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Shipment planning at oil refineries using column generation and valid inequalities

Jan A. Persson [a,*], Maud Göthe-Lundgren [b]

[a] *School of Engineering, Blekinge Institute of Technology, Box 214, SE-374 24 Karlshamn, Sweden*
[b] *Department of Mathematics, Linköpings Universitet, SE-581 83 Linköping, Sweden*

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Abstract

In this paper we suggest an optimization model and a solution method for a shipment planning problem. This problem concerns the simultaneous planning of how to route a fleet of ships and the planning of which products to transport in these ships. The ships are used for moving products from oil refineries to storage depots. There are inventory levels to consider both at the refineries and at the depots. The inventory levels are affected by the process scheduling at the refineries and demand at the depots. The problem is formulated using an optimization model including an aggregated representation of the process scheduling at the refineries. Hence, we integrate the shipment planning and the process scheduling at the refineries. We suggest a solution method based on column generation, valid inequalities, and constraint branching. The solution method is tested on data provided by the Nynas oil refinery company and solutions are obtained within 4 hours, for problem instances of up to 3 refineries, 15 depots, and 4 products when considering a time horizon of 42 days.

Keywords: Shipment planning; Integrated planning; Valid inequalities; Column generation; Oil refinery

1. Introduction

In this paper we consider shipment planning of bitumen products from a set of refineries to a set of depots. The problem concerns the design of the tanker routes as well as the planning of which quantities to deliver to the depots. The planning is about making sure that the given demand is satisfied at lowest possible cost. Deliveries of multiple products to one or two depots are allowed in a single journey from a refinery. The costs to be minimized are related to production, transportation, inventory holding, violation of inventory level restrictions, and deviation from target inventories. We consider a planning horizon of up to six weeks.

* Corresponding author.
E-mail address: jan.persson@bth.se (J.A. Persson).

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We suggest a shipment planning model that includes considerations of production, by representing the production (process scheduling) by a linear programming (LP) model. This is done in order to obtain better overall schedules for both shipping and production than the ones obtained when the two steps are planned separately; Sometimes costly process schedules are required in order to meet the planned shipments.

We formulate a mixed integer linear programming model, representing the combined process scheduling and shipment planning problem. The process scheduling problem is discussed in Göthe-Lundgren et al. (2002), and solution methods are suggested in Persson (2002). Other related planning and scheduling problems at different levels (operational to strategic) within the refinery industry are discussed in Bodington and Baker (1990), Coxhead (1994), Nygren et al. (1998), and Persson (1999). Further, in Chandra and Fisher (1995) and Coxhead (1994), the integration of production planning and shipment planning is discussed. For the shipment planning when using ships, some reviews can be found, e.g., Ronen (1983, 1993) and Fagerholt (1999).

The ship routing problems often differ from the more investigated vehicle routing problem, in which trucks are typically considered. In ship routing problems, the capacities of ships, the demand, and the storage capacities at ports are often in the same range and hence must all be considered when constructing routes. Our problem is in a mode, which can be referred to as industrial mode in which the cargos and the ships are controlled by the same owner or organization.

Many of the problems studied in the literature in the area of operation in industrial mode concern the transportation of cargoes for which specifications of the quantities to ship, the loading ports, and the unloading ports, are given. Often time windows are specified for the dates of loading and unloading of the cargoes. Sometimes cargoes may be split among different ships and sometimes the cargoes must be sent with a particular ship. See Bausch (1998) and Cho and Perakis (2001) for examples of this type of problems. In Fagerholt and Christiansen (2000) the allocation of cargos to cargo compartments (which are flexible) in a multi-ship pickup and delivery problem with time windows is studied. Another related application is ship scheduling using soft time windows in Fagerholt (2001). In the references given above related to operation in industrial mode, the inventory levels are given no matter which feasible ship schedules are used.

In the literature, the inventory routing problem includes consideration of inventory levels at customers (and possibly at a supply depot). This means that the inventory routing problems concern the simultaneous inventory control and routing. For reviews and definitions of inventory routing problems, see Dror and Ball (1987), Federgruen and Simchi-Levi (1995), Christiansen (1996), and Baita et al. (1998).

When it comes to inventory routing problems concerning ships, few references can be found according to Christiansen (1999). Miller (1987) is one example of an inventory routing problem concerning ship transportations, in which fluid products are delivered to depots (warehouses) at which inventory levels are considered. In Miller (1987) the solution procedure is based on a network flow formulation of the problem and combines manual and automatic improvement heuristics. An interesting application is the Inventory Pickup and Delivery Problem with Time Windows (IPDPTW) which is considered in Christiansen and Nygreen (1998a,b) and Christiansen (1999), which concerns the ship routing/scheduling of a fleet of ships while considering inventory levels of a single product (ammonia). The objective was to minimize transportation costs while meeting forecasted (fixed) production and demand. For this problem, inventory level restrictions are used for deriving time windows for port visits. The solution procedure applied to the IPDPTW problem is based on Dantzig–Wolfe decomposition. The problem is decomposed into a ship route subproblem for each ship and a port inventory subproblem for each port.

An important aspect of the problem studied in this paper, compared to the problems in Miller (1987) and Christiansen (1999) is that production at loading ports is variable. Compared to Christiansen (1999) we also consider more than one product. Further, we cannot convert inventory level restrictions into reasonable tight time windows, since we will not see inventory level requirements as hard constraints and we are also considering target levels of the inventories.

We suggest a solution procedure based on column generation and constraint branching. The suggested branching strategy implies that a ship must visit a port in a given time period or it must not. See Lasdon (1970) and Dantzig and Wolfe (1961) for a general description of column generation (or Dantzig–Wolfe decomposition). In Barnhart et al. (1998), issues related to branch-and-bound together with column generation (branch-and-price) are discussed and an application of limited branch search and column generation is presented in Bredström et al. (2001). We use a column generation procedure for approximately solving the linear relaxation of the shipment planning problem. A similar approach can be found in the paper by van den Akker et al. (2000), in which a column generation approach is suggested for solving the linear relaxation of the problem. An objective for using a column generation approach is to allow for fast re-optimization when performing constraint branching. Further, by using a column generation procedure, partial integer solutions are generated which can be useful when looking for a good integer solution to the problem.

A main contribution of the paper is the development of an optimization approach for solving a real life shipment planning problem at a refinery company. In addition to deliver a low cost shipment plan, the approach should provide a shipment plan which can be integrated with the process schedules at refineries. In order to achieve this, an optimization model is suggested which integrates shipment planning and process scheduling. This has, to our knowledge, not been attempted before in this problem domain. Further, a contribution of the paper is the valid inequalities suggested, which play an important role for the quality of the solutions obtained.

In Section 2, we describe the shipment planning problem at the oil refinery company Nynas and how it connects to the aggregated production planning and to the scheduling of the refineries. The problem is formulated as an optimization model in Section 3. As a first step in developing a solution approach, valid inequalities are presented in Section 4. In Section 6, details of how to generate columns, i.e., ship schedules, to the master problem formulation presented in Section 5, are presented. In Section 7, we suggest a heuristic for finding integer solutions to the problem, which are based on limited tree search and computational results are presented in Section 8. We test our solution approach on real life scenarios from Nynas covering up to a six-week planning horizon, one to three refineries, and 6–15 depots. The paper is concluded in Section 9, where we also suggest directions for future research.

2. Problem description—shipment planning

The considered problem concerns shipment planning of bitumen from three refineries of the Nynas Group [1] to a number of depots. The refineries are located in Nynäsham and Gothenburg in Sweden and in Antwerp in Belgium.

Nynas owns two tankers (ships), which are used for the distribution of the bitumen products from the refineries to the about 15 depots located in northern Europe. Additional tankers or barges are hired when needed. Since the requirements on ships carrying bitumen are rather special due to the necessity of keeping the product warm, opportunities for acquiring transportations on the spot market is rather limited. Each of the self-owned tankers has five separate tanks of different sizes (with a total capacity of 5000 tonnes), which allows for the delivery of multiple products on the same journey. At the refinery, up to three main qualities of bitumen products are produced, which are blended into up to four different qualities before they are shipped. At a depot up to three different bitumen products are stored. The bitumen products are delivered from the depots to the asphalt producers by truck.

[1] The Nynas Group has in total five refineries. Three product types are produced: Bitumen, naphthenic special oils and fuels (naphtha), of which bitumen products constitute the largest volume in tonnes and around half of the sales volume.

In Fig. 1, we show the two Swedish refineries and a number of bitumen depots located along the Swedish coastline. We illustrate a part of a possible ship schedule which starts at the Nynäshamn refinery and ends after about five days when the ship has loaded in Gothenburg and is moving towards, for example, a Danish depot.

The shipment planning problem is the problem of simultaneously planning ship routes and planning the quantities to ship in order to satisfy the demand at the depots, i.e., the aggregated forecasted demand at the asphalt producers. The plan has a time horizon of at least a month. According to the contracts between Nynas and the customers, Nynas may face a penalty cost if a product is not available at a depot when the product is required by a customer. The tank capacities at the depots are limited, and it is not economical to load more in the ships than can actually be unloaded at the depots visited.

The ship operating costs are based on costs for capital, manpower, fuel, heating, and port fees. Further, we have costs for running out of products at a depot and, possibly, also costs incurred in the production process (due to the choice of the shipment plan). A "skewed" shipment plan, such as one where the planned deliveries of a single bitumen product are relatively many, may cause the production costs to increase. There are also capital costs (inventory costs) associated with the delivery of unnecessarily large quantities to the depots.

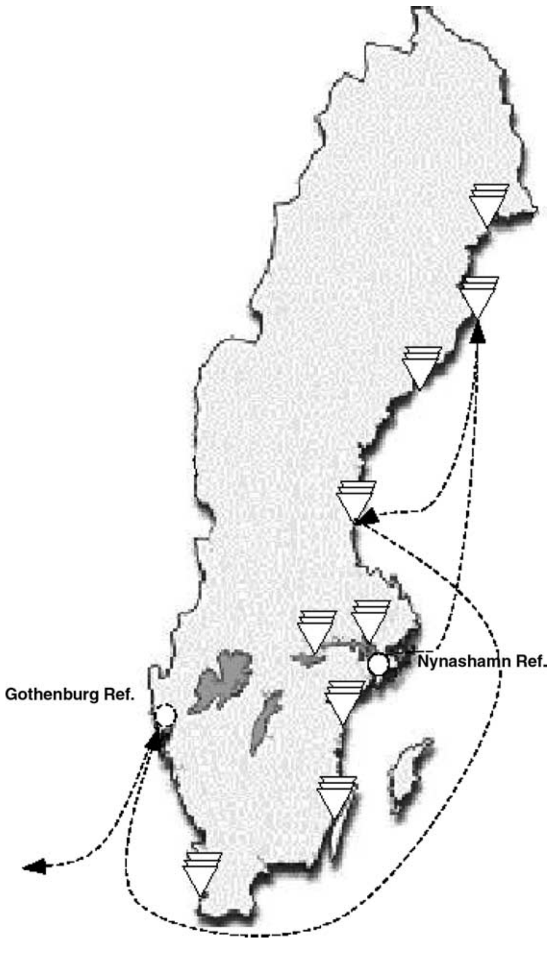

Fig. 1. A schematic picture of the distribution of bitumen products at Nynas.

The shipment plan needs to be realizable in terms of production, and it is important to make a cross-checking between the shipment plan and the corresponding process schedules to make sure that the products can be produced in time. The production at an oil refinery is carried out using a set of processing units, where the output from one unit may be used as input for another unit. Each processing unit can normally be operated in different modes of operation, or run-modes, where each mode is defined by the set of input/output products and the rate at which the products are produced and consumed. The production process at the Nynäshamn refinery, which is the main facility, consists of three processing units; the central distillation unit (CDU), which transforms crude oil into bitumen, distillates, and naphtha, and two hydro-treatment processes which transform distillates into naphthenic special oils. In the CDU, one bitumen product, four different distillates, and some naphtha are concurrently produced. The CDU is operated in about 10 different run-modes, and whenever the run-mode is changed a changeover cost is incurred. At a refinery, the bitumen products produced are typically not the same as the products shipped, due to blending of products. The process scheduling is about deciding which mode of operation to use in a particular time period.

A main difficulty for the planners of the shipments and the production process is how to devise a shipment plan and a process schedule which complement each other well. In particular, it would be useful to be able to evaluate the overall effect of making changes in one of the plans. A challenge is also to deal with inconsistencies from previous planning and inaccuracies in the demand forecast. Actually, it might not be possible to create a process schedule and a shipment plan, which meet the forecasted demand.

3. Mathematical modeling

For the modeling of the inventory levels and of production, we let the planning horizon consist of discrete uniform time periods with the length of 24 hours. The last time period of the planning horizon is denoted $\bar{t}$ and the planning horizon consists of the sequence of periods $1, \ldots, \bar{t}$. The corresponding set of time periods is denoted by T, i.e., $T = \{1, \ldots, \bar{t}\}$. For the modeling of ship routes, we use time periods of 6 hours and hence, there are four such time periods in each 24-hours period. We let the set $I = \{1, \ldots, 4\bar{t}\}$ denote the set of such 6-hours time periods. Let the set R denote the set of refineries and D the set of depots. Let B be the set of ships that can be used for the distribution of the products and let $P^{\mathcal{D}}$ be the set of products that are distributed to the depots.

A ship schedule can be represented by a path in a time expanded network, where each node represents a port (at refinery or at a depot) in a given time period and each arc represents either a move from one port in a time period to another port in a later time period or that the ship is not moved between two subsequent time periods. We denote an arc, representing a move from a refinery to one or more depots, an "outbound trip". In this application we allow for unloading at two depots at most. In Fig. 2, an outbound trip representing two-port unloading (first in "Sandarne" and then in "Holmsund") is illustrated. The solid line represents the arc of the outbound trip and the dotted line shows the actual ship route. Each node represents a port (refineries correspond to rhomboids and depots correspond to ovals) in a particular time period (6-hours period).

We let an *inbound trip* denote a move from the last depot of the outbound trip to a refinery. Additionally we have *stand-still trips* representing that the ship remains at a refinery from one time period to the next without loading or unloading. Further, there are also arcs representing a move from one refinery to another (refinery-to-refinery). A ship schedule is defined as a sequence of outbound and inbound trips, possibly combined with stand-still and refinery-to-refinery trips. Let J_b denote the set of all outbound trips for a ship b and $\overline{J}_b$ the set of inbound trips. The stand-still trips and refinery-to-refinery trips are also included in the set $\overline{J}_b$.

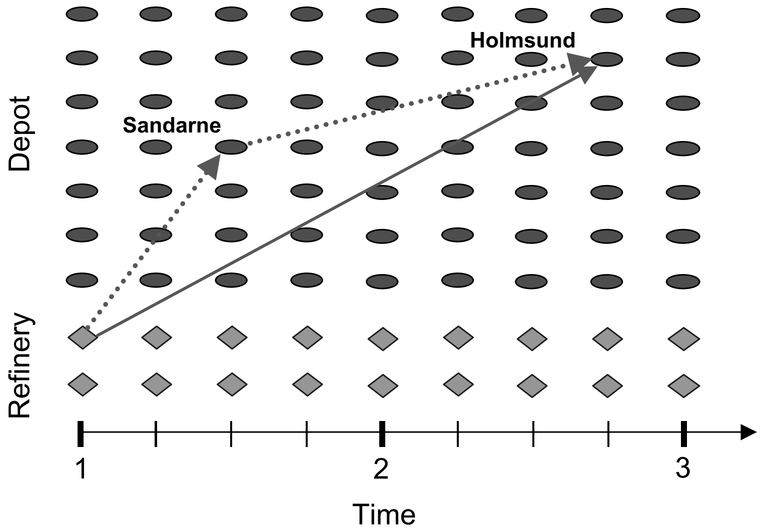

Fig. 2. Illustration of an outbound trip.

The number of outbound trips that can start at a refinery in a time period is bounded by $|D| + |D| * (|D| - 1)$, when unloading at one or at two depots is considered. The number of inbound trips is small, since the number of refineries is typically small (here, at most three). In practice some of the outbound trips can be ignored since they are low efficient, see Section 8.1.

Since we use 6-hours time periods when representing the ship schedule and 24-hours time periods representing everything else, we aggregate subsequent four 6-hours periods of the ship schedule into a single time period. This is illustrated in Fig. 3, where four nodes are included for each 24-hours time period. Note that for the outbound trip illustrated in Fig. 2, the unloading will occur at Sandarne and Holmsund in time periods one and two, respectively, as illustrated by the large ovals in Fig. 3.

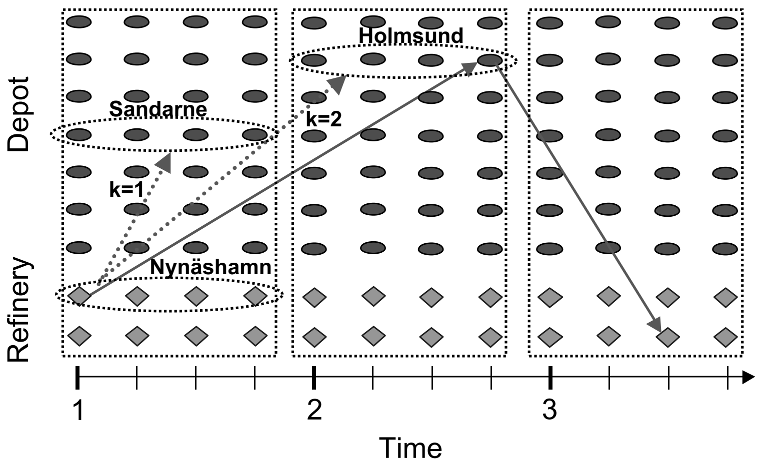

Fig. 3. Illustration of a trip with corresponding loading and unloading.

For an outbound trip j, which visits n depots, we define a set $K_j = \{1, \dots, k, \dots, n\}$, where the element k represents the number of order of the visited depots for the outbound trip j. In our application no more than two depots are visited which means that K_j is a set of either one element $\{1\}$ or two elements $\{1, 2\}$. Further, we define a set K, which includes the elements of the trip which visits the maximal number of depots. In our application this means that $K = \{1, 2\}$.

For a given outbound trip $j \in J_b$ and ship b, we want to express at which port and in which time period unloading may occur. Let r_0 denote the refinery started from and d_k the depot delivered to for visit $k \in K_j$. Further, let $t_k \in I$, $k \in K_j$, for a given trip j, denote the corresponding number of time periods in I, it takes from starting to load the tanker at a refinery until it finishes unloading at the kth visited depot. Observe, that the notation has been simplified, since r_0, d_k, and t_k all depend on j (the considered trip). The loading of product p at refinery r in time period $t \in T$ is represented by variable $x^{\mathcal{JP}}_{ijbkp}$ if the trip j commences in time period i, such that, $\lceil i/4 \rceil = t$ and that the trip starts at the refinery $r_0 = r$. For the loading at refineries we define the set of pairs of time periods $i \in I$ and outbound trips j representing loading at refinery r in time period $t \in T$, according to,

$$S^{\mathcal{R}}_{rtb} = \{(i, j) : i \in I, \lceil i/4 \rceil = t, j \in J_b, r_0 = r\}.$$

Equivalently, the unloading of product p at depot d in time period $t \in T$ is represented by variable $x^{\mathcal{JP}}_{ijbkp}$ if $k \in K_j : \lceil (i + t_k)/4 \rceil = t$, $d_k = d$ and we define the set,

$$S^{\mathcal{D}}_{dtbk} = \{(i, j) : i \in I, \lceil (i + t_k)/4 \rceil = t, j \in J_b, d_k = d\}$$

for such pairs of time periods i and trips j.

Next we introduce some notation necessary for modeling the process scheduling problem. At each refinery $r \in R$, there is a set of processing units Q_r. The set of possible run-modes at processing unit q and refinery r, is denoted by M_q, $q \in Q_r$. Next, let M be the set of possible run-modes at all refineries and at all processing units, i.e., $M = \cup_{r \in R, q \in Q_r} M_q$. Further, let $P^{\mathcal{R}}$ denote the set of products that are produced at the refineries.

The parameter definitions, which are needed for modeling the shipment planning and process scheduling problem, are given below.

$c^{\mathcal{ID}}_{dpt}$	inventory cost for product $p \in P^{\mathcal{D}}$ at depot $d \in D$ in time period $t \in T$
$c^{\mathcal{R}+}_{rpt}$, $c^{\mathcal{R}-}_{rpt}$	penalty costs for generating and removing product $p \in P^{\mathcal{R}}$ at refinery $r \in R$ in time period $t \in T$
$c^{\mathcal{D}+}_{d}$	penalty cost for generating product $p \in P^{\mathcal{D}}$ at depot $d \in D$ in time period $t \in T$
$I^{\mathcal{ITL}}_{dpt}$, $I^{\mathcal{ITU}}_{dpt}$	lower and upper target levels for product $p \in P^{\mathcal{D}}$ at depot $d \in D$ and time period $t \in T$
$\underline{I}^{\mathcal{D}}_{dpt}$, $\overline{I}^{\mathcal{D}}_{dpt}$	lower and upper inventory level limits for product $p \in P^{\mathcal{D}}$ at depot $d \in D$ and time period $t \in T$
$c^{\mathcal{IR}}_{rpt}$	inventory cost for product $p \in P^{\mathcal{R}}$ at refinery $r \in R$ in time period $t \in T$
C_b	capacity for ship $b \in B$
$e^{\mathcal{D}}_{dpt}$	demand for product $p \in P^{\mathcal{D}}$ at depot $d \in D$ in time period $t \in T$
$c^{\mathcal{J}}_{ijb}$, $c^{\overline{\mathcal{J}}}_{ijb}$	transportation cost incurred by using ship $b \in B$ on an trip (outbound or inbound) $j \in J_b$ or $j \in \overline{J}_b$ in time period $i \in I$
$c^{\mathcal{M}}_{mt}$	production cost for using run-mode $m \in M$ in time period $t \in T$
$e^{\mathcal{R}}_{rpt}$	direct demand at refinery $r \in R$ for product $p \in P^{\mathcal{R}}$ in time period $t \in T$
$\underline{I}^{\mathcal{R}}_{rpt}$	safety stock level for product $p \in P^{\mathcal{R}}$ at refinery $r \in R$ in time period $t \in T$
$\overline{I}^{\mathcal{R}}_{rpt}$	maximum inventory level of product $p \in P^{\mathcal{R}}$ at refinery $r \in R$ in time period $t \in T$
$\alpha^{\mathcal{R}}_{pm}$	production yield of product $p \in P$ when operating run-mode $m \in M$
$\beta^{\mathcal{R}}_{pm}$	consumption of product $p \in P$ when operating run-mode $m \in M$
$\gamma^{\mathcal{R}}_{p\acute{p}}$	share used of product $p \in P^{\mathcal{R}}$ for blending and shipping product $\acute{p} \in P^{\mathcal{D}}$

Next we present the variables.

$\omega^{\mathcal{R}}_{rpt}$ quantity of product $p \in P^{\mathcal{D}}$ shipped from refinery $r \in R$ in time period $t \in T$
$I^{\mathcal{D}}_{dpt}$ inventory level at depot $d \in D$ of product $p \in P^{\mathcal{D}}$ at the end of time period $t \in T$
$x^{\mathcal{JP}}_{ijbkp}$ quantity shipped of product $p \in P^{\mathcal{D}}$ on ship $b \in B$, in time period $i \in I$, on trip $j \in J_b$, for visit $k \in K_j$
$x^{\mathcal{J}}_{ijb}$ $\begin{cases} 1 & \text{if outbound trip } j \in J_b \text{ is used in period } i \in I \text{ for ship } b \in B \\ 0 & \text{otherwise} \end{cases}$
$x^{\overline{\mathcal{J}}}_{ijb}$ $\begin{cases} 1 & \text{if inbound trip } j \in \overline{J}_b \text{ is used in period } i \in I \text{ for ship } b \in B \\ 0 & \text{otherwise} \end{cases}$
$o^{\mathcal{R}+}_{rpt}, o^{\mathcal{R}-}_{rpt}$ quantity of product $p \in P^{\mathcal{D}}$ created and removed, respectively, at refinery $r \in R$ in time period $t \in T$
$o^{\mathcal{D}+}_{dpt}$ quantity created of product $p \in P^{\mathcal{D}}$ at depot $d \in D$ in time period $t \in T$
$o^{\mathcal{D}++}_{dt}$ quantity created totally at depot $d \in D$ in time period $t \in T$
$I^{\mathcal{IL}}_{dpt}, I^{\mathcal{IU}}_{dpt}$ positive and negative deviation from target inventory levels, respectively, for product $p \in P^{\mathcal{D}}$ at depot $d \in D$ in time period $t \in T$
$x^{\mathcal{R}}_{rpt}$ production of product $p \in P^{\mathcal{R}}$ at refinery $r \in R$ in time period $t \in T$
$z^{\mathcal{R}}_{rpt}$ consumption at refinery $r \in R$ of product $p \in P^{\mathcal{R}}$ in time period $t \in T$
$I^{\mathcal{R}}_{rpt}$ inventory level at refinery $r \in R$ of product $p \in P^{\mathcal{R}}$ at the end of time period $t \in T$
$y^{\mathcal{R}}_{mt}$ $\begin{cases} 1 & \text{if run-mode } m \in M \text{ is used in time period } t \in T \\ 0 & \text{otherwise.} \end{cases}$

For short hand notation, we also refer to variables $x^{\mathcal{J}}_{ijb}$ and $x^{\overline{\mathcal{J}}}_{ijb}$ as the vector variables $x^{\mathcal{J}}_{b}$ and $x^{\overline{\mathcal{J}}}_{b}$, for each ship b.

Using these parameters and variables allows us to formulate the "complete" shipment planning model:

$$
\begin{aligned}
\text{[SP-C]} \quad \min \quad & \sum_{p\in P^{\mathcal{R}}}\sum_{t\in T} c^{\mathcal{IR}}_{rpt} I^{\mathcal{R}}_{rpt} + \sum_{q\in Q_r}\sum_{m\in M_q}\sum_{t\in T} c^{\mathcal{M}}_{mt} y^{\mathcal{R}}_{mt} + \sum_{b\in B}\sum_{i\in I}\sum_{j\in J_b} c^{\mathcal{J}}_{ijb} x^{\mathcal{J}}_{ijb} + \sum_{b\in B}\sum_{i\in I}\sum_{j\in \overline{J}_b} c^{\overline{\mathcal{J}}}_{ijb} x^{\overline{\mathcal{J}}}_{ijb} \\
& + \sum_{d\in D}\sum_{p\in P^{\mathcal{D}}}\sum_{t\in T} c^{\mathcal{ID}}_{dpt} I^{\mathcal{D}}_{dpt} + \sum_{r\in R}\sum_{p\in P^{\mathcal{R}}}\sum_{t\in T} (c^{\mathcal{R}+}_{rpt} o^{\mathcal{R}+}_{rpt} + c^{\mathcal{R}-}_{rpt} o^{\mathcal{R}-}_{rpt}) + \sum_{d\in D}\sum_{t\in T} c^{\mathcal{D}+}_{d} o^{\mathcal{D}++}_{dt} \\
& + \sum_{d\in D}\sum_{p\in P^{\mathcal{D}}}\sum_{t\in T} (c^{\mathcal{ITL}}_{dpt} I^{\mathcal{IL}}_{dpt} + c^{\mathcal{ITU}}_{dpt} I^{\mathcal{IU}}_{dpt}) \qquad (1a)
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad & \omega^{\mathcal{R}}_{rpt} + o^{\mathcal{R}+}_{rpt} - o^{\mathcal{R}-}_{rpt} \\
& \quad - \sum_{b\in B}\sum_{(i,j)\in S^{\mathcal{R}}_{rtb}}\sum_{k\in K_j} x^{\mathcal{JP}}_{ijbkp} = 0, \quad r\in R,\ p\in P^{\mathcal{D}},\ t\in T, \qquad (1b) \\
& \quad \sum_{p\in P^{\mathcal{D}}}\sum_{k\in K_j} x^{\mathcal{JP}}_{ijbkp} \leqslant C_b x^{\mathcal{J}}_{ijb}, \quad b\in B,\ i\in I,\ j\in J_b, \qquad (1c) \\
& I^{\mathcal{D}}_{dp,t-1} - e^{\mathcal{D}}_{dpt} + \sum_{b\in B}\sum_{k\in K}\sum_{(i,j)\in S^{\mathcal{D}}_{dtbk}} x^{\mathcal{JP}}_{ijbkp} + o^{\mathcal{D}+}_{dpt} = I^{\mathcal{D}}_{dpt}, \quad d\in D,\ p\in P^{\mathcal{D}},\ t\in T, \qquad (1d) \\
& (x^{\mathcal{J}}_{b}, x^{\overline{\mathcal{J}}}_{b}) \in \mathcal{A}^{\mathcal{J}\overline{\mathcal{J}}}_{b}, \quad b\in B, \qquad (1e) \\
& o^{\mathcal{D}++}_{dt} \geqslant \sum_{p\in P^{\mathcal{D}}} o^{\mathcal{D}+}_{dpt}, \quad d\in D,\ t\in T, \qquad (1f) \\
& \underline{I}^{\mathcal{D}}_{dpt} \leqslant I^{\mathcal{D}}_{dpt} \leqslant \overline{I}^{\mathcal{D}}_{dpt}, \quad d\in D,\ p\in P^{\mathcal{D}},\ t\in T, \qquad (1g)
\end{aligned}
$$

$$I^{\mathcal{D}}_{dpt} + I^{\mathcal{IL}}_{dpt} \geqslant I^{\mathcal{ITL}}_{dpt}, \quad p \in P^{\mathcal{D}}, \ d \in D, \ t \in T, \tag{1h}$$

$$I^{\mathcal{D}}_{dpt} - I^{\mathcal{IU}}_{dpt} \leqslant I^{\mathcal{ITU}}_{dpt}, \quad p \in P^{\mathcal{D}}, \ d \in D, \ t \in T, \tag{1i}$$

$$\omega^{\mathcal{R}}_{rpt}, o^{\mathcal{R}+}_{rpt}, o^{\mathcal{R}-}_{rpt} \geqslant 0, \quad r \in R, \ p \in P^{\mathcal{D}}, \ t \in T, \tag{1j}$$

$$o^{\mathcal{D}+}_{dpt} \geqslant 0, \quad d \in D, \ p \in P^{\mathcal{D}}, \ t \in T, \tag{1k}$$

$$I^{\mathcal{IL}}_{dpt}, I^{\mathcal{IU}}_{dpt} \geqslant 0, \quad d \in D, \ p \in P^{\mathcal{D}}, \ t \in T, \tag{1l}$$

$$x^{\mathcal{R}}_{rpt} - \sum_{q \in Q_r} \sum_{m \in M_q} \alpha^{\mathcal{R}}_{pm} y^{\mathcal{R}}_{mt} = 0, \quad p \in P^{\mathcal{R}}, \ t \in T, \tag{1m}$$

$$z^{\mathcal{R}}_{rpt} - \sum_{q \in Q_r} \sum_{m \in M_q} \beta^{\mathcal{R}}_{pm} y^{\mathcal{R}}_{mt} = 0, \quad p \in P^{\mathcal{R}}, \ t \in T, \tag{1n}$$

$$I^{\mathcal{R}}_{rp,t-1} + x^{\mathcal{R}}_{rpt} - z^{\mathcal{R}}_{rpt} - e^{\mathcal{R}}_{rpt} - \sum_{\acute{p} \in P^{\mathcal{D}}} \gamma^{\mathcal{R}}_{p\acute{p}} \omega^{\mathcal{R}}_{r\acute{p}t} = I^{\mathcal{R}}_{rpt}, \quad p \in P^{\mathcal{R}}, \ t \in T, \tag{1o}$$

$$\underline{I}^{\mathcal{R}}_{rpt} \leqslant I^{\mathcal{R}}_{rpt} \leqslant \overline{I}^{\mathcal{R}}_{rpt}, \quad p \in P^{\mathcal{R}}, \ t \in T, \tag{1p}$$

$$\sum_{m \in M_q} y^{\mathcal{R}}_{mt} = 1, \quad q \in Q_r, \ t \in T, \tag{1q}$$

$$y^{\mathcal{R}}_{mt} \geqslant 0, \quad q \in Q_r, \ m \in M_q, \ t \in T, \tag{1r}$$

$$x^{\mathcal{R}}_{rpt}, z^{\mathcal{R}}_{rpt} \geqslant 0, \quad p \in P^{\mathcal{R}}, \ t \in T, \tag{1s}$$

$$x^{\mathcal{JP}}_{ijbkp} \geqslant 0, \quad b \in B, \ i \in I, \ j \in J_b, \ k \in K_j, \ p \in P^{\mathcal{D}}, \tag{1t}$$

where $I^{\mathcal{D}}_{dp0}$, $I^{\mathcal{R}}_{rp0}$ and $y^{\mathcal{R}}_{m0}$ are given values.

The cost terms in (1a) include costs of making the products available at the refineries (cost of the inventories and the cost of using the run-modes). Further it includes costs of transportation, inventory costs at the depots, costs of relaxing inventory constraints, and cost of not meeting target inventory constraints.

Note that the production costs may be different at different refineries. Also, note that no changeover costs are included in the model.

The transportation costs ($c^{\mathcal{J}}_{ijb}$ and $c^{\overline{\mathcal{J}}}_{ijb}$) represent the cost for letting a ship use an outbound or an inbound trip. These costs are based on real transportation costs and they are assumed to be fixed for a given trip, i.e., the transportation cost does not depend on the quantities loaded, shipped, and unloaded. Without loss of generality we may however include costs proportional to the quantities shipped. The costs of outbound and inbound trips are based on costs for capital, manpower, fuel, heating, and port fees.

We include consideration of inventory costs, even though all the inventories are "owned" by Nynas. The inventory costs are slightly higher at the depots than at the refineries, representing the extra value put into the products due to the transportation. The inventory costs may make some shipments occur later, and hence the production of bitumen products and the buying of crude oil can be done at a later date.

We introduce penalty costs into the objective function for generating and removing products with the cost parameters $c^{\mathcal{R}+}_{rpt}$, $c^{\mathcal{R}-}_{rpt}$, and $c^{\mathcal{D}+}_{d}$, at the refineries and depots, respectively. We assume that all these cost parameters are non-negative and that they are based on crude estimates of what are realistic costs of "generating" and "removing" products. The costs estimates have been derived from discussions with planners at the refinery and the estimates are based on how they can solve such situations in real life. For example, the cost of a land based transportation to a customer depot can be used for estimating the cost of "generating" products at a depot. Further, we introduce cost parameters ($c_{dpt}^{\mathcal{ITL}}$ and $c_{dpt}^{\mathcal{ITU}}$) of not meeting the target levels. Note that these costs of not meeting target levels are much lower than the costs of generating and removing products.

In constraints (1b), the quantities shipped from a refinery are connected to the quantities sent by the ships. The loaded quantities are connected to the use of outbound trips in constraints (1c). Observe that we assume that any mix of products can be shipped. This implies that we make a simplification since in our application there are five separate tanks in each ship which restricts the possible quantities that can be shipped. This simplification has been discussed with planners at the refinery and agree very much with current practise of planning.

The inventory levels at depots are modeled with constraints (1d). For a real life scenario, it might happen that no shipment plan can be found due to imbalances between shipment capacities and shipment requirements. Further, a shipment plan may be found with a relatively high cost compared to a plan where, for example, some safety stock levels have been reduced. In order to find a good but slightly "infeasible" shipment plan, both constraints (1b) and (1d) are relaxed by using variables $o_{rpt}^{\mathcal{R}+}$, $o_{rpt}^{\mathcal{R}-}$ and $o_{dpt}^{\mathcal{D}+}$. Removal of products at the depots is not included since it is always possible to ship nothing to the depots.

That the ships follow a ship route is guaranteed by constraints (1e), which imply binary restrictions on the trip vector variables ($x_b^{\mathcal{J}}$ and $x_b^{\overline{\mathcal{J}}}$). The set $\mathcal{A}_b^{\mathcal{J}\overline{\mathcal{J}}}$ is the set of feasible solutions for the trip vector variables $x_b^{\mathcal{J}}$ and $x_b^{\overline{\mathcal{J}}}$ for a ship b, with respect to the time expanded network for each ship. Hence, the node balancing constraints of the time expanded network are not included explicitly in the model.

The individual creation of products at depots are connected to the total creation of products at the depots by constraints (1f). This way of modeling was chosen since it becomes easier to use valid inequalities, which are introduced in Section 4.

Constraints (1g) ensure that the inventory levels at the depots are within specified upper and lower limits. In constraints (1h) and (1i) target levels of inventories are considered. These target levels can be determined based on the results of other planning steps at the refinery, e.g., the production planning in which a planning horizon of 12 months is considered. Further, we have a number of non-negativity constraints (1j)–(1l) and (1r)–(1t).

The constraints (1m)–(1s) define the feasible set representing the production at the refineries. Constraints (1m) and (1n) connect the production and consumption at the refineries with the use of the run-modes ($y_{mt}^{\mathcal{R}}$). The inventory levels at the refineries are modeled by constraints (1o) and (1p). Here, we relax the binary restrictions on variables $y_{mt}^{\mathcal{R}}$, and let them take values between zero and one. This is done since the aim here is not to decide on a particular process schedule at the refinery. Defining $y_{mt}^{\mathcal{R}}$ as continuous, we believe, gives an accurate enough representation of the production, while using the model for devising a shipment plan. Constraints (1q) ensure that the sum of run-mode usage equals one. Note, that one run-mode representing zero production is included in the set M_q for each processing unit q.

4. Valid inequalities

In this section, valid inequalities are introduced for the integer formulation of the shipment planning problem (SP-C). The valid inequalities ensure that the number of times ships visit a port is greater than a certain integer number. These inequalities are introduced since they may strengthen the linear relaxation of the formulation used, and hence, a stronger lower bound might be obtained for the integer problem. Further, the columns generated using a column generation procedure may have better integer characteristics when valid inequalities are included. Hence, the valid inequalities might be rather useful for the solution approach suggested later.

We first derive valid inequalities for a special version of formulation SP-C, where the constraints (1b) and (1d) are not relaxed, such that $o^{\mathcal{R}+}_{rpt} = o^{\mathcal{R}-}_{rpt} = 0$ and $o^{\mathcal{D}++}_{dt} = 0$. Then we show how the inequalities can be modified to also hold for formulation SP-C. The inequalities,

$$\sum_{b\in B}\sum_{\hat{t}\in T:\hat{t}\leqslant t}\sum_{k\in K}\sum_{(i,j)\in S^{\mathcal{D}}_{d\hat{t}bk}} x^{\mathcal{J}}_{ijb} \geqslant D^{\text{int}}_{dt} \quad d\in D,\ t\in T, \tag{2}$$

are valid for SP-C when $o^{\mathcal{R}+}_{rpt} = o^{\mathcal{R}-}_{rpt} = 0$ and $o^{\mathcal{D}++}_{dt} = 0$, and where $D^{\text{int}}_{dt} = \lceil D^{\text{low}}_{dt} \rceil$ and

$$D^{\text{low}}_{dt} = \frac{\sum_{p\in P^{\mathcal{D}}}\left[\sum_{\hat{t}\in T:\hat{t}\leqslant t} e^{\mathcal{D}}_{dp\hat{t}} + \underline{I}^{\mathcal{D}}_{dpt} - I^{\mathcal{D}}_{dp0}\right]^{+}}{\max_{b\in B} C_b}. \tag{3}$$

The numerator of (3) represents the total quantity to be delivered in or before time period t at depot d. Hence, D^{low}_{dt} denotes the minimum number of times ships must visit the depot d before the time period t.

The validity of inequalities (2) in SP-C can be justified by deriving the inequalities from the constraints of SP-C. For details see Persson (2002).

Next we suggest valid inequalities for SP-C (where inventory constraints (1b) and (1d) are relaxed). We suggest a modification of inequalities (2), such that they hold also for SP-C. In effect, the inequalities are relaxed with a penalty cost.

For formulation SP-C, we may formulate the valid inequalities,

$$\sum_{b\in B}\sum_{\hat{t}\in T:\hat{t}\leqslant t}\sum_{k\in K}\sum_{(i,j)\in S^{\mathcal{D}}_{d\hat{t}bk}} x^{\mathcal{J}}_{ijb} \geqslant D^{\text{int}}_{dt} - \frac{\sum_{\hat{t}\in T:\hat{t}\leqslant t} o^{\mathcal{D}++}_{d\hat{t}}}{c^{\text{v}}_{dt}\max_{b\in B} C_b} \quad d\in D,\ t\in T, \tag{4}$$

where parameters c^{v}_{dt}, for $d \in D$ and $t \in T$, are given values according to, $c^{\text{v}}_{dt} = (D^{\text{low}}_{dt} - (D^{\text{int}}_{dt} - 1))$. For the details of justifying the validity of (4), we refer to Persson (2002).

Next inequalities (4) are illustrated by an example. Assume that at a depot at a future point in time ($\hat{t}$) there will be a deficit of 2.3 times the ship capacity C (here we assume all ship capacities are the same) if no delivery occurs before time $\hat{t}$. Then we will pay a penalty of totally $2.3C \cdot c^{\mathcal{D}+}_{d}$. In Fig. 4, we illustrate this situation with the penalty cost on the vertical axis and the times any ship arrives at the depot on the horizontal axis. The solid line represents the actual penalty. We obtain a zero penalty if 2.3 ships go to the depot in a linear relaxation to the problem. In the figure, the "X" denotes the penalty costs for some integer solutions. Since we are interested only in integer solutions, we may include an extra cost, such that the dotted line constitutes the new cost for solutions representing the delivery of between 2 and 3 ship capacities. Introducing this cost increases the cost for the solution of delivering 2.3 times the capacities, which is a typical solution obtained to the LP-relaxation. This increase in cost is what formulation SP-C and inequalities (4) achieve.

5. Master problem formulation

In this section, we introduce a formulation which is equivalent to the linear relaxation of SP-C. We assume that all possible paths through the time expanded network, presented in Section 3, are known (i.e., all schedules are known). A path starts in the first time period at the port where the ship is available and ends in the last time period of the time horizon at a refinery. Since, the planning horizon is rather long compared to the time it takes to complete an outbound and an inbound trip, the ship can make many trips during the planning horizon. Hence, the number of possible paths (schedules) is huge. The variables $x^{\mathcal{JP}}_{ijbkp}$,

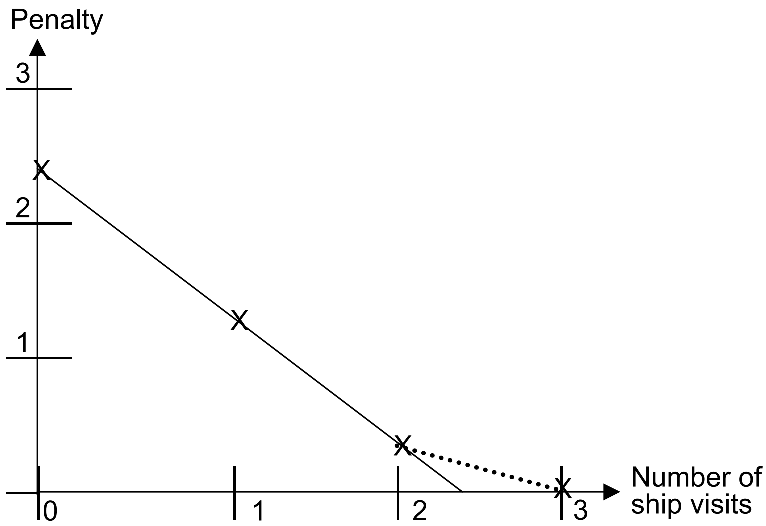

Fig. 4. Illustration of a valid inequality.

which are also used in SP-C, represent the flow of products from refineries to depots. We assume that all ship schedules, defined by the set $S_b^{\mathcal{FL}}$ for a ship b, are known. A schedule s represents the path through the time expanded network. Given a schedule $s \in S_b^{\mathcal{FL}}$, we define a set containing all outbound trips ($j \in J_b$) commencing in time periods ($i \in I$), which define the path corresponding to the schedule. That is, let the set $S_{bs}^{\mathcal{J}}$ contain all pairs (i,j) of time periods and outbound trips which are used in schedule s.

Given a ship schedule $s \in S_b^{\mathcal{FL}}$, the cost of using the schedule is denoted $c_{bs}^{\mathcal{B}}$ and can be identified by the cost of the corresponding sequence of trips ($c^{\mathcal{J}}$ and $c^{\overline{\mathcal{J}}}$). In addition to the variables specified for SP-C, we next specify the variables (υ_{bs}) to be used in the following formulation:

$$\upsilon_{bs} = \begin{cases} 1 & \text{if schedule } s \in S_b^{\mathcal{FL}} \text{ is used for ship } b \in B, \\ 0 & \text{otherwise.} \end{cases}$$

By using the above parameter and variable definitions, we may formulate the shipment planning problem with variable delivery pattern,

$$\begin{aligned} \text{[SP-FL]} \quad \min \quad & \sum_{p \in P^{\mathcal{R}}} \sum_{t \in T} c_{rpt}^{\mathcal{IR}} I_{rpt}^{\mathcal{R}} + \sum_{q \in Q_r} \sum_{m \in M_q} \sum_{t \in T} c_{mt}^{\mathcal{M}} y_{mt}^{\mathcal{R}} + \sum_{b \in B} \sum_{s \in S_b^{\mathcal{FL}}} c_{bs}^{\mathcal{B}} \upsilon_{bs} + \sum_{d \in D} \sum_{p \in P^{\mathcal{D}}} \sum_{t \in T} c_{dpt}^{\mathcal{ID}} I_{dpt}^{\mathcal{D}} \\ & + \sum_{r \in R} \sum_{p \in P^{\mathcal{R}}} \sum_{t \in T} (c_{rpt}^{\mathcal{R}+} o_{rpt}^{\mathcal{R}+} + c_{rpt}^{\mathcal{R}-} o_{rpt}^{\mathcal{R}-}) + \sum_{d \in D} \sum_{t \in T} c_d^{\mathcal{D}+} o_{dt}^{\mathcal{D}++} \\ & + \sum_{d \in D} \sum_{p \in P^{\mathcal{D}}} \sum_{t \in T} (c_{dpt}^{\mathcal{ITL}} I_{dpt}^{\mathcal{IL}} + c_{dpt}^{\mathcal{ITU}} I_{dpt}^{\mathcal{IU}}), \end{aligned} \tag{5a}$$

$$\omega_{rpt}^{\mathcal{R}} + o_{rpt}^{\mathcal{R}+} - o_{rpt}^{\mathcal{R}-} - \sum_{b \in B} \sum_{s \in S_b^{\mathcal{FL}}} \sum_{(i,j) \in S_{rtb}^{\mathcal{R}} \cap S_{bs}^{\mathcal{J}}} \sum_{k \in K_j} x_{ijbkp}^{\mathcal{JP}} = 0, \quad r \in R, \ p \in P^{\mathcal{D}}, \ t \in T, \tag{5b}$$

$$I_{dp,t-1}^{\mathcal{D}} - e_{dpt}^{\mathcal{D}} + \sum_{b \in B} \sum_{s \in S_b^{\mathcal{FL}}} \sum_{k \in K} \sum_{(i,j) \in S_{dtbk}^{\mathcal{D}} \cap S_{bs}^{\mathcal{J}}} x_{ijbkp}^{\mathcal{JP}} + o_{dpt}^{\mathcal{D}+} = I_{dpt}^{\mathcal{D}}, \quad d \in D, \ p \in P^{\mathcal{D}}, \ t \in T, \tag{5c}$$

$$\sum_{s \in S_b^{\mathcal{FL}}} \upsilon_{bs} = 1, \quad b \in B, \tag{5d}$$

$$\sum_{p \in P^{\mathcal{D}}} \sum_{k \in K_j} x_{ijbkp}^{\mathcal{JP}} \leqslant C_b \upsilon_{bs} \quad b \in B, \ s \in S_b^{\mathcal{FL}}, \ (i,j) \in S_{bs}^{\mathcal{J}}, \tag{5e}$$

$$x_{ijbkp}^{\mathcal{JP}} \geqslant 0, \quad b \in B, \ s \in S_b^{\mathcal{FL}}, \ (i,j) \in S_{bs}^{\mathcal{J}}, \ k \in K_j, \ p \in P^{\mathcal{D}}, \tag{5f}$$

$$\upsilon_{bs} \in \{0,1\} \quad b \in B, \ s \in S_b^{\mathcal{FL}}, \tag{5g}$$

$$\text{(1f)–(1l)}. \tag{5h}$$

Compared to SP-C, SP-FL includes costs in (5a) for using a schedule. Exactly one ship schedule can be used for each ship, which is enforced with constraints (5d). Additionally, the constraints (5e) ensure that the sum of the product flows to the first and second port visited on an outbound trip is less or equal to the ship capacity (C_b). Obviously, formulation SP-FL is equivalent to SP-C, since by definition, all schedules which can be represented in SP-C, are included in SP-FL with equivalent costs. We let the linear relaxation of SP-FL, denote the master problem $\overline{\text{SP-FL}}$ in a column generation context.

Also for formulation SP-FL, we formulate a valid inequality similar to (4). Then, the left-hand-side of (4) is expressed in variables υ_{bs}. We obtain the valid inequalities,

$$\sum_{b \in B} \sum_{\hat{t} \in T: \hat{t} \leqslant t} \sum_{s \in S_b^{\mathcal{FL}}} \sum_{k \in K} \sum_{(i,j) \in S_{d\hat{t}bk}^{\mathcal{D}} \cap S_{bs}^{\mathcal{J}}} \upsilon_{bs} \geqslant D_{dt}^{\text{int}} - \frac{\sum_{\hat{t} \in T: \hat{t} \leqslant t} o_{d\hat{t}}^{\mathcal{D}++}}{c_{dt}^{\text{v}} \max_{b \in B} C_b}, \quad d \in D, \ t \in T. \tag{6}$$

6. Column generation—cost updating and the subproblem

In this section we discuss the issue of using column generation for generating schedules to formulation $\overline{\text{SP-FL}}$. Using column generation for formulation $\overline{\text{SP-FL}}$ corresponds to construct the set $S_{bs}^{\mathcal{J}}$. That is to include new variables $x_{ijbkp}^{\mathcal{JP}}$ and υ_{bs} at each iteration. We denote a formulation including only a subset of the columns of the master problem, the restricted master problem.

Assuming we know a subset of the columns ($\subset S_b^{\mathcal{FL}}$) of the master problem $\overline{\text{SP-FL}}$ and a corresponding optimal solution, then the column generation problem is about finding a new schedule $\hat{s}$ for a ship $\hat{b}$ (or for all ships), such that the reduced cost is negative for variable $\upsilon_{\hat{b}\hat{s}}$.

In $\overline{\text{SP-FL}}$ the cost coefficient of a new variable $\upsilon_{\hat{b}\hat{s}}$ is denoted $c_{\hat{b}\hat{s}}^{\mathcal{B}}$, which in turn depends on the particular path of the corresponding schedule $\hat{s}$. This cost is given by the costs of the arcs corresponding to the path in the time expanded network. These arc costs are denoted $c_{ijb}^{\mathcal{J}}$ and $c_{ijb}^{\overline{\mathcal{J}}}$, and are defined in Section 3. The reduced cost of a new variable is affected by the dual prices of the constraints (5e). The dual prices of (5e), in turn, are affected by the dual prices of (5b) denoted μ_{rpt}^{R} and (5c) denoted μ_{dpt}^{D}, via the new variables $x_{ijbkp}^{\mathcal{JP}}$ corresponding to the new schedule $\hat{s}$.

We specify a partial reduced cost, $\bar{c}_{ijb}^{\mathcal{J}}$, associated with each outbound trip. The partial reduced cost corresponds to the cost which is incurred when the outbound trip j is included in the schedule and is based on the transportation cost ($c_{ijb}^{\mathcal{J}}$) and the dual prices of constraints (5b) and (5c) where loading and unloading may occur. The partial reduced cost of an inbound trip is equal to the transportation cost of an inbound trip ($c_{ijb}^{\overline{\mathcal{J}}}$).

Then for each boat $b \in B$, time period $i \in I$, and outbound trip $j \in J_b$, the partial reduced cost ($\bar{c}_{ijb}^{\mathcal{J}}$) can be computed by minimizing cost over all possible products to deliver to the first or to the second port or to deliver nothing at all. Correspondingly, compute

$$\bar{c}_{ijb}^{\mathcal{J}} = c_{ijb}^{\mathcal{J}} + C_b \cdot \min\left[0, \min_{p\in P^{\mathcal{D}}, k\in K_j}\left(\mu^R_{r_0,p,\lceil i/4\rceil} - \mu^D_{d_k,p,\lceil (i+t_k)/4\rceil}\right)\right], \quad i \in I, \ b \in B, \ j \in J_b. \tag{7}$$

Observe that t_k, r_0 and d_k depend on the trip $j \in J_b$. Here we multiply the dual prices by the ship capacity (C_b) in order to get the cost unit of the partial cost in units of using the schedule according to constraints (5e). The contribution from the dual prices to the partial reduced cost ($\bar{c}_{ijb}^{\mathcal{J}}$) is less than or equal to zero, since it is always possible to deliver zero quantities of the products.

We let the cost parameters $\bar{c}_{ijb}^{\mathcal{J}}$ for outbound trips and $c_{ijb}^{\overline{\mathcal{J}}}$ for inbound trips be the arc costs in the time expanded network. Then, given a path through the time expanded network and the corresponding new schedule $\hat{s}$, the cost of the path corresponds to the reduced cost of the variable $v_{b\hat{s}}$.

The column generation problem of finding the minimum reduced cost for ship $b \in B$ is a shortest path problem and can be formulated by using the variables defined in Section 3, as follows:

$$[\mathrm{SUB(b)}] \quad z_b^{\mathrm{SUB}} = \min \quad \sum_{b\in B}\sum_{j\in J_b}\sum_{i\in I} \bar{c}_{ijb}^{\mathcal{J}} x_{ijb}^{\mathcal{J}} + \sum_{b\in B}\sum_{j\in \overline{J}_b}\sum_{i\in I} c_{ijb}^{\overline{\mathcal{J}}} x_{ijb}^{\overline{\mathcal{J}}} \tag{8a}$$

$$\text{s.t.} \quad (x_b^{\mathcal{J}}, x_b^{\overline{\mathcal{J}}}) \in \mathcal{A}_b^{\mathcal{J}\overline{\mathcal{J}}}, \tag{8b}$$

where $\mathcal{A}_b^{\mathcal{J}\overline{\mathcal{J}}}$ defines the time expanded network of the subproblem.

Formulation SUB(b) is a shortest path problem, and hence, it can be solved efficiently by the use of standard algorithms. We utilize the fact that there are only forward arcs in the time dimension in this network. The reduced cost of a new column ($\bar{c}_b^{\mathcal{B}}$) can be computed as $\bar{c}_b^{\mathcal{B}} = z_b^{\mathrm{SUB}} - \gamma_b$, where γ_b is the dual price of (5d) for ship b.

We use a standard approach for generating columns to the restricted master problem ($\overline{\text{SP-FL}}$). However, each time the restricted master problem has been solved, we compute arc costs and solve the subproblem for a single ship (b) which results in one (and only one) new column (ship schedule) for the ship (b). This strategy is more favorable, than generating a column for every ship in our implementation, since the re-optimization of the restricted master problem in this case is done fairly quickly when only one new column is added (in general this may not be the case). In order to guarantee the existence of a feasible solution to the initial restricted master problem, we include for each ship a schedule (column) with zero delivery.

As in Dantzig–Wolfe decomposition a lower bound (LBD) to the objective function of the master problem can be computed, by adding the objective function value of the restricted master and the sum of the reduced costs for all ships, i.e., $\mathrm{LBD}^{\mathrm{LP}} = \mathcal{V} + \sum_{b\in B} \bar{c}_b^{\mathcal{B}}$, where $\mathcal{V}$ denotes the optimal objective function value of the restricted master problem. However, the computation of a $\mathrm{LBD}^{\mathrm{LP}}$ holds only if all subproblems are solved using the same dual prices. This is enforced in our procedure whenever $\mathrm{LBD}^{\mathrm{LP}}$ is computed.

The valid inequalities (6) need to be accounted for when generating columns to $\overline{\text{SP-FL}}$. Let the dual prices of (6) be denoted μ_{dt}^V, for $d \in D$ and $t \in T$. Then, formula (7) can be modified, such that the dual prices μ_{dt}^V are considered when computing the value of the partial reduced cost,

$$\bar{c}_{ijb}^{\mathcal{J}} = c_{ijb}^{\mathcal{J}} + C_b \cdot \min\left[0, \min_{p\in P^{\mathcal{D}}, k\in K_j}\left(\mu^R_{r_0,p,\lceil i/4\rceil} - \mu^D_{d_k,p,\lceil (i+t_k)/4\rceil}\right)\right] - \sum_{k\in K_j}\sum_{t\in T: t \geqslant (i+t_k)/4} \mu^V_{d_k,t},$$
$$i \in I, \ b \in B, \ j \in J_b. \tag{9}$$

The last term of (9) ensures that the dual prices (μ_{dt}^V) of a visited port in time periods after the visit has occurred contribute to the reduced cost of the column when the outbound trip indicates a visit to the port.

7. Finding an integer solution

Next we discuss how to decide on a particular schedule for each ship, i.e., to find an optimal or at least a good integer solution in the vector variables $x_b^{\mathcal{J}}$ and $x_b^{\overline{\mathcal{J}}}$ for SP-C or SP-FL. A natural solution approach is to use a branch-and-bound procedure. However, the solution time required for solving the linear relaxation is too large for performing a complete branch-and-bound procedure, which is illustrated with computational results in Section 8. We suggest to perform a partial (or limited) tree search by interchangeably branching and generating new columns in order to find a solution with integer characteristics. We branch on a "constraint" or on characteristics enforcing that a tanker either must visit a port (depot or refinery) in a time period, $t \in T$, or it must not. In our limited tree search we only consider the branch saying that the port must be visited. Since we only investigate one of the branches, we refer to branching as "fixing". This is illustrated in Fig. 5, where it is assumed that the port "Holmsund" must be visited in the second time period. Then, an outbound trip representing unloading at this port must be used. In the figure we have included a number of actual ship routes which pass Holmsund in time period $t = 2$ (depicted using arrows). One of the corresponding arcs representing outbound trips has to be used in the network when solving SUB(b). For illustration, we have in Fig. 5 included an outbound trip and its actual ship route (depicted using thick dotted arrows).

The port and time period to branch on is determined based on the fractional solution to the restricted master problem (possibly including branching constraints). The branch constraint is selected by identifying the port and time period with the largest sum of the fractional usage of such columns saying that the port is visited at that time period by the ship. This fractional solution is often not optimal, since in general it is time consuming to solve the linear relaxation to optimality. The fractional usage is denoted by f_{dt} for $d \in D$ and $t \in T$ (for a ship b). In order to prioritize early time periods (t) we add a constant (w_t) to the fractional usage, such that $w_t = 1/(3 + t)$, $t \in T$. Then we look for d^* and t^* with the largest adjusted fractional usage, i.e., find $d^* \in D$, $t^* \in T$, such that, $(d^*, t^*) = \text{argmax}_{d \in D, t \in T: 0 < f_{dt} < 1}(f_{dt} + w_t)$. In our strategy we include the branch constraint only if the sum of the fractional usage (without the adjustment by w_t) is strictly less than 1 and strictly greater than 0. Fixing the visit to a refinery is only done if the maximum fractional usage at a depot port is below a certain threshold value, i.e., we prioritize the fixing to depots. This value was set to $1/|D|$, where $|D|$ is the number of depots.

This strategy of fixing is illustrated in Fig. 6, where the box in bold corresponds to the column generation of the linear relaxation. In the column generation process, possible branch constraints are obeyed. The

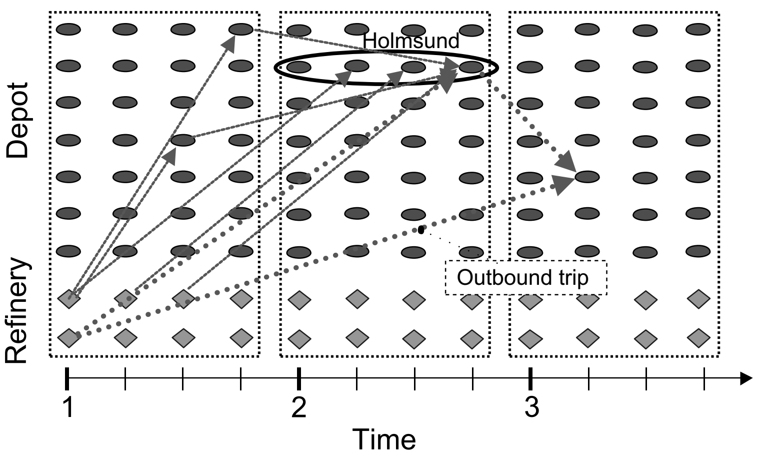

Fig. 5. Partial paths allowed if fixing occurs with regard to Holmsund in time period 2.

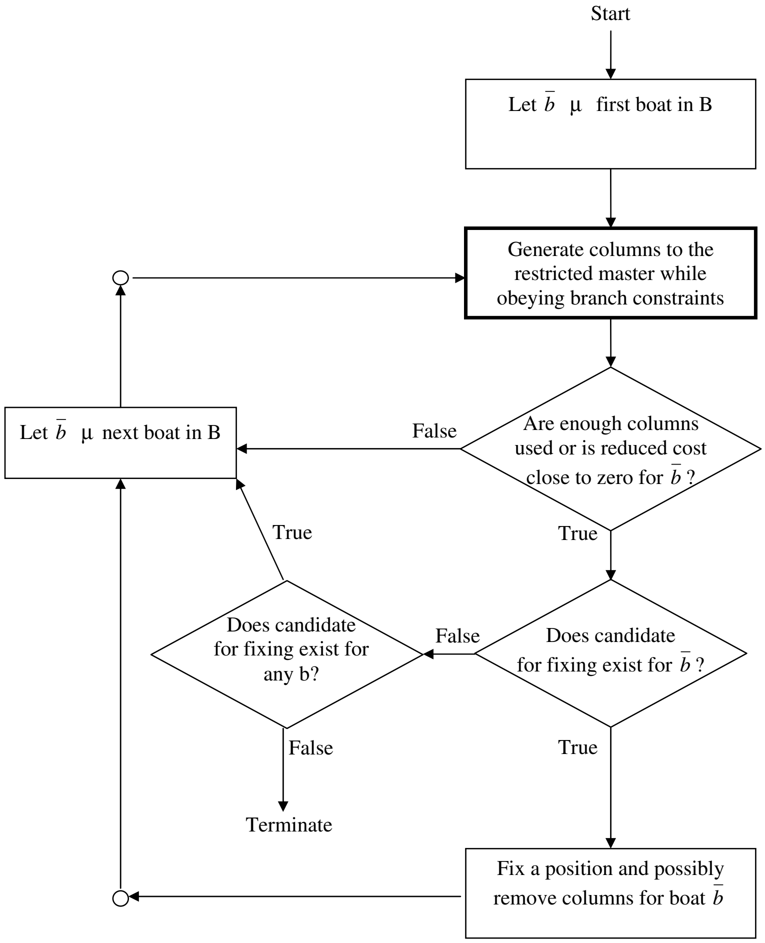

Fig. 6. Flow chart of the implemented search for solving the integer problem with fixing strategy.

column generation procedure is terminated when at least one column has been added for each ship in B. Additionally, it is enforced that a certain number of columns has been generated before the first branch constraint is included.

In order to enforce the branch constraint in the column generation approach, a large premium (negative cost) is included in SUB(b) for such arcs representing such a visit in any of the corresponding time periods $i \in I$. Since the shortest outbound trip together with the shortest inbound trip takes more than one time period in T, at most one of the premium arcs can be used for each branch constraint in a solution to SUB(b). The premium is naturally ignored when the optimal objective function value z_b^{SUB} of SUB(b) is computed. This method of implicitly including branch constraints allows a subproblem to remain a shortest path problem.

Since the restricted master problem is not necessarily solved to optimality, we want to guarantee that some stability has been reached in the column generation procedure before a new branch constraint is included for a ship. Including a branch constraint for a ship is only allowed if the number of columns used for a ship $\bar{b}$ is greater than a threshold value T^{incl}, which typically is 3 or 4 in our tests, i.e., $|s \in S_{\bar{b}} : \upsilon_{s\bar{b}} > 0| \geqslant T^{\mathrm{incl}}$. This is illustrated in the upper decision rhomboids in Fig. 6. This criteria is of course ignored if no more columns can be generated, i.e., the reduced cost of the last generated column for the ship is close to zero.

Next, a check is run to see if a candidate for fixing can be found for the ship. If this is the case, a new branch constraint is included (i.e., fixing is applied). If no candidate can be found for any of the ships the procedure is terminated, since this implies that a solution with integer characteristics has been found. After a new branch constraint has been included, columns which do not satisfy this constraint are removed from the restricted master problem.

A branch-and-bound solution procedure can be applied after the procedure of fixing has terminated. Then the branch-and-bound procedure finds the best combination of the schedules generated during the process. Note that each column represents a partial integer solution, and whenever columns are generated it might be worth exploring this possibility of finding the best combination.

We utilize "warm starts", i.e., the previous basis is used when the restricted master problem is re-optimized. The strategy of fixing reduces the potential for using warm starts, since this strategy implies the removal of columns when branching, which makes previous solutions infeasible. However, there are columns available, such that, a feasible solution will exist to the restricted master problem.

8. Computational results and conclusions

In this section we present computational results of using column generation together with the strategy of fixing on different problem scenarios of the shipment planning problem.

8.1. Problem scenarios and setup of the tests

We have created four test scenarios. For two of these, denoted SP1 and SP2, we use real life data from Nynas of the demand of products at the depots. A larger scenario, denoted SP3, includes additional real depots for which random demand patterns have been generated. Additionally, we have created SP4, which is similar to SP3, for which we have created six problem instances. In all the scenarios, the Nynäshamn refinery is included. In SP3 and SP4, a total of three and two refineries are considered, respectively. The additional refineries included in SP3 and SP4 are modeled rudimentarily, and essentially only the production capacity and the production costs are considered. In SP3 and SP4, a third ship is included with the same capacity as the two other ships. This third ship has a lower cruising speed than the other two ships. With regard to the ships, SP3 and SP4 mimic the situation the company is currently facing (2002).

The problem instances of SP4 have been obtained by simulating combined shipment planning and process scheduling using a rolling time horizon, where the planning horizon has moved forward a week each time before re-planning occurred. Each time the planning horizon has been moved, the forecasted demand has been randomly distorted and additional demand has been included for the last week of the planning horizon. For the problem instances of SP4, the process scheduling problem at the Nynäshamn refinery has been solved using a tabu search heuristic, see Persson et al. (2004) for details. In the process scheduling problem changeover costs and binary requirements on the usage of the run-modes ($y_{mt}^{\mathcal{R}}$) were included. When solving the process scheduling problem, the deliveries from the refineries are assumed to be fixed and given by the solution to the shipment planning problem (i.e., it is given by $\omega_{rpt}^{\mathcal{R}}$). When planning the shipments of the problem instances of SP4, the production at the Nynäshamn refinery is assumed to be fixed for the first week. The process schedule is fixed as suggested by the solution to the process scheduling problem. The characteristics of the scenarios are given in Table 1.

The number of outbound trips can be reduced since low efficient trips, from a practical point of view, can be excluded. An example of a low efficient outbound trip is when the outbound trip (including unloading at two depots) implies that the tanker, when moving between the shortest way between depots, will pass just outside the refinery where it loaded. Such an outbound trip will be excluded in our implementation.

Table 1
Characteristics for the scenarios

Characteristics	SP1	SP2	SP3	SP4
Depots ($\|D\|$)	6	9	15	13
Refineries ($\|R\|$)	1	1	3	2
Shipped products ($\|P^{\mathcal{D}}\|$)	4	4	4	4
Produced products ($\|P^{\mathcal{R}}\|$)	25	25	3	11
Run-modes ($\|M\|$)	23	23	7	17
Time periods ($\|T\|$)	31	42	42	42
Outbound trips ($\|J_b\|$)	14	29	115	60
Number of ships ($\|B\|$)	2	2	3	3

The length of the outbound trips ranges from 1 day and 6 hours to 11 days, that is 5–44 time periods in I. The loading and unloading procedures requires each about 6 hours.

For a number of products p and for some depots d, we include target levels of the inventories at the last time period. For such products and depots, we give a negative value for the inventory cost parameter $c^{\mathcal{ID}}_{dp\bar{t}}$ instead of the normal low positive inventory cost.

The computational tests were performed on an Intel based PC, with 1700 MHz processor and 1 Gbyte of RAM. We used AMPL (version 10.6.16—Win32) for maintaining the restricted master problem and Cplex 7.0 for solving the restricted master problem. The shortest path problem was solved using a program developed in C-programming language.

8.2. The linear relaxation

In the first set of computational tests, we solve the linear relaxation of the different formulations of the shipment planning problem. Further, we test the effect of including valid inequalities (4) or (6), which in Table 2 are denoted "+V". In the columns of Table 2, we present the objective function value of the best found solution, denoted $\mathcal{V}^{\mathrm{LP}}$ (given in a specific cost unit), the relative gap between $\mathcal{V}^{\mathrm{LP}}$ and $\mathrm{LBD}^{\mathrm{LP}}$ given in percentage, the number of columns generated, the final number of variables identified, and the CPU-time

Table 2
Results when solving the linear relaxation of the different formulations

Scenario	Formulation	$\mathcal{V}^{\mathrm{LP}}$	$\frac{\mathcal{V}^{\mathrm{LP}}-\mathrm{LBD}^{\mathrm{LP}}}{\mathcal{V}^{\mathrm{LP}}}100$	$\sum_{b\in B}\|S_b\|$	#Var.	CPU (m)
SP1	$\overline{\text{SP-C}}$	5138	0		33219	1
	$\overline{\text{SP-C}}$ + V	5154	0		33219	1.2
	$\overline{\text{SP-FL}}$	5138	0	177	14223	11
	$\overline{\text{SP-FL}}$ + V	5154	0	178	14490	12
SP2	$\overline{\text{SP-C}}$	7147	0		88101	2
	$\overline{\text{SP-C}}$ + V	7209	0		88101	3
	$\overline{\text{SP-FL}}$	7148	0.06	304	27954	67
	$\overline{\text{SP-FL}}$ + V	7209	0	178	21174	25
SP3	$\overline{\text{SP-C}}$	9406	0		450779	156
	$\overline{\text{SP-C}}$ + V	9719	0		450779	384
	$\overline{\text{SP-FL}}$	9406	0.01	393	27286	150
	$\overline{\text{SP-FL}}$ + V	9719	0.01	412	31941	144
SP4	$\overline{\text{SP-C}}$	9074	0		235008	15
	$\overline{\text{SP-C}}$ + V	9269	0		235008	158
	$\overline{\text{SP-FL}}$	9086	0.5	416	32007	137
	$\overline{\text{SP-FL}}$ + V	9280	0.5	410	34210	140

(CPU (m)) in minutes. LBD^{LP} refers to the lower bound obtained for each test run. The number of variables refers to the number identified by Cplex after pre-processing has been performed on the complete formulation or on the restricted master problem at hand at termination. The tests are terminated either when an optimal solution has been found or when the number of columns has reached a critical level with respect to memory requirements. The CPU-times are given in minutes and refer to the time including all types of processing and, hence, equals the clock time used when the computer is dedicated to these tests.

The results presented in Table 2 indicate that we can solve the linear relaxation to optimality or near optimality within reasonable computational times for all scenarios. The use of inequalities (4) increases the objective function value by 0.3%, 0.9%, 3.3%, and 2.1% for scenarios SP1, SP2, SP3, and SP4, respectively. Using the complete formulation $\overline{\text{SP-C}}$ appears to be a good alternative when solving the linear relaxation, at least for the smaller scenarios (SP1 and SP2) and for SP4 if optimality is important. For scenario SP3, the column generation approach appears to be slightly more efficient. Note that when we solve the integer problem, as documented in Section 8.3, the column generation approach is crucial for the larger problem scenarios SP3 and SP4.

It can be noted that the number of columns generated is rather small compared to the number of columns required for many other problems found in the literature. One possible explanation is that the dual prices obtained from the master problem have good values due to "bounding". Since we relax constraints (1b) and (1d), the dual prices are bounded by the values of generating and removing products.

8.3. Solving the integer problem

In this section, the results of using the strategy of fixing for finding integer solutions is studied. For comparison, we first use a branch-and-bound solution procedure (Cplex) for solving the complete formulation SP-C with valid inequalities (4). Recall that in SP-C, binary requirements are specified for the trip vector variables $x_b^{\mathcal{J}}$ and $x_b^{\overline{\mathcal{J}}}$. The results are presented in Table 3, where the obtained lower bounds (LBD), the objective function value of the best found solution ($\mathcal{V}$), and the CPU-time in minutes (CPU (m)) are presented. For SP4, the average values of the six problem instances are presented. We do not present the average of the best found objective function value for SP4, since for two out of six problem instances Cplex was unable to find an integer solution before termination. The branch-and-bound solution procedure has been terminated due to memory limitations (700 Mb) for SP1 and SP2 and time limitations for SP3 and SP4. For the smaller scenarios SP1 and SP2, the number of nodes in the search tree increases relatively fast, and hence, the procedure must be stopped earlier than for the two larger scenarios.

Next we study the strategy of fixing for formulations SP-C and SP-FL. The parameter T^{incl} determines the minimum number of columns which must be in use before a branching constraint can be added for formulation SP-FL. For formulation SP-FL, we let the parameter T^{incl} be equal to 3 for scenario SP1 and SP2 and equal to 4 for scenario SP3 and SP4. When using formulation SP-FL for SP1 and SP2, the process of fixing is started when at least 15 columns have been generated for each ship. For SP3 and SP4, we enforce that at least 20 columns must be generated.

In Table 4, the objective function value is given for the best found solution ($\mathcal{V}$) after the strategy of fixing has terminated. Further, we present in parentheses the objective function value obtained after performing a

Table 3
Results when solving the integer problem using standard branch-and-bound on SP-C with valid inequalities

Scenario	LBD	$\mathcal{V}$	CPU (m)
SP1	5171	5332	562
SP2	7214	7667	609
SP3	9723	13,901	2880
SP4	9314	–	1440

Table 4
Results of solving the integer problem of the scenarios using the strategy of fixing

Scenario	Formulation	$\mathcal{V}$	$\sum_{b\in B}\lvert S_b\rvert$	CPU (m)
SP1 (LBD = 5171)	SP-C	5538		10
	SP-C + V	5385		14
	SP-FL	5628 (5487)	127	9 + 2 = 11
	SP-FL + V	5376 (5366)	117	9 + 4 = 13
SP2 (LBD = 7214)	SP-C	8084		90
	SP-C + V	8167		84
	SP-FL	8164 (7784)	163	29 + 5 = 34
	SP-FL + V	7891 (7822)	135	20 + 20 = 40
SP3 (LBD = 9723)	SP-FL	13,242 (11,987)	290	96 + 147 = 243
	SP-FL + V	10,877 (10,444)	282	111 + 63 = 174
SP4 (LBD = 9314)	SP-FL	12,309 (11,605)	291	92 + 78 = 170
	SP-FL + V	11,184 (10,578)	274	93 + 102 = 195

branch-and-bound search procedure on the columns generated during the process. The table also includes a reference to the number of columns generated. The CPU-times in minutes required for generating columns and for performing the B&B procedure are also included. When applying the strategy of fixing to formulation SP-C, the problem is solved to optimality each time before a new branch constraint is added to the problem. For scenario SP3 and SP4, formulation SP-C could not be utilized when using this strategy, since the CPU-time becomes too large. The reason for this is that extensive CPU-time was required for re-optimizing the problem after fixing has occurred.

From the results in Table 4, it can be seen that including inequalities (4), often improves the performance significantly (although the computational time is somewhat increased). Using formulation SP-FL gives integer solutions with typically better objective function values than when formulation SP-C is used.

It is clear that if extensive computation time is utilized in a branch-and-bound procedure with binary requirements on variables $x_b^{\mathcal{J}}$ and $x_b^{\overline{\mathcal{J}}}$, competitive solutions can be found for the two smaller scenarios SP1 and SP2. This approach, however, is not only time consuming but also requires a lot of computer memory.

Regarding the quality of the solutions/plans obtained, some solutions have been studied and approved as reasonable by planner at Nynas. Additionally, based on our experience using the setup of rolling time horizon for SP4, it appears that penalty costs occurring for violating inventory levels, mainly come from time periods at the end of the planning horizon. Our, strategy of putting extra weight on the fixing of the ship schedule in early time periods appears to deliver a good schedule in the beginning of the planning horizon, which is important when using a rolling time horizon.

In our model we have not enforced that only a single ship can unload at a depot, which is the case at most depots. In the obtained shipment plans, it rarely occurred that ships were scheduled to the same port at the same time. If additional tests indicate that this is a problem, it can be included in the solution procedure without too much troubles (additional constraints are included in the master formulation and the associated dual prices are used in the subproblems).

From computational tests on SP4, when rolling time horizon is used, it has been seen that feasible and probably good production schedules can be found for the shipment plans obtained in this approach. This indicates that the integration of the process scheduling problem into the model serves it purpose.

The column generation approach would probably become more competitive if another implementation had been used. For example, when solving scenario SP3 using formulation $\overline{\text{SP-FL}}$ + V, approximately 4% of the CPU-time is used for solving the restricted master problem with Cplex and about 1% of the CPU-time is used for solving the subproblems. The rest of the CPU-time is spent within AMPL processing the output and input to the restricted master problem and the subproblems. One of the most time consuming phases is to add columns to the restricted master problem and to communicate this to Cplex. We believe that by using Cplex callable library, for example, the CPU-time needed for solving the scenarios using column generation would decrease significantly. In contrast, when the complete formulation $\overline{\text{SP-C}} + \text{V}$ is used for scenario SP3, less than 1% of the CPU-time is spent within AMPL and the rest within Cplex.

9. Summary and future research

In this paper a shipment planning problem is presented and a solution method based on column generation is suggested. A formulation of the master problem is introduced, which is suitable for column generation. A standard branch-and-bound approach is unsuccessful in solving the problems of realistic sizes. A "fixing" strategy, based on a limited branch search, for solving the shipment problem is suggested. This strategy includes column generation and constraint branching. The solution strategy is improved by using valid inequalities for the shipment planning formulation and by performing a branch-and-bound search for selecting the best combination of columns generated throughout the process of fixing. That is, the branch-and-bound procedure is applied after the fixing procedure has terminated. In essence the valid inequalities are relaxed, such that, the cost for non-integer solutions is increased.

An interesting issue for future research is to use the setup of rolling time horizon further and to study, for example, the effect on total cost of using different levels of the safety stocks. Additionally, one may also study the effect of using different time steps of the time horizon and its effect on the overall cost. Naturally, in such case additional real life data is required.

Concerning the implementation of the solution strategy, the computation time could probably be reduced by, for example, using Cplex callable library. Additionally, with some modifications, an analysis of removing or adding depots can be performed. One interesting issue is to include transportation activities to customers, which collect bitumen from the depots, into the model. Then whenever customers may collect their products at alternative depots, it is interesting to study the effects of closing depots or possibly opening new ones. Customers will expect to be compensated for the longer travel distances for collecting the products if depots are closed. This can be considered in the model, by including costs of delivering products to customers from depots.

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