# FINITE-ELEMENT ANALYSIS
# OF
# RADIATIVE TRANSPORT IN GRAY PARTICIPATING MEDIA

by

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*Vahan Kalenderoğlu*

## ABSTRACT

A finite-element solution scheme for radiative transport problems in gray participating media is devised, and its validity is substantiated through application to representative problems involving plane-parallel geometry with azimuthal symmetry.

The governing boundary value problem of radiative transport in gray participating media is first posed, and its simplification in the case of plane-parallel geometry with azimuthal symmetry is considered in great detail with emphasis on the physics assumed in the simplification process. To provide the necessary basis for the application of the finite-element approximation technique, the governing boundary value problem is formulated in the weak sense, and subsequently the Galerkin approximation of the resulting weak formulation is stated. In the weak formulation, and therefore in its corresponding Galerkin approximation, boundary conditions are incorporated as natural rather than essential conditions. The advantages of such an approach are clear and are discussed briefly. After a short discussion of the relevant concepts of the finite-element approximation technique, the finite-element model of the Galerkin approximation of the weak formulation of the governing boundary value problem is developed. The resulting equations describing this model are simple, well-conditioned algebraic equations. With the general underlying theory thus established, a specific finite-element model applicable to any radiative transport problem in plane-parallel

azimuthally symmetric gray participating media is derived, with particular emphasis laid on accounting for the angular discontinuities in the intensity distribution. For the sake of simplicity, linear rectangular finite elements are incorporated in this specific model. For this particular choice of elements, specific expressions for the relevant element matrices and the element vectors are derived and presented with the intention of facilitating the tailoring of a finite-element solution scheme to problems in the application range of the aforementioned specific model. Finally, the application of this specific model to selected homogeneous as well as nonhomogeneous problems, which are well-documented, is considered. Numerical results obtained for these problems are tabulated and compared with the corresponding exact results reported in the literature. The agreement between the finite-element and the corresponding exact results is seen to be highly satisfactory.

On the basis of the aforementioned theoretical and numerical results, it is found that the finite-element approximation technique provides an efficient and reliable solution scheme for radiative transport problems in gray participating media.

# ÖZET

Yayıcı, emici ve saçıcı gri ortamlarda ışıma problemlerinin çözümü için bir sonlu eleman yöntemi geliştirilmiş, yöntemin geçerliliği, paralel düzlem geometrisi ile belirlenen örnek problemlerdeki uygulamalar ile gösterilmiştir. Bu yöntemin uygulanışında seçilen problemlerin tümünde açısal simetrinin olduğu varsayılmıştır.

Yayıcı, emici ve saçıcı gri ortamlarda ışınım ile taşınma olayının matematiksel modelini oluşturan genel sınır değer problemi verilmiştir. Daha sonra, bu problem paralel düzlem geometrisi ve açısal simetri varsayımı ile indirgenmiş ve sonuçlar ayrıntılı bir şekilde incelenmiştir. İndirgeme sürecinde yapılan kabuller üzerinde önemle durulmuştur. Sonlu elemanlar yönteminin uygulanabilmesi için gerekli temeli hazırlayabilmek amacıyla söz konusu sınır değer probleminin zayıf şekli türetilmiş ve bu şeklin Galerkin yaklaşıklılığı verilmiştir. Zayıf şeklin ve buna karşı gelen Galerkin yaklaşıklılığında sınır koşulları çözüm sınıfını kısıtlayan koşullar olmaktan çok doğal koşullar olarak düşünülmüştür. Böyle bir yaklaşımın üstünlükleri kısa olarak ayrıca tartışılmıştır. Sonlu eleman yöntemi ile ilgili temel kavramlar özet olarak verildikten sonra ilgili sınır değer probleminin zayıf şeklinin Galerkin yaklaşıklılığının sonlu eleman modeli geliştirilmiştir. Bu modeli belirleyen denklemler basit

ve çözülebilirlilik açısından uygun davranışlı cebirsel denklemlerdir. Bu temel modele dayanarak paralel düzlem geometrisi ve açısal simetrisi olan ışıma problemlerine uygulanabilir bir model türetilmiş, paralel düzlem problemlerine özgü yeğinlik dağılımının sınır düzlemlerindeki açısal süreksizliliğinin içerilmesi üzerinde önemle durulmuştur. Modelde doğrusal sonlu elemanlar kullanılmış ve bu eleman türü için eleman matrisleri ve vektörleri türetilmiştir. Bu model daha önce başka yöntemlerle incelenmiş olan homojen ve homojen olmayan problemlere uygulanmış, elde edilen yaklaşık sonuçlar literatürde verilmiş olan analitik çözümler ile karşılaştırıldığında çok iyi bir uyum sağlandığı görülmüştür.

Bu çalışmada elde edilen sonuçlar geliştirilen sonlu eleman yönteminin yayıcı, emici ve saçıcı gri ortamlarda ışıma problemlerinin çözümünde kullanılabilecek etkin ve güvenilir bir yöntem olduğunu kanıtlamıştır.

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| | |
|---|---|
| $E$ | Three-dimensional Euclidean Space |
| $R$ | A closed and bounded region in $E$ |
| $\dot{R}$ | Interior of the region $R$ |
| $\partial R$ | Boundary of the region $R$ |
| $\partial R^s$, $s = 1,2,\ldots,S$ | Complementary regular subsurfaces of the boundary $\partial R$, s being a positive scalar |
| $\overline{R}$ | Closure of the region $R$ |
| $\tilde{R}$ | Finite-element model of the region $R$ |
| $\partial\tilde{R}$ | Finite-element model of the boundary $\partial R$ |
| $\partial\tilde{R}^s$, $s = 1,2,\ldots,S$ | Finite-element models of the complementary regular subsurfaces $\partial R^s$, $s = 1,2,\ldots,S$ |
| $\underset{\sim}{x} = (x_1,x_2,x_3)$ | A point in $R$ having coordinates $x_1$, $x_2$ and $x_3$ with respect to an orthogonal coordinate system |
| $\underset{\sim}{x} = (x,y,z)$ | A point in $R$ having coordinates x, y and z with respect to a Cartesian coordinate system |
| $\underset{\sim}{n}$ | Unit outward normal of the boundary $\partial R$ at a point $\underset{\sim}{x} \in \partial R$ |
| $z_o$ | Thickness or half thickness |
| $\tau$ | Optical distance |
| $\tau_o$ | Optical thickness or optical half thickness |

| | |
|---|---|
| $[0,t_o)$ | A time interval, $t_o$ being a positive scalar |
| $t$ | Time variable |
| $\Omega$ | Collection of points on the surface of the unit ball with center at a point $\underset{\sim}{x} \varepsilon \bar{R}$ |
| $\theta$ | Polar angle about a fixed axis |
| $\mu$ | Cosine of the polar angle $\theta$ |
| $\phi$ | Azimuthal angle about a fixed axis |
| $\underset{\sim}{\omega} = (\omega_1,\omega_2,\omega_3)$, $\underset{\sim}{\omega}' = (\omega_1',\omega_2',\omega_3')$ | Points, with spherical coordinates $(\theta,\phi)$ and $(\theta',\phi')$, on the surface of the unit ball centered at a point $\underset{\sim}{x} \varepsilon \bar{R}$; $\omega_1 = \sin\theta\cos\phi$, $\omega_2 = \sin\theta\sin\phi$ and $\omega_3 = \cos\theta$, and $\omega_1' = \sin\theta'\cos\phi'$, $\omega_2' = \sin\theta'\sin\phi'$ $\omega_3' = \cos\theta'$ |
| $\underset{\sim}{\omega}^i$ | Direction of incident radiation |
| $\underset{\sim}{\omega}^s$ | Direction of scattered radiation |
| $\underset{\sim}{\omega}^r$ | Direction of reflected radiation |
| $\mu_o = \underset{\sim}{\omega}^i \cdot \underset{\sim}{\omega}^s$ | Scattered angle |
| $\Omega^+$ | Collection of points defined as $\{\underset{\sim}{\omega} \mid \underset{\sim}{\omega} \varepsilon \Omega$, $\Omega$ being the unit ball with center at a point $\underset{\sim}{x} \varepsilon \partial R$, and $\underset{\sim}{n}\cdot\underset{\sim}{\omega} > 0$, $\underset{\sim}{n}$ being the unit outward normal of $\partial R$ at $\underset{\sim}{x} \varepsilon \partial R\}$ |
| $\Omega^-$ | Collection of points defined as $\{\underset{\sim}{\omega} \mid \underset{\sim}{\omega} \varepsilon \Omega$, $\Omega$ being the unit ball with center at a point $\underset{\sim}{x} \varepsilon \partial R$, and $\underset{\sim}{n}\cdot\underset{\sim}{\omega} < 0$, $\underset{\sim}{n}$ being the unit outward normal of $\partial R$ at $\underset{\sim}{x} \varepsilon \partial R\}$ |
| $\tilde{\Omega}$ | Finite-element model of the region $\Omega$ |
| $\tilde{\Omega}^+$ | Finite-element model of the region $\Omega^+$ |
| $\tilde{\Omega}^-$ | Finite-element model of the region $\Omega^-$ |
| $d\upsilon$ | An infinitesimal element in the region $R$ |

| | |
|---|---|
| $ds$ | An infinitesimal element on the boundary $\partial R$ |
| $d\tau$ | An infinitesimal element in the $\tau$-region |
| $d\omega$ | An infinitesimal element in the region $\Omega$ |
| $d\mu$ | An infinitesimal element in the $\mu$-region |
| $d\phi$ | An infinitesimal element in the $\phi$-region |
| $\Psi$ | Intensity distribution |
| $\alpha$ | Reduced absorption coefficient |
| $\sigma$ | Scattering coefficient |
| $\beta$ | Extinction coefficient |
| $\gamma$ | Albedo |
| $K$ | Scattering Kernel |
| $K_q = K_q(\cdot)$ | Legendre polynomial of order q |
| $a_q$, $q = 0,1,2,\ldots,Q$ | Expansion coefficients, Q being a positive integer |
| $K_q^p = K_q^p(\cdot)$, $p = 1,2,\ldots,q$; $q = 0,1,2,\ldots,Q$ | Associated Legendre polynomials, Q being a positive integer |
| $n$ | Index of refraction |
| $\Theta$ | Temperature distribution |
| $f$ | Blackbody intensity distribution |
| $s_o$ | Source strength |
| $c_o$ | Velocity of light in vacuum |
| $h$ | Planck constant |
| $k$ | Boltzmann constant |
| $T_{(s)}$, $s = 1,2,\ldots,S$ | Intensity distributions over the surfaces $\partial R^s$, $s = 1,2,\ldots,S$, due to transmission |
| $\varepsilon_{(s)}$, $s = 1,2,\ldots,S$ | Emissivities of the surfaces $\partial R^s$, $s = 1,2,\ldots,S$ |

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| $g_{(s)}$, $s = 1,2,\ldots,S$ | Reflectivities of the surfaces $\partial R^s$, $s = 1,2,\ldots,S$ |
| $\rho$ | Hemispherical reflectivity of a surface |
| $\rho^d_{(1)}$, $\rho^d_{(1)}$ | Hemispherical diffuse reflectivities of the surfaces 1 and 2 |
| $\rho^s_{(2)}$, $\rho^s_{(2)}$ | Hemispherical specular reflectivities of the surfaces 1 and 2 |
| $H$, $H^*$, $G$ | Solution spaces |
| $H^m$ | An m-dimensional subspace in the space $H$, m being a positive integer |
| $G^m$ | An m-dimensional subspace in the space $G$, m being a positive integer |
| $\tilde{H}$ | Finite-element model of the space $H$ |
| $\tilde{G}$ | Finite-element model of the space $G$ |
| $\tilde{H}^P$ | A P-dimensional finite-element subspace in the space $H$, P being a positive integer |
| $\tilde{G}^P$ | A P-dimensional finite-element subspace in the space $G$, P being a positive integer |
| $\tilde{\Psi} = \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega})$, $\tilde{\Phi} = \tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega})$ | Functions in the finite-dimensional spaces $H^m$ and $\tilde{H}^P$ |
| $\tilde{\Psi} = \tilde{\Psi}(\tau,\mu)$, $\tilde{\Phi} = \tilde{\Phi}(\tau,\mu)$ | Functions in the finite-dimensional spaces $G^m$ and $\tilde{G}^P$ |
| $\tilde{\Phi}_i = \tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega})$, $i = 1,2,\ldots,m$ | A set of basis functions spanning the space $H^m$, m being a positive integer |
| $\tilde{\Phi}_i = \tilde{\Phi}_i(\tau,\mu)$, $i = 1,2,\ldots,m$ | A set of basis functions spanning the space $G^m$, m being a positive integer |
| $\tilde{\psi}_i$, $\tilde{\phi}_i$, $i = 1,2,\ldots,m$ | Expansion coefficients, m being a positive integer |
| $\tilde{e},\hat{e},e$, $\tilde{e}',\hat{e}',e'$, $\tilde{e}^*,\hat{e}^*,e^*$ | Element labels |
| $\tilde{E}$ | Number of elements in a spatial region |
| $\hat{E}$ | Number of elements in an angular region |

| | |
|---|---|
| $E$ | Number of elements in a region which is the Cartesian product of spatial and angular regions |
| $\tilde{P}$ | Number of nodal points in a spatial region |
| $\hat{P}$ | Number of nodal points in an angular region |
| $P$ | Number of nodal points in a region which is the Cartesian product of spatial and angular regions |
| $P_{\tilde{e}}, \ \tilde{e} = 1,2,\ldots,E$ | Number of nodal points in an element $r_{\tilde{e}}$ or $\tau_{\tilde{e}}$, $\tilde{e} = 1,2,\ldots,\tilde{E}$ |
| $P_{\hat{e}}, \ \hat{e} = 1,2,\ldots,\hat{E}$ | Number of nodal points in an element $\omega_{\hat{e}}$ or $\mu_{\hat{e}}$, $\hat{e} = 1,2,\ldots,\hat{E}$ |
| $P_e, \ e = 1,2,\ldots,E$ | Number of nodal points in an element $d_e$, $e = 1,2,\ldots,E$ |
| $r_{\tilde{e}}, \tau_{\tilde{e}}, \ \tilde{e} = 1,2,\ldots,\tilde{E}$ | Finite-elements in a spatial region |
| $\omega_{\hat{e}}, \mu_{\hat{e}}, \ \hat{e} = 1,2,\ldots,\hat{E}$ | Finite-elements in an angular region |
| $d_e, \ e = 1,2,\ldots,E$ | Finite-elements in a region which is the Cartesian product of spatial and angular regions |
| $r_e, \tau_e, \ e = 1,2,\ldots,E$ | Spatial subregions in an element $d_e$, $e = 1,2,\ldots,E$ |
| $\omega_e, \mu_e, \ e = 1,2,\ldots,E$ | Angular subregions in an element $d_e$, $e = 1,2,\ldots,E$ |
| $\partial r_e^{\,s}, \omega_e^{\pm}, \omega_{\hat{e}'}^{\pm}, \mu_e^{\pm}, \mu_{\hat{e}'}^{\pm}, \ \hat{e}, \hat{e}' = 1,2,\ldots,\hat{E}$ | Subregions in boundary elements |
| $\underset{\sim}{x}^{\tilde{\Delta}}, \tau^{\tilde{\Delta}}, \ \tilde{\Delta} = 1,2,\ldots,\tilde{P}$ | Coordinates of global nodes in spatial regions |
| $\underset{\sim}{\omega}^{\hat{\Delta}}, \mu^{\hat{\Delta}}, \ \hat{\Delta} = 1,2,\ldots,\hat{P}$ | Coordinates of global nodes in angular regions |
| $(\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta}, \ (\tau,\mu)^{\Delta}, \Delta = 1,2,\ldots,P$ | Coordinates of global nodes in regions which are the Cartesian product of spatial and angular regions |

| | | |
|---|---|---|
| $\underset{\sim}{x}_{(\tilde{e})}{}^{\tilde{N}}$, $\tau_{(\tilde{e})}{}^{\tilde{N}}$, | $\tilde{e} = 1,2,\ldots,\tilde{E}$, $N = 1,2,\ldots,P_{\tilde{e}}$ | Coordinates of local nodes in spatial regions |
| $\underset{\sim}{\omega}_{(\hat{e})}{}^{\hat{N}}$, $\mu_{(\hat{e})}{}^{\hat{N}}$, | $\hat{e} = 1,2,\ldots,\hat{E}$ $\hat{N} = 1,2,\ldots,P_{\hat{e}}$ | Coordinates of local nodes in angular regions |
| $(\underset{\sim}{x},\underset{\sim}{\omega})_{(e)}{}^{N}$, $(\tau,\mu)_{(e)}{}^{N}$, | $e = 1,2,\ldots,E$ $N = 1,2,\ldots,P_e$ | Coordinates of the local nodes in regions which are the Cartesian product of spatial and angular regions |
| $\tilde{C}$, $\hat{C}$, $B$ | | Correspondance relations |
| $I_{(e)}$, $I_{(e)}{}^{-1}$ | | Incidence relations |
| $\underset{\sim}{I}_{(e)}$, $\underset{\sim}{I}_{(e)}{}^{-1}$, | $e = 1,2,\ldots,E$ | Incidence matrices |
| $\tilde{\psi}^{(e)}$, $\tilde{\phi}^{(e)}$, | $e = 1,2,\ldots,E$ | Restrictions of the functions $\tilde{\Psi}$ and $\tilde{\Phi}$ to an element $d_e$, $e = 1,2,\ldots,E$ |
| $\tilde{\Phi}_{\Delta}$ , $\Delta = 1,2,\ldots,P$ | | Global basis functions |
| $\tilde{\Phi}_{\Delta}{}^{(e)}$, | $\Delta = 1,2,\ldots,P$, $e = 1,2,\ldots,E$ | Restrictions of global basis functions to an element $d_e$, $e = 1,2,\ldots,E$ |
| $\tilde{\Phi}^{(e)}{}_{N}$, | $N = 1,2,\ldots,P_e$, $e = 1,2,\ldots,E$ | Local basis functions |
| $\tilde{\psi}^{\Delta}$, $\tilde{\phi}^{\Delta}$, $\Delta = 1,2,\ldots,P$ | | Global expansion coefficients |
| $\tilde{\psi}_{(e)}{}^{N}$, $\tilde{\phi}_{(e)}{}^{N}$, | $e = 1,2,\ldots,E$ $N = 1,2,\ldots,P_e$ | Local expansion coefficients |
| $\underset{\sim}{\tilde{\Phi}}_{(e)}$, | $e = 1,2,\ldots,E$ | Local basis vectors |
| $\underset{\sim}{\tilde{\psi}}$, $\underset{\sim}{\tilde{\phi}}$, | | Coefficient vectors |
| $\underset{\sim}{T}$ | | Transport matrix |
| $\underset{\sim}{T}_{(e)}$, | $e = 1,2,\ldots,E$ | Element transport matrices |
| $\underset{\sim}{E}$ | | Extinction matrix |
| $\underset{\sim}{E}_{(e)}$, | $e = 1,2,\ldots,E$ | Element extinction matrices |
| $\underset{\sim}{S}$ | | Scattering matrix |
| $\underset{\sim}{S}_{(e,e')}$, | $e,e' = 1,2,\ldots,E$ | Element scattering matrices |
| $\underset{\sim}{B}^{(s)}$, | $s = 1,2,\ldots,S$ | Boundary matrices |

| | | |
|---|---|---|
| $\underset{\sim}{B}^{(s)}{}_{(e)}$, | $s = 1,2,\ldots,S,$ <br> $e = 1,2,\ldots,E$ | Element boundary matrices |
| $\underset{\sim}{R}^{(s)}$, | $s = 1,2,\ldots,S$ | Reflection matrices |
| $\underset{\sim}{R}^{(s)}{}_{(e,e')}$, | $s = 1,2,\ldots,S,$ <br> $e,e' = 1,2,\ldots,E$ | Element reflection matrices |
| ${}^{d}\underset{\sim}{R}^{(s)}$, | $s = 1,2,\ldots,S$ | Diffuse reflection matrices |
| ${}^{d}\underset{\sim}{R}^{(s)}{}_{(e,e')}$, | $s = 1,2,\ldots,S,$ <br> $e,e' = 1,2,\ldots,E$ | Element diffuse reflection matrices |
| ${}^{s}\underset{\sim}{R}^{(s)}$, | $s = 1,2,\ldots,S$ | Specular reflection matrices |
| ${}^{s}\underset{\sim}{R}^{(s)}{}_{(e,e')}$, | $s = 1,2,\ldots,S,$ <br> $e,e' = 1,2,\ldots,E$ | Element specular reflection matrices |
| $\underset{\sim}{s}$ | | Source vector |
| $\underset{\sim}{s}_{(e)}$, | $e = 1,2,\ldots,E$ | Element source vectors |
| $\underset{\sim}{t}^{(s)}$, | $s = 1,2,\ldots,S$ | Transmission vectors |
| $\underset{\sim}{t}^{(s)}{}_{(e)}$, | $s = 1,2,\ldots,S,$ <br> $e = 1,2,\ldots,E$ | Element transmission vectors |
| $\underset{\sim}{e}^{(s)}$, | $s = 1,2,\ldots,S$ | Emission vectors |
| $\underset{\sim}{e}^{(s)}{}_{(e)}$, | $s = 1,2,\ldots,S,$ <br> $e = 1,2,\ldots,E$ | Element emission vectors |
| $\delta = \delta(\cdot)$ | | Delta function |
| $\delta_i^{\,j}$ | | Kronecker Delta |

# INTRODUCTION

Under the impact of modern technological developments in hypersonic flight, power plants for space exploration needs, high temperature energy conversion devices, high temperature chemical processing equipments, astrophysics and meteorology, there has been a dramatic growth of research activity in various aspects of heat transfer. The resulting applications have necessitated the solution of extremely complex problems involving radiative transport in a participating medium as well as the interaction of radiation with other modes of heat transfer, namely conduction and convection. Although extensive effort has been applied in solving problems in this field, even the solution of relatively simple problems involving radiative transport in the absence of the interaction of radiation with other modes of heat transfer is still quite complicated due to the difficul in the solution of the radiative transport equation. The treatment of prob lems involving radiative transport in the presence of the interaction of radiation with other modes of heat transfer, on the other hand, is even more involved because they require a simultaneous solution of the radiative transport equation and the classical energy equation. It is our purpose here to develop a finite-element solution scheme for the radiative transpor equation, and thus to alleviate the major difficulty encountered in solving both the aforementioned problems. It is of course, not possible in a study of this nature to examine all of the variations and refinements of the method. We have, therefore, considered only the fundamental concepts and a few relatively simple applications. However, irrespective of the degree and nature of complexity, the transport problems governed by the radiative transport equation are open to analysis by the present method without undue complications.

Since the classical works of Lord Rayleigh in 1871 [1]†, the theory of radiative transport and its applications have been of major concern to astrophysicists and engineers. Chandrasekhar [2], Kourganoff [3], and Sobolev [4] have introduced analytical methods to solve the radiative transport equation in a limited number of idealized applications. Their methods, however, are rather complicated analytically and can only deal with extremely idealized problems of radiative transport. Recently, Case's normal mode expansion technique has been applied with considerable success to the solution of the radiative transport equation [5-10]. Although this method is both mathematically and numerically involved, it provides a systematic and powerful approach for the analytical solution of the radiative transport equation in applications involving plane-parallel media. However, its extension to more complex problems is extremely difficult, if not impossible. The mathematical difficulties inherent in these analytical methods on the other hand have prompted a number of approximations, and consequently the development of approximate

† Numbers enclosed in brackets refer to the references cited at the end.

solution schemes based on these approximations has paralleled that of the analytical methods. The most widely used approximate methods of solution are that of Schuster and Schwarzchild introduced in 1905 and 1906 [12,13], that of Jeans introduced in 1917 [14], that of Roseland introduced in 1936 [15], and that of Eddington introduced in 1960 [16]. Equally widespread are the methods of discrete ordinates [2,3] and moments [17]. As in the case of the analytical methods, these approximate solution schemes are also limited to relatively simple problems of radiative transport, and, therefore, are inadequate to deal with the highly complex problems encountered in real life applications. A concise discussion of the aforementioned analytical methods as well as the approximate solution schemes is given in reference [10].

Recent rapid advances together with the ever-spreading use of the finite-element approximation technique in the solution of problems in structural mechanics and elasticity, and the recognition that this technique is applicable to any field problem which can be formulated in terms of an integral law have raised the possibility that it can be extended to problems of radiative transport. Although the development of the finite-element approximation technique has paralleled the development of the theory of radiative transport and the consequent need for numerical solution of the radiative transport equation, it has largely been ignored by engineers and scientists working in the field of radiative transport. On hindsight, this is quite surprising in view of the superior results obtained from the applications in various fields such as elasticity [18], heat conduction [19], thermoelasticity [20], thermoviscoelasticity [21], neutron transport and diffusion [22-24], and fluid flow [25].

The finite-element approximation technique has emerged from the classical works of the early investigators in the fields of structural and solid mechanics as one of the most powerful techniques devised for the approximate solution of boundary value problems of engineering science. A relatively complete historical account is given in [27], and the present state of the art can be found in the works of Zienkiewicz [26], Oden [27], Strang and Fix [28], Oden and Reddy [30], and Norrie and Vries [29,31].

Though the finite-element approximation technique would appear to be rather involved, the underlying principle is in fact relatively simple and parallels that of the classical methods of the Ritz-Galerkin type, where the solution of a given boundary value problem is approximated in terms of a function in a finite dimensional function space spanned by a finite number of basis functions and where the expansion coefficients of the approximation function are determined by some integral law, either of residual or variational nature. In the cases where the integral law is of residual nature, the expansion coefficients are determined by requiring the residual associated with the field equation involved in the boundary problem to vanish identically in some weighted average sense over the region of interest. On the other hand, in the cases where the integral law is of variational nature, the expansion coefficients are determined by forcing the approximating function to extremize a functional that has the field equation involved in the boundary value problem as its Euler equation. As in any application of the aforementioned classical methods, the finite-element approximation technique, therefore, unavoidably requires a global statement of the relevant boundary value problem in terms of an integral law. Accordingly, this statement can be of residual or variational nature, the choice depending largely on the specific type of the boundary value problem.

Assuming that such a global statement is achieved, then the finite-element approximation technique consists of representing the region of interest by a collection of a finite number of geometrically simple subregions connected together at certain points, and subsequently of identifying for each subregion a set of basis functions defined locally over the relevant subregion. The local basis functions are generally chosen to be simple polynomials. With the identification of the local basis functions, the integral law is formulated approximately over each subregion. The global approximation of the integral law is then obtained by simply assembling the subregions together to depict the region of interest, and summing the contributions of each element.

The significance of this process of formulating a global approximation from a number of local ones is that it embodies a systematic method for constructing basis functions for arbitrary regions, consequently alleviating the traditional difficulties due to irregular geometries and boundary conditions that are inherent in the aforementioned classical methods [32,33]. Moreover, the basis functions generated in this way are usually piecewise polynomials with local compact support;that is, they assume nonzero values only in a relatively small neighborhood of certain points. This feature normally leads to computational advantages in that the resulting equations are well conditioned and in that the system matrices involved in the resultant set of equations are banded - highly desirable advantages in the development of a solution algorithm and in its subsequent computer implementation. It is on account of these features that the finite-element approximation technique derives its power and lends itself to extensive application in virtually every area of engineering science.

The present study is divided into three major parts. The first part deals with the mathematical foundations where all the groundwork necessary for the subsequent application of the finite-element approximation technique is laid down. It starts with the statement of the boundary value problem governing radiative transport in participating media and proceeds to the development of an equivalent weak formulation in terms of an integral law. It concludes with the development of the corresponding approximate integral law, which provides a basis for the application of the finite-element approximation technique. The second part is devoted to the corresponding finite-element formulation. It starts with the construction of the finite-element models of the analytical region and the solution space associated with the governing boundary value problem, and proceeds with the derivation of the finite-element equations. The last part is concerned with some selected applications of the theoretical results obtained in the second part.

To keep the present study reasonably self-contained and at the same time to preserve the continuity of the text, it is felt necessary at this point to define the terminology and the notation adopted.

The standard set theoretics notation is employed throughout. That is, the symbol $\varepsilon$ is used to indicate membership; $\not\varepsilon$ to negate $\varepsilon$; $\{0\}$ to denote the null or the empty set; and, given two sets A and B, $A \subset B$ and $A \subseteq B$ are written to indicate that A is a subset of B and A is a proper subset of B, respectively. $A = \{a | \text{ a has property P}\}$ is written to identify a set A of quantities all having the defining property P, and similarly, $A = \{a_i\}_{i=1}^{N}$ to identify a set A of a finite number N of quantities $a_i$, $i = 1,2,\ldots,N$, N being a positive scalar. The union, intersection, sum and difference of two sets A and B are denoted by $A \cup B$, $A \cap B$, $A+B$, and $A-B$, respectively. $A\times B$ stands for the Cartesian product of two sets A and B, where $A\times B = \{(a,b) | \ a \in A \text{ and } b \in B\}$.

Let $A_1$ and $A_2$ denote nonempty subsets of a set A. If $A_1$ and $A_2$ have no elements in common, then $A_1 \cap A_2 = \{0\}$ and the sets $A_1$ and $A_2$ are said to be disjoint.

The usual notation $R$ is adopted to denote a closed and bounded region in three-dimensional Euclidean space $E$, with the interior $\dot{R}$ and boundary $\partial R$. $\bar{R}$ is written for the closure of the region $R$. Unless otherwise specified, the boundary $\partial R$ is to be understood as the union of a finite number of complementary regular subsurfaces $\partial R_s \subset \partial R$, $s = 1,2,\ldots,S$, S being a positive integer. $\underset{\sim}{x} = (x_1,x_2,x_3)$ is written for a generic point of $\bar{R}$ if the coordinate system is a general one, and, in particular, $\underset{\sim}{x} = (x,y,z)$ is written if the coordinate system is Cartesian. The infinitesimal elements of $\dot{R}$ and $\partial R$ are denoted by $dv$ and $ds$, respectively, and the unit outward normal of $\partial R$ at any point $x \in \partial R$, by $n$.

By a time interval is meant an interval of the form $[0,t_o)$ where $t_o$ is any positive scalar. $t$ is written to denote any value in the interval $[0,t_o)$.

$\Omega$ is written to denote the collection of points on the surface of the unit ball with center at a point $\underset{\sim}{x} \varepsilon R$, and the notation $\underset{\sim}{\omega} = (\omega_1,\omega_2,\omega_3)$ is used to identify a point in $\Omega$ with spherical coordinates $(\theta,\phi)$ with respect to any fixed orthogonal coordinate system. Then $\omega_1 = \sin\theta \cos\phi$, $\omega_2 = \sin\theta \sin\phi$ and $\omega_3 = \cos\theta$. For any point $\underset{\sim}{x} \varepsilon \partial R$, $\Omega^+$ and $\Omega^-$ are written to denote the sets $\{\underset{\sim}{\omega}|\underset{\sim}{\omega} \varepsilon \Omega$, $\Omega$ being the unit ball with center at $\underset{\sim}{x} \varepsilon \partial R$, and $\underset{\sim}{n}\cdot\underset{\sim}{\omega} > 0$, $\underset{\sim}{n}$ being the unit outward normal of $\partial R$ at $\underset{\sim}{x} \varepsilon \partial R\}$, and $\{\underset{\sim}{\omega}|\underset{\sim}{\omega} \varepsilon \Omega$ $\Omega$ being the unit ball with center at $\underset{\sim}{x} \varepsilon \partial R$, and $\underset{\sim}{n}\cdot\underset{\sim}{\omega} < 0$, $\underset{\sim}{n}$ being the unit outward normal of $\partial R$ at $\underset{\sim}{x} \varepsilon \partial R\}$, respectively. The notation $d\omega$ is used to denote an infinitesimal element in $\Omega$.

We will be dealing with real valued functions of position and time with real valued functions of position and direction. If $f = f(\underset{\sim}{x},t)$ is a function defined over $R\times[0,t_o)$, and if $g = g(\underset{\sim}{x},\underset{\sim}{\omega})$ is a function defined over $R\times\Omega$, then $f(.,t)$, $f(\underset{\sim}{x},.)$, $g(.,\underset{\sim}{\omega})$ and $g(\underset{\sim}{x},.)$ respectively are written for the subsidiary mappings of $R$ holding $t$ fixed, of $[0,t_o)$ holding $\underset{\sim}{x}$ fixed, of $R$ holding $\underset{\sim}{\omega}$ fixed and of $\Omega$ holding $\underset{\sim}{x}$ fixed. Let $I$ and $J$ be nonnegative integers. Then, $f$ is said to be of class $C^{I,J}$ over $R\times[0,t_o)$ if $f(\underset{\sim}{x},.)$ is of class $C^I$ over $R$ and $f(.,t)$ is of class $C^J$ over $[0,t_o)$, and $g$ is said to be of class $C^{I,J}$ over $R\times\Omega$ if $g(\underset{\sim}{x},.)$ is of class $C^I$ over $R$ and $g(.,\underset{\sim}{\omega})$ is of class $C^J$ over $\Omega$. Let $h = h(\underset{\sim}{x},\underset{\sim}{\omega},\underset{\sim}{\omega}')$ be a real valued function having $R\times\Omega^+\times\Omega^-$ as its domain of definition. Likewise, $h$ is said to be of class $C^{I,J,K}$ if the subsidiary mappings $h(.,\underset{\sim}{\omega},\underset{\sim}{\omega}')$ of $R$

holding $\underset{\sim}{\omega}$ and $\underset{\sim}{\omega}'$ fixed, $h(\underset{\sim}{x},.,\underset{\sim}{\omega}')$ of $\Omega^+$ holding $\underset{\sim}{x}$ and $\underset{\sim}{\omega}'$ fixed and $h(\underset{\sim}{x},\underset{\sim}{\omega},.)$ of $\Omega^-$ holding $\underset{\sim}{x}$ and $\underset{\sim}{\omega}$ fixed are respectively of class $C^I$ over $R$, of class $C^J$ over $\Omega^+$ and of class $C^K$ over $\Omega^-$, where I, J and K are any nonnegative integers.

Usual vector and matrix notation is adopted throughout. Any symbol with a tilde underneath should be understood to be either a vector or a matrix. Whether such a symbol stands for a vector or a matrix ought to be clear from the context. The notations {.} and [.] are respectively used to denote column vectors and matrices. The transpose of a vector or a matrix is indicated by a superscript T.

Whenever appropriate, the indicial notation and the summation convention are used. Regarding the use of indices, lower case Latin indices are employed either to indicate an entry in an array or to label a physical quantity. Whenever confusion is likely to occur, the indices used for labelling purposes are enclosed in parenthesis. Upper case Latin indices are used to indicate entries in system arrays, whereas upper case Greek indices are used to incate entries in element arrays. Likewise, upper case Latin and Greek indices are used to indicate quantities defined at the nodal points - Latin indices pertaining to local nodes and the Greek indices pertaining to global nodes. Regarding the use of the summation convention, in the case of indices enclosed in parenthesis or in the case of indices with a dot underneath, the summation convention does not apply.

Terminology and notation not covered in this short discussion but used throughout the text are defined and explained where they first appear.

## PART ONE

## WEAK AND GALERKIN FORMULATIONS OF THE BOUNDARY VALUE PROBLEM ASSOCIATED WITH RADIATIVE TRANSPORT IN GRAY PARTICIPATING MEDIA

Due to the inherent nature of the finite element approximation technique, finite element analysis of any physical problem requires a translation of the associated local mathematical model, which is generally formulated in terms of local field equations subject to certain initial and/or boundary conditions, into a global model formulated in terms of integral laws that hold over the entire region. It is our intention to present, in this part, a translation of this sort in the case of problems involving radiative transport within a gray participating medium, and, thus, to lay down the necessary framework on which the finite element analysis of such problems can be based. To this end, we first pose the corresponding boundary value problem, and, for future use, derive its particular form as applicable to problems in plane-parallel gray participating media with azimuthal symmetry. Having thus established the formulation of the local mathematical model, we then proceed to formulate an equivalent global model in terms of an integral law. We finally cast this integral law into an approximate form which provides a basis for the application of the finite element approximation technique.

## 1.1 The Boundary Value Problem Associated with Radiative Transport in Gray Participating Media

Let us consider a gray participating medium confined into a closed and bounded region $R \subset E$ with boundary $\partial R$. By a gray participating medium we mean a material continuum, which is, in general, capable of absorbing, emitting and scattering radiation, with radiative properties independent of frequency. Let the boundary $\partial R$ be composed of complementary regular subsurfaces $\partial R^s$ with unit outward normal $\underset{\sim}{n}$ at any point $\underset{\sim}{x} \in \partial R^s$, $s = 1,2,\ldots S$ being a positive integer. If the participating medium is characterized by

(i) a distribution of reduced absorption coefficient
$\alpha = \alpha(x)$ over $R$,

(ii) a distribution of scattering coefficient $\sigma = \sigma(x)$ over $R$,

(iii) a scattering kernel $K = K(\underset{\sim}{\omega}^i \cdot \underset{\sim}{\omega}^s)$ normalized such that

$$\int_\Omega K(\underset{\sim}{\omega}^i \cdot \underset{\sim}{\omega}^s)d\omega^s = 1,$$

where $\underset{\sim}{\omega}^i$, $\underset{\sim}{\omega}^s \in \Omega$ and denote, respectively, the unit vectors along the propagation directions of incident and scattered radiation at a point $\underset{\sim}{x} \in R$, where $\underset{\sim}{\omega}^i \cdot \underset{\sim}{\omega}^s$ is the scattering angle at the point $\underset{\sim}{x} \in R$, and where the infinitesimal element $d\omega$ in $\Omega$ is labeled with the superscript $s$ to indicate that the integration over the region $\Omega$ is with respect to the variable $\underset{\sim}{\omega}^s$ [10, p. 28; 11, p.7],

and (iv) a constant index of refraction $n$;

and if each subsurface $\partial R^s$, $s = 1,2,\ldots,S$, is characterized by

(i) a distribution of emissivity $\varepsilon_{(s)} = \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})$ over $\partial R_s \times \Omega$, and

(ii) a distribution of reflectivity $g_{(s)} = g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}^i,\underset{\sim}{\omega}^r)$ over $\partial R^s \times \Omega$, where $\underset{\sim}{\omega}^i \varepsilon \Omega^+$ and $\underset{\sim}{\omega}^r \varepsilon \Omega^-$ denote, respectively, the unit vectors along the propagation directions of incident and reflected radiation at a point $\underset{\sim}{x} \varepsilon \partial R^s$ ,

then, assuming that a local thermodynamic equilibrium is established at every point $\underset{\sim}{x} \varepsilon \overline{R}$ and, furthermore, that polarization effects are negligible, the governing equations of radiative transport within a gray participating medium can be written in the $\underset{\sim}{x}$-$\underset{\sim}{\omega}$ coordinate system, Figure 1.1.1, as follows [10, pp.244-253 and pp. 273-275; 11, pp. 10-16].

$$\underset{\sim}{\omega}\cdot\text{grad}\Psi^t(\underset{\sim}{x},\underset{\sim}{\omega}) + \beta(\underset{\sim}{x})\Psi^t(\underset{\sim}{x},\underset{\sim}{\omega}) = \alpha(\underset{\sim}{x})f[\Theta(\underset{\sim}{x},t)] + \sigma(\underset{\sim}{x})\int_\Omega K(\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})\Psi^t(\underset{\sim}{x},\underset{\sim}{\omega}')d\omega' \quad \text{over } R \times \Omega, \tag{1.1.1}$$

and

$$\Psi^t(\underset{\sim}{x},\underset{\sim}{\omega}) = T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) + \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})f[\Theta(\underset{\sim}{x},\underset{\sim}{t})] + \int_{\Omega^+} g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega})\Psi^t(\underset{\sim}{x},\underset{\sim}{\omega}')|\underset{\sim}{n}\cdot\underset{\sim}{\omega}'|d\omega' \quad \text{over } \partial R^s \times \Omega^-, \quad s = 1,2,\ldots,S, \tag{1.1.2}$$

where $\Psi = \Psi(\underset{\sim}{x},\underset{\sim}{\omega})$ is the intensity distribution over $\overline{R} \times \Omega$ ,

$$\beta = \beta(\underset{\sim}{x}) = \alpha(\underset{\sim}{x}) + \sigma(\underset{\sim}{x}) \tag{1.1.3}$$

and is referred to as the extinction coefficient [10, p. 253], $T_{(s)} = T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})$ is the intensity distribution over $\partial R^s \times \Omega$ due to transmission, $s = 1,2,\ldots,S$, and

$$f = f[\Theta(\underset{\sim}{x},t)] = \frac{2}{15}\,\frac{n^2}{c_o^2}\,\frac{\pi^4 k^4}{h^3}\,\Theta^4(\underset{\sim}{x},t)$$

and is referred to as the blackbody intensity distribution [10, p.20]. In Eq. (1.1.4) $c_o$ is the velocity of light in vacuum, $k$ is the Boltzmann constant, $h$ is the Planck constant, and $\Theta = \Theta(\underset{\sim}{x},t)$ is the distribution of temperature over $\bar{R} \times [0,t_o)$.

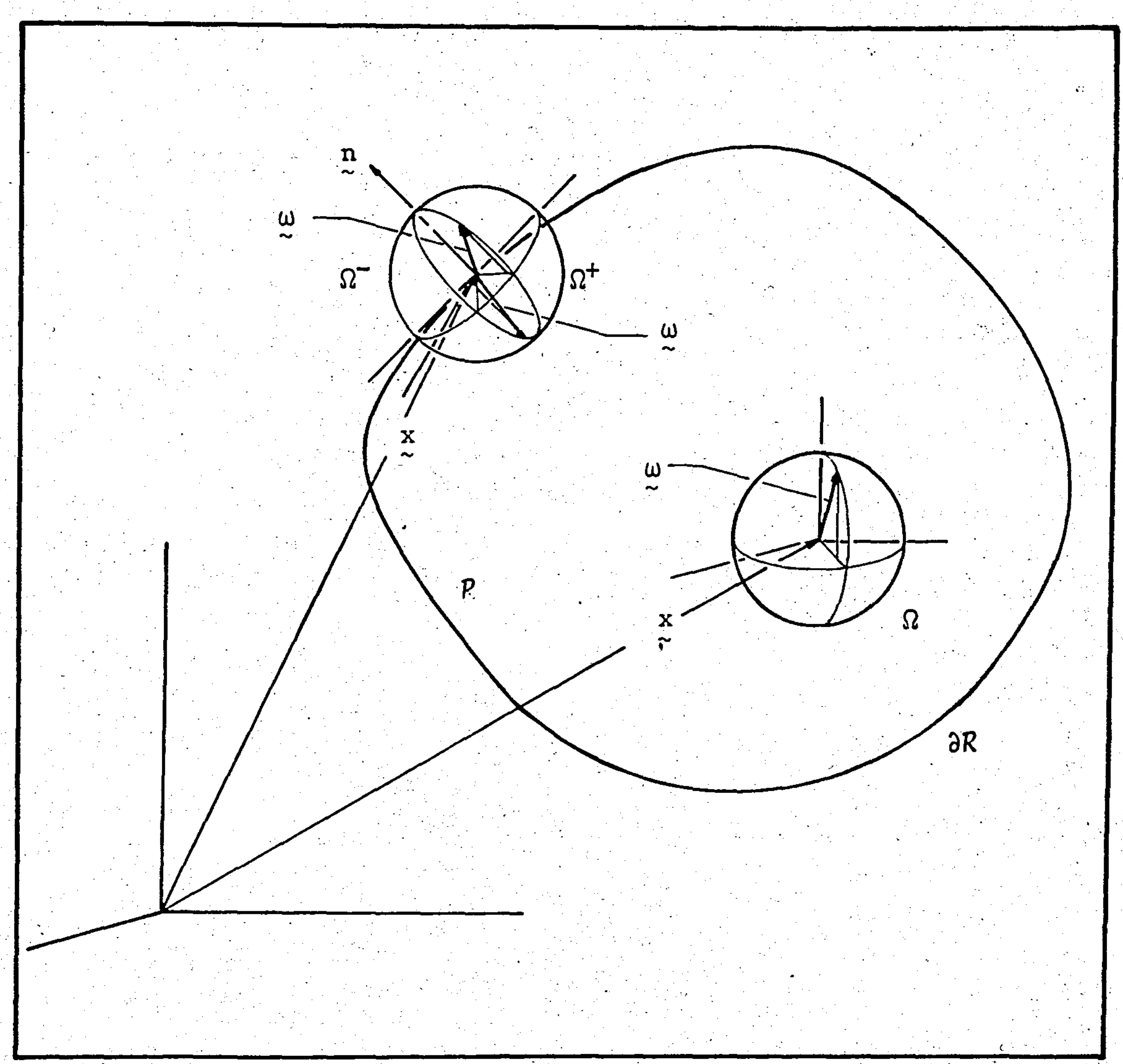


FIGURE 1.1.1 - The $\underset{\sim}{x}$-$\underset{\sim}{\omega}$ coordinate system

In Eqs. (1.1.1) and (1.1.2) the notation $(\cdot)^t$ is adopted to emphasize that the intensity distribution may, in fact, depend on time. It is worthwhile noting that this time dependence is simply a consequence of the time dependence of the temperature distribution, and, therefore, it is parametric in nature rather than being explicit. For the sake of simplicity in notation, we will suppress the superscript $t$. However, it must be kept in mind that an implicit time dependence may exist.

To avoid repeated hypotheses, in general, we assume once and for all that

(i) $\alpha = \alpha(\underset{\sim}{x})$ and $\sigma = \sigma(\underset{\sim}{x})$ are of class $C^o$ over $R$,

(ii) $K = K(\underset{\sim}{\omega}^i \cdot \underset{\sim}{\omega}^s)$ is a continuous function of scattering angle $\underset{\sim}{\omega}^i \cdot \underset{\sim}{\omega}^s$ where $\underset{\sim}{\omega}^i, \underset{\sim}{\omega}^s \varepsilon\ \Omega$,

(iii) $T_{(s)} = T_{(s)}(\underset{\sim}{x}, \underset{\sim}{\omega})$ is of class $C^{o,o}$ over $\partial R^s \times \Omega^-$, $s = 1,2,\ldots,S$,

(iv) $\varepsilon_{(s)} = \varepsilon_{(s)}(\underset{\sim}{x}, \underset{\sim}{\omega})$ is of class $C^{o,o}$ over $\partial R^s \times \Omega^-$, $s = 1,2,\ldots,S$,

(v) $g_{(s)} = g_{(s)}(\underset{\sim}{x}, \underset{\sim}{\omega}^i, \underset{\sim}{\omega}^r)$ is of class $C^{o,o,o}$ over $\partial R^s \times \Omega^+ \times \Omega^-$, $s = 1,2,\ldots,S$,

(vi) $\Theta = \Theta(\underset{\sim}{x}, t)$ is of class $C^{o,o}$ over $R \times [0, t_o)$.

The mathematical model based on the integro-partial differential equation (1.1.1) and the boundary conditions (1.1.2) is general enough, at least within the limits of the physics assumed thus far, to characterize radiative transport involved, in a very large class of engineering problems dealing with heat transfer within a gray participating medium, provided that, at any given instant of time, the temperature distribution is somehow known a priori. Hereafter, we will restrict our attention to the boundary value problem stated in terms of Eqs. (1.1.1) and (1.1.2) for a given temperature distribution over $\bar{R} \times [0,t_o)$, and briefly refer to this problem as the radiative transport problem.

Referring to a Cartesian coordinate system, for a plane-parallel gray participating medium of infinite extent in the x- and y- directions and of finite extent in the z-direction, $0 \leq z \leq z_o$, $z_o$ being an arbitrary positive scalar, Figure 1.1.2, we have

$$\Psi = \Psi(z,\mu,\phi)$$

and, therefore, the boundary value problem (1.1.1-2) reduces to

$$\mu \frac{\partial}{\partial z} \Psi(z,\mu,\phi) + \beta(z)\Psi(z,\mu,\phi) = \alpha(z)f[\Theta(z,t]$$

$$+ \sigma(z)\int_0^{2\pi}\int_{-1}^{1} K(\mu_o)\Psi(z,\mu',\phi)d\mu'd\phi' \qquad (1.1.5)$$

for $0 < z < z_o$, $-1 \leq \mu \leq 1$ and $0 \leq \phi < 2\pi$,

$$\Psi(0,\mu,\phi) = T_{(1)}(\mu,\phi) + \varepsilon_{(1)}(\mu,\phi)f[\Theta(0,t)]$$

$$- \int_0^{2\pi}\int_{-1}^{0} g_{(1)}(\mu',\phi',\mu,\phi)\Psi(0,\mu',\phi')\mu'd\mu'd\phi' \qquad (1.1.6)$$

for $0 < \mu \leq 1$ and $0 \leq \phi < 2\pi$,

$$\Psi(z_o,\mu,\phi) = T_{(2)}(\mu,\phi) + \varepsilon_{(2)}(\mu,\phi)f[\Theta(z_o,t)]$$

$$+ \int_0^{2\pi}\int_0^{1} g_{(2)}(\mu',\phi',\mu,\phi)\Psi(z_o,\mu',\phi')\mu'd\mu'd\phi' \qquad (1.1.7)$$

for $-1 \leq \mu < 0$ and $0 \leq \phi < 2\pi$.

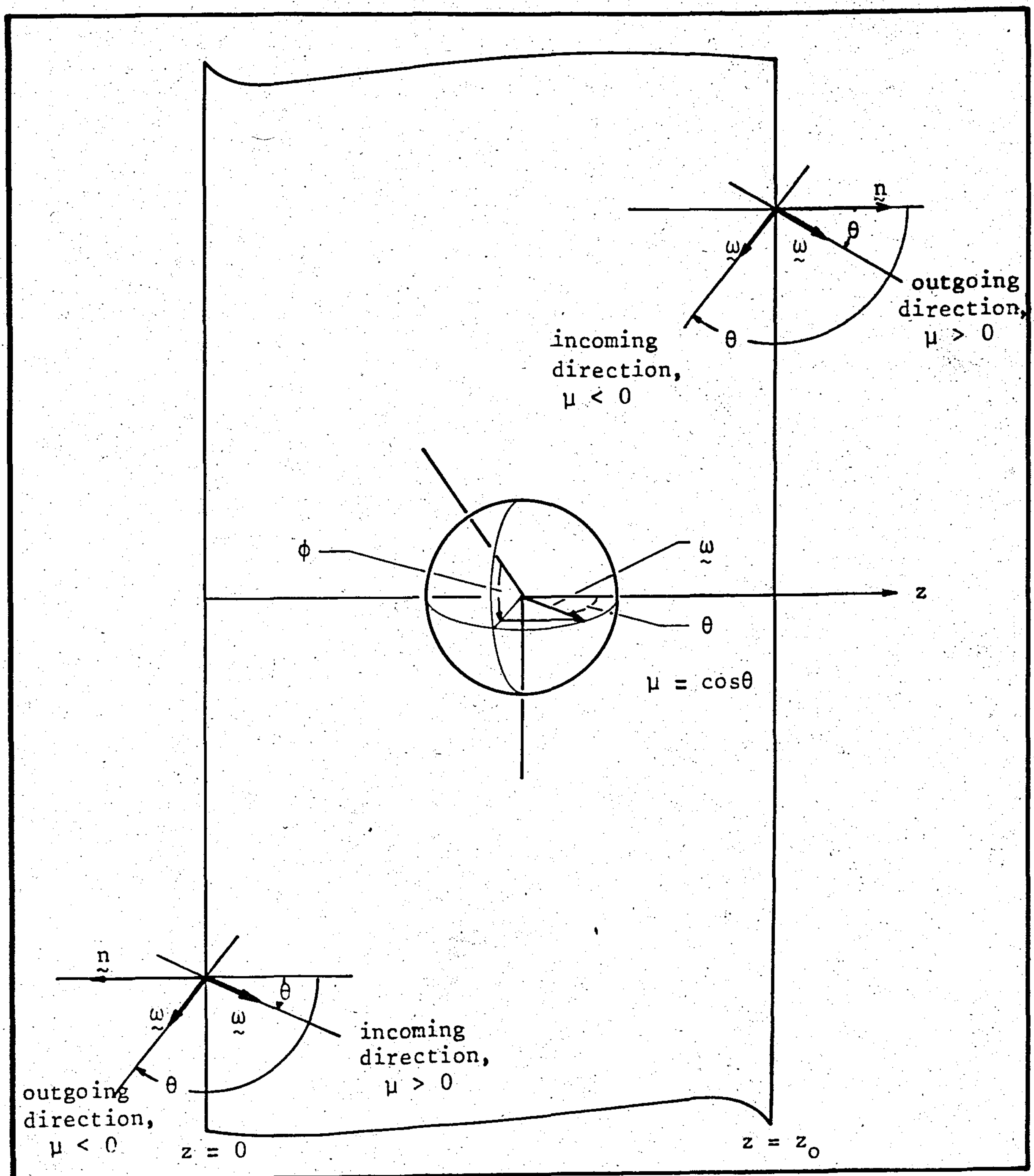


FIGURE 1.1.2 - The coordinate system for a plane-parallel medium

In Eq. (1.1.5)

$$\mu_o = \mu\mu' + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2}\cos(\phi - \phi') , \quad \mu,\mu'\varepsilon[-1,1], \qquad (1.1.8)$$

[10, p.29] and, in Eqs. (1.1.6) and (1.1.7), the subscripts 1 and 2 refer to the surfaces at $z = 0$ and $z = z_o$, respectively.

If, furthermore, the reflectivities of the surfaces at $z = 0$ and $z = z_o$ can be expressed as a sum of diffuse and specular components[10,p.79], then

$$g_{(1)}(\mu',\phi',\mu,\phi) = g^d_{(1)} + \frac{1}{\mu} g^s_{(1)}(\mu',\phi')\delta(\mu' + \mu)\delta(\phi' - \phi), \qquad (1.1.9)$$

and

$$g_{(2)}(\mu',\phi',\mu,\phi) = g^d_{(2)} + \frac{1}{\mu} g^s_{(2)}(\mu',\phi')\delta(\mu' + \mu)\delta(\phi' - \phi), \qquad (1.1.10)$$

where the superscripts d and s refer, respectively, to diffuse and specular reflections; $g^d_{(1)}$ and $g^d_{(2)}$ are constants; and $\delta(\cdot)$ is the Delta function.
Using, respectively, Eqs. (1.1.9) and (1.1.10) in Eqs. (1.1.6) and (1.1.7) we get

$$\Psi(0,\mu,\phi) = T_{(1)}(\mu,\phi) + \varepsilon_{(1)}(\mu,\phi)f[\Theta(0,t)] - g^d_{(1)}\int_0^{2\pi}\int_{-1}^{0} \Psi(0,\mu',\phi')\mu'd\mu'd\phi'$$

$$+ g^s_{(1)}(-\mu,\phi)\Psi\ 0,-\mu,\phi) \qquad (1.1.11)$$

$$\text{for} \quad 0 < \mu \leq 1 \quad \text{and} \quad 0 \leq \phi < 2\pi,$$

and

$$\Psi(z_o,\mu,\phi) = T_{(2)}(\mu,\phi) + \varepsilon_{(2)}(\mu,\phi) f\, \Theta(z_o,t) + g^d_{(2)}\int_0^{2\pi}\int_0^{-1} \Psi(z_o,\mu',\phi')\mu' d\mu' d\phi'$$

$$- g^s_{(2)}(-\mu,\phi)\Psi(z_o,-\mu,\phi) \qquad (1.1.12)$$

for $-1 \leq \mu < 0$ and $0 \leq \phi < 2\pi$.

On the other hand, from the definition of the hemispherical reflectivity $\phi$ of a surface [10,p. 39] it follows that

$$\rho^d_{(1)} = g^d_{(1)} \int_0^{2\pi}\int_0^1 \mu d\mu d\phi = \pi g^d_{(1)}, \qquad (1.1.13)$$

$$\rho^d_{(2)} = -g^d_{(2)}\int_0^{2\pi}\int_{-1}^0 \mu d\mu d\phi = \pi g^d_{(2)}, \qquad (1.1.14)$$

$$\rho^s_{(1)} = \frac{\int_0^{2\pi}\int_0^1 g^s_{(1)}(-\mu,\phi)\Psi(0,-\mu,\phi)\mu d\mu d\phi}{\int_0^{2\pi}\int_{-1}^0 \Psi(0,-\mu,\phi)\mu d\mu d\phi}, \qquad (1.1.15)$$

and

$$\rho^s_{(2)} = \frac{\int_0^{2\pi}\int_{-1}^0 g^s_{(1)}(-\mu,\phi)\Psi(z_o,-\mu,\phi)\mu d\mu d\phi}{\int_0^{2\pi}\int_{-1}^0 \Psi(z_o,-\mu,\phi)\mu d\mu d\phi}, \qquad (1.1.16)$$

where $\rho^d_{(1)}$ and $\rho^s_{(1)}$ are, respectively, the hemispherical diffuse and specular reflectivities of the surface at $z = 0$; and $\rho^d_{(2)}$ and $\rho^s_{(2)}$ are, respectively, the hemispherical diffuse and specular reflectivities of the surface at $z = z_o$. Now, if $g^s_{(1)}$ and $g^s_{(2)}$ are independent of $\mu$ and $\phi$, then from Eqs. (1.1.15) and (1.1.16), respectively, it follows that

$$\rho^s_{(1)} = g^s_{(1)}, \tag{1.1.17}$$

and

$$\rho^s_{(2)} = -g^s_{(2)}. \tag{1.1.18}$$

Using Eqs. (1.1.13) and (1.1.17) in Eq. (1.1.11), and Eqs. (1.1.14) and (1.1.18) in Eq. (1.1.12) we obtain, respectively,

$$\Psi(0,\mu,\phi) = T_{(1)}(\mu,\phi) + \varepsilon_{(1)}(\mu,\phi)f[\Theta(0,t)] - \frac{\rho^d_{(1)}}{\pi}\int_0^{2\pi}\int_{-1}^{0} \Psi(0,\mu',\phi')\mu' d\mu' d\phi' + \rho^s_{(1)}\Psi(0,-\mu,\phi)$$

$$\text{for} \quad 0 < \mu \leq 1 \quad \text{and} \quad 0 \leq \phi < 2\pi, \tag{1.1.19}$$

and

$$\Psi(z_o,\mu,\phi) = T_{(2)}(\mu,\phi) + \varepsilon_{(2)}(\mu,\phi)f[\Theta(z_o,t)] + \frac{\rho^d_{(2)}}{\pi}\int_0^{2\pi}\int_0^{1} \Psi(z_o,\mu',\phi')\mu' d\mu' d\phi' + \rho^s_{(2)}\Psi(z_o,-\mu,\phi)$$

$$\text{for} \quad -1 \leq \mu < 0 \quad \text{and} \quad 0 \leq \phi < 2\pi. \tag{1.1.20}$$

Now, if, in addition, there exists azimuthal symmetry, then

$$\Psi = \Psi(z,\mu),$$

$$T_{(1)} = T_{(1)}(\mu),$$

$$\varepsilon_{(1)} = \varepsilon_{(1)}(\mu),$$

$$T_{(2)} = T_{(2)}(\mu),$$

and

$$\varepsilon_{(2)} = \varepsilon_{(2)}(\mu).$$

Thus, the set of equations (1.1.5), (1.1.19) and (1.1.20) further reduces to

$$\mu \frac{\partial}{\partial z} \Psi(z,\mu) + \beta(z)\Psi(z,\mu) = \alpha(z)f[\Theta(z,t)]$$
$$+ \sigma(z) \int_{-1}^{1}\left[\int_{0}^{2\pi} K[\mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi-\phi')]d\phi'\right]\Psi(z,\mu')d\mu'$$
$$\text{for} \quad 0 < z < z_o \quad \text{and} \quad -1 \leq \mu \leq 1, \qquad (1.1.21)$$

$$\Psi(0,\mu) = T_{(1)}(\mu) + \varepsilon_{(1)}(\mu)f[\Theta(0,t)] - 2\rho^{d}_{(1)}\int_{-1}^{0} \Psi(0,\mu')\mu'd\mu'$$
$$+ \rho^{s}_{(1)} \Psi(0,-\mu) \qquad (1.1.22)$$
$$\text{for} \quad 0 < \mu \leq 1,$$

$$\Psi(z_o,\mu) = T_{(2)}(\mu) + \varepsilon_{(2)}(\mu)f[\Theta(z_o,t)] + 2\rho^{d}_{(2)}\int_{0}^{1} \Psi(z_o,\mu')\mu'd\mu'$$
$$+ \rho^{s}_{(2)} \Psi(z_o,-\mu) \qquad (1.1.23)$$
$$\text{for} \quad -1 \leq \mu < 0.$$

To simplify Eq. (1.1.21) further, it is possible to integrate the scattering kernel over the azimuthal region $[0,2\pi)$ by expanding $K = K(\mu_o)$ in Legendre polynomials in the form

$$K(\mu_o) = \sum_{q=o}^{Q} a_q K_q(\mu_o), \quad a_o = 1, \qquad (1.1.24)$$

where Q is some positive integer, $a_q$, $q = 0,1,2,\ldots,Q$, are the expansion coefficients, and $K_q(\cdot)$ is the Legendre polynomial of order $q$, $q = 0,1,2,\ldots,Q$. Indeed, it can easily be shown that

$$K_q(\mu_o) = K_q(\mu)K_q(\mu') + 2\sum_{p=1}^{q}\frac{(q-p)!}{(q+p)!}K_q^{\,p}(\mu)K_q^{\,p}(\mu')\cos p(\phi-\phi'), \quad (1.1.25)$$

$$q = 0,1,\ldots,Q,$$

where $K_q^{\,p}(\cdot)$, $q = 0,1,\ldots,Q$, $p = 1,2,\ldots,q$, are the associated Legendre polynomials [40,pp. 326-328]. Substituting Eq. (1.1.25) in Eq. (1.1.24) we get

$$K(\mu_o) = \sum_{q=o}^{Q} a_q K_q(\mu)K_q(\mu') + 2\sum_{q=1}^{Q}\sum_{p=1}^{q} a_q^{\,p}K_q^{\,p}(\mu)K_q^{\,p}(\mu')\cos p(\phi-\phi'), \quad (1.1.26)$$

where

$$a_o = 1,$$

and

$$a_q^{\,p} = a_q\frac{(q-p)!}{(q+p)!}, \qquad q = 1,2,\ldots,Q, \quad p = 1,2,\ldots,q.$$

Now, integrating Eq. (1.1.26) over $\phi'$ from 0 to $2\pi$, and noting that

$$\int_0^{2\pi}\cos p(\phi-\phi')d\phi' = 0$$

for integral values of $p$, we get

$$\int_0^{2\pi} K[\mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi-\phi')]d\phi' =$$

$$2\pi\sum_{q=1}^{Q} K_q(\mu)K_q(\mu'), \qquad a_o = 1. \quad (1.1.27)$$

Using Eq. (1.1.27) in Eq. (1.1.21), we finally obtain

$$\mu \frac{\partial}{\partial z} \Psi(z,\mu) + \beta(z)\Psi(z,\mu) = \alpha(z)f[\Theta(z,t)]$$

$$+ 2\pi\sigma(z) \sum_{q=o}^{Q} a_q K_q(\mu) \int_{-1}^{1} K_q(\mu')\Psi(z,\mu')d\mu' \ , \ a_o = 1, \quad (1.1.28)$$

$$\text{for} \quad 0 < z < z_o \quad \text{and} \quad -1 \leq \mu \leq 1.$$

It is worthwhile noting that the case of $Q = 0$ corresponds to isotropic scattering, and the cases of $Q > 0$ correspond to anisotropic scattering.

For convenience, we define a new variable $\tau = \tau(z)$ such that

$$d\tau = \beta(z)dz, \quad (1.1.29)$$

or, equivalently,

$$\tau = \int_0^z \beta(z)dz, \quad (1.1.30)$$

where it has been arbitrarily assumed that $\tau(0) = 0$. Assuming that the inverse function $z = z(\tau)$ exists we can, then, express Eq. (1.1.28) as follows.

$$\mu \frac{\partial}{\partial \tau} \Psi(\tau,\mu) + \Psi(\tau,\mu) = [1 - \gamma(\tau)]f[\Theta(\tau,t)]$$

$$+ 2\pi\gamma(\tau) \sum_{q=1}^{Q} a_q K_q(\mu) \int_{-1}^{1} K_q(\mu')\Psi(\tau,\mu')d\mu', \quad a_o = 1, \quad (1.1.31)$$

where we have defined

$$\gamma(\tau) = \frac{\sigma(\tau)}{\beta(\tau)} . \quad (1.1.32)$$

In radiative transport literature it has become a common practice to refer, respectively, to the variables $\tau = \tau(z)$ and $\gamma = \gamma(\tau)$ as the optical distance and the albedo [10, p.259 and p. 253].

In terms of the optical distance, Eqs. (1.1.22) and (1.1.23) can, respectively, be expressed as follows.

$$\Psi(0,\mu) = T_{(1)}(\mu) + \varepsilon_{(1)}(\mu)f[\Theta(0,t)] - 2\rho^{d}_{(1)}\int_{-1}^{0} \Psi(0,\mu')\mu' d\mu' + \rho^{s}_{(1)}\Psi(0,-\mu) \tag{1.1.33}$$

$$\text{for} \quad 0 < \mu \leq 1,$$

and

$$\Psi(\tau_0,\mu) = T_{(2)}(\mu) + \varepsilon_{(2)}(\mu)f[\Theta(\tau_0,t)] + 2\rho^{d}_{(2)}\int_{0}^{1} \Psi(\tau_0,\mu')\mu' d\mu' + \rho^{s}_{(2)}\Psi(\tau_0,-\mu) \tag{1.1.34}$$

$$\text{for} \quad -1 \leq \mu < 0.$$

## 1.2 Weak Formulation of the Radiative Transport Problem

In Section 1.1 the radiative transport problem has been formulated in terms of the equations

$$\underset{\sim}{\omega}\cdot \text{grad}\Psi(\underset{\sim}{x},\underset{\sim}{\omega}) + \beta(\underset{\sim}{x})\Psi(\underset{\sim}{x},\underset{\sim}{\omega}) = \alpha(x)f[\Theta(\underset{\sim}{x},t)] + \sigma(\underset{\sim}{x})\int_{\Omega} K(\underset{\sim}{\omega}' \cdot \underset{\sim}{\omega})\Psi(\underset{\sim}{x},\underset{\sim}{\omega}')d\omega' \quad \text{over } R \times \Omega \tag{1.1.1}$$

and

$$\Psi(\underset{\sim}{x},\underset{\sim}{\omega}) = T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) + \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})f[\Theta(\underset{\sim}{x},t)] + \int_{\Omega^+} g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega})\Psi(\underset{\sim}{x},\underset{\sim}{\omega}')|\underset{\sim}{n} \cdot \underset{\sim}{\omega}'|d\omega' \tag{1.1.2}$$

over $\partial R^s \times \Omega^-$, $s = 1,2,\ldots,S.$

Within the limits of the physics assumed thus far, the field equation (1.1.1) together with the boundary conditions (1.1.2) provides the problem of radiative transport in gray participating media with a local mathematical model. In this section we proceed to formulate an equivalent global model through a weak formulation of the boundary value problem (1.1.1-2).

Let the solution space of the boundary value problem given by Eqs. (1.1.1) and (1.1.2) be $H$. Then $H$ is simply the space of functions of class $C^{1,0}$ defined over the region $R \times \Omega$. Multiplying Eq. (1.1.1) by an arbitrary function $\Phi \in H$, and integrating over the spatial and angular regions we get

$$\int_R\int_\Omega [\underset{\sim}{\omega}\cdot \text{grad}\Psi(\underset{\sim}{x},\underset{\sim}{\omega})]\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega d\upsilon + \int_R\int_\Omega \beta(\underset{\sim}{x})\Psi(\underset{\sim}{x},\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega d\upsilon =$$
$$\int_R\int_\Omega \alpha(\underset{\sim}{x})f[\Theta(\underset{\sim}{x},t)]\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega d\upsilon + \int_R\int_\Omega \sigma(\underset{\sim}{x})[\int_\Omega K(\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})\Psi(\underset{\sim}{x},\underset{\sim}{\omega}')d\omega']\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega d\upsilon. \tag{1.2.1}$$

On the other hand, from Green's theorem it follows that

$$\int_R\int_\Omega [\underset{\sim}{\omega}\cdot \text{grad}\Psi(\underset{\sim}{x},\underset{\sim}{\omega})]\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega d\upsilon = -\int_R\int_\Omega \Psi(\underset{\sim}{x},\underset{\sim}{\omega})[\underset{\sim}{\omega}\cdot \text{grad}\Phi(\underset{\sim}{x},\underset{\sim}{\omega})]d\omega d\upsilon$$
$$+ \int_{\partial R}\int_{\Omega^+} \Psi(\underset{\sim}{x},\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n}\cdot\underset{\sim}{\omega}|d\omega ds - \int_{\partial R}\int_{\Omega^-} \Psi(\underset{\sim}{x},\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n}\cdot\underset{\sim}{\omega}|d\omega ds. \tag{1.2.2}$$

Substituting Eq. (1.2.2) in Eq. (1.2.1) we get

$$-\int_R \int_\Omega \Psi(\underset{\sim}{x},\underset{\sim}{\omega})\underset{\sim}{\omega}\cdot \mathrm{grad}\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv + \int_R \int_\Omega \beta(\underset{\sim}{x})\Psi(\underset{\sim}{x},\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv =$$
$$\int_R \int_\Omega \alpha(\underset{\sim}{x})f[\Theta(\underset{\sim}{x},t)]\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv$$
$$+ \int_R \int_\Omega \sigma(\underset{\sim}{x})[\int_\Omega K(\underset{\sim}{\omega}' \cdot \underset{\sim}{\omega})\Psi(\underset{\sim}{x},\underset{\sim}{\omega}')d\omega']\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv$$
$$- \int_{\partial R} \int_{\Omega^+} \Psi(\underset{\sim}{x},\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds$$
$$+ \sum_{s=1}^{S} \int_{\partial R^s} \int_{\Omega^-} \Psi(\underset{\sim}{x},\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds. \qquad (1.2.3)$$

Using Eq. (1.1.2) in Eq. (1.2.3) we finally obtain

$$-\int_R \int_\Omega \Psi(\underset{\sim}{x},\underset{\sim}{\omega})\underset{\sim}{\omega}\cdot \mathrm{grad}\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv + \int_R \int_\Omega \beta(\underset{\sim}{x})\Psi(\underset{\sim}{x},\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv =$$
$$\int_R \int_\Omega \alpha(\underset{\sim}{x})f[\Theta(\underset{\sim}{x},t)]\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv$$
$$+ \int_R \int_\Omega \sigma(\underset{\sim}{x})[\int_\Omega K(\underset{\sim}{\omega}' \cdot \underset{\sim}{\omega})\Psi(\underset{\sim}{x},\underset{\sim}{\omega}')d\omega']\Phi(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv$$
$$- \int_{\partial R} \int_{\Omega^+} \Psi(\underset{\sim}{x},\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds$$
$$+ \sum_{s=1}^{S} \int_{\partial R^s} \int_{\Omega^-} T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds$$
$$+ \int_{\partial R^s} \int_{\Omega^-} \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})f[\Theta(\underset{\sim}{x},t)]\Phi(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds$$
$$+ \int_{\partial R^s} \int_{\Omega^-} [\int_{\Omega^+} g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega})\Psi(\underset{\sim}{x},\underset{\sim}{\omega}')|\underset{\sim}{n} \cdot \underset{\sim}{\omega}'|d\omega']\Phi(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds. \qquad (1.2.4)$$

This proves the fact that if $\Psi \in H$ is a solution to the boundary value problem (1.1.1-2), then Eq. (1.2.4) should be valid for all $\Phi \in H$. Furthermore, it can easily be shown that the converse of this statement is also true. That is, if $\Psi \in H$ is such a function that Eq. (1.2.4) is valid for all $\Phi \in H$, then $\Psi$ is necessarily the solution to the boundary value problem (1.1.1-2). Thus, we can immediately conclude that solving the system of equations (1.1.1) and (1.1.2) is completely equivalent to finding a function $\Psi \in H$ such that Eq. (1.2.4) is valid for all $\Phi \in H$. Clearly, the problem of finding such a function provides the boundary value problem (1.1.1-2) with an equivalent weak formulation. In what follows, we will refer to Eq. (1.2.4) as the weak equation associated with the system of equations (1.1.1) and (1.1.2), and to the problem of finding a function in $H$ such that Eq. (1.2.4) is valid for all functions in $H$ as the weak radiative transport problem.

Before proceeding any further, it is worthwhile noting that Eq. (1.2.3) also provides the boundary value problem (1.1.1-2) with a weak formulation, provided that the solution space $H$ is constrained to a space $H^{*}$ defined as

$$H^{*} = \{\Phi(\underset{\sim}{x},\underset{\sim}{\omega}) \,|\, \Phi \in H \text{ and } \Phi(\underset{\sim}{x},\underset{\sim}{\omega}) = T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) + \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) f[\Theta(\underset{\sim}{x},t)]$$
$$+ \int_{\Omega} g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega})\Phi(\underset{\sim}{x},\underset{\sim}{\omega}')\,|\underset{\sim}{n}\cdot\underset{\sim}{\omega}'|\,d\omega'$$
$$\partial R^{s} \times \Omega^{-}\ , \quad s = 1,2,\ldots,S\}.$$

In weak formulations of this nature the boundary conditions are directly imposed, and, therefore, treated as essential conditions. Such formulations, however, tend to complicate the subsequent mathematical and computational analyses. In formulations based on a weak equation such as Eq. (1.2.4), on the other hand, boundary conditions are considered to be natural conditions, that is, they are directly incorporated into the weak equation rather being imposed as a constraint on the solution space. Since formulations of this type avoid the complications encountered in the aforementioned analyses, they are generally preferred. This will, indeed be our attitude in the present study.

## 1.3 Galerkin Formulation of the Radiative Transport Problem

In Section 1.1 we have posed the boundary value problem governing radiative transport in gray participating media, and, in Section 1.2, we have formulated an equivalent weak problem. A comparison of the system of equations (1.1.1) and (1.1.2) with Eq. (1.2.4) may erroneously lead to the conclusion that no progress has been achieved so far, since, as it stands, solving the weak problem stated in terms of Eq. (1.2.4) appears to be as formidable as solving the boundary value problem (1.1.1-2). A careful examination of Eq. (1.2.4), on the other hand, reveals that, within the context of the weak radiative transport problem, Eqs. (1.1.1) and (1.1.2) are required to be valid globally over $R \times \Omega$, but, within the context of the original boundary value problem, locally over $R \times \Omega$. The fact that global formulations have been found to be much more amenable to the development of an approximate solution scheme as compared to local formulations, thus, clarifies our progress achieved so far. With the intention of establishing such a solution scheme, we, now proceed to develop an approximate formulation of the weak radiative transport problem on which the subsequent finite element analysis can be based.

Let $\mathcal{H}^m \subset \mathcal{H}$ be some finite dimensional subspace of dimension m, m being a positive integer. Let $\tilde{\Phi}_i$, i = 1,2,...,m, be the basis functions in $\mathcal{H}^m$. We will write a tilde on top of a symbol representing a function whenever the function belongs to a finite dimensional subspace of a given function space. If, now, the solution space of the boundary value problem (1.1.1-2) is constrained to $\mathcal{H}^m$, then Eq. (1.2.4) can be written as

$$\begin{aligned}
-\int_R \int_\Omega \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega})\underset{\sim}{\omega}\cdot\text{grad}\tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv + \int_R \int_\Omega \beta(\underset{\sim}{x})\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv = \\
\int_R \int_\Omega \alpha'(\underset{\sim}{x})f[\Theta(\underset{\sim}{x},t)]\tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv \\
+ \int_R \int_\Omega \sigma(\underset{\sim}{x})[\int_\Omega K(\underset{\sim}{\omega}' \cdot \underset{\sim}{\omega})\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}')d\omega']\tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv \\
- \int_{\partial R} \int_{\Omega^+} \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds \\
+ \sum_{s=1}^{S} \int_{\partial R^s}\int_{\Omega^-} T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds \\
+ \int_{\partial R^s}\int_{\Omega^-} \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})f[\Theta(\underset{\sim}{x},t)]\tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds \\
+ \int_{\partial R^s}\int_{\Omega^-} [\int_{\Omega^+} g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega})\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}')|\underset{\sim}{n} \cdot \underset{\sim}{\omega}'|d\omega']\tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds,
\end{aligned} \tag{1.3.1}$$

where $\tilde{\Psi}$, $\tilde{\Phi} \in \mathcal{H}^m$, and the weak problem associated with it reads as follows: find a particular function $\tilde{\Psi} \in \mathcal{H}^m$ such that Eq. (1.3.1) is valid for all $\tilde{\Phi} \in \mathcal{H}^m$. But, since $\mathcal{H}^m$ is finite dimensional,

$$\tilde{\Phi}(\underset{\sim}{x},\underset{\sim}{\omega}) = \sum_{i=1}^{m} \tilde{\phi}_i\tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega}) \tag{1.3.2}$$

for any $\tilde{\Phi} \varepsilon H^m$ , where $\tilde{\phi}_i$, i = 1,2,...,m, are the expansion coefficients. It then follows that it is sufficient to require Eq. (1.3.1) to hold for the basis functions $\tilde{\Phi}_i$, i = 1,2,...,m, in order to ensure that it is valid for all functions $\tilde{\Phi} \varepsilon H^m$. Therefore, Eq. (1.3.1) reduces to

$$- \int_R \int_\Omega \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\omega} \cdot \mathrm{grad} \tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega}) d\omega dv + \int_R \int_\Omega \beta(\underset{\sim}{x}) \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}) \tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega}) d\omega dv =$$

$$+ \int_R \int_\Omega \alpha(\underset{\sim}{x}) f[\Theta(\underset{\sim}{x},t)] \tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega}) d\omega dv$$

$$+ \int_R \int_\Omega \sigma(\underset{\sim}{x}) [\int_\Omega K(\underset{\sim}{\omega}' \cdot \underset{\sim}{\omega}) \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}') d\omega' ] \tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega}) d\omega dv$$

$$- \int_{\partial R} \int_{\Omega^+} \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}) \tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega}) |\underset{\sim}{n} \cdot \underset{\sim}{\omega}| d\omega ds$$

$$+ \sum_{s=1}^{S} \int_{\partial R^s} \int_{\Omega^-} T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) \tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega}) |\underset{\sim}{n} \cdot \underset{\sim}{\omega}| d\omega ds$$

$$+ \int_{\partial R^s} \int_{\Omega^-} \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) f[\Theta(\underset{\sim}{x},t)] \tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega}) d\omega ds$$

$$+ \int_{\partial R^s} \int_{\Omega^-} [\int_{\Omega^+} g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega}) \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}') |\underset{\sim}{n} \cdot \underset{\sim}{\omega}'| d\omega' ] \tilde{\Phi}_i(\underset{\sim}{x},\underset{\sim}{\omega}) |\underset{\sim}{n} \cdot \underset{\sim}{\omega}| d\omega ds,$$

$$i = 1,2,\ldots,m. \tag{1.3.3}$$

Clearly, Eq. (1.3.3) approximates the weak equation (1.2.4), and thus provides the weak radiative transport problem with an approximate formulation. Formulations of this type are commonly referred to as Galerkin formulations. In what follows, we will refer to Eq. (1.3.3) as the Galerkin equation associated with the system of equations (1.1.1) and (1.1.2), and to the problem of finding a function in $H^m$ such that Eq. (1.3.3) is valid for all $\tilde{\Phi}_i$, i = 1,2,...,m, as the Galerkin radiative transport problem.

By expanding $\tilde{\Psi} \in H^m$ according to Eq. (1.3.2), and substituting the result in Eq. (1.3.3), we conclude that the Galerkin formulation of the radiative transport problem automatically leads to an approximate solution scheme involving a system of linear algebraic equations for the variables $\tilde{\psi}_i$, $i = 1,2,\ldots,m$, provided that it is possible to construct a finite dimensional subspace $H^m \subset H$ . In fact, this is where the finite element approximation technique comes in, and its application will be the subject of Part 2.

Before concluding this section, referring to a Cartesian coordinate system we should state, for future reference, the Galerkin radiative transport problem in the case of a plane-parallel gray azimuthally symmetric participating medium of infinite extent in the x- and y- directions and of finite extent in the z-direction, $0 \leq z \leq z_o$, $z_o$ being an arbitrary positive scalar. Let the solution space of the corresponding boundary value problem given by Eqs. (1.1.31), (1.1.33) and (1.1.34) be $G$. Then $G$ is simply the space of functions of class $C^{o,o}$ defined over the region $[0,\tau_o] \times [-1,1]$. For this particular case, it can then be shown that within the limitations of pertinent assumptions made in deriving Eqs. (1.1.31), (1.1.33) and (1.1.34), Eq. (1.3.3) reduces to

$$-\int_0^{\tau_o}\int_{-1}^{1} \mu\tilde{\Psi}(\tau,\mu) \frac{\partial}{\partial\tau} \tilde{\Phi}_i(\tau,\mu)d\mu d\tau + \int_0^{\tau_o}\int_{-1}^{1} \tilde{\Psi}(\tau,\mu)\tilde{\Phi}_i(\tau,\mu)d\mu d\tau =$$

$$\int_0^{\tau_0}\int_{-1}^{1} [1 - \gamma(\tau)]f[\Theta(\tau,t)]\tilde{\Phi}_i(\tau,\mu)d\mu d\tau$$

$$+ 2\pi \int_0^{\tau_0}\int_{-1}^{1} \gamma(\tau)[\sum_{q=o}^{Q} a_q K_q(\mu)\int_{-1}^{1} K(\mu')\tilde{\Psi}(\tau,\mu')d\mu']\tilde{\Phi}_i(\tau,\mu)d\mu d\tau$$

$$- \int_0^1 \tilde{\Psi}(\tau_0,\mu)\tilde{\Phi}_i(\tau_0,\mu)\mu d\mu + \int_{-1}^{0} \tilde{\Psi}(0,\mu)\tilde{\Phi}_i(0,\mu)\mu d\mu$$

$$- \int_{-1}^{0} T_{(2)}(\mu)\tilde{\Phi}_i(\tau_0,\mu)\mu d\mu - \int_{-1}^{0} \varepsilon_{(2)}(\mu)f[\Theta(\tau_0,t)]\tilde{\Phi}_i(\tau_0,\mu)\mu d\mu$$

$$- 2\rho^d_{(2)}\int_{-1}^{0} [\int_0^1 \tilde{\Psi}(\tau_0,\mu')\mu' d\mu']\tilde{\Phi}_i(\tau_0,\mu)\mu d\mu$$

$$- \rho^s_{(2)}\int_{-1}^{0} \tilde{\Psi}(\tau_0,-\mu)\tilde{\Phi}_i(\tau_0,\mu)\mu d\mu$$

$$+ \int_0^1 T_{(1)}(\mu)\tilde{\Phi}_i(0,\mu)\mu d\mu + \int_0^1 \varepsilon_{(1)}(\mu)f[\Theta(0,t)]\tilde{\Phi}_i(0,\mu)\mu d\mu$$

$$- 2\rho^d_{(1)}\int_0^1 [\int_{-1}^{0} \tilde{\Psi}(0,\mu')\mu' d\mu']\tilde{\Phi}_i(0,\mu)\mu d\mu$$

$$+ \rho^s_{(1)}\int_0^1 \tilde{\Psi}(0,-\mu)\tilde{\Phi}_i(0,\mu)\mu d\mu, \quad i = 1,2,\ldots,m, \qquad (1.3.4)$$

where m is some positive integer, $\tilde{\Phi}_i = \tilde{\Phi}_i(\tau,\mu)$, $i = 1,2,\ldots,m$, are the basis functions spanning a finite dimensional subspace $G^m \subset G$, and $\tilde{\Psi}$ is any function in $G^m$. Consequently, the corresponding Galerkin problem is simply the problem of finding a function in $G^m$ such that Eq. (1.3.4) is valid for all $\tilde{\Phi}_i$, $i = 1,2,\ldots,m$.

*PART TWO*

*FINITE-ELEMENT ANALYSIS OF THE RADIATIVE TRANSPORT PROBLEM*

*In the first part of the present study we have shown that the Galerkin formulation of the radiative transport problem automatically leads to an approximate solution scheme, provided that it is possible to construct a finite dimensional subspace in the solution space of this boundary value problem. In this part we proceed to apply the finite-element approximation technique in developing such an approximate solution scheme. To this end, we first develop a finite-element model* $\tilde{H}$ *approximating the solution space* $H$ *of the radiative transport problem, and, subsequently, construct a finite-dimensional subspace in the space* $\tilde{H}$. *We then proceed to formulate a finite-element model of the Galerkin radiative transport problem, which, as will be seen, leads to an approximate solution scheme for the radiative transport problem.*

## 2.1 Finite Element Model of the Solution Space $H$

In Section 1.2 we have defined $H$ to be the space of functions of class $C^{1,0}$ defined over the region $R \times \Omega$. Its corresponding first order finite-element model $\tilde{H}$ then is defined to be the space of functions of class $C^{0,0}$ defined over the finite-element model of the region $R \times \Omega$. Now, if $\tilde{R}$ and $\tilde{\Omega}$ denote, respectively, the finite-element models approximating the regions $R$ and $\Omega$, then from the definition of the region $R \times \Omega$ it follows that its corresponding finite-element model is simply the region $\tilde{R} \times \tilde{\Omega}$. Thus, $\tilde{H}$ is the space of functions of class $C^{0,0}$ defined over the region $\tilde{R} \times \tilde{\Omega}$; and, therefore, the construction of the models $\tilde{R}$ and $\tilde{\Omega}$ is sufficient to define the region $\tilde{R} \times \tilde{\Omega}$ as well as the space $\tilde{H}$. The approach and the notation we adopt in constructing the finite-element models $\tilde{R}$ and $\tilde{\Omega}$ will essentially be that of Oden described in great detail in [27] and [30].

To construct the finite-element model $\tilde{R}$ approximating the region $R$, let us identify

(i) a finite number $\tilde{P}$ of points $\underset{\sim}{x}^{\tilde{\Delta}}$, $\tilde{\Delta} = 1,2,\ldots,\tilde{P}$, in $R$,

(ii) a finite number $\tilde{E}$ of closed and disjoint subregions $r_{\tilde{e}} \subset R$, $\tilde{e} = 1,2,\ldots,\tilde{E}$,

and

(iii) a finite number $P_{\tilde{e}}$ of points $\underset{\sim}{x}_{(\tilde{e})}^{\tilde{N}}$, $\tilde{N} = 1,2,\ldots,P_{\tilde{e}}$, in each subregion $r_{\tilde{e}}$ such that local compatibility conditions are satisfied [27, p. 36].

The region $\tilde{R}$ is then defined to be the connected region constructed by assembling the subregions $r_{\tilde{e}}$, $\tilde{e} = 1,2,\ldots,\tilde{E}$, together.

To construct the finite-element model $\tilde{\Omega}$ approximating the region $\Omega$, let us similarly identify

(i) a finite number $\hat{P}$ of points $\underset{\sim}{\omega}^{\hat{\Delta}}$, $\hat{\Delta} = 1,2,\ldots,\hat{P}$, in $\Omega$,

(ii) a finite number $\hat{E}$ of closed and disjoint subregions $\omega_{\hat{e}} \subset \Omega$, $\hat{e} = 1,2,\ldots,\hat{E}$,

and

(iii) a finite number $P_{\hat{e}}$ of points $\underset{\sim}{\omega}_{(\hat{e})}{}^{\hat{N}}$, $\hat{N} = 1,2,\ldots,P_{\hat{e}}$, in each subregion $\omega_{\hat{e}}$ such that local compatibility conditions are satisfied [27, p. 36].

The region $\tilde{\Omega}$ is then defined to be the connected region constructed by assembling the subregions $\omega_{\hat{e}}$, $\hat{e} = 1,2,\ldots,\hat{E}$, together.

From the structure of the regions $\tilde{R}$ and $\tilde{\Omega}$, it then follows that the region $\tilde{R} \times \tilde{\Omega}$ is the connected region characterized by a finite number $P = \tilde{P}\hat{P}$ of points $(\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta}$, $\Delta = 1,2,\ldots,P$, and a finite number $E = \tilde{E}\hat{E}$ of subregions $d_e$, $e = 1,2,\ldots,E$, each subregion $d_e$ being defined as

$$d_e = r_e \times \omega_e \tag{2.1.1}$$

with the condition that

$$r_e = \tilde{C}_e{}^{\tilde{e}} r_{\tilde{e}} \quad , \qquad \tilde{e} = 1,2,\ldots,\tilde{E}, \tag{2.1.2}$$

and

$$\omega_e = \hat{C}_e{}^{\hat{e}} \omega_{\hat{e}} \quad , \qquad \hat{e} = 1,2,\ldots,\hat{E}, \tag{2.1.3}$$

where

$$\tilde{C}_e^{\tilde{e}} = \begin{cases} 1 & \text{if e corresponds to } \tilde{e} \in \{1,2,\ldots,\tilde{E}\}, \\ 0 & \text{if otherwise,} \end{cases} \tag{2.1.4}$$

and

$$\hat{C}_e^{\hat{e}} = \begin{cases} 1 & \text{if e corresponds to } \hat{e} \in \{1,2,\ldots,\hat{E}\}, \\ 0 & \text{if otherwise.} \end{cases} \tag{2.1.5}$$

Henceforth, we will refer to the points $(\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta}$, $\Delta = 1,2,\ldots,P$, as global nodes, and to the subregions $d_e$, $e = 1,2,\ldots,E$, as finite elements or, simply, as elements.

From the definition of the element $d_e$, $e = 1,2,\ldots,E$, it is clear that there exists within each element $d_e$ a finite number $P_e = \tilde{C}_e^{\tilde{e}}\, \hat{C}_e^{\hat{e}}\, P_{\tilde{e}}P_{\hat{e}}$, $\tilde{e} = 1,2,\ldots,\tilde{E}$ and $\hat{e} = 1,2,\ldots,\hat{E}$, of points $(\underset{\sim}{x},\underset{\sim}{\omega})_{(e)}^{N}$, $N = 1,2,\ldots,P_e$; and, since the subregions $\kappa_{\tilde{e}}$, $\tilde{e} = 1,2,\ldots,\tilde{E}$, and $\omega_{\hat{e}}$, $\hat{e} = 1,2,\ldots,\hat{E}$, are locally compatible, the elements $d_e$, $e = 1,2,\ldots,E$, are also locally compatible. Thus, for each element $d_e$, $e = 1,2,\ldots,E$, there exists an incidence relation

$$I^{(e)}:\{(\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta}\}_{\Delta=1}^{P} \rightarrow \{(\underset{\sim}{x},\underset{\sim}{\omega})_{(e)}^{N}\}_{N=1}^{P_e} \tag{2.1.6}$$

defined by

$$(\underset{\sim}{x},\underset{\sim}{\omega})_{(e)}^{N} = I^{(e)}{}_{\Delta}^{N}\, (\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta} \tag{2.1.7}$$

where

$$I^{(e)}{}_{\Delta}^{N} = \begin{cases} 1 & \text{if the point } (\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta} \text{ is incident on the point } (\underset{\sim}{x},\underset{\sim}{\omega})_{(e)}^{N}, \\ 0 & \text{if otherwise.} \end{cases} \tag{2.1.8}$$

Clearly, the inverse relation

$$\mathcal{I}_{(e)}^{-1} : \{(\underset{\sim}{x},\underset{\sim}{\omega})_{(e)}^{N}\}_{N=1}^{P_e} \rightarrow \{(\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta}\}_{\Delta=1}^{P} \tag{2.1.9}$$

also exists, and is defined by

$$(\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta} = \mathcal{I}_{(e)}^{-1}{}_{N}^{\Delta}\,(\underset{\sim}{x},\underset{\sim}{\omega})_{(e)}^{N} \tag{2.1.10}$$

where

$$\mathcal{I}_{(e)}^{-1}{}_{N}^{\Delta} = \begin{cases} 1 & \text{if the point } (\underset{\sim}{x},\underset{\sim}{\omega})_{(e)}^{N} \text{ is incident on the point } (\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta}, \\ 0 & \text{if otherwise.} \end{cases} \tag{2.1.11}$$

It then follows that [30, p. 203]

(i) the matrix $\mathcal{I}_{(e)}^{-1}{}_{N}^{\Delta}$ is the transpose of the matrix $\mathcal{I}^{(e)}{}_{\Delta}^{N}$, $N = 1,2,\ldots,P_e$ and $\Delta = 1,2,\ldots,P$,

(ii) $\mathcal{I}^{(e)}{}_{\Delta}^{N}\mathcal{I}_{(e)}^{-1}{}_{M}^{\Delta} = \delta_M^N$, $M,N = 1,2,\ldots,P_e$ and $\Delta = 1,2,\ldots,P$,

and

(iii) $\mathcal{I}_{(e)}^{-1}{}_{N}^{\Delta}\mathcal{I}^{(e)}{}_{\Gamma}^{N} = \begin{cases} \delta_\Gamma^\Delta & \text{if } (\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta}, (\underset{\sim}{x},\underset{\sim}{\omega})^{\Gamma} \in d_e, \quad \Delta,\Gamma = 1,2,\ldots,P, \; N = 1,2,\ldots,P_e \\ 0 & \text{if otherwise,} \end{cases}$

where $\delta_M^N$ and $\delta_\Gamma^\Delta$ are the Kronecker deltas. Hereafter, we will refer to points $(\underset{\sim}{x},\underset{\sim}{\omega})_{(e)}^{N}$, $e = 1,2,\ldots,E$ and $N = 1,2,\ldots,P_e$, as local nodes.

It is worthwhile noting that the collection of the relations

$$\mathcal{I} = \{\mathcal{I}^{(1)},\mathcal{I}^{(2)},\ldots,\mathcal{I}^{(E)}\}$$

effectively decomposes the connected region into a collection of E disjoint subregions, whereas the collection of the inverse relations

$$I^{-1} = \{I_{(1)}^{-1}, I_{(2)}^{-1}, \ldots, I_{(E)}^{-1}\}$$

effectively assembles the set of E disjoint subregions into a connected region.

## 2.2 Construction of a Finite Dimensional Subspace in the Finite-Element Model of the Solution Space $H$

Having established the finite-element model $\tilde{H}$ of the space $H$, we proceed to construct a finite dimensional subspace in the space $\tilde{H}$. In the subsequent analysis our viewpoint will be that of Oden, described briefly in [34].

Let $\tilde{\Phi}_1 = \tilde{\Phi}_1(\underset{\sim}{x},\underset{\sim}{\omega})$, $\tilde{\Phi}_2 = \tilde{\Phi}_2(\underset{\sim}{x},\underset{\sim}{\omega}), \ldots, \tilde{\Phi}_\Delta = \tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega}), \ldots, \tilde{\Phi}_P = \tilde{\Phi}_P(\underset{\sim}{x},\underset{\sim}{\omega})$, $\tilde{\Phi}_\Delta \in \tilde{H}$, $\Delta = 1,2,\ldots,P$, be a set of P basis functions such that for any point $(\underset{\sim}{x},\underset{\sim}{\omega})^\Gamma \in \tilde{R} \times \tilde{\Omega}$, $\Gamma = 1,2,\ldots,P$,

$$\tilde{\Phi}_\Delta[(\underset{\sim}{x},\underset{\sim}{\omega})^\Gamma] = \delta_\Delta{}^\Gamma, \qquad \Delta,\Gamma = 1,2,\ldots,P, \tag{2.2.1}$$

where $\delta_\Delta{}^\Gamma$ is the Kronecker delta. Eq. (2.2.1) simply implies that the basis functions are normalized with respect to the points $(\underset{\sim}{x},\underset{\sim}{\omega})^\Gamma \in \tilde{R} \times \tilde{\Omega}$ $\Gamma = 1,2,\ldots,P$. Let $\tilde{H}^P$ denote the P-dimensional subspace of the space $\tilde{H}$ spanned by the set of functions $\tilde{\Phi}_\Delta \in \tilde{H}$, $\Delta = 1,2,\ldots,P$. Then, for any function $\tilde{\Psi} \in \tilde{H}^P$

$$\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}) = \tilde{\psi}^{\Delta}\tilde{\Phi}_{\Delta}(\underset{\sim}{x},\underset{\sim}{\omega}) \ , \quad \Delta = 1,2,\dots,P, \tag{2.2.2}$$

where

$$\tilde{\psi}^{\Delta} = \tilde{\Psi}[(\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta}].$$

If, now, $\tilde{\Psi}^{(e)} = \tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})$ is the restriction of a function $\tilde{\Psi} \in \tilde{H}^P$ to the element $d_e$, $e = 1,2,\dots,E$, then from the structure of the region $\tilde{R} \times \tilde{\Omega}$ it follows that

$$\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}) = \sum_{e=1}^{E} \tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \tag{2.2.3}$$

almost everywhere and, moreover, that

$$\tilde{\psi}_{(e)}{}^{N} = I^{(e)}{}_{\Delta}{}^{N}\tilde{\psi}^{\Delta} \ , \qquad N = 1,2,\dots,P_e \qquad \Delta = 1,2,\dots,P, \tag{2.2.4}$$

and

$$\tilde{\psi}^{\Delta} = I_{(e)}{}^{-1}{}_{N}{}^{\Delta}\tilde{\psi}_{(e)}{}^{N} \ , \qquad N = 1,2,\dots,P_e \qquad \Delta = 1,2,\dots,P, \tag{2.2.5}$$

where

$$\tilde{\psi}_{(e)}{}^{N} = \tilde{\Psi}^{(e)}[(\underset{\sim}{x},\underset{\sim}{\omega})^{N}] \ , \quad N = 1,2,\dots,P_e,$$

$$\tilde{\psi}^{\Delta} = \tilde{\Psi}[(\underset{\sim}{x},\underset{\sim}{\omega})^{\Delta}] \qquad , \quad \Delta = 1,2,\dots,P \ ;$$

and where $I^{(e)}$ and $I_{(e)}{}^{-1}$ are, respectively, the incidence relations defined by Eqs. (2.1.7) and (2.1.10). For each element $d_e$, $e = 1,2,\dots,E$, identifying a set of $P_e$ basis functions $\tilde{\Phi}^{(e)}{}_{N} = \tilde{\Phi}^{(e)}{}_{N}(\underset{\sim}{x},\underset{\sim}{\omega})$, $N = 1,2,\dots,P_e$, such that

(i) $\tilde{\Phi}^{(e)}{}_N$, $N = 1,2,\ldots,P_e$, are of class $C^{o,o}$ over $d_e$,

(ii) for any point $(\underset{\sim}{x},\underset{\sim}{\omega})^M \varepsilon\, d_e$

$$\tilde{\Phi}^{(e)}{}_N(\underset{\sim}{x},\underset{\sim}{\omega})^M = \delta_N{}^M, \qquad M,N = 1,2,\ldots,P_e, \tag{2.2.6}$$

where $\delta_N{}^M$ is the Kronecker delta,

and

(iii) for $(\underset{\sim}{x},\underset{\sim}{\omega})^M \notin d_e$

$$\tilde{\Phi}^{(e)}{}_N(\underset{\sim}{x},\underset{\sim}{\omega}) \equiv 0 \quad , \qquad N = 1,2,\ldots,P_e, \tag{2.2.7}$$

we set

$$\tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) = \tilde{\psi}_{(e)}{}^N \Phi^{(e)}{}_N(\underset{\sim}{x},\underset{\sim}{\omega}), \qquad N = 1,2,\ldots,P_e, \tag{2.2.8}$$

where

$$\tilde{\psi}_{(e)}{}^N = \tilde{\Psi}^{(e)}[(\underset{\sim}{x},\underset{\sim}{\omega})^N].$$

Henceforth, we will refer to the conditions (i), (ii) and (iii), respectively, as the continuity, normality and locality conditions. From Eqs. (2.2.3), (2.2.4) and (2.2.8), it then follows that almost everywhere,

$$\begin{aligned}
\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}) &= \sum_{e=1}^{E} \tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \\
&= \sum_{e=1}^{E} \tilde{\Psi}_{(e)}{}^N \tilde{\Phi}^{(e)}{}_N(\underset{\sim}{x},\underset{\sim}{\omega}) \\
&= \sum_{e=1}^{E} I^{(e)}{}_\Delta{}^N \tilde{\psi}^\Delta \, \tilde{\Phi}^{(e)}{}_N(\underset{\sim}{x},\underset{\sim}{\omega}) \ , \qquad N = 1,2,\ldots,P_e \text{ and } \Delta = 1,2,\ldots,P,
\end{aligned}$$

where it is implicitly assumed that the elements and the basis functions $\tilde{\Phi}^{(e)}{}_N$, $N = 1,2,\ldots,N_e$ and $e = 1,2,\ldots,E$, are so identified that $\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega})$ is also of class $C^{0,0}$ over the interelement boundaries. Equivalently, we can write

$$\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}) = \sum_{e=1}^{E} [I^{(e)}{}_{\Delta}{}^{N}\tilde{\Phi}^{(e)}{}_N(\underset{\sim}{x},\underset{\sim}{\omega})]\tilde{\psi}^{\Delta} , \quad \begin{matrix} N = 1,2,\ldots,P_e \text{ and} \\ \Delta = 1,2,\ldots,P, \end{matrix} \tag{2.2.9}$$

almost everywhere. Comparing Eq. (2.2.9) with Eq. (2.2.2) we finally conclude that

$$\tilde{\Phi}_{\Delta}(\underset{\sim}{x},\underset{\sim}{\omega}) = \sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{N}\tilde{\Phi}^{(e)}{}_N(\underset{\sim}{x},\underset{\sim}{\omega}) , \quad \begin{matrix} N = 1,2,\ldots,P_e \text{ and} \\ \Delta = 1,2,\ldots,P, \end{matrix} \tag{2.2.10}$$

almost everywhere. Eq. (2.2.10) clearly provides a systematic method for generating a set of basis functions in terms of sets of basis functions defined locally over each subregion $d_e$, $e = 1,2,\ldots,E$, thus resolving the problem of constructing a finite dimensional subspace in the space $\tilde{H}$. Hereafter, we will refer respectively to $\tilde{\Phi}^{(e)}{}_N$, $e = 1,2,\ldots,E$ and $N = 1,2,\ldots,P_e$, to $\tilde{\Phi}_{\Delta}$, $\Delta = 1,2,\ldots,P$, and to $\tilde{\Psi}^{(e)}$, $e = 1,2,\ldots,E$, as the local basis functions, the global basis functions and the local approximation of $\tilde{\Psi}^{(e)} \in \tilde{H}^P$ over the element $d_e$, $e = 1,2,\ldots,E$.

## 2.3 Finite-Element Model of the Galerkin Radiative Transport Problem

Let us approximate the solution space $H$ of the radiative transport problem with the corresponding finite-element model $\tilde{H}$, and identify a finite dimensional subspace $\tilde{H}^P \subset \tilde{H}$ in terms of a set of finite-element basis functions $\tilde{\Phi}_{\Delta} \in \tilde{H}$, $\Delta = 1,2,\ldots,P$, defined by Eq. (2.2.10). The finite-element model of the Galerkin radiative transport problem is then basically the problem of finding a function $\tilde{\Psi} \varepsilon \tilde{H}^P$ such that

$$-\int_{\tilde{R}} \int_{\tilde{\Omega}} \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega})\underset{\sim}{\omega}\cdot \mathrm{grad}\tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv + \int_{\tilde{R}} \int_{\tilde{\Omega}} \beta(\underset{\sim}{x})\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv =$$

$$\int_{\tilde{R}} \int_{\tilde{\Omega}} \alpha(\underset{\sim}{x})f[\Theta(\underset{\sim}{x},t)]\tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv$$

$$+ \int_{\tilde{R}} \int_{\tilde{\Omega}} \sigma(\underset{\sim}{x})[\int_{\tilde{\Omega}} K(\underset{\sim}{\omega}' \cdot \underset{\sim}{\omega})\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}')d\omega']\tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega})d\omega dv$$

$$- \int_{\partial\tilde{R}}\int_{\tilde{\Omega}^+} \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds$$

$$+ \sum_{s=1}^{S}\{\int_{\partial\tilde{R}^s} \int_{\tilde{\Omega}^-} T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds$$

$$+ \int_{\partial\tilde{R}^s} \int_{\tilde{\Omega}^-} \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})f[\Theta(\underset{\sim}{x},t)]\tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n} \cdot \underset{\sim}{\omega}|d\omega ds$$

$$+ \int_{\partial\tilde{R}^s} \int_{\tilde{\Omega}^-} [\int_{\tilde{\Omega}^+} B_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega})\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}')|\underset{\sim}{n} \cdot \underset{\sim}{\omega}'|d\omega']\tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega})|\underset{\sim}{n}\cdot\underset{\sim}{\omega}|d\omega$$

$$\Delta = 1,2,\ldots,P, \qquad (2.3.1)$$

is valid for all $\tilde{\Phi}_\Delta \varepsilon \tilde{H}^P$, $\Delta = 1,2,\ldots,P$. In Eq. (2.3.1) $\tilde{R}$, $\tilde{\Omega}$, $\partial\tilde{R}^s$, $s = 1,2,\ldots,S$, $\tilde{\Omega}^+$ and $\tilde{\Omega}^-$ denote, respectively, the finite-element approximations of the regions $R$, $\Omega$, $\partial R^s$, $s = 1,2,\ldots,S$, $\Omega^+$ and $\Omega^-$; and $\underset{\sim}{n}$ is the unit outward normal at a point $(\underset{\sim}{x},\underset{\sim}{\omega}) \varepsilon \partial\tilde{R}^s$, $s = 1,2,\ldots,S$. Henceforth, we will refer to Eq. (2.3.1) as the finite-element Galerkin equation.

Since the region $\tilde{R} \times \tilde{\Omega}$ is an assembly of a finite number E of connected subregions $d_e$, e = 1,2,...,E, where each subregion $d_e$ is defined as the Cartesian product of the subregions $r_e$ and $\omega_e$ given, respectively, by Eqs. (2.1.2) and (2.1.3), it then follows that Eq. (2.3.1) can be written as

$$
\begin{aligned}
& -\sum_{e=1}^{E} \int_{r_e}\int_{\omega_e} \tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\underset{\sim}{\omega}\cdot\mathrm{grad}\tilde{\Phi}_{\Delta}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\,d\omega dv + \sum_{e=1}^{E}\int_{r_e}\int_{\omega_e} \beta(\underset{\sim}{x})\tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}_{\Delta}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\,d\omega dv \\
& \sum_{e=1}^{E}\int_{r_e}\int_{\omega_e} \alpha(\underset{\sim}{x})f[\Theta(\underset{\sim}{x},t)]\tilde{\Phi}_{\Delta}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\,d\omega dv \\
& + \sum_{e=1}^{E}\int_{r_e}\int_{\omega_e} \sigma(\underset{\sim}{x})\Big[\sum_{\hat{e}=1}^{\hat{E}}\int_{\omega_{\hat{e}'}}\big[K(\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega})\tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega}')\,d\omega'\big]^{(e)}\Big]\tilde{\Phi}_{\Delta}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\,d\omega dv \\
& - \sum_{s=1}^{S}\sum_{e=1}^{E}\int_{\partial r_e^{s}}\int_{\omega_e^{+}} \tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}_{\Delta}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\,|\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\,d\omega ds \\
& + \sum_{s=1}^{S}\sum_{e=1}^{E}\int_{\partial r_e^{s}}\int_{\omega_e^{-}} T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}_{\Delta}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\,|\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\,d\omega ds \\
& + \sum_{s=1}^{S}\sum_{e=1}^{E}\int_{\partial r_e^{s}}\int_{\omega_e^{-}} \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})f[\Theta(\underset{\sim}{x},t)]\tilde{\Phi}_{\Delta}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\,|\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\,d\omega ds \\
& + \sum_{s=}^{S}\sum_{e=1}^{E}\int_{\partial r_e^{s}}\int_{\omega_e^{-}} \Big[\sum_{\hat{e}'=1}^{\hat{E}}\int_{\omega_{\hat{e}'}^{+}}\big[g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega})\tilde{\psi}(\underset{\sim}{x},\underset{\sim}{\omega}')\,|\underset{\sim}{n}\cdot\underset{\sim}{\omega}'|\,d\omega'\big]^{(e)}\Big]\cdot \\
& \qquad \cdot\,\tilde{\Phi}_{\Delta}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})\,|\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\,d\omega ds\ , \qquad (2.3.2)
\end{aligned}
$$

$$\Delta = 1,2,\ldots,P,$$

where $\tilde{\Phi}_{\Delta}^{(e)} = \tilde{\Phi}_{\Delta}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})$, $\Delta = 1,2,\ldots,P$, and $\tilde{\Psi}^{(e)} = \tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})$ are, respectively, the restrictions of the functions $\tilde{\Phi}_{\Delta} = \tilde{\Phi}_{\Delta}(\underset{\sim}{x},\underset{\sim}{\omega})$, $\Delta = 1,2,\ldots,P$, and $\tilde{\Psi} = \tilde{\Psi}(\underset{\sim}{x},\underset{\sim}{\omega})$ to the element $d_e$, e = 1,2,...,E; and where $\partial r_e^{s} = r_e \cap \partial\tilde{R}^{s}$, e = 1,2,...,E and s = 1,2,...,S, $\omega_e^{\pm} = \omega_e \cap \tilde{\Omega}^{\pm}$, e = 1,2,...,E, $\omega_{\hat{e}'} \subset \tilde{\Omega}$,

$\hat{e}' = 1,2,\ldots,\hat{E}$, and $\omega_{\hat{e}'}{}^{+} = \omega_{\hat{e}'} \cap \tilde{\Omega}^{+}$. In accordance with Eqs. (2.2.10) and (2.2.9), on the other hand, for any $e = 1,2,\ldots,E$ the functions $\tilde{\Phi}_{\Delta}{}^{(e)} = \tilde{\Phi}_{\Delta}{}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})$, $\Delta = 1,2,\ldots,P$, and $\tilde{\Psi}^{(e)} = \tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega})$ are, respectively, given as

$$\tilde{\Phi}_{\Delta}{}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) = I^{(e)}{}_{\Delta}{}^{N}\,\tilde{\Phi}^{(e)}{}_{N}(\underset{\sim}{x},\underset{\sim}{\omega}) \quad , \quad \begin{array}{l} N = 1,2,\ldots,P_e \text{ and} \\ \Delta = 1,2,\ldots,P, \end{array} \tag{2.3.3}$$

and

$$\tilde{\Psi}^{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) = I^{(e)}{}_{\Delta}{}^{N}\tilde{\Phi}^{(e)}{}_{N}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\psi}^{\Delta} \quad , \quad \begin{array}{l} N = 1,2,\ldots,P_e \\ \Delta = 1,2,\ldots,P. \end{array} \tag{2.3.4}$$

Substituting Eqs. (2.3.3) and (2.3.4) in Eq. (2.3.2) we get

$$\begin{aligned}
&-\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M} I^{(e)}{}_{\Lambda}{}^{N}\int_{r_e}\int_{\omega_e} \underset{\sim}{\omega}\cdot\mathrm{grad}\tilde{\Phi}^{(e)}{}_{M}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}^{(e)}{}_{N}(\underset{\sim}{x},\underset{\sim}{\omega})\,d\omega dv\,\tilde{\psi}^{\Lambda} \\
&+\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M} I^{(e)}{}_{\Lambda}{}^{N}\int_{r_e}\int_{\omega_e} \beta(\underset{\sim}{x})\tilde{\Phi}^{(e)}{}_{M}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}^{(e)}{}_{N}(\underset{\sim}{x},\underset{\sim}{\omega})\,d\omega dv\,\tilde{\psi}^{\Lambda} = \\
&\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}\int_{r_e}\int_{\omega_e} \alpha(\underset{\sim}{x}) f[\Theta(\underset{\sim}{x},t)]\tilde{\Phi}^{(e)}{}_{M}(\underset{\sim}{x},\underset{\sim}{\omega})\,d\omega dv \\
&+\sum_{e=1}^{E}\sum_{e'=1}^{E} B(e,e') I^{(e)}{}_{\Delta}{}^{M} I^{(e')}{}_{\Lambda}{}^{N}\int_{r_e}\int_{\omega_e}\int_{\omega_{e'}} \sigma(\underset{\sim}{x})K(\underset{\sim}{\omega}\cdot\underset{\sim}{\omega}')\tilde{\Phi}^{(e)}{}_{M}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}^{(e')}{}_{N}(\underset{\sim}{x},\underset{\sim}{\omega})\,d\omega' d\omega d \\
&-\sum_{s=1}^{S}\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M} I^{(e)}{}_{\Lambda}{}^{N}\int_{\partial r_e{}^{s}}\int_{\omega_e{}^{+}} \tilde{\Phi}^{(e)}{}_{M}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}^{(e)}{}_{N}(\underset{\sim}{x},\underset{\sim}{\omega})\,|\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\,d\omega ds\,\tilde{\psi}^{\Lambda} \\
&+\sum_{s=1}^{S}\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}\int_{\partial r_e{}^{s}}\int_{\omega_e{}^{-}} T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega})\tilde{\Phi}^{(e)}{}_{M}(\underset{\sim}{x},\underset{\sim}{\omega})\,|\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\,d\omega ds \\
&+\sum_{s=1}^{S}\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}\int_{\partial r_e{}^{s}}\int_{\omega_e{}^{-}} \varepsilon_{(s)}(x,\omega) f[\Theta(x,t)]\tilde{\Phi}^{(e)}{}_{M}(x,\omega)\,|n\cdot\omega|\,d\omega ds
\end{aligned}$$

$$+ \sum_{s=1}^{S} \sum_{e=1}^{E} \sum_{e'=1}^{E} \mathcal{B}(e,e') \mathcal{I}^{(e)}{}_{\Delta}{}^{M} \mathcal{I}^{(e')}{}_{\Lambda}{}^{N} \int_{\partial r_e{}^s} \int_{\omega_e^-} \int_{\omega_{e'+}} [\, g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega}) \cdot$$

$$\cdot \tilde{\Phi}^{(e)}{}_M(\underset{\sim}{x},\underset{\sim}{\omega}) \tilde{\Phi}^{(e')}{}_N(\underset{\sim}{x},\underset{\sim}{\omega}') |\underset{\sim}{n} \cdot \underset{\sim}{\omega}'| |\underset{\sim}{n} \cdot \underset{\sim}{\omega}| d\omega' d\omega ds ] \tilde{\psi}^{\Lambda} , \quad (2.3.5)$$

$$M,N = 1,2,\ldots,P_e \qquad \Delta,\Lambda = 1,2,\ldots,P,$$

where

$$\mathcal{B} : \{(e,e') \,|\, e = e' = 1,2,\ldots,E\} \rightarrow \{0,1\}$$

defines a mapping such that

$$\mathcal{B}(e,e') = \begin{cases} 1 & \text{if } r_e \text{ coincides with } r_{e'}, \quad e = e' = 1,2,\ldots,E, \\ 0 & \text{if otherwise.} \end{cases}$$

Let

$$\underset{\sim}{\tilde{\psi}} = \{\tilde{\psi}^{\Delta}\} \; , \; \Delta = 1,2,\ldots,P \; ;$$

and for any $e = 1,2,\ldots,E$

$$\underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) = \{\tilde{\Phi}^{(e)}{}_M(\underset{\sim}{x},\underset{\sim}{\omega})\} \; , \qquad M = 1,2,\ldots,P_e,$$

and

$$\underset{\sim}{\mathcal{I}}_{(e)} = [\mathcal{I}^{(e)}{}_{\Delta}{}^{M}] \; , \qquad M = 1,2,\ldots,P_e \text{ and } \Delta = 1,2,\ldots,P.$$

Then Eq. (2.3.5) can be written in matrix form as follows.

$$
\begin{aligned}
&- \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e)}^{T} \int_{\mathcal{R}_e} \int_{\omega_e} \omega_i \frac{\partial \underset{\sim}{\tilde{\Phi}}_{(e)}}{\partial x_i} (\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e)}^{T}(\underset{\sim}{x},\underset{\sim}{\omega})\, d\omega d\upsilon \underset{\sim}{\tilde{\psi}} \\
&+ \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e)}^{T} \int_{\mathcal{R}_e} \int_{\omega_e} \beta(\underset{\sim}{x}) \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e)}^{T}(\underset{\sim}{x},\underset{\sim}{\omega})\, d\omega d\upsilon \underset{\sim}{\tilde{\psi}} = \\
&\sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \int_{\mathcal{R}_e} \int_{\omega_e} \alpha(\underset{\sim}{x}) f[\Theta(\underset{\sim}{x},t)] \underset{\sim}{\tilde{\Phi}}_{(e)} \\
&+ \sum_{e=1}^{E} \sum_{e'=1}^{E} B(e,e') \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e')}^{T} \int_{\mathcal{R}_e} \int_{\omega_e} \int_{\omega_{e'}} \sigma(\underset{\sim}{x}) K(\underset{\sim}{\omega}\cdot\underset{\sim}{\omega}') \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e')}^{T}(\underset{\sim}{x},\underset{\sim}{\omega})\, d\omega' d\omega \ldots \\
&- \sum_{s=1}^{S} \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e)}^{T} \int_{\partial \mathcal{R}_e s} \int_{\omega_e +} \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e)}^{T}(\underset{\sim}{x},\underset{\sim}{\omega}) |\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\, d\omega d\delta \underset{\sim}{\tilde{\psi}} \\
&+ \sum_{s=1}^{S} \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \int_{\partial \mathcal{R}_e s} \int_{\omega_e -} T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) |\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\, d\omega d\delta \\
&+ \sum_{s=1}^{S} \sum_{e=1}^{E} I_{(e)} \int_{\partial \mathcal{R}_e s} \int_{\omega_e -} \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) f[\Theta(\underset{\sim}{x},t)] \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) |\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\, d\omega d\delta \\
&+ \sum_{s=1}^{S} \sum_{e=1}^{E} \sum_{e'=1}^{E} B(e,e') \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e')}^{T} \int_{\partial \mathcal{R}_e s} \int_{\omega_e -} \int_{\omega_{e'} +} g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{e}(\underset{\sim}{x},\underset{\sim}{\omega}) \cdot \\
&\qquad \cdot \underset{\sim}{\tilde{\Phi}}_{(e')}^{T}(\underset{\sim}{x},\underset{\sim}{\omega}') |\underset{\sim}{n}\cdot\underset{\sim}{\omega}'| |\underset{\sim}{n}\cdot\underset{\sim}{\omega}|\, d\omega' d\omega d\delta \underset{\sim}{\tilde{\psi}}, \qquad (2.3.6)
\end{aligned}
$$

$$i = 1,2,3.$$

Let us define

$$\underset{\sim}{T} = \sum_{e=1}^{E} \underset{\sim}{T}_{(e)} = \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e)}^{T} \int_{\mathcal{R}_e} \int_{\omega_e} \omega_i \frac{\partial \underset{\sim}{\tilde{\Phi}}_{(e)}}{\partial x_i} (\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e)}^{T}(\underset{\sim}{x},\underset{\sim}{\omega})\, d\omega d\upsilon$$

where $\underset{\sim}{T}$ and $\underset{\sim}{T}_{(e)}$, $e = 1,2,\ldots,E$, are, respectively, referred to as the transport and the element transport matrices;

$$\underset{\sim}{E} = \sum_{e=1}^{E} \underset{\sim}{E}_{(e)} = \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e)}^{T} \int_{\mathcal{R}_e} \int_{\omega_e} \beta(\underset{\sim}{x}) \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e)}^{T}(\underset{\sim}{x},\underset{\sim}{\omega})\, d\omega dv$$

where $\underset{\sim}{E}$ and $\underset{\sim}{E}_{(e)}$, $e = 1,2,\ldots,E$, are, respectively, referred to as the extinction and the element extinction matrices;

$$\underset{\sim}{S} = \sum_{e=1}^{E} \sum_{e'=1}^{E} \mathcal{B}(e,e') \underset{\sim}{S}_{(e,e')}$$

$$= \sum_{e=1}^{E} \sum_{e'=1}^{E} \mathcal{B}(e,e') \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e')}^{T} \int_{\mathcal{R}_e} \int_{\omega_e} \int_{\omega_{e'}} \sigma(\underset{\sim}{x}) K(\underset{\sim}{\omega} \cdot \underset{\sim}{\omega}') \underset{\sim}{\tilde{\Phi}}_{(e)} \underset{\sim}{\tilde{\Phi}}_{(e')}^{T} d\omega' d\omega dv$$

where $\underset{\sim}{S}$ and $\underset{\sim}{S}_{(e,e')}$, $e = e' = 1,2,\ldots,E$, are, respectively, referred to as the scattering and the element scattering matrices;

$$\underset{\sim}{B}^{(s)} = \sum_{e=1}^{E} \underset{\sim}{B}^{(s)}_{(e)} = \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e)}^{T} \int_{\partial \mathcal{R}_e^{s}} \int_{\omega_e^{+}} \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e)}^{T}(\underset{\sim}{x},\underset{\sim}{\omega}) |\underset{\sim}{n} \cdot \underset{\sim}{\omega}|\, d\omega ds$$

where $\underset{\sim}{B}^{(s)}$ and $\underset{\sim}{B}^{(s)}_{(e)}$, $e = 1,2,\ldots,E$, are, respectively, referred to as the boundary and the element boundary matrices, $s = 1,2,\ldots,S$;

$$\underset{\sim}{R}^{(s)} = \sum_{e=1}^{E} \sum_{e'=1}^{E} \mathcal{B}(e,e') \underset{\sim}{R}^{(s)}_{(e,e')}$$

$$= \sum_{e=1}^{E} \sum_{e'=1}^{E} \mathcal{B}(e,e') \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e')}^{T} \int_{\partial \mathcal{R}_e^{s}} \int_{\omega_e^{-}} \int_{\omega_{e'}^{+}} g_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}',\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \cdot$$

$$\cdot \underset{\sim}{\tilde{\Phi}}_{(e')}^{T}(\underset{\sim}{x},\underset{\sim}{\omega}') |\underset{\sim}{n} \cdot \underset{\sim}{\omega}'| |\underset{\sim}{n} \cdot \underset{\sim}{\omega}|\, d\omega' d\omega ds$$

where $\underset{\sim}{R}^{(s)}$ and $\underset{\sim}{R}^{(s)}_{(e,e')}$, $e = e' = 1,2,\ldots,E$, are, respectively, referred to as the reflection and the element reflection matrices, $s = 1,2,\ldots,S$;

$$\underset{\sim}{s} = \sum_{e=1}^{E} \underset{\sim}{s}_{(e)} = \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \int_{\mathcal{R}_e} \int_{\omega_e} \alpha(\underset{\sim}{x}) f[\Theta(\underset{\sim}{x},t)] \tilde{\Phi}_{(e)}$$

where $\underset{\sim}{s}$ and $\underset{\sim}{s}_{(e)}$, $e = 1,2,\ldots,E$, are, respectively, referred to as the source and the element source vectors;

$$\underset{\sim}{t}^{(s)} = \sum_{e=1}^{E} \underset{\sim}{t}^{(s)}_{(e)} = \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \int_{\partial r_e{}^s} \int_{\omega_e^-} T_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \, |\underset{\sim}{n} \cdot \underset{\sim}{\omega}| \, d\omega ds$$

where $\underset{\sim}{t}^{(s)}$ and $\underset{\sim}{t}^{(s)}_{(e)}$, $e = 1,2,\ldots,E$, are, respectively, referred to as the transmission and element transmission vectors, $s = 1,2,\ldots,S$;

$$\underset{\sim}{e}^{(s)} = \sum_{e=1}^{E} \underset{\sim}{e}^{(s)}_{(e)} = \sum_{e=1}^{E} \underset{\sim}{I}_{(e)} \int_{\partial r_e{}^s} \int_{\omega_e^-} \varepsilon_{(s)}(\underset{\sim}{x},\underset{\sim}{\omega}) f[\Theta(\underset{\sim}{x},t)] \underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) \, |\underset{\sim}{n} \cdot \underset{\sim}{\omega}| \, d\omega ds$$

where $\underset{\sim}{e}$ and $\underset{\sim}{e}^{(s)}_{(e)}$, $e = 1,2,\ldots,E$, are, respectively, referred to as the emission and the element emission vectors, $s = 1,2,\ldots,S$. We can then write Eq. (2.3.6) in a compact form as follows.

$$[-\underset{\sim}{T} + \underset{\sim}{E} - \underset{\sim}{S} + \sum_{s=1}^{S} \underset{\sim}{B}^{(s)} - \sum_{s=1}^{S} \underset{\sim}{R}^{(s)}] \underset{\sim}{\tilde{\psi}} = [\underset{\sim}{s} + \sum_{s=1}^{S} \underset{\sim}{t}^{(s)} + \sum_{s=1}^{S} \underset{\sim}{e}^{(s)}] \quad (2.3.7)$$

Let $\underset{\sim}{\tilde{\psi}}^* = \{\tilde{\psi}^*_1, \tilde{\psi}^*_2, \ldots, \tilde{\psi}^*_P\}$ be the solution of Eq. (2.3.7). Then it immediately follows that the function

$$\tilde{\Psi}^*(\underset{\sim}{x},\underset{\sim}{\omega}) = \tilde{\psi}^{*\Delta} \tilde{\Phi}_\Delta(\underset{\sim}{x},\underset{\sim}{\omega}) \quad (2.3.8)$$

is the particular function sought such that Eq. (2.3.1) is valid for all $\tilde{\Phi}_\Delta \in \tilde{H}^P$, $\Delta = 1,2,\ldots,P$, and therefore, is the solution of the finite-element model of the Galerkin radiative transport problem. Since this model approximates the weak radiative transport problem which is an equivalent statement of the radiative transport problem, its solution provides the boundary value problem (1.1.1-2) with an approximate solution.

## PART THREE

## APPLICATION TO PROBLEMS OF RADIATIVE TRANSPORT IN PLANE-PARALLEL GRAY PARTICIPATING MEDIA WITH AZIMUTHAL SYMMETRY

In the previous parts of the present study we have theoretically developed a general finite-element model providing the radiative transport problem with an approximate solution scheme. With the general theory thus established, we proceed in this part with its applications.for the sake of simplicity, the attempted applications are constrained to radiative transport in plane-parallel gray participating media with azimuthal symmetry. It should be emphasized that our intention is not to provide a solution for a specific problem, but rather to substantiate the validity of the model developed, and at the same time the success of the finite-element approximation technique.

In Section 1 we derive the relevant finite-element equations, and in Section 2, confining our attention to linear rectangular elements, we give the explicit expressions for the corresponding system matrices and the system vectors. Finally, in Section 3, we present and discuss the results obtained for the attempted problems.

## 3.1 Finite Element Analysis

In Section 1.1, referring to a Cartesian coordinate system, we have shown that the boundary value problem governing radiative transport in a plane-parallel azimuthally symmetric gray participating medium of infinite extent in the x- and y- directions and of finite extent in the z-direction, $0 \leq z \leq z_o$, $z_o$ being an arbitrary positive scalar, is given in terms of the optical distance $\tau$ as follows.

$$\mu \frac{\partial}{\partial \tau} \Psi(\tau,\mu) + \Psi(\tau,\mu) = [1 - \gamma(\tau)] f[\Theta(\tau,t)]$$

$$+ 2\pi\gamma(\tau) \sum_{q=o}^{Q} a_q K_q(\mu) \int_{-1}^{1} K_q(\mu')\Psi(\tau,\mu')d\mu', \quad a_o = 1, \tag{1.1.31}$$

$$\text{for} \quad 0 < \tau < \tau_o \text{ and } -1 \leq \mu \leq 1,$$

$$\Psi(0,\mu) = T_{(1)}(\mu) + \varepsilon_{(1)}(\mu) f[\Theta(0,t)] - 2\rho^d_{(1)} \int_{-1}^{0} \Psi(0,\mu')\mu' d\mu' + \rho^s_{(1)} \Psi(0,-\mu)$$

$$\text{for} \quad 0 < \mu \leq 1, \tag{1.1.33}$$

and

$$\Psi(\tau_o,\mu) = T_{(2)}(\mu) + \varepsilon_{(2)}(\mu) f[\Theta(\tau_o,t)] + 2\rho^d_{(2)} \int_{0}^{1} \Psi(\tau_o,\mu')\mu' d\mu' + \rho^s_{(2)} \Psi(\tau_o,$$

$$\text{for} \quad -1 \leq \mu < 0. \tag{1.1.34}$$

The physical and analytical regions associated with the Eqs. (1.1.31), (1.1.33) and (1.1.34) are shown in Figure 3.1.1.

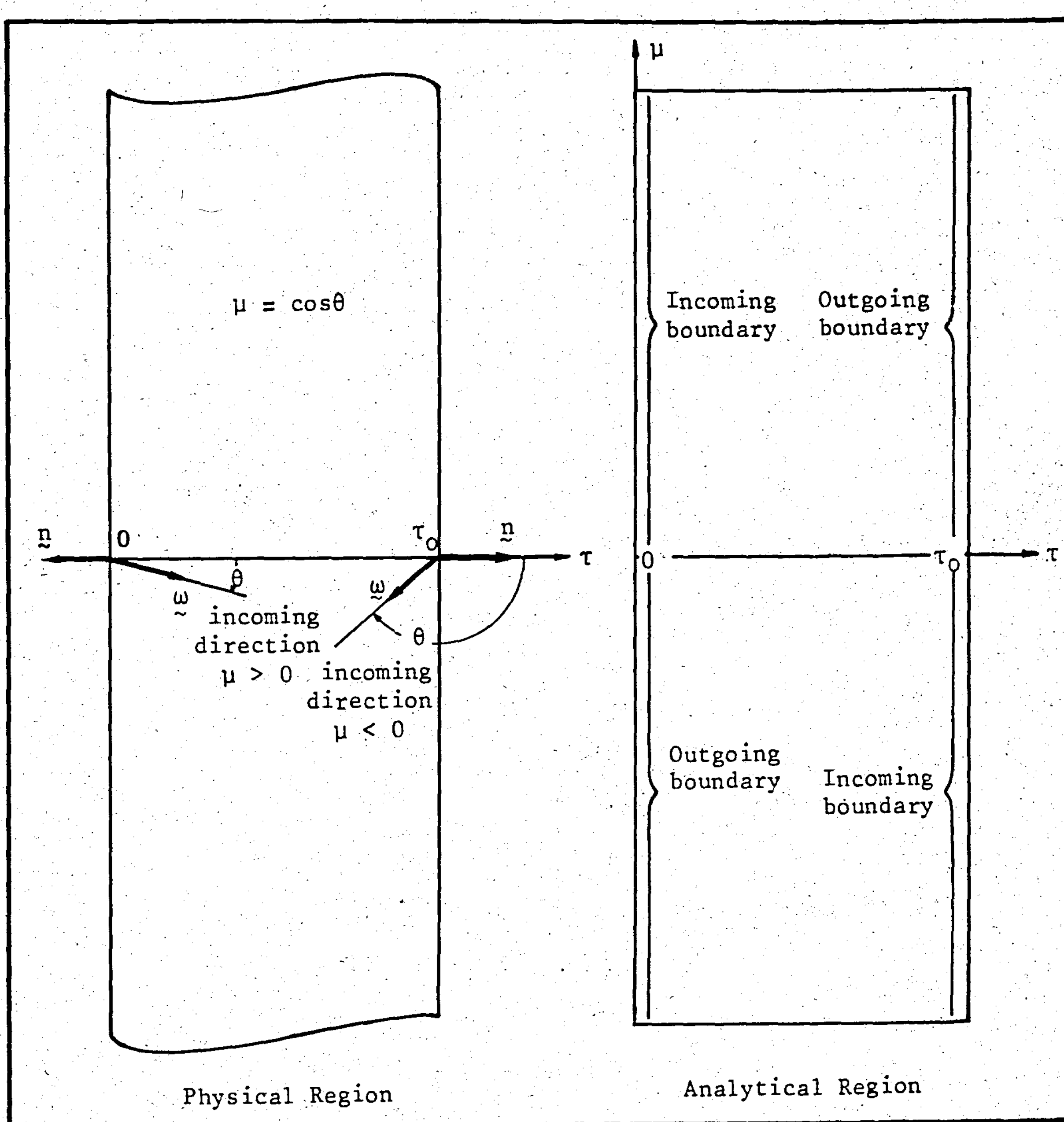


FIGURE 3.1.1 - Physical and analytical regions for the boundary value problem (1.1.31,33,34)

In Section 1.3 we have defined the solution space $G$ of the boundary value problem (1.1.31,33,34) to be the space of functions of class $C^{1,0}$ defin over the region $[0,\tau_o]\times[-1,1]$. Its corresponding first order finite-element model $\tilde{G}$ is then defined to be the space of functions of class $C^{o,o}$ defined over the finite-element model of the region$[0,\tau_o]x[-1,1]$. The finite element model of the region$[0,\tau_o]x[-1,1]$, on the other hand, is defined to be the Cartesian product of the individual finite-element models approximating the regions $[0,\tau_o]$and $[-1,1]$. Thus, in order to be able to specify $\tilde{G}$, we must first construct these individual models. Identification of the model approximating the region $[0,\tau_o]x[-1,1]$ is then straightforward.

To construct the finite-element model approximating the region $[0,\tau_o]$, let us identify

(i) a finite number $\tilde{P}$ of points $\tau^{\tilde{\Delta}}$, $\tilde{\Delta} = 1,2,\ldots,\tilde{P}$, in $[0,\tau_o]$,

(ii) a finite number $\tilde{E}$ of closed and disjoint subregions $\tau_{\tilde{e}} \subset [0,\tau_o]$, $\tilde{e} = 1,2,\ldots,\tilde{E}$,

(iii) a finite number of $P_{\tilde{e}}$ of points $\tau_{(\tilde{e})}^{\tilde{N}}$, $\tilde{N} = 1,2,\ldots,P_{\tilde{e}}$, in each subregion $\tau_{\tilde{e}}$ such that local compatibility conditions are satisfied [27, p. 36].

Then the finite-element model of the region $[0,\tau_o]$is the connected region constructed by assembling the subregions $\tau_{\tilde{e}}$ , $\tilde{e} = 1,2,\ldots,\tilde{E}$, together.

To construct the finite element model approximating the region $[-1,1]$, let us identify

(i) a finite number $\hat{P}$ of points $\mu^{\hat{\Delta}}$, $\hat{\Delta} = 1,2,\ldots,\hat{P}$, in $[-1,1]$ with the restriction that at least one of these $\hat{P}$ points coincides with the point $\mu = 0$,

(ii) a finite number $\hat{E}$ of closed and disjoint subregions $\mu_{\hat{e}} \subset [-1,1]$, $\hat{e} = 1,2,\ldots,\hat{E}$, with the restriction that for any subregion $\mu_{\hat{e}}$ there exists a subregion which is the mirror image of $\mu_{\hat{e}}$ with respect to the point $\mu = 0$,

(iii) a finite number $P_{\hat{e}}$ of points $\mu_{\hat{e}}^{\hat{N}}$, $\hat{N} = 1,2,\ldots,P_{\hat{e}}$, in each subregion $\mu_{\hat{e}}$ such that local compatibility conditions are satisfied [27, p. 36].

Then the finite element model of the region [-1,1] is the connected region constructed by assembling the subregions $\mu_{\hat{e}}$, $\hat{e} = 1,2,\ldots,\hat{E}$, together. The restriction on the choice of the points $\mu^{\hat{\Delta}}$, $\hat{\Delta} = 1,2,\ldots,\hat{P}$, is essential in order to be able to handle the integrals defined over the regions (-1,0) and (0,1). The restriction on the choice of the subregions $\mu_{\hat{e}}$, $\hat{e} = 1,2,\ldots,\hat{E}$, on the other hand, is essential only if it is necessary to consider specular reflection, and, therefore, it can be suspended if reflection is not specular.

Having established the individual finite-element models approximating the regions $[0,\tau_o]$ and $[-1,1]$, we can now identify the finite-element model approximating the region $[0,\tau_o]\times[-1,1]$ to be the connected region characterized by a finite number $P = \tilde{P}\hat{P}$ of points $(\tau,\mu)^{\Delta}$, $\Delta = 1,2,\ldots,P$, and a finite number $E = \tilde{E}\hat{E}$ of subregions $d_e$, $e = 1,2,\ldots,E$, each subregion $d_e$ being defined as

$$d_e = \tau_e \times \mu_e$$

with the condition that

$$\tau_e = C_{\tilde{e}}^{\tilde{e}}\, \tau_{\tilde{e}}\,, \qquad \tilde{e} = 1,2,\ldots,\tilde{E},$$

and

$$\mu_e = \hat{C}_e^{\hat{e}} \mu_{\hat{e}} , \qquad \hat{e} = 1,2,\ldots,\hat{E},$$

where $\tilde{C}_e^{\tilde{e}}$ and $\hat{C}_e^{\hat{e}}$ are, respectively, defined by Eqs. (2.1.4) and (2.1.5). From the definition of the subregions $d_e$, $e = 1,2,\ldots,E$, it follows that there exists within each subregion $d_e$ a finite number $P_e = \tilde{C}_e^{\tilde{e}} \hat{C}_e^{\hat{e}} P_{\tilde{e}} P_{\hat{e}}$, $e = 1,2,\ldots,E$ and $\hat{e} = 1,2,\ldots,\hat{E}$, of points $(\tau,\mu)_{(e)}^N$, $N = 1,2,\ldots,P_e$; and, since the subregions $\tau_{\tilde{e}}$, $\tilde{e} = 1,2,\ldots,\tilde{E}$ and $\mu_{\hat{e}}$, $\hat{e} = 1,2,\ldots,\hat{E}$, are locally compatible, the subregions $d_e$, $e = 1,2,\ldots,E$ are also locally compatible. Hence, for $e = 1,2,\ldots,E$

$$(\tau^N,\mu^N)_{(e)} = I^{(e)}{}_{\Delta}^{N} (\tau^\Delta,\mu^\Delta) , \qquad N = 1,2,\ldots,P_e \text{ and } \Delta = 1,2,\ldots,P,$$

and

$$(\tau\Delta,\mu\Delta) = I_{(e)}{}^{-1}{}_{N}{}^{\Delta}(\tau^N,\mu^N)_{(e)} , \quad N = 1,2,\ldots,P_e \text{ and } \Delta = 1,2,\ldots,P,$$

where

$$I^{(e)}{}_{\Delta}^{N} = \begin{cases} 1 & \text{if the point } (\tau,\mu)^\Delta \text{ is incident on the point } (\tau,\mu)_{(e)}^N, \\ 0 & \text{if otherwise,} \end{cases}$$

$$I_{(e)}{}^{-1}{}_{N}{}^{\Delta} = \begin{cases} 1 & \text{if the point } (\tau,\mu)_{(e)}^N \text{ is incident on the point } (\tau,\mu)^\Delta, \\ 0 & \text{if otherwise.} \end{cases}$$

A typical finite element model approximating the region $[0,\tau_o]\times[-1,1]$ and its constructional details are shown in Figure 3.1.2.

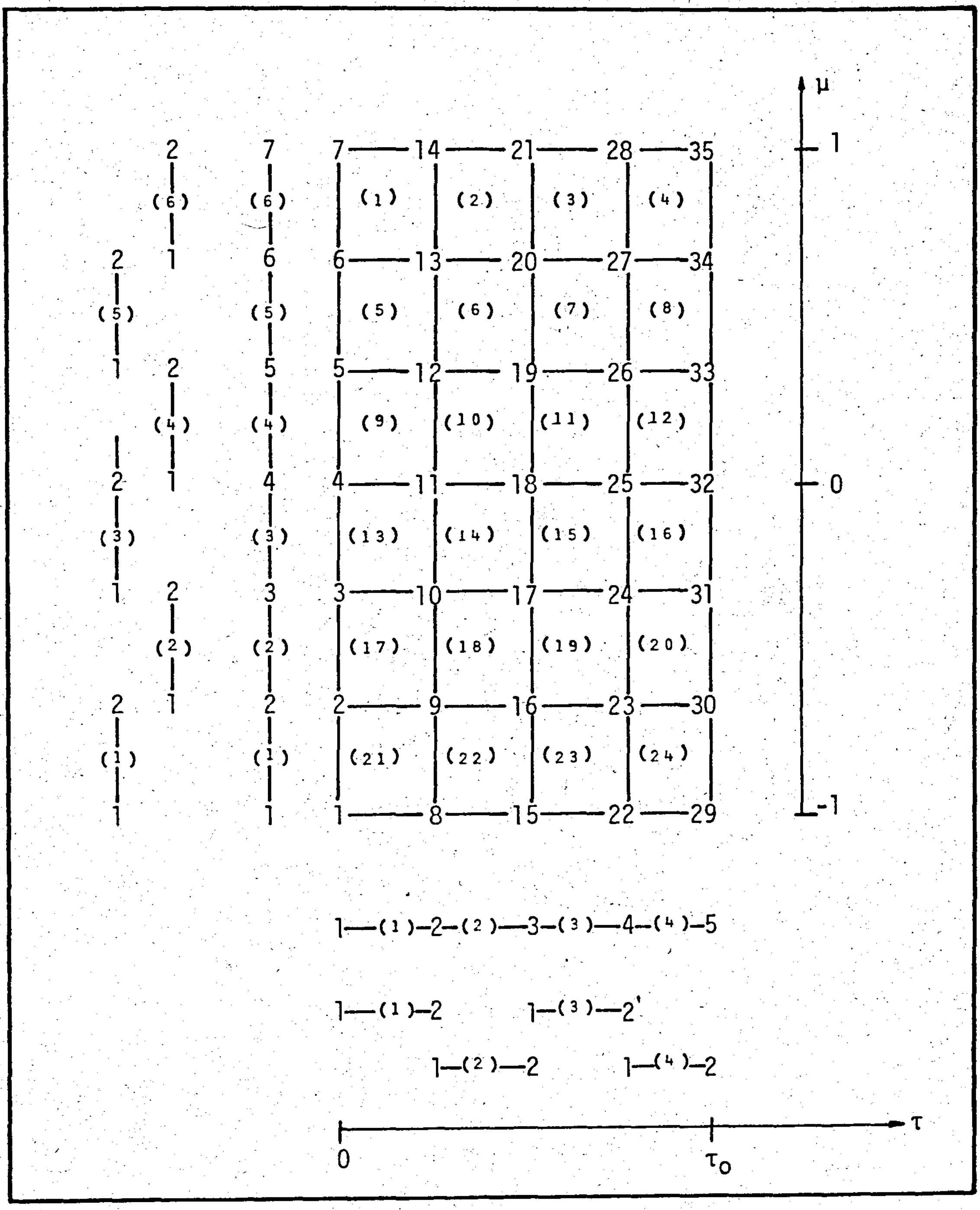


FIGURE 3.1.2 - A typical finite-element model approximating the region $[0,\tau_o]x[-1,1]$

A connected region approximating the region $[0,\tau_o]x[-1,1]$ having thus been constructed, we can now identify the space $\tilde{G}$ approximating the solution space $G$ of the boundary value problem (1.1.31,33,34) to be the space of functions of class $C^{o,o}$ defined over this connected region.

With the space $\tilde{G}$ now defined, we proceed in an analogous manner with the general formulation and generate a set of finite-element basis functions $\tilde{\Phi}_\Delta = \tilde{\Phi}_\Delta(\tau,\mu)$, $\Delta = 1,2,\ldots,P$, spanning a finite dimensional subspace $\tilde{G}^P \subset \tilde{G}$. Such a set of basis functions having been obtained, it then follows that for $e = 1,2,\ldots,E$

$$\tilde{\Phi}_\Delta^{(e)}(\tau,\mu) = I^{(e)}{}_\Delta^N \, \tilde{\Phi}^{(e)}{}_N(\tau,\mu) \;, \qquad N = 1,2,\ldots,P_e \text{ and } \Delta = 1,2,\ldots,P,$$

and

$$\tilde{\Psi}^{(e)}(\tau,\mu) = I^{(e)}{}_\Delta^N \, \tilde{\Phi}^{(e)}{}_N(\tau,\mu)\psi^\Delta, \qquad N = 1,2,\ldots,P_e \text{ and } \Delta = 1,2,\ldots,P,$$

where $\tilde{\Phi}^{(e)}{}_N = \tilde{\Phi}^{(e)}{}_N(\tau,\mu)$, $N = 1,2,\ldots,P_e$, are the local basis functions defined over the subregion $d_e$; where $\tilde{\Phi}_\Delta{}^{(e)} = \tilde{\Phi}_\Delta{}^{(e)}(\tau,\mu)$, $\Delta = 1,2,\ldots,P$, and $\tilde{\Psi}^{(e)} = \tilde{\Psi}^{(e)}(\tau,\mu)$ are, respectively, the restrictions of the global basis functions $\tilde{\Phi}_\Delta = \tilde{\Phi}_\Delta(\tau,\mu)$, $\Delta = 1,2,\ldots,P$, and of any function $\tilde{\Psi} = \tilde{\Psi}(\tau,\mu)$ in $\tilde{G}^P$ to the subregion $d_e$; and where $\psi^\Delta = \tilde{\Psi}[(\tau,\mu)^\Delta]$, $\Delta = 1,2,\ldots,P$.

Constraining the solution space of the boundary value problem (1.1.31,33,34) to the space $\tilde{G}$, it can easily be shown that the finite element model of the corresponding Galerkin problem reduces to the problem of solving the following system of algebraic equations.

$$- \sum_{e=1}^{E} I^{(e)}{}_\Delta^M I^{(e)}{}_\Lambda^N \int_{\tau_e}\int_{\mu_e} \frac{\partial}{\partial\tau} \tilde{\Phi}^{(e)}{}_M(\tau,\mu)\tilde{\Phi}^{(e)}{}_N(\tau,\mu)\mu\, d\mu\, d\tau\, \tilde{\psi}^\Lambda$$

$$+ \sum_{e=1}^{E} I^{(e)}{}_\Delta^M I^{(e)}{}_\Lambda^N \int_{\tau_e}\int_{\mu_e} \tilde{\Phi}^{(e)}{}_M(\tau,\mu)\tilde{\Phi}^{(e)}{}_N(\tau,\mu)\, d\mu\, d\tau\, \tilde{\psi}^\Delta =$$

$$\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}\int_{\tau_e}\int_{\mu_e}[1-\gamma(\tau)]f[\Theta(\tau,t)]\tilde{\Phi}^{(e)}{}_{M}(\tau,\mu)\,d\mu\,d\tau$$

$$+\ 2\pi\sum_{q=0}^{Q}\sum_{e=1}^{E}\sum_{e'=1}^{E} B(e,e')I^{(e)}{}_{\Delta}{}^{M}I^{(e')}{}_{\Lambda}{}^{N}\int_{\tau_e}\int_{\mu_e}\gamma(\tau)a_qK_q(\mu)\tilde{\Phi}^{(e)}{}_{M}(\tau,\mu)\ \cdot$$

$$\cdot\int_{\mu_{e'}}K_q(\mu')\tilde{\Phi}^{(e')}{}_{N}(\tau,\mu')\,d\mu'\,d\mu\,d\tau\,\tilde{\psi}^{\Lambda}$$

$$-\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}I^{(e)}{}_{\Lambda}{}^{N}\int_{\mu_e^{+}}\tilde{\Phi}^{(e)}{}_{M}(\tau_0,\mu)\tilde{\Phi}^{(e)}{}_{N}(\tau_0,\mu)\,\mu\,d\mu\,\tilde{\psi}^{\Lambda}$$

$$+\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}I^{(e)}{}_{\Lambda}{}^{N}\int_{\mu_e^{-}}\tilde{\Phi}^{(e)}{}_{M}(0,\mu)\tilde{\Phi}^{(e)}{}_{N}(0,\mu)\,\mu\,d\mu\,\tilde{\psi}^{\Lambda}$$

$$-\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}\int_{\mu_e^{-}}T_{(2)}(\mu)\tilde{\Phi}^{(e)}{}_{M}(\tau_0,\mu)\,\mu\,d\mu$$

$$-\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}\int_{\mu_e^{-}}\varepsilon_{(2)}(\mu)f[\Theta(\tau_0,t)]\tilde{\Phi}^{(e)}{}_{M}(\tau_0,\mu)\,\mu\,d\mu$$

$$-\ 2\rho^{d}_{(2)}\sum_{e=1}^{E}\sum_{e'=1}^{E} I^{(e)}{}_{\Delta}{}^{M}I^{(e')}{}_{\Lambda}{}^{N}\int_{\mu_e^{-}}\tilde{\Phi}^{(e)}{}_{M}(\tau_0,\mu)\int_{\mu_{e'}^{+}}\tilde{\Phi}^{(e')}{}_{N}(\tau_0,\mu')\mu'\,d\mu'\,\mu\,d\mu\,\tilde{\psi}^{\Lambda}$$

$$-\ \rho^{s}_{(2)}\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}I^{(e^*)}{}_{\Lambda}{}^{N}\int_{\mu_e^{-}}\tilde{\Phi}^{(e)}{}_{M}(\tau_0,\mu)\tilde{\Phi}^{(e^*)}{}_{N}(\tau_0,-\mu)\,\mu\,d\mu\,\tilde{\psi}^{\Lambda}$$

$$+\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}\int_{\mu_e^{+}}T_{(1)}(\mu)\tilde{\Phi}^{(e)}{}_{M}(0,\mu)\,\mu\,d\mu$$

$$+\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}\int_{\mu_e^{+}}\varepsilon_{(1)}(\mu)f[\Theta(0,t)]\tilde{\Phi}^{(e)}{}_{M}(0,\mu)\,\mu\,d\mu$$

$$-\ 2\rho^{d}_{(1)}\sum_{e=1}^{E}\sum_{e'=1}^{E} I^{(e)}{}_{\Delta}{}^{M}I^{(e')}{}_{\Lambda}{}^{N}\int_{\mu_e^{+}}\tilde{\Phi}^{(e)}{}_{M}(0,\mu)\int_{\mu_{e'}^{-}}\tilde{\Phi}^{(e')}{}_{N}(0,\mu')\mu'\,d\mu'\,\mu\,d\mu\,\tilde{\psi}^{\Lambda}$$

$$+\ \rho^{s}_{(1)}\sum_{e=1}^{E} I^{(e)}{}_{\Delta}{}^{M}I^{(e^*)}{}_{\Lambda}{}^{N}\int_{\mu_e^{+}}\tilde{\Phi}^{(e)}{}_{M}(0,\mu)\tilde{\Phi}^{(e^*)}{}_{N}(0,-\mu)\,\mu\,d\mu\,\tilde{\psi}^{\Lambda} \qquad (3.1.1)$$

where $\mu_e^+ = \mu_e \cap (0,1]$ and $\mu_e^- = \mu_e \cap [-1,0)$ , $\mathcal{B}$ is a mapping from $\{(e,e') | e = e' = 1,2,\ldots,E\}$ to $\{0,1\}$ defined such that $\mathcal{B}(e,e') = 1$ if $\tau_e$ coincides with $\tau_{e'}$ and $\mathcal{B}(e,e') = 0$ if otherwise, and the label $e^*$ identifies the mirror image with respect to the line $\mu = 0$ of the element labeled as $e$. Let

$$\underset{\sim}{\tilde{\psi}} = \{\tilde{\psi}^\Delta\} , \qquad \Delta = 1,2,\ldots,P;$$

and for any $e = 1,2,\ldots,E$

$$\underset{\sim}{\tilde{\Phi}}_{(e)}(\underset{\sim}{x},\underset{\sim}{\omega}) = \{\tilde{\Phi}^{(e)}{}_M(\tau,\mu)\} , \qquad M = 1,2,\ldots,P_e,$$

and

$$\underset{\sim}{I}_{(e)} = [I^{(e)}{}_\Delta^M] , \qquad M = 1,2,\ldots,P_e \qquad \Delta = 1,2,\ldots,P.$$

Then the system of equations (3.1.1) can be written in matrix form as follows.

$$(-\underset{\sim}{T}\ \underset{\sim}{E}-\underset{\sim}{S}+\underset{\sim}{B}^{(2)}-\underset{\sim}{B}^{(1)}+\underset{\sim}{d_R}^{(2)}+\underset{\sim}{s_R}^{(2)}+\underset{\sim}{d_R}^{(1)}-\underset{\sim}{s_R}^{(1)})\underset{\sim}{\tilde{\psi}} = \underset{\sim}{s}-\underset{\sim}{t}^{(2)}-\underset{\sim}{e}^{(2)}+\underset{\sim}{t}^{(1)}+\underset{\sim}{e}^{(1)} \quad (3.1.2)$$

In Eq. (3.1.2) $\underset{\sim}{T}$ is the transport matrix defined as

$$\underset{\sim}{T} = \sum_{e=1}^{E} \underset{\sim}{T}_{(e)}$$

where $\underset{\sim}{T}_{(e)}$ denotes the element transport matrix and is given by

$$\underset{\sim}{T}_{(e)} = \underset{\sim}{I}_{(e)}\underset{\sim}{I}_{(e)}{}^T\int_{\tau_e}\int_{\mu_e} \frac{\partial}{\partial\tau} \underset{\sim}{\tilde{\Phi}}_{(e)}(\tau,\mu)\underset{\sim}{\tilde{\Phi}}_{(e)}{}^T(\tau,\mu)\mu d\mu d\tau; \qquad (3.1.3)$$

$\underset{\sim}{E}$ is the extinction matrix defined as

$$\underset{\sim}{E} = \sum_{e=1}^{E} \underset{\sim}{E}_{(e)} ,$$

where $\underset{\sim}{E}_{(e)}$ denotes the element extinction matrix and is given by

$$\underset{\sim}{E}_{(e)} = \underset{\sim}{I}_{(e)}\underset{\sim}{I}_{(e)}^{T}\int_{\tau_e}\int_{\mu_e} \tilde{\underset{\sim}{\Phi}}_{(e)}(\tau,\mu)\tilde{\underset{\sim}{\Phi}}_{(e)}^{T}(\tau,\mu)\mu d\mu \; ; \tag{3.1.4}$$

$\underset{\sim}{S}$ is the scattering matrix defined as

$$\underset{\sim}{S} = \sum_{e=1}^{E}\sum_{e'=1}^{E} B(e,e')\underset{\sim}{S}_{(e,e')} \; ,$$

where $\underset{\sim}{S}_{(e,e')}$ denotes the element scattering matrix and is given by

$$\underset{\sim}{S}_{(e,e')} = 2\pi\sum_{q=0}^{Q} \underset{\sim}{I}_{(e)}\underset{\sim}{I}_{(e')}^{T}\int_{\tau_e}\int_{\mu_{e'}} \gamma(\mu)a_q K_q(\mu)\tilde{\underset{\sim}{\Phi}}_{(e)}(\tau,\mu) \cdot \tag{3.1.5}$$
$$\cdot \int_{\mu_{e'}} K_q(\mu')\tilde{\underset{\sim}{\Phi}}_{(e')}^{T}(\tau,\mu')d\mu' d\mu d\tau \; ;$$

$\underset{\sim}{B}^{(1)}$, $\underset{\sim}{B}^{(2)}$ are the boundary matrices defined as

$$\underset{\sim}{B}^{(1)} = \sum_{e=1}^{E} \underset{\sim}{B}^{(1)}_{(e)}$$

and

$$\underset{\sim}{B}^{(2)} = \sum_{e=1}^{E} \underset{\sim}{B}^{(2)}_{(e)} \; ;$$

where $\underset{\sim}{B}^{(1)}_{(e)}$, $\underset{\sim}{B}^{(2)}_{(e)}$ denote the element boundary matrices and are given by

$$\underset{\sim}{B}^{(1)}_{(e)} = \underset{\sim}{I}_{(e)}\underset{\sim}{I}_{(e)}^{T}\int_{\mu_e^-} \tilde{\underset{\sim}{\Phi}}_{(e)}(0,\mu)\tilde{\underset{\sim}{\Phi}}_{(e)}^{T}(0,\mu)\mu d\mu \tag{3.1.6}$$

and

$$\underset{\sim}{B}^{(2)}_{(e)} = \underset{\sim}{I}_{(e)}\underset{\sim}{I}_{(e)}^{T}\int_{\mu_e^+} \tilde{\underset{\sim}{\Phi}}_{(e)}(\tau_0,\mu)\tilde{\underset{\sim}{\Phi}}_{(e)}^{T}(\tau_0,\mu)\mu d\mu \; ; \tag{3.1.7}$$

${}^{d}\underset{\sim}{R}^{(1)}$, ${}^{d}\underset{\sim}{R}^{(2)}$ are the diffuse reflection matrices defined as

$${}^{d}\underset{\sim}{R}^{(1)} = \sum_{e=1}^{E}\sum_{e'=1}^{E} {}^{d}\underset{\sim}{R}^{(1)}_{(e,e')}$$

and

$$\underset{\sim}{{}^{d}R}^{(2)} = \sum_{e=1}^{E} \sum_{e'=1}^{E} \underset{\sim}{{}^{d}R}^{(2)}_{(e,e')} ,$$

where $\underset{\sim}{{}^{d}R}^{(1)}_{(e,e')}$, $\underset{\sim}{{}^{d}R}^{(2)}_{(e,e')}$ denote the element diffuse reflection matrices and are given by

$$\underset{\sim}{{}^{d}R}^{(1)}_{(e,e')} = 2\rho^{d}_{(1)} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e')}^{T} \int_{\mu_e+} \underset{\sim}{\tilde{\Phi}}_{(e)}(0,\mu) \int_{\mu_{e'}-} \underset{\sim}{\tilde{\Phi}}_{(e')}^{T}(0,\mu')\mu' d\mu' \mu d\mu \quad (3.1.8)$$

and

$$\underset{\sim}{{}^{d}R}^{(2)}_{(e,e')} = 2\rho^{d}_{(2)} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e')}^{T} \int_{\mu_e-} \underset{\sim}{\tilde{\Phi}}_{(e)}(\tau_0,\mu) \int_{\mu_{e'}+} \underset{\sim}{\tilde{\Phi}}_{(e')}^{T}(\tau_0,\mu')\mu' d\mu' \mu d\mu; \quad (3.1.9)$$

$\underset{\sim}{{}^{s}R}^{(1)}$, $\underset{\sim}{{}^{s}R}^{(2)}$ are the specular reflection matrices defined as

$$\underset{\sim}{{}^{s}R}^{(1)} = \sum_{e=1}^{E} \underset{\sim}{{}^{s}R}^{(1)}_{(e)}$$

and

$$\underset{\sim}{{}^{s}R}^{(2)} = \sum_{e=1}^{E} \underset{\sim}{{}^{s}R}^{(2)}_{(e)} ,$$

where $\underset{\sim}{{}^{s}R}^{(1)}_{(e)}$, $\underset{\sim}{{}^{s}R}^{(2)}_{(e)}$ denote the element specular matrices and are given by

$$\underset{\sim}{{}^{s}R}^{(1)}_{(e)} = \rho^{s}_{(1)} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e*)}^{T} \int_{\mu_e+} \underset{\sim}{\tilde{\Phi}}_{(e)}(0,\mu) \underset{\sim}{\tilde{\Phi}}_{(e*)}^{T}(0,-\mu)\mu d\mu \quad (3.1.10)$$

and

$$\underset{\sim}{{}^{s}R}^{(2)}_{(e)} = \rho^{s}_{(2)} \underset{\sim}{I}_{(e)} \underset{\sim}{I}_{(e*)}^{T} \int_{\mu_e-} \underset{\sim}{\tilde{\Phi}}_{(e)}(\tau_0,\mu) \underset{\sim}{\tilde{\Phi}}_{(e*)}^{T}(\tau_0,-\mu)\mu d\mu ; \quad (3.1.11)$$

$\underset{\sim}{S}$ is the source vector defined as

$$\underset{\sim}{S} = \sum_{e=1}^{E} \underset{\sim}{S}_{(e)} ,$$

where $\underset{\sim}{S}_{(e)}$ denotes the element source vector and is given by

$$\underset{\sim}{S}_{(e)} = \underset{\sim}{I}_{(e)}\int_{\tau_e}\int_{\mu_e} [1 - \gamma(\tau)]f[\Theta(\tau,t)]\underset{\sim}{\tilde{\Phi}}_{(e)}(\tau,\mu)d\mu d\tau \; ; \qquad (3.1.12)$$

$\underset{\sim}{t}^{(1)}$, $\underset{\sim}{t}^{(2)}$ are the transmission vectors defined as

$$\underset{\sim}{t}^{(1)} = \sum_{e=1}^{E} \underset{\sim}{t}^{(1)}{}_{(e)}$$

and

$$\underset{\sim}{t}^{(2)} = \sum_{e=1}^{E} \underset{\sim}{t}^{(2)}{}_{(e)} ,$$

where $\underset{\sim}{t}^{(1)}{}_{(e)}$, $\underset{\sim}{t}^{(2)}{}_{(e)}$ denote the element transmission vectors and are given by

$$\underset{\sim}{t}^{(1)}{}_{(e)} = \underset{\sim}{I}_{(e)}\int_{\mu_e^+} T_{(1)}(\mu)\underset{\sim}{\tilde{\Phi}}_{(e)}(0,\mu)\mu d\mu \qquad (3.1.13)$$

and

$$\underset{\sim}{t}^{(2)}{}_{(e)} = \underset{\sim}{I}_{(e)}\int_{\mu_e^-} T_{(2)}(\mu)\underset{\sim}{\tilde{\Phi}}_{(e)}(\tau_0,\mu)\mu d\mu \; ; \qquad (3.1.14)$$

$\underset{\sim}{e}^{(1)}$, $\underset{\sim}{e}^{(2)}$ are the emission vectors defined as

$$\underset{\sim}{e}^{(1)} = \sum_{e=1}^{E} \underset{\sim}{e}^{(1)}{}_{(e)}$$

and

$$\underset{\sim}{e}^{(2)} = \sum_{e=1}^{E} \underset{\sim}{e}^{(2)}{}_{(e)} ,$$

where $\underset{\sim}{e}^{(1)}{}_{(e)}$, $\underset{\sim}{e}^{(2)}{}_{(e)}$ denote the element emission vectors and are given by

$$\underset{\sim}{e}^{(1)}{}_{(e)} = \underset{\sim}{I}_{(e)}\int_{\mu_e^+} \varepsilon_{(1)}(\mu)f[\Theta(0,t)]\underset{\sim}{\tilde{\Phi}}_{(e)}(0,\mu)\mu d\mu \qquad (3.1.15)$$

and

$$\underset{\sim}{e}^{(2)}{}_{(e)} = \underset{\sim}{I}_{(e)} \int_{\mu_e^-} \varepsilon_{(2)}(\mu) f[\Theta(\tau_o,t)] \underset{\sim}{\tilde{\Phi}}_{(e)}(\tau_o,\mu)\mu d\mu. \tag{3.1.16}$$

## 3.2 Element Matrices and Element Vectors

In Section 3.1 we have formulated a finite-element model of the Galerkin approximation of the weak problem associated with the boundary value problem (1.1.31,33,34). The finite elements incorporated in this formulation, however, have been left arbitrary except for the requirement that they be the Cartesian product of the corresponding spatial and angular subregions. Consequently, the local basis functions and, therefore, the local approximations over each element have been treated as arbitrary with the assumption that the continuity, normality and locality requirements are fulfilled. In order to be able to compute, however, this generality must be abandoned, and specific types of elements as well as the local basis functions associated with them must be identified.

For the sake of simplicity, let us confine our attention to <u>linear</u> rectangular elements, Figure 3.2.1. Then it can be shown that

$$\tilde{\Psi}^{(e)}(\tau,\mu) = \tilde{\Phi}^{(e)}{}_1(\tau,\mu)\tilde{\psi}_{(e)}{}^1 + \tilde{\Phi}^{(e)}{}_2(\tau,\mu)\tilde{\psi}_{(e)}{}^2 + \tilde{\Phi}^{(e)}{}_3(\tau,\mu)\tilde{\psi}_{(e)}{}^3 +$$

$$+ \tilde{\Phi}^{(e)}{}_4(\tau,\mu)\tilde{\psi}_{(e)}{}^4 \tag{3.2.1}$$

where

$$\tilde{\Phi}^{(e)}{}_1 = \tilde{\Phi}^{(e)}{}_1(\tau,\mu) = \begin{cases} \dfrac{(\tau_2-\tau)(\mu_4-\mu)}{(\tau_2-\tau_1)(\mu_4-\mu_1)} & \text{for } \tau \in [\tau_1,\tau_2] \text{ and } \mu \in [\mu_1,\mu_4], \\ 0 & \text{for } \tau \notin [\tau_1,\tau_2] \text{ or/and } \mu \notin [\mu_1,\mu_4]; \end{cases}$$

$$\tilde{\Phi}^{(e)}{}_2 = \tilde{\Phi}^{(e)}{}_2(\tau,\mu) = \begin{cases} \dfrac{(\tau-\tau_1)(\mu_4-\mu)}{(\tau_2-\tau_1)(\mu_4-\mu_1)} & \text{for } \tau \in [\tau_1,\tau_2] \text{ and } \mu \in [\mu_1,\mu_4], \\ 0 & \text{for } \tau \notin [\tau_1,\tau_2] \text{ or/and } \mu \notin [\mu_1,\mu_4]; \end{cases}$$

$$\tilde{\Phi}^{(e)}{}_3 = \tilde{\Phi}^{(e)}{}_3(\tau,\mu) = \begin{cases} \dfrac{(\tau-\tau_1)(\mu-\mu_1)}{(\tau_2-\tau_1)(\mu_4-\mu_1)} & \text{for } \tau \in [\tau_1,\tau_2] \text{ and } \mu \in [\mu_1,\mu_4], \\ 0 & \text{for } \tau \notin [\tau_1,\tau_2] \text{ or/and } \mu \notin [\mu_1,\mu_4]; \end{cases}$$

$$\tilde{\Phi}^{(e)}{}_4 = \tilde{\Phi}^{(e)}{}_4(\tau,\mu) = \begin{cases} \dfrac{(\tau_2-\tau)(\mu-\mu_1)}{(\tau_2-\tau_1)(\mu_4-\mu_1)} & \text{for } \tau \in [\tau_1,\tau_2] \text{ and } \mu \in [\mu_1,\mu_4], \\ 0 & \text{for } \tau \notin [\tau_1,\tau_2] \text{ or/and } \mu \notin [\mu_1,\mu_4]; \end{cases}$$

and

$$\tilde{\psi}_{(e)}{}^N = \tilde{\Psi}_{(e)}[(\tau,\mu)^N] \quad , \quad N = 1,2,3,4.$$

Then for $(\tau,\mu) \in [\tau_1,\tau_2]\times[\mu_1,\mu_4]$

$$\tilde{\underset{\sim}{\Phi}}_{(e)} = \tilde{\underset{\sim}{\Phi}}_{(e)}(\tau,\mu) = \frac{1}{(\tau_2-\tau_1)}\frac{1}{(\mu_4-\mu_1)} \begin{bmatrix} (\tau_2-\tau)(\mu_4-\mu) \\ (\tau-\tau_1)(\mu_4-\mu) \\ (\tau-\tau_1)(\mu-\mu_1) \\ (\tau_2-\tau)(\mu-\mu_1) \end{bmatrix} . \tag{3.2.2}$$

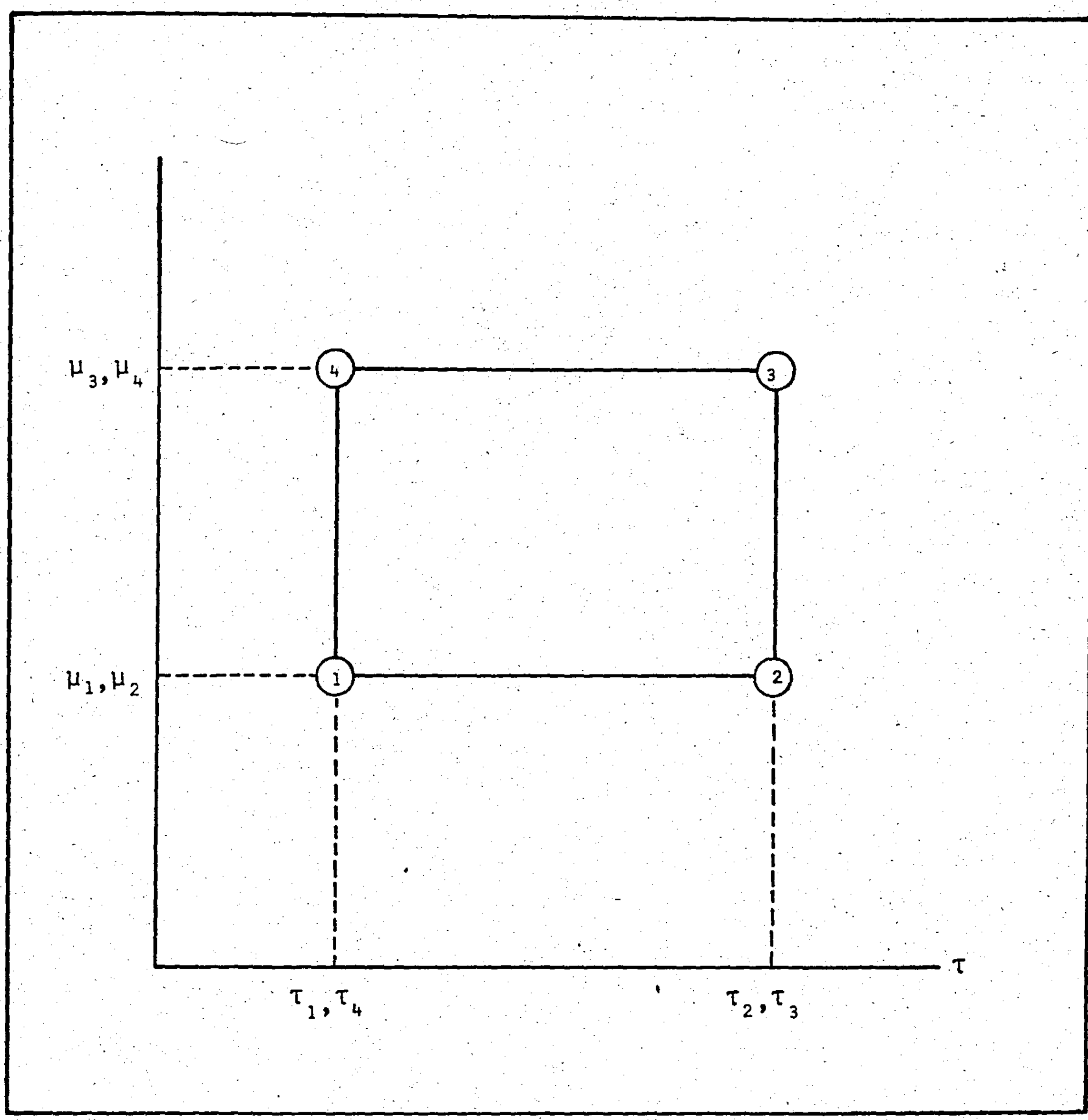


FIGURE 3.2.1 - A linear rectangular element

As can be seen the basis functions $\tilde{\Phi}^{(e)}_N$, $N = 1,2,3,4$ (i) are defined locally over the region $[\tau_1,\tau_2]\times[\mu_1,\mu_4]$, (ii) are normalized with respect to the local nodes $(\tau,\mu)^N$, $N = 1,2,3,4$, and (iii) are continuous at any point $(\tau,\mu) \in [\tau_1,\tau_2]\times[\mu_1,\mu_4]$. Equally important, since the element boundaries are parallel to the coordinate lines, the continuity along the interelement boundaries in the connected model is also guaranteed.

Having specified the particular type of the finite element to be used, and obtained the explicit expressions for the associated local basis functions, we now proceed to derive the corresponding expressions of the element matrices and the element vectors involved in Eq. (3.1.2). Let us assume that in the overall assembly the local nodes labeled as 1,2,3 and 4 in the element with label $(e)$ are, respectively, carried over to the global nodes with labels $i$, $j$, $k$ and $\ell$ in the connected model. Then

$$\underset{\sim}{I}_{(e)} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i\text{th row} \\ \\ \\ \\ \leftarrow j\text{th row} \\ \\ \\ \\ \leftarrow k\text{th row} \\ \\ \\ \\ \leftarrow \ell\text{th row} \\ \\ \\ \\ \end{matrix} \qquad (3.2.3)$$

and, by going through a simple yet tedious algebra, the explicit expressions for the element matrices and the element vectors can easily be derived. For general reference, in Appendix A we give a list of integrals used in the derivation. The results are as follows.

*Element Transport Matrix*

From Eqs. (3.1.3), (3.2.2) and (3.2.3) it follows that

$$\underset{\sim}{T}_{(e)} = \frac{1}{24(\mu_4 - \mu_1)} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \ldots 1 \ldots 0 \ldots 0 \ldots 0 \ldots \\ \ldots 0 \ldots 1 \ldots 0 \ldots 0 \ldots \\ \ldots 0 \ldots 0 \ldots 1 \ldots 0 \ldots \\ \ldots 0 \ldots 0 \ldots 0 \ldots 1 \ldots \end{bmatrix} \begin{bmatrix} -T_1 & -T_1 & -T_2 & -T_2 \\ T_1 & T_1 & T_2 & T_2 \\ T_2 & T_2 & T_3 & T_3 \\ -T_2 & -T_2 & -T_3 & -T_3 \end{bmatrix} \quad (3.2$$

where

$$T_1 = \mu_4^3 + \mu_1\mu_4^2 - 5\mu_1^2\mu_4 + 3\mu_1^3,$$

$$T_2 = (\mu_4 - \mu_1)^2(\mu_1 + \mu_4),$$

and

$$T_3 = 3\mu_4^3 - 5\mu_1\mu_4^2 + \mu_1^2\mu_4 + \mu_1^3 \quad .$$

*Element Extinction Matrix*

From Eqs. (3.1.4), (3.2.2) and (3.2.3) it follows that

$$\underset{\sim}{E}_{(e)} = \frac{(\tau_2-\tau_1)(\mu_4-\mu_1)}{9} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \ldots 1 \ldots 0 \ldots 0 \ldots 0 \ldots \\ \ldots 0 \ldots 1 \ldots 0 \ldots 0 \ldots \\ \ldots 0 \ldots 0 \ldots 1 \ldots 0 \ldots \\ \ldots 0 \ldots 0 \ldots 0 \ldots 1 \ldots \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/4 & 1/2 \\ 1/2 & 1 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1 & 1/2 \\ 1/2 & 1/4 & 1/2 & 1 \end{bmatrix} . \quad (3.2.$$

*Element Scattering Matrix*

From Eqs. (3.1.5), (3.2.2) and (3.2.3) it follows that

$$\underset{\sim}{S}_{(e,e')} = \frac{2\pi}{(\tau_2-\tau_1)^2(\mu_4-\mu_1)(\mu_{4'}-\mu_{1'})} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \ldots 1 \ldots 0 \ldots 0 \ldots 0 \ldots \\ \ldots 0 \ldots 1 \ldots 0 \ldots 0 \ldots \\ \ldots 0 \ldots 0 \ldots 1 \ldots 0 \ldots \\ \ldots 0 \ldots 0 \ldots 0 \ldots 1 \ldots \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix}$$

(3.2

where

$$T_{11} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau_2-\tau)^2(\mu_4-\mu)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu_4'-\mu')d\mu'd\mu d\tau \ ,$$

$$T_{12} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau_2-\tau)(\tau-\tau_1)(\mu_4-\mu)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu_4'-\mu')d\mu'd\mu d\tau,$$

$$T_{13} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau_2-\tau)(\tau-\tau_1)(\mu_4-\mu)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu'-\mu_1')d\mu'd\mu d\tau,$$

$$T_{14} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau_2-\tau)^2(\mu_4-\mu)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu'-\mu_1')d\mu'd\mu d\tau,$$

$$T_{21} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau-\tau_1)(\tau_2-\tau)(\mu_4-\mu)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu_4'-\mu')d\mu'd\mu d\tau,$$

$$T_{22} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau-\tau_1)^2(\mu_4-\mu)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu_4'-\mu')d\mu'd\mu d\tau,$$

$$T_{23} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau-\tau_1)^2(\mu_4-\mu)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu'-\mu_1')d\mu'd\mu d\tau,$$

$$T_{24} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau-\tau_1)(\tau_2-\tau)(\mu_4-\mu)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu'-\mu_1')d\mu'd\mu d\tau,$$

$$T_{31} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau-\tau_1)(\tau_2-\tau)(\mu-\mu_1)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu_4'-\mu')d\mu'd\mu d\tau,$$

$$T_{32} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau-\tau_1)^2(\mu-\mu_1)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu_4'-\mu')d\mu'd\mu d\tau,$$

$$T_{33} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau-\tau_1)^2(\mu-\mu_1)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu'-\mu_1')d\mu'd\mu d\tau,$$

$$T_{34} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau-\tau_1)(\tau_2-\tau)(\mu-\mu_1)\int_{\mu_1'}^{\mu_4'} K_q(\mu')(\mu'-\mu_1')d\mu'd\mu d\tau,$$

$$T_{41} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau_2-\tau)^2(\mu-\mu_1)\int_{\mu_{1'}}^{\mu_{4'}} K_q(\mu')(\mu_{4'}-\mu')d\mu'd\mu d\tau,$$

$$T_{42} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau_2-\tau)(\tau-\tau_1)(\mu-\mu_1)\int_{\mu_{1'}}^{\mu_{4'}} K_q(\mu')(\mu_{4'}-\mu')d\mu'd\mu d\tau,$$

$$T_{43} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau_2-\tau)(\tau-\tau_1)(\mu-\mu_1)\int_{\mu_{1'}}^{\mu_{4'}} K_q(\mu')(\mu'-\mu_{1'})d\mu'd\mu d\tau,$$

and

$$T_{44} = \sum_{q=o}^{Q} a_q \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} \gamma(\tau)K_q(\mu)(\tau_2-\tau)^2(\mu-\mu_1)\int_{\mu_{1'}}^{\mu_{4'}} K_q(\mu')(\mu'-\mu_{1'})d\mu'd\mu d\tau.$$

In the above expressions primed indices and variables refer to the element labeled as e'. If, however, scattering is isotropic and $\gamma$ is independent of $\tau$, then we get

$$\underset{\sim}{S}_{(e,e')} = \frac{\pi\gamma(\tau_2-\tau_1)(\mu_4-\mu_1)(\mu_{4'}-\mu_{1'})}{12} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \cdot \begin{bmatrix} \ldots 1 \ldots 0 \ldots 0 \ldots 0 \ldots \\ \ldots 0 \ldots 1 \ldots 0 \ldots 0 \ldots \\ \ldots 0 \ldots 0 \ldots 1 \ldots 0 \ldots \\ \ldots 0 \ldots 0 \ldots 0 \ldots 1 \ldots \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1 \\ 1/2 & 1 & 1 \\ 1 & 1/2 & 1/2 \end{bmatrix}$$

(3.2.7

*Element Boundary Matrices*

From Eqs. (3.1.6), (3.1.7), (3.2.2) and (3.2.3) it follows that

$$\underset{\sim}{B}^{(1)}{}_{(e)} = \frac{1}{12(\mu_4 - \mu_1)} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cdots 1 \cdots 0 \cdots 0 \cdots 0 \cdots \\ \cdots 0 \cdots 1 \cdots 0 \cdots 0 \cdots \\ \cdots 0 \cdots 0 \cdots 1 \cdots 0 \cdots \\ \cdots 0 \cdots 0 \cdots 0 \cdots 1 \cdots \end{bmatrix} \begin{bmatrix} B_1 & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B_2 & 0 & 0 & B_3 \end{bmatrix} \quad (3.2.8)$$

and

$$\underset{\sim}{B}^{(2)}{}_{(e)} = \frac{1}{12(\mu_4 - \mu_1)} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cdots 1 \cdots 0 \cdots 0 \cdots 0 \cdots \\ \cdots 0 \cdots 1 \cdots 0 \cdots 0 \cdots \\ \cdots 0 \cdots 0 \cdots 1 \cdots 0 \cdots \\ \cdots 0 \cdots 0 \cdots 0 \cdots 1 \cdots \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_1 & B_2 & 0 \\ 0 & B_2 & B_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.2.9)$$

where

$$B_1 = \mu_4^3 + \mu_1\mu_4^2 - 5\mu_1^2\mu_4 + 3\mu_1^3,$$

$$B_2 = (\mu_4 - \mu_1)^2(\mu_1 + \mu_4)\ ,$$

and

$$B_3 = 3\mu_4^3 - 5\mu_1\mu_4^2 + \mu_1^2\mu_4 + \mu_1^3.$$

*Element Diffuse Reflection Matrices*

From Eqs. (3.1.8), (3.1.9), (3.2.2) and (3.2.3) it follows that

$$ {}^{d}\underset{\sim}{R}^{(1)}{}_{(e,e')} = \frac{\rho^{d}{}_{(1)}}{18} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cdots 1 \cdots 0 \cdots 0 \cdots 0 \cdots \\ \cdots 0 \cdots 1 \cdots 0 \cdots 0 \cdots \\ \cdots 0 \cdots 0 \cdots 1 \cdots 0 \cdots \\ \cdots 0 \cdots 0 \cdots 0 \cdots 1 \cdots \end{bmatrix} \begin{bmatrix} R^{d}{}_{1} & 0 & 0 & R^{d}{}_{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ R^{d}{}_{3} & 0 & 0 & R^{d}{}_{4} \end{bmatrix} \qquad (3.2.10)$$

and

$$
{}^{d}\underset{\sim}{R}_{(2)}(e,e') = \frac{\rho^{d}_{(2)}}{18}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & 1 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & 0 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\begin{bmatrix}
\ldots 1 \ldots 0 \ldots 0 \ldots 0 \ldots \\
\ldots 0 \ldots 1 \ldots 0 \ldots 0 \ldots \\
\ldots 0 \ldots 0 \ldots 1 \ldots 0 \ldots \\
\ldots 0 \ldots 0 \ldots 0 \ldots 1 \ldots
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & R^{d}{}_{1} & R^{d}{}_{2} & 0 \\
0 & R^{d}{}_{3} & R^{d}{}_{4} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\qquad (3.2.1
$$

where

$$R^{d}{}_{1} = (\mu_4^2 + \mu_1\mu_4 - 2\mu_1^2)(\mu_{4'}^2 + \mu_{1'}\mu_{4'} - 2\mu_{1'}^2),$$

$$R^{d}{}_{2} = (\mu_4^2 + \mu_1\mu_4 - 2\mu_1^2)(2\mu_4^2 - \mu_{1'}\mu_{4'} - \mu_{1'}^2),$$

$$R^{d}{}_{3} = (2\mu_4^2 - \mu_1\mu_4 - \mu_1^2)(\mu_{4'}^2 + \mu_{1'}\mu_{4'} - 2\mu_{1'}^2),$$

and

$$R^{d}{}_{4} = (2\mu_4^2 - \mu_1\mu_4 - \mu_1^2)(2\mu_{4'}^2 - \mu_{1'}\mu_{4'} - \mu_{1'}^2).$$

In the above expressions primed indices refer to the element labeled as e'.

*Element Specular Reflection Matrices*

From Eqs. (3.1.10), (3.1.11), (3.2.2) and (3.2.3) it follows that

$$
\underset{\sim}{{}^{s}R}^{(1)}{}_{(e)} = \frac{\rho^{s}{}_{(1)}}{12(\mu_4 - \mu_1)}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & 1 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & 0 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\begin{bmatrix}
\ldots 1 \ldots 0 \ldots 0 \ldots 0 \ldots \\
\ldots 0 \ldots 1 \ldots 0 \ldots 0 \ldots \\
\ldots 0 \ldots 0 \ldots 1 \ldots 0 \ldots \\
\ldots 0 \ldots 0 \ldots 0 \ldots 1 \ldots
\end{bmatrix}
\begin{bmatrix}
R^{s}{}_{1} & 0 & 0 & R^{s}{}_{2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
R^{s}{}_{3} & 0 & 0 & R^{s}{}_{1}
\end{bmatrix}
\quad (3.2
$$

and

$$
\underset{\sim}{{}^{s}R}^{(2)}{}_{(e)} = \frac{\rho^{s}{}_{(2)}}{12(\mu_4 - \mu_1)}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & 1 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & 0 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\begin{bmatrix}
\ldots 1 \ldots 0 \ldots 0 \ldots 0 \ldots \\
\ldots 0 \ldots 1 \ldots 0 \ldots 0 \ldots \\
\ldots 0 \ldots 0 \ldots 1 \ldots 0 \ldots \\
\ldots 0 \ldots 0 \ldots 0 \ldots 1 \ldots
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & R^{s}{}_{1} & R^{s}{}_{2} & 0 \\
0 & R^{s}{}_{3} & R^{s}{}_{1} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad (3.2
$$

where

$$R^{s}{}_{1} = (\mu_4 - \mu_1)^2(\mu_1 + \mu_4),$$

$$R^{s}{}_{2} = \mu_4^3 + \mu_1\mu_4^2 - 5\mu_1^2\mu_4 + 3\mu_1^3,$$

and

$$R^{s}{}_{3} = 3\mu_4^3 - 5\mu_1\mu_4^2 + \mu_1^2\mu_4 + \mu_1^3.$$

*Element Source Vector*

From Eqs. (3.1.12), (3.2.2) and (3.2.3) it follows that

$$\underset{\sim}{S}_{(e)} = \frac{1}{(\tau_2-\tau_1)(\mu_4-\mu_1)} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \int_{\tau_1}^{\tau_2}\int_{\mu_1}^{\mu_4} [1 - \gamma(\tau)]f[\Theta(\tau,t)] \begin{bmatrix} (\tau_2-\tau)(\mu_4-\mu) \\ (\tau-\tau_1)(\mu_4-\mu) \\ (\tau-\tau_1)(\mu-\mu_1) \\ (\tau_2-\tau)(\mu-\mu_1) \end{bmatrix} d\tau d\mu. \tag{3.2.14}$$

For a constant source of strength $s_o$, we then get

$$\underset{\sim}{s}_{(e)} = s_o \frac{(\tau_2 - \tau_1)(\mu_4 - \mu_1)}{4} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3.2.15}$$

*Element Transmission Vectors*

From Eqs. (3.1.13), (3.1.14), (3.2.2) and (3.2.3) it follows that

$$\underset{\sim}{t}^{(1)}{}_{(e)} = \frac{1}{(\mu_4 - \mu_1)} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \int_{\mu_1}^{\mu_4} T_{(1)}(\mu) \begin{bmatrix} (\mu_4 - \mu) \\ 0 \\ 0 \\ (\mu - \mu_1) \end{bmatrix} \mu d\mu \tag{3.2.16}$$

and

$$\underset{\sim}{t}^{(2)}{}_{(e)} = \frac{1}{(\mu_4 - \mu_1)} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \int_{\mu_1}^{\mu_4} T_{(2)}(\mu) \begin{bmatrix} 0 \\ (\mu_4-\mu) \\ (\mu-\mu_1) \\ 0 \end{bmatrix} \mu d\mu. \tag{3.2.17}$$

*Element Emission Vectors*

From Eqs. (3.1.15), (3.1.16), (3.2.2) and (3.2.3) it follows that

$$\underset{\sim}{e}^{(1)}{}_{(e)} = \frac{f[\Theta(0,t)]}{(\mu_4 - \mu_1)} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \int_{\mu_1}^{\mu_4} \varepsilon_{(1)}(\mu) \begin{bmatrix} (\mu_4-\mu) \\ 0 \\ 0 \\ (\mu-\mu_1) \end{bmatrix} \mu d\mu \tag{3.2.18}$$

and

$$\underset{\sim}{e}^{(2)}{}_{(e)} = \frac{f[\Theta(\tau_0,t)]}{(\mu_4 - \mu_1)} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \int_{\mu_1}^{\mu_4} \varepsilon_{(2)}(\mu) \begin{bmatrix} 0 \\ (\mu_4-\mu) \\ (\mu-\mu_1) \\ 0 \end{bmatrix} \mu d\mu \ . \qquad (3.2.19)$$

## 3.3 Finite Element Solutions to Some Selected Problems

Having obtained the finite-element equation (3.1.2) in its explicit form, in this section we proceed to present finite-element solutions to some much-studied problems for which numerical solutions have been presented in the literature. Our intention is not necessarily to present more accurate solutions than have been presented before, but rather to provide a confirmation that the finite-element approximation technique is applied and implemented correctly, as well as evidence suggesting that this technique can be adopted to obtain reliable results efficiently.

The problems attempted fall into two categories. The problems of the first category are of a homogeneous type, leading to a generalized algebraic eigenvalue problem. Those of the second category, on the other hand, are nonhomogeneous problems, nonhomogeneity either being due to a source or due to boundary effects. Problems of this nature lead to problems involving a system of algebraic equations. The adopted solution scheme for the generalized algebraic eigenvalue problem is that of Moler and Stewart [36], implemented with a computer code in the form of a subroutine under the name EIGZF by IMSL [35]. The solution scheme for the system of algebraic equations is a standard one, based on the Crout algorithm [37]. Its computer implementation is in the form of a subroutine called LEQT1F, and developed by IMSL [35].

Computer implementation of the finite-element models approximating the problems considered is straightforward, and is based on the algorithm given in Table (3.3.1), which is common to almost any application of the finite-element approximation technique.

TABLE 3.3.1 - A typical finite-element algorithm

| | |
|---|---|
| Step 1 | Form the element matrices and the element vectors |
| Step 2 | Assemble the element matrices and the element vectors to form the system matrices and the system vectors |
| Step 3 | Form the system of equations and solve |

As can be seen, it is possible to develop a single code implementing the finite-element models of the problems considered, as well as the finite-element model of any problem stated in terms of Eqs. (1.1.31), (1.1.33) and (1.1.34). However, since our intention is to substantiate the application of the finite-element approximation technique to the class of problems considered in this study as well as to demonstrate its power in obtaining reliable results efficiently, the development of separate codes implementing the finite-element models approximating the problems considered is preferred over the development of such a single universal code. Listings of typical codes and of representative sets of data are given in Appendices B and C. Finite-element data listed in these appendices refer to the finite-element mesh shown in Figure 3.3.1. The code listed in Appendix B is a code typical of those used in the solution of homogeneous problems, whereas that listed in Appendix C is a code typical of those used in the solution of nonhomogeneous problems. UNIVAC 1106-I CPU† times of the former and the latter codes run with their associated sample data are respectively 46 seconds and 27 seconds. We should admit that both these codes and those used but

†Central Processor Unit

not listed are prototypal in nature and are merely developed for the sake of obtaining solutions with a minimum of programming effort; therefore, they should not be taken as professional codes.

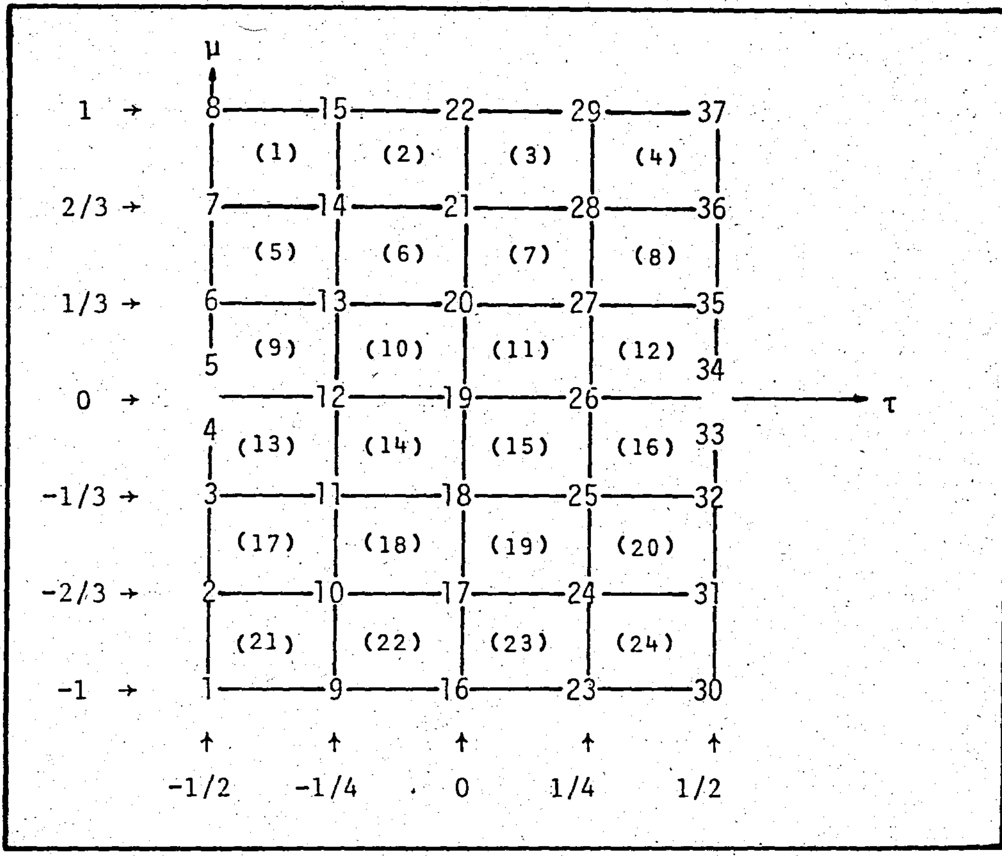


FIGURE 3.3.1 - A typical finite-element mesh

The problems attempted are solved for three different meshes: (4,4), (4,6) and (4,8). An (m,n) mesh refers to the finite-element model of the region $[0,\tau_0]\times[-1,1]$, or in some problems, of the region $[-\tau_0/2,\tau_0/2]\times[-1,1]$ constructed by using m linear elements in the spatial region and n linear elements in the angular region†. Whenever appropriate,

† It will be seen that the attempted problems are invariant under translations with respect to the variable τ. Therefore, for these particular class of problems whatever is applicable in the case of the region $[0,\tau_0]$ is also applicable in the case of the region $[-\tau_0/2,\tau_0/2]$.

on the boundaries a double node is used along the direction $\mu = 0$ to allow for the discontinuity of the intensity distribution along this direction. The incorporation of such a double node is perfectly legitimate since there are no angular derivatives involved. To facilitate the comparison and the evaluation of the solutions, the results are presented directly on their corresponding finite-element meshes by simply attaching the value of the intensity to the associated node.

*The Eigenvalue Problem with Isotropic Scattering*

The governing eigenvalue problem is given as follows.

$$\mu \frac{\partial}{\partial \tau} \Psi(\tau,\mu) + \Psi(\tau,\mu) = \frac{\gamma}{2} \int_{-1}^{1} \Psi(\tau,\mu')d\mu' \quad \text{for} \quad -\tau_0 \le \tau \le \tau_0 \quad \text{and} \quad -1 \le \mu \le 1, \tag{3.3.1}$$

$$\Psi(-\tau_0,\mu) = 0 \qquad \text{for} \qquad 0 < \mu \le 1, \tag{3.3.2}$$

$$\Psi(\tau_0,\mu) = 0 \qquad \text{for} \qquad -1 \le \mu < 0. \tag{3.3.3}$$

Its corresponding finite-element model then consists of the following generalized eigenvalue problem.

$$[\sum_{e=1}^{E} (-\underset{\sim}{T}_{(e)} + \underset{\sim}{E}_{(e)} + \underset{\sim}{B}^{(1)}_{(e)} - \underset{\sim}{B}^{(2)}_{(e)})]\underset{\sim}{\tilde{\psi}} =$$

$$[\frac{1}{4\pi} \sum_{e=1}^{E} \sum_{e'=1}^{E} B(e,e')\underset{\sim}{S}_{(e,e')}]\underset{\sim}{\tilde{\psi}}$$

where the element matrices $\underset{\sim}{T}_{(e)}$, $\underset{\sim}{E}_{(e)}$, $\underset{\sim}{B}^{(1)}_{(e)}$, $\underset{\sim}{B}^{(2)}_{(e)}$ and $\underset{\sim}{S}_{(e,e')}$ are respectively given by Eqs. (3.2.4), (3.2.5), (3.2.8), (3.2.9) and (3.2.7), and where $B(e,e')$ and $\underset{\sim}{\tilde{\psi}}$ are as defined in Section 3.1.

The eigenvalues and the corresponding intensity distributions are given in Tables 3.3.2-19 for different optical thicknesses and for different finite-element meshes.

TABLE 3.3.2 - (4,4) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 0.5$

| | | | | | |
|---|---|---|---|---|---|
| Mesh (4,4) | | | | | |
| Number of Elements: 16 | | | | Number of Nodal Points: 27 | |
| $\tau_o = 0.5$ | | | | | |
| Eigenvalue: 1.60938 | | | | | |
| μ | Intensity Distribution | | | | |
| 1 | -0.0157223 | 0.156896 | 0.287263 | 0.410481 | 0.445726 |
| 1/2 | -0.0281633 | 0.226468 | 0.436495 | 0.657439 | 0.644175 |
| 0 | 0.0322042<br>0.585775 | 0.835990 | 1.000000 | 0.835990 | 0.585775<br>0.0322043 |
| -1/2 | 0.644175 | 0.657439 | 0.436495 | 0.226468 | -0.0281634 |
| -1 | 0.445726 | 0.410481 | 0.287263 | 0.156896 | -0.0157223 |
| τ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.3 - (4,4) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 1$

| | | | | | |
|---|---|---|---|---|---|
| Mesh (4,4) | | | | | |
| Number of Elements: 16 | | | | Number of Nodal Points: 27 | |
| $\tau_o = 1$ | | | | | |
| Eigenvalue: 1.276626 | | | | | |
| μ | | | | | |
| 1 | -0.0355780 | 0.292595 | 0.501897 | 0.716267 | 0.673969 |
| 1/2 | -0.0643947 | 0.465502 | 0.742811 | 0.926228 | 0.744736 |
| 0 | 0.175310<br>0.483062 | 0.965908 | 1.000000 | 0.965908 | 0.483064<br>0.175311 |
| -1/2 | 0.744736 | 0.926228 | 0.742811 | 0.465502 | -0.0643947 |
| -1 | 0.673969 | 0.716267 | 0.501897 | 0.292596 | -0.0355779 |
| τ | -1 | -1/2 | 0 | 1/2 | 1 |

TABLE 3.3.4 - (4,4) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 2$

| Mesh (4,4) | | | | | |
|---|---|---|---|---|---|
| Number of Elements | | | | Number of Nodal Points | |
| 16 | | | | 27 | |
| $\tau_o = 2$ | | | | | |
| Eigenvalue | | | | | |
| 1.10812 | | | | | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | -0.0545461 | 0.439110 | 0.736630 | 0.926950 | 0.680545 |
| 1/2 | -0.0550171 | 0.644641 | 0.917282 | 0.986395 | 0.586071 |
| 0 | 0.214819<br>0.321269 | 0.924741 | 1.000000 | 0.924740 | 0.321271<br>0.214820 |
| -1/2 | 0.586071 | 0.986395 | 0.917282 | 0.644641 | -0.0550164 |
| -1 | 0.680545 | 0.926951 | 0.736630 | 0.439109 | -0.0545462 |
| $\tau$ | -2 | -1 | 0 | 1 | -2 |

TABLE 3.3.5 - (4,4) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 3$

| | | | | | |
|---|---|---|---|---|---|
| Mesh (4,4) | | | | | |
| Number of Elements | | | | Number of Nodal Points | |
| 16 | | | | 27 | |
| $\tau_o = 3$ | | | | | |
| Eigenvalue | | | | | |
| 1.05809 | | | | | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | -0.0604539 | 0.518155 | 0.844446 | 0.961403 | 0.579747 |
| 1/2 | -0.0362844 | 0.697901 | 0.960187 | 0.956119 | 0.453220 |
| 0 | 0.190089<br>0.240332 | 0.881067 | 1.000000 | 0.881067 | 0.240334<br>0.190089 |
| -1/2 | 0.453220 | 0.956119 | 0.960187 | 0.697901 | -0.0362845 |
| -1 | 0.579747 | 0.961403 | 0.844446 | 0.518155 | -0.0604537 |
| $\tau$ | -3 | -3/2 | 0 | 3/2 | 3 |

TABLE 3.3.6 - (4,4) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 4$

Mesh (4,4)

| Number of Elements | Number of Nodal Points |
|---|---|
| 16 | 27 |

$\tau_o = 4$

Eigenvalue

1.03626

| $\mu$ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | -0.0605472 | 0.565753 | 0.899373 | 0.952718 | 0.488503 |
| 1/2 | -0.0230507 | 0.716221 | 0.976035 | 0.923294 | 0.364780 |
| 0 | 0.163387<br>0.191893 | 0.850753 | 1.000000 | 0.850753 | 0.191892<br>0.163386 |
| -1/2 | 0.364780 | 0.923295 | 0.976034 | 0.716220 | -0.0230508 |
| -1 | 0.488504 | 0.952718 | 0.889372 | 0.565752 | -0.0605473 |
| $\tau$ | -4 | -2 | 0 | 2 | 4 |

TABLE 3.3.7 - (4,4) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 5$

| | | | | | |
|---|---|---|---|---|---|
| Mesh (4,4) | | | | | |
| Number of Elements: 16 | | | | Number of Nodal Points: 27 | |
| $\tau_o = 5$ | | | | | |
| Eigenvalue: 1.02478 | | | | | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | -0.0582409 | 0.596433 | 0.930241 | 0.934170 | 0.417128 |
| 1/2 | -0.0144737 | 0.722820 | 0.983683 | 0.896028 | 0.303735 |
| 0 | 0.141451<br>0.159594 | 0.829076 | 1.000000 | 0.829078 | 0.159594<br>0.141451 |
| -1/2 | 0.303734 | 0.896026 | 0.983683 | 0.722822 | -0.0144744 |
| -1 | 0.417127 | 0.934168 | 0.930242 | 0.596435 | -0.0582414 |
| $\tau$ | -5 | -5/2 | 0 | 5/2 | 5 |

TABLE 3.3.8 - (4,6) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 0.5$

Mesh (4,6)

Number of Elements: 24

Number of Nodal Points: 37

$\tau_o = 0.5$

Eigenvalue: 1.61399

| $\mu$ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | -0.0245821 | 0.172181 | 0.315459 | 0.474364 | 0.513886 |
| 2/3 | -0.0256927 | 0.246077 | 0.423193 | 0.608222 | 0.624331 |
| 1/3 | -0.0489716 | 0.392167 | 0.648062 | 0.846870 | 0.772178 |
| 0 | 0.169748<br>0.608315 | 0.973158 | 1.000000 | 0.973158 | 0.608316<br>0.169750 |
| -1/3 | 0.772178 | 0.846870 | 0.648062 | 0.392166 | -0.0489719 |
| -2/3 | 0.624332 | 0.608222 | 0.423193 | 0.246077 | -0.0256927 |
| -1 | 0.513886 | 0.474364 | 0.315460 | 0.172180 | -0.0245821 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.9 - (4,6) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 1$

Mesh (4,6)

Number of Elements: 24

Number of Nodal Points: 37

$\tau_o = 1$

Eigenvalue: 1.27653

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | -0.0445373 | 0.306273 | 0.530919 | 0.745136 | 0.696614 |
| 2/3 | -0.0425547 | 0.409586 | 0.667131 | 0.876489 | 0.748012 |
| 1/3 | -0.0499125 | 0.624878 | 0.883471 | 0.997799 | 0.722329 |
| 0 | 0.284024<br>0.464361 | 0.984326 | 1.000000 | 0.934326 | 0.464360<br>0.284023 |
| -1/3 | 0.722329 | 0.997800 | 0.883470 | 0.624878 | -0.0499122 |
| -2/3 | 0.748012 | 0.876489 | 0.667131 | 0.409585 | -0.0425548 |
| -1 | 0.696614 | 0.745136 | 0.530919 | 0.306272 | -0.0445337 |
| τ | -1 | -1/2 | 0 | 1/2 | 1 |

TABLE 3.3.10 - (4,6) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 2$

| Mesh (4,6) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 24 | | | | Number of Nodal Points: 37 | |
| $\tau_o = 2$ | | | | | |
| Eigenvalue: 1.10815 | | | | | |
| μ | Intensity Distribution | | | | |
| 1 | -0.0588303 | 0.449961 | 0.747105 | 0.929396 | 0.679985 |
| 2/3 | -0.0494703 | 0.565925 | 0.860982 | 0.980964 | 0.630759 |
| 1/3 | -0.0112123 | 0.750970 | 0.969417 | 0.979837 | 0.515338 |
| 0 | 0.261511<br>0.314766 | 0.919488 | 1.000000 | 0.919489 | 0.314765<br>0.261510 |
| -1/3 | 0.515338 | 0.979837 | 0.969418 | 0.750971 | -0.0112126 |
| -2/3 | 0.630759 | 0.980964 | 0.860982 | 0.565925 | -0.0494705 |
| -1 | 0.679985 | 0.929396 | 0.747106 | 0.449961 | -0.0588304 |
| τ | -2 | -1 | 0 | 1 | 2 |

TABLE 3.3.11 - (4,6) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 3$

| | | | | | |
|---|---|---|---|---|---|
| Mesh (4,6) | | | | | |
| Number of Elements: 24 | | | | Number of Nodal Points: 37 | |
| $\tau_o = 3$ | | | | | |
| Eigenvalue: 1.05809 | | | | | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | -0.0613314 | 0.524878 | 0.847271 | 0.959878 | 0.577731 |
| 2/3 | -0.0443428 | 0.631461 | 0.926279 | 0.966578 | 0.501903 |
| 1/3 | 0.0112175 | 0.769984 | 0.984181 | 0.936629 | 0.389702 |
| 0 | 0.214089<br>0.237937 | 0.876372 | 1.000000 | 0.876372 | 0.237935<br>0.214087 |
| -1/3 | 0.389702 | 0.936630 | 0.984180 | 0.769983 | 0.0112182 |
| -2/3 | 0.501903 | 0.996579 | 0.926279 | 0.631460 | -0.0443429 |
| -1 | 0.577732 | 0.959878 | 0.847271 | 0.524877 | -0.0613315 |
| $\tau$ | -3 | -3/2 | 0 | 3/2 | 3 |

TABLE 3.3.12 - (4,6) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 4$

Mesh (4,6)

Number of Elements: 24

Number of Nodal Points: 37

$\tau_o = 4$

Eigenvalue

1.03626

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | -0.0598836 | 0.569686 | 0.899816 | 0.951046 | 0.486999 |
| 2/3 | -0.0375210 | 0.662507 | 0.954452 | 0.938437 | 0.409580 |
| 1/3 | 0.0212225 | 0.786940 | 0.989607 | 0.902117 | 0.311088 |
| 0 | 0.177624<br>0.190822 | 0.847406 | 1.000000 | 0.847406 | 0.190821<br>0.177623 |
| -1/3 | 0.311088 | 0.902117 | 0.989607 | 0.768939 | 0.0212225 |
| -2/3 | 0.409581 | 0.938437 | 0.954453 | 0.662506 | -0.0375210 |
| 1 | 0.486999 | 0.951046 | 0.899816 | 0.569685 | -0.0598841 |
| τ | -4 | -2 | 0 | 2 | 4 |

TABLE 3.3.13 - (4,6) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 5$

| Mesh (4,6) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 24 | | | | Number of Nodal Points: 37 | |
| $\tau_o = 5$ | | | | | |
| Eigenvalue: 1.02478 | | | | | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | -0.0570080 | 0.598757 | 0.929994 | 0.932857 | 0.416097 |
| 2/3 | -0.0315631 | 0.678829 | 0.968995 | 0.912229 | 0.343691 |
| 1/3 | 0.0253143 | 0.763895 | 0.992489 | 0.875650 | 0.258050 |
| 0 | 0.150764<br>0.159052 | 0.826696 | 1.000000 | 0.826698 | 0.159050<br>0.150762 |
| -1/3 | 0.258051 | 0.875648 | 0.992489 | 0.763897 | 0.0253127 |
| -2/3 | 0.343691 | 0.912227 | 0.968996 | 0.678830 | -0.0315640 |
| -1 | 0.416097 | 0.932856 | 0.929995 | 0.598759 | -0.0570087 |
| $\tau$ | -5 | -5/2 | 0 | 5/2 | 5 |

TABLE 3.3.14 - (4,8) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o$ = 0.5

| Mesh (4,8) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 32 | | | | Number of Nodal Points: 47 | |
| $\tau_o$ = 0.5 | | | | | |
| Eigenvalue: 1.61457 | | | | | |
| μ | Intensity Distribution | | | | |
| 1 | -0.0231291 | 0.179827 | 0.324996 | 0.487582 | 0.528351 |
| 3/4 | -0.0288948 | 0.230602 | 0.405807 | 0.588672 | 0.613608 |
| 1/2 | -0.0285033 | 0.318393 | 0.528918 | 0.731477 | 0.714543 |
| 1/4 | -0.0491034 | 0.508700 | 0.770434 | 0.930452 | 0.797694 |
| 0 | 0.269765<br>0.592163 | 0.999999 | 0.988498 | 1.000000 | 0.592160<br>0.269765 |
| -1/4 | 0.797693 | 0.930452 | 0.770433 | 0.508700 | -0.0491039 |
| -1/2 | 0.714544 | 0.731477 | 0.528918 | 0.318393 | -0.0285034 |
| -3/4 | 0.613608 | 0.588672 | 0.405807 | 0.230603 | -0.0288951 |
| -1 | 0.528351 | 0.487582 | 0.324996 | 0.179827 | -0.0231289 |
| τ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.15 - (4,8) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 1$

Mesh (4,8)

| Number of Elements | Number of Nodal Points |
|---|---|
| 32 | 47 |

$\tau_o = 1$

Eigenvalue

1.27657

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | -0.0416656 | 0.305484 | 0.530591 | 0.744606 | 0.694303 |
| 3/4 | -0.0455010 | 0.380356 | 0.630686 | 0.839242 | 0.734601 |
| 1/2 | -0.0394776 | 0.495623 | 0.763859 | 0.941768 | 0.746035 |
| 1/4 | -0.0220261 | 0.716109 | 0.932489 | 1.000000 | 0.674497 |
| 0 | 0.336819<br>0.451037 | 0.969856 | 0.987263 | 0.969857 | 0.451034<br>0.336817 |
| -1/4 | 0.674497 | 1.000000 | 0.932489 | 0.716108 | -0.0220263 |
| -1/2 | 0.746035 | 0.941768 | 0.763859 | 0.495623 | -0.0394781 |
| -3/4 | 0.734601 | 0.839241 | 0.630687 | 0.380356 | -0.0455010 |
| -1 | 0.694303 | 0.744605 | 0.530590 | 0.305484 | -0.0416657 |
| τ | -1 | -1/2 | 0 | 1/2 | 1 |

TABLE 3.3.16 - (4,8) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 2$

Mesh (4,8)

| Number of Elements | Number of Nodal Points |
|---|---|
| 32 | 47 |

$\tau_o = 2$

Eigenvalue

1.10815

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | -0.0576818 | 0.451344 | 0.748943 | 0.930431 | 0.679277 |
| 3/4 | -0.0535508 | 0.535781 | 0.833325 | 0.970503 | 0.647067 |
| 1/2 | -0.0340464 | 0.650255 | 0.919885 | 0.989509 | 0.580692 |
| 1/4 | 0.0298890 | 0.804483 | 0.984460 | 0.967850 | 0.471616 |
| 0 | 0.281374<br>0.312978 | 0.914814 | 1.000000 | 0.914814 | 0.312977<br>0.281373 |
| -1/4 | 0.471617 | 0.967849 | 0.984461 | 0.804483 | 0.0298886 |
| -1/2 | 0.580692 | 0.989509 | 0.919885 | 0.650255 | -0.0340468 |
| -3/4 | 0.646067 | 0.970503 | 0.833325 | 0.535781 | -0.0535507 |
| -1 | 0.679277 | 0.930432 | 0.748943 | 0.451344 | -0.0576827 |
| τ | -2 | -1 | 0 | 1 | 2 |

TABLE 3.3.17 - (4,8) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 3$

| Mesh (4,8) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 32 | | | Number of Nodal Points: 47 | | |
| $\tau_o = 3$ | | | | | |
| Eigenvalue: 1.05810 | | | | | |
| μ | Intensity Distribution | | | | |
| 1 | -0.0604822 | 0.525849 | 0.847608 | 0.959507 | 0.576728 |
| 3/4 | -0.0500566 | 0.603654 | 0.907428 | 0.966742 | 0.523141 |
| 1/2 | -0.0218682 | 0.698000 | 0.959081 | 0.956067 | 0.449192 |
| 1/4 | 0.0479079 | 0.803192 | 0.990683 | 0.923027 | 0.354497 |
| 0 | 0.223493<br>0.237304 | 0.873600 | 1.000000 | 0.873600 | 0.237303<br>0.223492 |
| -1/4 | 0.354497 | 0.923027 | 0.990684 | 0.803192 | 0.0479083 |
| -1/2 | 0.449192 | 0.956067 | 0.959081 | 0.698001 | -0.0218693 |
| -3/4 | 0.523142 | 0.966742 | 0.907428 | 0.603654 | -0.0500558 |
| -1 | 0.576728 | 0.959508 | 0.847608 | 0.525849 | -0.0604841 |
| τ | -3 | -3/2 | 0 | 3/2 | 3 |

TABLE 3.3.18 - (4,8) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 4$

Mesh (4,8)

Number of Elements: 32

Number of Nodal Points: 47

$\tau_o = 4$

Eigenvalue

1.03627

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | -0.0591060 | 0.570319 | 0.899715 | 0.950599 | 0.486316 |
| 3/4 | -0.0444921 | 0.638596 | 0.941778 | 0.942291 | 0.430300 |
| 1/2 | -0.0128385 | 0.715084 | 0.974685 | 0.922835 | 0.362186 |
| 1/4 | 0.0520275 | 0.792631 | 0.993499 | 0.889406 | 0.282462 |
| 0 | 0.182986<br>0.190552 | 0.845743 | 1.000000 | 0.845743 | 0.190552<br>0.182985 |
| -1/4 | 0.282463 | 0.889406 | 0.993499 | 0.792632 | 0.0520265 |
| -1/2 | 0.362187 | 0.922834 | 0.974686 | 0.715083 | -0.0128382 |
| -3/4 | 0.430300 | 0.942921 | 0.941778 | 0.638596 | -0.0444922 |
| -1 | 0.486316 | 0.950598 | 0.899716 | 0.570319 | -0.0591063 |
| τ | -4 | -2 | 0 | 2 | 4 |

TABLE 3.3.19 - (4,8) finite-element solution of the eigenvalue problem (3.3.1-3) for $\tau_o = 5$

Mesh (4,8)

Number of Elements: 32

Number of Nodal Points: 47

$\tau_o = 5$

Eigenvalue

1.02478

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | -0.0562899 | 0.599150 | 0.929792 | 0.932510 | 0.415649 |
| 3/4 | -0.0391873 | 0.658457 | 0.960142 | 0.918341 | 0.362662 |
| 1/2 | -0.00694016 | 0.721481 | 0.982594 | 0.895593 | 0.302015 |
| 1/4 | 0.0512642 | 0.782310 | 0.995189 | 0.864110 | 0.234160 |
| 0 | 0.154196<br>0.158925 | 0.825629 | 1.000000 | 0.825629 | 0.158923<br>0.154194 |
| -1/4 | 0.234162 | 0.864110 | 0.995189 | 0.782310 | 0.0512632 |
| -1/2 | 0.302016 | 0.895593 | 0.982593 | 0.721481 | -0.00694157 |
| -3/4 | 0.362664 | 0.918341 | 0.960142 | 0.658457 | -0.0391877 |
| -1 | 0.415650 | 0.932511 | 0.929792 | 0.599149 | -0.0562906 |
| τ | -5 | -5/2 | 0 | 5/2 | 5 |

A comparison of the finite-element results for the eigenvalues with the corresponding exact results, as reported in [38], is given in Table 3.3.20.

TABLE 3.3.20 - Eigenvalues for the eigenvalue problem (3.3.1-3)

| $\tau_0$ | Finite-Element Results | | | Exact Results [38] |
|---|---|---|---|---|
| | Mesh (4,4)<br>Number of Elements 16<br>Number of Nodal Points 27 | Mesh (4,6)<br>Number of Elements 24<br>Number of Nodal Points 37 | Mesh (4,8)<br>Number of Elements 32<br>Number of Nodal Points 47 | |
| 0.5 | 1.609380 | 1.613990 | 1.614570 | 1.615378520 |
| 1 | 1.276626 | 1.27653 | 1.276570 | 1.277101824 |
| 2 | 1.108120 | 1.108150 | 1.108150 | 1.108467832 |
| 3 | 1.058090 | 1.058090 | 1.058100 | 1.058295896 |
| 4 | 1.03626 | 1.036260 | 1.036270 | 1.036402030 |
| 5 | 1.024780 | 1.024780 | 1.024780 | 1.024879373 |

As is seen from the results, the finite-element solutions perfectly reflect the inherent symmetry condition

$$\Psi(\tau,\mu) = \Psi(-\tau,-\mu), \qquad \tau_0 \leq \tau \leq \tau_0 \quad \text{and} \quad -1 \leq \mu \leq 1,$$

involved in the eigenvalue problem (3.3.1-3).

From the symmetry condition $\Psi(\tau,\mu) = \Psi(-\tau,-\mu)$, $-\tau_o \leq \tau \leq \tau_o$ and $-1 \leq \mu \leq 1$, on the other hand, it follows that the eigenvalue problem (3.3.1-3) can also be stated in terms of the following set of equations.

$$\mu \frac{\partial}{\partial\tau} \Psi(\tau,\mu) + \Psi(\tau,\mu) = \frac{\gamma}{2} \int_{-1}^{1} \Psi(\tau,\mu')d\mu' \quad \text{for } 0 \leq \tau \leq \tau_o \text{ and } -1 \leq \mu \leq 1, \qquad (3.3.4)$$

$$\Psi(0,\mu) = \Psi(0,-\mu) \quad \text{for } 0 \leq \mu \leq 1 \qquad (3.3.5)$$

$$\Psi(\tau_o,\mu) = 0 \quad \text{for } -1 \leq \mu \leq 0. \qquad (3.3.6)$$

Theoretically, there should be a perfect agreement between the solutions of the eigenvalue problems (3.3.1-3) and (3.3.4-6). To verify whether such an agreement can numerically be achieved or not, the finite-element solution of the eigenvalue problem (3.3.4-6) is next attempted.

The finite-element model associated with the eigenvalue problem (3.3.4-6) consists of the following generalized eigenvalue problem.

$$[\sum_{e=1}^{E} (-\underset{\sim}{T}_{(e)} + \underset{\sim}{E}_{(e)} + \underset{\sim}{B}^{(1)}_{(e)} - \underset{\sim}{B}^{(2)}_{(e)} - {}^{s}\underset{\sim}{R}^{(1)}_{(e)})]\underset{\sim}{\tilde{\psi}} =$$

$$[\frac{1}{4\pi} \sum_{e=1}^{E} \sum_{e'=1}^{E} B(e,e') S_{(e,e')}]\underset{\sim}{\tilde{\psi}}$$

where the element matrices $\underset{\sim}{T}_{(e)}$, $\underset{\sim}{E}_{(e)}$, $\underset{\sim}{B}^{(1)}_{(e)}$, $\underset{\sim}{B}^{(2)}_{(e)}$ and $\underset{\sim}{S}_{(e,e')}$ are respectively given by Eqs. (3.2.4), (3.2.5), (3.2.8), (3.2.9) and (3.2.7), where the element matrix ${}^{s}\underset{\sim}{R}^{(1)}_{(e)}$ is given by Eq. (3.2.12) for $\rho^{s}_{(1)} = 1$, and where $B(e,e')$ and $\underset{\sim}{\tilde{\Psi}}$ are as defined in Section 3.1.

The eigenvalues and the corresponding intensity distributions are given in Tables 3.3.21-23 for $\tau_o = 0.5$ and for different finite-element meshes.

TABLE 3.3.21 - (4,4) finite-element solution of the eigenvalue problem (3.3.4-6) for $\tau_o = 0.5$

| Mesh (4,4) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 16 | | | | Number of Nodal Points: 26 | |
| $\tau_o = 0.5$ | | | | | |
| Eigenvalue: 1.60444 | | | | | |
| μ | Intensity Distribution | | | | |
| 1 | 0.302782 | 0.362643 | 0.420497 | 0.437679 | 0.451713 |
| 1/2 | 0.461989 | 0.567235 | 0.625612 | 0.713165 | 0.677085 |
| 0 | 1.000000 | 0.955504 | 0.970308 | 0.568395 | 0.511224<br>-0.0459326 |
| -1/2 | 0.460548 | 0.345272 | 0.185108 | 0.0769642 | -0.00786693 |
| -1 | 0.301926 | 0.231748 | 0.163248 | 0.0742341 | -0.00427064 |
| τ | 0 | 1/8 | 1/4 | 3/8 | 1/2 |

TABLE 3.3.22 - (4,6) finite-element solution of the eigenvalue problem (3.3.4-6) for $\tau_o = 0.5$

Mesh (4,6)

| Number of Elements | Number of Nodal Points |
|---|---|
| 24 | 36 |

$\tau_o = 0.5$

Eigenvalue

1.61408

| $\mu$ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | 0.316628 | 0.393350 | 0.445497 | 0.492839 | 0.503490 |
| 2/3 | 0.425525 | 0.514318 | 0.572173 | 0.611806 | 0.609903 |
| 1/3 | 0.642561 | 0.748635 | 0.785678 | 0.822515 | 0.760988 |
| 0 | 1.000000 | 0.977691 | 0.932425 | 0.711195 | 0.573229<br>0.0306821 |
| -1/3 | 0.644841 | 0.519176 | 0.336813 | 0.156660 | -0.00568296 |
| -2/3 | 0.427222 | 0.331150 | 0.220348 | 0.109718 | -0.00394115 |
| -1 | 0.318574 | 0.239703 | 0.151249 | 0.0692577 | -0.00294550 |
| $\tau$ | 0 | 1/8 | 1/4 | 3/8 | 1/2 |

TABLE 3.3.23 - (4,8) finite-element solution of the eigenvalue problem (3.3.4-6) for $\tau_o = 0.5$

Mesh (4,8)

Number of Elements: 32

Number of Nodal Points: 46

$\tau_o = 0.5$

Eigenvalue

1.61517

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | 0.326523 | 0.405534 | 0.458561 | 0.505193 | 0.517577 |
| 3/4 | 0.406493 | 0.497049 | 0.552493 | 0.599065 | 0.601253 |
| 1/2 | 0.529708 | 0.630858 | 0.684401 | 0.720822 | 0.699898 |
| 1/4 | 0.760997 | 0.854672 | 0.869538 | 0.869355 | 0.783029 |
| 0 | 1.000000 | 0.985976 | 0.915787 | 0.773696 | 0.583253<br>0.111352 |
| -1/4 | 0.763739 | 0.641861 | 0.448646 | 0.226746 | -0.0138855 |
| -1/2 | 0.532357 | 0.420785 | 0.282734 | 0.145991 | -0.00409167 |
| -3/4 | 0.409183 | 0.314263 | 0.203981 | 0.0984488 | -0.00478156 |
| -1 | 0.328871 | 0.248386 | 0.159399 | 0.00764426 | -0.00333073 |
| τ | 0 | 1/8 | 1/4 | 3/8 | 1/2 |

A comparison of the finite-element results for the eigenvalue with the corresponding exact result, as reported in [38], is given in Table 3.3.24.

TABLE 3.3.24 - Eigenvalues for the eigenvalue problem (3.3.4-6)

| $\tau_0$ | Finite-Element Results | | | Exact Result [38] |
|---|---|---|---|---|
| | Mesh (4,4)<br>Number of Elements 16<br>Number of Nodal Points 26 | Mesh (4,6)<br>Number of Elements 24<br>Number of Nodal Points 36 | Mesh (4,8)<br>Number of Elements 32<br>Number of Nodal Points 46 | |
| 0.5 | 1.604440 | 1.614080 | 1.61570 | 1.615378520 |

As is seen from the results given in Tables 3.3.2-24, the agreement between the finite-element solutions of the eigenvalue problems (3.3.1-3) and (3.3.4-6) is highly satisfactory.

A comparison of the finite-element results† for the expression

$$\frac{\int_{-1}^{1} \Psi(1/4,\mu)d\mu}{\int_{-1}^{1} \Psi(0,\mu)d\mu}$$

†Finite-element results are based on the corresponding intensity distributions given in Tables 3.3.21-23.

with the corresponding exact result, as reported in[38], is given in Table 3.3.25.

TABLE 3.3.25 - Results for the expression $[\int_{-1}^{1}\Psi(1/4,\mu)d\mu]/[\int_{-1}^{1}\Psi(0,\mu)d\mu]$

| Finite-Element Results† | | | Exact Result [38] |
|---|---|---|---|
| Mesh (4,4)<br>Number of Elements 16<br>Number of Nodal Points 26 | Mesh (4,6)<br>Number of Elements 24<br>Number of Nodal Points 36 | Mesh (4,8)<br>Number of Elements 32<br>Number of Nodal Points 46 | |
| 0.916183 | 0.917134 | 0.896845 | 0.898694 |

*The Nonhomogeneous Problem with Isotropic Scattering and with Transparent Boundaries*

The governing boundary value problem is given as follows.

$$\mu \frac{\partial}{\partial \tau} \Psi(\tau,\mu) + \Psi(\tau,\mu) = \frac{\gamma(\tau)}{2}\int_{-1}^{1} \Psi(\tau,\mu')d\mu' \quad \text{for } -\tau_0 \le \tau \le \tau_0 \text{ and } -1 \le \mu \le 1, \tag{3.3.7}$$

$$\Psi(-\tau_0,\mu) = T_{(1)}(\mu) \quad \text{for } 0 \le \mu \le 1, \tag{3.3.8}$$

---

†Finite-element results are based on the corresponding intensity distributions given in Tables 3.3.21-23.

$$\Psi(\tau_0,\mu) = 0 \qquad \text{for} \qquad -1 \leq \mu < 0. \tag{3.3.9}$$

If $\gamma$ is assumed to be independent of $\tau$, then the finite-element model associated with the boundary value problem (3.3.7-9) consists of the following system of algebraic equations.

$$[\sum_{e=1}^{E} (-\underset{\sim}{T}_{(e)} + \underset{\sim}{E}_{(e)} - \frac{1}{4\pi} \sum_{e'=1}^{E} B(e,e')\underset{\sim}{S}_{(e,e')} + \underset{\sim}{B}^{(2)}{}_{(e)} - \underset{\sim}{B}^{(1)}{}_{(e)})]\underset{\sim}{\tilde{\Psi}} = [\sum_{e=1}^{E} \underset{\sim}{t}^{(1)}{}_{(e)}]$$

where the element matrices $\underset{\sim}{T}_{(e)}$, $\underset{\sim}{E}_{(e)}$, $\underset{\sim}{S}_{(e,e')}$, $\underset{\sim}{B}^{(1)}{}_{(e)}$ and $\underset{\sim}{B}^{(2)}{}_{(e)}$ are respectively given by Eqs. (3.2.4), (3.2.5), (3.2.7), (3.2.8) and (3.2.9), where the element vectors $\underset{\sim}{t}^{(1)}{}_{(e)}$ is given by Eq. (3.2.16), and where $B(e,e')$ and $\underset{\sim}{\tilde{\Psi}}$ are as defined in Section 3.1.

Three different cases are considered:

(i) $T_{(1)}(\mu) = \mu^{\beta}$, $\beta = 0,1,2,\ldots$

(ii) $T_{(1)}(\mu) = e^{\mu}$,

and

(iii) $T_{(1)}(\mu) = e^{-\mu}$.

The first case is handled by direct integration of the expressions obtained from the substitution of the functions $\mu^{\beta}$, $\beta = 0,1,2,\ldots$, in Eq. (3.2.16). The last two cases, on the other hand, are treated by first approximating the functions $e^{\mu}$ and $e^{-\mu}$ through a quadratic finite-element interpolation over the angular region, and then by integrating the expressions obtained from the substitution of the resulting quadratic polynomials in Eq. (3.2.16).

Incorporation of such an approximation, however, should not mean that the functions $e^{\mu}$ and $e^{-\mu}$ cannot be dealt with by direct integration, since both of these functions and the functions $\mu^{\beta}$, $\beta = 1,2,\ldots$, are of comparable complexity computation-wise. Direct integration works perfectly well in the case of the functions $e^{\mu}$ and $e^{-\mu}$. Nevertheless, the quadratic finite-element interpolation has been resorted to with the intention of demonstrating the ease of applicability of the finite-element approximation technique to problems involving specified intensity distributions at the boundaries given either in terms of complicated functions, or discrete numerical data as in most of the real-life applications. It is worthwhile noting that the order of the interpolation is free and can be increased at will to meet the requirements of a given level of accuracy. The results for the element vectors are given below.

*Case 1*

$$\underset{\sim}{t}^{(1)}{}_{(e)} = \frac{1}{(\beta+2)(\beta+3)} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \mu_4^{\beta+2} + \mu_1\mu_4^{\beta+1} + \mu_1^2\mu_4^{\beta} + \ldots + \mu_1^{\beta+1}\mu_4 - (\beta+2)\mu_1^{\beta+2} \\ 0 \\ 0 \\ (\beta+2)\mu_4^{\beta+2} - \mu_1\mu_4^{\beta+1} - \mu_1^2\mu_4^{\beta} - \ldots - \mu_1^{\beta+1}\mu_4 - \mu_1^{\beta+2} \end{bmatrix},$$

$$\beta = 0,1,2,\ldots\,.$$

*Case 2*

$$\underset{\sim}{t}^{(1)}(e) = \frac{2}{(\mu_4 - \mu_1)^2}\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \left[ \frac{C_o}{6}\begin{bmatrix} t_1 \\ 0 \\ 0 \\ t_2 \end{bmatrix} + \frac{C_1}{12}\begin{bmatrix} t_3 \\ 0 \\ 0 \\ t_4 \end{bmatrix} + \frac{C_2}{20}\begin{bmatrix} t_5 \\ 0 \\ 0 \\ t_6 \end{bmatrix} \right]$$

where

$$t_1 = \mu_4^2 + \mu_1\mu_4 - 2\mu_1^2,$$

$$t_2 = 2\mu_4^2 - \mu_1\mu_4 - \mu_1^2,$$

$$t_3 = \mu_4^3 + \mu_1\mu_4^2 + \mu_1^2\mu_4 - 3\mu_1^3,$$

$$t_4 = 3\mu_4^3 - \mu_1\mu_4^2 - \mu_1^2\mu_4 - \mu_1^3,$$

$$t_5 = \mu_4^4 + \mu_1\mu_4^3 + \mu_1^2\mu_4^2 + \mu_1^3\mu_4 - 4\mu_1^4,$$

$$t_6 = 4\mu_4^4 - \mu_1\mu_4^3 - \mu_1^2\mu_4^2 - \mu_1^3\mu_4 - \mu_1^4,$$

$$C_o = \frac{1}{2}\mu_4(\mu_1 + \mu_4)e^{\mu_1} - 2\mu_1\mu_4 e^{(\mu_1+\mu_4)/2} + \frac{1}{2}\mu_1(\mu_1 + \mu_4)e^{\mu_4},$$

$$C_1 = -\frac{1}{2}(\mu_1+3\mu_4)e^{\mu_1} + 2(\mu_1+\mu_4)e^{(\mu_1+\mu_4)/2} - \frac{1}{2}(3\mu_1 + \mu_4)e^{\mu_4},$$

and

$$C_2 = e^{\mu_1} - 2e^{(\mu_1+\mu_4)/2} + e^{\mu_4}.$$

*Case 3*

$$\underset{\sim}{t}^{(1)}(e) = \frac{2}{(\mu_4 - \mu_1)^2} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \left[\frac{C_o}{6} \begin{bmatrix} t_1 \\ 0 \\ 0 \\ t_2 \end{bmatrix} + \frac{C_1}{12} \begin{bmatrix} t_3 \\ 0 \\ 0 \\ t_4 \end{bmatrix} + \frac{C_2}{20} \begin{bmatrix} t_5 \\ 0 \\ 0 \\ t_6 \end{bmatrix}\right]$$

where

$$t_1 = \mu_4^2 + \mu_1\mu_4 - 2\mu_1^2,$$

$$t_2 = 2\mu_4^2 - \mu_1\mu_4 - \mu_1^2,$$

$$t_3 = \mu_4^3 + \mu_1\mu_4^2 + \mu_1^2\mu_4 - 3\mu_1^3,$$

$$t_4 = 3\mu_4^3 - \mu_1\mu_2^2 - \mu_1^2\mu_4 - \mu_1^3,$$

$$t_5 = \mu_4^4 + \mu_1\mu_4^3 + \mu_1^2\mu_4^2 \quad \mu_1^3\mu_4 - 4\mu_1^4,$$

$$t_6 = 4\mu_4^4 - \mu_1\mu_4^3 - \mu_1^2\mu_4^2 - \mu_1^3\mu_4 - \mu_1^4,$$

$$C_o = \frac{1}{2}\,\mu_4(\mu_1 + \mu_4)e^{-\mu_1} - 2\mu_1\mu_4 e^{-(\mu_1+\mu_4)/2} + \frac{1}{2}\,\mu_1(\mu_1 + \mu_4)e^{-\mu_4},$$

$$C_1 = -\frac{1}{2}(\mu_1 + 3\mu_4)e^{-\mu_1} + 2(\mu_1 + \mu_4)e^{-(\mu_1+\mu_4)/2} - \frac{1}{2}\,(3\mu_1 + \mu_4)e^{-\mu_4},$$

and

$$C_2 = e^{-\mu_1} - 2e^{-(\mu_1+\mu_4)/2} + e^{-\mu_4}.$$

The intensity distributions in case one are given in Tables 3.3.26-34 for $2\tau_o = 1$, for $\gamma = 0.8$, for $\beta = 0,1,2$ and for different finite-element meshes.

TABLE 3.3.26 - (4,4) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_0 = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = 1$

| Mesh (4,4) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 16 | | | | Number of Nodal Points: 27 | |
| $\gamma = 0.8$ | | $\tau_0 = 0.5$ | | $T_{(1)}(\mu) = 1$ | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 0.998034 | 0.874859 | 0.765030 | 0.656786 | 0.553081 |
| 1/2 | 1.00983 | 0.808150 | 0.617202 | 0.464749 | 0.351172 |
| 0 | 0.882861<br>0.502651 | 0.360096 | 0.265059 | 0.203797 | 0.124395<br>0.0353385 |
| -1/2 | 0.305388 | 0.211556 | 0.142975 | 0.0583963 | -0.00132135 |
| -1 | 0.205615 | 0.133942 | 0.0841641 | 0.0378734 | 0.00211277 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.27 - (4,6) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = 1$

Mesh (4,6)

Number of Elements: 24

Number of Nodal Points: 37

$\tau_o = 0.5$

$\gamma = 0.8$

$T_{(1)}(\mu) = 1$

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | 1.00348 | 0.878465 | 0.758670 | 0.646023 | 0.544026 |
| 2/3 | 0.997785 | 0.827326 | 0.679505 | 0.545723 | 0.431701 |
| 1/3 | 1.01039 | 0.716029 | 0.505501 | 0.361387 | 0.261412 |
| 0 | 0.783285<br>0.509279 | 0.339574 | 0.282085 | 0.215221 | 0.134812<br>0.0629470 |
| -1/3 | 0.364277 | 0.265135 | 0.182221 | 0.0876845 | -0.00212093 |
| -2/3 | 0.266496 | 0.179049 | 0.117350 | 0.0522662 | 0.00252094 |
| -1 | 0.209311 | 0.139047 | 0.0878664 | 0.0364484 | 0.000507361 |
| τ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.28 - (4,8) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = 1$

| Mesh (4,8) | | | | | |
|---|---|---|---|---|---|
| Number of Elements 32 | | $\tau_o = 0.5$ | | Number of Nodal Points 47 | |
| $\gamma = 0.8$ | | | | $T_{(1)}(\mu) = 1$ | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 1.00138 | 0.877484 | 0.759001 | 0.645952 | 0.543273 |
| 3/4 | 0.00230 | 0.841977 | 0.699712 | 0.571381 | 0.461232 |
| 1/2 | 0.995309 | 0.782200 | 0.610431 | 0.466771 | 0.353014 |
| 1/4 | 1.00361 | 0.643447 | 0.435849 | 0.308026 | 0.220513 |
| 0 | 0.716567<br>0.515210 | 0.338254 | 0.289093 | 0.219104 | 0.139031<br>0.0817977 |
| -1/4 | 0.397383 | 0.294759 | 0.208203 | 0.111366 | -0.000902147 |
| -1/2 | 0.309879 | 0.212979 | 0.143268 | 0.0664832 | 0.00315411 |
| -3/4 | 0.251418 | 0.169828 | 0.109425 | 0.0478127 | 0.000894933 |
| -1 | 0.209735 | 0.138590 | 0.0877234 | 0.0367510 | 0.00115376 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.29 - (4,4) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = \mu$

| Mesh (4,4) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 16 | | | | | Number of Nodal Points: 27 |
| | | $\tau_o = 0.5$ | | | |
| $\gamma = 0.8$ | | | | | $T_{(1)}(\mu) = \mu$ |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 0.999341 | 0.836302 | 0.695980 | 0.571795 | 0.469483 |
| 1/2 | 0.492592 | 0.378420 | 0.305581 | 0.247976 | 0.194054 |
| 0 | 0.0462582<br>0.283370 | 0.224145 | 0.201247 | 0.143970 | 0.0990572<br>0.0213542 |
| -1/2 | 0.190647 | 0.145754 | 0.0899279 | 0.0421329 | -0.00282323 |
| -1 | 0.128226 | 0.0902928 | 0.0580193 | 0.0272074 | 0.000308078 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.30 - (4,6) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = \mu$

| Mesh (4,6) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 24 | | $\tau_o = 0.5$ | | Number of Nodal Points: 37 | |
| $\gamma = 0.8$ | | | | $T_{(1)}(\mu) = \mu$ | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 0.996789 | 0.832331 | 0.696446 | 0.576824 | 0.476402 |
| 2/3 | 0.665243 | 0.535949 | 0.431759 | 0.345488 | 0.273840 |
| 1/3 | 0.322712 | 0.276993 | 0.240257 | 0.198589 | 0.155115 |
| 0 | 0.107351<br>0.287091 | 0.251216 | 0.195056 | 0.152837 | 0.100800<br>0.0401905 |
| -1/3 | 0.221655 | 0.171701 | 0.118787 | 0.0619559 | -0.00292225 |
| -2/3 | 0.165975 | 0.121864 | 0.0776347 | 0.0381112 | -0.000211747 |
| -1 | 0.130875 | 0.0930101 | 0.0578009 | 0.0265046 | -0.00100472 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.31 - (4,8) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = \mu$

Mesh (4,8)

Number of Elements: 32

Number of Nodal Points: 47

$\tau_o = 0.5$

$\gamma = 0.8$

$T_{(1)}(\mu) = \mu$

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | 0.997878 | 0.833549 | 0.697144 | 0.577934 | 0.477854 |
| 3/4 | 0.746570 | 0.608069 | 0.496446 | 0.400980 | 0.321749 |
| 1/2 | 0.498915 | 0.401229 | 0.323723 | 0.260142 | 0.204063 |
| 1/4 | 0.241027 | 0.246161 | 0.224698 | 0.185537 | 0.142778 |
| 0 | 0.150378<br>0.285320 | 0.258444 | 0.191429 | 0.154396 | 0.0988895<br>0.0542779 |
| -1/4 | 0.238939 | 0.188281 | 0.138214 | 0.0780044 | -0.00168025 |
| -1/2 | 0.191542 | 0.144486 | 0.0944960 | 0.0482390 | 0.000053678 |
| -3/4 | 0.156409 | 0.113599 | 0.0723973 | 0.0346131 | -0.000908652 |
| -1 | 0.131301 | 0.0934031 | 0.0579747 | 0.0268369 | -0.000609505 |
| τ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.32 - (4,4) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = \mu^2$

| | | | | | |
|---|---|---|---|---|---|
| Mesh (4,4) | | | | | |
| Number of Elements: 16 | | | | Number of Nodal Points: 27 | |
| $\tau_o = 0.5$ | | | | | |
| $\gamma = 0.8$ | | | | $T_{(1)}(\mu) = \mu^2$ | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 0.958487 | 0.776097 | 0.628809 | 0.505063 | 0.408859 |
| 1/2 | 0.203227 | 0.173532 | 0.159003 | 0.143135 | 0.117406 |
| 0 | -0.00264304<br>0.198764 | 0.178293 | 0.157462 | 0.112203 | 0.0833036<br>0.0135021 |
| -1/2 | 0.136056 | 0.105056 | 0.0650970 | 0.0315643 | -0.00202265 |
| -1 | 0.0930708 | 0.0684062 | 0.0436537 | 0.0210964 | -0.000277915 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.33 - (4,6) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_0 = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = \mu^2$

Mesh (4,6)

Number of Elements: 24 | Number of Nodal Points: 37

$\tau_0 = 0.5$

$\gamma = 0.8$ | $T_{(1)}(\mu) = \mu^2$

| μ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | 0.978366 | 0.797782 | 0.654284 | 0.531974 | 0.433610 |
| 2/3 | 0.424064 | 0.341417 | 0.276874 | 0.224953 | 0.180640 |
| 1/3 | 0.0882708 | 0.130698 | 0.143455 | 0.133248 | 0.109838 |
| 0 | 0.0737635<br>0.198182 | 0.193554 | 0.146669 | 0.116805 | 0.0785604<br>0.0297395 |
| -1/3 | 0.157725 | 0.124681 | 0.0881071 | 0.0469948 | -0.00229275 |
| -2/3 | 0.119525 | 0.0907946 | 0.0578221 | 0.0292610 | -0.000516433 |
| -1 | 0.0943681 | 0.0687705 | 0.0430039 | 0.0202980 | -0.000975323 |
| τ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.34 - (4,8) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = \mu^2$

Mesh (4,8)

Number of Elements: 32

Number of Nodal Points: 47

$\tau_o = 0.5$

$\gamma = 0.8$

$T_{(1)}(\mu) = \mu^2$

| $\mu$ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | 0.986827 | 0.805550 | 0.661344 | 0.539146 | 0.440662 |
| 3/4 | 0.548674 | 0.442512 | 0.359441 | 0.289968 | 0.232769 |
| 1/2 | 0.239655 | 0.211815 | 0.185248 | 0.160323 | 0.131701 |
| 1/4 | 0.0517010 | 0.133429 | 0.149567 | 0.133645 | 0.106318 |
| 0 | 0.111906<br>0.195550 | 0.194061 | 0.142095 | 0.116985 | 0.0754757<br>0.0411280 |
| -1/4 | 0.169393 | 0.137418 | 0.102978 | 0.0592807 | -0.00141559 |
| -1/2 | 0.137353 | 0.107216 | 0.0703815 | 0.0369033 | -0.000270667 |
| -3/4 | 0.112559 | 0.0841441 | 0.0539638 | 0.0264937 | -0.000933803 |
| -1 | 0.0947443 | 0.0693474 | 0.0432085 | 0.0205810 | -0.000692036 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

A comparison of the finite-element results† for the expression

$$(\beta + 2)\int_0^1 \Psi(1/2,\mu)\mu d\mu, \quad \beta = 0,1,2,$$

with the corresponding exact results, as reported in [39], is given in Table 3.3.35.

TABLE 3.3.35 - Results for the expression $(\beta+2)\int_0^1 \Psi(1/2,\mu)\mu d\mu$, $\beta = 0,1,2$

| | Finite-Element Results† | | | |
|---|---|---|---|---|
| $\beta$ | Mesh (4,4)<br>Number of Elements 16<br>Number of Nodal Points 27 | Mesh (4,6)<br>Number of Elements 24<br>Number of Nodal Points 37 | Mesh (4,8)<br>Number of Elements 32<br>Number of Nodal Points 47 | Exact Results [39] |
| 0 | 0.418475 | 0.415445 | 0.416749 | 0.4162 |
| 1 | 0.428796 | 0.436684 | 0.447485 | 0.4516 |
| 2 | 0.429114 | 0.439353 | 0.458997 | 0.4721 |

The intensity distributions in cases two and three are given in Tables 3.3.36-41 for $2\tau_0 = 1$, $\gamma = 0.8$ and for different finite-element meshes.

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† Finite-element results are based on the corresponding intensity distributions given in Tables 3.3.26-34.

TABLE 3.3.36 - (4,4) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = e^{\mu}$

| Mesh (4,4) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 16 | | Number of Nodal Points: 27 | | | |
| $\tau_o = 0.5$ | | | | | |
| $\gamma = 0.8$ | | $T_{(1)}(\mu) = e^{\mu}$ | | | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 2.67026 | 2.25213 | 1.89700 | 1.57723 | 1.30402 |
| 1/2 | 1.60804 | 1.28416 | 1.01746 | 0.801151 | 0.618895 |
| 0 | 0.941665<br>0.917723 | 0.706194 | 0.571834 | 0.423261 | 0.280406<br>0.0652275 |
| -1/2 | 0.585808 | 0.426257 | 0.276039 | 0.121451 | -0.00538968 |
| -1 | 0.395621 | 0.269999 | 0.171267 | 0.0792334 | 0.00216646 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.37 - (4,6) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = e^{\mu}$

| | | | | | |
|---|---|---|---|---|---|
| Mesh (4,6) | | | | | |
| Number of Elements: 24 | | | | Number of Nodal Points: 37 | |
| | | $\tau_o = 0.5$ | | | |
| $\gamma = 0.8$ | | | | $T_{(1)}(\mu) = e^{\mu}$ | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 2.69420 | 2.27374 | 1.91473 | 1.59497 | 1.32283 |
| 2/3 | 1.92381 | 1.57484 | 1.28418 | 1.03297 | 0.820102 |
| 1/3 | 1.38023 | 1.07472 | 0.838979 | 0.647965 | 0.489609 |
| 0 | 0.942588<br>0.926704 | 0.719461 | 0.574601 | 0.445940 | 0.288015<br>0.123017 |
| -1/3 | 0.690133 | 0.519517 | 0.359650 | 0.181008 | -0.00065810 |
| -2/3 | 0.511547 | 0.361247 | 0.233459 | 0.109918 | 0.00194934 |
| -1 | 0.402644 | 0.277742 | 0.174289 | 0.0765097 | -0.00116018 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.38 - (4,8) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = e^{\mu}$

Mesh (4,8)

Number of Elements: 32

Number of Nodal Points: 47

$\tau_o = 0.5$

$\gamma = 0.8$

$T_{(1)}(\mu) = e^{\mu}$

| $\mu$ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | 2.70260 | 2.28248 | 1.92338 | 1.60330 | 1.33032 |
| 3/4 | 2.10215 | 1.73530 | 1.42826 | 1.16004 | 0.933928 |
| 1/2 | 1.63477 | 1.31252 | 1.05009 | 0.829323 | 0.642183 |
| 1/4 | 1.27202 | 0.975652 | 0.759073 | 0.582300 | 0.434145 |
| 0 | 0.942488<br>0.929027 | 0.725238 | 0.574898 | 0.451462 | 0.288288<br>0.163565 |
| -1/4 | 0.748147 | 0.574216 | 0.414935 | 0.228934 | -0.00353862 |
| -1/2 | 0.592231 | 0.428670 | 0.284592 | 0.139356 | 0.00301420 |
| -3/4 | 0.482292 | 0.339325 | 0.217731 | 0.100116 | -0.00065085 |
| -1 | 0.403738 | 0.278071 | 0.174450 | 0.0773319 | 0.00006956 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.39 - (4,4) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = e^{-\mu}$

| Mesh (4,4) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 16 | | | | Number of Nodal Points: 27 | |
| $\tau_o = 0.5$ | | | | | |
| $\gamma = 0.8$ | | | | $T_{(1)}(\mu) = e^{-\mu}$ | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 0.357176 | 0.330573 | 0.306714 | 0.276571 | 0.239071 |
| 1/2 | 0.610180 | 0.505826 | 0.378933 | 0.275881 | 0.205076 |
| 0 | 0.830446<br>0.297350 | 0.204226 | 0.124987 | 0.103308 | 0.0571983<br>0.0194667 |
| -1/2 | 0.168297 | 0.107288 | 0.0785879 | 0.0286326 | 0.00067222 |
| -1 | 0.113861 | 0.0702827 | 0.0431915 | 0.0188587 | 0.00172916 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.40 - (4,6) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = e^{-\mu}$

| Mesh (4,6) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 24 | | $\tau_o = 0.5$ | | Number of Nodal Points: 37 | |
| $\gamma = 0.8$ | | | | $T_{(1)}(\mu) = e^{-\mu}$ | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 0.369547 | 0.343310 | 0.306876 | 0.268821 | 0.230748 |
| 2/3 | 0.507325 | 0.431522 | 0.360866 | 0.291611 | 0.230927 |
| 1/3 | 0.729028 | 0.493299 | 0.322722 | 0.215374 | 0.149329 |
| 0 | 0.702969<br>0.300439 | 0.163960 | 0.144391 | 0.107937 | 0.0646185<br>0.0343518 |
| -1/3 | 0.204663 | 0.142336 | 0.0978730 | 0.0440535 | -0.000091464 |
| -2/3 | 0.147471 | 0.0927099 | 0.0623171 | 0.0255556 | 0.00254066 |
| -1 | 0.115494 | 0.0729582 | 0.0468730 | 0.0178528 | 0.00113793 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

TABLE 3.3.41 - (4,8) finite-element solution of the boundary value problem (3.3.7-9) for $\tau_o = 0.5$, for $\gamma = 0.8$ and for $T_{(1)}(\mu) = e^{-\mu}$

| | | | | | |
|---|---|---|---|---|---|
| Mesh (4,8) | | | | | |
| Number of Elements: 32 | | | | Number of Nodal Points: 47 | |
| $\tau_o = 0.5$ | | | | | |
| $\gamma = 0.8$ | | | | $T_{(1)}(\mu) = e^{-\mu}$ | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 0.368282 | 0.342911 | 0.308157 | 0.269467 | 0.230442 |
| 3/4 | 0.474389 | 0.410244 | 0.346409 | 0.285737 | 0.231969 |
| 1/2 | 0.599670 | 0.469830 | 0.362894 | 0.271514 | 0.201757 |
| 1/4 | 0.786971 | 0.451049 | 0.270186 | 0.174838 | 0.119237 |
| 0 | 0.609202<br>0.307174 | 0.155919 | 0.153286 | 0.110371 | 0.0695710<br>0.0435355 |
| -1/4 | 0.225111 | 0.160328 | 0.110242 | 0.0564724 | 0.00023215 |
| -1/2 | 0.172323 | 0.110466 | 0.0762842 | 0.0326282 | 0.00300244 |
| -3/4 | 0.139226 | 0.0891630 | 0.0581197 | 0.0235237 | 0.00144714 |
| -1 | 0.115640 | 0.0723248 | 0.0466371 | 0.0179331 | 0.00150051 |
| $\tau$ | -1/2 | -1/4 | 0 | 1/4 | 1/2 |

Since for these cases no exact results have been reported in the literature, no comparison has been attempted.

*The Nonhomogeneous Problem with Isotropic Scattering, with Constant Source and with Transparent Boundaries.*

The governing boundary value problem is given as follows.

$$\mu \frac{\partial}{\partial\tau} \Psi(\tau,\mu) + \Psi(\tau,\mu) = \frac{\gamma(\tau)}{2} \int_{-1}^{1} \Psi(\tau,\mu')d\mu' + \frac{1}{8} \qquad (3.3.10)$$

for $0 \leq \tau \leq \tau_0$ and $-1 \leq \mu \leq 1$,

$$\Psi(0,\mu) = \frac{1}{8} \quad \text{for} \quad 0 < \mu \leq 1, \qquad (3.3.11)$$

$$\Psi(\tau_0,\mu) = \frac{1}{8} \quad \text{for} \quad -1 \leq \mu < 0. \qquad (3.3.12)$$

If $\gamma$ is assumed to be independent of $\tau$, then the finite-element model associated with the boundary value problem (3.3.10-12) consists of following system of algebraic equations.

$$[\sum_{e=1}^{E} (-\underset{\sim}{T}_{(e)} + \underset{\sim}{E}_{(e)} - \frac{1}{4\pi} \sum_{e'=1}^{E} B(e,e')\underset{\sim}{S}_{(e,e')} + \underset{\sim}{B}^{(2)}{}_{(e)} - \underset{\sim}{B}^{(1)}{}_{(e)})]\tilde{\underset{\sim}{\psi}} =$$

$$[\sum_{e=1}^{E} (\underset{\sim}{t}^{(1)}{}_{(e)} - \underset{\sim}{t}^{(2)}{}_{(e)})] + \sum_{e=1}^{E} \underset{\sim}{S}_{(e)}$$

where the element matrices $\underset{\sim}{T}_{(e)}$, $\underset{\sim}{E}_{(e)}$, $\underset{\sim}{S}_{(e,e')}$, $\underset{\sim}{B}^{(1)}{}_{(e)}$ and $\underset{\sim}{B}^{(2)}{}_{(e)}$ are respectively given by Eqs. (3.2.4), (3.2.5), (3.2.7), (3.2.8) and (3.2.9), where the element source vector $\underset{\sim}{S}_{(e)}$ is given by Eq. (3.2.15) for $s_0 = 1/8$, where the element vectors $\underset{\sim}{t}^{(1)}{}_{(e)}$ and $\underset{\sim}{t}^{(2)}{}_{(e)}$ are respectively given by Eqs. (3.2.16) and (3.2.17) for $T_{(1)} = T_{(2)} = 1/8$, and where $B(e,e')$ and $\tilde{\psi}$ are as defined in Section 3.1.

The intensity distributions are given in Tables 3.3.42-45 for $\tau_o = 1$, for $\gamma = 1$ and for different finite-element meshes.

TABLE 3.3.42 - (4,4) finite-element solution of the boundary value problem (3.3.10-12) for $\tau_o = 1$ and for $\gamma = 1$

| Mesh (4,4) | | | | | |
|---|---|---|---|---|---|
| Number of Elements: 16 | | | | Number of Nodal Points: 27 | |
| $\tau_o = 1$, $\gamma = 1$ | | | | | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 0.123469 | 0.196667 | 0.250199 | 0.299850 | 0.324134 |
| 1/2 | 0.116805 | 0.233408 | 0.322306 | 0.397422 | 0.409290 |
| 0 | 0.181821<br>0.423589 | 0.487721 | 0.524333 | 0.487721 | 0.423589<br>0.181821 |
| -1/2 | 0.409290 | 0.397422 | 0.322306 | 0.233408 | 0.116805 |
| -1 | 0.324134 | 0.299850 | 0.250199 | 0.196667 | 0.123469 |
| $\tau$ | 0 | 1/4 | 1/2 | 3/4 | 1 |

TABLE 3.3.43 - (4,6) finite-element solution of the boundary value problem (3.3.10-12) for $\tau_o = 1$ and for $\gamma = 1$

Mesh (4,6)

Number of Elements: 24

Number of Nodal Points: 37

$\tau_o = 1$

$\gamma = 1$

| $\mu$ | Intensity Distribution | | | | |
|---|---|---|---|---|---|
| 1 | 0.120536 | 0.195712 | 0.251464 | 0.304620 | 0.328689 |
| 2/3 | 0.122914 | 0.223854 | 0.292883 | 0.355058 | 0.373867 |
| 1/3 | 0.115924 | 0.285402 | 0.382869 | 0.438949 | 0.433474 |
| 0 | 0.240200<br>0.408240 | 0.495821 | 0.493080 | 0.495821 | 0.408240<br>0.240200 |
| -1/3 | 0.433474 | 0.438949 | 0.382869 | 0.285402 | 0.115924 |
| -2/3 | 0.373867 | 0.355058 | 0.292883 | 0.223854 | 0.122914 |
| -1 | 0.328689 | 0.304620 | 0.251464 | 0.194712 | 0.120536 |
| $\tau$ | 0 | 1/4 | 1/2 | 3/4 | 1 |

TABLE 3.3.44 - (4,8) finite-element solution of the boundary value problem (3.3.10-12) for $\tau_o = 1$ and for $\gamma = 1$

| | | | | | |
|---|---|---|---|---|---|
| Mesh (4,8) | | | | | |
| Number of Elements: 32 | | | | | Number of Nodal Points: 47 |
| $\tau_o = 1$ | | | | | |
| $\gamma = 1$ | | | | | |
| $\mu$ | Intensity Distribution | | | | |
| 1 | 0.121684 | 0.195337 | 0.251392 | 0.304929 | 0.328871 |
| 3/4 | 0.120840 | 0.215346 | 0.282481 | 0.341354 | 0.362097 |
| 1/2 | 0.124042 | 0.249189 | 0.328666 | 0.392954 | 0.402734 |
| 1/4 | 0.119785 | 0.325652 | 0.419852 | 0.459180 | 0.438823 |
| 0 | 0.279134<br>0.398931 | 0.493931 | 0.480866 | 0.493931 | 0.398931<br>0.279134 |
| -1/4 | 0.438824 | 0.459180 | 0.419852 | 0.325652 | 0.119785 |
| -1/2 | 0.402734 | 0.392954 | 0.328666 | 0.249189 | 0.124042 |
| -3/4 | 0.362097 | 0.341354 | 0.282481 | 0.215346 | 0.120840 |
| -1 | 0.328871 | 0.304929 | 0.251392 | 0.195337 | 0.121684 |
| $\tau$ | 0 | 1/4 | 1/2 | 3/4 | 1 |

It is worthwhile noting that the finite-element solutions perfectly reflect the inherent symmetry condition

$$\Psi(\tau,\mu) = \Psi(\tau_0-\tau,-\mu) , \qquad 0 \leq \tau \leq \tau_0/2 \text{ and } -1 \leq \mu \leq 1,$$

involved in the boundary value problem (3.3.7-9).

A comparison of the finite-element results† for the integral

$$\int_{-1}^{1} \Psi(1/2,\mu)d\mu$$

with the corresponding exact result, as reported in [7], is given in Table 3.3.45.

TABLE 3.3.45 - Results for the expression $\int_{-1}^{1}\Psi(1/2,\mu)d\mu$

| Finite-Element Results† | | | Exact Result [7] |
|---|---|---|---|
| Mesh (4,4)<br>Number of Elements 16<br>Number of Nodal Points 27 | Mesh (4,6)<br>Number of Elements 24<br>Number of Nodal Points 37 | Mesh (4,8)<br>Number of Elements 32<br>Number of Nodal Points 47 | |
| 0.687917 | 0.705532 | 0.699820 | 0.702056 |

From the finite-element solutions obtained so far and from the comparison of these results with their corresponding exact results, we conclude that in all of the problems attempted

(i) finite-element solutions approximately satisfy the boundary conditions,

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†Finite-element results are based on the corresponding intensity distributions given in Tables 3.3.42-44.

(ii) the convergence of the finite-element approximation technique is extremely rapid,

and

(iii) the finite-element results are in good agreement with the corresponding exact results even in the case of coarse meshes.

The solution of the eigenvalue problem (3.3.1-3) provides a verification for the scheme adopted to handle the basic terms appearing in the transport equation. The eigenvalue problem (3.3.4-6), although it governs essentially the same problem involved in the eigenvalue problem (3.3.1-3), verifies the method of accounting for specular reflection. The solution of the boundary value problem (3.3.7-9) is attempted to substantiate the method of treating specified boundary conditions, whereas the solution of the boundary value problem (3.3.10-12) is attempted to justify the way in which a source term can be treated. Since the way in which diffuse reflection boundary conditions can be treated is exactly the same as that of treating the scattering term, any further verification in this regard has not been resorted to. As is seen, the cases attempted provide a thorough verification of the solution algorithm and of its computer implementation.

It is worthwhile stressing that the approach adopted in solving the boundary value problem (3.3.7-9) for $T_{(1)}(\mu) = e^{\mu}$ and $T_{(1)}(\mu) = e^{-\mu}$ is particularly significant since it demonstrates the fact that the application of the finite-element approximation technique in no way suffers from any difficulty due to specified intensity distributions given either in terms of complicated functions or in terms of discrete numerical data. Through an exactly similar approach it can be demonstrated that the application of the finite-element approximation technique is also straightforward in the cases of problems involving anisotropy, blackbody intensity distributions and emissivity characterized either by complicated functions or by discrete numerical data.

# CONCLUSION

Fundamental concepts pertaining to the finite-element analysis of problems of radiative transport in gray participating media have been treated in detail. Subsequently, a finite-element model of the Galerkin approximation of the weak formulation of the relevant governing boundary value problem has been developed and applied to a selected set of problems involving plane-parallel geometry with azimuthal symmetry.

On the basis of the theoretical and numerical results presented in the preceding sections, we reach the following conclusions concerning the applicability of the finite-element approximation technique to problems of radiative transport in gray participating media.

The finite-element approximation technique provides an efficient and reliable method of obtaining highly accurate solutions to the radiative transport equation. The convergence with decreased mesh spacing is extremely rapid. For a given level of accuracy, this may allow solution of problems on a coarser mesh than is possible with conventional methods.

The finite-element approximation technique works equally well for problems characterized by pure absorption, pure emission, pure scattering or by combined absorption, emission and scattering processes, or for problems with arbitrarily high orders of anisotropy and/or with arbitrarily complex boundary conditions, with no increase in execution time for comparable mesh sizes. This is in sharp contrast to conventional methods, where solution accuracy and efficiency are quite sensitive to the physical problem being solved. Moreover, there should be no theoretical difficulty, apart from computational complexity, with its application to multidimensional problems, even when anisotropic scattering is included. This feature is also in sharp contrast to the conventional methods since their extension to such problems is extremely difficult, if not impossible.

The finite-element method is capable of treating problems with angular discontinuities in the intensity distribution as well as problems involving severe heterogeneities. Angular discontinuities involved in problems with Cartesian geometry can be accounted for in a routine manner by simply introducing an additional node at the point of discontinuity. In problems with cylindrical and spherical geometry, on the other hand, the situation is not so straightforward due to the presence of derivatives with respect to the angular variables in the governing equation. Angular discontinuities involved in these instances can be dealt with by incorporating discontinuous angular elements. Heterogeneous problems can likewise be dealt with by incorporating discontinuous spatial elements. Although the use of discontinuous elements eliminate the need for local mesh refinement, their incorporation is not an easy task and lies outside the scope of the present study. Recent investigations on the possibility of incorporating discontinuous elements can be found in

Due to the inherent nature of the finite-element approximation technique, higher order approximations may easily be incorporated, thus allowing the use of a coarse enough mesh to achieve a given level of accuracy. However, the accuracy level achieved in the solution of the particular problems attempted proves that first order approximations perform quite satisfactorily even with coarse meshes. This may imply that higher order approximations should not be incorporated unless absolutely essential.

The main disadvantage of the application of the finite-element approximation technique to radiative transport problems in gray participating media is the necessity of storing the system matrices and the system vectors and then of solving the corresponding set of equations. For large problems, especially in multidimensional applications, the storage requirement can become prohibitive for many computing installations. However, as advances are made in data management techniques and computer fast memory development, this drawback may be mitigated to a great extent. In addition, the required storage may be reduced extensively if the banded structure of the resulting system matrices are incorporated in the computer implementation.

Although the theoretical development of the finite-element model considered in this study has not been restricted to any specific geometry, its application has been restricted to plane-parallel geometry with azimuthal symmetry. The obvious generalizations are, therefore, to examine more complex geometries. The generalization to one-dimensional spherical geometry should be straightforward. Plane-parallel geometry without azimuthal symmetry or one-dimensional cylindrical geometry poses more of a challenge because two angular variables are required even though only one spatial variable is needed; nevertheless, the generalization to treat two angular variables is a logical step on the way to a solution scheme capable of dealing with multidimensional geometries. However, the application to multidimensional geometries poses a substantially greater challenge because of the size of the problems encountered. For example, a three-dimensional problem involves five variables - three spatial and two angular. Therefore, even a relatively coarse mesh with ten nodes along each region involves $10^5$ unknowns.

In summary then, the finite-element approximation technique as applied to the Galerkin approximation of the weak formulation of the boundary value problem governing radiative transport in gray participating media can be considered to be both efficient and reliable. Although multi-dimensional applications have not been considered, there is no theoretical reason why any of the above conclusions cannot be extended to multidimensional problems. Of course, there are substantial numerical and computing challenges associated with generalizations in this direction.

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# APPENDIX A

## Some Integration Formulae

$$\int_{\tau_1}^{\tau_2} (\tau - \tau_1)d\tau = \frac{1}{2} (\tau_2 - \tau_1)^2$$

$$\int_{\tau_1}^{\tau_2} \tau(\tau - \tau_1)d\tau = \frac{1}{6} (\tau_2 - \tau_1)(2\tau_2^2 - \tau_1\tau_2 - \tau_1^2)$$

$$\int_{\tau_1}^{\tau_2} (\tau - \tau_2)d\tau = - \frac{1}{2} (\tau_2 - \tau_1)^2$$

$$\int_{\tau_1}^{\tau_2} \tau(\tau - \tau_2)d\tau = \frac{1}{6} (\tau_2 - \tau_1)(2\tau_2^2 - \tau_1\tau_2 - \tau_1^2)$$

$$\int_{\tau_1}^{\tau_2} (\tau - \tau_1)^2 d\tau = \frac{1}{3} (\tau_2 - \tau_1)^3$$

$$\int_{\tau_1}^{\tau_2} (\tau - \tau_2)^2 d\tau = \frac{1}{3} (\tau_2 - \tau_1)^3$$

$$\int_{\tau_1}^{\tau_2} (\tau - \tau_1)(\tau - \tau_2)d\tau = - \frac{1}{6} (\tau_2 - \tau_1)^3$$

$$\int_{\mu_1}^{\mu_4} \mu^n(\mu - \mu_1)d\mu = \frac{1}{(n+1)(n+2)} (\mu_4 - \mu_1)[(n+1)\mu_4^{n+1} - \mu_1\mu_4^n - \mu_1^2\mu_4^{n-1} - \ldots - \mu_1^n\mu_4 - \mu_1^{n+1}$$

$$\int_{\mu_1}^{\mu_4} \mu^n(\mu - \mu_4)d\mu = - \frac{1}{(n+1)(n+2)} (\mu_4 - \mu_1)[\mu_4^{n+1} + \mu_1\mu_4^n + \mu_1^2\mu_4^{n-1} + \ldots + \mu_1^n\mu_4 - (n+1)\mu_1^{n}$$

$$\int_{\mu_1}^{\mu_4} (\mu - \mu_1)^2 d\mu = \frac{1}{3} (\mu_4 - \mu_1)^3$$

$$\int_{\mu_1}^{\mu_4} (\mu - \mu_4)^2 d\mu = \frac{1}{3} (\mu_4 - \mu_1)^3$$

$$\int_{\mu_1}^{\mu_4} \mu(\mu - \mu_1)^2 d\mu = \frac{1}{12} (\mu_4 - \mu_1)(3\mu_4^3 - 5\mu_1\mu_4^2 + \mu_1^2\mu_4 + \mu_1^3)$$

$$\int_{\mu_1}^{\mu_4} \mu(\mu - \mu_4)^2 d\mu = \frac{1}{12} (\mu_4 - \mu_1)(\mu_4^3 + \mu_1\mu_4^2 - 5\mu_1^2\mu_4 + 3\mu_1^3)$$

$$\int_{\mu_1}^{\mu_4} \mu(\mu^2 - \mu_1^2) d\mu = \frac{1}{4} (\mu_4 - \mu_1)^2(\mu_1 + \mu_4)^2$$

$$\int_{\mu_1}^{\mu_4} \mu(\mu^2 - \mu_4^2) d\mu = -\frac{1}{4} (\mu_4 - \mu_1)^2(\mu_1 + \mu_4)^2$$

$$\int_{\mu_1}^{\mu_4} (\mu - \mu_1)(\mu - \mu_4) d\mu = -\frac{1}{6} (\mu_4 - \mu_1)^3$$

$$\int_{\mu_1}^{\mu_4} \mu(\mu - \mu_1)(\mu - \mu_4) d\mu = -\frac{1}{12} (\mu_4 - \mu_1)^3(\mu_1 + \mu_4)$$

$$\int_{\mu_1}^{\mu_4} \mu(\mu - \mu_1)(\mu + \mu_4) d\mu = \frac{1}{12} (\mu_4 - \mu_1)(7\mu_4^3 - 3\mu_1^2\mu_4 - 3\mu_1\mu_4^2 - \mu_1^3)$$

$$\int_{\mu_1}^{\mu_4} \mu(\mu + \mu_1)(\mu - \mu_4) d\mu = -\frac{1}{12} (\mu_4 - \mu_1)(\mu_4^3 + 3\mu_1^2\mu_4 + 3\mu_1\mu_4^2 - 7\mu_1^3)$$

# APPENDIX B

## A Representative Code for Homogeneous Problems

```
GALERKIN-FINITE ELEMENT PROGRAM TO SOLVE THE EIGENVALUE PROBLEM
WITH ISTROPIC SCATTERING, AND WITH VACUUM BOUNDARY CONDITIONS
AT X=-L/2 AND AT X=L/2, L BEING THE SLAB THICKNESS


TERMINOLOGY


NE          NUMBER OF ELEMENTS
NNP         NUMBER OF NODAL POINTS
NER         NUMBER OF ELEMENT ROWS
NEC         NUMBER OF ELEMENT COLUMNS
EIGEN       EIGENVALUE
ICODE1      ELEMENT CODE
                 IF 0 ELEMENT IS AN INTERNAL ELEMENT
                 IF 1 ELEMENT IS A BOUNDARY ELEMENT
ICODE2      ELEMENT CODE
                 IF 0 ELEMENT IS AT X=-L/2 OR AT X=0., DEPENDING
                      ON THE PROBLEM
                      OR
                      ELEMENT IS AN INTERNAL ELEMENT
                 IF 1 ELEMENT IS AT X=L/2 OR AT X=L, DEPENDING ON
                      THE PROBLEM
ICODE3      ELEMENT CODE
                 IF 0 ELEMENT IS AN INCOMING BOUNDARY ELEMENT
                      OR
                      ELEMENT IS AN INTERNAL ELEMENT
                 IF 1 ELEMENT IS AN OUTGOING BOUNDARY ELEMENT
PRBID(I)    PROBLEM IDENTIFICATION ARRAY, I=1,...,120
X1(I)       X-COORDINATE OF THE ITH NODE, I=1,...,NNP
X2(I)       MU-COORDINATE OF THE ITH NODE, I=1,...,NUP
IND(I)      COLUMN NUMBER OF THE ITH ELEMENT, I=1,...,NE
IX(I,J)     NODE NUMBER OF THE NODE J OF THE ITH ELEMENT, I=1,...
            ,NE, J=1,...,4
IADM(I,J)   ELEMENT NUMBER AT THE ITH ROW AND JTH COLUMN, I=1,...
            ,NER, J=1,...,NEC
Z(I,J)      JTH COMPONENT OF THE ITH EIGENVECTOR, I=J=1,...,NNP


DECLARATIONS
```

```
      PARAMETER NNP=ℓ1, NE=ℓ2, NER=ℓ3, NEC=ℓ4
      PARAMETER NNJ=2*NNP*NNP
      PARAMETER NNL=NNP*NNP+1
      DIMENSION PRBID(120)
      DIMENSION SM1(NNP,NNP),SM2(NNP,NNP),EM1(4,4),EM2(4,4)
      DIMENSION EM3(4,4),X1(NNP),X2(NNP),BEM(4,4)
      DIMENSION IX(NE.4),IND(NE),IADM(NEC,NER)
      DIMENSION JX(4)
      DIMENSION BETA(NNP),WK(NNJ)
      COMPLEX Z(NNP,NNP),ALFA(NNP),EIGEN
      REAL MUIL,MUILM,MUILN
      DATA KI/5/
      DATA SM1,SM2/NNJ*0./
C
C
C
C        READ AND PRINT THE INPUT DATA
C
C
C
C
C        READ AND PRINT THE PROBLEM IDENTIFICATION DATA
C
      READ(KI,100)(PREID(I),I=1,120)
      WRITE(6,110)(PRBID(I),I=1,120)
C
C        READ AND/OR PRINT THE CONTROL DATA
C
      WRITE(6,202)NE,NNP
C
C        READ AND PRINT THE NODAL POINT DATA
C
      READ(KI,211) X1
      READ(KI,211) X2
      WRITE(6,601)
      WRITE(6,602)
      DO 1 I=1,NNP
      WRITE(6,203)I,X1(I),X2(I)
1     CONTINUE
C
C        READ AND PRINT THE ELEMENT DATA
C
      NNK=NE-1
      DO 1010 ICOUNT=1,NNK,2
      READ(KI,213)((IX(I,J),J=1,4),I=ICOUNT,ICOUNT+1),(IND(I),I=ICOUNT,I
     1COUNT+1)
1010  CONTINUE
      WRITE(6,601)
      WRITE(6,603)
      DO 210 M=1,NE
      WRITE(6,204)M,(IX(M,J),J=1,4),IND(M)
210   CONTINUE
```

```
C
C        READ AND PRINT THE ANGULAR DISTRIBUTION MATRIX
C
      WRITE(6,601)
      WRITE(6,604)
      DO 220 N=1,NEC
      READ(KI,201)(IADM(N,I),I=1,NER)
      WRITE(6,605)(IADM(N,I),I=1,NER)
220   CONTINUE
      WRITE(6,555)
C
C
C
C        GENERATE AND ASSEMBLE THE ELEMENT MATRICES
C
C
C
      DO 3 M=1,NE
C
C        SET UP THE ELEMENT INCIDENCE RELATION
C
      I=IX(M,1)
      J=IX(M,2)
      K=IX(M,3)
      L=IX(M,4)
C
C        CALCULATE GEOMETRICAL PARAMETERS OF THE ELEMENT
C
      XIJM=X1(J)-X1(I)
      MUILM=X2(L)-X2(I)
C
C        CALCULATE COMMON PARAMETERS OF THE ELEMENT
C
      CP1=X2(L)**3+X2(I)*X2(L)**2-5*X2(L)*X2(I)**2+3*X2(I)**3
      CP2=(X2(L)+X2(I))*MUILM**2
      CP3=3*X2(L)**3-5*X2(I)*X2(L)**2+X2(L)*X2(I)**2+X2(I)**3
      CP4=1./MUILM
      CP1=CP1*CP4
      CP2=CP2*CP4
      CP3=CP3*CP4
      CP5=8.*XIJM*MUILM/3.
C
C        FORM THE ELEMENT MATRICES EM1(I,J) AND EM2(I,J)
C
      EM1(1,1)=CP1
      EM1(1,2)=CP1
      EM1(1,3)=CP2
      EM1(1,4)=CP2
      EM1(2,1)=-CP1
      EM1(2,2)=-CP1
      EM1(2,3)=-CP2
      EM1(2,4)=-CP2
```

```
      EM1(3,1)=-CP2
      EM1(3,2)=-CP2
      EM1(3,3)=-CP3
      EM1(3,4)=-CP3
      EM1(4,1)=CP2
      EM1(4,2)=CP2
      EM1(4,3)=CP3
      EM1(4,4)=CP3
      EM2(1,1)=CP5
      EM2(1,2)=CP5/2
      EM2(1,3)=CP5/4
      EM2(1,4)=CP5/2
      EM2(2,1)=CP5/2
      EM2(2,2)=CP5
      EM2(2,3)=CP5/2
      EM2(2,4)=CP5/4
      EM2(3,1)=CP5/4
      EM2(3,2)=CP5/2
      EM2(3,3)=CP5
      EM2(3,4)=CP5/2
      EM2(4,1)=CP5/2
      EM2(4,2)=CP5/4
      EM2(4,3)=CP5/2
      EM2(4,4)=CP5
C
C        IMBED THE ELEMENT MATRICES EM1(I,J) AND EM2(I,J) INTO THE
C        SYSTEM MATRIX SM1(I,J)
C
      DO 5 I1=1,4
      I2=IX(M,I1)
      DO 5 J1=1,4
      J2=IX(M,J1)
      SM1(I2,J2)=SM1(I2,J2)+EM1(I1,J1)+EM2(I1,J1)
5     CONTINUE
C
C        IDENTIFY THE ELEMENT
C
      IDUMMY=IND(M)
      DO 2 N=1,NER
      IELM=IADM(IDUMMY,N)
C
C        SET UP THE ELEMENT INCIDENCE RELATION
C
      JX(1)=IX(IELM,1)
      JX(2)=IX(IELM,2)
      JX(3)=IX(IELM,3)
      JX(4)=IX(IELM,4)
      II=JX(1)
      JJ=JX(2)
      KK=JX(3)
      LL=JX(4)
```

```
C
C         CALCULATE GEOMETRICAL PARAMETERS OF THE ELEMENT
C
      MUILN=X2(LL)-X2(II)
C
C         CALCULATE COMMON PARAMETERS OF THE ELEMENT
C
      CP6=XIJM*MUILM*MUILN
C
C         FORM THE ELEMENT MATRIX EM3(I,J)
C
      EM3(1,1)=CP6
      EM3(1,2)=CP6/2
      EM3(1,3)=CP6/2
      EM3(1,4)=CP6
      EM3(2,1)=CP6/2
      EM3(2,2)=CP6
      EM3(2,3)=CP6
      EM3(2,4)=CP6/2
      EM3(3,1)=CP6/2
      EM3(3,2)=CP6
      EM3(3,3)=CP6
      EM3(3,4)=CP6/2
      EM3(4,1)=CP6
      EM3(4,2)=CP6/2
      EM3(4,3)=CP6/2
      EM3(4,4)=CP6
C
C         IMBED THE ELEMENT MATRIX EM3(I,J) INTO THE SYSTEM MATRIX
C         SM2(I,J)
C
      DO 4 I1=1,4
      I2=IX(M,I1)
      DO 4 J1=1,4
      J2=JX(J1)
      SM2(I2,J2)=SM2(I2,J2)+EM3(I1,J1)
4     CONTINUE
2     CONTINUE
3     CONTINUE
C
C
C
C         GENERATE AND ASSEMBLE THE BOUNDARY ELEMENT MATRICES
C
C
C
      WRITE(6,555)
      DO 6 M=1,NE
C
C         READ THE ELEMENT CODES
C
      READ(KI,201)ICODE1,ICODE2,ICODE3
```

```
C
C        CHECK WHETHER THE ELEMENTS IS AN INTERNAL ELEMENT OR NOT
C
      IF(ICODE1.EQ.0) GO TO 6
C
C        CHECK ON WHICH BOUNDARY THE ELEMENT IS
C
      IF(ICODE2.EQ.1) GO TO 8
C
C        CHECK WHETHER THE ELEMENT IS AN INCOMING BOUNDARY ELEMENT OR
C        NOT
C
      IF(ICODE3.EQ.0) GO TO 6
C
C        SET UP THE ELEMENT INCIDENCE RELATION
C
      I=IX(M,1)
      J=IX(M,2)
      K=IX(M,3)
      L=IX(M,4)
C
C        CALCULATE GEOMETRICAL PARAMETERS OF THE ELEMENT
C
      MUIL=X2(L)-X2(I)
C
C        INITIALIZE THE BOUNDARY ELEMENT MATRIX BEM(I,J)
C
      DO 7 N1=1,4
      DO 7 N2=1,4
7     BEM(N1,N2)=0.
C
C        CALCULATE COMMON PARAMETERS OF THE ELEMENT AND FORM THE
C        BOUNDARY ELEMENT MATRIX BEM(I,J)
C
      BCP0=2./MUIL
      BCP1=X2(L)**3+X2(L)**2*X2(I)-5*X2(I)**2*X2(L)+3*X2(I)**3
      BCP1=BCP1*BCP0
      BCP2=MUIL**2*(X2(I)+X2(L))
      BCP2=BCP2*BCP0
      BCP3=3*X2(L)**3-5*X2(L)**2*X2(I)+X2(I)**2*X2(L)+X2(I)**3
      BCP3=BCP3*BCP0
      BEM(1,1)=-BCP1
      BEM(1,4)=-BCP2
      BEM(4,1)=-BCP2
      BEM(4,4)=-BCP3
      GO TO 101
8     CONTINUE
C
C        CHECK WHETHER THE ELEMENT IS AN INCOMING BOUNDARY ELEMENT OR
C        NOT
C
```

```
      IF(ICODE3.EQ.0) GO TO 6
C
C        SET UP THE ELEMENT INCIDENCE RELATION
C
      I=IX(M,1)
      J=IX(M,2)
      K=IX(M,3)
      L=IX(M,4)
C
C        CALCULATE GEOMETRICAL PARAMETERS OF THE ELEMENT
C
      MUIL=X2(L)-X2(I)
C
C        INITIALIZE THE BOUNDARY ELEMENT MATRIX BEM(I,J)
C
      DO 9 N1=1,4
      DO 9 N2=1,4
9     BEM(N1,N2)=0.
C
C        CALCULATE COMMON PARAMETERS OF THE ELEMENT AND FORM THE
C        BOUNDARY ELEMENT MATRIX BEM(I,J)
C
      BCP4=2./MUIL
      BCP5=X2(L)**3+X2(L)**2*X2(I)-5*X2(I)**2*X2(L)+3*X2(I)**3
      BCP5=BCP5*BCP4
      BCP6=MUIL**2*(X2(I)+X2(L))
      BCP6=BCP6*BCP4
      BCP7=3*X2(L)**3-5*X2(L)**2*X2(I)+X2(I)**2*X2(L)+X2(I)**3
      BCP7=BCP7*BCP4
      BEM(2,2)=BCP5
      BEM(2,3)=BCP6
      BEM(3,2)=BCP6
      BEM(3,3)=BCP7
101   CONTINUE
C
C        IMBED THE BOUNDARY ELEMENT MATRIX BEM(I,J) INTO THE SYSTEM
C        MATRIX SM1(I,J)
C
      DO 10 I1=1,4
      I2=IX(M,I1)
      DO 10 J1=1,4
      J2=IX(M,J1)
      SM1(I2,J2)=SM1(I2,J2)+BEM(I1,J1)
10    CONTINUE
6     CONTINUE
C
C
C
C        SOLVE THE EIGENPROBLEM AND PRINT THE EIGENVALUES, EIGENVECTORS,
C        AND PERFORMANCE INDEX OF THE SOLUTION ROUTINE
C
C
C
```

```
      CALL EIGZF(SM1,NNP,SM2,NNP,NNP,2,ALFA,BETA,Z,NNP,WK,IER)
      WRITE(6,601)
      WRITE(6,8001)
      DO 3001 J=1,NNP
      WRITE(6,755)J
3001  WRITE(6,756)(Z(I,J),I=1,NNP)
      WRITE(6,601)
      DO 3002 I=1,NNP
      WRITE(6,4002)ALFA(I),BETA(I)
      IF(BETA(I).GT.(-0.000001).AND.BETA(I).LT.0.000001) GO TO 3002
      EIGEN=ALFA(I)/BETA(I)
      WRITE(6,2003)I,EIGEN
3002  CONTINUE
      WRITE(6,2004)IER,WK(1)
C
C
C
C         FORMATS
C
C
C
100   FORMAT(20A4)
110   FORMAT(1H1,////,(10X,20A4))
201   FORMAT( )
202   FORMAT(1H1,////,T20,,NUMBER OF ELEMENTS,,T50,,=,,I10,//,T20,
     1,NUMBER OF NODAL POINTS,,T50,,=,,I10)
203   FORMAT(/,T5,I10,T27,F10.6,T47,F10.6)
204   FORMAT(T8,I10,T30,4(I10,8X),I10)
211   FORMAT(7F10.0)
213   FORMAT(10I5)
555   FORMAT(1H1)
601   FORMAT(1H1,////)
602   FORMAT(T40,,NODAL POINT,,//,T10,,NUMBER,,T30,,X1,,T50,,X2,,//)
603   FORMAT(T10,,ELEMENT NUMBER,,T40,,IX(1),,T60,,IX(2),,T80,,IX(3),,T1
     100,,IX(4),,T120,,IND,,//)
604   FORMAT(T10,5X,,ANGULAR DISTRIBUTION MATRIX,,//)
605   FORMAT(T10,24(I5))
755   FORMAT(//,T20,,EIGENVECTOR,,I5)
756   FORMAT((5X,8(E13.6,2X)))
2003  FORMAT(T20,,EIGENVALUE,,I5,,  =,,2E13.6)
2004  FORMAT(T20,,IER,,T50,,=,,I5,//,T20,,PERFORMANCE INDEX,,I50,,=,,
     1E13.6)
4002  FORMAT(T20,,ALFA=,,2E13.6,10X,,BETA=,,E13.6)
8001  FORMAT(T40,,EIGENMATRIX,,//)
      STOP
      END
```

In the above listed code $\ell_1$,$\ell_2$,$\ell_3$ and $\ell_4$ stand for, respectively, number of nodal points, number of elements, number of element rows and number of element columns in the mesh. A list of typical data is given below.

*Mesh Parameters*

$\ell_1 = 37$

$\ell_2 = 24$

$\ell_3 = 6$

$\ell_4 = 4$

*Problem Identification Data*

```
        10        20        30        40        50        60        70
         ↓         ↓         ↓         ↓         ↓         ↓         ↓
........................................................................

GALERKIN-FINITE ELEMENT SOLUTION TO THE EIGENVALUE PROBLEM WITH ISOTROPI
C SCATTERING, AND WITH VACUUM BOUNDARY CONDITIONS AT X=-L/2 AND AT X=L/2
, L BEING THE SLAB THICKNESS


L = 1.0
........................................................................
         ↑         ↑         ↑         ↑         ↑         ↑         ↑
        10        20        30        40        50        60        70
```

*Nodal Point Data*

```
        10        20        30        40        50        60        70
         ↓         ↓         ↓         ↓         ↓         ↓         ↓
........................................................................

 -0.50     -0.50     -0.50     -0.50     -0.50     -0.50     -0.50
 -0.50     -0.25     -0.25     -0.25     -0.25     -0.25     -0.25
 -0.25      0.0       0.0       0.0       0.0       0.0       0.0
  0.0       0.25      0.25      0.25      0.25      0.25      0.25
  0.25      0.50      0.50      0.50      0.50      0.50      0.50
  0.50      0.50
```

```
 -1.0       -0.666667 -0.333333  0.0       0.0       0.333333  0.666667
  1.0       -1.0      -0.666667 -0.333333  0.0       0.333333  0.666667
  1.0       -1.0      -0.666667 -0.333333  0.0       0.333333  0.666667
  1.0       -1.0      -0.666667 -0.333333  0.0       0.333333  0.666667
  1.0       -1.0      -0.666667 -0.333333  0.0       0.0       0.333333
  0.666667   1.0
.......................................................................
         ↑         ↑         ↑         ↑         ↑         ↑         ↑
        10        20        30        40        50        60        70
```

*Element Data*

```
        10        20        30        40        50        60        70
         ↓         ↓         ↓         ↓         ↓         ↓         ↓
.......................................................................
     7   14   15    8   14   21   22   15    1    2
    21   28   29   22   28   36   37   29    3    4
     6   13   14    7   13   20   21   14    1    2
    20   27   28   21   27   35   36   28    3    4
     5   12   13    6   12   19   20   13    1    2
    19   26   27   20   26   34   35   27    3    4
     3   11   12    4   11   18   19   12    1    2
    18   25   26   19   25   32   33   26    3    4
     2   10   11    3   10   17   18   11    1    2
    17   24   25   18   24   31   32   25    3    4
     1    9   10    2    9   16   17   10    1    2
    16   23   24   17   23   30   31   24    3    4
.......................................................................
         ↑         ↑         ↑         ↑         ↑         ↑         ↑
        10        20        30        40        50        60        70
```

*Angular Distribution Data*

```
        10        20        30        40        50        60        70
         ↓         ↓         ↓         ↓         ↓         ↓         ↓
.......................................................................
1,5,9,13,17,21
2,6,10,14,18,22
3,7,11,15,19,23
4,8,12,16,20,24
.......................................................................
         ↑         ↑         ↑         ↑         ↑         ↑         ↑
        10        20        30        40        50        60        70
```

*Code Data*

```
         10        20        30        40        50        60        70
          ↓         ↓         ↓         ↓         ↓         ↓         ↓
........................................................................

1,0,0
0,0,0
0,0,0
1,1,1
1,0,0
0,0,0
0,0,0
1,1,1
1,0,0
0,0,0
0,0,0
1,1,1
1,0,1
0,0,0
0,0,0
1,1,0
1,0,1
0,0,0
0,0,0
1,1,0
1,0,1
0,0,0
0,0,0
1,1,0
........................................................................
          ↑         ↑         ↑         ↑         ↑         ↑         ↑
         10        20        30        40        50        60        70
```

# APPENDIX C

## A Representative Code for Nonhomogeneous Problems

```
C
C
C
C       GALERKIN-FINITE ELEMENT PROGRAM TO SOLVE THE NONHOMOGENEOUS
C       PROBLEM WITH ISTROPIC SCATTERING, AND WITH SPECIFIED INTENSITY
C       BOUNDARY CONDITION AT X=-L/2 AND VACUUM BOUNDARY CONDITION AT
C       X=L/2, L BEING THE SLAB THICKNESS
C
C
C
C       TERMINOLOGY
C
C
C
C       NE          NUMBER OF ELEMENTS
C       NNP         NUMBER OF NODAL POINTS
C       NER         NUMBER OF ELEMENT ROWS
C       NEC         NUMBER OF ELEMENT COLUMNS
C       OMEGA       EXTINCTION COEFFICIENT
C       KI          INPUT UNIT NUMBER
C       ICODE1      ELEMENT CODE
C                       IF 0 ELEMENT IS AN INTERNAL ELEMENT
C                       IF 1 ELEMENT IS A BOUNDARY ELEMENT
C       ICODE2      ELEMENT CODE
C                       IF 0 ELEMENT IS AT X=-L/2 OR AT X=., DEPENDING
C                            ON THE PROBLEM
C                            OR
C                            ELEMENT IS AN INTERNAL ELEMENT
C                       IF 1 ELEMENT IS AT X=L/2 OR AT X=L, DEPENDING
C                            ON THE PROBLEM
C       ICODE3      ELEMENT CODE
C                       IF 0 ELEMENT IS AN INCOMING BOUNDARY ELEMENT
C                            OR
C                            ELEMENT IS AN INTERNAL ELEMENT
C                       IF 1 ELEMENT IS AN OUTGOING BOUNDARY ELEMENT
C       PRBID(I)    PROBLEM IDENTIFICATION ARRAY, I=1,...,120
C       X1(I)       X-COORDINATE OF THE ITH NODE, I=1,...,NNP
C       X2(I)       MU-COORDINATE OF THE ITH NODE, I=1,...,NNP
C       IND(I)      COLUMN NUMBER OF THE ITH ELEMENT, I=1,...,NE
C       IX(I,J)     NODE NUMBER OF THE NODE J OF THE ITH ELEMENT, I=1,...
C                   ,NE, J=1,...,4
C       IADM(I,J)   ELEMENT NUMBER AT THE ITH ROW AND JTH COLUMN, I=1,...
C                   ,NER, J=1,...,NEC
C
C
C
C       DECLARATIONS
C
C
C
```

```
      PARAMETER NNP=ℓ1,NE=ℓ2,NER=ℓ3,NEC=ℓ4
      PARAMETER NNJ=NNP*NNP
      DIMENSION PRBID(120)
      DIMENSION SM(NNP,NNP),SV(NNP)
      DIMENSION IX(NE,4),IADM(NEC,NER),JX(4),IND(NE),X1(NNP),X2(NNP)
      DIMENSION EM1(4,4),EM2(4,4),EM3(4,4),BEM(4,4),EV(4),BEV(4)
      DIMENSION WKAREA(NNP)
      REAL MUIL,MUILM,MUILN
      DATA KI/5/
      DATA SM,SV/NNJ*0.,NNP*0./
C
C
C
C        READ AND PRINT THE INPUT DATA
C
C
C
C
C        READ AND PRINT THE PROBLEM IDENTIFICATION DATA
C
      READ(KI,100)(PRBID(I),I=1,120)
      WRITE(6,110)(PRBID(I),I=1,120)
C
C        READ AND/OR PRINT THE CONTROL DATA
C
      WRITE(6,202)NE,NNP
C
C        READ AND PRINT THE NODAL POINT DATA
C
      READ(KI,211) X1
      READ(KI,211) X2
      WRITE(6,601)
      WRITE(6,602)
      DO 1 I=1,NNP
      WRITE(6,203)I,X1(I),X2(I)
1     CONTINUE
C
C        READ AND PRINT THE ELEMENT DATA
C
      NNK=NE-1
      DO 1010 ICOUNT=1,NNK,2
      READ(KI,213)((IX(I,J),J=1,4),I=ICOUNT,ICOUNT+1),(IND(I),I=ICOUNT,I
     1COUNT+1)
1010  CONTINUE
      WRITE(6,601)
      WRITE(6,603)
      DO 210 M=1,NE
      WRITE(6,204)M,(IX(M,J),J=1,4),IND(M)
210   CONTINUE
```

```
C
C        READ AND PRINT THE ANGULAR DISTRIBUTION MATRIX
C
      WRITE(6,601)
      WRITE(6,604)
      DO 220 N=1,NEC
      READ(KI,201)(IADM(N,I),I=1,NER)
      WRITE(6,605)(IADM(N,I),I=1,NER)
220   CONTINUE
C
C        READ AND PRINT THE HEAT TRANSFER DATA
C
      READ(KI,201)GAMMA
      WRITE(6,601)
      WRITE(6,606)GAMMA
C
C
C
C        GENERATE AND ASSEMBLE THE ELEMENT MATRICES
C
C
C
      WRITE(6,555)
      DO 3 M=1,NE
C
C        SET UP THE ELEMENT INCIDENCE RELATION
C
      I=IX(M,1)
      J=IX(M,2)
      K=IX(M,3)
      L=IX(M,4)
C
C        CALCULATE GEOMETRICAL PARAMETERS OF THE ELEMENT
C
      XIJM=X1(J)-X1(I)
      MUILM=X2(L)-X2(I)
C
C        CALCULATE COMMON PARAMETERS OF THE ELEMENT
C
      CP1=X2(L)**3+X2(I)*X2(L)**2-5*X2(L)*X2(I)**2+3*X2(I)**3
      CP2=(X2(L)+X2(I))*MUILM**2
      CP3=3*X2(L)**3-5*X2(I)*X2(L)**2+X2(L)*X2(I)**2+X2(I)**3
      CP4=1./MUILM
      CP1=CP1*CP4
      CP2=CP2*CP4
      CP3=CP3*CP4
      CP5=8.*XIJM*MUILM/3.
C
C        FORM THE ELEMENT MATRICES EM1(I,J) AND EM2(I,J)
C
```

```
      EM1(1,1)=CP1
      EM1(1,2)=CP1
      EM1(1,3)=CP2
      EM1(1,4)=CP2
      EM1(2,1)=-CP1
      EM1(2,2)=-CP1
      EM1(2,3)=-CP2
      EM1(2,4)=-CP2
      EM1(3,1)=-CP2
      EM1(3,2)=-CP2
      EM1(3,3)=-CP3
      EM1(3,4)=-CP3
      EM1(4,1)=CP2
      EM1(4,2)=CP2
      EM1(4,3)=CP3
      EM1(4,4)=CP3
      EM2(1,1)=CP5
      EM2(1,2)=CP5/2
      EM2(1,3)=CP5/4
      EM2(1,4)=CP5/2
      EM2(2,1)=CP5/2
      EM2(2,2)=CP5
      EM2(2,3)=CP5/2
      EM2(2,4)=CP5/4
      EM2(3,1)=CP5/4
      EM2(3,2)=CP5/2
      EM2(3,3)=CP5
      EM2(3,4)=CP5/2
      EM2(4,1)=CP5/2
      EM2(4,2)=CP5/4
      EM2(4,3)=CP5/2
      EM2(4,4)=CP5
C
C        IMBED THE ELEMENT MATRICES EM1(I,J) AND EM2(I,J) INTO THE
C        SYSTEM MATRIX SM(I,J)
C
      DO 5 I1=1,4
      I2=IX(M,I1)
      DO 5 J1=1,4
      J2=IX(M,J1)
      SM (I2,J2)=SM (I2,J2)+EM1(I1,J1)+EM2(I1,J1)
5     CONTINUE
C
C        IDENTIFY THE ELEMENT
C
      IDUMMY=IND(M)
      DO 2 N=1,NER
      IELM=IADM(IDUMMY,N)
C
C        SET UP THE ELEMENT INCIDENCE RELATION
C
```

```
      JX(1)=IX(IELM,1)
      JX(2)=IX(IELM,2)
      JX(3)=IX(IELM,3)
      JX(4)=IX(IELM,4)
      II=JX(1)
      JJ=JX(2)
      KK=JX(3)
      LL=JX(4)
C
C        CALCULATE GEOMETRICAL PARAMETERS OF THE ELEMENT
C
      MUILN=X2(LL)-X2(II)
C
C        CALCULATE COMMON PARAMETERS OF THE ELEMENT
C
      CP6=-1.*GAMMA*XIJM*MUILM*MUILN
C
C        FORM THE ELEMENT MATRIX EM3(I,J)
C
      EM3(1,1)=CP6
      EM3(1,2)=CP6/2
      EM3(1,3)=CP6/2
      EM3(1,4)=CP6
      EM3(2,1)=CP6/2
      EM3(2,2)=CP6
      EM3(2,3)=CP6
      EM3(2,4)=CP6/2
      EM3(3,1)=CP6/2
      EM3(3,2)=CP6
      EM3(3,3)=CP6
      EM3(3,4)=CP6/2
      EM3(4,1)=CP6
      EM3(4,2)=CP6/2
      EM3(4,3)=CP6/2
      EM3(4,4)=CP6
C
C        IMBED THE ELEMENT MATRIX EM3(I,J) INTO THE SYSTEM MATRIX
C        SM(I,J)
C
      DO 4 I1=1,4
      I2=IX(M,I1)
      DO 4 J1=1,4
      J2=JX(J1)
      SM (I2,J2)=SM (I2,J2)+EM3(I1,J1)
4     CONTINUE
2     CONTINUE
3     CONTINUE
C
C
C
```

```
C          GENERATE AND ASSEMBLE THE BOUNDARY ELEMENT MATRICES AND THE
C          BOUNDARY ELEMENT VECTORS
C
C
C
      WRITE(6,555)
      DO 6 M=1,NE
C
C        READ THE ELEMENT CODES
C
      READ(KI,201)ICODE1,ICODE2,ICODE3
C
C        CHECK WHETHER THE ELEMENT IS AN INTERNAL ELEMENT OR NOT
C
      IF(ICODE1.EQ.0) GO TO 6
C
C        CHECK ON WHICH BOUNDARY THE ELEMENT IS
C
      IF(ICODE2.EQ.1) GO TO 20
C
C        SET UP THE ELEMENT INCIDENCE RELATION
C
      I=IX(M,1)
      J=IX(M,2)
      K=IX(M,3)
      L=IX(M,4)
C
C        CALCULATE GEOMETRICAL PARAMETERS OF THE ELEMENT
C
      MUIL=X2(L)-X2(I)
C
C        CHECK WHETHER THE ELEMENT IS AN OUTGOING BOUNDARY ELEMENT OR
C        NOT
C
      IF(ICODE3.EQ.1) GO TO 30
C
C        INITIALIZE THE BOUNDARY ELEMENT VECTOR BEV(I)
C
      DO 11 N1=1,4
11    BEV(N1)=0.
C
C        CALCULATE COMMON PARAMETERS OF THE ELEMENT AND FORM THE
C        BOUNDARY ELEMENT VECTOR BEV(I)
C
      BCP0=4.
      BCP1=X2(L)**2+X2(I)*X2(L)-2.*X2(I)**2
      BCP1=BCP1*BCP0
      BCP2=2.*X2(L)**2-X2(I)*X2(L)-X2(I)**2
      BCP2=BCP2*BCP0
      BEV(1)=BCP1
      BEV(4)=BCP2
```

```
C
C        IMBED THE BOUNDARY ELEMENT VECTOR BEV(I) INTO THE SYSTEM
C        VECTOR EV(I)
C
      DO 12 I1=1,4
      I2=IX(M,I1)
      SV(I2)=SV(I2) BEV(I1)
12    CONTINUE
      GO TO 6
30    CONTINUE
C
C        INITIALIZE THE BOUNDARY ELEMENT MATRIX BEM(I,J)
C
      DO 7 N1=1,4
      DO 7 N2=1,4
7     BEM(N1,N2)=0.
C
C        CALCULATE COMMON PARAMETERS OF THE ELEMENT AND FORM THE
C        BOUNDARY ELEMENT MATRIX
C
      BCP0=2./MUIL
      BCP1=X2(L)**3+X2(L)**2*X2(I)-5*X2(I)**2*X2(L)+3*X2(I)**3
      BCP1=BCP1*BCP0
      BCP2=MUIL**2*(X2(I)+X2(L))
      BCP2=BCP2*BCP0
      BCP3=3*X2(L)**3-5*X2(L)**2*X2(I)+X2(I)**2*X2(L)+X2(I)**3
      BCP3=BCP3*BCP0
      BEM(1,1)=-BCP1
      BEM(1,4)=-BCP2
      BEM(4,1)=-BCP2
      BEM(4,4)=-BCP3
      GO TO 50
20    CONTINUE
C
C        CHECK WHETHER THE ELEMENT IS AN INCOMING BOUNDARY ELEMENT OR
C        NOT
C
      IF(ICODE3.EQ.0) GO TO 6
C
C        SET UP THE ELEMENT INCIDENCE RELATION
C
      I=IX(M,1)
      J=IX(M,2)
      K=IX(M,3)
      L=IX(M,4)
C
C        CALCULATE GEOMETRICAL PARAMETERS OF THE ELEMENT
C
      MUIL=X2(L)-X2(I)
C
C        INITIALIZE THE BOUNDARY ELEMENT MATRIX BEM(I,J)
C
```

```
      DO 9 N1=1,4
      DO 9 N2=1,4
9     BEM(N1,N2)=0.
C
C        CALCULATE COMMON PARAMETERS OF THE ELEMENT AND FORM THE
C        BOUNDARY ELEMENT MATRIX BEM(I,J)
C
      BCP0=2./MUIL
      BCP1=X2(L)**3+X2(L)**2*X2(I)-5*X2(I)**2*X2(L)+3*X2(I)**3
      BCP1=BCP1*BCP0
      BCP2=MUIL**2*(X2(I)+X2(L))
      BCP2=BCP2*BCP0
      BCP3=3*X2(L)**3-5*X2(L)**2*X2(I)+X2(I)**2*X2(L)+X2(I)**3
      BCP3=BCP3*BCP0
      BEM(2,2)=BCP1
      BEM(2,3)=BCP2
      BEM(3,2)=BCP2
      BEM(3,3)=BCP3
50    CONTINUE
C
C        IMBED THE BOUNDARY ELEMENT MATRIX BEM(I,J) INTO THE SYSTEM
C        MATRIX SM(I,J)
C
      DO 10 I1=1,4
      I2=IX(M,I1)
      DO 10 J1=1,4
      J2=IX(M,J1)
      SM(I2,J2)=SM(I2,J2) BEM(I1,J1)
10    CONTINUE
6     CONTINUE
C
C
C
C        SOLVE THE SYSTEM OF EQUATIONS AND PRINT THE RESULT
C
C
C
      CALL LEQT1F(SM,1,NNP,NNP,SV,6,WKAREA,IER)
      WRITE(6,601)
      WRITE(6,8001)
      WRITE(6,655)(SV(I),I=1,NNP)
C
C
C
C        FORMATS
C
C
C
100   FORMAT(20A4)
110   FORMAT(1H1,////,(10X,20A4))
201   FORMAT( )
```

```
202   FORMAT(1H1,////,T20,,NUMBER OF ELEMENTS,,T50,,=,,I10,//,T20,
     1,NUMBER OF NODAL POINTS,,T50,,=I10)
203   FORMAT(/,T5,I10,T27,F10.6,T47,F10.6)
204   FORMAT(T8,I10,T30,4(I10,8X),I10)
211   FORMAT(7F10.0)
213   FORMAT(10I5)
555   FORMAT(1H1)
601   FORMAT(1H1,////)
602   FORMAT(T40,,NODAL POINT,,//,T10,,NUMBER,,T30,,X1,,T50,,X2,,//)
603   FORMAT(T10,,ELEMENT NUMBER,,T40,,IX(1),,T60,,IX(2),,T80,,IX(3),,T1
     100,,IX(4),,T120,,IND,,//)
604   FORMAT(T10,5X,,ANGULAR DISTRIBUTION MATRIX,,//)
605   FORMAT(T10,24(I5))
606   FORMAT(T20,,EXTINCTION COEFFICIENT = ,,F10.6)
655   FORMAT((5X,8(E13.6,2X)))
8001  FORMAT(T40,,INTENSITY DISTRIBUTION,,//)
      STOP
      END
```

In the above listed code $\ell_1$, $\ell_2$, $\ell_3$ and $\ell_4$ stand for, respectively, number of nodal points, number of elements, number of element rows and number of element columns in the mesh. A list of typical data is given below.

*Mesh Parameters*

$\ell_1$ = 37

$\ell_2$ = 24

$\ell_3$ = 6

$\ell_4$ = 4

*Problem Identification Data*

```
                10        20        30        40        50        60        70
                 ↓         ↓         ↓         ↓         ↓         ↓         ↓
.........................................................................

GALERKIN-FINITE ELEMENT PROGRAM TO SOLVE THE NONHOMOGENEOUS PROBLEM WITH
ISOTROPIC SCATTERING, AND WITH SPECIFIED BOUNDARY CONDITION AT X=-L/2 AN
D VACUUM BOUNDARY CONDITION AT X=L/2, L BEING THE SLAB THICKNESS

L = 1.
.........................................................................
                 ↑         ↑         ↑         ↑         ↑         ↑         ↑
                10        20        30        40        50        60        70
```

*Nodal Point Data*

```
                10        20        30        40        50        60        70
                 ↓         ↓         ↓         ↓         ↓         ↓         ↓
.........................................................................

 -0.50     -0.50     -0.50     -0.50     -0.50     -0.50     -0.50
 -0.50     -0.25     -0.25     -0.25     -0.25     -0.25     -0.25
 -0.25      0.0       0.0       0.0       0.0       0.0       0.0
  0.0       0.25      0.25      0.25      0.25      0.25      0.25
  0.25      0.50      0.50      0.50      0.50      0.50      0.50
  0.50      0.50
 -1.0      -0.666667 -0.333333  0.0       0.0       0.333333  0.666667
  1.0      -1.0      -0.666667 -0.333333  0.0       0.333333  0.666667
  1.0      -1.0      -0.666667 -0.333333  0.0       0.333333  0.666667
  1.0      -1.0      -0.666667 -0.333333  0.0       0.333333  0.666667
  1.0      -1.0      -0.666667 -0.333333  0.0       0.0       0.333333
  0.666667  1.0
.........................................................................
                 ↑         ↑         ↑         ↑         ↑         ↑         ↑
                10        20        30        40        50        60        70
```

*Element Data*

```
                10        20        30        40        50        60        70
                 ↓         ↓         ↓         ↓         ↓         ↓         ↓
.........................................................................

      7   14   15    8   14   21   22   15    1    2
     21   28   29   22   28   36   37   29    3    4
      6   13   14    7   13   20   21   14    1    2
     20   27   28   21   27   35   36   28    3    4
```

```
  5  12  13   6  12  19  20  13   1   2
 19  26  27  20  26  34  35  27   3   4
  3  11  12   4  11  18  19  12   1   2
 18  25  26  19  25  32  33  26   3   4
  2  10  11   3  10  17  18  11   1   2
 17  24  25  18  24  31  32  25   3   4
  1   9  10   2   9  16  17  10   1   2
 16  23  24  17  23  30  31  24   3   4
......................................................................
        ↑         ↑         ↑         ↑         ↑         ↑         ↑
       10        20        30        40        50        60        70
```

*Angular Distribution Data*

```
       10        20        30        40        50        60        70
        ↓         ↓         ↓         ↓         ↓         ↓         ↓
......................................................................
1,5,9,13,17,21
2,6,10,14,18,22
3,7,11,15,19,23
4,8,12,16,20,24
......................................................................
        ↑         ↑         ↑         ↑         ↑         ↑         ↑
       10        20        30        40        50        60        70
```

*Heat Transfer Data*

```
       10        20        30        40        50        60        70
        ↓         ↓         ↓         ↓         ↓         ↓         ↓
......................................................................
0.8
......................................................................
        ↑         ↑         ↑         ↑         ↑         ↑         ↑
       10        20        30        40        50        60        70
```

*Code Data*

```
         10        20        30        40        50        60        70
         ↓         ↓         ↓         ↓         ↓         ↓         ↓
........................................................................
1,0,0
0,0,0
0,0,0
1,1,1
1,0,0
0,0,0
0,0,0
1,1,1
1,0,0
0,0,0
0,0,0
1,1,1
1,0,1
0,0,0
0,0,0
1,1,0
1,0,1
0,0,0
0,0,0
1,1,0
1,0,1
0,0,0
0,0,0
1,1,0
........................................................................
         ↑         ↑         ↑         ↑         ↑         ↑         ↑
         10        20        30        40        50        60        70
```