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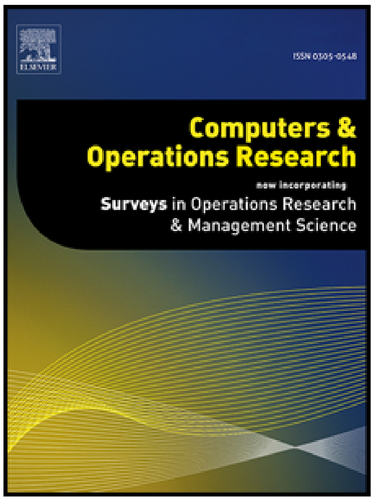

Analysis of an Improved Branch-and-Cut Formulation for the Inventory-Routing Problem with Transshipment

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Highlights

- We develop new sets of valid inequalities to strengthen the linear relaxation.
- We improve the coefficients in the inventory management constraints.
- We present a stronger formulation for the routing component of the problem.
- We prove that a large number of variables can be eliminated during a preprocessing stage.
- The experiments confirm the proposed improvements outperform the current best results.

Analysis of an Improved Branch-and-Cut Formulation for the Inventory-Routing Problem with Transshipment

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Abstract

The Inventory-Routing Problem with Transshipment (IRPT) is an extension of the Inventory-Routing Problem (IRP), in which not only the supplier can deliver goods to the retailers, but also transshipments between retailers or between the supplier and a retailer are possible. In this paper, we investigate the branch-and-cut (B&C) formulation of the IRPT and propose four different types of improvements for it. We first develop two new sets of valid inequalities for the problem which greatly strengthen the linear relaxation of the problem. We then improve the bounds on the continuous delivery variables and use these improved bounds to tighten the inventory management constraints. Next, we reformulate the routing component of the problem by exploiting the possible presence of direct shipments in the optimal solution. Finally, we prove that some integer and continuous variables can be eliminated out of the mathematical formulation. Experimental results demonstrate that these improvements drastically reduce the computational burden for solving the IRPT to optimality. On a large set of benchmark instances our proposed branch-and-cut algorithm, which integrates these improvements, outperforms the current best results in the literature. In addition, we are able to find the optimal solutions for two instances whose optimal solution was not known until now.

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1. Introduction

Since its successful application in the late 1980's by Wal-Mart and Procter & Gamble, Vendor-Managed Inventory (VMI) has gained in popularity [1]. Currently, it is one of the most investigated supply chain strategies. Research on the topic shows its potential for improving supply chain performance [2, 3]. In a VMI system the supplier manages the retailer's inventory. The freedom to choose the timing and the size of the deliveries allows him to combine orders of multiple retailers more efficiently. The only restriction the supplier has to respect is to make sure no stock-outs can occur at each retailer.

The suppliers that are engaged in a VMI system can integrate inventory management and transportation planning. The costs of both subsystems is optimized simultaneously. The mathematical problem that supports the decisions in a VMI system is known as the Inventory Routing Problem (IRP). The class of IRPs is fairly broad and based on the characteristics of the considered problem, IRPs are divided into categories. IRPs can be distinguished based on the considered time horizon, the demand, the topology, the fleet composition, etc. [4]

In this paper, we deal with a specific variant of the IRP: the Inventory-Routing Problem with Transshipment (IRPT). The concept of transshipment in an inventory-routing context was first formally introduced by Coelho et al. [5]. In this problem not only the supplier can deliver goods to the retailers, but also transshipments between retailers or between the supplier and a retailer are possible. These transshipments are carried out by a subcontractor. The objective is to determine the quantities delivered to all retailers by both the supplier and the subcontractor in the different periods of the planning horizon while minimizing the total cost. This total cost consists of inventory costs, transportation costs of the supplier and costs of the outsourced deliveries.

It is worthwhile to study the IRPT, from both an industrial as well as an academic point of view. In a recent and comprehensive survey on the industrial aspects of inventory-routing Andersson et al. [4] indicate the need for richer and more flexible models. The current models do not show in a sufficient way the increased collaboration that exists between the actors of a supply chain in reality. In the IRPT this increased collaboration is mostly reflected by the role of the retailers. In the basic IRP the retailers receive goods so that they can fulfill their demand. In the IRPT retailers can dispatch goods to other retailers as needed. Also an additional actor of the supply chain, the subcontractor, is included in the IRPT to ship these transshipments.

Another aspect the authors emphasize is the uncertain nature of the market conditions. Therefore, newer models should allow to incorporate flexibility in their logistic plans. The IRPT offers a great opportunity to build more flexible models. Retailers with uncertain demand can be left out of the fixed schedule of the supplier, but still be serviced through transshipments from other retailers or from the depot when their demand is revealed.

From an academic perspective research on the IRPT is scarce. Since its introduction by Coelho et al. [5], the IRPT has been applied in stochastic and dynamic inventory routing in Coelho et al. [6], but to the best of our knowledge there have been no further studies on modeling and solving the IRPT. Further investigation into the structure of this problem is thus definitely needed.

The aim of this paper is to present a stronger mathematical formulation for the IRPT. For this purpose we investigate four different aspects of the problem:

- We develop new sets of valid inequalities for the IRPT based on the existing valid inequalities for the IRP.
- We improve the bounds of the continuous delivery variables and tighten the inventory management constraints.
- We present a stronger formulation for the routing component of the problem that exploits the routing structure of the problem, in particular the

possible presence of direct shipments in the optimal solution.

- We prove that some integer and continuous variables can be eliminated out of the formulation.

The remainder of this paper is organized as follows. In Section 2, we present a literature survey on models and solution approaches for the IRP and the IRPT. In Section 3, we discuss the current mathematical model of the IRPT. In Section 4, we present and discuss the improved mathematical formulation of the IRPT (valid inequalities, improved bounds, strong routing component, eliminated variables). We also evaluate the effect of these improvements on the largest benchmark instances in literature and discuss their applicability on the IRP. In Section 5, we present the experimental results. Finally in Section 6, we conclude the paper with some remarks about further research.

2. Literature Review

The IRP was first introduced in 1983 by Bell et al. [7]. Since then, literature on this problem has grown a lot. Recent attempts to classify the literature have resulted in literature reviews on the industrial aspects of the IRP, by Andersson et al. [4], and on its methodological aspects, by Coelho et al. [8]. Andersson et al. [4] also proposed a classification for the class of IRPs. In their classification we can categorize the IRP on which the studied problem, the IRPT, is based as a finite-time horizon, one-to-many IRP where the demand is deterministic. The routing is done by a single vehicle that can visit more than one retailer on a trip. The inventory level of the retailers is fixed with the lowest inventory level equal to zero.

The roots of the investigated problem and its current formulation trace back to Bertazzi et al. [9]. They study a multiperiod IRP with an order-up-to (OU) level inventory policy for which they develop a heuristic algorithm and compare the solutions obtained with different objective functions. The first mixed-integer programming (MIP) model for this problem is presented by Archetti et al. [10]. Next to the MIP model, the authors also derive new valid inequalities to strengthen the linear relaxation. The improved model is solved to optimality using an exact branch-and-cut (B&C) algorithm. The MIP model of Archetti et al. [10] is further improved by the reformulating the inventory management component ([11, 12]) and by developing additional valid inequalities ([11, 12, 13, 14]).

Next to the work on the modeling and the effective solving of the IRP, there is a branch of research that focuses on the applicability of the IRP in a real-life context. Two important aspects are investigated: consistency and flexibility. Consistency is meant to reflect some common quality of service standards. For example ensuring that all deliveries to a retailer are roughly of the same size. On the other hand flexibility is introduced in the IRP so that the problem is more suited to manage stochastic elements.

Flexibility is introduced by allowing transshipments between retailers and between the supplier and retailers. The IRPT is first formally described in Coelho et al. [5]. The authors propose a heuristic based on the adaptive large neighborhood search (ALNS) framework presented by Pisinger and Ropke [15] to solve the IRPT.

The same authors demonstrate the flexibility of the IRPT in comparison to the IRP in Coelho et al. [6]. In this paper they investigate the Dynamic and Stochastic IRP (DSIRP), in which the retailer demands are gradually revealed over time. They propose four different policies to solve this problem which are reactive or proactive (using demand forecasts) and include or do not include transshipments. Results show that using demand forecasts and transshipments greatly improves the solutions quality.

In Coelho et al. [13] a B&C algorithm is proposed for several types of IRPs as well as for the IRPT. The B&C algorithm includes a solution improvement procedure that approximates the cost of a new solution resulting from vertex removals and/or insertions. This B&C algorithm is applied on 160 instances for the IRPT. Results show that 158 instances could be solved to optimality.

The value of transshipments in practical applications is illustrated in Peres et al. [16]. In this article a multi-period, multi-product IRPT is implemented in the Brazilian retail industry. The total logistics costs decrease with 9% compared to the initial situation. By introducing transshipments from one retailer to another the average inventory level decreases with 70%.

The use of transshipments has not only an economical impact, but it can also effect other aspects of the supply chain. In Mirzapour Al-e-heshem and Rekik [17] the ecological impact of transshipments is investigated. The results show that the use of transshipments results in more ecological solutions, since the supplier generally needs less vehicles to replenish every retailer. The total traveled distance is also reduced significantly. A study of Timajchi et al. [18] reports that the option to transship deteriorating goods not only decreases the overall costs, but also allows to keep the accident loss under control.

In this article, we will further investigate the IRPT. Our work can be situated closely to the work of Coelho et al. [5, 6, 13]. Computational results will be compared to the B&C algorithm [13].

3. Problem Description

The IRPT was first formally introduced in Coelho et al. [5]. We will present the general characteristics of the IRPT, but refer to this study for a detailed description of the problem. The problem is defined on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{0, \ldots, n\}$ is the vertex set and $\mathcal{E}$ is the edge set. Vertex 0 represents the supplier and the vertices of $\mathcal{V}' = \mathcal{V} \setminus \{0\}$ represent retailers. All the actors, supplier and retailers, have inventory holding costs of h_i per unit and per period. The retailers have a minimum inventory level, which is fixed at 0, and a maximum inventory level, which is fixed at its inventory holding capacity C_i. The supplier has a minimum inventory level of 0, but has no maximum inventory level.

In every period $p \in \mathcal{P} = \{1, \ldots, \rho\}$ of the planning horizon a quantity r^p of the product is made available at the supplier. The supplier will use this quantity and its initial inventory to ship quantities q_i^p to each retailer i so that he can fulfill his demand d_i^p. For this purpose a single vehicle of capacity Q is available. This vehicle can perform one route, beginning and ending at the depot, in every period p. The routing costs c_{ij} are associated with the length of the edge $(i, j) \in \mathcal{E}$.

In the IRPT transshipments w_{ij}^p can be made by a subcontractor later in the time period. A transshipment can start from the depot or from any retailer in a subset $\mathcal{R} \subseteq \mathcal{V}'$. There is no limit on the number of transshipments made. Goods can thus be transshipped from the supplier or from a retailer in the set $\mathcal{R}$ to multiple retailers in the same period, i.e. split deliveries. The transshipments are made by direct shipping and the unit cost associated with transshipping products from i to j is b_{ij}.

For the inventory management the maximum level (ML) policy is applied, i.e. the delivered quantities have to respect the minimum and maximum inventory level. The retailer's inventory I_i^p is calculated at the end of each period after deliveries, transshipments and satisfying the demand.

The objective of the problem is to minimize the total cost consisting of the inventory holding costs, the transportation costs of the supplier and the costs of the outsourced deliveries. Coelho et al. [13] present a model for the IRPT using the following binary variables: x_{ij}^p is equal to 1 iff retailer j immediately follows retailer i on the route of the supplier's vehicle in period p and 0 otherwise, and y_i^p is equal to 1 iff either the depot $(i = 0)$ or retailer i is visited in period p and 0 otherwise.

$$\text{minimize} \sum_{i \in \mathcal{V}} \sum_{p \in \mathcal{P}} h_i I_i^p + \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}, i<j} \sum_{p \in \mathcal{P}} c_{ij} x_{ij}^p + \sum_{i \in \mathcal{R} \cup \{0\}} \sum_{j \in \mathcal{V}'} \sum_{p \in \mathcal{P}} b_{ij} w_{ij}^p \tag{1}$$

$$\text{subject to } I_0^p = I_0^{p-1} + r^p - \sum_{i \in \mathcal{V}'} q_i^p - \sum_{i \in \mathcal{V}'} w_{0i}^p \qquad p \in \mathcal{P} \tag{2}$$

$$I_i^p = I_i^{p-1} + q_i^p + \sum_{j \in \mathcal{R} \cup \{0\}} w_{ji}^p - \sum_{j \in \mathcal{V}'} w_{ij}^p - d_i^p \qquad p \in \mathcal{P},\ i \in \mathcal{V}' \tag{3}$$

$$I_i^p \leq C_i \qquad p \in \mathcal{P},\ i \in \mathcal{V}' \tag{4}$$

$$q_i^p \leq C_i - I_i^{p-1} \qquad p \in \mathcal{P},\ i \in \mathcal{V}' \tag{5}$$

$$q_i^p \leq C_i y_i^p \qquad p \in \mathcal{P},\ i \in \mathcal{V}' \tag{6}$$

$$\sum_{i \in \mathcal{V}'} q_i^p \leq Q y_0^p \qquad p \in \mathcal{P} \tag{7}$$

$$\sum_{j \in \mathcal{V}', i<j} x_{ij}^p + \sum_{j \in \mathcal{V}', i>j} x_{ji}^p = 2 y_i^p \qquad p \in \mathcal{P},\ i \in \mathcal{V} \tag{8}$$

$$\sum_{i \in \mathscr{S}} \sum_{j \in \mathscr{S}, i<j} x_{ij}^p \leq \sum_{i \in \mathscr{S} \setminus \{m\}} y_i^p \qquad \mathscr{S} \subseteq \mathcal{V}',\ p \in \mathcal{P},\ m \in \mathscr{S} \tag{9}$$

$$x_{0j}^p \in \{0,1,2\} \qquad j \in \mathcal{V}',\ p \in \mathcal{P} \tag{10}$$

$$x_{ij}^p \in \{0,1\} \qquad i,j \in \mathcal{V}',\ p \in \mathcal{P} \tag{11}$$

$$y_i^p \in \{0,1\} \qquad i \in \mathcal{V},\ p \in \mathcal{P} \tag{12}$$

$$q_i^p \geq 0 \qquad i \in \mathcal{V}',\ p \in \mathcal{P} \tag{13}$$

$$I_i^p \geq 0 \qquad p \in \mathcal{P},\ i \in \mathcal{V} \tag{14}$$

$$w_{ij}^p \geq 0 \qquad i \in \mathcal{R} \cup \{0\}, j \in \mathcal{V}',\ p \in \mathcal{P} \tag{15}$$

In the IRPT, the total cost to be minimized is the sum of inventory holding costs at the supplier and at the retailers, the routing costs for the supplier's vehicle and the transshipment costs (1). Constraints (2) and (3) are inventory balance constraints for the supplier and the retailers. Constraints (4) limit the inventories of the retailer by their maximum capacity. Constraints (5)-(6) define the quantities delivered. If retailers i is not visited in period p, then constraints (6) enforce the quantity delivered to it to be zero. Otherwise, if retailer i is visited in period p, then constraints (6) limit the quantity delivered to the retailer's inventory holding capacity, and this bound is tightened by constraints (5), making it impossible to deliver more than what would fill this capacity. Constraints (7) state that the vehicle capacity is not exceeded. Constraints (8)-(9) guarantee that a feasible route is determined to visit all retailers served in period p. Finally, constraints (10)-(15) enforce integrality and non-negativity conditions on the variables.

One of the drawbacks of this mathematical model for the IRPT is the exponential number of subtour elimination constraints (9). In our implementation these constraints are added dynamically to the model.

4. Improvements on the Mathematical Model

In this section we will first introduce a number of notations and describe the existing valid inequalities for the IRPT. In the subsections 4.3, 4.4, 4.5 and 4.6 we will then discuss the different improvements we propose for the mathematical model. In subsection 4.7 we evaluate the effect of the improvements on the largest benchmark instances in literature. Finally, we discuss the applicability of the improvements on the IRP without transshipments in subsection 4.8.

4.1. Notations from Desaulniers et al. [11]

It is known that there exists an optimal solution to the IRP that respects the first-in, first-out principle. Based on this concept and in accordance with Desaulniers et al. [11] we introduce the following notations. $I_i^{0,s} = \max\left\{0, I_i^0 - \sum_{l=1}^{s} d_i^l\right\}$ represents the quantity remaining from the initial inventory at retailer $i \in \mathcal{V}'$ at the end of period $s \in \mathcal{P}$. Using $I_i^{0,s}$ the residual demands at retailer i can be defined:

$$\overline{d_i^s} \leq \begin{cases} \max\left\{0, d_i^1 - I_i^0\right\} & \text{if } s = 1 \\ \max\left\{0, d_i^s - I_i^{0,s-1}\right\} & \text{otherwise,} \end{cases} \qquad \forall s \in \mathcal{P}. \tag{16}$$

Using the residual demands $\overline{d_i^s}$, the demands d_i^s and the maximal holding capacity C_i the set of periods $\mathcal{P}_{ip}^+$ is determined. The set $\mathcal{P}_{ip}^+$ contains all periods in which a delivery of goods in period p can be used to satisfy demand or for the end inventory. Note that the artificial period $\rho + 1$ is introduced to handle the end inventories.

$$\mathcal{P}_{ip}^{+} = \Bigg\{ s \in \{p, p+1, \ldots, \rho+1\} \,\Bigg|\, (s \in \mathcal{P}, \overline{d_i^s} > 0 \text{ and } (s = p \text{ or } \sum_{l=p}^{s-1} d_i^l < C_i)) \tag{17}$$

$$\text{or } (s = \rho + 1 \text{ and} \sum_{l=p}^{s-1} d_i^l < C_i) \Bigg\}$$

$$= \left\{ p \,\middle|\, \overline{d_i^p} > 0 \right\} \cup \left\{ s > p \,\middle|\, \overline{d_i^s} > 0 \text{ and } \sum_{l=p}^{s-1} d_i^l < C_i \right\} \cup \left\{ \rho + 1 \,\middle|\, \sum_{l=p}^{\rho} d_i^l < C_i \right\} \tag{18}$$

Finally, the set $\mathcal{P}_{is}^{-} = \left\{ p \in \mathcal{P} | s \in \mathcal{P}_{ip}^{+} \right\}$ is defined which represents the set of periods at which a subdelivery can be made to fulfill the demand of retailer $i \in \mathcal{V}'$ at period $s \in \mathcal{P}$.

In Table 1 an example of the notations $I_i^{0,s}$, $\overline{d_i^s}$, $\mathcal{P}_{ip}^{+}$ and $\mathcal{P}_{is}^{-}$ is given. In this example the retailer faces a demand of 75 units in every period of the six-period long planning horizon. He can stock up to 225 units in his inventory. His initial inventory consists of 150 units of the considered product. Using his initial inventory he has enough stock to cover the demand of the first two periods of the planning horizon.

Taking the initial inventory and the inventory holding capacity into account, all units delivered in the first period can only be used in the third period, thus $\mathcal{P}_{i1}^{+} = \{3\}$. Because the initial inventory is being used in first and second periods, the sets $\mathcal{P}_{i2}^{+}$ and $\mathcal{P}_{i3}^{+}$ include the fourth and fifth period. The subsequent sets $\mathcal{P}_{ip}^{+}$ consist of three periods because units delivered in period p can only be used in period p, period $p + 1$ and period $p + 2$ due to the limited inventory capacity. At the end of the planning horizon the number of periods in the set $\mathcal{P}_{ip}^{+}$ logically decreases. Note that period 7 is the artificial period introduced to handle the end inventory. The sets $\mathcal{P}_{is}^{-}$ are then constructed using the sets $\mathcal{P}_{ip}^{+}$. $\mathcal{P}_{i3}^{-} = \{1, 2, 3\}$ is the first non-empty set and determines that subdeliveries to satisfy the demand in period three can be made in the first, second or third period. The subsequent sets $\mathcal{P}_{is}^{-}$ also always consist of the periods $p-2$, $p-1$ and p.

Period	C_i	d_i^s	$I_i^{0,s}$	$\overline{d_i^s}$	$\mathcal{P}_{ip}^{+}$	$\mathcal{P}_{is}^{-}$
initial	225	-	(150)	-	-	-
1	225	75	75	0	{3}	-
2	225	75	0	0	{3, 4}	-
3	225	75	0	75	{3, 4, 5}	{1, 2, 3}
4	225	75	0	75	{4, 5, 6}	{2, 3, 4}
5	225	75	0	75	{5, 6, 7}	{3, 4, 5}
6	225	75	0	75	{6, 7}	{4, 5, 6}

Table 1: Example of the notations $I_i^{0,s}$, $\overline{d_i^s}$, $\mathcal{P}_{ip}^{+}$ and $\mathcal{P}_{is}^{-}$

4.2. Existing Valid Inequalities

Archetti et al. [10] proposed the following valid inequalities for the IRP:

$$x_{0i}^{p} \leq 2y_i^p \quad i \in \mathcal{V}' \,, p \in \mathcal{P} \tag{19}$$

$$x_{ij}^{p} \leq y_i^p \quad i, j \in \mathcal{V}' \,, p \in \mathcal{P} \tag{20}$$

$$y_i^p \leq y_0^p \quad i \in \mathcal{V}' \,, p \in \mathcal{P} \tag{21}$$

Inequalities (19), (20) and (21) act purely on the routing component of the IRPT and are referred to as logical inequalities. Constraints (19) and (20) enforce the relation between the routing variables x_{ij}^p and the binary variables y_i^p which determine if the retailer i is visited in period p. The coefficient 2 before y_i^p in constraints (19) ensures that the inequality is valid in case of a direct shipment, when x_{0i}^p equals 2. Constraints (21) guarantee that the supplier is included in the route if any retailer is visited in that period.

4.3. New Valid Inequalities on the Inventory Management Component

Next to the logical inequalities of Archetti et al. [10], many other valid inequalities have been developed for the IRP. In this section we investigate the strongest of those inequalities and how they can be extended so that they are valid for the IRPT.

4.3.1. Minimum Number of Visits

Coelho and Laporte [14] proposed a set of valid inequalities for the IRP and the MIRP to determine the minimum number of times the supplier has to visit a retailer i over a period interval $[p_1, p_2]$. We propose to extend this set of valid inequalities for the IRPT in the following way:

$$\sum_{p=p_1}^{p_2} y_i^p + \frac{\sum\limits_{p=p_1}^{p_2} \sum\limits_{j \in \mathcal{R} \cup \{0\}} w_{ji}^p}{\sum\limits_{p=p_1}^{p_2} d_i^p} \geq \frac{\sum\limits_{p=p_1}^{p_2} d_i^p - I_i^{p_1-1}}{\sum\limits_{p=p_1}^{p_2} d_i^p} \quad i \in \mathcal{V}',\ p_1, p_2 \in \mathcal{P},\ p_2 \geq p_1 \quad (22)$$

As long as the left hand side of equation (22) is less or equal to zero, the inventory in period $p_1 - 1$ is sufficient to cover the demand over the interval $[p_1, p_2]$. If this is no longer the case, then either the retailer should be visited by the supplier or one or more transshipments should be made from the supplier and/or another retailer to cover the remaining demand of the period interval $[p_1, p_2]$.

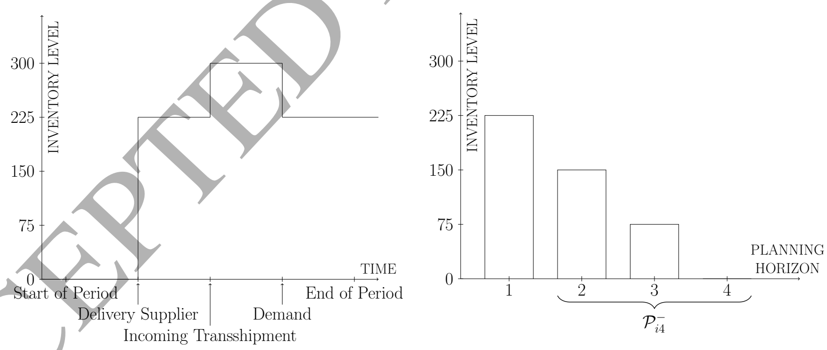
4.3.2. Minimum Number of Subdeliveries

In Desaulniers et al. [11] a set of valid inequalities is proposed based on the minimum number of subdeliveries per demand. It is possible to adapt these valid inequalities for the IRPT model. The set $\mathcal{P}_{is}^-$ contains all the periods in which a subdelivery can be made to satisfy the demand in period s. Ignoring the transshipments, at least one subdelivery should be made in one of the periods of $\mathcal{P}_{is}^-$:

$$\sum_{p \in P_{is}^-} y_i^p \geq 1 \quad i \in \mathcal{V}',\ s \in \mathcal{P} \,\Big|\, \mathcal{P}_{is}^- \neq \emptyset \quad (23)$$

To include transshipments into this valid inequality, one must take into account the sequence of events during a period. In this problem transshipments are delivered later in the period than deliveries by the supplier. Also, they are done according to the ML policy. Due to these conditions, the retailer's inventory capacity may temporarily be exceeded during the period. An example using the data of Table 1 is given in Figure 1 (a). At the start of the period the retailer has no stock. During the period the supplier delivers 225 units of the considered product. This is the maximum amount of units the supplier can deliver according to the ML policy constraints. Later in the period an additional delivery of 75 units is made via transshipments from other retailers or from the supplier. At this moment the inventory capacity of 225 units is exceeded. When the demand of the period is satisfied, the inventory level drops down back to 225 units which is also the inventory at the end of the period.

In Figure 1 (b) the evolution of the end inventory of the retailer is shown in case the end inventory in the first period is 225 units. At the end of the third period the retailer still has 75 units in stock, enough to cover the demand of the fourth period.



(a) Sequence of events during a period. (b) Evolution of the end inventory.

Figure 1: Illustrative example.

To include transshipments into the valid inequality (23) it does not suffice to just include the incoming transshipments of the periods in the set P_{is}^{-}. Since transshipments may result in a situation where the inventory at the end of a period is equal to the maximal capacity, one should also include the transshipments done in the period right before the first period of P^-_{is}, $s - |\mathcal{P}^-_{is}|$. This results in the following set of valid inequalities:

$$\sum_{p\in\mathcal{P}^-_{is}} y^p_i + \frac{\sum\limits_{p\in\mathcal{P}^-_{is}\cup\{s-|\mathcal{P}^-_{is}|\}} \sum\limits_{j\in\mathcal{R}\cup\{0\}} w^p_{ji}}{d^s_i} \geq 1 \quad i\in\mathcal{V}',\ s\in\mathcal{P}\ \Big|\ \mathcal{P}^-_{is}\neq\emptyset \tag{24}$$

4.4. Improved Bounds

In figure 1 (a) the inventory level at the start of the period is 0. However, in most instances of the IRPT there is some initial inventory level at the start of the first period. In those cases the maximum quantity that can be delivered is less than the inventory holding capacity. Taking these initial conditions into account, we can rewrite constraints (6) and (7) with C^*_i the maximal quantity that can be delivered and Q^* the maximal quantity that can be transported:

$$q^p_i \leq C^*_i y^p_i \qquad p\in\mathcal{P},\ i\in\mathcal{V}' \tag{25}$$

$$\sum_{i\in\mathcal{V}'} q^p_i \leq Q^* y^p_0 \qquad p\in\mathcal{P} \tag{26}$$

with

$$C^*_i = \begin{cases} C_i - I^0_i & \text{if } s=1 \\ C_i & \text{otherwise.} \end{cases} \tag{27}$$

and

$$Q^* = \min\left\{Q, \sum_{i\in\mathcal{V}'} C^*_i\right\} \tag{28}$$

In the first period the maximum quantity that can be delivered to retailer i is equal to the inventory holding capacity minus the initial inventory (27). During the first period transshipments may happen so that the inventory is empty at the end of the first period. For the other periods of the planning horizon the maximum quantity that can be delivered is thus the inventory holding capacity C_i.

The maximum quantity that can be transported is equal to the minimum of the transportation capacity Q and the sum over all retailers of the maximum
290 quantity that can be delivered C_i^*.

4.5. Reformulation of the Routing Component

In the current model of the IRPT the routing component is described by the following constraints:

$$\sum_{j\in\mathcal{V}',i<j} x_{ij}^p + \sum_{j\in\mathcal{V}',i>j} x_{ji}^p = 2y_i^p \qquad p\in\mathcal{P},\ i\in\mathcal{V} \tag{8}$$

$$\sum_{i\in\mathscr{S}}\sum_{j\in\mathscr{S},i<j} x_{ij}^p \le \sum_{i\in\mathscr{S}\setminus\{m\}} y_i^p \qquad \mathscr{S}\subseteq\mathcal{V}',\ p\in\mathcal{P},\ m\in\mathscr{S} \tag{9}$$

$$x_{0j}^p \in \{0,1,2\} \qquad j\in\mathcal{V}',\ p\in\mathcal{P} \tag{10}$$

$$x_{ij}^p \in \{0,1\} \qquad i,j\in\mathcal{V}',\ p\in\mathcal{P} \tag{11}$$

$$y_i^p \in \{0,1\} \qquad i\in\mathcal{V},\ p\in\mathcal{P} \tag{12}$$

A weakness in this model is the domain of the routing variables x_{0j}^p: {0,1,2}.
295 The value 2 is included in this domain because a direct shipment may occur in the optimal solution, i.e. the supplier delivers goods to only one customers and returns to his depot. This leads to a weaker linear relaxation. Another consequence is the presence of the coefficient 2 in the right hand side of the valid inequalities (19).

We present a new routing formulation which is stronger than the current formulation and is computationally not more demanding. First, we introduce a new variable z_i^p which is equal to 1 if the supplier makes a direct shipment to retailer i and 0 otherwise. Now there is no more need to include 2 in the domain of the routing variables x_{0j}^p. The routing component can be reformulated as:

$$\sum_{j \in \mathcal{V}'} x_{0j}^p + 2 \sum_{j \in \mathcal{V}'} z_j^p = 2y_0^p \qquad p \in \mathcal{P} \tag{29}$$

$$\sum_{j \in \mathcal{V}, i<j} x_{ij}^p + \sum_{j \in \mathcal{V}, i>j} x_{ji}^p + 2z_i^p = 2y_i^p \qquad p \in \mathcal{P},\ i \in \mathcal{V}' \tag{30}$$

$$\sum_{i \in \mathscr{S}} \sum_{j \in \mathscr{S}, i<j} x_{ij}^p \leq \sum_{i \in \mathscr{S} \setminus \{m\}} y_i^p \qquad \mathscr{S} \subseteq \mathcal{V}',\ p \in \mathcal{P},\ m \in \mathscr{S} \tag{31}$$

$$x_{ij}^p \in \{0,1\} \qquad i,j \in \mathcal{V},\ p \in \mathcal{P} \tag{32}$$

$$y_i^p \in \{0,1\} \qquad i \in \mathcal{V},\ p \in \mathcal{P} \tag{33}$$

$$z_i^p \in \{0,1\} \qquad i \in \mathcal{V}',\ p \in \mathcal{P} \tag{34}$$

Due to the introduction of the new variable z_i^p constraint (8) can no longer be written for every $i \in \mathcal{V}$ in the same way. We distinguish the case of the supplier (29) and the case of a retailer $i \in \mathcal{V}'$ (30). The supplier will be visited if either there is a route to two different retailers or if there is a direct shipment to a retailer j. A retailer will be visited if either there is a route to two different retailers (expressed similarly as in the original model) or there is a direct shipment from the supplier.

Since direct shipments cannot have subtours, the subtour elimination constraints (31) do not change. In the objective function the binary variables z_i^p should be added with a coefficient $2c_{0i}$.

This new formulation for the routing component is computationally not more demanding than the original formulation. The binary variable z_i^p replaces the value 2 in the domain of the original routing variables x_{0j}^p.

The new routing formulation also improves the logical valid inequalities (19) and (21). The valid inequalities (19) do not have to take the possibility of a direct shipment into account anymore, because these are expressed via the z_i^p variables, and becomes similar to valid inequality (20):

$$x_{0i}^p \leq y_i^p \qquad i \in \mathcal{V}'\ , p \in \mathcal{P} \tag{35}$$

It can be further strengthened: Since it is impossible for the supplier to make a direct shipment to one retailer and start a route to another retailer, the z_i^p variable can be incorporated into the inequality:

$$x_{0i}^p + z_i^p \leq y_i^p \quad i \in \mathcal{V}' \;, p \in \mathcal{P} \tag{36}$$

Valid inequalities (21) can be improved, since it is impossible to visit a retailer and visit another retailer via a direct shipment in the same period:

$$y_i^p + \sum_{j \in \mathcal{V}' \setminus \{i\}} z_j^p \leq y_0^p \quad i \in \mathcal{V}' \;, p \in \mathcal{P} \tag{37}$$

4.6. Elimination of Unnecessary Variables

In this section we investigate a number of different conditions that if respected, allow to eliminate variables out of the model. In the first three proofs we reason on the geographical locations of the retailers and the associated cost to eliminate routes, transshipments and direct shipments that will never be part of the optimal solution. In the fourth proof we demonstrate that it is sometimes cheaper to deliver goods to retailers via transshipments instead of through direct shipments. In the fifth, sixth, and seventh proof we reason on the geographical location and the inventory holding cost of retailers to eliminate possible transshipments.

4.6.1. Eliminated Links

Although in theory there can be a route between any two sites in the IRPT, practical considerations often prevent the use of certain routes. For example consider a situation in which three retailers i, j and k are located so that one always passes by retailer j when going from retailer i to retailer k. In this case, the routing variable x_{ik}^p can be eliminated out of the mathematical formulation, because x_{ij}^p and x_{jk}^p can be used at the same cost. Proposition 1 formalizes this concept:

Proposition 1 *The routing variable x_{ik}^p can be eliminated for all $p \in \mathcal{P}$ and $i, k \in \mathcal{V}'$, if there exists a $j \in \mathcal{V}'$ for which $c_{ij} + c_{jk} \leq c_{ik}$.*

Proof: Suppose there would exist an optimal solution that uses the route x^p_{ik}, then either this solution could be replaced by a solution that uses the routes x^p_{ij} and x^p_{jk} with the same cost (the equality-case) or with a lower cost (which contradicts the original assumption). □

A straightforward consequence of Proposition 1 is that all transshipment variables w^p_{ik} that respect the condition $c_{ij} + c_{jk} \leq c_{ik}$ can also be eliminated out of the mathematical model:

Proposition 2 *The transshipment variable w^p_{ik} can be eliminated for all $p \in \mathcal{P}$ and $i, k \in \mathcal{V}$, if there exists a $j \in \mathcal{V}$ for which $b_{ij} + b_{jk} \leq b_{ik}$.*

Proof: Suppose there would exist an optimal solution that uses the transshipment w^p_{ik}, then either this solution could be replaced by a solution that uses the transshipments w^p_{ij} and w^p_{jk} with the same cost (the equality-case) or with a lower cost (which contradicts the original assumption). □

Another consequence is that direct shipment variables can be eliminated in case there are 2 retailers located so that the supplier always passes by one when going to the furthest retailer:

Proposition 3 *The direct shipment variable z^p_k can be eliminated for all $p \in \mathcal{P}$ and $k \in \mathcal{V}'$, if there exists a $j \in \mathcal{V}'$ for which $c_{0j} + c_{jk} \leq c_{0k}$.*

Proof: Suppose there would exist an optimal solution that uses the direct shipment z^p_k, then either this solution could be replaced by a solution that uses the tour x^p_{0j}, x^p_{jk} and x^p_{0k} with the same cost (the equality-case) or with a lower cost (which contradicts the original assumption). □

There are two important remarks that one should take into account regarding these propositions. First, in a network where the triangle inequality is valid, the propositions will only effect the cases where $c_{ij} + c_{jk} = c_{ik}$. Secondly, it is not possible to include the supplier as one of the retailers i, j or k in

Proposition 1. In extreme cases, for example when all sites are located in a straight line, Proposition 1 would not be valid anymore. Another consequence is that including the supplier as one of the retailers i, j or k in Proposition 1 would no longer guarantee the validity of Proposition 3. In this case, the route x^p_{0k} could already be eliminated.

4.6.2. Eliminated Direct Shipments

Since transshipments from the supplier to a retailer are allowed, these can be used to replace direct shipments depending on their cost.

Proposition 4 *The direct shipment variable z^p_i can be eliminated, if $C^*_i b_{0i} \leq 2c_{0i}$.*

Proof: Suppose there would exist an optimal solution that uses the direct shipment z^p_i, then either this solution could be replaced by a solution that uses a transshipment from the supplier with the same cost (the equality-case) or with a lower cost, which contradicts the original assumption. □

In Proposition 4 it suffices to take into account the maximum amount that could be delivered to i C^*_i to determine if a transshipment is cheaper than a direct shipment or not. If this is the case, then the direct shipment variable z^p_i can be eliminated.

4.6.3. Eliminated Transshipments

In the IRPT transshipments may be used in different ways: to transship goods away from sites with high inventory costs, to transship goods to sites with low inventory costs, to deliver goods to sites that are geographically isolated, etc. We present three propositions that eliminate the number of possible transshipments without violating the problem definition. Proposition 5 is based on the idea that it is in some cases cheaper to transship from the supplier to a retailer j than to transship from retailer i to retailer j:

Proposition 5 *The transshipment variable w_{ij}^p from retailer i to retailer j can be eliminated for all $p \in \mathcal{P}$, if $h_i + b_{0j} \leq h_0 + b_{ij}$.*

Proof: When the condition $h_i + b_{0j} \leq h_0 + b_{ij}$ is satisfied, it is cheaper or equally costly to keep a unit in the inventory of retailer i at a cost of h_i and tranship a unit from the supplier to retailer j at a cost of b_{0j} than to keep a unit in the inventory of the supplier at a cost h_0 and to tranship a unit from the retailer i to the retailer j at a cost of b_{ij}. $\square$

Proposition 5 is valid for every period of the planning horizon. Proposition 6 is based on the same concept, but is only valid when the initial inventory is exhausted:

Proposition 6 *The transshipment variable w_{ij}^p from retailer i to retailer j can be eliminated for all $p \in \mathcal{P} \,\Big|\, \overline{d_i^p} > 0$, if $h_0 \leq h_i$ and $b_{0j} \leq b_{ij}$.*

Proof: When $\overline{d_i^p} > 0$, the initial inventory of retailer i has already been used. Every unit that is then delivered to retailer i can only be transshipped to retailer j at a higher or equal cost than if it were transshipped from the supplier, since transshipping in the same period is more expensive ($b_{0j} \leq b_{ij}$) and transshipping in a later period results in higher inventory holding costs ($h_0 \leq h_i$) and higher transshipping costs ($b_{0j} \leq b_{ij}$). $\square$

In Proposition 6 the retailer i cannot function as a satellite depot because it has higher inventory holding costs than the supplier. In Proposition 7, it is shown that even when a retailer has lower inventory holding costs than the supplier, it might still be a bad choice as satellite depot because of its geographical location.

Proposition 7 *The transshipment variable w_{ij}^s from retailer i to retailer j can be eliminated, if $|\mathcal{P}^-{}_{is}|h_0 + b_{0j} \leq |\mathcal{P}^-{}_{is}|h_i + b_{ij}$ and $h_i \leq h_0$.*

Proof: When $h_i \leq h_0$, it might be interesting for the supplier to store units in the inventory of retailer i at a lower inventory holding cost. The supplier will profit the most if the units are stored for the maximum duration. Since the optimal solution respects the FIFO-rule, the maximum duration the units can be stored in the inventory of i before using them in period s is $|\mathcal{P}^{-}{}_{is}|$. If the inequality $|\mathcal{P}^{-}{}_{is}|h_0 + b_{0j} \leq |\mathcal{P}^{-}{}_{is}|h_i + b_{ij}$ is respected, that means that goods that should be transshipped to retailer j can never be stored profitable at retailer i. □

An example of Proposition 7 based on the data from Table 1 is given in Figure 2. Suppose the supplier has delivered the maximum amount of goods to retailer i in the first period so that his inventory at the end of the period is equal to the inventory capacity, 225. Without other deliveries retailer i can hold these goods at most until period 4 when he will either use them to satisfy his own demand or transship them to another retailer j. For every unit that was delivered in the first period and is transshipped in fourth period $3h_i + b_{ij}$ needs to be paid. Since, it is cheaper or equally costly to transship the goods directly from the supplier, the transshipment from retailer i to retailer j will never happen.

Since the condition $h_i \leq h_0$ is satisfied in Proposition 7, $b_{0j} \leq b_{ij}$ is also satisfied. Therefore, every delivery that will eventually result in a transshipment in period s from retailer i to retailer j, will always be at least as costly as transshipping directly from the supplier: $\Delta h_0 + b_{0j} \leq \Delta h_i + b_{ij}$ with $\Delta \in [0, |\mathcal{P}^{-}{}_{is}|]$.

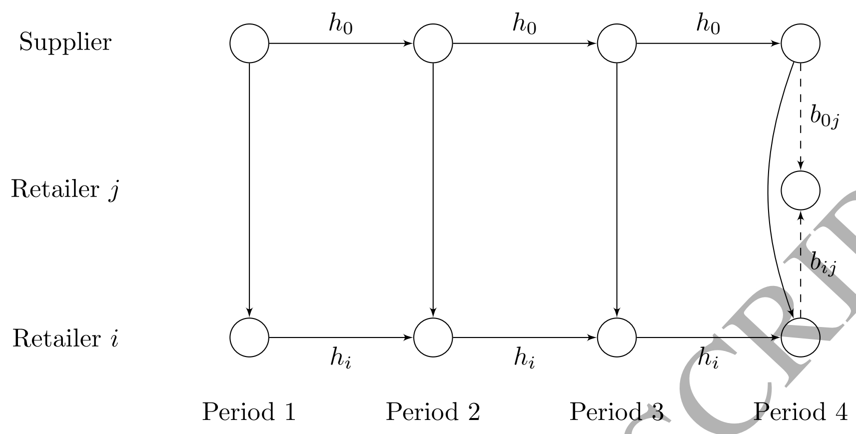


Figure 2: Example of an unnecessary transshipment based on Proposition 7.

Note that when the condition $h_i \leq h_0$ is not satisfied, the transshipment option from retailer i to retailer j cannot be eliminated. In this case, one cannot guarantee that $b_{0j} \leq b_{ij}$ which means that delivering in period p to retailer i and directly transshipping to retailer j might be cheaper than transshipping from the supplier. Guaranteeing $b_{0j} \leq b_{ij}$ equates to Proposition 6.

4.7. Evaluation of the Proposed Improvements on the IRPT

In Table 2 the impact of the proposed valid inequalities, bounds, reformulation and variable eliminations on the linear relaxation are shown. The impact is evaluated on the largest instances available in literature (Archetti et al. [10]). The sets 'LIPH3-absn50' and 'HIPH3-absn50' contain 5 instances with a planning horizon of 3 periods and 50 retailers. In the set 'LIPH3-absn50' the inventory holding costs are low, while in the set 'HIPH3-absn50' the inventory holding costs are high. Similarly, the sets 'LIPH6-absn30' and 'HIPH6-absn30' contain 5 instances with a planning horizon of 6 periods and 30 retailers with low and high inventory holding costs.

In the table the average of the linear relaxations of the 5 instances in each set is shown. In the column 'OF+EVI' the values for the original formulation including the existing valid inequalities are given. In the column 'NVI' the new developed valid inequalities are added to the formulation. This results in an average increase in the linear relaxation value of 2.92%. By improving the bounds ('IB') the linear relaxation value increases by 3.10%. When the routing component is replaced by the new routing component ('RORC') that includes the direct shipment variables, the linear relaxation value increases with 3.55% with respect to the original formulation. Finally, by eliminating unnecessary variables ('EOUV') the average linear relaxation increases with 3.70%.

	OF+EVI	NVI	IB	RORC	EOUV
LIPH3-absn50	3125.88	3438.47	3439.14	3458.20	3474.27
HIPH3-absn50	14040.31	14389.09	14406.67	14429.18	14447.95
LIPH6-absn30	5539.26	5722.58	5725.80	5796.88	5809.34
HIPH6-absn30	17941.08	18284.00	18335.89	18405.21	18417.92
Average increase	-	+2.92%	+3.10%	+3.55%	+3.70%

Table 2: Impact of the different improvements on the mathematical model.

The newly developed valid inequalities and new routing component result in the highest increase in linear relaxation. The impact of the valid inequalities on the linear relaxation is further detailed in Table 3. In the second column the impact on the linear relaxation is shown when only the set of valid inequalities that determine the minimum number of visits ('MNOV') is added to the original formulation. In the third column the impact on the linear relaxation is shown when only the set of valid inequalities that determine the minimum number of subdeliveries ('MNOS') is added to the original formulation. The impact of the MNOS-inequalities is in general larger than the impact of the MNOV-inequalities, but clearly the MNOV-inequalities are not dominated by the MNOS-inequalities. Both are needed to obtain an increase of 2.92% in linear relaxation.

	OF+EVI	MNOV (22)	MNOS (24)	NVI
LIPH3-absn50	3125.88	3143.09	3436.97	3438.47
HIPH3-absn50	14040.31	14161.14	14337.55	14389.09
LIPH6-absn30	5539.26	5562.43	5696.58	5722.58
HIPH6-absn30	17941.08	18124.88	18138.52	18284.00
Average increase	-	+0.85%	+2.37%	+2.92%

Table 3: Impact of the different valid inequalities.

The elimination of unnecessary variables does not necessarily have to result in an increase in linear relaxation. However, we often do observe a significant increase in linear relaxation in our computational results. In Table 4 the average percentage of eliminated transshipment variables, routing variables, and direct shipment variables is given.

	Eliminated w_{ij}^p (%)	Eliminated x_{ij}^p (%)	Eliminated z_i^p (%)
LIPH3-absn50	53.52	40.02	90.13
HIPH3-absn50	53.55	40.02	90.13
LIPH6-absn30	60.21	27.48	85.44
HIPH6-absn30	63.22	27.48	85.44

Table 4: Average percentage of eliminated variables.

From the results of Table 4 we notice that a large number of direct shipment variables is eliminated. The increase in linear relaxation we observe mostly originates from the elimination of these variables. Without their elimination the linear relaxation often has fractional values for both z_i^p and x_{0i}^p variables. Eliminating the direct shipments then results in an increase of the transportation variables x_{0i}^p and an increased LP value.

4.8. Applicability to the IRP

In this section we discuss the applicability of the four improvements to the IRP without transshipment: valid inequalities, improved bounds, reformulation and variable eliminations.

The presented sets of valid inequalities both originate from research on the IRP. The set of valid inequalities determining the minimum number of visits and its impact on the B&C formulation is discussed in Coelho and Laporte [14]. The notion of subdeliveries and the derived valid inequalities are discussed by Desaulniers et al. [11]. In this paper, the valid inequalities strengthen a set-partitioning-based formulation that is solved using a branch-price-and-cut method. To the best of our knowledge the impact of these valid inequalities on the B&C formulation has not been investigated yet. Further investigation of these valid inequalities may significantly improve the B&C formulation of the IRP.

To support this idea we conducted a small experiment in which we compare the LP values of the original B&C formulation ('BCF') and the B&C formulation strengthened by the different sets of valid inequalities developed for the IRP. In the column 'LI + MNOV' the inequalities of Archetti et al. [10], consisting of the logical inequalities and a set of MNOV-inequalities, are added to the formulation. The formulation is further strengthened by additional MNOV-inequalities of Coelho and Laporte [14] ('MNOV'). Finally, adding the MNOS-inequalities of Desaulniers et al. [11] ('MNOS') to the formulation results in an average increase in LP value of 16.17%.

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The results of Table 5 indicate that the notion of subdeliveries can indeed strengthen the B&C formulation of the IRP. We believe that an in-depth investigation of these valid inequalities complemented by the recent advances in reformulation of the inventory management component (Desaulniers et al. [11], Avella et al. [12]) is definitely worthwhile.

	BCF [10]	LI + MNOV [10]	MNOV [14]	MNOS [11]
LIPH3-absn50	2987.11	4008.50	4075.28	4095.28
HIPH3-absn50	13865.79	14943.70	15011.61	15084.59
LIPH6-absn30	5121.66	7111.98	7161.60	7161.84
HIPH6-absn30	17524.78	19347.80	19542.79	19542.98
Average increase	-	+14.97%	+15.93%	+16.17%

Table 5: Comparison of LP values of the B&C formulation of the IRP.

The presented improved bounds can be extended for the IRP by taking into account the quantity remaining from the initial inventory:

$$C_i^* = C_i - I_i^{0,p} \tag{38}$$

For the first period this bound is the same as in the IRPT, since $I_i^{0,0} = I_i^0$. In the next period, the bound for the IRPT rises directly to C_i as transshipments may have taken place. In the IRP the value of C_i^* will increase gradually until $I_i^{0,p}$ equals 0. For the IRP the bound C_i^* can be strengthened further. It is known that there exists an optimal solution for the IRP in which the end inventory of the retailers with higher inventory holding costs than the supplier is equal to zero. For these retailers C_i^* can also be limited to the cumulative future demand.

The reformulation of the routing component can be applied to the IRP without any modification, since the routing components of the IRPT and the IRP are equal and independent of the inventory management component of the problems. Also, the Propositions 1 and 3 which eliminate unnecessary routing and direct shipment variables remain valid.

5. Computational Experiments

5.1. Sets of Benchmark Instances

To evaluate the effectiveness of the proposed valid inequalities, the improved bounds, the new routing component and the eliminated variables we conducted experiments on a set of benchmark instances proposed by Archetti et al. [10] for the single-vehicle IRP. In total there are 160 instances that are solved under ML policy. Of these 160 instances 100 are solved for a planning horizon of three periods and the remaining 60 are solved for a planning horizon of six periods. These instances are further distinguished according to the inventory holding costs: high inventory holding costs, $h_i \in [0.1, 0.5]$, and low inventory holding costs, $h_i \in [0.01, 0.05]$. The final classification is based on the number of retailers involved in the problem. Every set counts 5 instances with the same characteristics.

To adapt these benchmark instances for the IRPT an appropriate transshipment cost per unit must be chosen. Like in Coelho et al. [13], we will choose a transshipment cost per unit of $0.01c_{ij}$.

The benchmark instances and a more detailed overview of the obtained solutions values, including the number of nodes visited during the optimization, are available on the website http://users.ugent.be/∼wlefever/.

5.2. Parameter Settings of the Branch-and-Cut Algorithm

Our B&C algorithm is implemented in Java™ using Eclipse Mars.2 (4.5.2). In the algorithm we use the following branching strategy: First, we branch on the fractional y_0^p variables, since these decide whether a tour is made or not. Secondly, we branch on the fractional y_i^p variables, since these decide whether a retailer is visited. Finally, we branch on the fractional z_i^p and x_{ij}^p variables.

The logical inequalities (19), (20) and (21)) and the developed valid inequalities (22) and (24) are all added at the beginning of the optimization procedure, since there is only a polynomial number of these inequalities. The subtour elimination constraints are added dynamically using the Gurobi callback routine. Whenever an integer solution is found that violates one of these constraints, the violated constraint is added to the problem.

The optimization problems are solved with Gurobi 7.0.2. The solver is run with its standard features enabled: presolve, heuristics, cutting planes, and parallelism. We put a time limit of 43200 seconds (12h) on the computation time of every instance. All computations were done on an Intel (R) Core(TM) i7-3770 3.4 GHz and 32.0 GB RAM PC.

5.3. Results

In Tables 6 and 7 the aggregated results are given for every set of instances. In the first column the name of the set is given, which also indicates the number of retailers involved in the problems. In the second column 'Linear Relaxation' the LP value and the gap with the MIP value are given. The third column 'B&C' shows the results of the B&C algorithm with the new valid inequalities, bounds, routing component and the eliminated variables are given. The performance of the proposed solution method is compared with the B&C algorithm ('B&C Coelho et al. [13]') of Coelho et al. [13]. Notice that the hardware used by Coelho et al. [13] (a grid of Intel Xeon$^{\text{TM}}$ processors 2.66 GHz and up to 48 GB RAM) is different from the one used in this article and that the results should be approached with caution.

In Table 6 the results are presented for the instances with a planning horizon of three periods. We are able to solve all instances like Coelho et al. [13], but in terms of computation time our implementation outperforms the B&C algorithm of Coelho et al. [13].

We notice that for small instances the gap between the LP value and the MIP value is very small. For these instances branching on the fractional values often suffices to find the optimal solution. For larger instances the integer solutions found after branching contain subtours. Hence, subtour elimination constraints need to be added in order to obtain a feasible solution. Consequently, the gap between the LP value and the MIP value is larger.

In Table 7 the results are given for the instances with a planning horizon of six periods. Our algorithm manages to solve all 60 instances within 43200 seconds of computation time. The B&C algorithm of Coelho et al. [13] manages to solve 58 instances. In terms of computation time our algorithm outperforms

	Linear Relaxation		B&C				B&C Coelho et al. [13]	
Instance	LP	gap (%)	LB	UB	#solved	Time (s)	#solved	Time(s)
Low inventory cost								
absn05	721.92	3.18	744.89	744.89	5	0.01	5	0.20
absn10	1497.32	5.34	1577.31	1577.31	5	0.05	5	0.80
absn15	1610.62	14.25	1840.03	1840.07	5	0.24	5	1.20
absn20	1994.76	14.20	2278.02	2278.04	5	0.46	5	4.00
absn25	2268.79	13.66	2578.75	2578.75	5	0.76	5	8.40
absn30	2664.44	11.25	2964.02	2964.08	5	1.54	5	12.60
absn35	2825.33	13.28	3200.62	3200.62	5	2.58	5	29.00
absn40	2845.74	16.32	3310.14	3310.14	5	18.82	5	58.60
absn45	3085.76	14.07	3519.90	3519.90	5	15.15	5	107.00
absn50	3474.27	11.14	3861.28	3861.28	5	352.30	5	520.00
Average		11.67				39.19		74.18
High inventory cost								
absn05	1634.03	1.61	1660.27	1660.27	5	0.01	5	0.20
absn10	3912.78	2.20	3999.03	3999.03	5	0.07	5	0.80
absn15	4826.42	4.73	5054.50	5054.50	5	0.22	5	1.20
absn20	6519.74	4.59	6818.76	6818.76	5	0.49	5	3.40
absn25	8234.54	3.93	8557.89	8557.89	5	0.94	5	6.40
absn30	10221.88	3.05	10533.38	10533.45	5	1.39	5	11.40
absn35	10739.24	3.56	11121.66	11121.66	5	2.78	5	23.80
absn40	11603.63	4.24	12095.25	12095.25	5	8.68	5	57.40
absn45	12988.52	3.62	13458.16	13458.36	5	16.67	5	121.60
absn50	14447.95	3.08	14892.09	14892.40	5	437.59	5	579.20
Average		3.46				46.88		80.54

Table 6: Average results for the IRPT-ML - $p = 3$ and transshipment cost $b_{ij} = 0.01 c_{ij}$

the B&C algorithm of Coelho et al. [13] as well. For the low inventory cost instances we need on average 1043.57 seconds, while Coelho et al. [13] need 6418.47 seconds. For the high inventory cost instances the difference between the two solution methods is even larger: 824.38 seconds versus 7704.37 seconds.

The average gaps between the LP values and the MIP values are much larger for these instances than for the short planning horizon instances. The main reason is that the presented valid inequalities are less effective when a larger planning horizon is considered. For the instances with a planning horizon of three periods the valid inequalities often result in integer values for the y_0^p

variables, while for the instances with a planning horizon of six periods the linear relaxation has fractional values for the y_0^p variables.

	Linear Relaxation		B&C				B&C Coelho et al. [13]	
Instance	LP	gap (%)	LB	UB	#solved	Time (s)	#solved	Time(s)
Low inventory cost								
absn05	2370.55	7.76	2554.45	2554.45	5	0.10	5	0.60
absn10	3480.32	14.32	3978.71	3978.71	5	0.62	5	3.80
absn15	3885.88	21.57	4724.20	4724.20	5	3.54	5	14.00
absn20	4708.92	21.39	5715.93	5715.93	5	29.92	5	348.00
absn25	5093.65	23.57	6293.84	6294.01	5	1128.26	5	6183.40
absn30	5809.34	21.07	7033.16	7033.53	5	5098.96	4	31961.00
Average		18.28				1043.57		6418.47
High inventory cost								
absn05	4565.88	3.86	4742.19	4742.19	5	0.09	5	0.6
absn10	7467.09	6.33	7940.05	7940.05	5	0.60	5	2.80
absn15	9985.79	8.35	10819.71	10819.71	5	2.77	5	9.80
absn20	12661.69	8.03	13678.22	13678.43	5	33.35	5	279.80
absn25	14707.78	8.36	15937.38	15937.70	5	1195.90	5	11677.60
absn30	18417.92	6.75	19659.64	19661.19	5	3713.57	4	34255.60
Average		6.95				824.38		7704.37

Table 7: Average results for the IRPT-ML - $p = 6$ and transshipment cost $b_{ij} = 0.01c_{ij}$

6. Conclusion

We have investigated the current mathematical formulation of the Inventory-Routing Problem with transshipments (IRPT). For this problem we developed a number of new valid inequality sets. These new valid inequalities greatly improve the linear relaxation of the IRPT. Also, we demonstrated that the bounds on some of the variables can be improved which results in better coefficients in the mathematical model. The routing component of the formulation was investigated as well. We developed a stronger formulation for the routing component that exploits the possibility of direct shipping in the solution. Finally, we showed that a lot of unnecessary variables can be eliminated out of the mathematical model.

The results have shown that a branch-and-cut algorithm in which the new valid inequalities, the improved bounds, the new routing component and the elimination of unnecessary variables are integrated outperforms the current best method in literature. We were able to find the optimal solution of two instances for which the optimal solution was not known yet.

We believe this work opens up to a number of meaningful extensions. First, as the IRPT is an extension of the Inventory-Routing Problem (IRP), the improvements presented in this article can also be applied to the IRP. We have briefly discussed the applicability of our improvements, but we believe that an in-depth analysis within the general context of inventory-routing would definitely be worthwhile. Secondly, the fact that the routing component and the inventory management component are independent opens up possibilities for decomposition. Finally, since we were able to solve the deterministic IRPT effectively, it might be worth while to investigate the stochastic version of the problem. We believe that a comparison of the stochastic IRP and the stochastic IRPT could emphasize the value of transshipments between retailers in a real-world context.

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