions attached to the upper Fock space. The matrix element of the elliptically deformed version of Equation (7.17) is

$$
\begin{aligned}
\begin{array}{c}\langle\varnothing| \\ \otimes \\ \langle\varnothing|\end{array} \Big(7.17\Big)^{(e)} \begin{array}{c}|\varnothing\rangle \\ \otimes \\ |\varnothing\rangle\end{array} &= A(w_1,w_2)B(w_1,w_2,y)\left(\prod_{i=1}^{2} w_i^{\log_q u+\beta(i-3)}\right) \\
&\qquad\times \frac{\Gamma_{q,p}\big(t\frac{\omega_1}{\omega_2}\big)}{\Gamma_{q,p}\big(\frac{\omega_1}{\omega_2}\big)}\int_{-\infty}^{\infty} d_q y\, y^{\log_q\frac{u_2}{u_1}-2\beta}\prod_{i=1}^{2}\frac{\Gamma_{q,p}\Big(\sqrt{\frac{q}{t}}\frac{\omega_i}{y}\Big)}{\Gamma_{q,p}\Big(t\sqrt{\frac{q}{t}}\frac{\omega_i}{y}\Big)}. \qquad (8.21)
\end{aligned}
$$

It has the same interpretation as the undeformed one; a $U(1)$ gauge theory with two fundamental hypers but for a 4d theory.

Elliptic deformation of general networks like Equation (7.19) and Equation (7.22) is easily computed and follows the same steps done before. The fact that zero modes do not change under the deformation makes the procedure amounts basically to introducing elliptic gammas instead of q-Pochhammers. Then, the resulting theories has the same interpretation as the undeformed ones in terms of the gauge group and matter content and the difference is just that the resulting theory is a 4d one and the gamma functions correspond to the 4d holomorphic blocks of $\mathbb{R}^2 \times \mathbb{T}^2$.

8.4. Gauge/Liouville Triality

Earlier we discussed the equivalence of instanton partition functions in 5d with the q-Liouville conformal blocks that was given in [28]. This is not the end of the story though and another equivalence was shown in that paper. This other equivalence identifies the DF representation of the q-Liouville conformal blocks on a sphere with $N+2$ punctures with the partition function of 3d $\mathcal{N}=2$ gauge theory defined on $\mathbb{R}^2_\epsilon \times S^1$ with N hypers in the fundamental, N in the anti-fundamental and one in the adjoint representation of $U(r)$ and a Fayet-Iliopoulos parameter (a supersymmetric D-term). This equivalence takes the form

$$\mathcal{B}^{3d}[U(r)] \simeq \langle \prod_{i=0}^{N+1} V_{\alpha_i}(z_i) \prod_{j=1}^{r} \oint dy_j S(y_j) \rangle_{q-Vir} \tag{8.22}$$

We use the notation $\mathcal{B}$ for the 3d partition function because we want to emphasize that they can be computed as factorized holomorphic blocks. The full *triality* of equivalences then takes the form

$$Z^{5d}_{inst}|_{a=a_r} \simeq \mathcal{B}^{3d}[U(r)] \simeq \langle \prod_{i=0}^{N+1} V_{\alpha_i}(z_i) \prod_{j=1}^{r} \oint dy_j S(y_j) \rangle_{q-Vir} \tag{8.23}$$

The Higgsed networks we used in the previous chapter gave us the 3d part of this triality which is related to the higher dimensional AGT extension. On the same footing, elliptically deformed networks gave us the 4d holomorphic blocks and it should be one of the vertices of the triality that includes the 6d/elliptic-Virasoro AGT-like relation. This triality was shown in [29] and it takes the form

$$Z^{6d}_{inst}|_{a=a(r)} \simeq \mathcal{B}^{4d}[U(r)] \simeq \langle \prod_{i=0}^{N+1} V_{\alpha_i}(z_i) \prod_{j=1}^{r} \oint dy_j S(y_j) \rangle_{ell-Vir} \tag{8.24}$$

Figure 8.1 shows the origianl triality as well as its extension to higher dimensions.

$$
\begin{tikzcd}
\text{5d} \arrow[rr, leftrightarrow, "\text{AGT-like}"] \arrow[dr, leftrightarrow] \arrow[ddd] & & q\text{-Virasoro} \arrow[dl, leftrightarrow] \arrow[ddd] & {} \arrow[ddd, "\text{elliptic deformation}" description] \\
& \text{3d} \arrow[d] & & \\
& \text{4d} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & & \\
\text{6d} \arrow[rr, leftrightarrow, "\text{AGT-like}"'] & & \text{elliptic Virasoro} & {}
\end{tikzcd}
$$

Figure 8.1. Gauge/Liouville Triality and its lift to higher dimensions

9. CONCLUSION

In this thesis we started with two seemingly different theories in different dimensions witth different type of symmetries. One was the 4d supersymmetric Yang-Mills with an $SU(2)$ gauge group while the other was the conformally invariant Liouville field theory in 2d. We studied the pure 4d theory and showed the Seiberg-Witten as well as Nekrasov's approaches to obtaining the partition functions of the theory. Then we studied the superconformal version of the theory where we had a matter sector resulting in the so-called class-$\mathcal{S}$ theories $\mathcal{T}_{g,n}$ living on some Riemann surface $\mathcal{C}_{g,n}$ known as the Gaiotto curve. Then we studied the Liouville theory living on the surface of the Gaiotto curve. The AGT then asserts that the instanton partition function of the 4d theory corresponds to the conformal blocks of Liouville theory. This correspondence has a natural extension with the 4d side lifted to a q-analogue; the 5d theory and Liouville theory has a q-deformation and we got a 5d/q-Virasoro AGT-like. A higher extension exists as well with a 6d/elliptic-Virasoro AGT-like correspondence. The 5d extension has a symmetry algebra known as the DIM algebra. It has an intertwining map that can be used to build networks of intertwiners with the VEV of the networks yielding the partition function of 3d quiver gauge theories. We then elliptically deform the algebra and the matrix element of the deformed network results in a 1d uplift of the 3d theory; a 4d quiver theory. The 3d theory is related to the poles of the 5d/q-Vir AGT correspondence via the so-called Gauge/Liouville triality and the resulting 4d quiver theory has an analogue of this triality with the poles of the 6d/ell-Vir AGT.

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APPENDIX A: YOUNG DIAGRAMS

A partition (or Young diagram) is a finite sequence of positive decreasing integers

$$Y = (\lambda_1, \lambda_2, \cdots, \lambda_r, \cdots) \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots \tag{A.1}$$

These non-zero λ_i's are called the parts of the partition Y and, in the graphical representation they are the number of boxes in the i-th column (we use the French notation where the boxes are justified from below as opposed to the English notation where they are justified from the left as shown in Figure A.1).

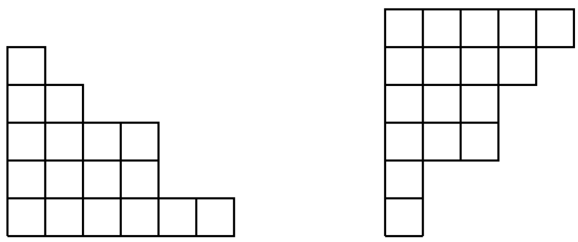

Figure A.1. French (left) vs English (right) notation

The number of parts (or columns in the diagram representation) is called the length of the partition $\ell(Y)$ and the sum of the parts is weight of the partition (or number of boxes in the diagram). When $r > \ell(Y)$ the r-th part vanishes and the sequence terminates.

For a box s sitting in the coordinate (i, j), its arm length $A_Y(s)$ and leg length $L_Y(s)$ are defined with respect to some Young diagram Y as

$$A_Y(s) = \lambda_i - j \qquad L_Y(s) = \lambda'_j - i \tag{A.2}$$

where λ'_j is the j-th part of the transposed diagram, which is the reflection of the original diagram about its main diagonal. The arm and leg length of the box s are exchanged from the original to the transposed diagram;

$$A_Y(s) = L_{Y'}(s), \qquad A_{Y'}(s) = L_Y(s) \tag{A.3}$$

A partition $Y = (5, 4, 3, 3, 1, 1)$ is shown in Figure A.2, along with its transpose $Y' = (6, 4, 4, 2, 1)$ and a box s with its arm length denoted by the white bullet and leg length by the black bullets for Y and the opposite for Y' according to the previous discussion.

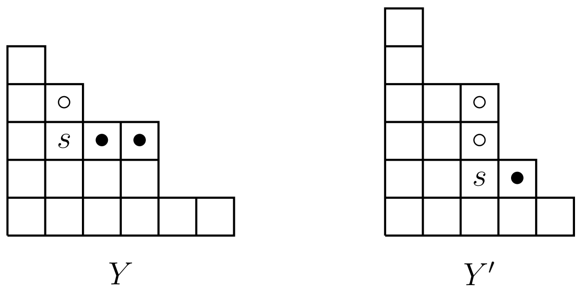

Figure A.2. Diagram and its transpose

APPENDIX B: SPECIAL FUNCTIONS

This appendix will serve the function of giving the definitions of some of the special functions appearing in the main body of the thesis in the second part as well as some relationships that these functions obey and which we used to get from one form of an expression to another.

The q-Pochhammer, also known as the quantum dilogarithm, $(x;q)_\infty$ is defined as

$$(x;q)_\infty \equiv \prod_{n=0}^{\infty}(1-xq^n) \qquad |q|<1 \tag{B.1}$$

The Jacobi theta function $\theta_q(x)$ is defined in terms of the q-Pochhammers as

$$\theta_q(x) \equiv (q;q)_\infty (x;q)_\infty (qx^{-1};q)_\infty \tag{B.2}$$

and it obeys the following relationships

$$\theta_q(qx) = -x^{-1}\theta_q(x), \qquad \theta_q(x) = -x\;\theta_q(x^{-1}) \tag{B.3}$$

The last two relationships, together, implies the following relationship

$$\theta_q(qx) = \theta_q(x^{-1}) \tag{B.4}$$

When elliptic deformation is present, another deformation parameter, the elliptic parameter p is introduced. The first function to introduce here is the double qp-Pochhammer $(x;q,p)_\infty$ which is defined as

$$(x;q,p)_\infty \equiv \prod_{n,m=0}^{\infty}(1-xq^np^m) \qquad |q|,|p|<1 \tag{B.5}$$

and we define the elliptic gamma function in terms of this double qp-Pochhammer as

$$\Gamma_{q,p}(x) \equiv \frac{(qpx^{-1};q,p)_\infty}{(x;q,p)_\infty} \tag{B.6}$$

It obeys the so-called reflection property

$$\Gamma_{q,p}(x) = \frac{1}{\Gamma_{q,p}(qpx^{-1})} \tag{B.7}$$

and, in the limit when $p \to 0$; that is, when the elliptic deformation is removed, the elliptic gamma function reduces to the inverse of the q-Pochhammer

$$\lim_{p\to 0} \Gamma_{q,p}(x) \to \frac{1}{(x;q)_\infty} \tag{B.8}$$

An elliptic theta function $\theta_p(x)$ is defined analogous to (B.2) but with the deformation parameter q replaced by the elliptic parameter p. It obeys the same relationships (B.3) and (B.4) that $\theta_q(x)$ obeys.

For the elliptic gamma function and the theta functions $\theta_q(x)$ and $\theta_p(x)$, the following two expressions hold

$$\Gamma_{q,p}(qx) = \frac{\theta_p(x)}{(p;p)_\infty}\Gamma_{q,p}(x), \qquad \Gamma_{q,p}(px) = \frac{\theta_q(x)}{(q;q)_\infty}\Gamma_{q,p}(x) \tag{B.9}$$