COMBINATORIAL COMPUTATION TECHNIQUES FOR HOMOLOGY COBORDISM INVARIANTS OF PLUMBED 3-MANIFOLDS

by

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B.S., Mathematics, Ege University, 2014

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2018

ACKNOWLEDGEMENTS

No words can express my deepest gratitude to my thesis supervisor Assoc. Prof. Çağrı Karakurt who politely and encouragingly taught almost everything I know about Heegaard Floer theory and beyond. I am grateful to him for generously devoting his time explaining mathematics word by word. Moreover, his persistent approach to comprehend mathematics in a concrete way taught me how to transform abstract notions into reality and vice versa. Thanks to his endless effort, my writings have exceeded to a standard work.

I would like to thank to Assoc. Prof. Ekin Özman for her invaluable encouragement to study mathematics sedulously during my inevitable darkest time. I am profoundly indebted to Prof. Dr. Turgut Önder for his inspiring support to study topology with Assoc. Prof. Çağrı Karakurt.

I owe much to Erman Işık for their friendship and support during almost all my life. I also thank Yavuz Köroğlu for sharing his time generously to create fascinating computer algorithms which help me to understand the flow and pattern of subtle data hiding in numbers.

I also owe special thank to Ebru Ataç for her considerable encouragement and firm guidance during my time at English Preparatory Division and beyond.

Last but not least, I would like to express my innermost thanks to my parents for their permanent love, tolerance and encouragement.

ABSTRACT

COMBINATORIAL COMPUTATION TECHNIQUES FOR HOMOLOGY COBORDISM INVARIANTS OF PLUMBED 3-MANIFOLDS

We study some invariants of homology cobordism classes in dimension three. These invariants have their roots in gauge theory but are computable combinatorially for plumbed 3-manifolds. We illustrate certain computation techniques for these invariants and using them we try to answer basic questions about three-dimensional homology cobordism group.

ÖZET

TESİSAT 3-MANİFOLDLARIN HOMOLOJİ KOBORDİZM DEĞİŞMEZLERİ İÇİN KOMBİNATORYAL HESAPLAMA TEKNİKLERİ

Üçüncü boyutta homoloji kobordizm sınıflarının bazı değişmezlerini inceliyoruz. Bu değişmezlerin kökleri ayar teorisine sahiptir, ancak tesisat 3-manifoldlar için kombinatoryal olarak hesaplanabilirler. Bu değişmezler için belirli hesaplama tekniklerini açıklıyoruz ve bunları kullanarak üç boyutlu homoloji kobordizm grubuyla ilgili temel soruları yanıtlamaya çalışıyoruz.

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LIST OF SYMBOLS

CF^{+}	Plus version of Heegaard Floer chain complex
CF^{-}	Minus version of Heegaard Floer chain complex
$\widehat{CF}$	Hat version of Heegaard Floer chain complex
CFI^{∞}	Infinity version of involutive Heegaard Floer chain complex
CFI^{+}	Plus version of involutive Heegaard Floer chain complex
CFI^{-}	Minus version of involutive Heegaard Floer chain complex
$\widehat{CFI}$	Hat version of involutive Heegaard Floer chain complex
CFI^{∞}	Infinity version of involutive Heegaard Floer chain complex
$\mathrm{Char}(G)$	The set of characteristic elements
d	Correction term
$\underline{d}$	Lower correction term
$\bar{d}$	Upper correction term
$d(v)$	Valency of vertex v on the plumbing graph
G	Plumbing graph
$\mathcal{H}$	Heegaard diagram
$\mathbb{H}^{+}$	Graded module
HF^{+}	Plus version of Heegaard Floer homology group
HF^{-}	Minus version of Heegaard Floer homology group
$\widehat{HF}$	Hat version of Heegaard Floer homology group
HF^{∞}	Infinity version of Heegaard Floer homology group
HF_{red}	Reduced version of Heegaard Floer homology group
HFI^{+}	Plus version of involutive Heegaard Floer homology group
HFI^{-}	Minus version of involutive Heegaard Floer homology group
$\widehat{HFI}$	Hat version of involutive Heegaard Floer homology group
HFI^{∞}	Infinity version of involutive Heegaard Floer homology group
HFI_{red}	Reduced version of involutive Heegaard Floer homology group
$I(G)$	Intersection matrix according to G
k	Knot

K	Characteristic vector
$\mathbb{K}^+$	Dual of $\mathbb{H}^+$
$N(k)$	Tubular neighbourhood of the knot k
$X(G)$	Plumbing 4-manifold obtained according to G
R	Graded root
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathfrak{s}$	Typical element of Spin^c-structure
S^n	n-dimensional unit sphere
$\mathrm{sign}(I(G))$	Sign of intersection matrix associated to G
Spin	Spin structure on a manifold
Spin^c	Spin^c structure on a manifold
$\mathrm{Sym}^g(\Sigma)$	Symmetric g-fold product of Σ
$\mathcal{T}^+$	graded algebra $\mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$
v_j, v	Vertex of plumbing graph
∂	Boundary
μ	Rokhlin invariant
$\bar{\mu}$	Neumann-Siebenmann invariant
$\Sigma(p, q, r)$	Brieskorn homology sphere
τ	Tau sequence
$\Theta^3_{\mathbb{Z}}$	Integral homology cobordism group

LIST OF ACRONYMS/ABBREVIATIONS

AR	Almost rational graph
det	Determinant of a matrix
Im	Image of a map
Ker	Kernel of a map
sign	Signature of a matrix

1. INTRODUCTION

Cobordism theory dates back to the substantial amount effort of great mathematicians such as Bernhard Riemann, Gustav Roch, Henri Poincaré, Lev Pontryagin, René Thom, Isadore Singer, Friedrich Hirzebruch and Micheal Atiyah. In general, a cobordism between manifolds Y_1 and Y_2 is a compact manifold X of dimension one higher whose boundary is the disjoint union of Y_1 and Y_2. In this case, Y_1 and Y_2 are said to be *cobordant*. In this thesis, we are mostly interested in homology cobordism theory.

A n-dimensional manifold Y is called *homology n-sphere* if it has same homology as n-dimensional sphere S^n. Let Y_1 and Y_2 be integral homology n-spheres. They are called *homology cobordant*, if there exists a smooth compact oriented $(n+1)$-manifold X with boundary $\partial X = (-Y_1) \cup Y_2$ such that $H_*(X, Y_i; \mathbb{Z}) = 0$ for $i = 0, 1$.

The set of integral homology spheres up to integral homology cobordism admits an abelian group structure with the operation induced by connected sum. This group is called *integral homology cobordism group* in dimension n and is denoted by $\Theta^n_{\mathbb{Z}}$. Here, a manifold represents the identity element if it is null cobordant, i.e., it is cobordant to S^n. The additive inverse of $[Y]$ is obtained by reversing orientation, i.e., $-[Y]$. Due to the work of Kervaire in [1], the integral n-homology cobordism group is trivial except $n = 3$.

One would extract the properties of $\Theta^3_{\mathbb{Z}}$ using sensitive topological invariants. In [2], Rokhlin proved that there is a non-trivial homomorphism $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$ sending the Poincáre homology sphere to 1. This yields a $\mathbb{Z}/2\mathbb{Z}$-valued invariant of homology 3-spheres, which is called *Rokhlin invariant*, does not change under homology cobordism.

Later Neumann and Siebenmann independently found an integer lift of Rokhlin invariant $\bar{\mu}$ for a special class of 3-manifolds, so called plumbed 3-manifolds, see [3] and [4]. This invariant $\bar{\mu}$ is also called *Neumann-Siebenmann invariant.* Saveliev proved that Neumann-Siebenmann invariant does not change under homology cobordism which shows that $\Theta^3_{\mathbb{Z}}$ has a subgroup isomorphic to $\mathbb{Z}$, see [Section 7.2.3, [5]].

Furuta and independently Fintushel and Stern proved that $\Theta^3_{\mathbb{Z}}$ has a subgroup isomorphic to $\mathbb{Z}^\infty$, see [6] and [7]. Currently it is not known whether $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ summand or not. But in [8], Frøyshov showed that there is a non-trivial group homomorphism $\delta : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$ which shows that $\Theta^3_{\mathbb{Z}}$ has at least a $\mathbb{Z}$ summand generated by the Poincaré homology sphere.

Works of Furuta and Frøyshov were based on gauge theory in which homology cobordism invariants are hardly difficult to compute. In 2000's, Ozsváth and Szabó introduced Heegaard Floer theory which is more computable substitute of gauge theory. They were able to recover Frøyshov's invariant in their Heegaard Floer homology. They also defined a new homology cobordism invariant, so called d-invariant. For plumbed 3-manifolds, computation of Heegaard Floer homology has purely combinatorial description. In [9], Ozsváth and Szabó described a tractable algorithm to partially compute plus flavor of Heegaard Floer homology together with d-invariant for plumbed 3-manifolds. Unfortunately, this algorithm determines only the part of the Heegaard Floer homology group that lies in the kernel of endomorphism between Heegaard Floer complexes. In order to find the whole group, one must find the minimal relations among characteristic vectors, see Chapter 5. Although these relations can be found for several families of plumbed 3-manidols, no general technique is known to find them in an algorithmic way. Later András Némethi fortunately published an article mainly building on the work of [9]. In [10], he constructed a complete algorithm that enables to compute full Heegaard Floer homology groups for not only family of plumbed 3-manifolds with at most two bad vertices but also family of plumbed 3-manifolds carrying AR-structure, see Section 5.2 and Section 5.3.

In [11], Manolescu defines Pin(2)-equivariant Seiberg Witten homology and homology cobordism invariants one of which is an integer lift of Rokhlin invariant. This powerful invariant was the main tool for his famous work disproving triangulation conjecture. In the paper [12], Hendricks and Manolescu defined involutive Heegaard Floer homology which can be considered $\mathbb{Z}/4\mathbb{Z}$-equivariant version of Manolescu's Seiberg Witten homology in Heegaard Floer setting. They also mimiced the construction of ordinary correction term to produce two new correction terms, called lower and upper correction term, for details see Chapter 6 and Section 6.1. Moreover, for the family of plumbed 3-manifolds, Dai and Manolescu described harmonious relation between these new correction terms and well-known homology cobordism invariants such as Neumann-Siebenmann invariant and d-invariant, see [13] and Theorem 6.6.

With the technology of Pin(2)-equivariant Seiberg Witten homology, Stoffregen reproved Furuta's result proving that $\Theta^3_{\mathbb{Z}}$ has a subgroup isomorphic to $\mathbb{Z}^\infty$, see [14]. The same thing was proved by Dai and Manolescu using involutive Heegaard Floer homology together with new correction terms, see [13] and Chapter 7.

The purpose of this thesis is to outline the argument of Dai and Manolescu reproving Furuta's work. Along the way, we describe how to compute the relevant homology cobordism invariants such as: Neumann-Siebenmann invariant, d-invariant, Heegaard Floer homology and involutive Heegaard Floer homology.

2. PRELIMINARIES

2.1. Homology Cobordism

A closed oriented 3-manifold Y is called *integral homology sphere* if it has the homology of S^3, that is, $H_*(Y,\mathbb{Z})$=$H_*(S^3,\mathbb{Z})$. If we choose our ring $\mathbb{Q}$ instead of $\mathbb{Z}$, then Y is called *rational homology sphere.*

Let Y_1 and Y_2 be integral homology spheres. They are called *homology cobordant*, if there exits a smooth compact oriented 4-manifold X with boundary $\partial X = (-Y_1) \cup Y_2$ such that such that $H_*(X, Y_i; \mathbb{Z}) = 0$ for $i = 0, 1$. A homology sphere is said to be *null homology cobordant* if it is homology cobordant to S^3.

The very basic example of homology cobordism is that any homology sphere Y is homology cobordant to itself via $X = Y \times [0,1]$. Non-trivial examples need some effort that we will consider soon, see Theorem 2.1.

The set of integral homology spheres up to integral homology cobordism forms an abelian group with the operation induced by connected sum. This group is called *integral homology cobordism group* and is denoted by $\Theta^3_{\mathbb{Z}}$. Here, the zero element is homology cobordism class of S^3, and the additive inverse is obtained by reversing orientation.

2.2. Dehn Surgery

Let Y be a closed oriented 3-manifold, k a knot in Y and $N(k)$ a tubular neighbourhood of k in Y. By cutting Y along the 2-torus $\partial N(k)$, we get two 3-manifolds:

- The *knot exterior* K which is the closure of $Y \setminus \partial N(k)$
- The solid torus $N(k)$ which is homeomorphic to $D^2 \times S^1$

Thus, the knot exterior K is a 3-manifold with boundary $\partial K = S^1 \times S^1$ and $Y = K \cup (D^2 \times S^1)$. One can use an arbitrary homeomorphism $h : \partial D^2 \times S^1 \to \partial K$ to glue $D^2 \times S^1$ back in K. Then, we obtain a closed oriented 3-manifold $M = K \cup_h (D^2 \times S^1)$. This manifold M is said to be obtained from Y by *surgery* on k.

Notice that there is a close relationship between surgery on link and cobodism as follows: Let k be a knot in 3-manifold Y with an integer framing defined by a curve $\gamma \subset \partial K$ such that $[\gamma] = [k] \in H_1(N(k))$. Let $z \in S^1 = \partial D^2$. Up to isotopy, there exists a unique homeomorphism $\phi : S^1 \times D^2 \to N(k)$ such that $h(S^1 \times \{1\}) = k$ and $\phi(S^1 \times \{z\}) = \gamma$. By gluing 2-handle $D^2 \times D^2$ to the thickened 4-manifold $Y \times [0,1]$ with the embedding $\phi : S^1 \times D^2 \to N(k)$, we obtain a 4-manifold $X = (Y \times \{1\}) \cup_\phi (D^2 \times D^2)$.

Theorem 2.1. *The manifold X is a cobordism between Y and the manifold obtained from Y surgery on k.*

Proof. The boundary of X consists of two components. One of these is $Y \times \{0\} \approx M$. Another one is the manifold obtained from Y surgery on k as follows: Gluing 2-handle $D^2 \times D^2$ to $Y \times [0,1]$ changes $Y \times \{1\}$ in the following way. The solid torus $N(k) = \phi(\partial D^2 \times D^2)$ is removed and replaced by the solid torus $D^2 \times \partial D^2$. Note that the meridian $\partial D^2 \times \{z\}$ is identified with the curve $\gamma = \phi(\partial D^2 \times \{z\})$, which means that an integral surgery is performed on $Y \times \{1\}$ along k with the framing $[\gamma]$. ☐

Now we shall state of one the classical theorems of 3-manifold theory:

Theorem 2.2 (Theorem 2, [15]; Theorem 1, [16])**.** *Every closed orientable 3-manifold Y can be obtained from S^3 by an integral surgery on a link $\mathcal{L} \subset S^3$.*

Corollary 2.3. *Any closed orientable* 3*-manifold is null cobordant.*

Proof. By combining Theorem 2.1 and 2.2, we say that closed orientable 3-manifold Y is cobordant to S^3 which bounds D^4. Thus, Y is null cobordant. ☐

2.3. Seifert Fibered Homology Spheres

Let $a_1, \dots, a_n$ be positive integers for $n \geq 3$. Let $A = (a_{ij})$ be a matrix in $\mathbb{M}_{n-2\times n}(\mathbb{C})$ such that each of the maximal minors of A is non-zero. The variety

$$V_A(a_1, \dots, a_n) = \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid a_{i1}z_1^{a_1} + \dots + a_{in}z_n^{a_n} = 0, i = 1, \dots, n-2\} \quad (2.1)$$

is a non-singular complex surface except perhaps at the origin. The link of singularity at the origin

$$\Sigma(a_1, \dots, a_n) = V_A(a_1, \dots, a_n) \cap S^{2n-1} \quad (2.2)$$

is a smooth 3-manifold with the induced orientation. The manifold $\Sigma(a_1, \dots, a_n)$ is an integral homology sphere if and only if $a_1, \dots, a_n$ are pairwise relatively prime. In this case, $\Sigma(a_1, \dots, a_n)$ called a *Seifert fibered homology sphere.*

Let p, q, r be pairwise relatively prime positive integers. The *Brieskorn homology sphere* $\mathit{\Sigma}(p, q, r)$ is defined as the link of Brieskorn singularity,

$$\Sigma(p, q, r) = \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0\} \cap S^5 \quad (2.3)$$

This complex surface induces a canonical orientation of the link $\mathit{\Sigma}(p, q, r)$. Brieskorn homology spheres are clearly a special case of Seifert fibered homology spheres.

Dehn surgery representation of Seifert fibered homology spheres can be described briefly as follows: Let $F = S^2 \setminus int(D_1^2 \cup \dots \cup D_n^2)$ be the n-punctured 2-sphere and consider an S^1-bundle $W \to F$ with Euler number b and with fixed trivialization over ∂F. The boundary of W consists n tori, $(\partial D_k^2) \times S^1$. Given n pairs of relatively prime integers, (a_k, b_k), paste n solid tori $D_k^2 \times S^1$ into W in such a way that the homology class $a_k(S^1 \times \{1\}) + b_k(\{1\} \times S^1)$ in the k the boundary component of W is null-homologous in $D_k^2 \times S^1$ after pasting. This construction results in a closed 3-manifold $Y(b; (a_1, b_1), \dots, (a_n, b_n))$ see Figure 2.1. The integers $b; (a_1, b_1), \dots, (a_n, b_n)$ are called

Seifert invariants of Y. The manifold Y is an integral homology sphere if and only if $det(I) = \pm 1$,or equivalently, $a_1 \dots a_n \cdot (-b + \sum_{k=1}^{n} b_k/a_k) = \pm 1$. For more detailed explanation, see [Section 1.1.4, [5]].

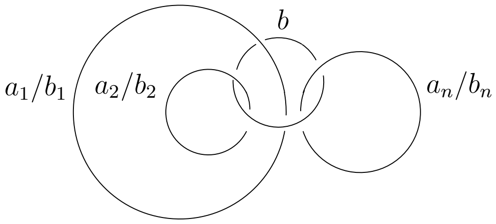


Figure 2.1. Dehn surgery description of a Seifert fibered homology sphere.

3. PLUMBING

A *plumbing graph* G is a weighted graph on which each of vertex v_i is assigned by integers e_i called the *weight* of v_i. Then one associates to each vertex v_i the disk bundle over S^2 classified by e_i. If the valency of vertex v_i is d_i in G, then one chooses d_i disjoint disks B_{ij} in the base of bundle so that $B_{ij} = D^2 \times D^2$. When two vertices v_i and v_j are connected by an edge, one then identifies B_{ij} with B_{kl} by exchanging the base and fiber coordinates.

By iterating this process one can construct simply-connected smooth 4-manifold $X(G)$ corresponding to any plumbing graph G. This process is called *plumbing* and the resulting manifold $X(G)$ is said to be obtained by *plumbing according to G.*

The zero sections of the plumbed bundles represents the natural basis of the group $H_2(X(G), \mathbb{Z})$. The intersection form on the second homology group is given by the matrix $I(G) = (a_{ij})_{i,j=1,\ldots,s}$ with the entries

$$a_{ij} = \begin{cases} e_i & \text{if } i = j; \\ 1 & \text{if } i \text{ is connected to } j \text{ by an edge;} \\ 0 & \text{otherwise.} \end{cases}$$

Further, a 3-manifold $Y = \partial X(G)$ is an integral homology sphere if and only if $det(I(G)) = \pm 1$. In this case, Y is called a *plumbed homology sphere.*

Every Seifert fibered homology sphere $\Sigma(a_1, \ldots, a_n)$ can be obtained by plumbing on a star-shaped graph shown in Figure 3.1, see [Example 1.11, [5]]. The integer weights in the graph are found as follows: First of all, one finds a collection of Seifert invariants

$b, (a_1, b_1), \ldots, (a_n, b_n)$ from the Diophantine equation

$$a_1 \ldots a_n \cdot (b + \sum_{i=1}^{n} b_i/a_i) = -1 \tag{3.1}$$

where $0 \leq b_i \leq a_i - 1$. Then the weights t_{ij} are found from the continued fractions $[t_{i1}, \ldots, t_{im_i}] = a_i/b_i$ where $i = 1, \ldots, n$ and

$$[t_{i1}, \ldots, t_{im_i}] = t_1 - \cfrac{1}{t_2 - \cfrac{1}{\cdots - \cfrac{1}{t_k}}}$$

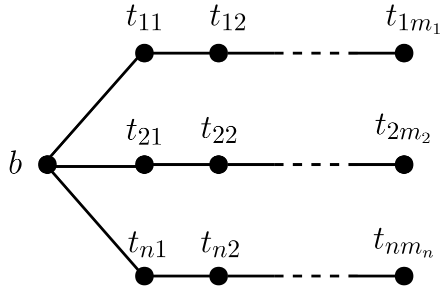


Figure 3.1. Star-shaped graph.

Example 3.1. *Let p, q and r be relatively prime positive integers satisfying*

$$pq + pr - qr = 1 \tag{3.2}$$

Then the Brieskorn homology sphere $Y = \Sigma(p, q, r)$ is the boundary of the plumbing graph shown in Figure 3.2.

First we are looking for integers b, p', q', r' satisfying the Diophantine equation 3.1

$$bpqr + p'qr + pq'r + pqr' = -1 \tag{3.3}$$

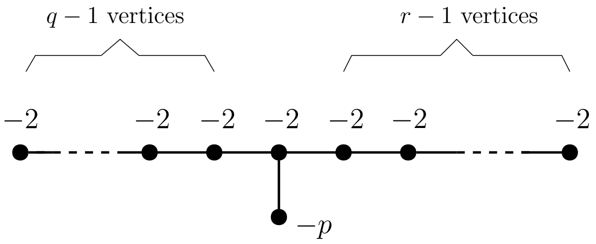


Figure 3.2. Plumbing graph of Y.

where $0 \leq p' \leq p-1$, $0 \leq q' \leq q-1$ *and* $0 \leq r' \leq r-1$.

Taking $\mod p$, $\mod q$ *and* $\mod r$ *reductions of both sides Equation 3.2 respectively, we immediately see*

$$qr \equiv -1 \mod p \tag{3.4}$$

$$pr \equiv 1 \mod q \tag{3.5}$$

$$pq \equiv 1 \mod r \tag{3.6}$$

Taking the $\mod p$ *reductions of both sides of equation 3.3, we get*

$$p'qr \equiv -1 \mod p.$$

Combining this with 3.4, we obtain

$$p' = 1 \tag{3.7}$$

Taking one more reduction of both sides of equation 3.3 with respect to $\mod q$, *we get*

$$pq'r \equiv -1 \mod q$$

Combining this with equation 3.5, we see

$$q' = q - 1 \tag{3.8}$$

Again taking the $\mod r$ *reduction of both sides of equation 3.3, we get*

$$pqr' \equiv -1 \mod r.$$

Combining above equation with 3.6, we see

$$r' = r - 1 \tag{3.9}$$

Plugging finally the values of p', q' *and* r' *in equation 3.3, we obtain*

$$b = -2 \tag{3.10}$$

Eventually, we find continued fraction expansion describing arms of plumbing graph:

$$\begin{aligned} \frac{p}{p'} &= \frac{p}{1} = [p], \\ \frac{q}{q'} &= \frac{q}{q-1} = [2:2:\ldots:2], \\ \frac{r}{r'} &= \frac{r}{r-1} = [2:2:\ldots:2]. \end{aligned}$$

where $[2:2:\ldots:2]$ *denotes* $q-1$ *and* $r-1$ *times 2 respectively.*

Remark 3.2. *There are many different triplets* (p,q,r) *satisfying the identity 3.2. For example,*

- $(p,q,r) = (2k, 4k-1, 4k+1)$ *for* $k \geq 1$,
- $(p,q,r) = (2k+1, 4k+1, 4k+3)$ *for* $k \geq 1$,
- $(p,q,r) = (2k+1, 3k+2, 6k+1)$ *for* $k \geq 1$,
- $(p,q,r) = (2k+1, 3k+1, 6k+5)$ *for* $k \geq 1$.

Example 3.3. *Let* a_n *be the* n^{th} *element of Fibonacci sequence, i.e., the unique solution of the difference equation*

$$a_{n+2} = a_{n+1} + a_n$$

with initial conditions $a_1 = a_2 = 1$.

For any $k \in \mathbb{N}$, *let* $p = a_{2k+1}$, $q = a_{2k+2}$ *and* $r = a_{2k+3}$. *So,* $r = p + q$. *By induction on* k, *one can easily verify that* p, q, r *are pairwise relatively prime integers. Let* $Y = \Sigma(p,q,r)$.

By Cassini's identity of Fibonacci numbers we know that $a_{n-1}.a_{n+1} - a_n^2 = (-1)^n$, *see [17]. Plugging* $n = 2k+2$ *in this identity, we obtain that* $pr - q^2 = 1$. *Then the* *equation 3.2 holds because*

$$pq + pr - qr = pr - q(r - p) = pr - q^2 = 1.$$

So, Y bounds the plumbing graph shown in Figure 3.1.

Example 3.4. *Let $Y_n = \Sigma(3, 3n+1, 9n+4)$ for $n \geq 1$. Then the family of Brieskorn spheres Y_n bounds the plumbing graph shown in Figure 3.3.*

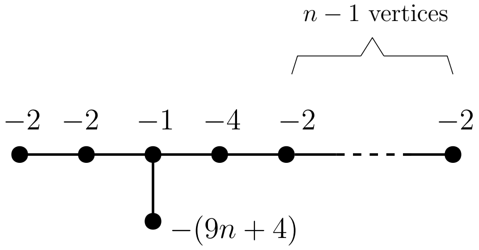


Figure 3.3. Plumbing graph of Y_n.

For $a_1 = 3$, $a_2 = 3n+1$ and $a_3 = 9n+4$, we are investigating for integers b, b_1, b_2, b_3 satisfying the Diophantine equation

$$b.3.(3n+1).(9n+4) + b_1.(3n+1)(9n+4) + 3.b_2.(9n+4) + 3.(3n+1).b_3 = -1$$

where $0 \leq b_1 \leq 2$, $0 \leq b_2 \leq 3n$ and $0 \leq b_3 \leq 9n+3$

By taking $\mod 3$, $\mod (3n+1)$ and $\mod (9n+4)$ respectively, it can be found that $b = -1$, $b_1 = 2$, $b_2 = n$ and $b_3 = 1$. Then the continued fraction expansions are

$$3/2 = [2, 2],$$
$$(3n+1)/n = [4, 2 : \ldots 2 : 3]$$
$$\textit{where} \quad 2 : \ldots : 2 \quad \textit{denotes} \quad n-1 \quad \textit{times} \quad -2,$$
$$(9n+4)/1 = [9n+4].$$

4. ROKHLIN INVARIANT AND NEUMANN-SIEBENMANN INVARIANT

To define Rokhlin and Neumann-Siebenmann invariants of plumbed homology spheres, we will show that integral homology cobordism group is not trivial and moreover it has a subgroup isomorphic to integers.

4.1. Rokhlin Invariant

Let X be compact, smooth, spin 4-manifold with boundary $Y = \partial X$. Then *Rokhlin homomorphism* is defined by the map

$$\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z},\ \mu(Y) = sign(X)/8.$$

Moreover, $\mu(Y)$ is called *Rokhlin invariant.*

Theorem 4.1 ([2]). *The Rokhlin invariant μ is a homology cobordism invariant.*

Fact 4.2. *Let Y be the plumbing homology sphere bounding $X(G)$. Let $I(G)$ be the intersection matrix. Then $sign(X) = sign(I(G))$.*

Fact 4.3. *$sign(I(G))$ is the difference between the number of positive and negative eigenvalues of $I(G)$. So, $sign(I(G))$ is equal to the minus number of vertices of G since the matrix $I(G)$ is negative-definite.*

Now, we are ready to give initial information about the group structure of $\Theta^3_{\mathbb{Z}}$.

Corollary 4.4. *The integral homology cobordism group is non-trivial, that is , $\Theta^3_{\mathbb{Z}} \neq 0$.*

Proof. When p is even, the class of Brieskorn spheres $Y = \Sigma(p, q, r)$ bounds a smooth simply connected spin manifold X shown in Figure 3.2. Let I be intersection matrix associated to plumbing graph. Thus, $sign(I) = -(q + r)$ and so $\mu(Y) = -(q + r)/8$, which is equal to 1 for infinitely many triples (p, q, r). This implies that Y is not homology cobordant to S^3 and hence $\Theta^3_{\mathbb{Z}} \neq 0$. □

4.2. Neumann-Siebenmann Invariant

Let G be a plumbing graph, $X = X(G)$ a 4-manifold obtained by plumbing, and Y a 3-manifold such that $Y = \partial X$.

An integral *Wu class* for X is a class $w \in H_2(X, \mathbb{Z})$ such that

$$w \cdot x = x \cdot x \mod 2 \quad for \quad all \quad x \in H_2(X, \mathbb{Z})$$

.

A *spherical* Wu class is a class which is representable by a smoothly embedded sphere in X. The spherical Wu class w of X is uniquely defined to be sum $w = \sum \epsilon_i \cdot w_i$ with $\epsilon_i = 0, 1$ satisfying $w \cdot w_i = e_i \mod 2$. Then the *Neumann-Siebenmann invariant* for Y is defined by

$$\bar{\mu}(Y) = (sign(X) - w \cdot w)/8 \tag{4.1}$$

Fact 4.5. *To determine Wu class, we must find the solution of the linear equation system* $I.\epsilon = \mathrm{diag}(I) \mod 2$ *where* I *is the intersection matrix,* $\mathrm{diag}(I)$ *denotes the column matrix including diagonal elements of* I *and* ϵ *also denotes column matrix including* ϵ_i*'s. It turns out that* $\epsilon_i = \mathrm{diag}(I^{-1}_{ii}) \mod 2$.

Theorem 4.6 (Theorem 1, [3]; Theorem 2, [4])**.** *The integer* $\bar{\mu}$ *is an invariant of plumbed homology spheres and it is additive with respect to connected sums. Moreover,* $\bar{\mu}(Y) \mod 2 = \mu(Y)$ *where* μ *denotes the Rokhlin invariant.*

Instead of solving big and complicated linear equation systems, Neumann and Raymond described an effective algorithm which enables to find the Wu set of a plumbing graph in order to compute Neumann-Siebenmann invariant $\bar{\mu}$, [Theorem 7.1, [18]]. Given a plumbing graph G shown in Figure 4.1, apply one of the following two moves:

$M1$. If e_i is even, replace G by the disjoint union G' of $G_1, \ldots, G_t$.
$M2$. If e_i is odd, replace G by G' as shown in Figure 4.1.

Repeating an arbitrary combination of these moves sufficiently many times, one can reach a collection of isolated points. If Y is plumbed homology sphere, there is a dialectical process which enables to define the subset of $\mathcal{S}(G)$ inductively as follows:

(i) If G_0 is a set of isolated points with odd weights, put $\mathcal{S}(G_0)$ equal to the set of all these points.
(ii) If $\mathcal{S}(G')$ is known and G reduces to G' by the move $M1$ put $\mathcal{S}(G) = \mathcal{S}(G') \cup \{v_i\}$ or $\mathcal{S}(G) = \mathcal{S}(G')$ according as the number of points in $\mathcal{S}(G')$ adjacent to vertex v_j is congruent to $e_j - 1$ or $e_j \mod 2$.
(iii) If $\mathcal{S}(G')$ is known and G reduces to G' by the move $M2$, put $\mathcal{S}(G) = \mathcal{S}(G') \cup \{v_i\}$ if v_j is not in $\mathcal{S}(G')$, and put $\mathcal{S}(G) = \mathcal{S}(G')$ if v_j in $\mathcal{S}(G')$.

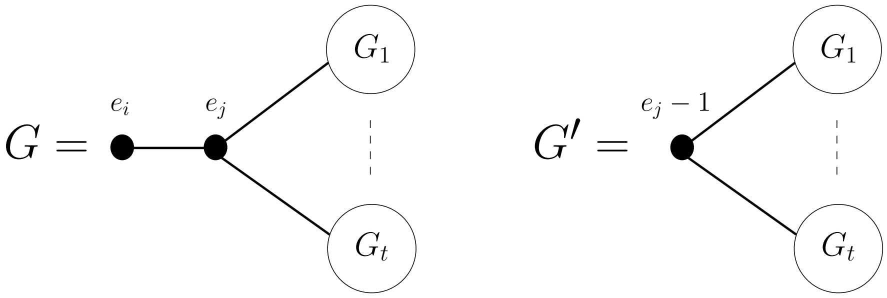


Figure 4.1. Plumbing diagrams of G and G'.

Then $\mathcal{S}(G)$ is the Wu set of the plumbing graph G. We shall call this algorithm as *Neumann-Raymond algorithm.*

Example 4.7. *Consider the family of Brieskorn spheres* $Y = \Sigma(p, q, r)$ *where* p, q, r *are relatively prime integers satisfying 3.2, i.e.,* $pq + pr - qr = 1$. *Then*

- $\bar{\mu}(Y) = -(q+r)/8$ *where* p *is even,*
- $\bar{\mu}(Y) = (p-r)/8$ *where* p *and* r *are odd and* q *is even,*
- $\bar{\mu}(Y) = (p-q)/8$ *where* p *and* q *are odd and* r *is even,*
- $\bar{\mu}(Y) = 0$ *where* p*,* q *and* r *are odd.*

Let G *be the plumbing graph of* Y *and let* I *be its intersection matrix. Since plumbing graph* G *has* $q+r$ *vertices,* $\operatorname{sign}(I) = -(q+r)$*.*

In general, we decorate the vertices of plumbing graph G *shown in Figure 3.2 as follows: central node with* v_1*, first branch including vertex weighted* $-p$ *with* v_2*, second branch including* $q-1$ *times* -2 *with* $v_3, v_4, \ldots, v_{q+1}$ *from left to right and third branch including* $r-1$ *times* -2 *with* $v_{q+2}, v_{q+3}, \ldots, v_{q+r}$ *from right to left.*

Suppose first that p *is even. Then the zero matrix is the solution of linear equation system* $I.\epsilon = \operatorname{diag}(I) \mod 2$*. Therefore,* $\bar{\mu}(Y) = -(q+r)/8$*.*

From now onward, we assume that p *is odd.*

Now suppose that q *is even and* r *is odd. On the left side of Figure 4.2, one can find the description of Neumann-Raymond algorithm for this case. Eventually, we reach a set of isolated point with odd weight by using the move* $M1$ $(q+r-3)/2$ *times. Then the Wu set can be found as* $\mathcal{S} = \{v_2, v_3, v_5, \ldots, v_{q+1}\}$*. Thus the self intersection of Wu class is*

$$w \cdot w = -p + (q/2) \cdot (-2) = -p - q.$$

Therefore,

$$\bar{\mu}(Y) = \frac{-(q+r) - (-p-q)}{8} = \frac{p-r}{8}.$$

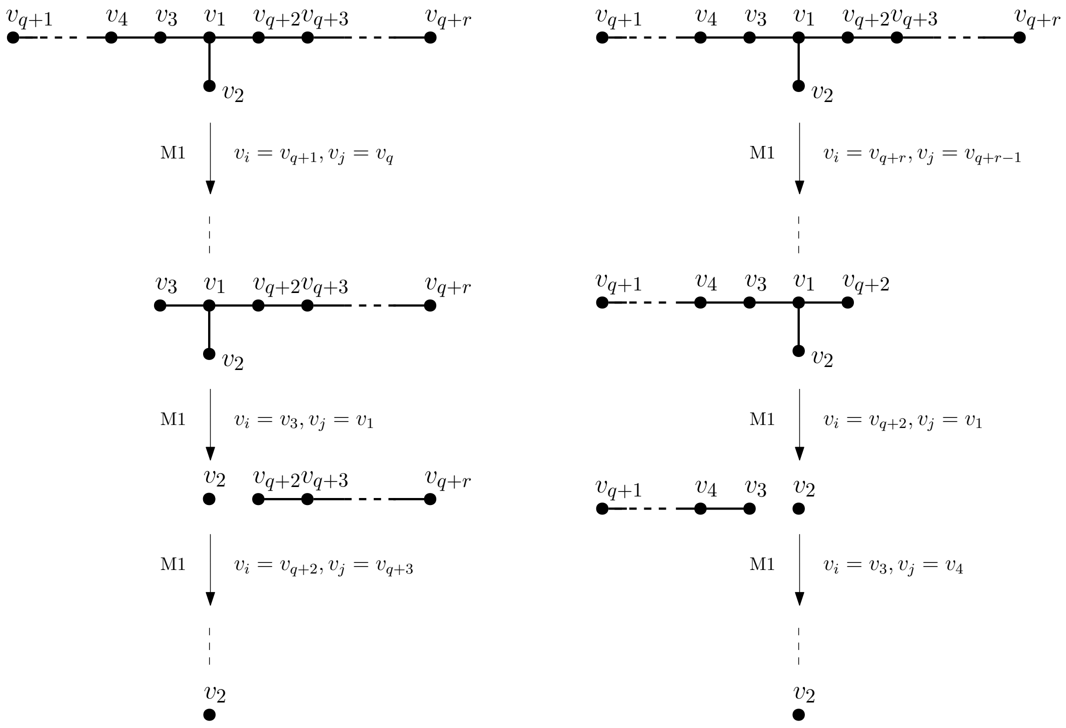


Figure 4.2. Algorithm steps for $\Sigma(p, q, r)$.

Next assume that q *is odd and* r *is even. For this case, the application of Neumann-Raymond algorithm is found on the right side of Figure 4.2. Ultimately, we reach a set of isolated point with odd weight by using the move* $M1$ *again* $(q + r - 3)/2$ *times. Thus, the Wu set is similarly* $\mathcal{S} = \{v_2, v_{q+2}, v_{q+4}, \ldots, v_{q+r}\}$. *Thus, the self intersection of Wu class is*

$$w \cdot w = -p + (r/2) \cdot (-2) = -p - r.$$

Hence,

$$\bar{\mu}(Y) = \frac{-(q + r) - (-p - r)}{8} = \frac{p - q}{8}.$$

Finally, suppose that both q *and* r *are odd. Since* $\bar{\mu}$ *is the multiple of* 8, *numerator of* $\bar{\mu}$ *must be even. Since* sign(I) *is also even, one can immediately realize that*

Neumann-Raymond algorithm cannot be applied to this case. Thus, we need to find the solution of linear equation system $I.\epsilon = \operatorname{diag}(I) \mod 2$. *It can be checked that* ϵ_i*'s* $(i = 0, 1, \ldots, (q+r))$ *are chosen respectively by the series*

$$1001\ldots01$$

where $01\ldots01$ *denotes* $(q+r-2)/2$ *times* 01.

Therefore, $w \cdot w = (-2).1 + (-2).(q+r-2)/2 = -q-r$. *Hence,* $\bar{\mu}(Y) = 0$.

Example 4.8. *Consider the family of Brieskorn spheres* $Y_n = \Sigma(3, 3n+1, 9n+4)$ *described in Example 3.4. Then* $\bar{\mu}(Y_n) = n$.

The plumbing graph G *is described in Figure 3.3. We shall decorate it as follows: The central node is with* v_0, *first branch including vertex weighted* $-(9n+4)$ *with* v_1, *second branch including two times* -2 *with* v_2, v_3 *from left to right and third branch including* -4 *and* $n-1$ *times* -2 *with* $v_4, v_5, \ldots, v_{n+4}$ *from right to left the weight* $(-9n+4)$. *It has* $n+4$ *vertices, so sign*$(I(G)) = -(n+4)$.

It can be checked that ϵ_i*'s* $(i = 0, 1, \ldots, n+4)$ *are chosen respectively by the series*

$$01000\ldots0$$

where $0\ldots0$ *denotes* $n+2$ *times* 0. *Then* $w \cdot w = -9n-4$. *Thus,* $\bar{\mu}(Y_n) = n$.

There is a tight relation between integral homology cobordism of plumbed homology spheres and links of singularity. Plumbing graphs inherently emerge in the singularity theory as good dual resolution graphs in the following sense.

Theorem 4.9 (Theorem 24.1, [19])**.** *A plumbed homology sphere* $Y(G)$ *is the algebraic link of singularity if and only if its plumbing graph* G *negative definite.*

Theorem 4.10 (Theorem 7.29, [5])**.** *Let a homology sphere* Y *be the algebraic link of singularity. If* Y *is null homology cobordant, then* $\bar{\mu}(Y) \geq 0$.

Corollary 4.11. *The integral homology cobordism group* $\Theta^3_{\mathbb{Z}}$ *has a subgroup isomorphic to* $\mathbb{Z}$.

Proof. As shown in Example 3.1, Y bounds a smooth simply connected X whose plumbing graph shown in Figure 3.2. By Example 4.7, we know that $\bar{\mu}(Y) = (p-q)/8 < 0$ where p and q are odd and r is even. By Theorem 4.10, this implies that Y is not homology cobordant to S^3.

Consider the n-times connected sum of Y with itself. Then $\bar{\mu}(Y\#\ldots\#Y) = n\bar{\mu}(Y) < 0$. Thus, $Y\#\ldots\#Y$ is not null homology cobordant for any n. Therefore, Y has infinite order in the group $\Theta^3_{\mathbb{Z}}$. □

Note that Corollary 4.11 was first proved by Fintushel and Stern using a numerical invariant defined by authors, see [Theorem 1.2, [20]].

5. HEEGAARD FLOER HOMOLOGY

Heegard Floer homology has been developed by Peter Ozsváth and Zoltán Szabó in [21], [22] as a topological invariant for closed oriented 3-manifolds. We shall give a brief explanation of their detailed work for 3-manifolds:

Let Y be a closed, connected, oriented 3-manifold, equipped with a Spin^c structure $\mathfrak{s} \in \text{Spin}^c(Y)$. Heegaard Floer homology is computed from a pointed Heegaard diagram for Y. A *pointed Heegaard diagram* is a set of data $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ where:

- $\Sigma \subset Y$ is an embedded, oriented surface of genus g that splits Y into two homeomorphic handlebodies U_0 and U_1;
- $\alpha = \{\alpha_1, \ldots, \alpha_g\}$ is a set of nonintersecting simple closed curves on Σ which bound disks in U_0, and span the kernel of $H_1(\Sigma; \mathbb{Z}) \to H_1(U_0; \mathbb{Z})$;
- $\beta = \{\beta_1, \ldots, \beta_g\}$ is a is a set of nonintersecting simple closed curves on Σ which bound disks in U_1, and span the kernel of $H_1(\Sigma; \mathbb{Z}) \to H_1(U_1; \mathbb{Z})$;
- α and β curves intersect transversely, and z is a basepoint on the surface Σ that does not lie on any of the α or β curves.

Note that above description comes from Morse theory. Because α and β curves can be considered as the intersection of Σ with the ascending submanifolds of critical points of index 1 and 2 respectively. In this case, the number of these critical points both equal the number of genus g of Σ. Moreover, there is one critical point of index 0, and one of index 3.

Let $\text{Sym}^g(\Sigma)$ be the symmetric g-fold product of Σ, which is naturally equipped with a complex structure J. Let $\mathbb{T}_\alpha = \alpha_1 \times \ldots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \ldots \times \beta_g$ be two g-tori in $\text{Sym}^g(\Sigma)$. The Heegaard Floer groups are variations of Lagrangian Floer homology for the two real tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ inside $\text{Sym}^g(\Sigma)$. There is a natural map $\mathfrak{s}_z\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \text{Spin}^c(Y)$, and we will focus on intersection points $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $\mathfrak{s}_z(x) = \mathfrak{s}$. These intersection points determine g trajectories in Y, by considering the gradient flow of a Morse function on Y that is compatible with the Heegaard decomposition of Σ. Each point $z \in \Sigma$ that stands outside α and β curves also determines the trajectory connecting critical points of index 1 and 3.

Let $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$. Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. A *Whitney disk* connecting x and y is a map $\phi\colon \mathbb{D} \to \mathrm{Sym}^g(\Sigma)$ such that $\phi(-i) = x, \phi(i) = y, \phi(z) \in \mathbb{T}_\alpha$ for $|z| = 1$ and $Re(z) \geq 0$, and $\phi(z) \in \mathbb{T}_\beta$ for $|z| = 1$ and $Re(z) \leq 0$. Define $\pi_2(x, y)$ to be the space of homotopy classes of Whitney disks connecting x and y. As we emphasized above, there is an induced complex structure on $\mathrm{Sym}^g(\Sigma)$ coming from Σ; so we can talk about holomorphic maps of $\mathbb{D}$ into $\mathbb{C}$. Let ϕ be a homotopy class in $\pi_2(x, y)$; certain representatives of ϕ may be homolomorphic. Similarly, if we choose a different almost complex structure J on $\mathrm{Sym}^g(\Sigma)$, we can consider pseudoholomorphic representatives of ϕ.

On the other hand, Spin^c structures have an important role in Heegaard Floer theory. In general, Spin^c structures are homology classes of nowhere vanishing vector fields on Y, where such vector fields are homologous only if they are homotopic in the complement of a 3-ball D^3 in Y. Note that all closed oriented 3-manifolds are parallelizable so that they admit nowhere vanishing vector fields and hence Spin^c structures.

Given a point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, gradient of a Morse function determines a Spin^c-structure, which is denoted by $s_z(x)$. Note that if there is a pseudoholomorphic disk connecting x and y, then $s_z(x) = s_z(y)$, see [Lemma 2.19, [21]]. In order to define the chain complex associated to a pointed Heegaard diagram $\mathcal{H}$, we also need to *admissibility condition* on $\mathcal{H}$ depending on $\mathfrak{s}$, see [Section 4.2.2, [21]].

Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in \pi_2(x, y)$. Then $\mathcal{M}(\phi)$ denotes the set of holomorphic representatives of ϕ with respect to some almost complex structure $J \in \mathrm{Sym}^g(\Sigma)$. One can identify infinite strip $\{z\colon\ Re(z) \in [0, 1]\}$ with the unit disk $\mathbb{D} \subset \mathbb{C}$. Translation by imaginary numbers in the infinite strip allows one to define $\mathbb{R}$-action on $\mathcal{M}(\phi)$. Let $\widehat{\mathcal{M}}(\phi)$ denote the space $\mathcal{M}(\phi)/\mathbb{R}$. For suitable perturbations of the almost complex structure on $\mathrm{Sym}^g(\Sigma)$, one can show that $\mathcal{M}(\phi)$ and $\widehat{\mathcal{M}}(\phi)$ are manifolds in a natural way. Furthermore, there exists a mapping $\mu\colon \pi_2(x, y) \to \mathbb{Z}$ called the *Maslow index.* Suppose that $\mu(\phi) = 1$. These manifolds are orientable manifold of dimension 1 and 0 respectively. Moreover, if $\mu(\phi) = 1$, then $\widehat{\mathcal{M}}(\phi)$ is compact zero-manifold, see [Section 3.8, [21]]. Finally, let $n_z(\phi)$ denote the algebraic intersection number of ϕ with the submanifold $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$.

Given such a pair $\mathcal{H}$, the Heegaard Floer chain complex $CF^\infty(\mathcal{H}, \mathfrak{s})$ is freely generated over $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ by pairs $[x, i]$ with $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i \in \mathbb{Z}$, such that $\mathfrak{s}_z(x) = \mathfrak{s}$. In general, the differential is given by

$$\partial[x, i] = \sum_{\{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta | \mathfrak{s}_z(y) = \mathfrak{s}\}} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi) = 1\}} \#\widehat{\mathcal{M}}(\phi) \cdot [y, i - n_z(\phi)].$$

There is an action of $\mathbb{F}[U, U^{-1}]$ on CF^∞, where U acts by $U \cdot [x, i] = [x, i - 1]$ and decreases relative grading by 2. The other complexes CF^+, CF^- and $\widehat{CF}$ are obtained from CF^∞ by considering only pairs $[x, i]$ with $i \geq 0, i < 0$ and $i = 0$ respectively. All three complexes have also an induced $\mathbb{F}[U]$-action, which is trivial in the case of $\widehat{CF}$. The differential satisfies $\partial^2 = 0$, so one can define $HF^\infty(Y) = Ker\partial/Im\partial$. There are four versions of *Heegaard Floer homology* groups $\widehat{HF}(Y, \mathfrak{t})$, $HF^+(Y, \mathfrak{t})$, $HF^-(Y, \mathfrak{t})$, and $HF^\infty(Y, \mathfrak{t})$. These groups are also $\mathbb{F}[U]$ modules where multiplication by U decreases degree by 2.

Theorem 5.1 (Theorem 1.1, [21]). *The groups* $\widehat{HF}(Y, \mathfrak{s})$*,* $HF^+(Y, \mathfrak{s})$*,* $HF^-(Y, \mathfrak{s})$*, and* $HF^\infty(Y, \mathfrak{s})$ *are topological invariants of* Y *and* $\mathfrak{s}$*, in the sense that they are independent of the Heegaard splitting, the choice of attaching circles, the basepoint* z*, and the complex structure.*

Now, we shall give our attention to describe Heegaard Floer homology groups of plumbed homology spheres following the recipe [9].

Let G denote the negative definite graph with vertices v_j weighted in integers. We know that this graph gives rise to a 4-manifold $X(G)$ with boundary by plumbing.

Assume that the Euler number of the sphere bundle corresponding to the vertex v_j is given by e_j. The *degree* of a vertex $v_j \in Vert(G)$, denoted $d(v_j)$, is the number of of edges which contain v_j. A vertex $v_j \in Vert(G)$ is said to be a *bad vertex* of the weighted graph if $e_j + d(v_j) > 0$. Let $Y(G)$ denote the oriented 3-manifold which is the boundary of $X(G)$.

The second cohomology $H^2(X(G), \mathbb{Z})$ is a free module generated by vertices of G. Let $\mathrm{Char}(G)$ denote the set of all characteristic vectors of this module, i.e., every element of K satisfies $\langle K, v_j \rangle = e_j \mod 2$ for each j. We shall work with $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients in order to avoid sign ambiguities.

Every element of $\mathrm{Char}(G)$ uniquely defines a Spin^c structure on X, because the first Chern class gives an identification of the set of Spin^c structures over X with the characteristic vectors $\mathrm{Char}(G)$. The image of $H^2(X, Y; \mathbb{Z})$ in $H^2(X; \mathbb{Z})$ is spanned by the Poincaré duals of spheres corresponding to vertices. Two such Spin^c structures induce the same Spin^c structure on Y if and only if the corresponding characteristic cohomology classes are in the same $H^2(X, Y; \mathbb{Z})$-orbit. For a fixed Spin^c structure $\mathfrak{t}$ of Y, let $\mathrm{Char}(G, \mathfrak{t})$ denote the set of all characteristic cohomology classes restricting to $\mathfrak{t}$ on Y. Since Y is a rational homology sphere, it has finitely many Spin^c structures. Moreover, there is a in one-to-one correspondence between Spin^c structures over Y and elements of the set of $2H^2(X, Y; \mathbb{Z})$-orbits in $\mathrm{Char}(G)$. So we can realize $\mathrm{Char}(G)$ as a disjoint union of finitely many $\mathrm{Char}(G, \mathrm{t})$'s.

Let $\mathcal{T}^+$ be the graded algebra $\mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$ where the formal variable U has degree -2 and let $\mathcal{T}_d^+$ be the one in which the lowest degree element is supported in degree d. Form the set $\mathbb{H}^+(G) \subset \mathrm{Hom}(\mathrm{Char}(G), \mathcal{T}^+)$ where any element ϕ of $\mathbb{H}^+(G)$ satisfies the following property, namely *adjunction relation*; if K is a characteristic vector and n is an integer such that

$$\langle K, v_j \rangle + e_j = 2n,$$

we have

$$U^{m+n}\phi(K+2\mathrm{PD}(v_j)) = U^m\phi(K) \text{ if } n \geq 0,$$

or

$$U^{m}\phi(K+2\mathrm{PD}(v_j)) = U^{m-n}\phi(K) \text{ if } n < 0.$$

The set $\mathbb{H}^+(G,\mathfrak{t})$ is the set of all maps in $\mathbb{H}^+(G)$ with support $\mathrm{Char}(G,\mathrm{t})$. Then we have a direct sum splitting $\mathbb{H}^+(G) = \bigoplus_{\mathrm{t}} \mathbb{H}^+(G,\mathfrak{t})$. The group $\mathbb{H}^+(G)$ is graded in the following way. An element $\phi \in \mathbb{H}^+(G)$ is said to be *homogeneous* of degree d if for every characteristic vector K with $\phi(K) \neq 0$, $\phi(K) \in \mathcal{T}^+$ is a homogeneous element with

$$\deg(\phi(K)) - \frac{K^2+|G|}{4} = d.$$

More generally when Y is a rational homology sphere, it is well-known fact that the Heegaard Floer homology is given by $HF^+(Y,\mathfrak{t}) = \mathcal{T}_d^+ \oplus HF_{\mathrm{red}}(Y,\mathfrak{t})$ where $HF_{\mathrm{red}}(Y,\mathfrak{t})$ is a finitely generated $\mathbb{F}[U]$-module, see [23].

With above construction, fundamental theorem of plus version of Heegaard Floer group of plumbed 3-manifolds occurs:

Theorem 5.2 (Theorem 1.2, [9]). *Let G be a negative-definite weighted graph with at most one bad vertex. Then, for each* Spin^c *structure $\mathfrak{t}$ over $-Y(G)$, there is an isomorphism of graded $\mathbb{Z}[U]$ modules, $HF^+(-Y(G),\mathfrak{t}) \cong \mathbb{H}^+(G,\mathfrak{t})$.*

Let $\mathbb{K}^+(G)$ be the set of equivalence classes of elements of $\mathbb{Z}^{\geq 0} \times \mathrm{Char}(G,\mathfrak{t})$. We shall write $U^m \otimes K$ for the typical element of $\mathbb{K}^+(G)$. We rewrite adjunction relation in terms of dual elements. For any vertex v_j, let

$$\langle K, v_j \rangle + e_j = 2n$$

If $n \geq 0$, then

$$U^{m+n} \otimes (K + 2PD(v_{)}])) \sim U^m \otimes K, \tag{5.1}$$

while if $n \leq 0$, then

$$U^m \otimes (K + 2PD(v_j)) \sim U^{m-n} \otimes K. \tag{5.2}$$

Beginning with a map

$$\phi : \mathrm{Char}(G) \to \mathcal{T}_0^+,$$

one consider the induced map

$$\tilde{\phi} : \mathbb{Z}^{\geq 0} \times \mathrm{Char}(G) \to \mathcal{T}_0^+$$

defined by the rule

$$\tilde{\phi}(U^n \otimes K) = U^n \cdot \phi(K)$$

Then the set of finitely supported functions $\phi : \mathrm{Char}(G) \to \mathcal{T}_0^+$ whose induced map $\tilde{\phi}$ descends to $\mathbb{K}^+(G)$ is $\mathbb{H}^+(G)$.

A *basic element* of $\mathbb{K}^+(G)$ is one whose equivalence class does not contain any of $U^m \otimes K$ with $m > 0$. Given two non-equivalent basic elements $K_1 = U^0 \otimes K_1$ and $K_2 = U^0 \otimes K_2$ in the same Spin^c structure, one can find positive integers n and m such that

$$U^n \otimes K_1 \sim U^m \otimes K_2.$$

If these n and m are minimal then this relation is said to be the *minimal relationship* between K_1 and K_2.

With above adoption, $\mathbb{K}^+(G)$ is precisely determined as soon as one finds its basic elements and minimal relationships among them. Now we shall describe the algorithm to calculate basic elements in a combinatorial way by following the recipe of [9].

Let K satisfy

$$e_j + 2 \leq \langle K, v_j \rangle \leq -e_j \tag{5.3}$$

for each j. Construct a sequence of vectors $K = K_0, K_1, \ldots, K_n$, where K_{n+1} is obtained from K_i by choosing any vertex v_{i+1} with

$$\langle K_i, v_{i+1} \rangle = -e_{i+1},$$

and letting $K_{i+1} = K_i + 2PD[v_{i+1}]$. This algorithm can terminate either

- the final vector $L = K_n$ satisfies the inequality

  $$e_j \leq \langle L, v_j \rangle \leq -e_j + 2 \tag{5.4}$$

  at each j or
- there is some m for which

  $$\langle L, v_m \rangle > -e_m. \tag{5.5}$$

It turns out that there is a one to one correspondence between elements of $\mathbb{K}^+(G)$ which have no representative of the form $U^m \otimes K'$ with $m > 0$ and initial vectors K satisfying inequality 5.3 for which the algorithm above terminates in a characteristic vector L satisfying inequality 5.4. We shall call vectors satisfying inequalities 5.4 and 5.5 *good full path* and *bad full path*, respectively. We shall call this algorithm as *Ozsváth-Szabó algorithm.*

Remark 5.3. *The degree of characteristic vectors in the same full path are always equal.*

Remark 5.4. *If a characteristic vector supports a good full path, then all full paths including it are good. On the other hand, bad full paths are hereditary. If a characteristic vector for a plumbing subgraph has a bad full path, then so does the containing characteristic vector and plumbing graph.*

Definition 5.5. *The lowest degree $d(Y, \mathfrak{t})$ of non-torsion elements in $HF^+(Y, \mathfrak{t})$ is called the Ozsváth-Szabó correction term, or sometimes $d-invariant$ for a* spin^c *3-manifold $(Y, \mathfrak{t})$.*

Proposition 5.6 (Corollary 1.4, [9]). *Let G be a negative-definite weighted graph with at most two bad vertices, and fix a* Spin^c *structure $\mathfrak{t}$ over Y. Then,*

$$d(-Y(G), \mathfrak{t}) = \max_{\{K \in \mathrm{Char}(G, \mathfrak{t})\}} \frac{K^2 + |G|}{4} \tag{5.6}$$

Remark 5.7. *Let $\mathcal{C}$ denote the set of characteristic vectors satisfying 5.3. Let $\mathcal{C}'$ denote the set of final characteristic vectors of good full paths. Then*

$$\begin{aligned} d(-Y(G), \mathfrak{t}) &= \max_{\mathcal{C}} \frac{K^2 + |G|}{4} &(5.7)\\ &= \max_{\mathcal{C}'} \frac{K^2 + |G|}{4} &(5.8) \end{aligned}$$

Note that the set $\{K \in \mathrm{Char}(G, \mathfrak{t})\}$ has infinitely many elements. On the other hand, $\mathcal{C}$ has finitely many elements but it is considerably large set. Among them, $\mathcal{C}'$ is the smallest set even though it is still not easy to describe all elements lying in $\mathcal{C}'$.

In order to calculate K^2 in the formula 5.6, one must find the inverse of intersection matrix. Due to work of Némethi and Niculaescu in [Section 5, [24]], there is a way to find entries of I^{-1} in the following fashion:

For any $v, w \in Vert(G)$, let I^{-1}_{vw} denotes the (v, w)-entry of the intersection matrix I^{-1}. Since I is negative definite and G is connected, $I^{-1}_{vw} < 0$ for each entry v, w. Also since I is described by integer weighted graph, we can interpret these entries in the following way.

For any two vertex $v, w \in Vert(G)$, let p_{vw} be the unique minimal path in G connecting v and w. Let $I_{(vw)}$ be the matrix obtained from I deleting all the rows and columns corresponding to the vertices on the path p_{vw}; that is, $I_{(vw)}$ is the intersection matrix of the complement graph of the path p_{vw}. Then $I^{-1}_{vw} = -|\det(I_{(vw)})/\det(I)|$. These determinants can be easily computed using the recipe in [Section 5.21, [19]].

Consider Brieskorn spheres $Y = \Sigma(p, q, r)$ described in Example 3.1. Let I be the intersection matrix of G. Note that $|G| = q + r$.

Proposition 5.8. *For an even* p*, we have that* $d(Y) = -(q + r)/4$.

Proof. When p is even, X has even intersection form and thus $K = 0$ is a characteristic vector. Since the intersection form is negative definite, $K = 0$ is clearly maximizes the expression 5.8. Hence, $d(-Y) = q + r/4$. Since $d(-Y) = -d(Y)$ by [Proposition 4.2, [23]], we are done. □

From now onward, we assume that p is odd.

Lemma 5.9. *For* Y*, the maximum of 5.8 is achieved among characteristic vectors* $k_{a,n}$ *shown in Figure 5.1 where* $-p \leq a \leq p - 2$ *and* $1 \leq n \leq r - 1$.

Proof. Let K and L denote the initial and final vectors of the good full path for the plumbing graph G.

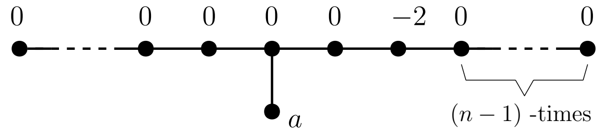


Figure 5.1. Characteristic vectors $k_{a,n}$.

Considering Ozsváth-Szabó algorithm for any characteristic vector K leading to good full path, one can easily see that K cannot include consecutive elements weighted by 2; otherwise, it leads a bad full path.

If K is an initial characteristic vector $(0, a, 0, \ldots, 0, 2, 0, \ldots, 0)$ where 2 is the $(q-n-1)^{th}$-entry of K, then under Ozsváth-Szabó algorithm L is a characteristic vector $(0, a, 0, \ldots, 0, -2, 0, \ldots, 0)$ where -2 is the $(r-n-1)^{th}$-entry of L.

Further, K cannot be the characteristic vector of the form $(0, a, 0, \ldots, 0)$ because the degree of this vector is less than the previous one.

We know that degrees of the initial vector $K = (0, a, 0, \ldots, 0, 2, 0, \ldots, 0)$ and the final vector $L = (0, 0, \ldots, 0, -2, 0, \ldots, 0)$ are same since they are equivalent characteristic vectors in the Ozsváth-Szabó sense. Thus, L is of the form $k_{a,n}$. □

Remark 5.10. *For simplicity, we describe the characteristic vector* $k_{a,n}$ *as a plumbing graph.*

Lemma 5.11. *Let* (a, n) *be in Lemma 5.9. Then*

$$f(a,n) = k_{a,n}^2 = -(q+r)a^2 + 4aqn - 4(q-p)n^2 - 4n. \tag{5.9}$$

Proof. Let $k_{a,n}$ be the final vector as shown in Figure 5.1. Since $k_{a,n}$ is decorated by almost all zeros, we do not need to describe the whole of I^{-1}.

Let a and u be the position of vertex in $k_{a,n}$ corresponding to vertex weighted by a and -2, respectively. Then

$$f(a,n) = k_{a,n}^2 = a^2 I_{aa}^{-1} - 4aI_{au}^{-1} + 4I_{uu}^{-1}.$$

Let p_{aa}, p_{au} and p_{uu} be unique minimal paths in G connecting vertices a and u, respectively. Deleting p_{aa} in G, we get that $I_{aa}^{-1} = -(q+r)$ where $-(q+r)$ is the determinant of remaining linear graph including $q+r-1$ vertices decorated by -2. Deleting p_{au} in G, we similarly obtain $I_{au}^{-1} = -qn$. Finally, I_{uu}^{-1} is the minus of product of $-n$ and the one which is equal to the numerator of the following quantity:

$$-2 + \frac{1}{p} + \frac{q-1}{q} + \frac{r-n-1}{r-n} = -\frac{n(q-p)+1}{pq(r-n)}.$$

Thus, $I_{uu}^{-1} = -(q-p)n^2 - n$. Hence, the equation 5.9 holds. ☐

Lemma 5.12. *Let $f(a,n)$ be in the Lemma 5.11. Then for $p = 2k+1$, $q = 4k+1$ and $r = 4k+3$, we have*

$$\max\{f(a,n)\} = f(1,1)$$

.

Proof. It suffices to show that $-f(a,n) + f(1,1) \geq 0$. Then

$$-f(a,n) + f(1,1) = (q+r)a^2 - 4aqn + 4(q-p)n^2 + 4n - [q+r-4p+4] \tag{5.10}$$

Plugging $p = 2k+1$, $q = 4k+1$ and $r = 4k+3$ in 5.10, we obtain that

$$\begin{aligned}-f(a,n)+f(1,1) =&(8k+4)a^2-4an(4k+1)+8kn^2+4n-[8k+4-4(2k+1)+4]\\ =&(8k+4)a^2-(16k+4)an+8kn^2+4n-4\end{aligned}$$

If $a \le 0$, then we are done. On the other hand, if $a > 0$, then

$$-f(a,n)+f(1,1) = 8k(a-n)^2+4a(a-n)+4n-4$$

If $a \ge n$, then we are done. Otherwise, $a < n$, i.e., $a \le n-1$. Then

$$\begin{aligned}-f(a,n)+f(1,1) =&8k(a-n)^2+4a(a-n)+4n-4\\ =&4(n-a)[2k(n-a)-a]+4n-4\\ \ge&4(2-n)+4n-4\\ \ge&0\end{aligned}$$

□

Remark 5.13. *For all triplets* (p,q,r), $\min\{f(a,n)\}$ *is not equal to* $f(1,1)$. *For example, if we choose* $(p,q,r) = (89,144,233)$, *then* $\min\{f(a,n)\} = f(-5,4)$.

Proposition 5.14. *Let* $Y_k = \Sigma(2k+1, 4k+1, 4k+3)$ *for* $k \ge 1$. *Then* $d(Y_k) = -2k$.

Proof. As an immediate corollary of above lemmas, the final vector $k_{1,1}$ maximizes the quantity K^2. By Corollary 5.6 and Remark 5.7, we find that

$$\begin{aligned}d(Y_k,\mathfrak{t}) &= \max_{C'}\frac{K^2+8k+4}{4}\\ &= -\frac{-k_{11}^2+8k+4}{4}\\ &= -\frac{-4+8k+4}{4}\\ &= -2k\end{aligned}$$

Since $d(-Y_k) = -d(Y_k)$ by [Proposition 4.2, [23]], we obtain the desired result. $\square$

From now onward, we adopt the notations, tools and descriptions of papers [10] and [25]. Note that the latter one is a user friendly paper.

We are now ready to decipher the complete relation between algebraic and combinatorial data associated to plumbed 3-manifolds.

5.1. Almost Rational Graphs

We know that $L = H_2(X(G); \mathbb{Z})$ is freely generated by the the fundamental classes of the zero-sections of the disk bundles in the plumbing construction. Thus, for each vertex of G, we have a generator of L, which we shall denote it corresponding to the vertex b_j with the same symbol. The intersection form $(\cdot, \cdot)$ on L is a symmetric bilinear form naturally characterized by G.

Plumbing graphs can be realized as the good dual resolution graph since its intersection form is negative definite, see [5]. Henceforward, we assume that all intersection matrices according to plumbing graphs are negative definite. Thus L is a lattice and we have the short exact sequence

$$0 \longrightarrow L \xrightarrow{PD} L' \longrightarrow H \longrightarrow 0 \tag{5.11}$$

where L' is the dual lattice $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong H^2(X; \mathbb{Z}) \cong H_2(X, Y; \mathbb{Z})$ with $PD(x) = (x, \cdot)$, and $H = H_1(Y; \mathbb{Z})$.

We say that $k \in L'$ is *characteristic* cohomology class if for every vertex b_j of G, we have $k(b_j) + e_j \equiv 0 \bmod 2$. There is a natural action of L on $\mathrm{Char}(G)$ given by the rule $x * k = k + 2PD(x)$ for every $x \in L$. We denote the orbit of any $k \in \mathrm{Char}(G)$ under this action with $[k]$. The characteristic cohomology class $K \in L'$ satisfying $K(b_j) = -e_j - 2$ for all $j \in J$ is called the *canonical class*. Fix $k \in \mathrm{Char}(G)$, and define the function

$\chi_k : L \to \mathbb{Z}$ by

$$\chi_k(x) = -(k(x) + (x,x))/2. \tag{5.12}$$

A plumbing graph G is called *rational* if $\chi(x) \geq 1$ for any $x > 0$. A graph is called *almost rational* (or AR for short) if there exists a vertex of G with weight b_0 such that by replacing b_0 with some $b_0^{'} \leq b_0$ we obtain a rational graph.

The class of AR graphs has a broad range that encompasses:

- Graphs with at most one bad vertex;
- negative-definite, star-shaped graphs (plumbing graphs of Seifert fibered rational homology spheres);
- rational graphs;
- elliptic graphs (corresponding to normal surface singularities of geometric genus equal to one).

One curious reader may find more information about AR graphs in [10].

A plumbing graph is said to be *proper almost rational* if it is almost rational but not rational. If a surface singularity admits a good resolution graph which is rational (respectively AR and proper AR), then it is called *rational (respectively, AR and proper AR)*. Finally notice that Brieskorn spheres are proper AR if $(p,q,r) \neq (2,3,5)$, see [26].

5.2. Graded Roots and Factorization of Main Isomorphism

Let R be an infinite tree, $\mathcal{V}(R)$ its vertex set and $\mathcal{E}(R)$ its edge set. A *graded root* consists of a pair (R,χ) where $\chi : \mathcal{V}(R) \to \mathbb{Z}$ satisfies the following properties:

(i) $\chi(u) - \chi(v) = \pm 1$, if $[u, v] \in \mathcal{E}(R)$.

(ii) $\chi(u) > \min\{\chi(v), \chi(w)\}$, if $[u, v] \in \mathcal{E}(R)$, and $[u, w] \in \mathcal{E}(R)$.

(iii) χ is bounded below.

(iv) $\chi^{-1}(n)$ is a finite set for every n.

(v) $|\chi^{-1}(n)| = 1$ for n large enough.

A graded root (R, χ) is generated by the function $\tau : \mathbb{Z}^{\geq 0} \to \mathbb{Z}$ as follows: For any $i \in \mathbb{Z}^{\geq 0}$ consider the $R_i := R_{\tau(i)}$ with vertices $\{v_i^k\}$ and the edges $\{[v_i^k, v_i^{k+1}]\}$ with grading $\chi(v_i^k) = k$, where $k \geq \tau(i)$. On the disjoint union $\coprod_i R_i$ of graded trees, for any (i, j), one can identify $v_i^k \sim v_j^l$ and $[v_i^k, v_i^{k+1}] \sim [v_j^l, v_j^{l+1}]$ only if $k = l$ and $k \geq \tau(m)$ for all $i \leq m \leq j$. Write $\overline{v_i^k}$ for the class of v_i^k. Then $(R, \chi) = R_\tau = \coprod_i R_i / \sim$ is a graded root with respect to the natural induced grading $\chi(\overline{v_i^k}) = k$.

For a graded root (R, χ), let $\mathbb{H}(R, \chi)$ (shortly $\mathbb{H}(R)$) be the set of functions $\phi : \mathcal{V}(\mathcal{R}) \to \mathcal{T}_0^+$ satisfying

$$U \cdot \phi(v) = \phi(w), \tag{5.13}$$

or equivalently,

$$U^{\chi(w) - \chi(v)} \cdot \phi(v) = \phi(w). \tag{5.14}$$

whenever $[v, w] \in \mathcal{E}(R)$ with $\chi(v) < \chi(w)$.

The U-action on $\mathbb{H}(R)$ is defined via $(U \cdot \phi)(v) = U(\phi(v))$ which makes $\mathbb{H}(R)$ a graded $\mathbb{F}[U]$-module. Furthermore, the grading on $\mathbb{H}(R, \chi)$ is defined as follows: An element $\phi \in \mathbb{H}(R, \chi)$ is homogeneous of degree $d \in \mathbb{Z}$ if for each $v \in \mathcal{V}$ with $\phi(v) \neq 0, \phi(v) \in \mathcal{T}_0^+$ is homogeneous of degree $d - 2\chi(v)$.

Building on the work of Ozsváth and Szabó, Némethi proved:

Theorem 5.15 (Theorem 8.3, [10])**.** *Let $Y(G)$ be an AR plumbed 3-manifold. For $k \in \mathrm{Char}(G)$, Heegaard Floer homology of $Y(G)$ is given by*

$$HF^+(-Y(G),[k]) \cong \mathbb{H}(R_{\tau_k})[\sigma]. \tag{5.15}$$

where $|G|$ denotes the number of vertices of G and $\sigma = \sigma(G,k) = -\frac{|G|+k^2}{4}$.

In particular, the d-invariants of $Y(G)$ are given by

$$d(Y(G),[k]) = \frac{|G|+k^2}{4} - 2\min \chi_k.$$

Consider the factorization of main isomorphism $\Phi\colon HF^+(-Y(G)) \to \mathbb{H}(R_\tau)$ described in the equation 5.15 by following the recipe in [25] as follows:

$$HF^+(-Y) \xrightarrow{\Phi_1} \mathbb{H}^+(G) \xrightarrow{\widetilde{\Phi}} (\mathbb{K}^+(G))^* \xrightarrow{\Phi_2} \mathbb{H}(R_\tau) \tag{5.16}$$

For any Heegaard Floer homology class $c \in HF^+(-Y,\mathfrak{t})$ and for any characteristic cohomology class k, the map

$$\Phi_1 : HF^+(-Y,\mathfrak{t}) \to \mathbb{H}^+(G,\mathfrak{t})$$

is defined by the rule $\Phi_1(c)(k) := F_{\tilde{X},k}(c)$ where

$$F_{\tilde{X},k} : HF^+(-Y,\mathfrak{t}) \to HF^+(S^3) = \mathcal{T}_0^+$$

is a map for each characteristic cohomology class $k \in \mathrm{Char}(G, \mathfrak{t})$. Φ_1 is a well-defined homomorphism due to adjunction relation and in particular is an isomorphism since the plumbing graph G is AR, Theorem 5.1.

Recall that $\mathbb{K}^+(G, \mathfrak{t})$ is the set of equivalence classes of elements of $\mathbb{Z}^{\geq 0} \times \mathrm{Char}(G, \mathfrak{t})$ and $(\mathbb{K}^+(G, \mathfrak{t}))^*$ is its dual. For any $n \in \mathbb{Z}_{\geq 0}$, let Ker U^{n+1} denote the subgroup of $\mathbb{H}^+(G, \mathfrak{t})$. Then we have the below identification:

Lemma 5.16 (Lemma 2.3, [9])**.** $(\mathrm{Ker}\,(U^{n+1}) \subset \mathbb{H}^+(G, \mathfrak{t})) \cong \mathrm{Hom}\left(\frac{\mathbb{K}^+(G,\mathfrak{t})}{\mathbb{Z}^{\geq l} \times \mathrm{Char}(G,\mathfrak{t})}, \mathbb{F}\right)$

We shall denote the isomorphism described in Lemma 5.16 by $\widetilde{\Phi}_n$. Since every element of $\mathbb{H}^+(G, \mathfrak{t})$ lies in some $\mathrm{Ker}\,(U^{n+1})$, the maps $\widetilde{\Phi}_n$ induce an isomorphism $\widetilde{\Phi} : \mathbb{H}^+(G, \mathfrak{t}) \to (\mathbb{K}^+(G, \mathfrak{t}))^*$.

5.3. Tau Sequences, Laufer Sequences and Detecting Root Vertices

Let R_τ be a graded root associated to a τ function, $\mathcal{V}(R_\tau)$ its vertex set and $\mathfrak{t}_{\mathrm{can}}$ the canonical Spin^c structure with respect to the canonical cohomology class.

We now describe the construction of the tau function τ. According to [25], one can form a sequence $(k(i))_{i=0}^{\infty}$ in $\mathrm{Char}(G, \mathfrak{t}_{\mathrm{can}})$ recursively as follows: Begin with the canonical class $k(0) = K$. Suppose $k(i)$ has already been constructed. Then we may find $k(i+1)$ in the following way.

(i) We construct a computational sequence $z_0, z_1, \cdots z_l$. Let $z_0 = k(i) + 2\mathrm{PD}(b_0)$. Suppose z_m has been found. If there exists $j \in J - \{0\}$ such that

$$z_m(b_j) = -e_j$$

then we let $z_{m+1} = z_m + 2\mathrm{PD}(\mathrm{b_j})$.

(ii) If there is no such j, stop. Set $l = m$ and $k(i+1) = z_l$.

The sequence $(k(i))_{i=0}^{\infty}$ is called the *Laufer sequence* of the AR graph G associated with $\mathfrak{t}_{\text{can}}$. Notice that Laufer sequence is dependent to the choice of the distinguished vertex b_0 but is independent from our choice of vertices in step 1 of the above recursive algorithm. Define the function $\chi_{i,0} = \chi_{k(i)}(b_0)$. Let $\tau(n) = \sum_{i=0}^{n-1} \chi_{i,0}$, with $\tau(0) = 0$.

Theorem 5.17 (Theorem 6.1, Theorem 9.3 [10]). *There exists an integer l such that $\tau(i+1) \geq \tau(i)$ for any $i \geq l$. In particular, the graded root (R_τ, χ_τ) is well-defined. Moreover, for the canonical Spin^c structure, $\tau(l) \geq 2$ and $\tau(i) \leq 1$ for all $i \leq l-1$.*

Therefore, τ defines a graded root R_τ and indices after l are redundant in the construction of R_τ. Furthermore, one can stop the computation process of the Laufer sequence when $\tau \geq 2$.

As a consequence of Theorem 5.17, the map $\Phi_3 : (\mathbb{K}^+(G))^* \to \mathbb{H}(R_\tau)$ can be defined by the rule $\Phi_3(\overline{v_i^m}) = U^{m-\tau(i)} \otimes k(i)$. Then Φ_3 is a bijection so that it induces an $\mathbb{F}[U]$-module isomorphism $\Phi_2 : (\mathbb{K}^+(G))^* \to \mathbb{H}(R_\tau)$, which shifts grading by $(K^2 + |G|)/4.$, for details see [25].

Remark 5.18 (Remark 5.1, [25]). *The Laufer sequence $\{x(i)\}$ in [10] resides in L while $\{k(i)\}$ here resides in* $\text{Char}(G, \mathfrak{t}_{\text{can}})$. *These two Laufer sequences are related by $k(i) = K + 2PD(x(i))$.*

Remark 5.19 (Remark 5.2, [25]). *The sequence $(\tau(i))_{i=0}^{\infty}$ contains a lot of redundant elements. The finite subsequence consisting of local maximum and local minimum values of τ is sufficient to construct the whole graded root. We shall call that sequence a reduced Tau sequence.*

In a graded root R, we say that a vertex is a *root vertex* if it has valency 1. The following lemma identifies root vertices of R_τ with elements of Ker$U$.

Lemma 5.20 (Lemma 5.4 [25]). *Given $k \in \mathbb{K}^+(G, \mathfrak{t}_{\text{can}})$ such that $k^* \in$ Ker$U$, there exists an element $k(i_0)$ of the Laufer sequence such that $k \sim k(i_0)$. This element is unique in the following sense: if $k(i_0) \sim k(i_1) \sim k$ and $i_0 < i_1$ then $\tau(i) = \tau(i_0)$ for* *all i satisfying $i_0 \leq i \leq i_1$. As a result $\Phi_3^{-1}(k)$ is the root vertex of the branch in the graded root R_τ corresponding to $\tau(i_0)$.*

Finally, we state the next lemma which makes computation of plus version of Heegaard Floer homology group much easier.

Lemma 5.21 (Lemma 5.5, [25]). *If $k \in \mathrm{Char}(G, \mathfrak{t}_{\mathrm{can}})$ satisfies*

$$e_j + 2 \leq k(b_j) \leq -e_j - 2, \text{ for all } j \in \mathcal{J},$$

then $k^* \in \mathrm{Ker}U$ and there exists a unique $i_0 \in \mathbb{Z}^{\geq 0}$ such that $k = k(i_0)$. Consequently $\Phi_3^{-1}(k)$ is the root vertex of the branch in the graded root R_τ corresponding to $\tau(i_0)$. Moreover, the index i_0 is the component of the vector $\mathrm{PD}^{-1}(k(i_0) - K)/2$ on b_0.

5.4. Computation of Plus Version of Heegaard Floer Homology

In this section, we shall give detailed explanation of computation of Heegaard Floer homology group of some Brieskorn homology spheres.

Example 5.22. *The Brieskorn sphere $\Sigma(2, 3, 7)$ is the boundary of the graph in Figure 5.22 which we denote it by G. We index the vertices $\{b_0, b_1, b_2, b_3, b_4\}$ of G so that b_0 is the one with adjacency 3 and b_1, b_2, b_3 correspond to weights $-2, -3, -7$, respectively.*

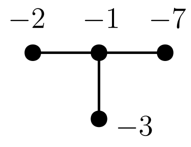


Figure 5.2. The plumbing graph G for $\Sigma(2, 3, 7)$

Then the intersection matrix $I = I(G)$ according to G is

$$I = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -7 \end{bmatrix}$$

Also we know that G is proper AR since $(2,3,7)$ are pairwise relatively prime.

The Heegaard Floer homology of $\Sigma(2,3,7)$ is well-known; we can readily compute it here by considering the Laufer sequence $k(n)$ and the corresponding values $\tau(n)$. Moreover from Lemma 5.21, we know that $k_{\pm}$ will appear in the Laufer sequence. Then $k(n)$ and $\tau(n)$ are as follows:

$$\begin{aligned} k_+ = k(0) &= (-1,0,1,5), & \tau(0) &= 0 \\ k(1) &= (3,-2,-3,-7), & \tau(1) &= 1 \\ k_- = k(2) &= (1,0,-1,-5), & \tau(2) &= 0 \end{aligned}$$

So, the Tau sequence associated to $\Sigma(2,3,7)$ is $\{0,1,0\}$. Hence we obtain the graded root R_{τ_K} as shown in Figure 5.5. Note that the absolute minumum grading is zero.

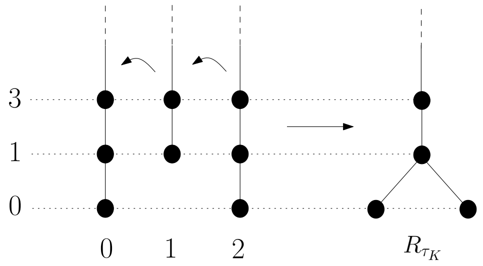


Figure 5.3. The graded root associated to $\Sigma(2,3,7)$

Finally we determine the Heegaard Floer homology by computing the homology of the graded root and shifting the degree by $(K^2+4)/4$ where

$$K^2 = KI^{-1}K^T = \begin{bmatrix} -1 & 0 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} -42 & -21 & -14 & -6 \\ -21 & -17 & -7 & -3 \\ -14 & -7 & -5 & -2 \\ -6 & -3 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 5 \end{bmatrix} = -4$$

. Hence we get that

$$HF^{+}(-\Sigma(2,3,7)) = \mathcal{T}^{+}_{(0)} \oplus \mathbb{F}_{(0)}.$$

Example 5.23. *The Brieskorn sphere $\Sigma(2,3,11)$ is the boundary of the graph in Figure 5.23 which we denote by G. We index the vertices $\{b_0, b_1, \dots, b_8\}$ of G so that b_0 is the one with adjacency 3 and b_8 is the one with weight -3.*

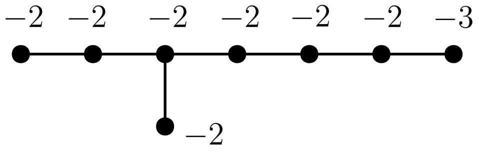


Figure 5.4. The plumbing graph G for $\Sigma(2,3,11)$

Then the intersection matrix $I = I(G)$ according to G is

$$I = \begin{bmatrix} -2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

Also we know that G is proper AR since $(2,3,11)$ are pairwise relatively prime.

Now we can readily compute The Heegaard Floer homology of $\Sigma(2,3,11)$ here by considering the Laufer sequence $k(n)$ and the corresponding values $\tau(n)$. By Lemma 5.21, we know that $k_{\pm}$ will appear in the Laufer sequence. Then $k(n)$ and $\tau(n)$ are as follows:

$$\begin{aligned}
k_+ = k(0) &= (0,0,0,0,0,0,0,0,1), & \tau(0) &= 0\\
k(1) &= (2,-2,0,-2,0,0,0,0,-3), & \tau(1) &= 1\\
k(2) &= (2,0,-2,0,0,0,0,-2,-1), & \tau(2) &= 1\\
k(3) &= (2,-2,0,0,0,0,-2,0,-1), & \tau(3) &= 1\\
k(4) &= (2,0,0,-2,0,-2,0,0,-1), & \tau(4) &= 1\\
k(5) &= (4,-2,-2,0,-2,0,0,0,-1), & \tau(5) &= 1\\
k_- = k(6) &= (0,0,0,0,0,0,0,0,-1), & \tau(6) &= 0
\end{aligned}$$

So, the Tau sequence associated to $\Sigma(2,3,11)$ *is* $\{0,1,1,1,1,1,0\}$. *Hence we obtain the graded root* R_{τ_K} *as shown in Figure 5.5. Note that the absolute minumum grading is zero.*

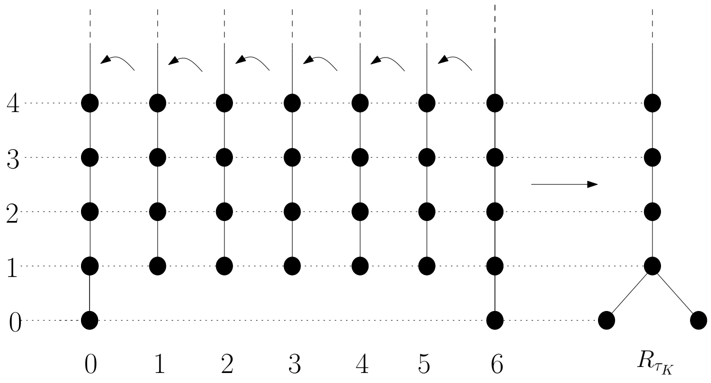


Figure 5.5. The graded root associated to $\Sigma(2,3,11)$

Finally we determine the Heegaard Floer homology by computing the homology of the graded root and shifting the degree by $(K^2+9)/4$ *where* $K^2 = KI^{-1}K^T = -17$. *Hence we get that*

$$HF^+(-\Sigma(2,3,11)) = \mathcal{T}^+_{(-2)} \oplus \mathbb{F}_{(-2)}.$$

Example 5.24. *The Brieskorn sphere* $\Sigma(2,7,15)$ *is the boundary of the graph in Figure 5.24 which we denote by* G. *We index the vertices* $\{b_0,b_1,b_2,b_3,b_4,b_5\}$ *of*

G so that b_0 *is the one with adjacency 3 and* b_1, b_2, b_3, b_4, b_5 *correspond to weights* $-2, -15, -3, -2, -2$, *respectively.*

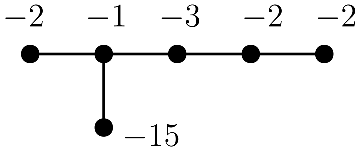


Figure 5.6. The plumbing graph G for $\Sigma(2, 7, 15)$

Then the intersection matrix $I = I(G)$ *according to G is*

$$I = \begin{bmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -15 & 0 & 0 & 0 \\ 1 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Also we know that G is proper AR since $(2, 7, 15)$ *are pairwise relatively prime.*

Now we can readily compute the Heegaard Floer homology of $\Sigma(2, 7, 15)$ *here by considering the Laufer sequence* $k(n)$ *and the corresponding values* $\tau(n)$. *By Lemma 5.21, we know that* $k_\pm$ *will appear in the Laufer sequence. Then* $k(n)$ *and* $\tau(n)$ *are as follows:*

$$k_+ = k(0) = (-1, 0, 13, 1, 0, 0), \qquad \tau(0) = 0$$

$$k(1) = (3, 2, -15, -1, 0, -2), \qquad \tau(1) = 1$$

$$k(2) = (1, 0, -13, 1, 0, -2), \qquad \tau(2) = 0$$

$$k(3) = (3, -2, -11, -1, -2, 0), \qquad \tau(3) = 0$$

$$k(4) = (1, 0, -9, 1, -2, 0), \qquad \tau(4) = -1$$

$$k(5) = (3, -2, -7, -3, 0, 0), \qquad \tau(5) = -1$$

$$k(6) = (1, 0, -5, -1, 0, 0), \qquad \tau(6) = -2$$

$$k(7) = (1, -2, -3, 1, 0, 0), \qquad \tau(7) = -2$$

$$
\begin{aligned}
k(8) &= (1, 0, -1, -1, 0, -2), & \tau(8) &= -2 \\
k(9) &= (1, -2, 1, 1, 0, -2), & \tau(9) &= -2 \\
k(10) &= (1, 0, 3, -1, -2, 0), & \tau(10) &= -2 \\
k(11) &= (1, -2, 5, 1, -2, 0), & \tau(11) &= -2 \\
k(12) &= (1, 0, 7, -3, 0, 0), & \tau(12) &= -2 \\
k(13) &= (1, -2, 9, -1, 0, 0), & \tau(13) &= -2 \\
k(14) &= (-1, 0, 11, 1, 0, 0), & \tau(14) &= -2 \\
k(15) &= (1, -2, 13, -1, 0, -2), & \tau(15) &= -1 \\
k(16) &= (1, 0, -15, 1, 0, -2), & \tau(16) &= -1 \\
k(17) &= (3, -2, -13, -1, -2, 0), & \tau(17) &= -1 \\
k(18) &= (1, 0, -11, 1, -2, 0), & \tau(18) &= -2 \\
k(19) &= (3, -2, -9, -3, 0, 0), & \tau(19) &= -2 \\
k(20) &= (1, 0, -7, -1, 0, 0), & \tau(20) &= -3 \\
k(21) &= (1, -2, -5, 1, 0, 0), & \tau(21) &= -3 \\
k(22) &= (1, 0, -3, -1, 0, -2), & \tau(22) &= -3 \\
k(23) &= (1, -2, -1, 1, 0, -2), & \tau(23) &= -3 \\
k(24) &= (1, 0, 1, -1, -2, 0), & \tau(24) &= -3 \\
k(25) &= (1, -2, 3, 1, -2, 0), & \tau(25) &= -3 \\
k(26) &= (1, 0, 5, -3, 0, 0), & \tau(26) &= -3 \\
k(27) &= (1, -2, 7, -1, 0, 0), & \tau(27) &= -3 \\
k(28) &= (-1, 0, 9, 1, 0, 0), & \tau(28) &= -3 \\
k(29) &= (1, -2, 11, -1, 0, -2), & \tau(29) &= -2 \\
k(30) &= (-1, 0, 13, 1, 0, -2), & \tau(30) &= -2 \\
k(31) &= (3, -2, -15, -1, -2, 0), & \tau(31) &= -1 \\
k(32) &= (1, 0, -13, 1, -2, 0), & \tau(32) &= -2 \\
k(33) &= (3, -2, -11, -3, 0, 0), & \tau(33) &= -2 \\
k(34) &= (1, 0, -9, -1, 0, 0), & \tau(34) &= -3 \\
k(35) &= (1, -2, -7, 1, 0, 0), & \tau(35) &= -3
\end{aligned}
$$

$$k(36) = (1, 0, -5, -1, 0, -2), \qquad \tau(36) = -3$$

$$k(37) = (1, -2, -3, 1, 0, -2), \qquad \tau(37) = -3$$

$$k(38) = (1, 0, -1, -1, -2, 0), \qquad \tau(38) = -3$$

$$k(39) = (1, -2, 1, 1, -2, 0), \qquad \tau(39) = -3$$

$$k(40) = (1, 0, 3, -3, 0, 0), \qquad \tau(40) = -3$$

$$k(41) = (1, -2, 5, -1, 0, 0), \qquad \tau(41) = -3$$

$$k(42) = (-1, 0, 7, 1, 0, 0), \qquad \tau(42) = -3$$

$$k(43) = (1, -2, 9, -1, 0, -2), \qquad \tau(43) = -2$$

$$k(44) = (-1, 0, 11, 1, 0, -2), \qquad \tau(44) = -2$$

$$k(45) = (1, -2, 13, -1, -2, 0), \qquad \tau(45) = -1$$

$$k(46) = (1, 0, -15, 1, -2, 0), \qquad \tau(46) = -1$$

$$k(47) = (3, -2, -13, -3, 0, 0), \qquad \tau(47) = -1$$

$$k(48) = (1, 0, -11, -1, 0, 0), \qquad \tau(48) = -2$$

$$k(49) = (1, -2, -9, 1, 0, 0), \qquad \tau(49) = -2$$

$$k(50) = (1, 0, -7, -1, 0, -2), \qquad \tau(50) = -2$$

$$k(51) = (1, -2, -5, 1, 0, -2), \qquad \tau(51) = -2$$

$$k(52) = (1, 0, -3, -1, -2, 0), \qquad \tau(52) = -2$$

$$k(53) = (1, -2, -1, 1, -2, 0), \qquad \tau(53) = -2$$

$$k(54) = (1, 0, 1, -3, 0, 0), \qquad \tau(54) = -2$$

$$k(55) = (1, -2, 3, -1, 0, 0), \qquad \tau(55) = -2$$

$$k(56) = (-1, 0, 5, 1, 0, 0), \qquad \tau(56) = -2$$

$$k(57) = (1, -2, 7, -1, 0, -2), \qquad \tau(57) = -1$$

$$k(58) = (-1, 0, 9, 1, 0, -2), \qquad \tau(58) = -1$$

$$k(59) = (1, -2, 11, -1, -2, 0), \qquad \tau(59) = 0$$

$$k(60) = (-1, 0, 13, 1, -2, 0), \qquad \tau(60) = 0$$

$$k(61) = (-3, -2, -15, -3, 0, 0), \qquad \tau(61) = 1$$

$$k_- = k(62) = (1, 0, -13, -1, 0, 0), \qquad \tau(62) = 0$$

As emphasized in Remark 4.4, this tau sequences include a lot of redundant elements. Thus, we describe the graded root R_{τ_K} *shown in Figure 5.7 according to reduced Tau sequence* $\{0, 1, -2, -1, -3, -1, -3, -1, -2, 1, 0\}$. *Note that the absolute minumum grading is* -3.

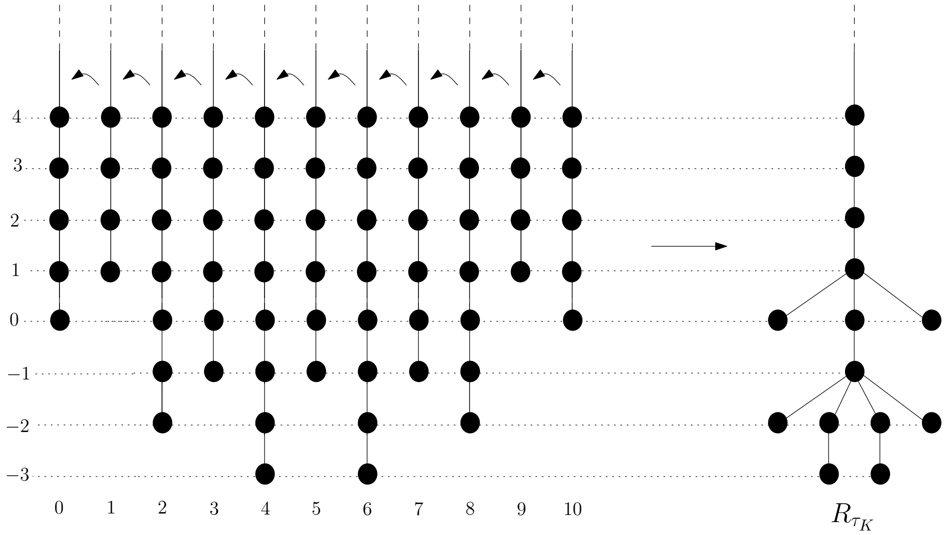


Figure 5.7. The graded root associated to $\Sigma(2, 7, 15)$

Finally we determine the Heegaard Floer homology by computing the homology of the graded root and shifting the degree by $(K^2 + 6)/4 - 2(-3)$ *where*

$$K^2 = \begin{bmatrix} -1 & 0 & 13 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -210 & -105 & -14 & -90 & -60 & -30 \\ -105 & -53 & -7 & -45 & -30 & -15 \\ -14 & -7 & -1 & -6 & -4 & -2 \\ -90 & -45 & -6 & -39 & -26 & -13 \\ -60 & -30 & -4 & -26 & -18 & -9 \\ -30 & -15 & -2 & -13 & -9 & -5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 13 \\ 1 \\ 0 \\ 0 \end{bmatrix} = -30$$

Hence $d(\Sigma(2, 7, 15)) = (-30 + 6)/4 - 2(-3) = 0$

Therefore, we get that

$$HF^{+}(-\Sigma(2,7,15)) = \mathcal{T}^{+}_{(0)} \oplus \mathbb{F}_{(0)}(2) \oplus \mathbb{F}_{(2)}(1) \oplus \mathbb{F}_{(2)}(1) \oplus \mathbb{F}_{(6)}(1) \oplus \mathbb{F}_{(6)}(1).$$

5.5. Brieskorn Homology Spheres of Projective Type

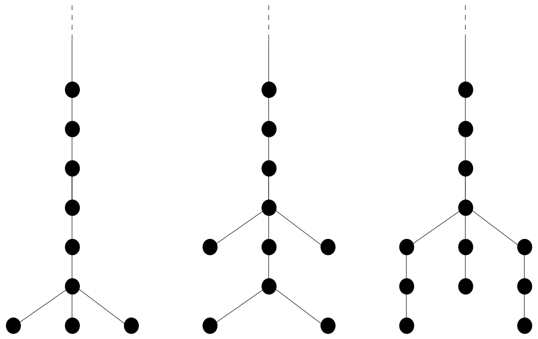
A Brieskorn homology sphere with spin structure $(Y, \mathfrak{s})$ is called *of projective type* if, for some constants d, n, m, a_i, m_i and some index set I, its Heegaard Floer homology is of the form

$$HF^{+}(Y, \mathfrak{s}) = \mathcal{T}^{+}_{(d)} \oplus \mathcal{T}^{+}_{(-2n+1)}(m) \oplus \bigoplus_{i \in I} \mathcal{T}^{+}_{(a_i)}(m_i)^{\oplus 2} \tag{5.17}$$

Example 5.25. *Following computations in the Section 5.4, we see that $\Sigma(2,3,7)$, $\Sigma(2,3,11)$ and $\Sigma(2,7,15)$ are all of projective type.*

Lemma 5.26 (Fact 8.8, [14])**.** *Let Y be a Brieskorn homology sphere with spin structure $\mathfrak{s}$. Let (R, χ) be the graded root associated to $(-Y, \mathfrak{s})$, and ι be the associated involution of R. Let $v \in R$ be the vertex of minimal grading which is invariant under ι. The space $(Y, \mathfrak{s})$ is of projective type if and only if there exists a vertex w, and a path from v to w in R which is grading decreasing at each step, with $\chi(w) = \min_{x \in R} \chi(x)$.*

A graded root is called *of projective type* if its homology is of the form 5.17, so that a Briekorn homology sphere is of projective type if and only if its graded root is. After the Lemma 5.26, we may give some explicit examples of graded roots of projective type, by referring to Figure 5.9.



(a) (b) (c)

Figure 5.8. The graded roots (a) and (b) are of projective type while (c) is not.

Example 5.27. *From the graded roots of* $\Sigma(2,3,7)$, $\Sigma(2,3,11)$ *and* $\Sigma(2,7,15)$ *described in Section 5.4, one can easily see that they are all of projective type.*

Theorem 5.28 (Theorem 8.9, [14])**.** *For* $k \geq 1$, *the family of* $\Sigma(2k+1, 4k+1, 4k+3)$ *is of projective type.*

Let $\mathcal{S}_{p,q} \subset \mathbb{Z}_{\geq 0}$ denote the semigroup

$$\mathcal{S}_{p,q} = \{ap + bq | (a,b) \in \mathbb{Z}^2_{\geq 0}\} \quad \text{and} \quad \alpha_i = \#\{s \notin \mathcal{S}_{p,q} | s > i\},$$

for relatively prime integers p and q. Also, set $g = \frac{(p-1)(q-1)}{2}$. Then Némethi showed that

Theorem 5.29 (Example 5.6.2, [27])**.** *For the family of Brieskorn sphere* $\Sigma(p,q,pqn+1)$,

$$HF^+(-\Sigma(p,q,pqn+1)) = \mathcal{T}_0^+ \oplus \mathcal{T}_0^+(\alpha_{g-1})^{\oplus n} \oplus \bigoplus_{i=1}^{n(g-1)} \mathcal{T}^+_{(\lfloor \frac{i}{n} \rfloor + 1)(\{\frac{i}{n}\}n+i)}(\alpha_{g-1+\lceil \frac{i}{n} \rceil})^{\oplus 2}$$

By reversing orientation, we get

$$\begin{aligned} HF^+(\Sigma(p,q,pqn+1)) = & \mathcal{T}_0^+ \oplus \mathcal{T}^+_{1-2\alpha_{g-1}}(\alpha_{g-1})^{\oplus n} \oplus \\ & \bigoplus_{i=1}^{n(g-1)} \mathcal{T}^+_{1-(\lfloor \frac{i}{n} \rfloor + 1)(\{\frac{i}{n}\}n+i) - 2\alpha_{g-1+\lceil \frac{i}{n} \rceil}}(\alpha_{g-1+\lceil \frac{i}{n} \rceil})^{\oplus 2} \end{aligned}$$

Then definition 5.17 implies that $\Sigma(p, q, pqn + 1)$ is of projective type. Note that $\Sigma(p, q, pqn + 1)$ can be obtained by $(-1/n)$ surgery on the right-handed (p, q)-torus knot, where $n \geq 1$, see [5].

Remark 5.30. *Let* $Y_n = \Sigma(3, 3n + 1, 9n + 4)$*. Taking* $p = 3, = 3n + 1$ *in Theorem 5.29, we conclude that* Y_n *is of projective type and* $d(Y_n) = 0$*.*

6. INVOLUTIVE HEEGAARD FLOER HOMOLOGY

In [12], Hendricks and Manolescu define *involutive Heegaard Floer homology* building on the work of Manolescu's Pin(2)-equivariant Seiberg-Witten Floer homology in the ordinary Heegaard Floer homology setting. Concretely, involutive Heegaard Floer homology is a Heegaard Floer analog of $\mathbb{Z}/4\mathbb{Z}$-equivariant Seiberg-Witten Floer homology in the following fashion.

Let Pin(2) be the group consisting of two copies of the complex unit circle with a map j interchanging them such that $ij = -ji$ and $j^2 = -1$, i.e.

$$\mathrm{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}$$

where $\mathbb{H}$ denotes quaternions. Note that $\mathbb{Z}/4\mathbb{Z}$ is the subgroup of Pin(2) generated by the element j. The $\mathbb{Z}/4\mathbb{Z}$-equivariant theory does not have the power of Pin(2)-equivariant one. For example, one cannot use it to disprove triangulation conjecture, because its homology cobordism invariants do not capture Rokhlin invariant, for more details see [12].

Now we shall describe the cornerstones of involutive Heegaard Floer theory:

Recall that the definition of Heegaard Floer theory starts with a pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ representing a 3-manifold Y, and there are four versions of this construction, denoted $\widehat{HF}$, HF^+, HF^-, and HF^∞. In general, the notation HF° uses to denote any of them, with $\circ \in \{\widehat{\ }, +, -, \infty\}$.

In Heegaard Floer theory, the set of Spin^c structures comes equipped with a natural involution. It is a conjugation symmetry $\mathfrak{s} \to \bar{\mathfrak{s}}$ switching the orientation of the surface Σ, as well as swapping the α and the β curves in the following fashion:

$$(\Sigma, \alpha, \beta, z) \to (-\Sigma, \beta, \alpha, z).$$

As noted in [Theorem 2.4, [22]], this induces an isomorphism

$$\mathcal{J}\colon\ HF^\circ(Y,\mathfrak{s}) \xrightarrow{\cong} HF^\circ(Y,\bar{\mathfrak{s}})$$

for any Spin^c structure $\mathfrak{s}$ on Y. We have $\mathcal{J}^2 = \mathrm{id}$, so $\mathcal{J}$ is a natural involution on Heegaard Floer homology groups $HF^\circ(Y)$.

Let $\varpi \in [\mathrm{Spin}^c(Y)]$ be an orbit of the form $\{\mathfrak{s},\bar{\mathfrak{s}}\}$ with $\mathfrak{s} \neq \bar{\mathfrak{s}}$. Recall that Heegaard Floer homology decomposes as direct sums, indexed by the Spin^c structures on Y, and there is a map $\iota\colon\ CF^\circ(Y) \to CF^\circ(Y)$ that induces the map $\mathcal{J} = \iota_*$ on homology. Set

$$CF^\circ(\mathcal{H},\varpi) := \bigoplus_{\mathfrak{s}\in\varpi} CF^\circ(\mathcal{H},\mathfrak{s}).$$

One can define *involutive Heegaard Floer complex* $CFI^\circ(\mathcal{H},\varpi)$ to be the mapping cone complex

$$1+\iota\colon\ CF^\circ(\mathcal{H},\varpi) \longrightarrow CF^\circ(\mathcal{H},\varpi). \tag{6.1}$$

The cone complex is

$$CF^\circ(\mathcal{H},\varpi)[-1] \oplus CF^\circ(\mathcal{H},\varpi)$$

with the first factor being the domain of $1+\iota$ and the second the target, where brackets denotes a degree shift in grading, i.e. $C[n]_k = C_{k+n}$. Let Q be a formal variable of degree -1, with $Q^2 = 0$. Then one can write 6.1 as

$$Q(1+\iota)\colon\ CF^\circ(\mathcal{H},\varpi) \longrightarrow Q\cdot CF^\circ(\mathcal{H},\varpi)[-1]. \tag{6.2}$$

Note that in the target shift $[-1]$ cancels out the shift due to the variable Q, so $Q\cdot CF^\circ(\mathcal{H},\varpi)[-1]$ is isomorphic to $CF^\circ(\mathcal{H},\varpi)$ as a graded module. More precisely,

$CFI^\circ(\mathcal{H},\varpi)$ denotes the complex with the underlying space $CF^\circ(\mathcal{H},\varpi)[-1] \oplus Q \cdot CF^\circ(\mathcal{H},\varpi)[-1]$. The differential map associated to this complex is

$$\partial^\iota = \begin{bmatrix} \partial & 0 \\ Q(1+\iota) & \partial \end{bmatrix}$$

where ∂ denotes the ordinary Heegaard Floer differential.

Recap that if one can work with coefficients in $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$, then Heegaard Floer groups are equipped with $\mathbb{F}[U]$-module structures, and thus one obtains a $\mathbb{F}[Q,U]/(Q^2)$-module structure on $HFI^\circ(Y)$ by construction with Q and U decreasing the grading by 1 and 2 respectively. Finally, we define the involutive Heegaard Floer homology $HFI^\circ(Y,\varpi)$ to be the homology of the complex $CFI^\circ(\mathcal{H},\varpi)$, i.e. $HFI^\circ(Y,\varpi) = Ker\partial^\iota / Im\partial^\iota$. Summing over all Spin^c-orbits, one sets

$$HFI^\circ(Y) := \bigoplus_{\varpi \in [\mathrm{Spin}^c(Y)]} HFI^\circ(Y,\varpi).$$

Then the main theorem of involutive Heegaard Floer homology occurs.

Theorem 6.1 (Theorem 1.1, [12]). *For any flavor $\circ \in \{\hat{}, +, -, \infty\}$, the isomorphism class of the involutive Heegaard Floer homology $HFI^\circ(Y)$, as a $\mathbb{Z}_2[Q,U]/(Q^2)$-module, is a 3-manifold invariant.*

As might be expected, there is a closed relationship between involutive Heegaard Floer homology groups and Heegaard Floer homology groups as an isomorphism of graded modules:

Proposition 6.2 (Proposition 4.5, [12]). *Let $\varpi \in [\mathrm{Spin}^c(Y)]$ be an orbit of the form $\{\mathfrak{s}, \bar{\mathfrak{s}}\}$ with $\mathfrak{s} \neq \bar{\mathfrak{s}}$. Then there is an isomorphism of graded $\mathbb{Z}_2[Q,U]/(Q^2)$-modules*

$$HFI^\circ(Y,\varpi) \cong HF^\circ(Y,\mathfrak{s})[-1] \oplus HF^\circ(Y,\mathfrak{s}).$$

6.1. New Correction Terms

Recap that the correction term $d(Y,\mathfrak{s})$ defined in [23] associated to a rational homology 3-sphere Y and a Spin^c structure $\mathfrak{s}$ is the minimal homological degree of non-zero element in the infinite tower of HF^+; or equivalently, the maximal homological degree of non-zero element in the infinite tail of HF^-, plus two:

$$\begin{aligned} d(Y,\mathfrak{s}) &= \min\{r \mid \exists\ x \in HF^+_r(Y,\mathfrak{s}), \forall\ n \geq 0, U^n x \neq 0\} \\ &= \max\{r \mid \exists\ x \in HF^-_r(Y,\mathfrak{s}), \forall\ n \geq 0, U^n x \neq 0\} + 2 \end{aligned}$$

In the paper [12], Hendricks and Manolescu mimic this construction to produce two new correction terms, namely *lower and upper involutive correction term*, denoted by $\underline{d}$ and $\bar{d}$, respectively. For $\mathfrak{s}$ a spin structure on Y, we consider the exact triangle (6.3),

$$\begin{tikzcd} HF^+(Y,\mathfrak{s}) \arrow[rr, "Q(1+\iota_*)"] & & Q \cdot HF^+(Y,\mathfrak{s})[-1] \arrow[dl, "g"] \\ & HFI^+(Y,\mathfrak{s}) \arrow[ul, "h"] & \end{tikzcd} \tag{6.3}$$

consisting of U-equivariant maps. For $r \gg 0$, we have that $HF^+_r(Y,\mathfrak{s})$ is either trivial or a one-dimensional $\mathbb{F}[U,U^{-1}]$. Since ι_* is an isomorphism, this implies that ι_* is the identity, so $Q(1+\iota_*)$ is trivial. Hence, the elements of $HF^+_r(Y,\mathfrak{s})$ for r large are of the form $h(x)$ for $x \in HFI^+(Y,\mathfrak{s})$, necessarily such that $x \in \mathrm{Im}(U^n), x \notin \mathrm{Im}(U^n Q)$ for $n \gg 0$. This allows us to define the *lower correction term* as follows.

$$\begin{aligned} \underline{d}(Y,\mathfrak{s}) =& \min\{r \mid \exists\ x \in HFI^+_r(Y,s), x \in \mathrm{Im}(U^n), x \notin \mathrm{Im}(U^n Q) \text{ for } n \gg 0\} - 1, \\ =& \max\{r \mid \exists\ x \in HFI^-_r(Y,\mathfrak{s}), \forall\ n \geq 0,\ U^n x \neq 0 \text{ and } U^n x \notin \mathrm{Im}(Q)\} + 1 \end{aligned}$$

On the other hand, if $y \in QHF^+_r(Y,\mathfrak{s})[-1]$ for r large, then $y \in \mathrm{Im}(U^n)$ and y is not in the image of $Q(1+\iota_*)$, so it must map to a non-zero element $x = g(y) \in \mathrm{Im}(U^n Q)$.

So, we define the *upper correction term* as

$$\begin{aligned}\bar{d}(Y,\mathfrak{s}) &= \min\{r \mid \exists\, x \in \mathit{HFI}^+_r(Y,s), x \neq 0,\ x \in \operatorname{Im}(U^n Q) \text{ for } n \gg 0\},\\ &= \max\{r \mid \exists\, x \in \mathit{HFI}^-_r(Y,\mathfrak{s}), \forall\, n \geq 0, U^n x \neq 0; \exists\, m \geq 0 \text{ s.t. } U^m x \in \operatorname{Im}(Q)\} + 2\end{aligned}$$

Note that the above shifts by 1 and 2 in the definitions of new correction terms are chosen so that $d = \underline{d} = \bar{d} = 0$ for $Y = S^3$.

Theorem 6.3 (Theorem 1.3, [12]). *The correction terms* $\underline{d}, \bar{d}$ *are invariants of* $\mathbb{Z}/2\mathbb{Z}$-*homology cobordism, i.e., they descend to (non-additive) maps*

$$\underline{d}, \bar{d}\colon\ \Theta^3_{\mathbb{Z}_2} \to \mathbb{Q}$$

When Y *is an integral homology sphere then* $\underline{d}$ *and* $\bar{d}$ *take even integer values, that is,*

$$\underline{d}, \bar{d}\colon\ \Theta^3_{\mathbb{Z}} \to 2\mathbb{Z}.$$

We shall list some fundamental properties of new correction terms:

Proposition 6.4 (Proposition 5.1, Proposition 5.2, [12]). *Let* $(Y,\mathfrak{s})$ *be a closed orientable* 3-*manifold equipped with a* spin *structure* $\mathfrak{s}$ *on* Y. *Then we have*

(i) $\bar{d}(Y,\mathfrak{s}) \equiv \underline{d}(Y,\mathfrak{s}) \equiv d(Y,\mathfrak{s}) \pmod{2\mathbb{Z}}$;
(ii) $\underline{d}(Y,\mathfrak{s}) \leq d(Y,\mathfrak{s}) \leq \bar{d}(Y,\mathfrak{s})$;
(iii) $\underline{d}(Y,\mathfrak{s}) = -\bar{d}(-Y,\mathfrak{s})$.

6.2. Involutive Heegaard Floer Homology of Plumbed 3-Manifolds

For the sake of exposition, Dai and Manelescu choose HFI^- (instead of HFI^+) while describing the involutive Heegaard Floer Homology of plumbed 3-Manifolds.

From now onward, we adopt the minus flavour of Heegaard Floer homology, as well as minus version of combinatorial module in the following way.

Let $Y = Y(G)$ be the plumbed 3-manifold associated to an AR graph G. Then the lattice isomorphism described in Theorem 5.15 transforms into minus flavor as follows: To each plumbing graph G and characteristic element k one can associate a graded root R_k. Then to each graded root R_k one can associate a lattice homology group $\mathbb{H}^-(G,k) = \mathbb{H}^-(R_k)$. Thus we have a lattice isomorphism

$$HF^-(Y(G),[k]) \cong \mathbb{H}^-(R_k)[\sigma + 2]$$

where $[k]$ denotes the Spin^c structure on $Y(G)$ associated to k, and $\sigma + 2$ denotes a grading shift, where $\sigma = -(|G| + k^2)/4$.

To understand the involutive Heegaard Floer homology, one can focus on self-conjugate $[k]$. In that case, there is a natural involution J_0 on $\mathbb{H}^-(R_k)$ coming from reflecting the graded root along its vertical axis.

Theorem 6.5 (Theorem 1.1, [13]). *Let $Y = Y(G)$ be the plumbed 3-manifold associated to an AR graph G. Let $\mathfrak{s} = [k]$ be a self-conjugate Spin^c structure on Y. Let also R_k be the graded root associated to (G,k), and let J_0 be the reflection involution on the lattice homology $\mathbb{H}^-(R_k)[\sigma + 2]$.*

Then, there is an isomorphism of graded $\mathbb{F}[U]$-modules

$$HFI^-(Y,\mathfrak{s}) \cong \ker(1 + J_0)[-1] \oplus \operatorname{coker}(1 + J_0).$$

Under this isomorphism, the action of Q on $HFI^-(Y,\mathfrak{s})$ is given by the quotient map

$$\ker(1 + J_0) \to \ker(1 + J_0)/\operatorname{im}(1 + J_0) \subseteq \operatorname{coker}(1 + J_0).$$

For the class of plumbed 3-manifolds, the following result holds for involutive Heegaard Floer correction terms: .

Theorem 6.6 (Theorem 1.2, [13]). *Let* $Y = Y(G)$ *be the plumbed* 3*-manifold associated to an AR graph* G. *Let* $\mathfrak{s}$ *be a self-conjugate* Spin^c *structure on* Y. *Then, the involutive Heegaard Floer correction terms are given by*

$$\underline{d}(Y,\mathfrak{s}) = -2\bar{\mu}(Y,\mathfrak{s}), \quad \bar{d}(Y,\mathfrak{s}) = d(Y,\mathfrak{s}),$$

where $\bar{\mu}(Y,\mathfrak{s})$ *is the Neumann-Siebenmann invariant and* $d(Y,\mathfrak{s})$ *is the d-invariant.*

On the other hand, there are nice connected sum formulas for these new homology cobordism invariants.

Theorem 6.7 (Theorem 1.3, Corollary 1.4, [13]). *Let* $Y_1, \dots, Y_k$ *be AR plumbed* 3*-manifolds equipped with self-conjugate* Spin^c *structures* $\mathfrak{s}_1, \dots, \mathfrak{s}_k$. *Suppose that*$(Y_i, \mathfrak{s}_i)$ *are all of projective type. Then we have*

(*a*) $\bar{d}(Y_1\#\dots\#Y_k, \mathfrak{s}_1\#\dots\#\mathfrak{s}_k) = d(Y_1\#\dots\#Y_k, \mathfrak{s}_1\#\dots\#\mathfrak{s}_k) = \sum_{i=1}^{k} d(Y_i, \mathfrak{s}_i)$,

(*b*) $\underline{d}(Y_1\#\dots\#Y_k, \mathfrak{s}_1\#\dots\#\mathfrak{s}_k) = \sum_{i=1}^{k-1} d(Y_i, \mathfrak{s}_i) - 2\bar{\mu}(Y_k, \mathfrak{s}_k)$.

7. MAIN THEOREM

Theorem 7.1 ([6], [7]). *The homology cobordism group $\Theta^3_{\mathbb{Z}}$ is infinitely generated, that is, it contains a $\mathbb{Z}^\infty$ subgroup.*

There are many infinite families of Seifert fibered homology spheres which are linearly independent in homology cobordism group which constitute $\mathbb{Z}^\infty$ subgroup. Originally Furuta proved in [6] that $\Sigma(2,3,6n-1)$ are linearly independent in $\Theta^3_{\mathbb{Z}}$ using Chern-Simons invariant.

Recently Stoffregen showed in [14] that $\Sigma(2k+1,4k+1,4k+3)$ are linearly independent in $\Theta^3_{\mathbb{Z}}$ using Pin(2)-equivariant Seiberg-Witten Floer homology. The same family was shown to be linearly independent in $\Theta^3_{\mathbb{Z}}$ by Dai and Manolescu using involutive Heegaard Floer homology. Their argument goes as follows:

Let Y_k denote the family of Brieskorn spheres $= \Sigma(2k+1,4k+1,4k+3)$ where $k \geq 1$.

By Theorem 5.28, we know that Y_k is of projective type. Furthermore, $d(Y_k) = -2k$ and $\bar{\mu}(Y_k) = 0$ by Proposition 5.14 and Example 4.7, respectively.

Using Theorem 6.7, we can calculate the involutive correction terms of connected sums of Y_k's. Indeed, if $k_1 \leq k_2 \leq \cdots \leq k_n$, then

$$\bar{d}(Y_{k_1}\# \dots \# Y_{k_n}) = \sum_{i=1}^{n} d(Y_{k_i}), \quad \underline{d}(Y_{k_1}\# \dots \# Y_{k_n}) = \sum_{i=1}^{n-1} d(Y_{k_i}).$$

Hence,

$$(\bar{d} - \underline{d})(Y_{k_1}\# \dots \# Y_{k_n}) = d(Y_{k_n}) = -2k_n. \tag{7.1}$$

We claim that the classes $[Y_k] \in \Theta^3_{\mathbb{Z}}$ are linearly independent, thus they generate a $\mathbb{Z}^\infty$ group isomorphic to subgroup of $\Theta^3_{\mathbb{Z}}$. Assume for a contradiction that we had a linear relation between the classes $[Y_k]$. By grouping the terms according to the sign of their coefficients, we can write the relation as

$$[Y_{k_1}] + [Y_{k_2}] + \cdots + [Y_{k_n}] = [Y_{k'_1}] + [Y_{k'_2}] + \cdots + [Y_{k'_m}], \tag{7.2}$$

where

$$k_1 \leq k_2 \leq \cdots \leq k_n, \quad k'_1 \leq k'_2 \leq \cdots \leq k'_m,$$

and

$$k_i \neq k'_j \text{ for any } i, j.$$

Since $\bar{d}$ and $\underline{d}$ are homology cobordism invariants, so is their difference $\bar{d} - \underline{d}$. By applying (7.1) to both sides of the identity (7.2), we obtain that $k_n = k'_m$, which leads a contradiction.

Here, we show another linearly independent family by using the argument of Dai and Manolescu. Our results are mainly based on Némethi's computations presented in Section 5.5.

Consider the family of Brieskorn spheres $Y_n = \Sigma(3, 3n+1, 9n+4)$ where $n \geq 1$. By Remark 5.30, we know that Y_n is of projective type and $d(Y_n) = 0$. Moreover, $\bar{\mu}(Y_k) = n$ by Example 4.8.

Suppose that $n_1 \leq n_2 \leq \cdots \leq n_k$. Then by Theorem 6.7, we write

$$\bar{d}(Y_{n_1} \# \ldots \# Y_{n_k}) = \sum_{i=1}^{k} d(Y_{n_i}), \quad \underline{d}(Y_{n_1} \# \ldots \# Y_{n_k}) = \sum_{i=1}^{k-1} d(Y_{n_i}) - 2\bar{\mu}(Y_{n_k}).$$

Thus, we obtain

$$(\bar{d} - \underline{d})(Y_{n_1}\# \ldots \# Y_{n_k}) = 2\bar{\mu}(Y_{n_k}) = 2n_k. \tag{7.3}$$

We claim that the classes $[Y_n] \in \Theta^3_{\mathbb{Z}}$ are linearly independent. Suppose for a contradiction that we had a linear relation between the classes $[Y_n]$. By grouping the terms according to the sign of their coefficients, we can write the relation as

$$[Y_{n_1}] + [Y_{n_2}] + \cdots + [Y_{n_k}] = [Y_{n'_1}] + [Y_{n'_2}] + \cdots + [Y_{n'_l}], \tag{7.4}$$

where

$$n_1 \leq n_2 \leq \cdots \leq n_k, \quad n'_1 \leq n'_2 \leq \cdots \leq n'_l,$$

and

$$n_i \neq n'_j \text{ for any } i, j.$$

Since $\bar{d}$ and $\underline{d}$ are homology cobordism invariants, so is their difference $\bar{d} - \underline{d}$. By applying (7.3) to both sides of the identity (7.4), we get that $n_k = n'_l$, which is a contradiction.

8. CONCLUSION

In this thesis, we demonstrate several combinatorial techniques for computing homology cobordism invariants of plumbed 3-manifolds. These invariants are powerful enough to prove linear independence of several families of Brieskorn spheres. Some computer search suggest that these families exist in abundance. Curiously, one cannot prove linear independence of Furuta's original family using Heegaard Floer type invariants. Hence there are still room for improvement.

On the other hand, basic group theoretic questions related to $\Theta^3_{\mathbb{Z}}$ are still unanswered; for example, we do not know whether $\Theta^3_{\mathbb{Z}}$ has any torsion or not. Our current homology cobordism invariants are all additive with respect to connected sum. Therefore, we cannot detect any torsion in $\Theta^3_{\mathbb{Z}}$ using them. Furthermore, we do not know whether $\Theta^3_{\mathbb{Z}}$ is in fact $\mathbb{Z}^\infty$ or contains $\mathbb{Z}^\infty$ summand. But in virtue of Frøyshov's work, we know that it has at least a $\mathbb{Z}$ summand generated by the Poincaré homology sphere.

Answering these crucial and precious questions will continue to be related to these homology cobordism invariants. To derive more information about the group structure of $\Theta^3_{\mathbb{Z}}$, we will undoubtedly need new and unusual homology cobordism invariants.

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