SMOOTHING ESTIMATES FOR THE PERIODIC KdV EQUATION

by

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ABSTRACT

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In this thesis we aim at understanding an article of M. B. Erdoğan and N. Tzirakis on the famous KdV (Korteweg-de Vries) equation, entitled "Global smoothing for the periodic KdV", which appeared in *International Mathematics Research Notices* in 2012. The article establishes *smoothing estimates* in the case of the periodic KdV equation. Roughly speaking, smoothing estimates indicate that the solutions to the equation turn out to be smoother than the initial data, and constitute a subject closely related to *well-posedness* problems. This smoothing effect of a *dispersive* partial differential equation (PDE) on its solutions has been studied extensively, but the global smoothing effect in the periodic case was inaccessible prior to the paper of Erdoğan and Tzirakis. The most important tools they have used are the so-called *Bourgain spaces*, introduced by J. Bourgain, that are defined specifically for each equation and reflect the dispersion relation of the equation, and the so-called *differentiation by parts method.*

ÖZET

PERİYODİK KdV DENKLEMİ İÇİN YUMUŞATMA KESTİRİMLERİ

Bu tez çalışmamızda M. B. Erdoğan ve N. Tzirakis'in ortaklaşa yazmış oldukları ve 2012 yılında yayınlanan "Global smoothing for the periodic KdV" başlıklı makalelerini, detayları ile anlamak hedeflenmiştir. Bu makalede, periyodik KdV denklemi için *yumuşatma kestirimleri* elde edilmiştir. Kabaca anlatmak gerekirse, yumuşatma kestirimleri, kısmi türevli bir diferansiyel denklemin çözümlerinin, ilk şart fonksiyonundan daha yumuşak fonksiyonlar olduklarını gösterirler. Bu konu *iyi konmuşluk problemleri* ile de yakından bağlantılıdır. Daha önceleri dispersif kısmi türevli diferansiyel denklemlerin çözümleri üzerinde gözlemlenen bu yumuşatma etkisi hakkında çokça araştırmalar yapılmıştır, fakat Erdoğan ve Tzirakis'in makalelerinden önce periyodik denklemler için global yumuşatma etkisi adına pek bir ilerleme kaydedilememiştir. Bu makalede kullanılan en önemli araçlar olarak, ilk kez J. Bourgain tarafından, her denklem için denklemin dispersiyon bağıntısını yansıtacak şekilde tanımlanan, Bourgain uzaylarından, ve kısmi türevleme metodundan sözedilebilir.

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LIST OF SYMBOLS

$C^\infty(\mathbb{T})$	The space of all infinitely many differentiable functions on the torus $\mathbb{T}$
$C_t^0 H_x^s$	The Banach space of H^s valued continuous functions with the norm $\sup_t \|u(t,.)\|_{H^s}$
$C([-\delta,\delta]; H_x^s(\mathbb{T}))$	The Banach space of $H^s(\mathbb{T})$ valued continuous functions with the norm $\sup_{t\in[-\delta,\delta]} \|u(t,.)\|_{H^s(\mathbb{T})}$
$H^s(\mathbb{T})$	The Sobolev space of index s on $\mathbb{T}$
L	Linear differential operator
l^p	The space of p-th power summable sequences, with the p-norm $\|u_k\|_p$
$X^{s,b}$	Bourgain spaces of index s, b for the KdV equation
$A \lesssim B$	There exists a constant C with $A \leq C \cdot B$
$A \gtrsim B$	There exists a constant C with $A \geq C \cdot B$
$A \approx B$	$A \lesssim B$ and $A \gtrsim B$
$\mathcal{D}'(\mathbb{T})$	The dual space of $C^\infty(\mathbb{T})$
$\mathcal{L}^p$	The space of p-th power integrable functions with the p-norm $\|u\|_{\mathcal{L}^p}$
$\Re(z)$	The real part of $z \in \mathbb{C}$
$\langle t \rangle$	$1 + \|t\|$
$\mathbb{T}$	The one-dimensional torus or the circle
$\hat{u}$	The Fourier transform of u
$\|u\|_{H^s}$	The Sobolev norm of u

LIST OF ACRONYMS/ABBREVIATIONS

PDE	Partial differential equation
KdV	Korteweg-de Vries

1. INTRODUCTION

We mainly focus on the *periodic Korteweg-de Vries (KdV) equation*:

$$\begin{aligned} u_t + u_{xxx} + 2uu_x = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) = g(x) \in H^s(\mathbb{T}), \end{aligned} \tag{1.1}$$

as well as on the perturbed version with a smooth mean-zero spacetime potential:

$$\begin{aligned} u_t + u_{xxx} + (u^2 + \lambda u)_x = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) = g(x) \in H^s(\mathbb{T}). \end{aligned} \tag{1.2}$$

As an introduction let us first make some remarks about the history of the general problem we will be dealing with in this thesis, namely the so-called smoothing effect of a non-linear dispersive PDE on its solutions. We shall begin by talking about linear dispersive partial differential equations.

Firstly, a linear PDE is *dispersive* if different frequency components of the initial data propagate with different velocities. As an example, in one dimension, if we assume that there is a simple wave solution of the form $u(x,t) = Ae^{i(kx-wt)}$ of a given linear PDE, we arrive at a dependence relation $w = w(k)$, where w becomes a function of k. This relation is then referred to as the dispersion relation of the equation. The differential equation itself is dispersive if the velocity $v = w(k)/k$ is not a constant. For example, Schrödinger's equation $iu_t + u_{xx} = 0$ and Airy's equation (the linear part of the KdV equation) $u_t + u_{xxx} = 0$ have the dispersion relations $w(k) = k^2$ and $w(k) = k^3$ respectively. Hence, they have non-constant velocities $v = k$ and $v = k^2$, and are dispersive. On the other hand, the transport equation $u_t - u_x = 0$ and the wave equation $u_{tt} - u_{xx} = 0$ have the dispersion relations $w = k$ and $w = \pm k$ respectively. Hence they have constant velocities and are not dispersive.

Secondly, a non-linear PDE is dispersive if its linear part is dispersive. Hence the KdV equation is dispersive since its linear part, Airy's equation, is dispersive as we have mentioned above.

Now let us talk about the *smoothing effect*. The main theme here is that the solution to a dispersive PDE is smoother than its initial data. A groundbreaking study in this direction was published by R. S. Strichartz in 1977, see [1]. There it was shown that the solutions to the linear Schrödinger's equation on $\mathbb{R}^d$ are smoother than the initial data. Among the most important results thereafter were T. Kato's results about the KdV equation, see [2]. In 1988, it was proven by P. Constantin and J.-C. Saut that a similar smoothing effect holds for every dispersive differential equation under some mild conditions on the equation, see [3].

However, for non-linear PDEs, the smoothing effect can also be studied from another point of view. Here the effect is that the Duhamel part of the solution of the dispersive non-linear PDE is smoother than its initial data, where the Duhamel part is the difference of the solution of the original PDE and the solution of the linear part of the PDE. This approach to the dispersion effect on non-linear PDEs was pioneered by J. Bourgain, see [4]. The tools he introduced, Bourgain spaces among other things, enabled an investigation of the problem for periodic dispersive PDEs.

In this thesis we study the periodic KdV equation following Bourgain's approach to the smoothing effect. Smoothing estimates in this case were recently established by M. B. Erdoğan and N. Tzirakis, see [5]. Understanding this article is the main objective of this thesis.

We point out here two methods they use, namely the *normal form method through differentiation by parts* and the *restricted norm method*, introduced by A. V. Babin, A. A. Ilyin and E. S. Titi [6], and Bourgain [7], respectively. The importance of these methods are that they do not use the integrability structure of the KdV (1.1), and they thereby can also be applied to the KdV with a mean-zero spacetime potential (1.2), as Erdoğan and Tzirakis did in their paper, and perhaps to other dispersive models as well. It is also worth noting here that these results are highly related to *well-posedness* results. Indeed, as a corollary to their main estimates, Erdoğan and Tzirakis also improved upon some well-posedness results for the periodic KdV (1.2).

2. PRELIMINARIES

2.1. Some Basic Definitions Concerning PDEs

In this section, we will try to explain *Fourier transforms* of a PDE. With the usage of these transforms, we shall be able to represent the solutions of *Airy's equation* in a compact form, which corresponds to the linear part of the KdV equation. For more details see [8].

Definition 2.1. *Let* $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$.

- *We define its* spatial Fourier transform $\hat{u} : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ *by*

$$\hat{u}(k,t) = \frac{1}{2\pi}\int_{\mathbb{T}} u(x,t)\mathrm{e}^{-ikx}\mathrm{d}x \tag{2.1}$$

with the inversion formula

$$u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}(k,t)\mathrm{e}^{\mathrm{i}kx} \tag{2.2}$$

- *We define its* spacetime Fourier transform $\hat{u} : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ *by*

$$\hat{u}(k,\tau) = \int_{\mathbb{R}}\int_{\mathbb{T}} u(x,t)\mathrm{e}^{-\mathrm{i}(t\tau+kx)}\mathrm{d}x\mathrm{d}t \tag{2.3}$$

with the inversion formula

$$u(x,t) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}(t\tau+kx)}\hat{u}(k,\tau)\mathrm{d}\tau \tag{2.4}$$

Definition 2.2. *A constant coefficient linear PDE is called* dispersive *if the solutions of this equation are waves that spread out spatially as long as no boundary conditions are imposed.*

Consider the general evolution equation

$$u_t + Lu = 0, \tag{2.5}$$

where L is the linear differential operator in the spatial variable whose Fourier transform corresponds to a polynomial $h(k)$. If we take the spacetime Fourier transform of this evolution equation, we get

$$(\tau - h(k))\hat{u}(\tau, k) = 0. \tag{2.6}$$

Hence, the Fourier transform of the solution is supported on the *characteristic surface*

$$\{(\tau, k) : \tau = h(k)\}.$$

Informally, the evolution equation is called dispersive if this surface is "curved". This amounts to $h''(k)$ being non-zero in one dimension.

For example, in our case, if we take the Fourier transform of Airy's equation

$$u_t + u_{xxx} = 0 \quad x \in \mathbb{T}, t \in \mathbb{R}, \tag{2.7}$$

we get

$$(\tau - k^3)\hat{u}(\tau, k) = 0. \tag{2.8}$$

The characteristic surface of Airy's equation is then $\{(\tau, k) : \tau = k^3\}$, and so this equation is seen to be dispersive since $h''(k) \neq 0$. At this point, we also note that, if we take the spatial Fourier transform of Airy's equation, we get

$$\partial_t \widehat{u(t)}(k) = \mathrm{i}k^3 \widehat{u(t)}(k), \tag{2.9}$$

which has the unique solution

$$\widehat{u(t)}(k) = \mathrm{e}^{\mathrm{i}tk^3}\widehat{u_0}(k). \tag{2.10}$$

Then, applying the inverse Fourier transform we get

$$u(x,t) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}tk^3+\mathrm{i}xk}\widehat{u_0}(k), \tag{2.11}$$

and we shall denote this formula by $\mathrm{e}^{tL}u_0(x)$.

Definition 2.3. *For $s \in \mathbb{R}$, the* Hilbert space*, $H^s(\mathbb{T})$, is defined as follows:*

$$H^s(\mathbb{T}) := \{u \in \mathcal{D}'(\mathbb{T}) : \|u\|_{H^s(\mathbb{T})} < \infty\} \tag{2.12}$$

where the norm *is defined by*

$$\|u\|^2_{H^s(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |\hat{u}(k)|^2\langle k\rangle^{2s} = \|\hat{u}(k)\langle k\rangle^s\|^2_{l^2} \tag{2.13}$$

with $\mathcal{D}'(\mathbb{T})$ denoting the dual space of $C^\infty(\mathbb{T})$.

Note that for a mean-zero L^2 function u, $\|u\|_{H^s} \approx \|\hat{u}(k)|k|^s\|_{l^2}$. For a sequence u_k, with $u_0 = 0$, we will use $\|u\|_{H^s}$ to denote $\|u_k|k|^s\|_{l^2}$.

2.2. Some Important Results from Analysis

In this section, we recall some well-known results from analysis that are frequently used in the rest of this thesis. Let $\mathbb{K}$ stand for $\mathbb{R}$ or $\mathbb{T}$ in the following.

Theorem 2.4. (Generalized Hölder's Inequality) *Let $1 \leq p_1, p_2, ..., p_m \leq \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_m} = 1$, and assume $u_k \in \mathcal{L}^{p_k}(\mathbb{K})$ for $k = 1, ..., m$. Then*

$$\int_{\mathbb{K}} |f_1 f_2 ... f_m| \mathrm{d}x \leq \prod_{i=1}^{m} \|f_k\|_{\mathcal{L}^{p_k}(\mathbb{K})} \tag{2.14}$$

Theorem 2.5. (Young's Inequality) *Suppose* $f_1 \in \mathcal{L}^p(\mathbb{K})$, $f_2 \in \mathcal{L}^q(\mathbb{K})$ *and* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$ *with* $1 \leq p,q,r \leq \infty$. *Then*

$$\|f_1 * f_2\|_{\mathcal{L}^r(\mathbb{K})} \leq \|f_1\|_{\mathcal{L}^p(\mathbb{K})}\|f_2\|_{\mathcal{L}^q(\mathbb{K})} \tag{2.15}$$

where $*$ *denotes convolution.*

Theorem 2.6. (Grönwall's Inequality) *Assume that* u *and* β *are continuous and*

$$u(t) \leq C + \int_0^t \beta(s)u(s)\mathrm{d}s \text{ for all } t \in [0,\infty)$$

then

$$u(t) \leq C\mathrm{e}^{\int_0^t \beta(s)\mathrm{d}s}, \ t \in [0,\infty) \tag{2.16}$$

Theorem 2.7. (Parseval's Identity) *For* $u, v \in \mathcal{L}^2(\mathbb{K})$, *we have*

$$\langle u, v\rangle = \int_{\mathbb{K}} u(x)\overline{v(x)}\mathrm{d}x = \int_{\mathbb{K}} \hat{u}(\tau)\overline{\hat{v}(\tau)}\mathrm{d}\tau = \langle \hat{u}, \hat{v}\rangle \tag{2.17}$$

Theorem 2.8. (Plancherel's Theorem) *For* $u \in \mathcal{L}^2(\mathbb{K})$, *we have*

$$|u\|_{\mathcal{L}^2(\mathbb{K})} = \|\hat{u}\|_{\mathcal{L}^2(\mathbb{K})} \tag{2.18}$$

2.3. Bourgain Spaces

In this section, we define the spaces that are commonly used in the theory of the periodic KdV equation. We define the $X^{s,b}$ spaces for the 2π-periodic KdV equation via the norm

$$\|u\|_{X^{s,b}} = \|\langle k\rangle^s \langle \tau - k^3\rangle^b \hat{u}(k,\tau)\|_{\mathcal{L}^2(\mathrm{d}k\mathrm{d}\tau)}. \tag{2.19}$$

We also define the restricted norm

$$\|u\|_{X^{s,b}_\delta} = \inf_{\tilde{u}=u \ on \ [-\delta,\delta]} \|\tilde{u}\|_{X^{s,b}}. \tag{2.20}$$

The local well-posedness theory for the periodic KdV equation was established in the space $X^{s,1/2}$. Unfortunately, these spaces fail to control the $\mathcal{L}^\infty_t H^s_x$ norm of the solution. To settle this problem and ensure the continuity of the KdV flow, the Y^s and Z^s spaces are defined via the norms

$$\|u\|_{Y^s} = \|u\|_{X^{s,1/2}} + \|\langle k\rangle^s \hat{u}(k,\tau)\|_{\mathcal{L}^2(\mathrm{d}k)\mathcal{L}^1(\mathrm{d}\tau)}, \tag{2.21}$$

$$\|u\|_{Z^s} = \|u\|_{X^{s,-1/2}} + \left\|\frac{\langle k\rangle^s \hat{u}(k,\tau)}{\langle \tau - k^3\rangle}\right\|_{\mathcal{L}^2(\mathrm{d}k)\mathcal{L}^1(\mathrm{d}\tau)}. \tag{2.22}$$

One defines Y^s_δ and Z^s_δ accordingly. Note that if $u \in Y^s$, then $u \in \mathcal{L}^\infty_t H^s_x$.

2.4. Well-posedness Results on the Periodic KdV Equation

Definition 2.9. (Well-posedness) *Let X be a Banach space. We say that KdV* (1.1) *or* (1.2) *is* locally well-posed in $H^s(\mathbb{T})$, *if for a given initial data $g \in H^s(\mathbb{T})$, there exists $T > 0$ and a unique solution $u \in X \cap C^0_t H^s_x([0,T] \times \mathbb{T})$. We also demand that there is continuity with respect to the initial data in an appropriate topology. If T can be taken arbitrarily large, then we say that the problem is* globally well-posed.

Theorem 2.10. (Colliander, et. al., [9]) *For any $s \geq -\frac{1}{2}$, the initial value problem (1.1) is locally well-posed in H^s. In particular, $\exists \delta \approx (1 + \|g\|_{H^s})^{-\frac{6}{3+2s}-\epsilon}$ such that there exists a unique solution*

$$u \in C([-\delta,\delta]; H^s_x(\mathbb{T})) \cap Y^s_\delta$$

with

$$\|u\|_{X^{s,\frac{1}{2}}_\delta} \leq \|u\|_{Y^s_\delta} \leq C\|g\|_{H^s} \tag{2.23}$$

Theorem 2.11. (Colliander, et. al., [9]) *For any* $s \geq -\frac{1}{2}$, *the initial value problem (1.1) is globally well-posed in* H^s. *Moreover, for* $-\frac{1}{2} \leq s < 0$, *there is a growth bound (with some* $\alpha(s) > 0$*)*

$$\|u\|_{H^s} \leq C(1+|t|)^{\alpha(s)} \tag{2.24}$$

where C *depends on* $\|g\|_{H^s}$.

Theorem 2.12. *For any* $s \geq 0$, *the initial value problem (1.2) is locally well-posed in* H^s. *In particular,* $\exists \delta \approx (1+\|g\|_{H^s})^{-6}$ *such that there exists a unique solution*

$$u \in C([-\delta, \delta]; H^s_x(\mathbb{T})) \cap Y^s_\delta$$

with

$$\|u\|_{X^{s,\frac{1}{2}}_\delta} \leq \|u\|_{Y^s_\delta} \leq C\|g\|_{H^s} \tag{2.25}$$

2.5. Further Results on the Periodic KdV Equation

In this section we present some results that were proven recently. These shall be useful when obtaining the smoothing estimates for the periodic KdV equation.

Proposition 2.13. (Bourgain, [7]) *For any* $\epsilon > 0$ *and* $b > \frac{1}{2}$, *we have*

$$\|\chi_{[-\delta,\delta]}(t)u\|_{\mathcal{L}^6_{t,x}(\mathbb{R}\times\mathbb{T})} \leq C_{\epsilon,b}\|u\|_{X^{s,b}_\delta} \tag{2.26}$$

Theorem 2.14. (Oskolkov, [10]) *Let* $L = -\partial_x^3 + (\frac{1}{2\pi}\int_{-\pi}^{\pi} g)\partial_x$, *and assume that* g *is of bounded variation. Then* $\mathrm{e}^{tL}g$ *is a continuous function of* x, *if* $t/2\pi$ *is an irrational number. For rational values of* $t/2\pi$, *it is a bounded function with at most countably many discontinuities. Moreover, if* g *is also continuous then* $\mathrm{e}^{tL}g \in C^0_t C^0_x$.

Lemma 2.15. (Ginibre, [11]) *For* $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$, *we have*

$$\|\psi_\delta(t)\int_0^t \mathrm{e}^{L(t-r)}F(r)\mathrm{d}r\|_{X^{s,b}} \lesssim \delta^{1-b+b'}\|F\|_{X_\delta^{s,b'}} \tag{2.27}$$

where $\psi_\delta(t) := \psi(t/\delta)$, $\psi \in C^\infty$, ψ *supported on* $[-2,2]$ *and* $\psi(t) = 1$ *for* $t \in [-1,1]$.

3. MAIN RESULTS

3.1. Preparation

We return to our initial discussion of the KdV equation on the torus. From this point on we shall continue our discussion under the assumption that the initial data, and hence the solution at each time by momentum conservation, is mean-zero. To remove this assumption one changes the equation, introducing two terms of the form $[\frac{1}{\pi}\int_{-\pi}^{\pi} g]u_x$ and $(\frac{1}{2\pi}\int_{-\pi}^{\pi})\lambda_x$. The first term changes the linear operator from $-\partial_x^3$ to L, as is stated in the Theorems 3.4 and 3.5 below. This assumption can be justified as follows (note that on $\mathbb{R}$ this idea fails):

Let $u_t + u_{xxx} + uu_x = 0$ with $u(x,0) = g(x)$. Now set $v(x,t) = u(x,t) + c$ and observe that v solves

$$v_t + v_{xxx} - cv_x + vv_x = 0 \tag{3.1}$$

with $v(x,0) = g(x) + c$. Then choose $v = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\mathrm{d}x$, so that v is mean-zero by momentum conservation:

$$\int_{-\pi}^{\pi} v(x,t)\mathrm{d}x = \int_{-\pi}^{\pi} u(x,0)\mathrm{d}x + c = \int_{-\pi}^{\pi} g(x)\mathrm{d}x + c = 0. \tag{3.2}$$

Although (1.2) does not have integrability structure, integrating the equation with respect to x, one can easily see that the solution of this equation also satisfies momentum conservation, see [12]. We note that after this change, the resonances and the multilinear estimates remain the same. The second term is in H^s for any s, and in the calculation below it will only go into the B operator defined there, which satisfies the same estimates. Using the notation $u(x,t) = \sum_{k\in\mathbb{Z}} u_k(t)\mathrm{e}^{\mathrm{i}kx}$ and $\lambda(x,t) = \sum_{k\in\mathbb{Z}} \lambda_k(t)\mathrm{e}^{\mathrm{i}kx}$, we now write (1.2) on the Fourier side:

$$\partial_t u_k = \mathrm{i}k^3 u_k - \mathrm{i}k \sum_{k_1+k_2=k} (\lambda_{k_1} + u_{k_1}) u_{k_2}, \quad u_k(0) = \hat{g}(k). \tag{3.3}$$

Because of the mean-zero assumption on u and λ, there are no zero harmonics in this equation, i.e., $\lambda_0(t) = \int_{-\pi}^{\pi} \lambda(x,t)\mathrm{d}x = 0$ and $u_0(t) = \int_{-\pi}^{\pi} u(x,t)\mathrm{d}x = 0$. Using the transformations

$$u_k(t) = v_k(t)\mathrm{e}^{\mathrm{i}k^3}, \tag{3.4}$$

$$\lambda_k(t) = \Lambda_k(t)\mathrm{e}^{\mathrm{i}k^3 t}, \tag{3.5}$$

together with the identity

$$(k_1 + k_2)^3 - k_1^3 - k_2^3 = 3(k_1 + k_2)k_1 k_2, \tag{3.6}$$

the equation can be rewritten in the form

$$\partial_t v_k = -\mathrm{i}k \sum_{k_1+k_2=k} \mathrm{e}^{-3\mathrm{i}kk_1k_2t} (v_{k_1} + \Lambda_{k_1}) v_{k_2}. \tag{3.7}$$

Proposition 3.1. *The system* (3.7) *can be written in the following form:*

$$\partial_t [v + B(\Lambda + v, v)]_k = \rho_k + B(\partial_t \Lambda, v)_k + R(\Lambda + 2v, \Lambda + v, v)_k \tag{3.8}$$

where we define $B(f,g)_0 = \rho_0 = R(f,g,h)_0 = 0$, *and for* $k \neq 0$

$$B(f,g)_k := -\frac{1}{3} \sum_{k_1+k_2=k} \frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{k_1k_2} f_{k_1} g_{k_2} \tag{3.9}$$

$$\rho_k := \frac{\mathrm{i}}{3} \Lambda_k \sum_{|j|\neq|k|} \frac{\Lambda_j \overline{v_j}}{j} - \frac{\mathrm{i}}{3} \frac{(\overline{\Lambda_k} + 2\overline{v_k})(\Lambda_k + v_k)v_k}{k} \tag{3.10}$$

$$R(f,g,h)_k := \frac{\mathrm{i}}{3} \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_1+k_3)(k_2+k_3)\neq 0}} \frac{\mathrm{e}^{-3\mathrm{i}t(k_1+k_2)(k_2+k_3)(k_1+k_3)}}{k_1} f_{k_1} g_{k_2} h_{k_3} \tag{3.11}$$

Proof. Since $\mathrm{e}^{-3\mathrm{i}kk_1k_2t} = \partial_t(\frac{\mathrm{i}}{3kk_1k_2}\mathrm{e}^{-3\mathrm{i}kk_1k_2t})$, using differentiation by parts we have

$$\begin{aligned}
&\partial_t\left(\sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{3k_1k_2}(\Lambda_{k_1}+v_{k_1})v_{k_2}\right) \\
&= \mathrm{i}k\sum_{k_1+k_2=k}\mathrm{e}^{-3\mathrm{i}kk_1k_2t}(v_{k_1}+\Lambda_{k_1})v_{k_2} + \sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{3k_1k_2}\partial_t\left[(\Lambda_{k_1}+v_{k_1})v_{k_2}\right].
\end{aligned} \tag{3.12}$$

Using this equality we can write (3.7) as

$$\begin{aligned}
\partial_t v_k &= \partial_t\left(\sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{3k_1k_2}(\Lambda_{k_1}+v_{k_1})v_{k_2}\right) - \sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{3k_1k_2}\partial_t\left[(\Lambda_{k_1}+v_{k_1})v_{k_2}\right] \\
&= \partial_t\left(\sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{3k_1k_2}(\Lambda_{k_1}+v_{k_1})v_{k_2}\right) - \sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{3k_1k_2}\partial_t\Lambda_{k_1}v_{k_2} \\
&\quad + \sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{3k_1k_2}\left[v_{k_2}\partial v_{k_1} + (\Lambda_{k_1}+v_{k_1})\partial_t v_{k_2}\right].
\end{aligned} \tag{3.13}$$

Recalling the definition of B, we can rewrite this equation in the form:

$$\begin{aligned}
\partial_t v = &-\partial_t B(\Lambda+v,v)_k + B(\partial_t\Lambda,v)_k \\
&- \sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{3k_1k_2}\left[v_{k_2}\partial v_{k_1} + (\Lambda_{k_1}+v_{k_1})\partial_t v_{k_2}\right].
\end{aligned} \tag{3.14}$$

Note that, since $v_0 = 0$, in the sums above k_1 and k_2 are not zero. If we insert $\partial_t v_{k_1}$ back again, using the Equation 3.7, we have

$$\begin{aligned}
&-\frac{1}{3}\sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{k_1k_2}v_{k_2}\partial_t v_{k_1} \\
&= \frac{\mathrm{i}}{3}\sum_{k_1+k_2=k}\frac{\mathrm{e}^{-3\mathrm{i}tkk_1k_2t}}{k_1}v_{k_2}\sum_{l_1+l_2=k_1\neq 0}\mathrm{e}^{-3\mathrm{i}tk_1l_1l_2}(\Lambda_{l_1}+v_{l_1})v_{l_2} \\
&= \frac{\mathrm{i}}{3}\sum_{\substack{l_1+l_2+k_2=k \\ l_1+l_2\neq 0}}\frac{\mathrm{e}^{-3\mathrm{i}t[kk_2(l_1+l_2)+l_1l_2(l_1+l_2)]}}{k_2}v_{k_2}(\Lambda_{l_1}+v_{l_1})v_{l_2}.
\end{aligned} \tag{3.15}$$

Using the identity

$$kk_2 + l_1 l_2 = (k_2 + l_1 + l_2)k_2 + l_1 l_2 = (k_2 + l_1)(k_2 + l_2) \tag{3.16}$$

and renaming the variables $k_2 \to k_1$, $l_1 \to k_2$, $l_2 \to k_3$, we get

$$\begin{aligned} &-\frac{1}{3} \sum_{k_1+k_2=k} \frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{k_1k_2} v_{k_2} \partial_t v_{k_1} \\ &= \frac{\mathrm{i}}{3} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}} \frac{\mathrm{e}^{-3\mathrm{i}t(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1} (\Lambda_{k_2} + v_{k_2}) v_{k_3}. \end{aligned} \tag{3.17}$$

Similarly,

$$\begin{aligned} &-\frac{1}{3} \sum_{k_1+k_2=k} \frac{\mathrm{e}^{-3\mathrm{i}kk_1k_2t}}{k_1k_2} (\Lambda_{k_1} + v_{k_1}) \partial_t v_{k_2} \\ &= \frac{\mathrm{i}}{3} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}} \frac{\mathrm{e}^{-3\mathrm{i}t(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} (\Lambda_{k_1} + v_{k_1}) (\Lambda_{k_2} + v_{k_2}) v_{k_3}. \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we can write (3.14) as

$$\begin{aligned} &\partial_t \left[v + B(\Lambda + v, v) \right]_k = B(\partial_t \Lambda, v)_k + \\ &+ \frac{\mathrm{i}}{3} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}} \frac{\mathrm{e}^{-3\mathrm{i}t(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} (\Lambda_{k_1} + 2v_{k_1}) (\Lambda_{k_2} + v_{k_2}) v_{k_3}. \end{aligned} \tag{3.19}$$

Note that the set on which the phase on the right-hand side vanishes is the disjoint union of the following sets:

$$\begin{aligned} S_1 &= \{k_1 + k_2 = 0\} \cap \{k_3 + k_1 = 0\} \cap \{k_2 + k_3 \neq 0\} \\ &= \{k_1 = -k,\ k_2 = k,\ k_3 = k\}, \end{aligned} \tag{3.20}$$

$$S_2 = \{k_1 + k_2 = 0\} \cap \{k_3 + k_1 \neq 0\} \cap \{k_2 + k_3 \neq 0\}$$
$$= \{k_1 = j,\ k_2 = -j,\ k_3 = k,\ |j| \neq |k|\}, \tag{3.21}$$
$$S_3 = \{k_1 + k_2 \neq 0\} \cap \{k_3 + k_1 = 0\} \cap \{k_2 + k_3 \neq 0\}$$
$$= \{k_1 = j,\ k_2 = k,\ k_3 = -j,\ |j| \neq |k|\}. \tag{3.22}$$

Since $\mathrm{e}^{-3\mathrm{i}t(k_1+k_2)(k_2+k_3)(k_3+k_1)} = 1$ on each $S_l,\ l = 1, 2, 3$, and since the set $\{k_1 + k_2 + k_3 = k, k_2 + k_3 \neq 0\}$ is equal to the disjoint union of the sets $\{k_1 + k_2 + k_3 = k, (k_1 + k_2)(k_2 + k_3)(k_3 + k_1) \neq 0\}$ and S_l, $l = 1, 2, 3$, we have:

$$\partial_t \left[v + B(\Lambda + v, v)\right]_k = B(\partial_t \Lambda, v)_k +$$
$$+\frac{\mathrm{i}}{3} \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \frac{\mathrm{e}^{-3\mathrm{i}t(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} \left(\Lambda_{k_1} + 2v_{k_1}\right)\left(\Lambda_{k_2} + v_{k_2}\right) v_{k_3}$$
$$+\frac{\mathrm{i}}{3} \sum_{l=1}^{3} \sum_{S_l} \frac{\left(\Lambda_{k_1} + 2v_{k_1}\right)\left(\Lambda_{k_2} + v_{k_2}\right) v_{k_3}}{k_1}. \tag{3.23}$$

Thus, using the definition of $R(u, v, w)$, we have

$$\partial_t \left[v + B(\Lambda + v, v)\right]_k = B(\partial_t \Lambda, v)_k + +R(\Lambda + 2v, \Lambda + v, v)_k$$
$$+\frac{\mathrm{i}}{3} \sum_{l=1}^{3} \sum_{S_l} \frac{\left(\Lambda_{k_1} + 2v_{k_1}\right)\left(\Lambda_{k_2} + v_{k_2}\right) v_{k_3}}{k_1}. \tag{3.24}$$

The proposition follows, if we can show that the second line above is equal to ρ_k. Note that

$$\sum_{l=1}^{3} \sum_{S_l} \frac{\left(\Lambda_{k_1} + 2v_{k_1}\right)\left(\Lambda_{k_2} + v_{k_2}\right) v_{k_3}}{k_1} = -\frac{(\Lambda_{-k} + 2v{-k})(\Lambda_k + v_k)v_k}{k} +$$
$$+v_k \sum_{|j|\neq|k|} \frac{(\Lambda_j + 2v_j)(\Lambda_{-j} + v_{-j})}{j} + (\Lambda_k + v_k) \sum_{|j|\neq|k|} \frac{(\Lambda_j + 2v_j)v_{-j}}{j}, \tag{3.25}$$

and that using $v_j = \overline{v_{-j}}$ and $\Lambda_j = \overline{\Lambda_{-j}}$, we can rewrite the second line above as

$$\begin{aligned} & v_k \sum_{|j|\neq|k|} \frac{|\Lambda_j + v_j|^2 + |v_j|^2 + v_j\overline{\Lambda_j}}{j} + (\Lambda_k + v_k) \sum_{|j|\neq|k|} \frac{(\Lambda_j\overline{v_j} + 2|v_j|^2)}{j} \\ &= v_k \sum_{|j|\neq|k|} \frac{v_j\overline{\Lambda_j}}{j} + (\Lambda_k + v_k) \sum_{|j|\neq|k|} \frac{\Lambda_j\overline{v_j}}{j} \\ &= 2v_k \sum_{|j|\neq|k|} \frac{\Re(v_j\overline{\Lambda_j})}{j} + \Lambda_k \sum_{|j|\neq|k|} \frac{\Lambda_j\overline{v_j}}{j}. \end{aligned} \tag{3.26}$$

Indeed, the first equality follows from the symmetry relation $j \leftrightarrow -j$. By the same token, since $\Re(v_j\overline{\Lambda_j}) = \Re(\overline{v_j}\Lambda_j) = \Re(v_{-j}\overline{\Lambda_{-j}})$, the first summand in the last line above vanishes. Using this in (3.25), we obtain

$$\begin{aligned} \sum_{l=1}^{3}\sum_{S_l} \frac{(\Lambda_{k_1} + 2v_{k_1})(\Lambda_{k_2} + v_{k_2})\, v_{k_3}}{k_1} &= -\frac{(\Lambda_{-k} + 2v{-}k)(\Lambda_k + v_k)v_k}{k} + \Lambda_k \sum_{|j|\neq|k|} \frac{\Lambda_j\overline{v_j}}{j} \\ &= \frac{3}{\mathrm{i}}\rho_k, \end{aligned} \tag{3.27}$$

which yields the assertion of the proposition.

□

Integrating (3.8) form 0 to t, we obtain

$$\begin{aligned} v_k(t) - v_k(0) = &- B(\Lambda + v, v)_k(t) + B(\Lambda + v, v)_k(0) + \int_0^t B(\partial_r\Lambda, v)_k(r)\mathrm{d}r + \\ &+ \int_0^t \rho_k(r)\mathrm{d}r + \int_0^t R(\Lambda + 2v, \Lambda + v, v)_k(r)\mathrm{d}r. \end{aligned} \tag{3.28}$$

Transforming back to the u, λ variables, we have

$$\begin{aligned} u_k(t) - \mathrm{e}^{\mathrm{i}k^3t}u_k(0) = &- \mathcal{B}(\lambda + u, u)_k(t) + \mathrm{e}^{\mathrm{i}k^3t}\mathcal{B}(\lambda + u, u)_k(0) \\ &+ \int_0^t \mathrm{e}^{\mathrm{i}k^3(t-r)}\mathcal{B}(\mathrm{e}^{rL}\partial_r(\mathrm{e}^{-rL}\lambda), u)_k(r)\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \mathrm{e}^{\mathrm{i}k^3(t-r)}\tilde{\rho}_k(r)\mathrm{d}r \\
&+\int_0^t \mathrm{e}^{\mathrm{i}k^3(\mathrm{t}-\mathrm{r})}\mathcal{R}(\lambda+2u,\lambda+u,u)_k(r)\mathrm{d}r, \qquad (3.29)
\end{aligned}
$$

where

$$
\mathcal{B}(f,g)_k = -\frac{1}{3}\sum_{k_1+k_2=k}\frac{f_{k_1}g_{k_2}}{k_1k_2}, \qquad (3.30)
$$

$$
\tilde{\rho_k} = \frac{\mathrm{i}}{3}\lambda_k\sum_{|j|\neq|k|}\frac{\lambda_k\overline{u_j}}{j} - \frac{\mathrm{i}}{3}\frac{\overline{(\lambda_k}+2\overline{u_k})(\lambda_k+u_k)(u_k)}{k}, \qquad (3.31)
$$

$$
\mathcal{R}(f,g,h)_k = \frac{\mathrm{i}}{3}\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}}\frac{f_{k_1}g_{k_2}h_{k_3}}{k_1}. \qquad (3.32)
$$

Lemma 3.2. *(i) For $s>-\frac{1}{2}$ and $s_1\le s+1$, we have*

$$
\|\mathcal{B}(u,v)\|_{H^{s_1}} \lesssim \|u\|_{H^s}\|v\|_{H^s} \qquad (3.33)
$$

(ii) For $s>-\frac{1}{2}$ and $s\le s_1\le 3s+1$, we have

$$
\|\tilde{\rho}\|_{H^{s_1}} \lesssim \|u\|_{H^s}\left(\|\lambda\|_{H^s}^2+\|u\|_{H^s}^2\right) \qquad (3.34)
$$

Proof. By symmetry, we can assume in the estimate for $\mathcal{B}(u,v)$ that $|k_1|\ge|k_2|$. Thus, for $s_1<s+1$ and $s>-\frac{1}{2}$, we have

$$
\begin{aligned}
||\mathcal{B}(u,v)||_{H^{s_1}} &= \left\|-\frac{1}{3}\sum_{k_1+k_2=k}\frac{u_{k_1}v_{k_2}}{k_1k_2}\right\| \lesssim \left\|\sum_{k_1+k_2=k,|k_1|\ge|k_2|}\frac{|u_{k_1}v_{k_2}|}{|k_1k_2|}\right\|_{H^{s_1}} \\
&= \left\|\sum_{k_1+k_2=k,|k_1|\ge|k_2|}\frac{|u_{k_1}v_{k_2}||k|^{s_1}}{|k_1k_2|}\right\|_2 \\
&\lesssim \left\|\sum_{k_1+k_2=k,|k_1|\ge|k_2|}\frac{|k_1|^{s_1}|u_{k_1}||k_2|^s|v_{k_2}||k|^{s_1-s-1}}{|k_2|^{s+1}}\right\|_2. \qquad (3.35)
\end{aligned}
$$

(3.36)

In the last step above we used $|\frac{k_1}{k}|^{s+1} \gtrsim 1$ and this follows by using the inequalities $|k| = |k_1 + k_2| \le |k_1| + |k_2| \le 2|k_1|$ and $s + 1 > 0$. Since $s_1 - s - 1 < 0$, we also have $|k|^{s_1-s-1} \lesssim 1$. So, we get

$$\|\mathcal{B}(u,v)\| \lesssim \left\| \sum_{k_1+k_2=k, |k_1|\ge|k_2|} \frac{|k_1|^s |u_{k_1}||k_2|^s|v_{k_2}|}{|k_2|^{s+1}} \right\|_2$$
$$\lesssim \left\| \frac{|k|^s v_k}{|k|^{1+s}} \right\|_{l^1} \||k|^s u_k\|_2 \lesssim \||k|^s v_k\|_2 \||k|^{-1-s}\|_2 \|u\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{H^s}. \tag{3.37}$$

In the second inequality above, using the convolution structure, we employed Young's inequality with parameters $p = 1$, $q = r = 2$. The last two inequalities follow from Hölder's inequality and the convergence of the norm $\||k|^{-1-s}\|_2$ for $s > -1/2$. Now, note that for $s_1 < 3s + 1$,

$$\left\| \frac{u_k v_k w_k}{k} \right\|_{H^{s_1}} = \left\| |u_k v_k w_k| |k|^{3s} |k|^{s_1-3s-1} \right\|_2 \lesssim \left\| |u_k v_k w_k| |k|^{3s} \right\|_2$$
$$\lesssim \|u\|_{H^s} \|v\|_{H^s} \|w\|_{H^s}. \tag{3.38}$$

In the last inequality above we applied generalized Hölder's inequality. Also note for any $-\frac{1}{2} \le s$,

$$\left\| \lambda_k \sum \frac{|\lambda_j||u_j|}{|j|} \right\|_{H^{s_1}} \le \|\lambda\|_{H^{s_1}} \sum \frac{|\Lambda_j|}{|j|^{1+s}} |u_j||j|^s = \|\lambda\|_{H^{s_1}} \| \frac{|\lambda_j||u_j||j|^s}{|j|^{1+s}} \|_{l^1}$$
$$\le \|\lambda\|_{H^{s_1}} \|\lambda\|_{H^{-s-1}} \|u\|_{H^s} \le \|\lambda\|^2_{H^{s_1}} \|u\|_{H^s}. \tag{3.39}$$

In the last line above, we again used Hölder's inequality. These two estimates imply the asserted bound for $\tilde{\rho}$. □

Proposition 3.3. *For $s > -\frac{1}{2}$, $s_1 < \min(s + 1, 3s + 1)$, and $\epsilon > 0$ sufficiently small, we have*

$$\|\mathcal{R}(u,v,w)\|_{X_\delta^{s_1,-\frac{1}{2}+\epsilon}} \le C\|u\|_{X_\delta^{s,1/2}} \|v\|_{X_\delta^{s,1/2}} \|w\|_{X_\delta^{s,1/2}} \tag{3.40}$$

Proof. As usual, this follows by considering the $X^{s,b}$ norms instead of the restricted versions. Note that the dual space of $X^{s,b}$ is $X^{-s,-b}$ for any s, b. Using this duality and Plancherel's theorem, it will be enough to show that for any $h \in X^{s_1,1/2-\epsilon}$

$$\left|\int_{\mathbb{R}\times\mathbb{T}} \mathcal{R}(u,v,w)h\right| = \left|\sum_k \int_{\mathbb{R}} \hat{\mathcal{R}}(k,\tau)\hat{h}(-k,-\tau)\mathrm{d}\tau\right| \\ \lesssim \|u\|_{X^{s,1/2}}\|v\|_{X^{s,1/2}}\|w\|_{X^{s,1/2}}\|h\|_{X^{-s_1,1/2-\epsilon}} \tag{3.41}$$

hold true. For this, recall that

$$\mathcal{R}(u,v,w)(x,t) = \sum_{k\neq 0} \mathcal{R}(u,v,w)_k(t)\mathrm{e}^{\mathrm{i}kx}, \tag{3.42}$$

where

$$\mathcal{R}(u,v,w)_k = \frac{\mathrm{i}}{3} \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}} \frac{u_{k_1}v_{k_2}w_{k_3}}{k_1}. \tag{3.43}$$

If we take the Fourier transform of $\mathcal{R}(u,v,w)$, we get

$$\hat{\mathcal{R}}(k,\tau) = \frac{\mathrm{i}}{3}\int_{\tau_1+\tau_2+\tau_3=\tau} \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}} \frac{u(k_1,\tau_1)v(k_2,\tau_2)w(k_3,\tau_3)}{k_1}. \tag{3.44}$$

With this formulation, we can bound the integral in (3.41) as

$$|\sum_k \int_{\mathbb{R}} \hat{\mathcal{R}}(k,\tau)\hat{h}(-k,-\tau)\mathrm{d}\tau| = \\ |\sum_{\substack{k_1+k_2+k_3+k_4=0 \\ (k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \frac{u(k_1,\tau_1)v(k_2,\tau_2)w(k_3,\tau_3)h(k_4,\tau_4)}{k_1}| \\ \leq \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ (k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}\prod_{i=1}^4 f_i(k_i,\tau_i)}{|k_1|\prod_{i=1}^4\langle\tau_i - k_i^3\rangle^{1/2-\epsilon}}, \tag{3.45}$$

where the functions f_i are defined as follows:

$$f_1(k,\tau) = |\hat{u}(k,\tau)||k|^s \langle \tau - k^3 \rangle^{1/2}, \tag{3.46}$$

$$f_2(k,\tau) = |\hat{v}(k,\tau)||k|^s \langle \tau - k^3 \rangle^{1/2}, \tag{3.47}$$

$$f_3(k,\tau) = |\hat{w}(k,\tau)||k|^s \langle \tau - k^3 \rangle^{1/2}, \tag{3.48}$$

$$f_4(k,\tau) = |\hat{h}(k,\tau)||k|^{-s_1} \langle \tau - k^3 \rangle^{1/2-\epsilon}. \tag{3.49}$$

Notice that, if we can show the next following inequality, i.e. the Inequality 3.50, then the Inequality 3.41 follows using the above bound.

Claim:

$$\sum_{\substack{k_1+k_2+k_3+k_4=0 \\ (k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}\prod_{i=1}^4 f_i(k_i,\tau_i)}{|k_1|\prod_{i=1}^4 \langle \tau_i - k_i^3 \rangle^{1/2-\epsilon}} \lesssim \prod_{i=1}^4 \|f_i\|_2, \tag{3.50}$$

where

$$\prod_{i=1}^4 \|f_i\|_2 = \|u\|_{X^{s,1/2}} \|v\|_{X^{s,1/2}} \|w\|_{X^{s,1/2}} \|h\|_{X^{-s_1,1/2-\epsilon}} \tag{3.51}$$

by the definitions of f_i's and the $X^{s,b}$ norms.

Proof of (3.50) : Using $\tau_1+\tau_2+\tau_3+\tau_4=0$ and $k_1+k_2+k_3+k_4=0$, we have

$$\sum_{i=1}^4 \tau_i - k_i^3 = -k_1^3 - k_2^3 - k_3^3 - k_4^3 = 3(k_1+k_2)(k_1+k_3)(k_2+k_3) \leq 4\max_i(\tau_i - k_i^3).$$

Therefore, $\max_{i=1,2,3,4}\langle \tau_i - k_i^3 \rangle \gtrsim |k_1+k_2||k_1+k_3||k_2+k_3|$. Since the inequality in (3.50) is symmetric in the f_i's, it does not matter which of these terms is the actual maximum. We can thus assume, without loss of generality, that

$$\langle \tau_1 - k_1^3 \rangle = \max_{i=1,2,3,4} \langle \tau_1 - k_1^3 \rangle \gtrsim |k_1+k_2||k_1+k_3||k_2+k_3|.$$

Using this, we have

$$\begin{aligned}\prod_{i=1}^{4}\langle\tau_i-k_i^3\rangle^{1/2-\epsilon}&=\langle\tau_1-k_1^3\rangle^{1/2-\epsilon}\prod_{i=2}^{4}\langle\tau_1-k_1^3\rangle^{-2\epsilon}\prod_{i=2}^{4}\langle\tau_i-k_i^3\rangle^{1/2+\epsilon}\\&\gtrsim\langle\tau_1-k_1^3\rangle^{1/2-7\epsilon}\prod_{i=2}^{4}\langle\tau_i-k_i^3\rangle^{1/2+\epsilon}\\&\gtrsim(|k_1+k_2||k_1+k_3||k_2+k_3|)^{1/2-7\epsilon}\prod_{i=2}^{4}\langle\tau_i-k_i^3\rangle^{1/2+\epsilon}.\end{aligned}\tag{3.52}$$

Also note that, since all factors $k_i+k_j\neq 0$ for $i\neq j$, $i,j=1,2,3$ and since the sum $k_1+k_2+k_3+k_4=0$, we have

$$|k_1+k_2||k_1+k_3||k_2+k_3|\gtrsim|k_i|,\qquad i=1,2,3,4.\tag{3.53}$$

Now we shall prove that for $s>-\frac{1}{2}$, $s_1<\min(s+1,3s+1)$ and for ϵ sufficiently small,

$$\frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}}{|k_1|(|k_1+k_2||k_1+k_3||k_2+k_3|)^{1/2-7\epsilon}}\lesssim|k_1k_2k_3k_4|^{-\epsilon}.\tag{3.54}$$

Indeed, by (3.53), this follows if we can establish

$$\frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}}{|k_1|(|k_1+k_2||k_1+k_3||k_2+k_3|)^{1/2-11\epsilon}}\lesssim 1.\tag{3.55}$$

In order to show this last inequality, let us consider two separate cases:

Case 1: $s>-\frac{1}{3}$, $s_1<\min(3s+1,s+1)$. Without loss of generality we can assume that $s_1\geq 0$. (Since otherwise the conclusion obviously holds.) Let us define $M=\max(|k_1|,|k_2|,|k_3|)$. Then employing the inequalities $|k_1||k_1+k_2|\gtrsim|k_2|$ and $|k_1||k_1+k_3||k_3+k_2|\gtrsim|k_3||k_3+k_2|\gtrsim|k_2|$, and by symmetry of k_2 and k_3, we have

$$\begin{aligned}&|k_1|(|k_1+k_2||k_1+k_3||k_2+k_3|)^{1/2-11\epsilon}\\&=|k_1|^{22\epsilon}(|k_1||k_1+k_2||k_1||k_1+k_3||k_2+k_3|)^{1/2-11\epsilon}\\&\gtrsim|k_2|^{1-22\epsilon}\gtrsim M^{1-22\epsilon}.\end{aligned}\tag{3.56}$$

Thus,

$$\frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}}{|k_1|(|k_1+k_2||k_1+k_3||k_2+k_3|)^{1/2-11\epsilon}} \lesssim \frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}}{M^{1-22\epsilon}}. \tag{3.57}$$

Since $|k_1k_2k_3|^{-s} \leq M^{-3s}$ for $s < 0$ and $|k_1k_2k_3|^{-s} \leq M^{-s}$ for $s \geq 0$, we have $|k_1k_2k_3|^{-s} \leq \max(M^{-s}, M^{-3s}) = M^{-\min(3s,s)}$. And, since $k_1 + k_2 + k_3 + k_4 = 0$, we also have $|k_4| \lesssim M$. Using these two inequalities, for $0 \leq s_1 < \min(3s + 1, s + 1)$, we get

$$\begin{aligned}\frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}}{|k_1|(|k_1+k_2||k_1+k_3||k_2+k_3|)^{1/2-11\epsilon}} &\lesssim M^{-\min(s,3s)}M^{s_1}M^{22\epsilon-1}\\ &= M^{s_1+22\epsilon-\min(3s+1,s+1)} \lesssim 1 \end{aligned} \tag{3.58}$$

for sufficiently small ϵ.

Case 2: $-\frac{1}{2} < s \leq -\frac{1}{3}$, $s_1 < 3s + 1 = \min(3s + 1, s + 1) \leq 0$. Again, since $k_1 + k_2 + k_3 + k_4 = 0$, we have $|k_1 + k_2 + k_3||k_2 + k_3| = |k_4||k_1 + k_4| \gtrsim |k_1|$. Thus,

$$\begin{aligned}&\frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}}{|k_1|(|k_1+k_2||k_1+k_3||k_2+k_3|)^{1/2-11\epsilon}}\\ \lesssim &\frac{|k_2k_3|^{-s}}{|k_1|^{1+s-s_1}(|k_1+k_2||k_1+k_3|)^{1/2-11\epsilon}|k_2+k_3|^{1/2+s_1-11\epsilon}}.\end{aligned} \tag{3.59}$$

Now, using $|k_1||k_1 + k_i| \gtrsim |k_i|$, we bound the multiplier by

$$\frac{|k_2k_3|^{-s-\frac{1+s-s_1}{2}}}{(|k_1+k_2||k_1+k_3|)^{\frac{s_1-s}{2}-11\epsilon}|k_2+k_3|^{1/2-s_1-11\epsilon}} \lesssim |k_2k_3|^{\frac{s_1-(3s+1)}{2}} \lesssim 1. \tag{3.60}$$

This completes the proof of (3.54). Using (3.54) and (3.53), and eliminating the term $|k_1|^{-\epsilon}$, we obtain

$$\sum_{\substack{k_1+k_2+k_3+k_4=0\\(k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}\prod_{i=1}^4 f_i(k_i,\tau_i)}{|k_1|\prod_{i=1}^4\langle\tau_i-k_i^3\rangle^{1/2-\epsilon}} \lesssim$$

$$\lesssim \sum \int \frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}\prod_{i=1}^4 f_i(k_i,\tau_i)}{|k_1|(|k_1+k_2||k_1+k_3||k_2+k_3|)^{1/2-7\epsilon}\prod_{i=2}^4\langle\tau_i-k_i^3\rangle^{1/2+\epsilon}}$$
$$\lesssim \sum_{\substack{k_1+k_2+k_3+k_4=0\\(k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \frac{|k_2k_3k_4|^{-\epsilon}\prod_{i=1}^4 f_i(k_i,\tau_i)}{\prod_{i=2}^4\langle\tau_i-k_i^3\rangle^{1/2+\epsilon}}. \tag{3.61}$$

By Plancherel's theorem, we can rewrite this as

$$\int_{\mathbb{R}\times\mathbb{T}} \check{f}_1(x,t)\left(\int_{\tau_1+\tau_2+\tau_3=\tau}\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}}\prod_{i=2}^4\frac{f_i(k_i,\tau_i)|k_i|}{\langle\tau_i-k_i^3\rangle^{1/2+\epsilon}}\mathrm{d}\tau\right)^\vee(x,t). \tag{3.62}$$

By the convolution structure, this term can also be written as

$$\int_{\mathbb{R}\times\mathbb{T}} \check{f}_1(x,t)\prod_{i=2}^4\left(\frac{f_i|k|^{-\epsilon}}{\langle\tau-k^3\rangle}^{1/2+\epsilon}\right)^\vee(x,t). \tag{3.63}$$

Using generalized Hölder's inequality with $p_1 = 1/2$, $p_2 = p_3 = p_4 = 1/6$, we obtain the following inequality

$$\int_{\mathbb{R}\times\mathbb{T}} \check{f}_1(x,t)\prod_{i=2}^4\left(\frac{f_i|k|^{-\epsilon}}{\langle\tau-k^3\rangle}^{1/2+\epsilon}\right)^\vee(x,t)$$
$$\leq \|f_1\|_{L^2(\mathbb{R}\times\mathbb{T})}\prod_{i=2}^4\|\left(\frac{f_i|k|^{-\epsilon}}{\langle\tau-k^3\rangle^{1/2+\epsilon}}\right)^\vee\|_{L^6(\mathbb{R}\times\mathbb{T})}. \tag{3.64}$$

Now, it is time to use (2.13). Choose $b = 1/2+\epsilon$, and $s = \epsilon$, then we have

$$\|\left(\frac{f_i|k|^{-\epsilon}}{\langle\tau-k^3\rangle^{1/2+\epsilon}}\right)^\vee\|_{L^6(\mathbb{R}\times\mathbb{T})} \lesssim \|\left(\frac{f_i|k|^{-\epsilon}}{\langle\tau-k^3\rangle^{1/2+\epsilon}}\right)^\vee\|_{X^{\epsilon,1/2+\epsilon}}$$
$$= \|\frac{f_i|k|^{-\epsilon}|k|^{\epsilon}\langle\tau-k^3\rangle^{1/2+\epsilon}}{\langle\tau-k^3\rangle^{1/2+\epsilon}}\|_2 = \|f_i\|_2. \tag{3.65}$$

Using (3.64) and (3.65), we prove that

$$\sum_{\substack{k_1+k_2+k_3+k_4=0\\(k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0}} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \frac{|k_1k_2k_3|^{-s}|k_4|^{s_1}\prod_{i=1}^4 f_i(k_i,\tau_i)}{|k_1|\prod_{i=1}^4\langle\tau_i-k_i^3\rangle^{1/2-\epsilon}} \lesssim \prod_{i=1}^4\|f_1\|_2, \tag{3.66}$$

as we wanted. $\square$

3.2. The Results

Theorem 3.4. (Erdoğan and Tzirakis, [5]) *Fix* $s > -\frac{1}{2}$ *and* $s_1 < \min(3s+1, s+1)$. *Consider the real-valued solution of the KdV equation* (1.1) *on* $\mathbb{T} \times \mathbb{R}$ *with initial data* $u(x,0) = g(x) \in H^s$. *Assume we have a growth bound* $\|u(t)\|_{H^s} \leq C(\|g\|_{H^s})(1+|t|)^{\alpha(s)}$ *for some* $\alpha(s)$. *Then* $u(t) - \mathrm{e}^{tL}g \in C_t^0 H_x^{s_1}$ *and*

$$\|u(t) - \mathrm{e}^{tL}g\|_{H^{s_1}} \leq C(s, s_1, \|g\|_{H^s})(1+|t|)^{1+\beta(s)\alpha(s)} \tag{3.67}$$

Here, $\beta(s) = \frac{15+6s}{3+2s} + \epsilon$ *and* $L = -\partial_x^3 + (\frac{1}{2\pi}\int_{-\pi}^{\pi} g)\partial_x$.

Theorem 3.5. (Erdoğan and Tzirakis, [5]) *Fix* $s \geq 0$ *and* $s_1 < s+1$. *Consider the KdV equation* (1.2), *where* $\lambda \in C^\infty(\mathbb{T} \times \mathbb{R})$ *is a mean-zero real-valued potential with bounded derivatives and initial data* $u(x,0) = g(x) \in H^s$. *Assume that we have a growth bound* $\|u(t)\|_{H^s} \leq C(\|g\|_{H^s})T(t)$ *for some non-decreasing function* T *on* $[0,\infty)$. *Then* $u(t) - \mathrm{e}^{tL}g \in C_t^0 H_x^{s_1}$ *and*

$$\|u(t) - \mathrm{e}^{tL}g\|_{H^{s_1}} \leq C(s, s_1, \|g\|_{H^s})(1+|t|)T(t)^9 \tag{3.68}$$

where $L = -\partial_x^3 + (\frac{1}{2\pi}\int_{-\pi}^{\pi} g)\partial_x$.

Proof. We merely give a proof for the second theorem, then it will imply the first theorem by using a special case of the former. With the tools we presented in the previous section, we are ready for the proof. Recalling the Equation 3.29, we have

$$\begin{aligned}\|u_k(t) - \mathrm{e}^{tL}g\|_{H^{s_1}} \leq \|\mathcal{B}(\lambda(t)+u(t), u(t))\|_{H^{s_1}} + \|\mathcal{B}(\lambda(0)+g, g)\|_{H^{s_1}} + \\ \left\| \int_0^t \mathrm{e}^{\mathrm{i}k^3(t-r)} \mathcal{B}(\mathrm{e}^{rL}\partial_r(\mathrm{e}^{-rL}\lambda), u)_k(r)\mathrm{d}r \right\|_{H^{s_1}} + \left\| \int_0^t \tilde{\rho}_k(r)\mathrm{d}r \right\|_{H^{s_1}} + \\ \left\| \int_0^t \mathrm{e}^{\mathrm{i}k^3(t-r)} \mathcal{R}(\lambda+2u, \lambda+u, u)_k(r)\mathrm{d}r \right\|_{H^{s_1}}.\end{aligned} \tag{3.69}$$

First using the monotonicity of the integral and then using the estimates in Lemma 3.2 in the above inequality, we obtain (for $s > -1/2$ and $s_1 < \min(3s+1, s+1)$)

$$\begin{aligned}
\|u_k(t) - \mathrm{e}^{tL}g\|_{H^{s_1}} &\lesssim \|\mathcal{B}(\lambda+u,u)\|_{H^{s_1}} + \|\mathcal{B}(\lambda(0)+g,g)\|_{H^{s_1}} \\
&+ \int_0^t \|\mathcal{B}(\mathrm{e}^{rL}\partial_r(\mathrm{e}^{-rL}\lambda),u)\|_{H^{s_1}}\mathrm{d}r + \int_0^t \|\tilde{\rho}\|_{H^{s_1}}\mathrm{d}r \\
&+ \left\| \int_0^t \mathrm{e}^{(t-r)L}\mathcal{R}(\lambda+2u,\lambda+u,u)_k(r)\mathrm{d}r \right\|_{H^{s_1}} \\
&\lesssim \|\lambda(t)+u(t)\|_{H^s}\|u(t)\|_{H^s} + \|\lambda(0)+g\|_{H^s}\|g\|_{H^s} \\
&+ \int_0^t \|\mathrm{e}^{rL}\partial_r\mathrm{e}^{-rL}\lambda(r)\|_{H^s}\|u(r)\|_{H^s}\mathrm{d}r \\
&+ \int_0^t \|u(r)\|_{H^s}(\|\lambda(r)\|_{H^{s_1}}^2 + \|u(r)\|_{H^s}^2)\mathrm{d}r \\
&\lesssim \|u(t)\|_{H^s} + \|u(t)\|_{H^s}^2 + \|g\|_{H^s} + \|g\|_{H^s}^2 \\
&+ \int_0^t (\|u(r)\|_{H^s} + \|u(r)\|_{H^s}^3)\mathrm{d}r \\
&+ \left\| \int_0^t \mathrm{e}^{L(t-r)}\mathcal{R}(\lambda+2u,\lambda+u,u)_k(r)\mathrm{d}r \right\|_{H^{s_1}}. \qquad (3.70)
\end{aligned}$$

In these estimates, we also made use of the fact that $\|\mathrm{e}^{tL}g\|_{H^s} = \|g\|_{H^s}$ for any $s \in \mathbb{R}$. For more details see [8]. Now, in the last inequality of the previous display, we have some bounds in terms of H^s norms at hand, except for the last term $\left\|\int_0^t \mathrm{e}^{L(t-r)}\mathcal{R}(\lambda+2u,\lambda+u,u)_k(r)\mathrm{d}r\right\|_{H^{s_1}}$. Actually, exactly to bound this resonant term, we will work with so-called *Bourgain Spaces*. For $t \in [-\delta/2, \delta/2]$ and for $b > 1/2$, we have

$$\begin{aligned}
&\left\| \int_0^t \mathrm{e}^{L(t-r)}\mathcal{R}(\lambda+2u,\lambda+u,u)_k(r)\mathrm{d}r \right\|_{H^{s_1}} \\
&\leq \left\| \psi_\delta(t) \int_0^t \mathrm{e}^{L(t-r)}\mathcal{R}(\lambda+2u,\lambda+u,u)(r)\mathrm{d}r \right\|_{L_t^\infty H_x^{s_1}} \\
&\lesssim \left\| \psi_\delta(t) \int_0^t \mathrm{e}^{L(t-r)}\mathcal{R}(\lambda+2u,\lambda+u,u)(r)\mathrm{d}r \right\|_{X^{s_1,b}} \\
&\lesssim \delta^{\epsilon/2}\, \|\mathcal{R}(\lambda+2u,\lambda+u,u)(r)\mathrm{d}r\|_{X_\delta^{s_1,-1/2+\epsilon}} \qquad (3.71)
\end{aligned}$$

for sufficiently small $\epsilon > 0$. Here, we used the embedding $X^{s_1,b} \subset L_t^\infty H_x^{s_1}$ for $b > 1/2$ and employed the Lemma 2.15. Using (3.71) and the Proposition 3.3 in (3.70), we see that for $t \in [-\delta/2, \delta/2]$, we have (with an implicit constant depending on λ)

$$\begin{aligned}\|u(t) - \mathrm{e}^{tL} g\|_{H^{s_1}} \lesssim \|u(t)\|_{H^s} + \|u(t)\|_{H^s}^2 + \|g\|_{H^s} + \|g\|_{H^s}^2 \\ + \int_0^t (\|u\|_{H^s} + \|u\|_{H^s}^3)\mathrm{d}r + \|u\|_{X_\delta^{s,1/2}} + \|u\|_{X_\delta^{s,1/2}}^3.\end{aligned} \tag{3.72}$$

In this equation, we do not have to write $\|u(t)\|^2$, because it is dominated by either $\|u(t)\|^3$ or $\|u(t)\|$.

In the rest of the proof implicit constants shall also depend on $\|g\|_{H^s}$. Fix t large. For $r \leq t$, by the assumptions of the theorem, we have the bound

$$\|u(r)\|_{H^s} \lesssim T(r) \leq T(t). \tag{3.73}$$

We assume $T(t) \geq 1$ without loss of generality. Thus, by using the local theory, with $\delta \approx T(t)^{-6}$, for any $j \leq t/\delta \approx tT(t)^6$, we have the bound

$$\|u\|_{X_{[(j-1)\delta, j\delta} s,1/2} \lesssim \|u((j-1)\delta)\|_{H^s}. \tag{3.74}$$

Using this bound and the bound we have obtained in (3.72), we have

$$\|u(j\delta) - \mathrm{e}^{\delta L} u((j-1)\delta)\|_{H^{s_1}} \lesssim T(t)^3. \tag{3.75}$$

Using this one, we now obtain (with $J = t/\delta \approx tT(t)^6$)

$$\begin{aligned}\|u(t) - \mathrm{e}^{tL} u(0)\|_{H^{s_1}} &= \|u(J\delta) - \mathrm{e}^{J\delta L} u(0)\|_{H^{s_1}} \\ &\leq \sum_{j=1}^{J} \|\mathrm{e}^{(J-j)\delta L} u(j\delta) - \mathrm{e}^{(J-j+1)\delta L} u((j-1)\delta)\|_{H^{s_1}} \\ &= \sum_{j=1}^{J} \|u(j\delta) - \mathrm{e}^{\delta L} u((j-1)\delta)\|_{H^{s_1}} \lesssim JT(t)^3 \approx tT(t)^6.\end{aligned} \tag{3.76}$$

This completes the proof of the growth bound stated in the Theorem 3.5.

In the case of KdV without potential, the local theory bound gives $\delta \approx T(t)^{-\frac{6}{3+2s}-\epsilon}$ instead of $T(t)^{-6}$. Also taking $T(t) = \langle t\rangle^{\alpha(s)}$ into account, we obtain the growth bound in the Theorem 3.4.

Now we shall prove the continuity of $N(t) := u(t) - \mathrm{e}^{tL}g$ in H^{s_1}. Using (3.29), we obtain

$$\begin{aligned}
&N(t) - N(\tau) = \mathcal{B}(\lambda + u, u)_k(\tau)) - \mathcal{B}(\lambda + u, u)_k(t)\\
&+ (\mathrm{e}^{\mathrm{i}k^3t} - \mathrm{e}^{\mathrm{i}k^3t})\mathcal{B}(\lambda + u, u)_k(0)\\
&+ \int_0^t \mathrm{e}^{\mathrm{i}k(t-r)}\mathcal{B}(\mathrm{e}^{rL}\partial_r(\mathrm{e}^{-rL}\lambda), u)_k(r)\mathrm{d}r - \int_0^\tau \mathrm{e}^{\mathrm{i}k(\tau-r)}\mathcal{B}(\mathrm{e}^{rL}\partial_r(\mathrm{e}^{-rL}\lambda), u)_k(r)\mathrm{d}r\\
&+ \int_0^t \mathrm{e}^{\mathrm{i}k^3(t-r)}\tilde{\rho}_k(r)\mathrm{d}r - \int_0^\tau \mathrm{e}^{\mathrm{i}k^3(\tau-r)}\tilde{\rho}_k(r)\mathrm{d}r\\
&+ \int_0^t \mathrm{e}^{\mathrm{i}k(t-r)}\mathcal{R}\lambda + 2u, \lambda + u, u)_k(r)\mathrm{d}r - \int_0^\tau \mathrm{e}^{\mathrm{i}k(\tau-r)}\mathcal{R}(\lambda + 2u, \lambda + u, u)_k(r)\mathrm{d}r. \quad (3.77)
\end{aligned}$$

For fixed τ, we shall show that the H^{s_1} norm of each line in the formula above converges to zero as $t \to \tau$. We assume, without loss of generality, that $t > \tau$. For the first line this follows by using the difference $u(\tau) - u(t)$, the continuity of the solution in H^s, and the a priori bounds in Lemma 3.2 for $\mathcal{B}$. For the second line, we use the following inequality, (for any $\epsilon \in [0, 1]$)

$$|\mathrm{e}^{\mathrm{i}k^3t} - \mathrm{e}^{\mathrm{i}k^\tau}| \lesssim \min(1, |k|^3|t-\tau|) \le (|k|^3|t-\tau|)^\epsilon, \tag{3.78}$$

and the a priori bounds for $\mathcal{B}$ in $H^{s_1+3\epsilon}$ for sufficiently small $\epsilon > 0$, to obtain a bound of the form $|t-\tau|^\epsilon$. We now explain how to bound the fifth line. The third and the fourth lines can be treated similarly using H^s norms instead of $X^{s,b}$ norms. We write the fifth line as

$$(\mathrm{e}^{\mathrm{i}k^3(t-\tau)} - 1)\int_0^\tau \mathrm{e}^{\mathrm{i}k(t-r)}\mathcal{R}\lambda + 2u, \lambda + u, u)_k(r)\mathrm{d}r \quad + \tag{3.79}$$

$$+\int_{\tau}^{t} \mathrm{e}^{\mathrm{i}k(t-r)}\mathcal{R}(\lambda+2u,\lambda+u,u)_k(r)\mathrm{d}r. \tag{3.80}$$

To estimate the H^{s_1} norm of (3.79), we use (3.78) with sufficiently small $\epsilon > 0$ to obtain

$$\|(3.79)\|_{H^{s_1}} \lesssim |t-\tau|^{\epsilon} \left\| \int_0^{\tau} \mathrm{e}^{\mathrm{i}k^3(\tau-r)}\mathcal{R}(\lambda+2u+\lambda+u,u)_k(r)\mathrm{d}r \right\|_{H^{s_1+3\epsilon}}. \tag{3.81}$$

To estimate the last norm, we divide the integral into τ/δ pieces, where δ is given by the local $X^{s,1/2}$ theory. Here, δ depends on $\sup_{r\in[0,\tau]} \|u(r)\|_{H^s}$, which is finite due to the global well-posedness. Then, we use (3.71) and the Proposition 3.3 (with $s_1+3\epsilon$) to estimate each integral. Finally, the bound for the H^{s_1} norm of (3.65) follows from the gain in δ in (3.71) and by using Proposition 3.3.

□

Corollary 3.6. (Erdoğan and Tzirakis, [5]) *Let u be the real-valued solution of (1.2) with initial data $g \in BV \subset \mathcal{L}^2$. Then, u is a continuous function of x, if $t/2\pi$ is an irrational number. For rational values of $t/2\pi$, it is a bounded function with at most countably many discontinuities. Moreover, if g is also continuous then $u \in C_t^0 C_x^0$.*

Theorem 3.7. (Erdoğan and Tzirakis, [5]) *The initial value problem (1.2) is globally well-posed in $\mathcal{L}^2$. In particular, any H^s norm for $s \geq 0$ grows at most exponentially. Moreover, if $\partial_x\lambda \in \mathcal{L}_t^1\mathcal{L}_x^\infty$, then the $\mathcal{L}^2$ norm remains bounded, and any H^s norm for $s > 0$ grows at most polynomially.*

Proof. Using the Equation 1.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} u^2\mathrm{d}x = 2\int_{\mathbb{T}} uu_t\mathrm{d}x = -2\left(\int_{\mathbb{T}} uu_{xxx}\mathrm{d}x + 2\int_{\mathbb{T}} u^2u_x\mathrm{d}x + \int_{\mathbb{T}} \lambda uu_x\mathrm{d}x\right). \tag{3.82}$$

Let us evaluate each three integrals separately.

$$\int_{\mathbb{T}} uu_{xxx} = uu_{xx}\Big|_{x=-\pi}^{x=\pi} - \int_{\mathbb{T}} u_xu_{xx}\mathrm{d}x = \int_{\mathbb{T}} (\frac{u_x^2}{2})_x = \frac{(u_x)^2}{2}\Big|_{x=-\pi}^{x=\pi} = 0, \tag{3.83}$$

$$\int_{\mathbb{T}} 2u^2 u_x \mathrm{d}x = \int_{\mathbb{T}} \frac{2}{3}(u^3)_x \mathrm{d}x = \frac{2}{3}u^3\Big|_{x=-\pi}^{x=\pi} = 0, \tag{3.84}$$

$$\int_{\mathbb{T}} \lambda u u_x \mathrm{d}x = \frac{1}{2}\lambda u^2\Big|_{x=-\pi}^{x=\pi} - \frac{1}{2}\int_{\mathbb{T}} \lambda_x u^2 \mathrm{d}x = -\frac{1}{2}\int_{\mathbb{T}} \lambda_x u^2 \mathrm{d}x. \tag{3.85}$$

So, using these equations, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} u^2 \mathrm{d}x = \int_{\mathbb{T}} \lambda_x u^2 \mathrm{d}x. \tag{3.86}$$

Integrating this equation in time, we get

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 \mathrm{d}t = \int_0^t \int_{\mathbb{T}} u^2 \lambda_x \mathrm{d}x \mathrm{d}s. \tag{3.87}$$

By the fundamental theorem of calculus we get

$$\|u(t)\|_2^2 - \|u(0)\|_2^2 \leq \int_0^t \|\lambda_x(s)\|_{L^\infty} \left(\int_{\mathbb{T}} u^2 \mathrm{d}x\right) \mathrm{d}s. \tag{3.88}$$

So, we obtain

$$\|u(t)\|_2^2 \leq \|g\|_2^2 + \int_0^t \|\lambda_x(s)\|_{L^\infty} \|u(s)\|_2^2 \mathrm{d}s. \tag{3.89}$$

Thus, by Grönwall's inequality, we now get

$$\|u\|_2 \leq \|g\|_2 \mathrm{e}^{\int_o^t \|\lambda_x(s)\|_{L^\infty} \mathrm{d}s}. \tag{3.90}$$

Since λ_x is bounded, this gives

$$\|u\|_2 \leq \|g\|_2 \mathrm{e}^{Ct}. \tag{3.91}$$

Moreover, if $\lambda_x \in L_t^1 L_x^\infty$, then

$$\|u\|_2 \lesssim \|g\|_2. \tag{3.92}$$

In both cases, we can iterate the local solution and obtain a global-in-time solution evolving from an L^2 data.

To obtain the growth bound for the higher order norms, we use Theorem 3.5 repeatedly as follows. For $s \in (0,1)$, using the theorem with $s = 0$, $s_1 = s$, and $T(t) = \mathrm{e}^{Ct}$ or $T(t) = C$ depending on the assumptions on λ_x, we obtain

$$\|u(t) - \mathrm{e}^{tL} g\|_{H^s} \leq C_{\|g\|_2} \langle t \rangle T(t)^9, \tag{3.93}$$

which implies, by the unitarity of the linear evolution, that

$$\|u(t)\|_{H^s} \leq C_{\|g\|_2} \langle t \rangle T(t)^9. \tag{3.94}$$

One can proceed in this fashion and iteratively reach any index s. □

4. CONCLUSION

As a conclusion, the theory of non-linear dispersive PDEs with periodic boundary conditions was improved upon by this paper of Erdoğan and Tzirakis, see [5]. Indeed, this was the first global smoothing effect to be shown on the torus. Although the theory for unbounded domains has already been well-developed, it took a long time to achieve global results in the periodic case. The tools for unbounded domains were simply not applicable here.

The result of the paper is somewhat unexpected, since the bilinear estimates in Bourgain spaces $X^{s,b}$ are known to be optimal for the KdV equation. The smoothing effect was obtained by passing over to the Fourier side. The proof actually relies on the normal form method in [6], a trilinear smoothing estimate in $X^{s,b}$ spaces and Bourgain's $\mathcal{L}^6$ restriction estimate for the linear part of the KdV equation. It is worth noting that this proof does not make use of the celebrated integrability property of the KdV equation (1.1). It is also remarkable that this smoothing result improves the well-posedness statements in [7, 9, 13]. This indicates that smoothing estimates are of extreme help in proving well-posedness results with rough initial data.

Another highly related subject in this direction is the near-linear behavior of non-linear dispersive equations, which is currently an active area of research, see [14, 15].

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