Computers and Chemical Engineering 35 (2011) 2113–2123

Contents lists available at ScienceDirect

Computers and Chemical Engineering

journal homepage: www.elsevier.com/locate/compchemeng

A Lagrangian relaxation approach for a multi-mode inventory routing problem with transshipment in crude oil transportation

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ARTICLE INFO

Article history:
Received 29 January 2010
Received in revised form 15 November 2010
Accepted 5 January 2011
Available online 12 January 2011

Keywords:
Crude oil transportation
Logistics
Inventory routing problem
Optimization
Lagrangian relaxation

ABSTRACT

An inventory routing problem in crude oil transportation is studied, in which crude oil is transported from a supply center to multiple customer harbors to satisfy their demands over multiple periods. In the problem, a heterogeneous fleet of tankers consisting of tankers owned by a distributor and tankers rented from a third party, a pipeline, and multiple types of routes are considered; both inventory level and shortage level at each customer harbor are limited. The objective is to determine for each period over a given time horizon the number of tankers of each type to be rented/returned at the supply center, the number of tankers of each type to be dispatched on each route, and the quantity of crude oil flowing through the pipeline that minimizes the total logistics cost.

After formulating the problem as a mixed integer programming problem, a Lagrangian relaxation approach is developed for finding a near optimal solution of the problem. The approach is also applied to a variant of the problem in which both fully and partially loaded tankers are allowed in the transportation of crude oil. Numerical experiments show that this approach outperforms an existing meta-heuristic algorithm, especially for the instances of large sizes.

1. Introduction

As we well know, crude oil is an important strategic material to most countries in the world. Unfortunately the production and the consumption of crude oil are usually geographically separated, that is, a crude oil extraction plant is usually far away from the final consumption market of its oil products. For example, most oil fields are located in Middle East, West Africa, South America and Russia; but the main consumption markets lie in East Asia, Europe and North America, the average distance from an extraction plant to a consumption market reaches several thousands sea miles. At the same time, the transportation of crude oil is expensive; its cost may reach several dollars per barrel. Thus, effective transportation planning is a very important issue in the trade of crude oil. In real life, crude oil can be transported in multiple transportation modes including pipeline, tanker, train and tank truck, in which pipeline and tanker are more suitable for a long distance transportation of crude oil. As the distance between an oil export country and its corresponding import country is usually quite long, tanker and pipeline are considered as two major transportation modes of crude oil in this paper. Because of the existence of multiple transportation modes, a heterogeneous fleet of tankers and various logistics costs, the effective transportation planning of crude oil has a high complexity and is a challenge issue in petroleum logistics.

This paper studies a crude oil transportation planning problem, in which crude oil is transported from a supply center to a set of customer harbors with dynamic demands and limited inventory and shortage capacities. The crude oil is transported by an oil distributor using a heterogeneous fleet of tankers and a pipeline. The tankers may be owned by the oil distributor itself or rented from a third-party at the supply center with rental costs. There are multiple types of routes between the supply center and the customers, including general marine routes, marine routes with canal and marine routes combined with the pipeline. The inventory cost of holding crude oil at each customer harbor is taken in account. Because of the limited transportation capacity or the need for the reduction of transportation costs, backlogging of part of a customer's demand is allowed but with a penalty (backlogging cost). The problem is to determine for each period over a given time horizon the number of tankers of each type to be rented/returned and the number of tankers of each type to be dispatched on each route to min-

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doi:10.1016/j.compchemeng.2011.01.005

imize the total logistics cost including the transportation costs, the inventory and backlogging costs at the customers and other related costs.

In the literature, several papers have studied various crude oil transportation problems. Brown, Graves, and Ronen (1987) considered a tanker routing and scheduling problem for transporting crude oil between an export terminal and an import terminal, where a single type of tankers with a single compartment was used for each type of crude oil in each delivery with a single loading and unloading port. The problem was formulated and solved as an elastic set partitioning problem which determines the least expensive schedule for each cargo. Similar to the problem studied by Brown et al. (1987) and motivated by a crude oil transportation problem of Kuwait Petroleum Corporation (KPC), Sherali, Al-Yakoob, Mohammed, and Hassan (1999) studied a tanker routing and scheduling problem with delivery time-windows, in which a supply port, several demand ports and multiple types of ships with various compartments were considered. The problem was modeled as a complex mixed-integer linear programming (MILP) and solved with CPLEX for small instances. Later on, an aggregate model that retains the principal features of the MILP model was formulated and solved for practical size instances by using a specialized rolling horizon heuristic. To minimize the environmental pollution caused by oil spill incidents and to optimize the logistic costs of the maritime transportation of oil products, Iakovou (2001) presented a multi-objective network flow model to describe a problem of maritime transportation of oil products between multiple origins and destination locations, with the aim to minimize transportation costs and expected risk costs (due to oil spills). The model was then decomposed into two subproblems, one is a risk problem and the other is a transportation problem. An interactive solution methodology was proposed to solve the subproblems. Chu, Chu, Shen, and Chen (2008) transformed a point-to-point oil transportation problem with a homogeneous fleet of tankers into a single item lot sizing problem, and developed a polynomial dynamic programming algorithm to solve it.

As the problem studied in this paper integrates transportation planning with inventory management of crude oil, it can be regarded as a multiple-period inventory routing problem (IRP) with dynamic demands. The articles studying inventory routing problems are numerous. Chien, Balakrishnan, and Wong (1989) studied a multiple periods IRP and solved the problem using a comprehensive decomposition scheme in a rolling horizon framework, in which a vehicle routing problem is repeatedly solved over a 2-week moving period. Campbell and Savelsbergh (2004) proposed a decomposition approach to a multiple periods IRP in a rolling-horizon framework as well, where a delivery schedule is first created by solving an integer programming model and a set of delivery routes is then constructed based on the schedule. For more work on multiple period IRPs, readers can refer to Campbell, Clarke, and Savelsbergh (2001) and Kleywegt, Nori, and Savelsbergh (2002). A general introduction of IRP is provided by Campbell, Clarke, Kleywegt, and Savelsbergh (1998).

Compared with the problems studied in the literature and cited above, our problem is more complex. Firstly, the tankers dispatching decisions are made over multiple periods with the constraints of satisfying customers' demands and the upper and lower bounds of the inventory level at each customer, the problem is thus a multiple-period inventory routing problem rather than a single period delivery planning problem; and as crude oil can be transported from a central depot directly or via input and output ports of a pipeline to a set of demand harbors, it is in fact an inventory routing problem with multiple transshipment ports. Secondly, our problem involves multiple transportation modes, which include pipeline and tanker, and multiple types of routes, which include general maritime routes and maritime routes with canal. Thirdly, in our problem, the lead time for each tanker to transport crude oil from one port to another is considered. Finally, various capacity restrictions, e.g., the capacity of each tanker, the capacity of pipeline, the canal limitation for tanker tonnage and the renting of tankers from a third-party are also considered in our problem.

Very few papers studied a similar or the same crude oil transportation problem as the one we consider in this paper. Cheng and Duran (2004) studied a crude oil transportation and inventory problem with multiple transportation modes including tankers and pipelines but all tankers have identical capacity, they developed a decision support system which integrates discrete event system simulation and stochastic optimal control to evaluate and improve decisions for the problem. Shen, Chen, Chu, and Zhou (2009) proposed a mixed-integer programming model to the same problem studied in this paper and developed a metaheuristic method, Greedy Randomized Adaptive Search Procedure enhanced by Path Relinking method (GRASP/PR), to find a near-optimal solution of the problem. The method performs well for randomly generated instances of small-to-medium sizes. However, for large instances, the computation time of the method for finding a high quality solution becomes quite long. Thus, a more efficient approach is needed to solve large instances of the problem.

This paper tries to develop an efficient optimization-based approach for the crude oil transportation problem based on Lagrangian relaxation. Lagrangian relaxation method was first presented by Geoffrion (1974), it was then widely applied in solving large integer and mixed integer programming problems. Fisher (2004) reviewed the method and showed that it was quite effective in solving some hard integer optimization problems, such as TSP, scheduling problems, location problems, generalized assignment problems, etc. This method was also successfully applied in inventory routing problems. Fisher et al. (1982) and Bell et al. (1983) applied the method to solve an inventory routing problem of air product distribution with the objective to maximize the profit of product distribution over multiple periods. Yu, Chen, and Chu (2006) proposed a new model for an IRP with split delivery and developed a Lagrangian relaxation method for quickly finding a near-optimal solution of the IRP.

The traditional way of using Lagrangian relaxation to solve resource constrained optimization problems usually relaxes linear capacity constraints of the resources. But for our problem, this kind of relaxation will lead to the Lagrangian relaxed problem containing little information about the optimal solution of the original problem. As a result, it is very difficult to construct a high quality feasible solution of the original problem based on the solution of the relaxed problem even if we have got optimal Lagrange multipliers. For this reason, we first reformulate the capacity constraints of our problem in a nonlinear way and then relax the nonlinear capacity constraints by introducing Lagrange multipliers. The resultant Lagrangian relaxed problem is solved approximately with its corresponding dual problem solved by using the surrogate subgradient method (SSG) proposed by Zhao, Luh, and Wang (1999). Under some mild conditions, the method can solve the Lagrangian dual problem in case of an approximate resolution of the relaxed problem. The advantage of this reformulation and relaxation is that the relaxed problem contains more useful information about the optimal solution of the original problem, which permits effective construction of a high quality solution of the latter problem from the solution of the former problem.

Compared with our previous work (Shen et al., 2009), the main contribution of this paper is the development of an optimization based Lagrangian relaxation approach which is more efficient and effective in solving large instances of the studied problem, numerical experiments show that the new approach can find better solutions in shorter computation time for large instances. Moreover, we also study a variant of the crude oil transportation problem in which both fully and partially loaded tankers are permitted in the transportation of crude oil and develop a variant of the Lagrangian relaxation approach which is effective and efficient in solving the new problem.

The rest of this paper is organized as follows. The crude oil transportation problem with fully loaded tankers is described in Section 2 and its mixed integer programming model (MILP) is presented in Section 3. In Section 4, some additional and valid constraints are introduced to tighten the MILP model and its Lagrangian relaxation approach is presented with algorithms for the relaxed problem and the dual problem. The heuristic for constructing a feasible solution of the original problem from the solution of the relaxed problem is described in Section 5. The resolution of a variant of the crude oil transportation problem is discussed in Section 6. The computational results are presented in Section 7. Section 8 concludes this paper with research perspectives.

2. Problem description and notation

The problem considered involves one unlimited crude oil supply center, several transshipment ports with limited inventory capacity and multiple customer harbors with limited inventory capacity geographically dispersed in a petroleum logistics network. Crude oil can be transported from the supply center, either directly or via transshipment ports at the input and output ports of a pipeline to customer harbors to satisfy dynamic demands of the customers, as shown in Fig. 1 below. There are three types of routes existing between the supply center and a demand harbor: general marine route, marine route with canal and marine route with pipeline, and multiple routes may exist from the supply center to a demand harbor. A marine route with canal can only be taken by tankers with lower capacity since tanks with higher capacity cannot pass the canal; for a marine route with pipeline, the capacity of the pipeline and the inventory capacity of the depot at the input or the output port of the pipeline are limited. After crude oil is unloaded at a customer harbor, the empty tanker can moor at the harbor or return to the supply center or to the output port of the pipeline through a marine route or a marine route with canal. It is assumed that the oil distributor considered manages a heterogeneous fleet of tankers composed of its own tankers and rental tankers, tankers can only be rented at the supply center from a third-party with rental costs and rental tankers can be returned to the third-party later at the same place. Various cost components are considered in the problem, such as the pierage when a tanker moors at a harbor, fixed usage cost of a tanker for each delivery, operating cost of a moving tanker in each period, rental cost when a tanker is rented, pipeline toll when crude oil is transported through the pipeline, canal toll when a tanker moves through a canal, inventory holding and backlogging costs at each customer harbor and at the input and output ports of each pipeline.

In practice, crude oil transportation planning is usually made in a rolling horizon way. Here, the term "horizon" refers to a number of periods in the future for which demand forecasting and transportation planning are made (Sethi & Sorger, 1991). The main reason for the rolling horizon planning is that forecasting the future demand in a dynamic stochastic environment is neither accurate nor cheap. Because of this, the rolling horizon planning method only implements the most immediate decisions made for the first period although planning decisions are made for all the periods within a given time horizon based on available information. At the beginning of the second period, the second period decisions become most immediate. In order to make these decisions, the demand forecast for an additional period in the future is usually required, and in some cases existing forecasts may also be revised or updated, based on newly available information. This procedure is repeated every period. It is this horizon, that gets "rolled over" each period (Sethi & Sorger, 1991). For a given horizon, our problem is to determine for each period the number of tankers of each type to be rented/returned and the number of tankers of each type to be dispatched on each route to minimize the total logistics cost including transportation costs, inventory and backlogging costs and other related costs. Taking as an example, if the planning horizon is taken as 12 months and each month corresponds to a period, the crude oil transportation problem has to be solved 12 times every year, that is, the problem is solved once at the beginning of each month.

Note that the assumption of an unlimited capacity of the crude oil supply center is only for simplifying the description and the formulation of the problem, our Lagrangian relaxation approach to be presented hereafter is also applicable to the problem with a supply center with limited capacity after some modifications.

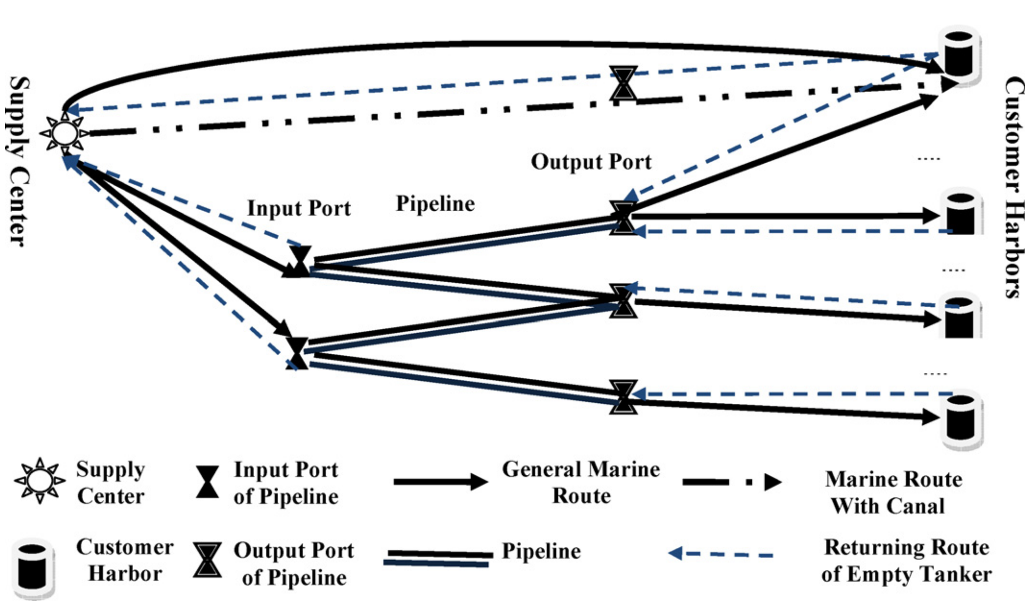

Fig. 1. Logistics network for transportation of crude oil.

The indices, parameters, and variables to be used in the model are first defined in the following:

Indices and sets:

i,j:	Index of harbor;
k:	Index of tanker type;
K:	Set of tanker types;
K_r:	Set of tanker types that can take route r;
l_{ijr}:	Transportation lead time from harbor i to harbor j through route r, it is assumed that $l_{ijr} \le \lvert T \rvert$, where $\lvert T \rvert$ is the number of periods in a given planning horizon;
L_{ijr}:	A set of periods, $L_{ijr} = \{0,1,2,\ldots,l_{ijr}-1\}$, $\lvert L_{ijr} \rvert \le \lvert T \rvert$;
N_D:	Set of customer harbors;
N_{PI}:	Set of input ports of pipelines;
N_{PO}:	Set of output ports of pipelines;
N:	Set of all harbors, including N_D, N_{PI}, N_{PO} and the supply center with index 0, $N = N_D \cup N_{PI} \cup N_{PO} \cup \{0\}$;
P:	Set of pipelines, $P \subseteq N_{PI} \times N_{PO}$;
PI_i:	The set of input ports of pipelines that are connected to output port i by a pipeline;
PO_i:	The set of output ports of pipelines that are connected to input port i by a pipeline;
pl_{ij}:	Transportation lead time of the pipeline with input port i and output port j;
r:	Index of route;
R_{ij}:	Set of routes from harbor i to harbor j;
RC_{ij}:	Set of routes from harbor i to harbor j with canal;
t:	Index of time period;
T:	Set of time periods in a given planning horizon

Parameters:

AP_{ij}:	Capacity of the pipeline with input port i and output port j per period;
CA_k:	The capacity of each tanker of type k;
CF_i:	Backlogging cost per unit quantity of crude oil at harbor i, and $CF_0 = 0$;
CH_{ik}:	Pierage at harbor i per tanker of type k per period;
CI_i:	Storage cost per unit quantity of crude oil per period at harbor i, and $CI_0 = 0$;
CM_k:	Canal toll per pass of a tanker of type k through a route with canal;
CO_k:	Operating cost of each tanker of type k per period;
CP_{ij}:	Toll of the pipeline with input port i and output port j for the transportation of each unit quantity of crude oil;
D_{it}:	Crude oil demand at customer harbor i in period t;
F_{ijrk}:	Fixed usage cost per tanker of type k leaving from harbor i to harbor j through route r;
FH_{ik}:	Fixed rental cost per tanker of type k at harbor i;
LS_i, US_i:	Inventory lower bound and upper bound of the depot at harbor i, respectively, and $US_0 = +\infty$, $LS_0 = -\infty$.

Variables:

h_{ikt}:	Number of tankers of type k rented (if $h_{ikt} \ge 0$, denoted by h^+_{ikt}) or returned (if $h_{ikt} < 0$, denoted by h^-_{ikt}) at harbor i in period t, integer variable;
p_{ijt}:	Quantity of crude oil flowing through the pipeline with input port i and output port j in period t, continuous variable;
q_{ijrkt}:	The total quantity of crude oil transported from the supply center or output port i of a pipeline to customer harbor j or input port j of a pipeline on route r by tankers of type k in period t, continuous variable
s_{it}:	Inventory level of crude oil at harbor i in period t, s_{it} is denoted by s^+_{it}, if $s_{it} \ge 0$ or by s^-_{it}, if $s_{it} < 0$, continuous variable;
x_{ikt}:	Number of tankers of type k mooring at harbor i in period t, integer variable;
y_{ijrkt}:	Number of tankers of type k departing from harbor i to harbor j on route r in period t, integer variable.

In the model, variables y_{ijrkt}, h_{ikt}, p_{ijt} and q_{ijrkt} are decision variables; x_{ikt} and s_{it} are state variables.

3. Mathematical model

Under the assumption that all tankers are fully loaded in transportation, a mixed integer programming model for the crude oil transportation problem (*COTP*) can be given as follows:

Model *COTP*:

$$
\begin{aligned}
MinC = & \sum_{t\in T}\sum_{i\in N}\sum_{k\in K}(CH_{ik}\times x_{ikt}) \\
& +\sum_{t\in T}\sum_{i\in N}\sum_{j\in N, j\neq i}\sum_{r\in R_{ij}}\sum_{k\in K}(F_{ijrk}\times y_{ijrkt}) \\
& +\sum_{t\in T}\sum_{i\in N}\sum_{j\in N, j\neq i}\sum_{r\in R_{ij}}\sum_{k\in K}\sum_{\tau\in L_{ijr}}(CO_k\times y_{ijrk(t-\tau)}) \\
& +\sum_{t\in T}\sum_{i\in N}\sum_{k\in K}(FH_{ik}\times h^+_{ikt}) \\
& +\sum_{t\in T}\sum_{i\in N}\sum_{j\in N, j\neq i}\sum_{r\in RC_{ij}}\sum_{k\in K}(CM_k\times y_{ijrkt}) \\
& +\sum_{t\in T}\sum_{(i,j)\in P}(CP_{ij}\times p_{ijt})+\sum_{t\in T}\sum_{i\in N}(CI_i\times s^+_{it}) \\
& +\sum_{t\in T}\sum_{i\in N}(CF_i\times s^-_{it}) \qquad (1)
\end{aligned}
$$

s.t.

$$
\begin{aligned}
x_{ik(t+1)} = & x_{ikt} - \sum_{j\in N, j\neq i}\sum_{r\in R_{ij}} y_{ijrkt} + \sum_{j\in N, j\neq i}\sum_{r\in R_{ji}} y_{jirk(t-l_{jir})} \\
& + h_{ikt} \quad \forall i\in N, k\in K, t\in T \qquad (2)
\end{aligned}
$$

$$s_{i(t+1)} = s_{it} + \sum_{j\in N, j\neq i}\sum_{r\in R_{ji}}\sum_{k\in K} q_{jirk(t-l_{jir})} - D_{it} \quad \forall i\in N_D, t\in T \qquad (3)$$

$$s_{i(t+1)} = s_{it} - \sum_{j\in PO_i} p_{ijt} + \sum_{j\in N, j\neq i}\sum_{r\in R_{ji}}\sum_{k\in K} q_{jirk(t-l_{jir})} \quad \forall i\in N_{PI}, t\in T \qquad (4)$$

$$s_{i(t+1)} = s_{it} + \sum_{j\in PI_i} p_{ji(t-pl_{ji})} - \sum_{j\in N, j\neq i}\sum_{r\in R_{ij}}\sum_{k\in K} q_{ijrkt} \quad \forall i\in N_{PO}, t\in T \qquad (5)$$

$$q_{ijrkt} = y_{ijrkt}\times CA_k, \quad \forall i\in(0\cup N_{PO}), j\in(N_{PI}\cup N_D), r\in R_{ij}, k\in K, t\in T \qquad (6)$$

$$0 \le p_{ijt} \le AP_{ij}, \quad \forall (i,j)\in P, t\in T \qquad (7)$$

$$LS_i \le s_{it} \le US_i, \quad \forall i\in N, t\in T \qquad (8)$$

$$y_{ijrkt} \ge 0 \text{ integer}, \quad \text{and} \quad y_{ijrkt} = 0 \text{ if } k\notin K_r \quad \forall i,j\in N, j\neq i, r\in R_{ij}, k\in K, t\in T \qquad (9)$$

$$h_{ikt} \text{ integer}, \quad \forall i\in N, k\in K, t\in T, \quad \text{and} \quad h_{ikt} = 0 \text{ if } i\neq 0 \qquad (10)$$

$$x_{ikt} \ge 0 \text{ integer}, \quad \forall i\in N, k\in K, t\in T \qquad (11)$$

$$p_{ijt} \ge 0, \quad \forall (i,j)\in P, t\in T \qquad (12)$$

$$q_{ijrkt} \ge 0, \quad \text{and} \quad q_{ijrkt}=0 \text{ if } (i,j)\notin(0\cup N_{PO})\times(N_{PI}\cup N_D) \quad \forall i,j\in N, j\neq i, r\in R_{ij}, k\in K, t\in T \qquad (13)$$

$$s_{it} \ge 0, \quad \forall i\in N_{PI}\cup N_{PO}, t\in T \qquad (14)$$

$$s^+_{it} \ge s_{it}, s^+_{it} \ge 0, \quad \forall i\in N, t\in T \qquad (15)$$

$$s^-_{it} \ge -s_{it}, s^-_{it} \ge 0, \quad \forall i\in N, t\in T \qquad (16)$$

$$h^+_{ikt} \ge h_{ikt}, h^+_{ikt} \ge 0 \text{ integer}, \ \forall i\in N, k\in K, t\in T, \quad \text{and} \quad h^+_{ikt} = 0 \text{ if } i\neq 0 \qquad (17)$$

$$h_{ikt}^{-} \geq -h_{ikt},\, h_{ikt}^{-} \geq 0 \text{ integer}, \quad \forall i \in N, k \in K, t \in T, \quad \text{and}$$
$$h_{ikt}^{-} = 0 \text{ if } i \neq 0 \tag{18}$$

In the model, all variables are continuous except for the variables whose types are explicitly indicated. The objective function (1) is to minimize the total logistics cost that consists of pierages at harbors, fixed usage costs of tankers departing from harbors, operating costs of moving tankers, rental costs of tankers, canal tolls, pipeline tolls, inventory holding costs and backlogging costs at customer harbors. Constraints (2) describe the tanker number balance at each harbor, taking account of tanker renting and returning. Constraints (3)–(5) are the inventory balance constraints for each customer harbor, the input port and the output port of each pipeline, respectively. Constraints (6) model the logical relationship between decision variables q_{ijrkt} and y_{ijrkt}, and ensure that the quantity of crude oil q_{ijrkt} can be transported by y_{ijrkt} fully loaded tankers. Constraints (7) and (8) are the flow capacity constraints of pipelines and the inventory level constraints for the depot at each harbor, respectively. The inventory/shortage level of crude oil at each harbor and the number of tankers of each type rented/returned at the supply center in each period are defined by Eqs. (15)–(18).

Note that: 1) for variable $y_{jirk(t-l_{jir})}$, if $t-l_{jir}<0$, it is initialized with a nonnegative integer number which represents the number of tankers of type k departed on route r from harbor j to harbor i before the considered planning horizon; similarly, for variable $p_{ji(t-pl_{ji})}$, if $t-pl_{ji}<0$, it is initialized with a nonnegative real number which represents the quantity of crude oil flowing through the pipeline from input port j to output port i before the considered planning horizon. 2) In the model, the values of variables h_{ikt}^{+}, h_{ikt}^{-} and s_{it}^{+}, s_{it}^{-} can be derived from the values of variables h_{ikt} and s_{it}, respectively. In fact, in an optimal solution of the model, the equations $h_{ikt}^{+}=\max(h_{ikt},0)$, $h_{ikt}^{-}=\max(-h_{ikt},0)$, $s_{it}^{+}=\max(s_{it},0)$, and $s_{it}^{-}=\max(-s_{it},0)$ hold. The introduction of these variables and constraints (15)–(18) in the above model is just for making it a linear MIP model. Because without these variables, the objective function (1) will contain nonlinear terms like $\max(h_{ikt},0)$ and $\max(s_{it},0)$. The constraints (15)–(18) ensure that in an optimal solution of the model, the equations always hold. For simplification, in the rest of this paper, in case of no confusion, the auxiliary variables h^{+}, h^{-} and s^{+}, s^{-} will be omitted in the model. The model *COTP* contains integer variables, continuous variables, and $|N|\times|K|\times|T|+5|N|\times|T|(\sum_{i\in(0\cup N_{PO})}\sum_{j\in(N_{PI}\cup N_D)}|R_{ij}|)\times|K|\times|T|+|P|\times|T|+2|K|\times|T|$ constraints.

4. Lagrangian relaxation approach

In this section, a Lagrangian relaxation (LR) approach is developed to find a near optimal solution of the crude oil transportation problem. In order to improve the performance of the LR approach, we first add some additional and valid constraints to the basic model *COTP*.

In the model, we first extend the definition of variables q_{ijrkt} to the returning routes of empty tankers, i.e., the routes from customer harbors or the input ports of pipelines to the supply center or the output ports of pipelines, namely, q_{ijrkt}, $\forall i\in(N_{PI}\cup N_D)$, $j\in(0\cup N_{PO})$, $r\in R_{ij}$, $k\in K$, $t\in T$, which are virtual variables. Each virtual variable q_{ijrkt} represents the total transportation capacity of y_{ijrkt} empty tankers, that is, $q_{ijrkt}=y_{ijrkt}\times CA_k$, $\forall i\in(N_{PI}\cup N_D)$, $j\in(0\cup N_{PO})$, $r\in R_{ij}$, $k\in K$, $t\in T$. As these virtual variables q_{ijrkt} do not appear in the objective function of the model and hence have no effect on its optimal solution. With this extension, the number of tankers of type k departing in period t on route r from harbor i to harbor j can always be represented by q_{ijrkt}/CA_k no matter whether the tankers are empty or not. With these virtual variables and from constraints (2), the following constraints are also valid for model *COTP*.

$$x_{ik(t+1)} = x_{ikt} - \sum_{j\in N, j\neq i}\sum_{r\in R_{ij}}\left(\frac{q_{ijrkt}}{CA_k}\right) + \sum_{j\in N, j\neq i}\sum_{r\in R_{ji}}\left(\frac{q_{jirk(t-l_{jir})}}{CA_k}\right) + h_{ikt} \quad \forall i\in N, k\in K, t\in T \tag{19}$$

From these constraints, the following constraints can be deduced:

$$x_{ik(t+1)} = x_{ik1} + \sum_{\theta=1}^{t} h_{ik\theta} - \sum_{\theta=1}^{t}\sum_{j\in N, j\neq i}\sum_{r\in R_{ij}}\left(\frac{q_{ijrk\theta}}{CA_k}\right) + \sum_{\theta=1}^{t}\sum_{j\in N, j\neq i}\sum_{r\in R_{ji}}\left(\frac{q_{jirk(\theta-l_{jir})}}{CA_k}\right) \quad \forall i\in N, k\in K, t\in T \tag{20}$$

Since $x_{ik(t+1)}\geq 0$, we have

$$\sum_{\theta=1}^{t}\sum_{j\in N, j\neq i}\sum_{r\in R_{ij}}\left(\frac{q_{ijrk\theta}}{CA_k}\right) - \sum_{\theta=1}^{t}\sum_{j\in N, j\neq i}\sum_{r\in R_{ji}}\left(\frac{q_{jirk(\theta-l_{jir})}}{CA_k}\right) \leq x_{ik1} + \sum_{\theta=1}^{t} h_{ik\theta} \quad \forall i\in N, k\in K, t\in T \tag{21}$$

where x_{ik1} is the number of tankers of type k in harbor i at the beginning of period 1, which is given. Let $\bar{h}_{ikt}$ be the maximal value of $\sum_{\theta=1}^{t} h_{ik\theta}$ obtained by solving the linear programming relaxation of model *COTP* but with the objective function $\text{Max}(\sum_{\theta=1}^{t} h_{ik\theta})$. The maximal value is finite since the inventory level of crude oil at each customer harbor or each input or output port of a pipeline is limited and unused tankers will be not rented. Then, the following constraints are valid for model *COTP*.

$$\sum_{\theta=1}^{t}\sum_{j\in N, j\neq i}\sum_{r\in R_{ij}}\left(\frac{q_{ijrk\theta}}{CA_k}\right) - \sum_{\theta=1}^{t}\sum_{j\in N, j\neq i}\sum_{r\in R_{ji}}\left(\frac{q_{jirk(\theta-l_{jir})}}{CA_k}\right) \leq x_{ik1} + \bar{h}_{ikt} \quad \forall i\in N, k\in K, t\in T \tag{22}$$

The model *COTP* can thus be reformulated by adding the constraints (22) above. This reformulation can improve the performance of the Lagrangian relaxation approach to be proposed in the next section.

4.1. Relaxation scheme

In model *COTP*, integer variables and real variables are coupled through constraints (6). Obviously, (6) can be relaxed by introducing a set of Lagrange multipliers and the model can then be decomposed into independent subproblems. However, as mentioned in Section 1, this kind of relaxation will lead to the Lagrangian relaxed problem containing little information about the optimal solution of the original problem and the quality of the feasible solution constructed based on the solution of the relaxed problem will be poor.

In order to have an effective relaxation scheme, constraints (6) are first equivalently reformulated as the following constraints:

$$\max\{q_{ijrkt} - CA_k\times y_{ijrkt},\, CA_k\times y_{ijrkt} - q_{ijrkt}\} \leq 0, \quad \forall i\in(0\cup N_{PO}), j\in(N_{PI}\cup N_D), r\in R_{ij}, k\in K, t\in T \tag{23}$$

Constraints (23) are then relaxed and incorporated into the objective function by introducing a set of nonnegative Lagrange multipliers $\{\lambda_{ijrkt},\, i\in(0\cup N_{PO}), j\in(N_{PI}\cup N_D), r\in R_{ij}, k\in K, t\in T\}$. The relaxed problem can then be formulated as follows:

$$\mathrm{Min}\, Z_\lambda(y,h,q,p) = C + \left\{ \sum_{t\in T}\sum_{i\in(0\cup N_{PO})}\sum_{j\in(N_{PI}\cup N_{PO})}\sum_{r\in R_{ij}}\sum_{k\in K}(\lambda_{ijrkt} \times \max\{q_{ijrkt} - CA_k \times y_{ijrkt}, CA_k \times y_{ijrkt} - q_{ijrkt}\}) \right\} \tag{24}$$

subject to constraints (2)–(5), (7)–(18) and (22), where C is the objective function of model *COTP*. Note that the values of state variables x and s can be derived from the values of the decision variables y, h, q and p, so they do not appear in the objective function Z_λ of the relaxed problem.

As the second term of (24) is a coupling term of variables q_{ijrkt} and y_{ijrkt}, the relaxed problem can not be decomposed into independent subproblems, which can be solved optimally. Instead, the relaxed problem is solved approximately by using a Gauss–Seidel like method. Let $IR(\hat{y},\hat{h},q,p)$ and $NF(y,h,\hat{q},\hat{p})$ denote two subproblems derived from the relaxed problem, by fixing (y,h) to the given value ($\hat{y}$, $\hat{h}$), and fixing (q,p) to the given value ($\hat{q}$, $\hat{p}$), respectively. If (y,h) is given as ($\hat{y}$, $\hat{h}$), by solving $IR(\hat{y},\hat{h},q,p)$, the variables q and p can be determined optimally; similarly, if (q,p) is given as ($\hat{q}$, $\hat{p}$), by solving $NF(y,h,\hat{q},\hat{p})$), the variables y and h can be determined optimally. The two subproblems IR and NF are solved alternatively with the solution of NF used as ($\hat{y}$, $\hat{h}$) of IR and the solution of IR used as ($\hat{q}$, $\hat{p}$) of NF. The alternative solution procedure stops if the solutions of both IR and NF do not change. In this way, the relaxed problem is approximately solved. For the corresponding Lagrangian dual problem, it can be solved by using the surrogate subgradient method proposed by Zhao et al. (1999), which ensures that its solution converges to the optimal Lagrange multipliers of the dual problem under some conditions even if the relaxed problem is approximately solved.

4.2. Resolution of the subproblems

4.2.1. Subproblem IR

The subproblem *IR* of the relaxed problem with given values $\hat{y}$ and $\hat{h}$ of variables y and h, can be formulated as follows:

$$\text{Submodel}\quad IR(\hat{y},\hat{h},q,p):\quad \mathrm{Min}\, Z^1_\lambda(\hat{y},\hat{h},q,p) = \sum_{t\in T}\sum_{(i,j)\in P}(CP_{ij}\times p_{ijt}) + \sum_{t\in T}\sum_{i\in N}(CI_i \times s^+_{it}) + \sum_{t\in T}\sum_{i\in N}(CF_i\times s^-_{it}) + \sum_{t\in T}\sum_{i\in(0\cup N_{PO})}\sum_{j\in(N_{PI}\cup N_{PO})}\sum_{r\in R_{ij}}\sum_{k\in K}(\lambda_{ijrkt}\times \max\{q_{ijrkt} - CA_k\times\hat{y}_{ijrkt}, CA_k\times \hat{y}_{ijrkt} - q_{ijrkt}\}) \tag{25}$$

subject to constraints (3)–(5), (7) and (8), (12)–(16) and (22), where $Z^1_\lambda(\hat{y},\hat{h},q,p)$ is obtained from $Z_\lambda(y,h,q,p)$ by replacing y_{ijrkt} with $\hat{y}_{ijrkt}$ and removing all terms that do not depend on the decision variables q and p. Note that $Z^1_\lambda(\hat{y},\hat{h},q,p)$ does not depend on $\hat{h}$. The retaining of $\hat{h}$ in objective function Z^1_λ of subproblem IR is only for facilitating the understanding of the subproblem.

The subproblem IR is to determine the total quantity transported by tankers of each type on each route in each period over a given planning horizon to minimize the sum of the inventory holding cost and the backlogging cost at each customer harbor, the pipeline toll when crude oil is transported through pipelines and the Lagrange penalty, subject to the inventory balance constraints for the depot at each customer harbor or the depot at the input or output port of pipelines ((3)–(5)), the flow capacity constraints of pipelines and the inventory capacity constraints for each depot ((7) and (8)). Because of the existence of nonlinear term $\max(q_{ijrkt} - CA_k\times\hat{y}_{ijrkt}, CA_k\times\hat{y}_{ijrkt} - q_{ijrkt})$, the subproblem IR is a nonlinear programming model, but it can be transformed into a linear programming model in the following way: replace $\max(q_{ijrkt} - CA_k\times\hat{y}_{ijrkt}, CA_k\times\hat{y}_{ijrkt} - q_{ijrkt})$ by ε_{ijrkt} and add constraints $\varepsilon_{ijrkt} \geq q_{ijrkt} - CA_k\times\hat{y}_{ijrkt}$ and $\varepsilon_{ijrkt} \geq CA_k\times\hat{y}_{ijrkt} - q_{ijrkt}$. Since the Lagrange multipliers $\lambda_{ijrkt} \geq 0$ (deduced from (23)), this replacement is equivalent. The subproblem IR can then be solved by using the simplex method implemented in a commercial optimization software, such as LINGO, CPLEX.

4.2.2. Subproblem NF

The subproblem NF of the relaxed problem with given values $\hat{q}$ and $\hat{p}$ of variables q and p, can be formulated as:

$$\text{Submodel}\quad NF(y,h,\hat{q},\hat{p}):$$

$$\mathrm{Min}\, Z^2_\lambda(y,h,\hat{q},\hat{p}) = \sum_{t\in T}\sum_{i\in N}\sum_{k\in K}(CH_{ik}\times x_{ikt}) + \sum_{t\in T}\sum_{i\in N}\sum_{j\in N, j\neq i}\sum_{r\in R_{ij}}\sum_{k\in K}(F_{ijrk}\times y_{ijrkt}) + \sum_{t\in T}\sum_{i\in N}\sum_{j\in N, j\neq i}\sum_{r\in R_{ij}}\sum_{k\in K}\sum_{\tau\in L_{ijr}}(CO_k\times y_{ijrk(t-\tau)}) + \sum_{t\in T}\sum_{i\in N}\sum_{k\in K}(FH_{ik}\times h^+_{ikt}) + \sum_{t\in T}\sum_{i\in N}\sum_{j\in N, j\neq i}\sum_{r\in RC_{ij}}\sum_{k\in K}(CM_k\times y_{ijrkt}) + \sum_{t\in T}\sum_{i\in(0\cup N_{PO})}\sum_{j\in(N_{PI}\cup N_{PO})}\sum_{r\in R_{ij}}\sum_{k\in K}\lambda_{ijrkt}\times\max\{\hat{q}_{ijrkt} - CA_k\times y_{ijrkt}, CA_k\times y_{ijrkt} - \hat{q}_{ijrkt}\} \tag{26}$$

subject to constraints (2), (9)–(11) and (17) and (18), where $Z^2_\lambda(y,h,\hat{q},\hat{p})$ is obtained from $Z_\lambda(y,h,q,p)$ by replacing q_{ijrkt} with $\hat{q}_{ijrkt}$ and removing all terms that do not depend on the decision variables y and h. Note that $Z^2_\lambda(y,h,\hat{q},\hat{p})$ does not depend on $\hat{p}$. The retaining of $\hat{p}$ in objective function Z^2_λ of subproblem NF is only for facilitating the understanding of the subproblem.

The problem NF is to determine the dispatching schedule of tankers for every route and the number of tankers rented or returned at the supply center at the minimal total cost including the pierages at all harbors, fixed usage costs and operating costs of moving tankers, rental costs of tankers and canal tolls. Constraints (2) are the tanker number balance equation at each harbor.

This subproblem can be represented graphically as shown in Fig. 2. In the graph, each harbor in each period is represented by a node, each possible route with given departure harbor, arrival harbor, departure period, and arrival period is represented by an arc connecting two nodes: one node represents the departure harbor in the departure period and the other represents the arrival harbor in the arrival period; the tankers (including empty tankers and fully loaded tankers) departing in each period on each route (namely decision variables y_{ijrkt}) can be represented by the discrete flow associated with the corresponding arc, and the tankers mooring at each harbor in each period (namely state variables x_{ikt}) can be represented by the discrete flow associated with the arc from the node representing the harbor in one period to the node representing the same harbor in the next period. In addition, for each period, a virtual node which is only connected with the node representing the supply center in the period is introduced to represent the renting and the returning of tankers at the supply center in the period.

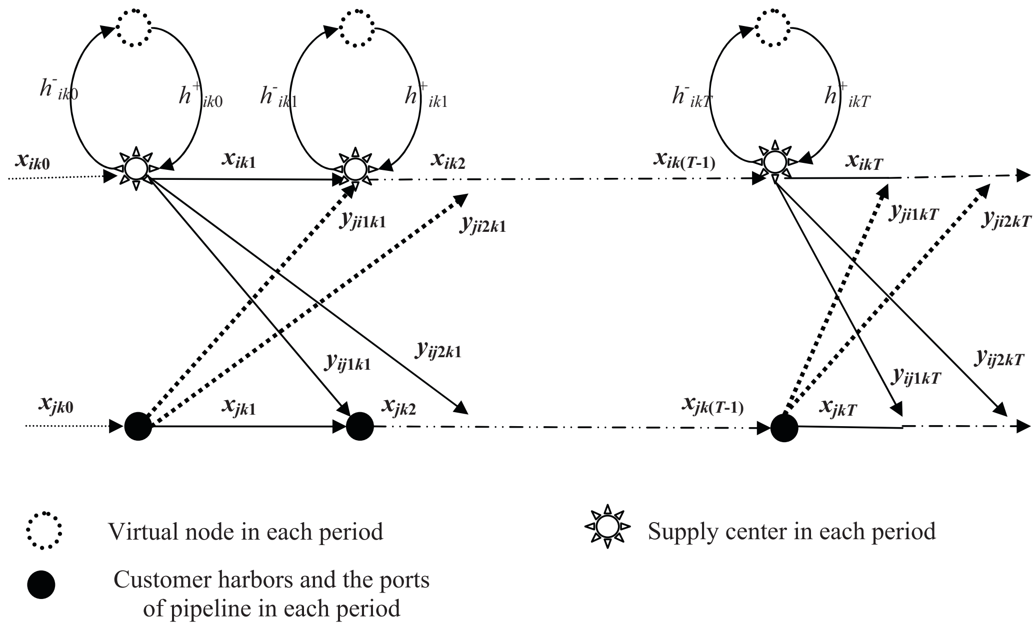

Fig. 2. The graphical representation of the subproblem NF.

According to this graphical interpretation, the constraints (2), (17) and (18) of NF can be reformulated as follows:

$$x_{ikt}+\sum_{j=0,j\neq i}^{N}\sum_{r=1}^{R}y_{jirk(t-l_{jir})}+h_{ikt}^{+}=x_{ik(t+1)}+\sum_{j=0,j\neq i}^{N}\sum_{r=1}^{R}y_{ijrkt}+h_{ikt}^{-}\quad \forall i\in N,k\in K,t\in T \tag{27}$$

$$y_{ijrkt}\geq 0,\ \text{integer},\quad \forall i,j\in N,j\neq i,r\in R_{ij},k\in K,t\in T,$$
$$x_{ikt},h_{ikt}^{+},h_{ikt}^{-}\geq 0,\ \text{integer},\quad \forall i\in N,k\in K,t\in T \tag{28}$$

in which $h_{ikt}^{+}=h_{ikt}^{-}=0$ if $i\neq 0$, i.e., harbor i is not the supply center.

Then, the subproblem NF is a network flow model if we ignore the nonlinear term $\max\{\widehat{q}_{ijrkt}-CA_k\times y_{ijrkt},CA_k\times y_{ijrkt}-\widehat{q}_{ijrkt}\}$ in its objective function. This term can be linearized in a way similar to that for subproblem IR: replace the term by ε_{ijrkt} and add constraints:

$$\varepsilon_{ijrkt}\geq \widehat{q}_{ijrkt}-CA_k\times y_{ijrkt} \tag{29}$$

$$\varepsilon_{ijrkt}\geq CA_k\times y_{ijrkt}-\widehat{q}_{ijrkt} \tag{30}$$

From (29) we can deduce the equation $y_{ijrkt}\geq(\widehat{q}_{ijrkt}/CA_k)-(\varepsilon_{ijrkt}/CA_k)$. Inspired by the integrality property of the classical network flow model, we want to solve the subproblem NF as a linear programming problem. In order to do so, we round $\widehat{q}_{ijrkt}/CA_k$ down to its nearest integer $\left\lfloor\frac{\widehat{q}_{ijrkt}}{CA_k}\right\rfloor$ and replace ε_{ijrkt}/CA_k by ε'_{ijrkt}, the constraint (29) becomes

$$y_{ijrkt}+\varepsilon'_{ijrkt}\geq\left\lfloor\frac{\widehat{q}_{ijrkt}}{CA_k}\right\rfloor \tag{31}$$

Similarly, we round $\widehat{q}_{ijrkt}/CA_k$ up to its nearest integer $\left\lceil\frac{\widehat{q}_{ijrkt}}{CA_k}\right\rceil$ and replace ε_{ijrkt}/CA_k by ε'_{ijrkt}, the constraint (30) becomes

$$y_{ijrkt}-\varepsilon'_{ijrkt}\leq\left\lceil\frac{\widehat{q}_{ijrkt}}{CA_k}\right\rceil \tag{32}$$

Although our sub model NF is not the classical network flow model, we still solve it as a linear programming model by using the simplex method after the replacement of constraints (29) and (30) by (31) and (32), respectively. In our numerical experiments, for all randomly generated instances, the solution of the sub model obtained in this way is always integer. However, mathematically we cannot prove that the solution is always integer. If this situation happens, as our Lagrangian relaxation algorithm only requires an approximate resolution of the sub model NF, the noninteger solution can be modified to get a near-optimal feasible (integer) solution of the sub model. Suppose that the noninteger solution is $(y^*,h^{+*},h^{-*},x^*,\varepsilon'^*)$. In the modification, we first ignore the constraints (31) and (32) and solve the sub model NF with additional constraints $y_{ijrkt}\geq y^*_{ijrkt}$, $h^+_{ikt}\geq h^{+*}_{ikt}$, $h^-_{ikt}\geq h^{-*}_{ikt}$ and $x_{ikt}\geq x^*_{ikt}$ to get an integer solution $(\bar{y},\bar{h}^+,\bar{h}^-,\bar{x},\bar{\varepsilon}')$. Note that the new model solved is always unimodular, so the existence of such an integer solution is guaranteed. We then modify $\bar{\varepsilon}'$ to $\bar{\varepsilon}'=\{\bar{\varepsilon}'_{ijrkt}\}$ with $\bar{\varepsilon}'_{ijrkt}=\max\left(\left\lfloor\frac{\widehat{q}_{ijrkt}}{CA_k}\right\rfloor-\bar{y}_{ijrkt},\bar{y}_{ijrkt}-\left\lfloor\frac{\widehat{q}_{ijrkt}}{CA_k}\right\rfloor\right)$ to get an integer solution which also satisfies the constraints (31) and (32).

4.3. Resolution of the Lagrangian dual problem

Let $D(\lambda)$ be the objective value of the Lagrangian dual problem, where λ is the vector of Lagrange multipliers, the dual problem can be formulated as:

$$\max_{\lambda\geq 0}D(\lambda)$$

where $D(\lambda)=\min Z_\lambda(y,\ h,\ q,\ p)$ subject to constraints: (2)–(5), (7)–(18) and (22).

As the relaxed problem in Section 4.1 is only approximately solved, the dual problem cannot be solved by using the classical subgradient method which requires the relaxed problem to be solved optimally. Instead, it is solved by using the surrogate subgradient (SSG) method proposed by Zhao et al. (1999). The surrogate subgradient method ensures that its solution converges to the optimal multipliers of the dual problem even if the relaxed problem is approximately solved under some conditions. To present the method, we first introduce some notations:

The notations used in the subgradient method:

- D^* The optimal objective value of the dual problem
- D^m The surrogate dual at iteration m, $D^m = D(\lambda^m) = Z_{\lambda^m}(y^m, h^m, q^m, p^m)$
- $D^{\lceil m \rceil}$ The best surrogate dual obtained prior to iteration m
- g^m The surrogate subgradient at iteration m, $g(q^m, y^m) = \{g^m_{ijrkt}\}$, $g^m_{ijrkt} = \max\{q^m_{ijrkt} - CA_k \times y^m_{ijrkt}, CA_k \times y^m_{ijrkt} - q^m_{ijrkt}\}$
- m Index of Lagrangian iteration
- s^m The step size at iteration m
- $Z_\lambda(y, h, q, p)$ The objective function of the Lagrangian relaxed problem for a given multiplier vector λ
- $Z^1_\lambda(\widehat{y}, \widehat{h}, q, p)$ The objective function of the subproblem IR for a given multiplier vector λ and given values $\widehat{y}$ and $\widehat{h}$ of variables y and h
- $Z^2_\lambda(y, h, \widehat{p}, \widehat{q})$ The objective function of the subproblem NF for a given multiplier vector λ and given values $\widehat{q}$ and $\widehat{p}$ of variables q and p
- λ^m Lagrange multipliers at iteration m

The algorithm procedure of the SSG method with adaptive step sizing scheme can now be given as follows:

Step 1. Initialization: $m = 0$; $\{\lambda^0\} = 0$; as $\lambda^0 = 0$, the coupling term of variables y and q does not appear in subproblems IR and NF in the first iteration, the subproblems can then be solved independently to initialize the values y^0, h^0, q^0 and p^0.

Step 2. Solve the subproblem $IR_{\lambda^m}(y^{m-1}, h^{m-1}, q, p)$ to determine the values q^m and p^m; then solve the subproblem $NF_{\lambda^m}(y, h, q^m, p^m)$ to determine the values y^m and h^m.

Step 3. Set step size: $s^m = \beta^*(D^* - D^m)/||g^m||^2$, where parameter β is a scalar with $0 < \beta < 1$; D^* is estimated by $(1 + w/\theta^\rho) \times D^{\lceil m \rceil}$, where ω, ρ and θ are three parameters, in which ω is taken from interval [0.1,1.0]; ρ is taken from interval [1.1,1.5]; θ is an integer parameter and is initialized with value 1; it is updated in the following way: at iteration m, if $D^m > D^{\lceil m \rceil}$, then $\theta = \max(1, \theta - 1)$, otherwise $\theta = \theta + 1$.

Step 4. Update the Lagrange multipliers at iteration m: $\lambda^{m+1} = \max(0, \lambda^m + s^m \times g^m)$.

Step 5. Check the stopping criterions: (1) a given maximal iteration number is reached or (2) no improvement of D^m is observed for a given number of iterations. If one of the two criterions is met, the procedure is stopped; otherwise $m = m + 1$ and go to step 1.

The convergence conditions for the SSG (Zhao et al., 1999) are:

$$Z_{\lambda^0}(y^0, h^0, q^0, p^0) < D^*;$$

$$Z_{\lambda^m}(y^m, h^m, q^m, p^m) < Z_{\lambda^m}(y^{m-1}, h^{m-1}, q^{m-1}, p^{m-1})$$

where (y^m, h^m, q^m, p^m) is the solution of the relaxed problem at multiplier vector λ^m. It is easy to show that the first condition always holds since (y^0, h^0, q^0, p^0) is the optimal solution of the relaxed problem at $\lambda = \lambda^0 = 0$. For the second condition, it holds if we replace '<' by '≤' in the inequality, this is because, at iteration m of the SSG, if (y,h) is fixed to the given value (y^{m-1},h^{m-1}), the optimal value (q^m,p^m) of (q,p) can be obtained by solving $IR_{\lambda^m}(y^{m-1}, h^{m-1}, q, p)$, since the objective function $Z_{\lambda^m}(y^{m-1}, h^{m-1}, q, p)$ is $Z^1_{\lambda^m}(y^{m-1}, h^{m-1}, q, p)$ of IR_{λ^m} plus a constant term that does not depend on q and p, and $Z^1_{\lambda^m}(y^{m-1}, h^{m-1}, q^m, p^m) \le Z^1_{\lambda^m}(y^{m-1}, h^{m-1}, q^{m-1}, p^{m-1})$, we have $Z_{\lambda^m}(y^{m-1}, h^{m-1}, q^m, p^m) \le Z_{\lambda^m}(y^{m-1}, h^{m-1}, q^{m-1}, p^{m-1})$; similarly, if (q,p) is fixed to the given value (q^m,p^m), the optimal value (y^m,h^m) of (y,h) can be obtained by solving $NF_{\lambda^m}(y, h, q^m, p^m)$, since the objective function $Z_{\lambda^m}(y, h, q^m, p^m)$ is $Z^2_{\lambda^m}(y, h, q^m, p^m)$ of NF_{λ^m} plus a constant term that does not depend on y and h, and $Z^2_{\lambda^m}(y^m, h^m, q^m, p^m) \le Z^2_{\lambda^m}(y^{m-1}, h^{m-1}, q^m, p^m)$, we have $Z_{\lambda^m}(y^m, h^m, q^m, p^m) \le Z_{\lambda^m}(y^{m-1}, h^{m-1}, q^m, p^m)$, hence we have $Z_{\lambda^m}(y^m, h^m, q^m, p^m) \le Z_{\lambda^m}(y^{m-1}, h^{m-1}, q^{m-1}, p^{m-1})$. The case '=' rarely happens. If it happens, the relaxed sub-problems IR and NF can be solved once again with the updated values of $\{q,p\}$ and $\{y,h\}$ obtained from the resolution of the subproblems in the last time to improve the current solution until the second condition holds.

Note that the surrogate subgradient dual is not a strict Lagrangian dual; its value may exceed the minimum objective value of the original problem, so the best surrogate subgradient dual $D^{\lceil m \rceil}$ obtained by the SSG may not be a lower bound for the original problem.

5. Construction and evaluation of feasible solutions

The best feasible solution of the original problem is constructed based on the solutions of the relaxed problem of the Lagrangian relaxation approach in the last several iterations or in all iterations. For simplification but without loss of generality, the algorithm for constructing a feasible solution is explained for only one iteration.

5.1. Construction of a feasible solution

Let us denote the feasible solution of the original problem to be constructed by (y',h',x',q',p',s'), where x' and s' are derived from (y',h') and (q',p') according to Eqs. (2) and (3)–(5), respectively. At each iteration of Lagrangian relaxation approach, suppose that the solution of the subproblem IR is given by (q,p,s) and the solution of the subproblem NF is given by (y,h,x), where s and x are derived from (q,p) and (y,h) according to Eqs. (3)–(5) and (2), respectively. If we put the two solutions together to form a solution (y,h,x,q,p,s), the obtained solution may be not feasible to the original problem because the capacity constraints (6) of the model *COTP* may be violated. However, if we set $y' = y$, $h' = h$, $x' = x$ and $p' = p$, according to the constraints (6), the value q' in the feasible solution must be set as $q'_{ijrkt} = y'_{ijrkt} \times CA_k$; similarly, the value s' in the feasible solution must be calculated according to the constraints (3)–(5) of the model based on the values q' and p'.

However, some values s'_{it} calculated in this way may violate the inventory upper bound and lower bound of a customer harbor or the input or output port of a pipeline; we have to modify the solution (y',h',x',q',p',s') to make it feasible to the original problem. Let US_i and LS_i denote the inventory upper bound and lower bound of the harbor (or the port) i, respectively. We successively examine the violation of the inventory bounds of each harbor from the first period to the final period. Suppose that the harbor and the period under consideration are harbor i and period t, respectively, three cases may happen: Case 1: $LS_i \le s'_{it} \le US_i$; Case 2: $s'_{it} < LS_i$; Case 3: $s'_{it} > US_i$. For case 1, we do nothing. For case 2, the inventory shortage quantity of harbor i in period t, denoted by $\Delta^- s_{it}$, is $LS_i - s'_{it}$. For this case, the number of tankers dispatched from harbor j to harbor i in period $t - l_{jir} - 1$ on route r must be increased by $\Delta y_{jirk(t-l_{jir}-1)}$, where $\Delta y_{jirk(t-l_{jir}-1)}$ is the minimal integer number satisfying the equation $\Delta y_{jirk(t-l_{jir}-1)} \times CA_k \ge \Delta^- s_{it}$. Accordingly, the number of tankers of type k mooring at harbor j in each period $t' \in [t - l_{jir}, T]$, $x'_{jkt'}$ must be reduced by $\Delta y_{jirk(t-l_{jir}-1)}$. Note that if $\Delta y_{jirk(t-l_{jir}-1)}$ exceeds the minimal number of tankers mooring at harbor j during the periods belonging to $[t - l_{jir}, T]$, denoted by $x'_{jkt''}$, then $\Delta y_{jirk(t-l_{jir}-1)} - x'_{jkt''}$ tankers have to be rented from the third-party and dispatched immediately on route r in period $t - l_{jir} - 1$ at harbor j. For case 3, the inventory surplus quantity of harbor i, denoted by $\Delta^+ s_{it}$, is $s'_{it} - US_i$. In this case, the number of tankers dispatched from harbor j to harbor i in period $t - l_{jir} - 1$ on route r must be reduced by $\Delta y_{jirk(t-l_{jir}-1)}$, where

$\Delta y_{jirk(t-ljir-1)}$ is the minimal integer number satisfying the equation $\Delta y_{jirk(t-ljir-1)} \times CA_k \geq \Delta^+ s_{it}$. Accordingly, the number of tankers of type k mooring at harbor j in each period belonging to $[t - l_{jir}, T]$ must be increased by $\Delta y_{jirk(t-ljir-1)}$. When all values s'_{it} violating the inventory bounds are repaired by the above procedure, the feasible solution (y',h',x',q',p',s') of the original problem is obtained.

The above heuristic for the construction of a feasible solution is based on the observation that the transportation costs of tankers are usually higher than the inventory holding costs and backlogging costs at customer harbors, so using the information on tanker dispatching schedule provided by the solution of subproblem NF may lead to a good feasible solution of the original problem.

5.2. Evaluation of feasible solutions

In order to evaluate the performance of a feasible solution found by our algorithm, a lower bound for the objective value of model *COTP* is required. As the best surrogate dual value obtained by the SSG method may not be a lower bound of the original problem, we use the optimal objective value of the linear relaxation of model *COTP* as the lower bound to evaluate the quality of the solution.

6. Lagrangian relaxation approach to a variant of the crude oil transportation problem

In the above study, it is assumed that all tankers are fully loaded in transportation. However, in reality, if the demand of a customer harbor in some period or the storage capacity of the depot at the harbor is smaller than the capacity of a tanker, or the unit inventory holding cost at the depot is higher than the unit transportation cost from the supply center to the depot, it is possible that fully loaded transportation is not the best choice. In this case, a combination of fully loaded transportation and partially loaded transportation may be better. For this reason, we generate model *COTP* to allow also the transportation of crude oil by partially loaded tankers. The generalized model, denoted by *COTPV*, where *COTP* stands for crude oil transportation problem and *V* stands for Variant. The model is a variant of model *COTP*. The model *COTPV* is similar to the basic model *COTP* in Section 3 except that constraints (6) are replaced by the constraints (33) below.

$$q_{ijrkt} \leq CA_k \times y_{ijrkt}, \quad \forall i \in (0 \cup N_{PO}), j \in (N_{PI} \cup N_D), r \in R_{ij}, k \in K, t \in T \tag{33}$$

To solve the new model by using a Lagrangian relaxation approach similar to that for model *COTP*, constraints (33) are first reformulated as:

$$\max\{0, q_{ijrkt} - CA_k \times y_{ijrkt}\} \leq 0, \quad \forall i \in (0 \cup N_{PO}),\ j \in (N_{PI} \cup N_D),\ r \in R_{ij},\ k \in K,\ t \in T \tag{34}$$

The constraints (34) are then relaxed and incorporated into the objective function by introducing a set of nonnegative Lagrange multipliers $\{\lambda_{ijrkt},\ i \in (0 \cup N_{PO}),\ j \in (N_{PI} \cup N_D),\ r \in R_{ij},\ k \in K,\ t \in T\}$. The relaxed problem of model *COTPV* and its two subproblems (denoted by VIR and VNF, respectively) are similar to those for model *COTP* and can be solved by using the same algorithms for subproblems IR and NF, respectively. For the corresponding dual problem, it can also be solved by using the surrogate subgradient method, but the surrogate subgradient is now given by $g(q,y) = \{g_{ijrkt}\}$, $g_{ijrkt} = \max\{0, q_{ijrkt} - CA_k \times y_{ijrkt}\}$.

The construction of a feasible solution for model *COTPV* based on the solution of its relaxed problem is similar to that for model *COTP* but more straightforward.

Using the same notation presented in Section 5.1, at any iteration of the Lagrangian relaxation approach, suppose that the solution of subproblem VIR is given by (q,p,s) and the solution of subproblem VNF is given by (y,h,x). Similarly, we cannot directly put the two solutions together to form a feasible solution (y,h,x,q,p,s) of the original problem, because the solution may violate some capacity constraints of (33). We propose a new heuristic to construct a feasible solution (y',h',x',q',p',s') of the original problem. Firstly, we set $y' = y$, $h' = h$ and $x' = x$; the values q', p' and s' in the feasible solution can then be obtained by solving a linear programming model $LP(y',q,p,s)$ defined as follows:

$$LP(y', q, p, s): \quad \text{Min}\, C' = \sum_{t \in T} \sum_{(i,j) \in P} (CP_{ij} \times p_{ijt}) + \sum_{t \in T} \sum_{i \in N} (CI_i \times s^+_{it}) + \sum_{t \in T} \sum_{i \in N} (CF_i \times s^-_{it}) \tag{35}$$

Subject to constraints (3)–(5), (7) and (8) and (12)–(16), and constraints (36) given as follows:

$$q_{ijrkt} \leq CA_k \times y'_{ijrkt}, \quad \forall i \in (0 \cup N_{PO}), j \in (N_{PI} \cup N_D), r \in R_{ij}, k \in K, t \in T \tag{36}$$

The linear programming depends on y'.

In model $LP(y',q,p,s)$, the constraints (36) imply that in each period and on each route, the total quantity of crude oil transported by tankers of type k cannot exceed the total transportation capacity of the tankers. Since the inventory shortage at each customer harbor is allowed, the model always has a solution.

Suppose that the optimal solution of $LP(y',q,p,s)$ is (q^*,p^*,s^*). As q^* and y' satisfy constraints (33), a feasible solution (y',h',x',q',p',s') of the original problem can be constructed by setting $q' = q^*$, $p' = p^*$, $s' = s^*$.

The main feature of the heuristic is that the information of the optimal solution of subproblem VNF is kept in the construction of a feasible solution of model *COTPV*.

Both models *COTP* and *COTPV* are solved in a rolling horizon way. One issue for the rolling horizon based planning is the determination of the ending inventory level at each stocking location (the depot at each customer harbor). In practice, as how long the planning horizon must be taken to make first period decisions optimal is typically unknown, the ending inventory level could be either given or treated as a decision variable. The readers can refer to a review paper of Van Den Heuvel and Wagelmans (2005) for this issue.

For the crude oil transportation problem studied in this paper, if only fully loaded tankers are permitted in transportation, i.e., if model *COTP* is considered, the ending inventory level should be taken as a decision variable, this is because, if the inventory level is set to a fixed value, the problem may be infeasible. However, if both fully loaded and partially loaded tankers are permitted in transportation as considered by model *COTPV*, the ending inventory level can be taken either as a decision variable or a given value.

7. Numerical results

In this section, we first compare the performance of our Lagrangian relaxation approach for model *COTP* with that of the *GRASP/PR* approach proposed in our previous work on large scale instances of the crude oil transportation problem. Ten instances with a planning horizon of 12 months (each month corresponds to a period) are randomly generated based on a crude oil transportation problem in real life. Because only fully loaded tankers are permitted in the model, the ending inventory level s_{iT} for each customer harbor is taken as a decision variable. The performance of our LR approach for model *COTPV* which allows partially loaded tankers in crude oil transportation is also evaluated on a set of ten instances

Table 1
Numerical results for the instances of model *COTP*.

Instances	$Cost_{LP}$	*LR*					*GRASP/PR*		*LINGO*	
		$Cost_{LR}$	Gap (%)	Time (s)	Imp1 (%)	Imp2 (%)	$Cost_{GP}$	Time (s)	$Cost_{LG}$	Time (s)
1	55734	59842	7.37	3383	14.53	2.19	70015	6673	61154	3600
2	59688	63783	6.86	3363	15.41	3.23	75402	5295	65848	3600
3	59973	64951	8.3	3370	6.72	1.97	69630	5146	66237	3600
4	61357	65683	7.05	3361	6.74	5.59	70430	3937	69358	3600
5	58091	61856	6.48	3411	10.94	4.78	69454	5225	64818	3600
6	58323	62913	7.87	3442	9.55	4.44	69556	5363	65712	3600
7	60642	65336	7.74	3379	10.42	5.78	72936	5218	69116	3600
8	58289	63173	8.38	2512	3.62	4.95	65546	4969	66301	3600
9	54488	58608	7.56	3366	16.99	9.19	70603	2296	63999	3600
10	52003	56283	8.23	2542	13.87	6.81	65346	4176	60121	3600
Average			7.58	3213	10.91	4.89		4829		3600

generated randomly with a planning horizon of 12 months. However, for this model, two cases are considered in the setting of its ending inventory level s_{iT}: Case 1: s_{iT} is taken as a decision variable; Case 2: s_{iT} is set to zero.

To illustrate the applicability of our model and algorithm to real problems, the parameters of the instances mentioned above are devised based on a case of crude oil transportation of a Chinese oil company and are set in the following way: the number of customer harbors is taken as 10 for each instance of both sets, all instances have one pipeline, one canal, and two types of tanker tonnages (fifty thousand tons and hundred thousand tons). For any $i \in (0 \cup N_{PO})$, $j \in (N_{PI} \cup N_D)$ and any $j \in (0 \cup N_{PO})$, $i \in (N_{PI} \cup N_D)$, R_{ij} contains two routes with transportation lead time 1 month and 2 months, respectively. Other parameters are generated randomly and uniformly from some intervals: $CH_{ik} \in [5.6,21.6]$, $F_{irk} \in [8,44]$, $CO_k \in [8,24]$, $FH_{ik} \in [16,96]$, $CM_k \in [0.8,2.4]$, $CP_{ij} \in [0.4,0.6]$, $CI_i \in [0.8,1.2]$, $CF_i \in [6.4,14.4]$, $D_{it} \in [40,100]$, $CA_k = 5$ or 10, $AP_{ij} \in [36,44]$, $LS_i = 50$ for all demand harbors and 0 for the output and input port of the pipeline, $US_i = 80$ for all demand harbors and $US_i \in [45,55]$ for the output and input port of the pipeline. The parameters of the SSG method β, ω and ρ are taken as 0.5, 0.5 and 1.3, respectively, and the number of iterations of our Lagrangian relaxation algorithm is set to 150.

Our algorithm is coded in C++ using a callable library of Lingo8.0, and the numerical test is performed on a Pentium IV 1.6 GHz PC with 512 MB RAM.

7.1. Results for model COTP

As mentioned above, by using our LR algorithm, the relaxed problem is only solved approximately and the best Lagrange dual value obtained by the SSG method may be not a lower bound of the original problem. Moreover, for each testing instance, as it contains 2496 integer variables, 1536 real variables and 2208 constraints, its optimal cost is too time-consuming to be obtained by using a branch and bound algorithm of Lingo8.0. We attempted to solve a testing instance by the MIP solver of Lingo8.0, but it could not find its optimal solution within a running time of 1 day. For this reason, we compare the best feasible solution obtained by our algorithm with that obtained by Lingo8.0 for a running time of 1 h for each instance. Note that for all instances, the computation time of our algorithm is less than 1 h.

For the reasons above, the optimal cost of the linear programming relaxation model of each instance calculated by using the simplex algorithm in Lingo8.0 is taken as the lower bound of the instance; the cost of the best solution obtained by our algorithm is taken as an upper bound and is compared with the LP lower bound, the best cost obtained by the *GRASP/PR* approach, and the best cost obtained by the MIP solver of Lingo8.0 for a running time of 1 h, respectively.

The numerical results are given in Table 1 below, where $Cost_{LP}$ represents the optimal cost of the linear programming relaxation of model *COTP*, $Cost_{LR}$ represents the best cost obtained by our LR algorithm, $Cost_{GP}$ represents the best cost obtained by the *GRASP/PR* approach, $Cost_{LG}$ represents the best cost obtained by the MIP solver of LINGO, Gap is defined as $(Cost_{LR} - Cost_{LP})/Cost_{LP} \times 100\%$, Imp1 is defined as $(Cost_{GP} - Cost_{LR})/Cost_{GP} \times 100\%$, Imp2 is defined as $(Cost_{LG} - Cost_{LR})/Cost_{LG} \times 100\%$, and Time is the computation time in seconds.

From Table 1, we can observe that:

(1) Our algorithm is effective in solving the randomly generated instances of model *COTP*. The average gap with linear relaxation for all instances is 7.58%, and the minimal gap is 6.48%. Note that the lower bound provided by the linear relaxation of the model *COTP* may be poor, a tighter lower bound may be found to reduce the gap.
(2) The performance of our LR algorithm in solving these instances is much better than that of *GRASP/PR*. The newly developed algorithm can improve the solutions obtained by *GRASP/PR* by 3.62–16.99% with the average improvement 10.91% in terms of cost at two-third running time of *GRASP/PR* on average. One possible reason for this result is that the LR approach is an optimization-based approach which uses some global information in searching a near-optimal solution of the model *COTP*. On the other hand, the *GRASP/PR* approach uses local information when generating initial solutions and improving the solutions by local search. Although a path-relinking strategy is used to diversify the search in the approach, its implementation is quite time consuming, which prevents the approach from finding a solution better than that found by the LR algorithm within a running time of 1 h. If we remove the computation time limitation, the *GRASP/PR* approach may get better results than the LR approach.
(3) The numerical results demonstrate that our LR algorithm can improve the best feasible solution obtained by Lingo8.0 from 1.97% to 9.19% in terms of cost with the average improvement 4.89%. Thus our LR algorithm is more effective than Lingo8.0 for all tested instances for a given computation time of 1 h for each instance.

7.2. Results for model COTPV

Since model *COTPV* allows the transportation of crude oil by partially loaded tankers, as mentioned, two cases for the setting of the ending inventory level s_{iT} of the model were tested: Case 1: s_{iT} is taken as a decision variable; Case 2: s_{iT} is set to zero. Similar to that for model *COTP*, the best feasible solution obtained by our LR algorithm for each instance is compared with the best feasible solution obtained by the MIP solver of Lingo8.0 with a running time of 1 h and the LP lower bound of model *COTPV*.

Table 2
Numerical results for the instances of model *COTPV*.

Instances	LR/Case1			LG/Case1		LR/Case2			LG/Case2	
	Gap (%)	Imp (%)	Time (s)	Gap (%)	Time (s)	Gap (%)	Imp (%)	Time (s)	Gap (%)	Time (s)
1	7.37	11.39	2584	21.17	3600	5.84	12.58	2612	21.07	3600
2	5.39	15.42	2637	24.56	3600	6.33	14.04	2644	23.7	3600
3	5.53	9.73	2257	16.89	3600	6.14	8.89	2630	16.51	3600
4	5.33	11.23	2605	18.66	3600	8.75	4.61	2662	13.99	3600
5	7.61	11.66	2612	21.81	3600	8.48	7.02	2740	16.67	3600
6	6.65	8.89	2714	17.06	3600	5.24	9.74	2590	16.6	3600
7	7.46	6.14	2583	14.49	3600	6.55	6.44	2686	13.89	3600
8	6.93	4.93	2627	12.47	3600	5.85	5.80	2634	12.37	3600
9	8.1	8.22	2654	17.78	3600	6.60	12.26	2673	21.47	3600
10	5.47	9.56	2673	16.61	3600	5.15	9.52	2664	16.21	3600
Average	6.58	9.72	2595	18.15	3600	6.49	9.09	2654	17.25	3600

The numerical results are given in Table 2 below. In the table, *LR*/*Case1* and *LR*/*Case2* correspond to model *COTPV* with two ending inventory level settings, respectively, Gap (%) is defined as (the upper bound − the lower bound)/the lower bound × 100%, Imp is defined as $(\text{Cost}_{\text{LG}} - \text{Cost}_{\text{LR}})/\text{Cost}_{\text{LG}} \times 100\%$, and Time is the computation time in seconds.

Table 2 shows that our algorithm can also effectively solve model *COTPV* in a reasonable computation time no matter whether the ending inventory level s_{iT} is fixed or not. The average gap with linear relaxation for the two cases is 6.58% and 6.49%, respectively, and the minimal gap is 5.33% and 5.15%, respectively. As mentioned in the testing of model *COTP*, the lower bound provided by the linear programming relaxation of model *COTPV* may be poor, and a tighter lower bound may be found to reduce the gap. This table also shows that our LR algorithm can improve the best feasible solution obtained by Lingo from 4.93% to 15.42% in terms of cost with the average improvement 9.72% for all tested instances in Case 1 and from 4.61% to 14.04% with the average improvement 9.09% for all tested instances in Case 2, respectively. Thus our LR algorithm is more effective than Lingo8.0 for the resolution of model *COTPV* in both cases if the computation time for each instance is limited to 1 h.

From Tables 1 and 2, we can observe that our LR algorithm can effectively solve large instances of the crude oil transportation problem and its variant, with a reasonable computation time. The algorithm performs better in solving *COTPV* than in solving *COTP*, this may be because *COTPV* only requires that the total quantity of crude oil transported by tankers of type k is less than the total capacity of the tankers, its relaxed subproblem in which the crude oil transportation quantity is determined may contain more information about the optimal solution of the corresponding original model than that of *COTP*.

8. Conclusion

In this paper, an inventory routing problem in crude oil transportation is studied, in which multiple transportation modes, a heterogeneous fleet of tankers and various logistics costs are considered. The problem is first reformulated by adding more valid constraints to its basic model *COTP* proposed in our previous work, and then an algorithm based on Lagrangian relaxation is developed to solve the model. Our *LR* algorithm is also extended to a variant of model *COTP* which permits both fully loaded tankers and partially loaded tankers to be used in the transportation of crude oil.

In order to improve the performance of our *LR* algorithm, the capacity constraints of model *COTP* are first reformulated in a nonlinear way and are then relaxed by introducing Lagrange multipliers; the relaxed problem is solved approximately with its corresponding dual problem solved by using the surrogate subgradient method. A feasible solution is constructed based on the solution of the relaxed problem using a heuristic.

Our numerical experiments in this paper demonstrate that compared with the *GRASP*/*PR* algorithm proposed in our previous work, our *LR* algorithm can get better solutions for large instances in less computation time with the average cost improvement 10.91%. For model *COTPV* which is the variant of *COTP*, the average gap with the linear relaxation is about 6.5% and the average running time is 2500 s. Thus our *LR* algorithm is effective for both *COTP* and *COTP*V.

Our future work is to find a tighter lower bound for the crude oil transportation problem and to reduce the computational time of our LR algorithm by optimizing its program code.

References

Bell, W., Dalberto, L., Fisher, M., Greenfield, A., Jaikumar, R., Kedia, P., et al. (1983). Improving the distribution of industrial gases with an on-line computerized routing and scheduling optimizer. *Interfaces*, *13*(6), 4–23.

Brown, G. G., Graves, G. W., & Ronen, D. (1987). Scheduling ocean transportation of crude oil. *Management Science*, *33*(3), 335–346.

Campbell, A. M., Clarke, L. W., Kleywegt, A. J., & Savelsbergh, M. W. P. (1998). The inventory routing problem. In *Fleet management and logistics*. Kluwer Academic Publishers., pp. 95–113.

Campbell, A. M., Clarke, L., & Savelsbergh, M. W. P. (2001). Inventory routing in practice. In *The vehicle routing problem*. Society for Industrial and Applied Mathematics., pp. 309–330.

Campbell, A. M., & Savelsbergh, M. W. P. (2004). A decomposition approach for the inventory-routing problem. *Transportation Science*, *38*(4), 488–502.

Cheng, L., & Duran, M. A. (2004). Logistics for world-wide crude oil transportation using discrete event simulation and optimal control. *Computers and Chemical Engineering*, *28*(6–7), 897–911.

Chien, T., Balakrishnan, A., & Wong, R. (1989). An integrated inventory allocation and vehicle routing problem. *Transportation Science*, *23*(2), 67–76.

Chu, F., Chu, C., Shen, Q., & Chen, H. (2008). An O(T3) polynomial algorithm for crude oil transportation. In *Proceedings of the 4th IEEE Conference on Automation Science and Engineering* (pp. 303–308).

Fisher, M. L. (2004). The Lagrangian relaxation method for solving integer programming problems. *Management Science*, *50*(Suppl. 12), 1861–1871.

Fisher, M., Greenfield, A., Jaikumar, R., Kedia, P. (1982). Real-time scheduling of a bulk delivery fleet: Practical application of Lagrangean relaxation. *Technical report*, The Wharton School, University of Pennsylvania.

Geoffrion, A. M. (1974). Lagrangian relaxation and its uses in integer programming. *Mathematical Programming Study*, *2*, 82–114.

Iakovou, E. T. (2001). An interactive multiobjective model for the strategic maritime transportation of petroleum products: Risk analysis and routing. *Safety Science*, *39*(1–2), 19–29.

Kleywegt, A. J., Nori, V. S., & Savelsbergh, M. W. P. (2002). The stochastic inventory routing problem with direct deliveries. *Transportation Science*, *36*(1), 94–118.

Sethi, S., & Sorger, G. (1991). A theory of rolling horizon decision marking. *Annals of Operations Research*, *29*, 387–416.

Shen, Q., Chen, H., Chu, F., & Zhou, M. (2009). Multi-mode transportation planning of crude oil via Greedy Randomized Adaptive Search and Path Relinking. In *Transactions of the Institute of Measurement and Control* , first published on line on 7 September, 2009 with doi:10.1177/0142331208100105

Sherali, H. D., Al-Yakoob, Mohammed, S., & Hassan, M. M. (1999). Fleet management models and algorithms for an oil-tanker routing and scheduling problem. *IIE Transactions*, *31*(5), 395–406.

Van Den Heuvel, W., & Wagelmans, A. P. M. (2005). A comparison of methods for lot-sizing in a rolling horizon environment. *Operations Research Letters*, *33*(5), 486–496.

Yu, Y. G., Chen, H. X., & Chu, F. (2006). Large scale inventory routing problem with split delivery: A new model and Lagrangian relaxation approach. *International Journal of Services Operations and Informatics*, *1*(3), 304–320.

Zhao, X., Luh, P. B., & Wang, J. (1999). Surrogate gradient algorithm for Lagrangian relaxation. *Journal of Optimization Theory and Applications*, *100*(3), 699–712.