THE MEAN VALUES OF THE FUNCTIONAL EQUATION FACTOR χ_ψ AT THE ZEROS OF $\zeta^{(k)}(s)$ AND $L^{(k)}(s,\psi_2)$

by

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ABSTRACT

THE MEAN VALUES OF THE FUNCTIONAL EQUATION FACTOR χ_ψ AT THE ZEROS OF $\zeta^{(k)}(s)$ AND $L^{(k)}(s,\psi_2)$

In this work, we calculate some mean values of the functional equation factor χ_ψ at the zeros of derivatives of Riemann zeta function and Dirichlet L-functions.

ÖZET

FONKSİYONEL DENKLEM ÇARPANI χ_ψ'NİN $\zeta^{(k)}(s)$ VE $L^{(k)}(s,\psi_2)$'NİN SIFIRLARINDAKİ ORTALAMA DEĞERLERİ

Bu çalışmada fonksiyonel denklem çarpanı χ_ψ'nin Riemann zeta fonksiyonu ve Dirichlet L-fonksiyonlarının türevlerinin sıfırlarındaki ortalama değerleri hesaplanmaktadır.

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LIST OF SYMBOLS

$s = \sigma + it$	A complex variable with σ, $t \in \mathbb{R}$
$e(\theta)$	$= e^{2\pi i\theta}$ for $\theta \in \mathbb{R}$
$f^{(j)}(s)$	j^{th} derivative of $f(s)$, $f^{(0)} = f$
$L(s, \psi)$	A Dirichlet L-function
$N(T)$	The number of zeros $\rho_\zeta = \sigma_\zeta + i\gamma_\zeta$ of $\zeta(s)$ with $0 < \gamma_\zeta \leqslant T$
$N(T, \psi)$	The number of zeros $\rho_\psi = \sigma_\psi + i\gamma_\psi$ of $L(s, \psi)$ with $0 < \gamma_\psi \leqslant T$
$N_k(T)$	The number of zeros $\rho_{\zeta,k} = \sigma_{\zeta,k} + i\gamma_{\zeta,k}$ of $\zeta^{(k)}(s)$ with $0 < \gamma_{\zeta,k} \leqslant T$
$N_k(T, \psi)$	The number of zeros $\rho_{\psi,k} = \sigma_{\psi,k} + i\gamma_{\psi,k}$ of $L^{(k)}(s, \psi)$ with $0 < \gamma_{\psi,k} \leqslant T$
$\mu(n)$	The Möbius function
$\Gamma(s)$	The Gamma Function
$\zeta(s)$	The Riemann Zeta Function
$\Lambda(n)$	The von-Mangoldt Function
$\tau(\psi)$	The Gauss sum of ψ
$\varphi(n)$	The number of a, $1 \leqslant a \leqslant n$, for which $(a, n) = 1$; known as Euler's totient function
$\psi(n)$	A Dirichlet character
$\chi_\zeta(s)$	The factor in the unsymmetric form of the functional equation of $\zeta(s)$
$\chi_\psi(s)$	The factor in the unsymmetric form of the functional equation of $L(s, \psi)$
$f(x) = O(g(x))$	$\lvert f(x)\rvert \leqslant Cg(x)$ where C is an absolute constant
$f(x) = O_k(g(x))$	$\lvert f(x)\rvert \leqslant Cg(x)$ where C is a constant depending on k
$f(x) = O_q(g(x))$	$\lvert f(x)\rvert \leqslant Cg(x)$ where C is a constant depending on q
$f(x) = O_{k,q}(g(x))$	$\lvert f(x)\rvert \leqslant Cg(x)$ where C is a constant depending on k and q
$f(x) \ll_k g(x)$	$f(x) = O_k(g(x))$
$f(x) \ll_q g(x)$	$f(x) = O_q(g(x))$
$f(x) \ll_{k,q} g(x)$	$f(x) = O_{k,q}(g(x))$

$f(x) \asymp g(x)$	$cf(x) \leqslant g(x) \leqslant Cf(x)$ for some positive absolute constants c and C
$f(x) \sim g(x)$	$\lim_{x\to\infty} f(x)/g(x) = 1$

LIST OF ACRONYMS/ABBREVIATIONS

GRH	Generalized Riemann Hypothesis
RH	Riemann Hypothesis

1. INTRODUCTION AND STATEMENT OF RESULTS

First, we state some well-known results about the Riemann zeta function and Dirichlet L-functions. These results can be found in [1–7].

1.1. The Riemann Zeta Function

For a complex number $s = \sigma + it$, $\sigma > 1$, the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Riemann, in his only paper on the theory of numbers, proved two main results: The function $\zeta(s)$ can be analytically continued over the complex plane, and the resulting function is meromorphic having only a simple pole at $s = 1$ with residue 1. Thus, the Laurent expansion of $\zeta(s)$ around $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \sum_{i=0}^{\infty} \gamma_i (s-1)^i,$$

and by differentiation, we have

$$\zeta^{(j)}(s) = \frac{(-1)^j j!}{(s-1)^{j+1}} + O_j(1), \quad (s \to 1). \tag{1.1}$$

Moreover, $\zeta(s)$ satisfies the functional equation:

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma[\frac{1}{2}(1-s)]}{\Gamma(\frac{s}{2})} \zeta(1-s).$$

From the functional equation, it can be inferred that the only zeros of $\zeta(s)$ for $\sigma < 0$ are at the poles of $\Gamma(\frac{s}{2})$ that is at the negative even integers. These are called the *trivial zeros* of the Riemann zeta function.

In this work we use the notation:

$$\chi_\zeta(s) := \pi^{s-\frac{1}{2}} \frac{\Gamma[\frac{1}{2}(1-s)]}{\Gamma(\frac{s}{2})}.$$

Throughout this work, $N_k(T)$ denotes the number of zeros of the k^{th} derivative of $\zeta(s)$ inside the horizontal strip $0 < \gamma_{\zeta,k} < T$. It is known that (See Chapter 10 in [3])

$$N_0(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{1.2}$$

Moreover, by the work of Berndt [8], it is known that for $k \geq 1$

$$N_k(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O_k(\log T). \tag{1.3}$$

The famous Riemann Hypothesis (RH) asserts that all the zeros of the function $\zeta(s)$ in the the strip $0 \leq \sigma \leq 1$, are on the line $\sigma = \frac{1}{2}$.

1.2. Dirichlet L-functions

Let $q \geqslant 1$ be a given integer. Dirichlet character $\psi \pmod q$ is a complex valued, totally multiplicative arithmetic function with the following properties: $|\psi(n)| = 1$ or 0, $\psi(n) = \psi(m)$ if $n \equiv m \pmod q$, $\psi(n) = 0$ if and only if $(n, q) > 1$. There exist $\varphi(q)$ distinct Dirichlet characters $\pmod q$, and they form an abelian group under multiplication of functions. Let q be a positive integer and let $\psi(n)$ be a Dirichlet character $\pmod q$. If for the values $\psi(n)$ restricted to n with $(n, q) = 1$ have a period less than q, then ψ is said to be an *imprimitive* character $\pmod q$. Otherwise ψ is said to be a *primitive* character $\pmod q$. A special character is the *principal character*, denoted by $\psi_0(n)$, defined by $\psi_0(n) = 1$ if $(n, q) = 1$, and zero otherwise. Note that if $q > 1$, the principal character ψ_0 is not primitive. Dirichlet characters $\pmod q$ satisfy the relations:

$$\sum_{n \pmod q} \psi(n) = \begin{cases} \varphi(q) & \text{if } \psi = \psi_0, \\ 0 & \text{if } \psi \neq \psi_0, \end{cases}$$

and

$$\sum_{\psi \pmod q} \psi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv 1 \pmod q, \\ 0 & \text{if } n \not\equiv 1 \pmod q, \end{cases}$$

that gives an important identity for a given residue class $a \pmod q$, where $(a,q) = 1$:

$$\sum_{\psi \pmod q} \overline{\psi}(a)\psi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv a \pmod q, \\ 0 & \text{if } n \not\equiv a \pmod q. \end{cases}$$

For a Dirichlet character $\psi \pmod q$, the Gauss sum $\tau(\psi)$ is defined as

$$\tau(\psi) := \sum_{n=1}^{q} \psi(n) e\left(\frac{n}{q}\right)$$

where $e(\alpha) := e^{2\pi i \alpha}$. Note that $\overline{\tau(\psi)} = \psi(-1)\tau(\overline{\psi})$. In particular, we have

$$\tau(\psi_0) = \sum_{n=1}^{q} \psi_0(n) e\left(\frac{n}{q}\right) = \sum_{\substack{n=1 \\ (n,q)=1}}^{q} e\left(\frac{n}{q}\right) = \mu(q).$$

For a given Dirichlet character $\psi \pmod q$, the Dirichlet L-function is defined for $s = \sigma + it$, $\sigma > 1$, as:

$$L(s,\psi) = \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s}.$$

For a primitive Dirichlet character $\psi \pmod q$, $L(s,\psi)$ satisfies the functional equation (see Chapter 9 in [2]):

$$\left(\frac{\pi}{q}\right)^{-\frac{1}{2}(s+\mathfrak{a})}\Gamma\left[\frac{1}{2}(s+\mathfrak{a})\right]L(s,\psi)=\frac{\tau(\psi)}{i^{\mathfrak{a}}q^{\frac{1}{2}}}\left(\frac{\pi}{q}\right)^{-\frac{1}{2}(1-s+\mathfrak{a})}\Gamma\left[\frac{1}{2}(1-s+\mathfrak{a})\right]L(1-s,\overline{\psi}),$$

where $\mathfrak{a}$ is defined as

$$\mathfrak{a}:=\begin{cases}0 & \text{if } \psi(-1)=1,\\ 1 & \text{if } \psi(-1)=-1.\end{cases}$$

From the functional equation it can be inferred that for $\sigma<0$, $L(s,\psi)$ has zeros at the poles of $\Gamma(\frac{1}{2}s+\mathfrak{a})$, that is when $\frac{1}{2}(s+\mathfrak{a})$ is a negative integer. These are called the *trivial zeros* of $L(s,\psi)$. The trivial zeros of $L(s,\psi)$ are the negative even integers if $\psi(-1)=1$, and are the negative odd integers if $\psi(-1)=-1$.
In this work we will use the notation:

$$\chi_{\psi}(s):=\frac{\frac{\tau(\psi)}{i^{\mathfrak{a}}q^{\frac{1}{2}}}\left(\frac{\pi}{q}\right)^{-\frac{1}{2}(1-s+\mathfrak{a})}\Gamma\left[\frac{1}{2}(1-s+\mathfrak{a})\right]}{\left(\frac{\pi}{q}\right)^{-\frac{1}{2}(s+\mathfrak{a})}\Gamma\left[\frac{1}{2}(s+\mathfrak{a})\right]},$$

so that

$$\chi_{\psi}(s)=\begin{cases}\frac{\tau(\psi)}{q^{\frac{1}{2}}}\left(\frac{\pi}{q}\right)^{s-\frac{1}{2}}\frac{\Gamma\left[\frac{1}{2}(1-s)\right]}{\Gamma\left[\frac{1}{2}s\right]} & \text{if } \psi(-1)=1,\\ \frac{\tau(\psi)}{iq^{\frac{1}{2}}}\left(\frac{\pi}{q}\right)^{s-\frac{1}{2}}\frac{\Gamma\left[1-\frac{1}{2}s\right]}{\Gamma\left[\frac{1}{2}(s+1)\right]} & \text{if } \psi(-1)=-1.\end{cases}$$

We know that (see Section 10.2 in [5])

$$\frac{L'}{L}(s,\psi)=B(\psi)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)-\frac{1}{2}\log\frac{q}{\pi}+\sum_{\rho_{\psi}}\left(\frac{1}{s-\rho_{\psi}}+\frac{1}{\rho_{\psi}}\right) \tag{1.4}$$

and

$$\Re B(\psi)=-\sum_{\rho_{\psi}}\Re\frac{1}{\rho_{\psi}}. \tag{1.5}$$

The Generalized Riemann Hypothesis (GRH) asserts that all the zeros of the function $L(s,\psi)$ in the strip $0 \leq \sigma \leq 1$ are on the line $\sigma = \frac{1}{2}$. For a primitive Dirichlet character ψ modulo $q \geq 3$, if we denote the number of zeros of $L(s,\psi)$ in the rectangle $0 < \sigma < 1$, $|t| < T$ by $N_0(T,\psi)$, then it is known that (see [2]),

$$N_0(T,\psi) = \frac{T}{\pi} \log \frac{qT}{2\pi e} + O(\log qT).$$

In [9], Yıldırım proved that for any $\delta > 0$, $\exists M = M(k, \delta, \mathfrak{a})$ such that there is no zero of $L^{(k)}(s,\psi)$ in the region $|s| > q^M$, $\sigma < -\delta$, $|t| > \delta$. Moreover, there exists sufficiently large σ_k such that $L^{(k)}(s,\psi)$ has no zero in $\sigma > \sigma_k$. Now, for $k \geqslant 1$, let $N_k(T,\psi)$ denote the number of zeros of $L^{(k)}(s,\psi)$ in $-q^K < \sigma < \sigma_k$, $|t| < T$. It is shown in [9] that

$$N_k(T,\psi) = \frac{T}{\pi} \log \frac{qT}{2\pi em} + O\left(q^K \log T\right), \quad (T \to \infty) \tag{1.6}$$

where m denotes the smallest prime number that does not divide q.

1.3. Motivation

In [10], Conrey and Ghosh studied the sum $\sum_{0<\gamma_{\zeta,k}<T} \chi_\zeta(\rho_{\zeta,k})$, and in [11] the above sum is calculated as:

$$\sum_{0<\gamma_{\zeta,k}<T} \chi_\zeta(\rho_{\zeta,k}) = \mathcal{A}_k \frac{T}{2\pi} + O_k\left(\frac{T}{\log T}\right), \quad (T \to \infty), \tag{1.7}$$

where

$$\mathcal{A}_k := -\sum_{u=0}^{\infty} (-1)^u \sum_{v=1}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1, \ldots i_k}$$
$$\times \prod_{w=1}^{k} (-1)^w w! \binom{k}{w}^{i_w} \frac{(-1)^v (v+1)!}{(i_1 + 2i_2 + \ldots + ki_k + v)!}.$$

In [11], it is also shown that

$$\mathcal{A}_k = -\left(k + 1 - \sum_{r=1}^{k} e^{-z_r}\right),$$

where z_r, $r = 1, ..., k$ are the zeros of $P_k(z) := \sum_{j=0}^{k} \frac{z^j}{j!}$, and $\mathcal{A}_0 = -1$. In this work, following the method in [11], we calculate the following generalizations of the sum in (1.7):

$$\sum_{0<\gamma_{\zeta,k}\leqslant T} \chi_\psi(\rho_{\zeta,k}) \qquad \text{and} \qquad \sum_{0<\gamma_{\psi_1,k}\leqslant T} \chi_{\psi_2}(\rho_{\psi_1,k}).$$

1.4. Statement of Results

Theorem 1.1. *Assume that the Generalized Riemann Hypothesis is true. Let $k \geqslant 1$ be a fixed integer, q be a fixed odd prime number, ψ be a non-principal Dirichlet character (mod q), and $\chi_\psi(s)$ is the factor in the functional equation of $L(s,\psi)$. Moreover, let $\rho_{\zeta,k}$ denote a non-real zero of the k^{th} derivative $\zeta^{(k)}(s)$ of $\zeta(s)$ and $\gamma_{\zeta,k} := \Im\rho_{\zeta,k}$. Then*

$$\sum_{0<\gamma_{\zeta,k}<T} \chi_\psi(\rho_{\zeta,k}) = -\frac{\overline{\tau(\overline{\psi})}}{q-1}\mathcal{A}_k\frac{T}{2\pi} + O_{k,q}\left(\frac{T}{\log T}\right), \quad (T \to \infty). \tag{1.8}$$

Theorem 1.2. *Assume that the Generalized Riemann Hypothesis is true. Let $k \geqslant 1$ be a fixed integer, q_1, q_2 be distinct fixed odd prime numbers. Let ψ_1 be a non-principal Dirichlet character (mod q_1) and ψ_2 be a non-principal Dirichlet character (mod q_2), and $\chi_{\psi_2}(s)$ is the factor in the functional equation of $L(s,\psi_2)$. Moreover, let $\rho_{\psi_1,k}$ denote a non-real zero of the k^{th} derivative $L^{(k)}(s,\psi_1)$ of $L(s,\psi_1)$ and $\gamma_{\psi_1,k} := \Im\rho_{\psi_1,k}$. Then*

$$\sum_{0<\gamma_{\psi_1,k}<T} \chi_{\psi_2}(\rho_{\psi_1,k}) \ll_{k,q_1,q_2} T^{1-\frac{1}{\log\log T}} \log T \log\log T, \quad (T \to \infty). \tag{1.9}$$

We should mention here that the assumption of GRH in both Theorems can be removed, but this would mean larger and more involved calculations.

2. LEMMAS

The first two lemmas are well-known in the literature (see, for example [12]).

Lemma 2.1. *For any real numbers* α *and* β,

$$\chi_\zeta(s) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma} \exp\left(-it\log\frac{|t|}{2\pi e} + \frac{i\pi}{4}\operatorname{sgn}(t)\right)\left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{2.1}$$

uniformly in $\alpha \leqslant \sigma \leqslant \beta$, $|t| \geqslant 1$, *where*

$$\operatorname{sgn}(t) := \begin{cases} 1 & \text{if } \; t > 0, \\ 0 & \text{if } \; t = 0, \\ -1 & \text{if } \; t < 0. \end{cases}$$

Lemma 2.2. *For any real numbers* α *and* β,

$$\frac{\chi_\zeta'}{\chi_\zeta}(s) = -\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right), \tag{2.2}$$

uniformly in $\alpha \leqslant \sigma \leqslant \beta$, $|t| \geqslant 1$. *Moreover, for a positive integer* m *and for* $|t| \geqslant 1$, *we have*

$$\left(\frac{d}{ds}\right)^m \frac{\chi_\zeta'}{\chi_\zeta}(s) \ll |t|^{-m}. \tag{2.3}$$

Lemma 2.3. *For a positive integer* m, *and for* $|t| \geqslant 1$, *we have*

$$\frac{\chi_\zeta^{(m)}}{\chi_\zeta}(s) = \left(-\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^m. \tag{2.4}$$

Proof. The proof proceeds by induction on m. When $m = 1$ we have (2.2). Assume that the statement (2.4) holds for $m \geqslant 1$. By the identity

$$\frac{\chi_\zeta^{(m+1)}}{\chi_\zeta}(s) = \frac{d}{ds}\left(\frac{\chi_\zeta^{(m)}}{\chi_\zeta}(s)\right) + \frac{\chi_\zeta^{(m)}}{\chi_\zeta}(s)\frac{\chi_\zeta'}{\chi_\zeta},$$

and the induction hypothesis, we then have

$$\begin{aligned}\frac{\chi_\zeta^{(m+1)}}{\chi_\zeta}(s) =& m\left(-\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^{m-1} O\left(\frac{1}{|t|}\right) \\ &+ \left(-\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^{m}\left(-\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right) \\ =& \left(-\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^{m+1}.\end{aligned}$$

Thus by the induction principle, the result follows. □

The following two lemmas can be found in [6].

Lemma 2.4. *(Lemma 2.1 in [6]) Let ψ be a primitive Dirichlet character modulo $q \geqslant 3$. In any fixed half-strip $\alpha \leqslant \sigma \leqslant \beta$, $t \geqslant \delta > 0$, where δ is an arbitrary small positive number, we have*

$$\chi_\psi(s) = \frac{\tau(\psi)e^{\frac{\pi i}{4}}}{q^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\frac{1}{2}-\sigma}\exp\left(-it\log\left(\frac{qt}{2\pi e}\right)\right)\left\{1 + O\left(\frac{1}{t}\right)\right\} \tag{2.5}$$

and

$$\chi_\psi(1-s) = \frac{\tau(\psi)e^{-\frac{\pi i}{4}}}{q^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\sigma-\frac{1}{2}}\exp\left(it\log\left(\frac{qt}{2\pi e}\right)\right)\left\{1 + O\left(\frac{1}{t}\right)\right\}. \tag{2.6}$$

Lemma 2.5. *(Lemma 2.2 in [6]) For any fixed σ, and $|t| > 1$,*

$$\frac{\chi_\psi'}{\chi_\psi}(s) = -\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right), \tag{2.7}$$

Lemma 2.6. *For a positive integer m, and for $|t| \geqslant 1$, we have*

$$\frac{\chi_\psi^{(m)}}{\chi_\psi}(s) = \left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^m. \tag{2.8}$$

Proof. The proof proceeds by induction on m. When $m = 1$ we have (2.7). Assume that the statement (2.8) holds for $m \geqslant 1$. By the identity

$$\frac{\chi_\psi^{(m+1)}}{\chi_\psi}(s) = \frac{d}{ds}\left(\frac{\chi_\psi^{(m)}}{\chi_\psi}(s)\right) + \frac{\chi_\psi^{(m)}}{\chi_\psi}(s)\frac{\chi_\psi'}{\chi_\psi},$$

and the induction hypothesis, we then have

$$\begin{aligned}\frac{\chi_\psi^{(m+1)}}{\chi_\psi}(s) =& m\left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^{m-1} O\left(\frac{1}{|t|}\right)\\ &+ \left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^{m}\left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)\\ =& \left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^{m+1}.\end{aligned}$$

Thus by the induction principle, the result follows. □

The following is a result from literature, see pp. 119-120 in [13].

Lemma 2.7. *(Vinogradov-Korobov, 1958) There exists a positive absolute constant δ_1 such that $\zeta(s) \neq 0$ throughout the region*

$$\sigma \geqslant 1 - \frac{\delta_1}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}, \quad |t| \geqslant 3,$$

in which one has

$$\frac{\zeta'}{\zeta}(s) \ll (\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}.$$

Our next lemma is Theorem 2 in [9].

Lemma 2.8. *Let ψ be a primitive Dirichlet character modulo q, $q \geqslant 3$, and m be the smallest prime number that does not divide q. Then $|L^{(k)}(s,\psi)| > \frac{(\log m)^k}{2m^\sigma}$ for $\sigma > 1 + m\left(1 + \sqrt{1 + \frac{2k^2}{m\log m}}\right)$.*

Proof. Since m is the smallest prime number that does not divide q, m is the smallest natural number $n > 1$ such that $\psi(n) \neq 0$. Thus we have

$$L^{(k)}(s,\psi) = (-1)^k \sum_{n=m}^{\infty} \frac{\psi(n)(\log n)^k}{n^s} \qquad (\sigma > 0).$$

and

$$\begin{aligned} |L^{(k)}(s,\psi)| &\geq \frac{(\log m)^k}{m^\sigma} - \sum_{n=m+1}^{\infty} \frac{(\log n)^k}{n^\sigma} \quad (\sigma > 1) \\ &> \frac{(\log m)^k}{m^\sigma} - \int_m^{\infty} \frac{(\log x)^k}{x^\sigma}\, dx \quad (\sigma > \frac{k}{\log m}) \\ &= \frac{(\log m)^k}{m^\sigma} - \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}} \sum_{j=0}^{k} \frac{((\sigma-1)\log m)^j}{j!}. \end{aligned}$$

Now we put $z := (\sigma - 1)\log m$. If $\frac{(\log m)^k}{2m^\sigma} - \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}} \sum_{j=0}^{k} \frac{z^j}{j!} \geq 0$, then we obtain the desired result. Note that

$$\begin{aligned} \frac{(\log m)^k}{2m^\sigma} - \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}} \sum_{j=0}^{k} \frac{z^j}{j!} &= \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}} \left(\frac{(\log m)^k(\sigma-1)^{k+1}}{2mk!} - \sum_{j=0}^{k} \frac{z^j}{j!} \right) \\ &= \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}} \left(\frac{z^{k+1}}{k!.2m\log m} - \sum_{j=0}^{k} \frac{z^j}{j!} \right) \end{aligned}$$

and there is only one $z \in \mathbb{R}^+$ such that $\frac{z^{k+1}}{k!.2m\log m} - \sum_{j=0}^{k} \frac{z^j}{j!} = 0$ ([14] Part III, Problem 16). We can assume $\sigma > 1 + \frac{k}{\log m}$, so that $z > k$. Then

$$\sum_{j=0}^{k} \frac{z^j}{j!} \leq \frac{z^k}{k!} + \frac{kz^{k-1}}{(k-1)!} \leq \frac{z^{k+1}}{k!2m\log m}, \tag{2.9}$$

the last inequality being true for $z^k 2m\log m + z^{k-1}k^2 2m\log m \leq z^{k+1}$ which is equivalent to $0 \leq z^2 - (2m\log m)z - 2k^2 m\log m$. Here the discriminant of the quadratic expression $z^2 - (2m\log m)z - 2k^2 m\log m$ in z is $\Delta = 4m^2\log^2 m + 8k^2 m\log m$. Thus, for $z \geq m\log m + \sqrt{m^2\log^2 m + 2k^2 m\log m} = m\log m\left(1 + \sqrt{1 + \frac{2k^2}{m\log m}}\right)$ or in terms of σ, for $\sigma > 1 + m\left(1 + \sqrt{1 + \frac{2k^2}{m\log m}}\right)$ the result follows. □

We state the following and refer to [5] for the details:

Lemma 2.9. *(Riesz Typical Mean) For a sequence $\{a_n\}_{n\geqslant 1}$,a positive integer k, a positive real x, we have*

$$R_k(x) := \frac{1}{k!}\sum_{n\leqslant x} a_n(\log\frac{x}{n})^k = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s)\frac{x^s}{s^{k+1}}\,ds.$$

where $\alpha(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$, and $\sigma_0 > \max\{0, \sigma_c\}$, σ_c being the abscissa of convergence of the Dirichlet series $\sum_{n=1}^{\infty}\frac{a_n}{n^s}$.

The following is an exercise in [5] (See page 444):

Lemma 2.10. *Suppose that $L(s,\psi) \neq 0$ for $\sigma > 1/2$ i.e., assume GRH. Then*

$$\frac{L'}{L}(s,\psi) \ll ((\log q\tau)^{2-2\sigma} + 1)\min\left(\frac{1}{|\sigma - 1|}, \log\log q\tau\right),$$

uniformly for $1/2 + 1/\log\log q\tau \leq \sigma \leq 3/2$ where $\tau := |t| + 4$.

Proof. Suppose that $L(s,\psi) \neq 0$ for $\sigma > 1/2$.

For $x \geq 2$ and $y \geq 2$, define

$$w(u) := \begin{cases} 1 & \text{if } 1 \leqslant u \leqslant x, \\ 1 - \frac{\log\frac{u}{x}}{\log y} & \text{if } x \leqslant u \leqslant xy \\ 0 & \text{if } u \geqslant xy. \end{cases}$$

Now, consider the sum

$$\begin{aligned}\sum_{n\leqslant xy}\frac{w(n)\Lambda(n)\psi(n)}{n^s} &= \sum_{n\leqslant x}\frac{\Lambda(n)\psi(n)}{n^s}+\sum_{x<n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}-\sum_{x<n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\frac{\log\frac{n}{x}}{\log y}\\ &= \sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}+\frac{1}{\log y}\sum_{x<n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n},\end{aligned}$$

where $\Lambda(n)$ is the von Mangoldt function.

Since the Dirichlet coefficients of $\frac{L'}{L}(s,\psi)$ are given by $\Lambda(n)\psi(n)$, we use Lemma 2.9 to get

$$\sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}=\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}-\frac{L'}{L}(s+w,\psi)\frac{(xy)^w}{w}\,dw,$$

and

$$\begin{aligned}\frac{1}{\log y}\sum_{x<n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n} =&\frac{1}{\log y}\sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n}-\frac{1}{\log y}\sum_{n\leqslant x}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n}\\ =&\frac{1}{\log y}\sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{xy}{n}-\sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\\ &-\frac{1}{\log y}\sum_{n\leqslant x}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n}\\ =&\frac{1}{2\pi i\log y}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}-\frac{L'}{L}(s+w,\psi)\frac{(xy)^w}{w^2}\,dw\\ &-\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}-\frac{L'}{L}(s+w,\psi)\frac{(xy)^w}{w}\,dw\\ &-\frac{1}{2\pi i\log y}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}-\frac{L'}{L}(s+w,\psi)\frac{(x)^w}{w^2}\,dw.\end{aligned}$$

Combining these,

$$\sum_{n\leqslant xy}\frac{w(n)\Lambda(n)\psi(n)}{n^s}=-\frac{1}{2\pi i\log y}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{L'}{L}(s+w,\psi)\frac{(xy)^w-x^w}{w^2}\,dw.$$

Now, using Cauchy's residue theorem,

$$
\begin{aligned}
-\frac{1}{2\pi i\log y}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{L'}{L}(s+w,\psi)\frac{(xy)^w-x^w}{w^2}\,dw\\
&= -\frac{L'}{L}(s,\psi)-\sum_{\rho_\psi}\frac{(xy)^{\rho_\psi-s}-x^{\rho_\psi-s}}{(\rho_\psi-s)^2\log y}\\
&\quad -(1-\mathfrak{a})\sum_{k=1}^{\infty}\frac{(xy)^{-2k-s}-x^{-2k-s}}{(2k+s)^2\log y}\\
&\quad -\mathfrak{a}\sum_{k=1}^{\infty}\frac{(xy)^{-2k+1-s}-x^{-2k+1-s}}{(2k-1+s)^2\log y}
\end{aligned}
$$

If $\sigma \geq 1/2$ then $|y^{\rho_\psi-s}-1|\leqslant 2$. Thus, for $\sigma>1/2$ we have

$$
\left|\sum_{\rho_\psi}\frac{(xy)^{\rho_\psi-s}-x^{\rho_\psi-s}}{(\rho_\psi-s)^2\log y}\right|\leqslant\frac{2x^{1/2-\sigma}}{\log y}\sum_{\rho_\psi}\frac{1}{|s-\rho_\psi|^2}.
$$

From (1.4) and (1.5), it follows that

$$
\begin{aligned}
(\sigma-\frac{1}{2})\sum_{\rho_\psi}\frac{1}{|s-\rho_\psi|^2} &= (\sigma-\frac{1}{2})\sum_{\rho_\psi}\frac{1}{(\sigma-\frac{1}{2})^2+(t-\gamma_\psi)^2}\\
&= \Re\left(\sum_{\rho_\psi}\frac{1}{s-\rho_\psi}\right)\\
&= \Re\left(\frac{L'}{L}(s,\psi)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)+\frac{1}{2}\log\frac{q}{\pi}\right)\\
&= \Re\frac{L'}{L}(s,\psi)+\frac{1}{2}\log\tau+\frac{1}{2}\log\frac{q}{\pi}+O(1).
\end{aligned}
$$

So we have for a complex number θ with $|\theta|\leqslant 1$,

$$
\begin{aligned}
\frac{L'}{L}(s,\psi) = -\sum_{n\leqslant xy}\frac{w(n)\Lambda(n)\psi(n)}{n^s}+\frac{\theta 2x^{\frac{1}{2}-\sigma}}{(\sigma-\frac{1}{2})\log y}\left|\Re\frac{L'}{L}(s,\psi)\right|\\
+O\left(\frac{x^{\frac{1}{2}-\sigma}\log q\tau}{(\sigma-\frac{1}{2})\log y}\right).
\end{aligned}
\tag{2.10}
$$

Thus if

$$\frac{2x^{\frac{1}{2}-\sigma}}{(\sigma-\frac{1}{2})\log y}\leqslant c<1, \tag{2.11}$$

then we have

$$\frac{L'}{L}(s,\psi)\ll\left|\sum_{n\leqslant xy}\frac{w(n)\Lambda(n)\psi(n)}{n^s}\right|+\frac{x^{\frac{1}{2}-\sigma}\log q\tau}{(\sigma-\frac{1}{2})\log y}.$$

We take

$$y=\exp\left(\frac{1}{\sigma-\frac{1}{2}}\right),\quad x=\frac{(\log q\tau)^2}{y}.$$

Note that $\sigma\leq 3/2$ implies $y\geq 2$. Now we have ,

$$\frac{2x^{\frac{1}{2}-\sigma}}{(\sigma-\frac{1}{2})\log y}=2e(\log q\tau)^{1-2\sigma}.$$

Thus for $\sigma\geqslant\frac{1}{2}+\frac{1}{\log\log q\tau}$ and $c=\frac{2}{e}$, inequality (2.11) holds. We observe that

$$\sum_{n\leqslant xy}\frac{w(n)\Lambda(n)\psi(n)}{n^s}\ll\sum_{n\leqslant(\log q\tau)^2}\frac{\Lambda(n)}{n^{\frac{1}{2}}}\ll\log q\tau,$$

uniformly for $\sigma\geqslant\frac{1}{2}$. Thus we obtain

$$\frac{L'}{L}(s,\psi)\ll\log q\tau, \tag{2.12}$$

uniformly for $1/2+1/\log\log q\tau\leq\sigma\leq 3/2$. By using (2.12) in (2.10), we get

$$\left|\frac{L'}{L}(s,\psi)\right|\leq\sum_{n\leqslant(\log q\tau)^2}\frac{\Lambda(n)}{n^\sigma}+O((\log q\tau)^{2-2\sigma}) \tag{2.13}$$

uniformly for $1/2+1/\log\log q\tau\leq\sigma\leq 3/2$.

Now we write

$$\sum_{n\leqslant(\log q\tau)^2}\frac{\Lambda(n)}{n^\sigma}=\sum_{0\le k\le 2\log\log q\tau}\ \sum_{e^k\le n<e^{k+1}}\frac{\Lambda(n)}{n^\sigma}. \tag{2.14}$$

By Chebyshev's estimate on the von-Mangoldt function $\sum_{n\le z}\Lambda(n)\asymp z$, we get

$$\sum_{e^k\le n<e^{k+1}}\frac{\Lambda(n)}{n^\sigma}\ll e^{k(1-\sigma)}, \tag{2.15}$$

and

$$\sum_{0\le k\le 2\log\log q\tau} e^{k(1-\sigma)}\ll(\log q\tau)^{2-2\sigma},$$

if σ is away from 1 in the sense that $\frac{1}{|\sigma-1|}<\log\log q\tau$. If, on the other hand, σ is close to 1 such that $\frac{1}{|\sigma-1|}>\log\log q\tau$, then on the right hand side of (2.14) we sum at most $2\log\log q\tau$ terms which are $\ll 1$ by (2.15). Combining these results, we obtain

$$\frac{L'}{L}(s,\psi)\ll((\log q\tau)^{2-2\sigma}+1)\min\left(\frac{1}{|\sigma-1|},\log\log q\tau\right),$$

uniformly for $1/2+1/\log\log q\tau\le\sigma\le 3/2$. □

Lemma 2.11. *(Perron's Formula) Let* $A(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ *converge absolutely for* $\sigma=\Re s>1$ *and let* $|a_n|<C\Phi(n)$ *where* $C>0$ *and for* $x\ge x_0$, $\Phi(x)$ *is monotonically increasing. Let further* $\sum_{n=1}^{\infty}|a_n|n^{-s}\ll(\sigma-1)^{-\alpha}$ *as* $\sigma\to 1^+$ *for some* $\alpha>0$. *If* $w=u+iv$ *(u; v real) is arbitrary,* $b>0$, $T>0$, $u+b>1$, *then*

$$\begin{aligned}\sum_{n\le x}a_n n^{-w}=&\frac{1}{2\pi i}\int_{b-iT}^{b+iT}A(s+w)\frac{x^s}{s}\,ds+O\left(\frac{x^b}{T(u+b-1)^\alpha}\right)\\&+O\left(\frac{\Phi(x)x^{1-u}\log 2x}{T}\right)+O\left(\frac{\Phi(2x)}{x^u}\right),\end{aligned}$$

and the estimate is uniform in x, T, b, *and* u *provided that* b *and* u *are bounded.*

Lemma 2.12. *For $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, and $\sigma > 1$, define*

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v)}{n^s} := \frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s) \right)^{i_w}.$$

Then for a Dirichlet character ψ modulo $q \geqslant 1$ and for $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v)\psi(n)}{n^s} = \frac{L^{(v+1)}}{L}(s, \psi) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s, \psi) \right)^{i_w}.$$

Proof. For a natural number r, we have

$$\begin{aligned} \frac{\zeta^{(r)}}{\zeta}(s) = \zeta^{(r)}(s) \frac{1}{\zeta(s)} &= \left((-1)^r \sum_{n=1}^{\infty} \frac{(\log n)^r}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right) \\ &= (-1)^r \sum_{n=1}^{\infty} \frac{\Lambda_r(n)}{n^s}, \quad (\sigma > 1), \end{aligned} \tag{2.16}$$

where $\Lambda_r(n) := (\mu * \log^r)(n) = \sum_{d|n} \mu(d)(\log(n/d))^r$. So, we have

$$\frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s) \right)^{i_w} = (-1)^{K+1} \sum_{n=1}^{\infty} \frac{\Lambda_{v+1} * \Lambda_1^{i_1 *} * ... * \Lambda_k^{i_k *}(n)}{n^s},$$

where for a natural number t, $\Lambda_r(n)^{t*}$ denote the t times Dirichlet convolution of $\Lambda_r(n)$ with itself and $K := i_1 + 2i_2 + ... + ki_k + v$. Similarly, by using the total multiplicativity of $\psi(n)$, we have

$$\begin{aligned} \frac{L^{(r)}}{L}(s, \psi) &= \left((-1)^r \sum_{n=1}^{\infty} \frac{(\log n)^r \psi(n)}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{\mu(n)\psi(n)}{n^s} \right) \\ &= (-1)^r \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \mu(d)\psi(d) \log(n/d)\psi(n/d) \\ &= (-1)^r \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} \sum_{d|n} \mu(d) \log(n/d) \\ &= (-1)^r \sum_{n=1}^{\infty} \frac{\Lambda_r(n)\psi(n)}{n^s}, \quad (\sigma > 1). \end{aligned}$$

Thus, by using the total multiplicativity of $\psi(n)$ again,

$$\begin{aligned}\frac{L^{(v+1)}}{L}(s,\psi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi)\right)^{i_w} &= (-1)^{K+1}\sum_{n=1}^{\infty}\frac{\Lambda_{v+1}*\Lambda_1^{i_1*}*...*\Lambda_k^{i_k*}(n)\psi(n)}{n^s}\\ &= \sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v)\psi(n)}{n^s}, \quad (\sigma>1).\end{aligned}$$

□

Lemma 2.13. *For $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, ψ_1 a Dirichlet character modulo $q_1 \geq 1$ and $\sigma > 1$, define*

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v;\psi_1)}{n^s} := \frac{L^{(v+1)}}{L}(s,\psi_1)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_1)\right)^{i_w}.$$

Then for a Dirichlet character ψ_2 modulo $q_2 \geq 1$ and $\sigma > 1$,

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v;\psi_1)\psi_2(n)}{n^s} = \frac{L^{(v+1)}}{L}(s,\psi_1\psi_2)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_1\psi_2)\right)^{i_w}.$$

Proof. The proof uses the total multiplicativity of Dirichlet characters and it is almost the same proof as in Lemma 2.12, hence it is omitted. □

Lemma 2.14. *For $k, i_1, i_2, ..., i_k, n \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, we have*

$$\Lambda_{v+1}*\Lambda_1^{i_1*}*...*\Lambda_k^{i_k*}(n) \ll (\log n)^{K+1},$$

where $K := i_1 + 2i_2 + ... + ki_k + v$.

Proof. From (2.16)

$$\sum_{n=1}^{\infty}\Lambda_k(n)n^{-s} = (-1)^k\frac{\zeta^{(k)}}{\zeta}(s) \quad (\sigma>1), \tag{2.17}$$

where $\Lambda_k(n) = \sum_{d|n} \mu(d)(\log \frac{n}{d})^k$. By the Möbius inversion formula, we have

$$(\log n)^k = \sum_{d|n} \Lambda_k(d) \tag{2.18}$$

Taking $k = 1$ in (2.17) shows that $\Lambda_1(n) = \Lambda(n)$ for any natural number n. By considering the coefficients of the identity

$$\left(\frac{\zeta^{(k)}}{\zeta}(s)\right)' = \frac{\zeta^{(k+1)}}{\zeta}(s) - \frac{\zeta^{(k)}}{\zeta}(s)\frac{\zeta'}{\zeta}(s) \tag{2.19}$$

we see

$$\Lambda_{k+1}(n) = \Lambda_k(n)\log n + \sum_{d|n} \Lambda_k(d)\Lambda_1(n/d) \tag{2.20}$$

By definition $\Lambda_1(n) \le \log n$. Thus,

$$\begin{aligned}\Lambda_{k+1}(n) &\le \Lambda_k(n)\log n + \log n \sum_{d|n} \Lambda_k(d)\\ &= \Lambda_k(n)\log n + (\log n)^{k+1}\\ &\le 2(\log n)^{k+1},\end{aligned}$$

if we assume that $\Lambda_k(n) \le (\log n)^k$. This induction argument shows that

$$\Lambda_k(n) \ll (\log n)^k, \tag{2.21}$$

for any natural number k. Now, we observe that for any natural numbers m, t

$$\begin{aligned}\Lambda_m * \Lambda_t(n) &= \sum_{d|n} \Lambda_m(d)\Lambda_t(n/d)\\ &\ll (\log n)^t \sum_{d|n} \Lambda_m(d)\\ &= (\log n)^{m+t}.\end{aligned}$$

Hence by an induction argument again, the desired result can easily be obtained. $\square$

The following is Lemma 2 in [12]:

Lemma 2.15. *For large A and $A < r \leqslant B \leqslant 2A$,*

$$\int_A^B \exp\left(it\log\frac{t}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt$$
$$= (2\pi)^{1-a} r^a e^{-ir+i\pi/4} + E(r, A, B),$$

where a is fixed and where

$$E(r, A, B) = O(A^{a-\frac{1}{2}}) + O\left(\frac{A^{a+\frac{1}{2}}}{|A-r| + A^{\frac{1}{2}}}\right) + O\left(\frac{B^{a+\frac{1}{2}}}{|B-r| + B^{\frac{1}{2}}}\right).$$

For $r \leqslant A$ or $r > B$,

$$\int_A^B \exp\left(it\log\frac{t}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt = E(r, A, B).$$

The following is Lemma 2.1 in [11]:

Lemma 2.16. *Let A be large and $m \in \mathbb{Z}$ with $|m| = o(\log A)$. We have, for $A < r \leqslant B \leqslant 2A$,*

$$\int_A^B \exp\left(it\log\frac{t}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m dt$$
$$= (2\pi)^{1-a} r^a e^{-ir+i\pi/4}\left(\log\frac{r}{2\pi}\right)^m + E(r, A, B)(\log A)^m,$$

while for $r \leqslant A$ or $r > B$,

$$\int_A^B \exp\left(it\log\frac{t}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m dt = K_0^{|m|} E(r, A, B)(\log A)^m.$$

Here

$$E(r,A,B)=O(A^{a-\frac{1}{2}})+O\left(\frac{A^{a+\frac{1}{2}}}{|A-r|+A^{\frac{1}{2}}}\right)+O\left(\frac{B^{a+\frac{1}{2}}}{|B-r|+B^{\frac{1}{2}}}\right),$$

the constants implied in the O-terms do not depend on m, K_0 can be taken to be any fixed number >1 when m is negative and to be 1 when m is non-negative.

Now we restate Lemma 2.16 with a slight modification, and give the proof because the proof is not given in [11].

Lemma 2.17. *Let A be large and $m\in\mathbb{Z}$ with $|m|=o(\log A)$. We have, for $A\leqslant\frac{r}{q}\leqslant B\leqslant 2A$,*

$$\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$=q^{-a}(2\pi)^{1-a}r^a e^{-\frac{ir}{q}+\frac{i\pi}{4}}\left(\log\frac{r}{2\pi}\right)^m+E\left(\frac{r}{q},A,B\right)(\log qA)^m,$$

while for $\frac{r}{q}<A$ or $\frac{r}{q}>B$,

$$\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m dt=K_0^{|m|}E\left(\frac{r}{q},A,B\right)(\log qA)^m.$$

Here

$$E\left(\frac{r}{q},A,B\right)=O(A^{a-\frac{1}{2}})+O\left(\frac{A^{a+\frac{1}{2}}}{|A-r|+A^{\frac{1}{2}}}\right)+O\left(\frac{B^{a+\frac{1}{2}}}{|B-r|+B^{\frac{1}{2}}}\right),$$

the constants implied in the O-terms do not depend on m, K_0 can be taken to be any fixed number >1 when m is negative and to be 1 when m is non-negative.

Proof. The case when m is a positive integer is covered by [6]. The case when $m=0$ can be easily obtained from Lemma 2.15 by replacing r with $\frac{r}{q}$. Thus, we will only consider the case when m is negative. For $m=-1$, we have

$$
\begin{aligned}
&\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-1}dt\\
=&\frac{1}{\log\left(\frac{r}{2\pi}\right)}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}dt\\
&-\frac{1}{\log\left(\frac{r}{2\pi}\right)}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\log\frac{qt}{r}\left(\log\frac{qt}{2\pi}\right)^{-1}dt, \qquad (2.22)
\end{aligned}
$$

since $\left(\log\frac{qt}{2\pi}\right)^{-1}=\left(\log\frac{r}{2\pi}\right)^{-1}\left(1-\frac{\log\frac{qt}{r}}{\log\frac{qt}{2\pi}}\right)$. The first integral on the right hand side of (2.22) is just the case when $m=0$. We apply integration by parts to the second integral on the right hand side of (2.22) and obtain

$$
\begin{aligned}
&\frac{-1}{\log\left(\frac{r}{2\pi}\right)}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\log\frac{qt}{r}\left(\log\frac{qt}{2\pi}\right)^{-1}dt\\
&=-\frac{1}{\log\left(\frac{r}{2\pi}\right)}\left[-i\exp\left(it\log\frac{t}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-1}\right]_A^B\\
&\quad+\frac{-i}{\log\left(\frac{r}{2\pi}\right)}\int_A^B \exp\left(it\log\frac{t}{re}\right)\frac{t^{a-\frac{3}{2}}}{\log\left(\frac{qt}{2\pi}\right)(2\pi)^{a-1/2}}\left(a-\frac{1}{2}-\frac{1}{\log\left(\frac{qt}{2\pi}\right)}\right)dt\\
&\ll\left|\frac{1}{\log\left(\frac{r}{2\pi}\right)}\right|A^{a-\frac{1}{2}}\left(\log qA\right)^{-1}.
\end{aligned}
$$

In general, for $m\in\mathbb{Z}^+$, we have

$$
\begin{aligned}
&\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m}dt\\
=&\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m+1}\left(\log\frac{qt}{2\pi}\right)^{-1}dt\\
=&\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m+1}\left(\log\frac{r}{2\pi}\right)^{-1}\left(1-\frac{\log\frac{qt}{r}}{\log\frac{qt}{2\pi}}\right)dt.
\end{aligned}
$$

Now, let $I_m := \int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m} dt$, then

$$\begin{aligned}
I_m =& \left(\log\frac{r}{2\pi}\right)^{-1}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m+1} \\
&-\left(\log\frac{r}{2\pi}\right)^{-1}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\log\frac{qt}{r}\left(\log\frac{qt}{2\pi}\right)^{-m} dt, \\
=& \left(\log\frac{r}{2\pi}\right)^{-1} I_{m-1} \\
&-\left(\log\frac{r}{2\pi}\right)^{-1}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\log\frac{qt}{r}\left(\log\frac{qt}{2\pi}\right)^{-m} dt. \qquad (2.23)
\end{aligned}$$

Using $\frac{d}{dt}\left(\exp\left(it\log\frac{qt}{re}\right)\right) = i\left(\exp\left(it\log\frac{qt}{re}\right)\right)\log\frac{qt}{r}$ and integration by parts, we see that the last term in (2.23) is

$$\begin{aligned}
=& i\left(\log\frac{r}{2\pi}\right)^{-1}\left[\exp\left(it\log\frac{t}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m}\right]_A^B \\
&-i\left(\log\frac{r}{2\pi}\right)^{-1}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\frac{t^{a-3/2}}{(2\pi)^{a-1/2}}\left(\log\frac{qt}{2\pi}\right)^{-m}\left[a-\frac{1}{2}-m\left(\log\frac{qt}{2\pi}\right)^{-1}\right] \\
\ll& \left|\left(\log\frac{r}{2\pi}\right)^{-1}\right| A^{a-\frac{1}{2}}\left(\log qA\right)^{-m}.
\end{aligned}$$

Thus the induction principle completes the proof. □

The following lemma is Lemma 2.13 in [6].

Lemma 2.18. *Let A be large and $A < B < 2A$, $E(r, A, B)$ be defined as in Lemma 2.15. Assume that $(b_n)_{n\geqslant 1}$ is a sequence of complex numbers such that $b_n \ll n^\epsilon$ for any $\epsilon > 0$. Then if $a > 1$,*

$$\sum_{n=1}^{\infty}\frac{b_n}{n^a}E(2\pi n/q, A, B) \ll_q A^{a-\frac{1}{2}}.$$

Lemma 2.19. *Let T be large, ψ be a primitive Dirichlet character modulo $q \geqslant 3$ and*

$(b_n)_{n\geqslant 1}$ be a sequence of complex numbers such that $b_n \ll n^\epsilon$ for any $\epsilon > 0$. For $m \in \mathbb{Z}$ with $|m| = o(\log T)$ as $T \to \infty$ and $a > 1$, we have

$$\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right)\chi_\psi(1-(a+it))\left(\log\frac{t}{2\pi}\right)^m dt$$
$$=\frac{\tau(\psi)}{q}\sum_{\frac{qT}{4\pi}<n\leqslant\frac{qT}{2\pi}} b_n\left(\log\frac{n}{q}\right)^m e^{-\frac{2\pi in}{q}} + O_q(K_0^{|m|}T^{a-\frac{1}{2}}(\log T)^m).$$

Proof. By Lemma 2.4, we have

$$\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right)\chi_\psi(1-(a+it))\left(\log\frac{t}{2\pi}\right)^m dt$$
$$=\frac{\tau(\psi)e^{-\frac{\pi i}{4}}}{2\pi q^{\frac{1}{2}}}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{qt}{2\pi e}\right)\left(\log\frac{t}{2\pi}\right)^m dt$$
$$+O_q\left(\int_{T/2}^{T}\left|\sum_{n=1}^{\infty}\frac{b_n}{n^a}\right|t^{a-\frac{3}{2}}(\log t)^m dt\right) \tag{2.24}$$

Since $b_n \ll n^\epsilon$ and $a > 1$, we have

$$\sum_{n=1}^{\infty}\frac{|b_n|}{n^a} \ll 1.$$

So the error term in (2.24) is $\ll T^{a-\frac{1}{2}}(\log T)^m$. The main term in (2.24) is

$$=\frac{\tau(\psi)e^{-\frac{i\pi}{4}}q^{a-\frac{1}{2}}}{2\pi q^{\frac{1}{2}}}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^a}\int_{T/2}^{T}\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m\exp\left(it\log\frac{qt}{2\pi ne}\right)dt\right),$$

changing the order of summation and integration is justified by the absolute convergence of the above series.

By taking $A = T/2, B = T$ and $r = \frac{2\pi n}{q}$ in Lemma 2.16, we get,

$$
\begin{aligned}
&\frac{\tau(\psi)e^{-\frac{i\pi}{4}}q^{a-1}}{2\pi}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^a}\int_{T/2}^{T}\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m \exp\left(it\log\frac{qt}{2\pi ne}\right)\, dt\right)\\
&\qquad=\frac{\tau(\psi)e^{-\frac{i\pi}{4}}q^{a-1}}{2\pi}\left(\sum_{\frac{qT}{4\pi}<n\leqslant\frac{qT}{2\pi}}\frac{b_n}{n^a}(2\pi)^{1-a}\left(\frac{2\pi n}{q}\right)^a e^{-\frac{2\pi in}{q}+\frac{\pi i}{4}}\left(\log\frac{2\pi n}{q2\pi}\right)^m\right)\\
&\qquad\quad+\frac{\tau(\psi)e^{-\frac{i\pi}{4}}q^{a-1}}{2\pi}\left(\log\frac{T}{2}\right)^m\sum_{n=1}^{\infty}\frac{b_n}{n^a}E\left(\frac{2\pi n}{q},\frac{T}{2},T\right)\\
&\qquad=\frac{\tau(\psi)}{q}\sum_{\frac{qT}{4\pi}<n\leqslant\frac{qT}{2\pi}}b_n\left(\log\frac{n}{q}\right)^m e^{-\frac{2\pi in}{q}}\\
&\qquad\quad+O_q\left(\left(\log\frac{T}{2}\right)^m\sum_{n=1}^{\infty}\frac{b_n}{n^a}E\left(\frac{2\pi n}{q},\frac{T}{2},T\right)\right)
\end{aligned}\tag{2.25}
$$

Now applying Lemma 2.18,it follows that the last expression in 2.25 is

$$
=\frac{\tau(\psi)}{q}\sum_{\frac{qT}{4\pi}<n\leqslant\frac{qT}{2\pi}}b_n\left(\log\frac{n}{q}\right)^m e^{-\frac{2\pi in}{q}}+O_q(T^{a-\frac{1}{2}}(\log T)^m).
$$

Hence the result follows. □

Lemma 2.20. *Let $q_1 \geqslant 1$ be an integer and ψ_2 be a primitive Dirichlet character modulo $q_2 \geqslant 3$ and $(b_n)_{n\geqslant 1}$ be a sequence of complex numbers such that $b_n \ll n^\epsilon$ for any $\epsilon > 0$. For $m \in \mathbb{Z}$ with $|m| = o(\log T)$ as $T \to \infty$ and $a > 1$, we have*

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right)\chi_{\psi_2}(1-(a+it))\left(\log\frac{q_1 t}{2\pi}\right)^m dt\\
&\qquad=\frac{\tau(\psi_2)}{q_2}\sum_{\frac{q_2T}{4\pi}<n\leqslant\frac{q_2T}{2\pi}}b_n\left(\log\frac{q_1 n}{q_2}\right)^m e^{-\frac{2\pi in}{q_2}}\\
&\qquad\quad+O_{q_1,q_2}(K_0^{|m|}T^{a-\frac{1}{2}}(\log T)^m).
\end{aligned}
$$

Proof. The proof is quite similar to the proof of Lemma 2.19.

By Lemma 2.4, we have

$$\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right)\chi_{\psi_2}(1-(a+it))\left(\log\frac{q_1t}{2\pi}\right)^m dt$$
$$=\frac{\tau(\psi_2)e^{-\frac{\pi i}{4}}}{2\pi q_2^{\frac{1}{2}}}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right)\left(\frac{q_2t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{q_2t}{2\pi e}\right)\left(\log\frac{q_1t}{2\pi}\right)^m dt$$
$$+O_{q_2}\left(\int_{T/2}^{T}\left|\sum_{n=1}^{\infty}\frac{b_n}{n^a}\right|t^{a-\frac{3}{2}}(\log q_1t)^m dt\right).$$

Since $b_n \ll n^{\epsilon}$ and $a>1$, we have $\sum_{n=1}^{\infty}\frac{|b_n|}{n^a}\ll 1$. So the error term above is $\ll_{q_1,q_2}$ $T^{a-\frac{1}{2}}(\log T)^m$. The main term of the last expression is

$$=\frac{\tau(\psi_2)e^{-\frac{i\pi}{4}}q_2^{a-\frac{1}{2}}}{2\pi q_2^{\frac{1}{2}}}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^a}\int_{T/2}^{T}\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{q_2t}{2\pi ne}\right)\left(\log\frac{q_1t}{2\pi}\right)^m dt\right),$$

changing the order of summation and integration is justified by the absolute convergence of the series $\sum_{n=1}^{\infty}\frac{b_n}{n^a}$.

We put $A=T/2$, $B=T$, $q=q_1$, $r=\frac{q_12\pi n}{q_2}$ in Lemma 2.17 and we get,

$$\frac{\tau(\psi_2)e^{-\frac{i\pi}{4}}q_2^{a-1}}{2\pi}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^a}\int_{T/2}^{T}\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{q_2t}{2\pi ne}\right)\left(\log\frac{q_1t}{2\pi}\right)^m dt\right)$$
$$=\frac{\tau(\psi_2)e^{-\frac{i\pi}{4}}q_2^{a-1}}{2\pi}\left(\sum_{\frac{q_2T}{4\pi}<n\leqslant\frac{q_2T}{2\pi}}\frac{b_n}{n^a}(2\pi)^{1-a}\left(\frac{2\pi n}{q_2}\right)^a e^{-\frac{2\pi in}{q_2}+\frac{\pi i}{4}}\left(\log\frac{q_12\pi n}{q_22\pi}\right)^m\right)$$
$$+\frac{\tau(\psi_2)e^{-\frac{i\pi}{4}}q_2^{a-1}}{2\pi}\left(\log\frac{T}{2}\right)^m\sum_{n=1}^{\infty}\frac{b_n}{n^a}E\left(\frac{2\pi n}{q_2},\frac{T}{2},T\right)$$
$$=\frac{\tau(\psi_2)}{q_2}\sum_{\frac{q_2T}{4\pi}<n\leqslant\frac{q_2T}{2\pi}}b_n\left(\log\frac{q_1n}{q_2}\right)^m e^{-\frac{2\pi in}{q_2}}$$
$$+O_{q_1,q_2}\left(K_0^{|m|}T^{a-\frac{1}{2}}(\log T)^m\right),$$

where the last equation follows by Lemma 2.18. Hence the result follows. ∎

Lemma 2.21. *Let q_1, q_2 be two positive integers. Let ψ be a primitive Dirichlet character (mod q_1). For $\delta > 0$, and for large A and B with $A < B \leqslant 2A$,*

$$\int_A^B \chi_\psi(-\delta - it)\log\left(\frac{q_2 t}{2\pi}\right) dt \ll_{q_1,q_2} A^{\frac{1}{2}+\delta}\log A.$$

Proof. We take $m = 1$, $b_1 = 1$ and $b_n = 0$ for $n > 1$ in Lemma 2.20. ∎

Lemma 2.22. *Assume GRH. Let q be an odd prime and ψ be a primitive Dirichlet character modulo q. For $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, $K := i_1 + 2i_2 + ... + ki_k + v$ and $\sigma > 1$, define*

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v;\overline{\psi})}{n^s} := \frac{L^{(v+1)}}{L}(s,\overline{\psi})\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi})\right)^{i_w}.$$

If ψ' is a Dirichlet character modulo q such that $\psi' \neq \psi$, then we have

$$\sum_{n\leq x} b_n(i_1,...,i_k;v;\overline{\psi})\psi'(n) = O(x^{1-\frac{1}{\log\log q(x+4)}}(\log x)(A(k)\log\log q(x+4))^{K+1}).$$

Moreover, we have (unconditionally),

$$\sum_{n\leq x} b_n(i_1,...,i_k;v;\overline{\psi})\psi(n) = S(i_1,...,i_k;v)x(\log x)^K + E_b(i_1,...,i_k;v),$$

where

$$S(i_1,...,i_k;v) := \frac{(-1)^{K+1}(v+1)!\prod_{w=1}^{k}(w!)^{i_w}}{K!},$$
$$E_b(i_1,...,i_k;v) := O_q\left((A(k)^K)\left((\log x)^{K+2} + \frac{x(\log x)^{K-1}}{(K-1)!} + \frac{x(\log x)^{(\frac{2}{3}+\epsilon)(K+3)}}{e^{\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon}}}\right)\right).$$

Proof. Let ψ' be a Dirichlet character modulo q such that $\psi' \neq \psi$. By Lemma 2.13, we have

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v;\overline{\psi})\psi'(n)}{n^s}=\frac{L^{(v+1)}}{L}(s,\overline{\psi}\psi')\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi}\psi')\right)^{i_w},$$

for $\sigma>1$. By Lemma 2.14, we know that $|b_n(i_1,...,i_k,v;\overline{\psi})\psi'(n)|\le(\log n)^{K+1}$. Thus, we have

$$\sum_{n\le x}b_n(i_1,...,i_k,v;\overline{\psi})\psi'(n)=\frac{1}{2\pi i}\int\limits_{1+\frac{1}{\log x}-ix}^{1+\frac{1}{\log x}+ix}\frac{L^{(v+1)}}{L}(s,\overline{\psi}\psi')\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi}\psi')\right)^{i_w}\frac{x^s}{s}\,ds$$
$$+O\left((A(k)\log x)^{K+2}\right),\tag{2.26}$$

by Lemma 2.11. Note that $\overline{\psi}\psi'$ is a non-principal Dirichlet character modulo q. Thus, the integrand in (2.26) has no poles inside the rectangle with vertices $1+\frac{1}{\log x}-ix$, $1+\frac{1}{\log x}+ix$, $1-\frac{1}{\log\log q(x+4)}+ix$ and $1-\frac{1}{\log\log q(x+4)}-ix$. Now, by the residue theorem, we have

$$\sum_{n\le x}b_n(i_1,...,i_k,v;\overline{\psi})\psi'(n)=I_1+I_2+I_3+O\left((A(k)\log x)^{K+2}\right),$$

where

$$I_1:=-\frac{1}{2\pi i}\int\limits_{1+\frac{1}{\log x}+ix}^{1-\frac{1}{\log\log q(x+4)}+ix}\frac{L^{(v+1)}}{L}(s,\overline{\psi}\psi')\prod_{w=1}^{k}\frac{L^{(w)}}{L}(s,\overline{\psi}\psi')\frac{x^s}{s}\,ds,$$

$$I_2:=-\frac{1}{2\pi i}\int\limits_{1-\frac{1}{\log\log q(x+4)}+ix}^{1-\frac{1}{\log\log q(x+4)}-ix}\frac{L^{(v+1)}}{L}(s,\overline{\psi}\psi')\prod_{w=1}^{k}\frac{L^{(w)}}{L}(s,\overline{\psi}\psi')\frac{x^s}{s}\,ds,$$

$$I_3:=-\frac{1}{2\pi i}\int\limits_{1-\frac{1}{\log\log q(x+4)}-ix}^{1+\frac{1}{\log x}-ix}\frac{L^{(v+1)}}{L}(s,\overline{\psi}\psi')\prod_{w=1}^{k}\frac{L^{(w)}}{L}(s,\overline{\psi}\psi')\frac{x^s}{s}\,ds.$$

Along the line of integration in I_1, we have $s=\sigma+ix$ with $1-\frac{1}{\log\log q(x+4)}\le\sigma\le1+\frac{1}{\log x}$. Note that for large x, $\frac{1}{2}+\frac{1}{\log\log q(x+4)}\le1-\frac{1}{\log\log q(x+4)}$. Thus, we can use Lemma 2.10

and get

$$\begin{aligned}\frac{L'}{L}(s,\overline{\psi}\psi') &\ll ((\log q\tau)^{2-2\sigma}+1)\min\left(\frac{1}{|\sigma-1|},\log\log q\tau\right)\\ &\ll ((\log q(x+4))^{2-2\sigma}+1)\log\log q(x+4)\\ &\ll ((\log q(x+4))^{2-2(1-1/\log\log(x+4))}+1)\log\log q(x+4)\\ &\ll \log\log q(x+4).\end{aligned}$$

By Cauchy's theorem, for $r := 1/\log\log(x+4)$,

$$\frac{d}{ds}\frac{L'}{L}(s,\overline{\psi}\psi') = \frac{1}{2\pi i}\int_{|w-s|=r}\frac{\frac{L'}{L}(s,\overline{\psi}\psi)}{(w-s)^2}\,dw \ll \frac{\log\log(x+4)}{r^2}r = (\log\log(x+4))^2,$$

and

$$\begin{aligned}\frac{L''}{L}(s,\overline{\psi}\psi') &= \frac{d}{ds}\frac{L'}{L}(s,\overline{\psi}\psi') + \left(\frac{L'}{L}(s,\overline{\psi}\psi)\right)^2\\ &\ll (\log\log q(x+4))^2 + (\log\log q(x+4))^2\\ &\ll (\log\log q(x+4))^2.\end{aligned}$$

Now by using the identity $\frac{L^{(w+1)}}{L} = (\frac{L^{(w)}}{L})' + \frac{L^{(w)}}{L}\frac{L'}{L}$ and induction on w, we obtain

$$\frac{L^{(w)}}{L}(s,\overline{\psi}\psi') \ll_w (\log\log q(x+4))^w, \tag{2.27}$$

along the line of integration in I_1. Since $|\frac{x^s}{s}| \ll 1$ along this line, we have

$$I_1 \ll (A(k)\log\log q(x+4))^{K+1}.$$

By symmetry, we also have

$$I_3 \ll (A(k)\log\log q(x+4))^{K+1}.$$

Now, we will find an upper bound for I_2. Along the upper part of the line of integration in I_2, we have $s = 1 - \frac{1}{\log\log q(x+4)} + it$ with $0 \le t \le x$. Note that for large x, we have $\frac{1}{2} + \frac{1}{3} < 1 - \frac{1}{\log\log q(x+4)}$ and clearly, the inequality $\frac{1}{2} + \frac{1}{\log\log q(t+4)} \le \frac{1}{2} + \frac{1}{3}$ is equivalent to $t \ge \frac{\exp(\exp 3)}{q} - 4$. Thus, for $t \ge \left|\frac{\exp(\exp 3)}{q} - 4\right| =: t_0$, we can use Lemma 2.10. We split the upper part of the integral in I_2 as $\int_0^{t_0} + \int_{t_0}^{x}$. Note that for $t \ge t_0$, we have

$$\begin{aligned}\frac{L'}{L}(s, \overline{\psi}\psi') &\ll ((\log q\tau)^{2-2(1-\frac{1}{\log\log q(x+4)})} + 1) \min\left(\frac{1}{|1 - \frac{1}{\log\log q(x+4)} - 1|}, \log\log q\tau\right)\\ &\ll ((\log q(x+4))^{\frac{2}{\log\log q(x+4)}} + 1)(\log\log q(x+4))\\ &\ll \log\log q(x+4),\end{aligned}$$

by Lemma 2.10 and the same argument as in (2.27) shows that

$$\frac{L^{(w)}}{L}(s, \overline{\psi}\psi') \ll_w (\log\log q(x+4))^w,$$

for $t \ge t_0$. Thus, the part $\int_{t_0}^{x}$ in I_2 is

$$\begin{aligned}&\ll (A(k)\log\log q(x+4))^{K+1} x^{1-\frac{1}{\log\log q(x+4)}} \int_{\frac{\exp(\exp 3)}{q}-4}^{x} \frac{dt}{\sqrt{(\frac{1}{\log\log q(x+4)})^2 + t^2}}\\ &\ll (A(k)\log\log q(x+4))^{K+1} x^{1-\frac{1}{\log\log q(x+4)}} \left(\log\left(\sqrt{\frac{1}{\log\log q(x+4)})^2 + x^2} + x\right)\right)\\ &\ll x^{1-\frac{1}{\log\log q(x+4)}} (\log x)(A(k)\log\log q(x+4))^{K+1}.\end{aligned}$$

On the part $\int_0^{t_0}$ in I_2, we have $s = 1 - \frac{1}{\log\log q(x+4)} + it$, $0 \le t \le t_0$.

$$\frac{L^{(v+1)}}{L}(s, \overline{\psi}\psi') \prod_{w=1}^{k} \frac{L^{(w)}}{L}(s, \overline{\psi}\psi') \frac{x^s}{s} \ll x^{1-\frac{1}{\log\log q(x+4)}}.$$

Thus, the part $\int_0^{t_0}$ in I_2 is $\ll x^{1-\frac{1}{\log\log q(x+4)}}$. Combining the results for $\int_0^{t_0}$ and $\int_{t_0}^{x}$ and using symmetry for the lower part of I_2, we obtain

$$I_2 \ll x^{1-\frac{1}{\log\log q(x+4)}} (\log x)(A(k)\log\log q(x+4))^{K+1}.$$

Combining the results for I_1, I_2, and I_3 finishes the proof of the first statement in the lemma.

To prove the second statement in the lemma, we consider the sum $\sum_{n\le x} b_n(i_1,...,i_k;v;\overline{\psi})\psi(n)$. Note that $\overline{\psi}\psi = \psi_0$ where ψ_0 is the principal Dirichlet character modulo q and that

$$\begin{aligned} L(s,\psi_0) &= \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right) \\ &= \zeta(s)\left(1-\frac{1}{q^s}\right) \\ &=: \zeta(s)A_q(s), \quad (\sigma>1). \end{aligned} \tag{2.28}$$

By Lemma 2.13, we know

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v;\overline{\psi})\psi(n)}{n^s} = \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w} \quad (\sigma>1).$$

By Lemma 2.14 and 2.11, we have

$$\begin{aligned} \sum_{n\leqslant x} b_n(i_1,...,i_k;v;\overline{\psi})\psi(n) =& \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-ix}^{1+\frac{1}{\log x}+ix} \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds \\ &+ O((A(k)\log x)^{K+2}). \end{aligned}$$

Define $\delta_1(k) := \frac{\delta_1}{2^k}$ where δ_1 is the constant whose existence is given in Lemma 2.7 and let $\epsilon_0 > 0$ be a small fixed number. By the residue theorem, we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_C \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds \\ &\qquad = \mathrm{Res}_{s=1}\left\{\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right\}, \end{aligned} \tag{2.29}$$

where C is the positively oriented boundary of the rectangle with vertices $1+\frac{1}{\log x}-ix$,

$1+\frac{1}{\log x}+ix$, $1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+ix$ and $1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}-ix$. Note that the only pole of the integrand in (2.29) inside C is at $s=1$ and the order of this pole is $K+1$. Thus, for

$$I_1^0 := \int_{1+\frac{1}{\log x}+ix}^{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+ix} \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds,$$

$$I_2^0 := \int_{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+ix}^{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}-ix} \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds,$$

$$I_3^0 := \int_{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}-ix}^{1+\frac{1}{\log x}-ix} \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds,$$

we have

$$\sum_{n\leqslant x} b_n(i_1,...,i_k;v;\overline{\psi})\psi(n) = \mathrm{Res}_{s=1}\left\{\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right\}$$
$$-\frac{1}{2\pi i}\left(I_1^0+I_2^0+I_3^0\right)+O((A(k)\log x)^{K+2}). \qquad (2.30)$$

First we will find some upper bounds for I_1^0, I_2^0, I_3^0 and then evaluate the above residue. By (2.28) and Lemma 2.7, we have

$$\frac{L'}{L}(s,\psi_0) = \frac{\zeta'}{\zeta}(s)+\frac{A_q'}{A_q}(s) \ll_q (\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}} \ll_q (\log|t|)^{\frac{2}{3}+\epsilon_0}, \qquad (2.31)$$

in the region

$$\sigma \geqslant 1-\frac{\delta_1}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}, \quad |t|\geqslant 3. \qquad (2.32)$$

By (2.31) and Cauchy's theorem for $r=\frac{1}{((\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}})}$, we obtain

$$\frac{d}{ds}\frac{L'}{L}(s,\psi_0) = \frac{1}{2\pi i}\int_{|w-s|=r} \frac{\frac{L'}{L}(s,\psi_0)}{(w-s)^2}\,dw \ll \frac{(\log|t|)^{\frac{2}{3}+\epsilon_0}}{r^2}r \ll \left((\log|t|)^{\frac{2}{3}+\epsilon_0}\right)^2,$$

in the region given in (2.32). Now by using the identity $\frac{L^{(w+1)}}{L} = (\frac{L^{(w)}}{L})' + \frac{L^{(w)}}{L}\frac{L'}{L}$ and induction on w, we obtain

$$\frac{L^{(w)}}{L}(s,\psi_0) \ll_{w,q} (\log|t|)^{(\frac{2}{3}+\epsilon_0)w} \qquad \left(\sigma \geqslant 1 - \frac{\delta_1}{2^w(\log|t|)^{\frac{2}{3}+\epsilon_0}},\ |t| \geqslant 3\right).$$

Hence, on I_1^0, I_3^0 and the part of I_2^0 of distance $\geqslant 3$ from the real line, we have

$$\frac{L^{(w)}}{L}(s,\psi_0) = O_{w,q}\,(\log x)^{(\frac{2}{3}+\epsilon_0)w}\,. \tag{2.33}$$

Since $|\frac{x^s}{s}| \ll 1$ on the the horizontal parts of the contour, we get

$$I_1^0 \ll_q (A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+1)}\,,$$
$$I_3^0 \ll_q (A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+1)}\,.$$

For I_2^0, by symmetry it is enough to consider the part above the real axis which can be split up as $\int_0^3 + \int_3^x$. By (2.33), the second part is

$$\ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}(A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+1)}\int_3^x \frac{dt}{t}$$
$$\ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}(A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+3)}\,.$$

By (2.28), for $\sigma > 1$, we have $L^{(t)}(s,\psi_0) = \sum_{m=0}^{t}\binom{t}{m}\zeta^{(t-m)}(s)A_q^{(m)}(s)$, so that

$$\frac{L^{(t)}}{L}(s,\psi_0) = \sum_{m=0}^{t}\binom{t}{m}\frac{\zeta^{(t-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s), \tag{2.34}$$

and

$$\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}$$
$$=\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}.$$

Thus, by (1.1) as $s\to 1$.

$$\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\ll\left|\frac{\zeta^{(v+1)}}{\zeta}(s)\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}\right|$$
$$\ll\left(\frac{A(k)}{|s-1|}\right)^{K+1}. \tag{2.35}$$

Hence, by (2.35), the first part is

$$\ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}\,(A(k))^{K+1}\int\limits_{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}}^{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+3i}\frac{ds}{|s-1|^{K+1}}. \tag{2.36}$$

The integral in (2.36) is

$$\ll\int\limits_{0}^{\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}}\frac{dt}{\left(\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}\right)^{K+1}}+\int\limits_{\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}}^{3}\frac{dt}{t^{K+1}}$$
$$\ll\left(\frac{(\log x)^{\frac{2}{3}+\epsilon_0}}{\delta_1(k)}\right)^{K}.$$

Hence we have

$$I_2^0\ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}\,(A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+3)}.$$

Now by (2.30),

$$\sum_{n\leqslant x} b_n(i_1,...,i_k;v;\overline{\psi})\psi(n) = \mathrm{Res}_{s=1}\left\{\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right\}$$
$$+ O\left((A(k)\log x)^{K+2}\right)$$
$$+ O\left(xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}(A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+3)}\right). \quad (2.37)$$

The pole at $s=1$ is of order $K+1$, so the residue in (2.37) is

$$=\frac{1}{K!}\frac{d^K}{ds^K}\left\{(s-1)^{K+1}\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right\}_{s=1}$$
$$=\frac{1}{K!}\sum_{\substack{j_1+j_2+j_3=K\\ j_1,j_2,j_3\in\mathbb{N}}}\binom{K}{j_1,j_2,j_3}\frac{d^{j_1}}{ds^{j_1}}$$
$$\times\left\{(s-1)^{K+1}\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right\}_{s=1}$$
$$\times\frac{d^{j_2}}{ds^{j_2}}\left\{x^s\right\}_{s=1}\frac{d^{j_3}}{ds^{j_3}}\left\{\frac{1}{s}\right\}_{s=1} \quad (2.38)$$

Since $j_3 = K-j_1-j_2$, the expresssion in (2.38) is

$$=\frac{1}{K!}\sum_{j_1\leqslant K}\frac{K!}{j_1!j_2!(K-j_1-j_2)!}\frac{d^{j_1}}{ds^{j_1}}\left\{\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\right\}_{s=1}$$
$$\times\sum_{j_2\leqslant K-j_1}\frac{d^{j_2}}{ds^{j_2}}\left\{x^s\right\}_{s=1}\left\{\frac{(-1)^{K-j_1-j_2}(K-j_1-j_2)!}{s^{K-j_1-j_2+1}}\right\}_{s=1}$$
$$=(-1)^K\sum_{j_1\leqslant K}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\right\}_{s=1}$$
$$\times\sum_{j_2\leqslant K-j_1}\frac{(-1)^{j_2}}{j_2!}\left\{x^s(\log x)^{j_2}\right\}_{s=1}$$
$$=x(-1)^K\sum_{j_1\leqslant K}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\right\}_{s=1}$$
$$\times\sum_{j_2\leqslant K-j_1}\frac{(-1)^{j_2}}{j_2!}(\log x)^{j_2}. \quad (2.39)$$

Now, we consider three special cases for $i_t, 1\leq t\leq k$ and $v\in\mathbb{N}$.

If $i_1 = i_2 = ... = i_k = v = 0$, then the expression in (2.39) is

$$= x\left\{(s-1)\frac{L'}{L}(s,\psi_0)\right\}_{s=1} = x\left\{(s-1)\left(\frac{\zeta'}{\zeta}(s)+\frac{A_q'}{A_q}(s)\right)\right\}_{s=1} = -x.$$

If $i_1 = i_2 = ... = i_k = 0, v = 1$, then the expression in (2.39) is

$$\begin{aligned}
=&x(-1)^1\sum_{j_1\leqslant 1}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^2\frac{L^{(2)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\right\}_{s=1}\\
&\times\sum_{j_2\leqslant 1-j_1}\frac{(-1)^{j_2}}{j_2!}(\log x)^{j_2}\\
=&(-x+x\log x)\left\{(s-1)^2\frac{\zeta''A_q+2\zeta'A_q'+\zeta A_q''}{\zeta A_q}(s)\right\}_{s=1}\\
&+x\frac{d}{ds}\left\{(s-1)^2\frac{\zeta''A_q+2\zeta'A_q'+\zeta A_q''}{\zeta A_q}(s)\right\}_{s=1}\\
=&2x\log x-2x-x\frac{2A_q'}{A_q}(1).
\end{aligned}$$

If $i_1 = 1, i_2 = i_3 = ... = i_k = v = 0$, then the expression in (2.39) is

$$\begin{aligned}
=&-x\sum_{j_1\leqslant 1}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^2\frac{L'}{L}(s,\psi_0)\frac{L'}{L}(s,\psi_0\right\}_{s=1}\sum_{j_2\leqslant 1-j_1}\frac{(-1)^{j_2}\log^{j_2}x}{j_2!}\\
=&-x\left\{(s-1)^2\left(\frac{L'}{L}(s,\psi_0)\right)^2\right\}_{s=1}(1-\log x)\\
&+x\frac{d}{ds}\left\{(s-1)^2\left(\frac{L'}{L}(s,\psi_0)\right)^2\right\}_{s=1}\\
=&-x\left\{(s-1)^2\left(\frac{\zeta'}{\zeta}+\frac{A_q'}{A_q}\right)^2(s)\right\}_{s=1}(1-\log x)\\
&+x\frac{d}{ds}\left\{(s-1)^2\left(\frac{\zeta'}{\zeta}+\frac{A_q'}{A_q}\right)^2(s)\right\}_{s=1}\\
=&x\log x-x-2x\frac{A_q'}{A_q}(1).
\end{aligned}$$

Now, we consider the general case.

By (2.34), we have

$$
\begin{aligned}
&\operatorname{Res}_{s=1}\left(\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right)\\
=&x(-1)^K\sum_{j_1\leqslant K}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\right\}_{s=1}\\
&\times\sum_{j_2\leqslant K-j_1}\frac{(-1)^{j_2}}{j_2!}(\log x)^{j_2}\\
=&x(-1)^K\sum_{j_1\leqslant K}\frac{(-1)^{j_1}}{j_1!}\\
&\times\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\right.\\
&\qquad\left.\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\right\}_{s=1}\\
&\times\sum_{j_2\leqslant K-j_1}\frac{(-1)^{j_2}}{j_2!}(\log x)^{j_2}
\end{aligned}
$$

Now we split the double sum over j_1, j_2 into four parts: The term with $j_1 = 0$, $j_2 = K$ (which gives the main term) and $j_1 = 0$, $j_2 = K-1$ and $j_1 = 1$, $j_2 = K-1$ and the remaining terms. Thus,

$$
\begin{aligned}
&\operatorname{Res}_{s=1}\left(\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right)\\
=&x\frac{(\log x)^K}{K!}\left\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\right.\\
&\qquad\left.\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\right\}_{s=1}\\
&-x\frac{(\log x)^{K-1}}{(K-1)!}\left\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\right.\\
&\qquad\left.\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\right\}_{s=1}
\end{aligned}
$$

$$
\begin{aligned}
&+x\frac{(\log x)^{K-1}}{(K-1)!}\frac{d}{ds}\Bigg\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\\
&\quad\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\Bigg\}_{s=1}\\
&+x(-1)^K\sum_{j_1\leqslant K}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\Bigg\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\\
&\quad\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\Bigg\}_{s=1}\\
&\times\sum_{j_2\leqslant K-a(j_1)}\frac{(-1)^{j_2}}{j_2!}(\log x)^{j_2},
\end{aligned}
$$

where

$$
a(j_1)=\begin{cases}2, & \text{if } j_1=0,1\\ j_1, & \text{otherwise.}\end{cases}
$$

By (1.1) , we have

$$
\begin{aligned}
&\Bigg\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\\
&\quad\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\Bigg\}_{s=1}=(-1)^{K+1}(v+1)!\prod_{w=1}^{k}(w!)^{i_w}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathrm{Res}_{s=1}\left(\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right)\\
=&\frac{(-1)^{K+1}(v+1)!\prod_{w=1}^{k}(w!)^{i_w}}{K!}x(\log x)^K\\
&+\frac{(-1)^{K}(v+1)!\prod_{w=1}^{k}(w!)^{i_w}}{(K-1)!}x(\log x)^{K-1}\\
&+x\frac{(\log x)^{K-1}}{(K-1)!}\frac{d}{ds}\Bigg\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)
\end{aligned}
$$

$$\times \prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\Bigg\}_{s=1}$$
$$+ x(-1)^K\sum_{j_1\leqslant K}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\Bigg\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)$$
$$\times \prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\Bigg\}_{s=1}$$
$$\times \sum_{j_2\leqslant K-a(j_1)}\frac{(-1)^{j_2}}{j_2!}(\log x)^{j_2}. \tag{2.40}$$

Instead of the sum over j_2 in (2.40), we use Taylor remainder theorem and write

$$\frac{1}{x}-\frac{(-1)^{K-a(j_1)+1}(\log x)^{K-a(j_1)+1}}{(K-a(j_1)+1)!}x^{-\theta},$$

for some $\theta \in [0,1]$. Then the fourth term in (2.40) can be bounded as

$$\ll \sum_{j_1\leqslant K}\frac{1}{j_1!}\Bigg|\frac{d^{j_1}}{ds^{j_1}}\Bigg\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)$$
$$\times \prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\Bigg\}_{s=1}\Bigg|$$
$$+ x(\log x)^{K+1}\sum_{j_1\leqslant K}\frac{(\log x)^{-a(j_1)}}{j_1!(K-a(j_1)+1)!}$$
$$\times \Bigg|\frac{d^{j_1}}{ds^{j_1}}\Bigg\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)$$
$$\times \prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\Bigg\}_{s=1}\Bigg| \tag{2.41}$$

By Cauchy's estimate on a disk of radius 1 centered at $s=1$, we have

$$\Bigg|\frac{d^{j_1}}{ds^{j_1}}\Bigg\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)$$
$$\times \prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\Bigg\}_{s=1}\Bigg|$$

$$
\begin{aligned}
&\leqslant j_1! \max_{|s-1|=1} \left\{ \left| \sum_{m=0}^{v+1} \binom{v+1}{m} \frac{\zeta^{(v+1-m)}}{\zeta}(s) \frac{A_q^{(m)}}{A_q}(s) \right| \right. \\
&\qquad \left. \times \prod_{w=1}^{k} \left| \sum_{n=0}^{w} \binom{w}{n} \frac{\zeta^{(w-n)}}{\zeta}(s) \frac{A_q^{(n)}}{A_q}(s) \right| \right\} \\
&\ll_q j_1!(A(k))^K. \qquad (2.42)
\end{aligned}
$$

By using (2.42), the fourth term in (2.40) can be majorized by

$$
\begin{aligned}
&\ll (A(k))^K \left[\sum_{j_1 \leqslant K} 1 + x(\log x)^{K+1} \sum_{j_1 \leqslant K} \frac{(\log x)^{-a(j_1)}}{j_1!(K - a(j_1) + 1)!} \right] \\
&= (A(k))^K \left[K + x(\log x)^{K+1} \left(\frac{3(\log x)^{-2}}{(K-1)!} + \sum_{3 \leqslant j_1 \leqslant K} \frac{(\log x)^{j_1}}{(K - j_1 + 1)!} \right) \right].
\end{aligned}
$$

The condition $i_1 + ... + i_k \leqslant \frac{\log x}{\log\log x}$ implies

$$
K \leqslant \frac{k \log x}{\log\log x} + k, \qquad (2.43)
$$

so in the case that the last sum over j_1 is not void, we have

$$
\begin{aligned}
\sum_{3 \leqslant j_1 \leqslant K} \frac{(\log x)^{j_1}}{(K - j_1 + 1)!} &= \frac{(\log x)^{-3}}{(K-2)!} \left[1 + \frac{K-2}{\log x} + \frac{(K-2)(K-3)}{(\log x)^2} + ... + \frac{(K-2)!}{(\log x)^{K-3}} \right] \\
&< \frac{(\log x)^{-3}}{(K-2)!} \left[1 + \frac{2k}{\log\log x} + \left(\frac{2k}{\log\log x} \right)^2 + ... \right] \\
&< \frac{2(\log x)^{-3}}{(K-2)!} < \frac{3(\log x)^{-2}}{(K-1)! \log\log x},
\end{aligned}
$$

for large x. Thus, the expression in (2.41) is

$$
\ll \frac{(A(k))^K x(\log x)^{K-1}}{(K-1)!}. \qquad (2.44)
$$

By (2.42), the upper bound in (2.44) dominates the second and third terms in the right-hand side of (2.40) with an appropriate $A(k)$. Combining (2.37), (2.40) and (2.44) we obtain the desired result. □

Lemma 2.23. *Let ψ be a primitive Dirichlet character modulo q, $q \geqslant 3$. For $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, $K := i_1 + 2i_2 + ... + ki_k + v$ and $\sigma > 1$, define*

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v; \overline{\psi})}{n^s} := \frac{L^{(v+1)}}{L}(s, \overline{\psi}) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s, \overline{\psi}) \right)^{i_w}.$$

If ψ' is a Dirichlet character modulo q such that $\psi' \neq \psi$, then we have

$$\sum_{\frac{x}{2} < n \leq x} \frac{b_n(i_1, ..., i_k; v; \overline{\psi})\psi'(n)}{(\log n)^K} = O\left(\frac{x^{1 - \frac{1}{\log \log q(x+4)}} (A(k) \log \log q(x+4))^{K+1}}{(\log x)^{K-1}} \right).$$

Moreover, if also $i_1 + ... + i_k \leqslant \frac{\log x}{\log \log x}$, then we have

$$\sum_{\frac{x}{2} < n \leq x} \frac{b_n(i_1, ..., i_k; v; \overline{\psi})\psi(n)}{(\log n)^K} = S(i_1, ..., i_k; v)\frac{x}{2} + O\left(\frac{E_b(i_1, ..., i_k; v)}{(\log x)^K} \right),$$

where $S(i_1, ..., i_k; v)$ and $E_b(i_1, ..., i_k; v)$ are as in Lemma 2.22.

Proof. We apply partial summation to Lemma 2.22. □

Lemma 2.24. *For $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, $K := i_1 + 2i_2 + ... + ki_k + v$ and $\sigma > 1$, put*

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v)}{n^s} := \frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s) \right)^{i_w}.$$

If ψ is a nonprincipal Dirichlet character modulo q then we have

$$\sum_{n \leq x} b_n(i_1, ..., i_k, v)\psi(n) = O(x^{1 - \frac{1}{\log \log q(x+4)}} (\log x)(A(k) \log \log q(x+4))^{K+1}).$$

Moreover, if also $i_1 + ... + i_k \leqslant \frac{\log x}{\log \log x}$, then we have

$$\sum_{n \leq x} b_n(i_1, ..., i_k, v)\psi_0(n) = S(i_1, ..., i_k; v)x(\log x)^K + E_b(i_1, ..., i_k; v),$$

where $S(i_1, ..., i_k; v)$ and $E_b(i_1, ..., i_k; v)$ are as in Lemma 2.22.

Proof. This is a result of Lemma 2.22 together with Lemma 2.12. □

Lemma 2.25. *For $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, $K := i_1 + 2i_2 + ... + ki_k + v$ and $\sigma > 1$, put*

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v)}{n^s} := \frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s) \right)^{i_w}.$$

If ψ is a nonprincipal Dirichlet character modulo q then we have (unconditionally),

$$\sum_{\frac{x}{2}<n\leq x} \frac{b_n(i_1, ..., i_k, v)\psi(n)}{(\log \frac{n}{q})^K} = O\left(\frac{x^{1-\frac{1}{\log\log q(x+4)}} (A(k) \log\log q(x+4))^{K+1}}{(\log x)^K - 1} \right).$$

Moreover, if also $i_1 + ... + i_k \leqslant \frac{\log x}{\log\log x}$, then we have (unconditionally),

$$\sum_{\frac{x}{2}<n\leq x} \frac{b_n(i_1, ..., i_k, v)\psi_0(n)}{(\log \frac{n}{q})^K} = S(i_1, ..., i_k; v)\frac{x}{2} + O\left(\frac{E_b(i_1, ..., i_k; v)}{(\log x)^K} \right),$$

where $S(i_1, ..., i_k; v)$ and $E_b(i_1, ..., i_k; v)$ are as in Lemma 2.22.

Proof. We apply partial summation to Lemma 2.24.

$$\begin{aligned} &\sum_{\frac{x}{2}<n\leq x} \frac{b_n(i_1, ..., i_k, v)\psi(n)}{(\log \frac{n}{q})^K} \\ &= \left(\log \frac{x}{q}\right)^{-K} \sum_{n\leqslant x} b_n(i_1, ..., i_k, v)\psi(n) \\ &\quad - \left(\log \frac{x}{2q}\right)^{-K} \sum_{n\leqslant \frac{x}{2}} b_n(i_1, ..., i_k, v)\psi(n) \\ &\quad + \int_{\frac{x}{2}}^{x} \sum_{n\leqslant u} b_n(i_1, ..., i_k, v)\psi(n) \frac{K du}{u \left(\log \frac{u}{q}\right)^{K+1}} \end{aligned}$$

$$
\begin{aligned}
=&S(i_1,...,i_k;v)x\frac{(\log x)^K}{\left(\log\frac{x}{q}\right)^K}-S(i_1,...,i_k;v)\frac{x}{2}\frac{(\log\frac{x}{2})^K}{\left(\log\frac{x}{2q}\right)^K}\\
&+O_q\left(\frac{E_b(i_1,...,i_k;v)}{(\log x)^K}\right)\\
=&S(i_1,...,i_k;v)\frac{x}{2}\left(2\left(1+\frac{\log q}{\log x-\log q}\right)^K-\left(1+\frac{\log q}{\log x-\log q-\log 2}\right)^K\right)\\
&+O_q\left(\frac{E_b(i_1,...,i_k;v)}{(\log x)^K}\right)\\
=&S(i_1,...,i_k;v)\frac{x}{2}+O_q\left(\frac{E_b(i_1,...,i_k;v)}{(\log x)^K}\right).
\end{aligned}
$$

Hence, we obtain the desired result. □

Lemma 2.26. *Assume GRH. Let q_1, q_2 be two distinct and fixed odd prime numbers, ψ_1 be a primitive Dirichlet character modulo q_1 and ψ_{0,q_2} denote the principal Dirichlet character modulo q_2.*

For $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, $K := i_1 + 2i_2 + ... + ki_k + v$ and $\sigma > 1$, define

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v;\overline{\psi_1})}{n^s}:=\frac{L^{(v+1)}}{L}(s,\overline{\psi_1})\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{i_w}.$$

Then,

$$\sum_{n\le x}b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_{0,q_2}(n)=O_{q_2}(x^{1-\frac{1}{\log\log q_1(x+4)}}(\log x)(A(k)\log\log q_1(x+4))^{K+1}).$$

Proof. By Lemma 2.13, we have

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_{0,q_2}(n)}{n^s}=\frac{L^{(v+1)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\right)^{i_w},$$

for $\sigma > 1$.

By Lemma 2.14, we know that $|b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_{0,q_2}(n)| \le (\log n)^{K+1}$. Thus, we have

$$\sum_{n\le x} b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_{0,q_2}(n)$$
$$=\frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-ix}^{1+\frac{1}{\log x}+ix} \frac{L^{(v+1)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\right)^{i_w}\frac{x^s}{s}\,ds$$
$$+O_{q_1,q_2}\left((A(k)\log x)^{K+2}\right), \tag{2.45}$$

by Lemma 2.11. By Euler product, we see that for $\sigma>1$

$$L(s,\overline{\psi_1}\psi_{0,q_2})=\prod_p\left(1-\frac{\overline{\psi_1}\psi_{0,q_2}(p)}{p^s}\right)^{-1}$$
$$=\left(1-\frac{\overline{\psi_1}(q_2)}{q_2^s}\right)\prod_p\left(1-\frac{\overline{\psi_1}(p)}{p^s}\right)^{-1}=:F_{\overline{\psi_1},q_2}(s)L(s,\overline{\psi_1}).$$

Thus, the integrand in (2.45) has no poles inside the rectangle with vertices $1+\frac{1}{\log x}-ix$, $1+\frac{1}{\log x}+ix$, $1-\frac{1}{\log\log q_1(x+4)}+ix$ and $1-\frac{1}{\log\log q_1(x+4)}-ix$. Now, by the residue theorem, for

$$I_1:=-\frac{1}{2\pi i}\int_{1+\frac{1}{\log x}+ix}^{1-\frac{1}{\log\log q_1(x+4)}+ix}\frac{L^{(v+1)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\prod_{w=1}^{k}\frac{L^{(w)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\frac{x^s}{s}\,ds,$$
$$I_2:=-\frac{1}{2\pi i}\int_{1-\frac{1}{\log\log q_1(x+4)}+ix}^{1-\frac{1}{\log\log q_1(x+4)}-ix}\frac{L^{(v+1)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\prod_{w=1}^{k}\frac{L^{(w)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\frac{x^s}{s}\,ds,$$
$$I_3:=-\frac{1}{2\pi i}\int_{1-\frac{1}{\log\log q_1(x+4)}-ix}^{1+\frac{1}{\log x}-ix}\frac{L^{(v+1)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\prod_{w=1}^{k}\frac{L^{(w)}}{L}(s,\overline{\psi_1}\psi_{0,q_2})\frac{x^s}{s}\,ds.$$

we have

$$\sum_{n\le x} b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_{0,q_2}(n)=I_1+I_2+I_3+O((A(k)\log x)^{K+2}).$$

Along the line of integration in I_1, we have $s = \sigma + ix$ with $1 - \frac{1}{\log\log q_1(x+4)} \le \sigma \le 1 + \frac{1}{\log x}$. Note that for large x, $\frac{1}{2} + \frac{1}{\log\log q_1(x+4)} \le 1 - \frac{1}{\log\log q_1(x+4)}$. Thus, we can use Lemma 2.10 and get

$$\begin{aligned}
\frac{L'}{L}(s, \overline{\psi_1}\psi_{0,q_2}) &= \frac{F'_{\overline{\psi_1},q_2}}{F_{\overline{\psi_1},q_2}}(s) + \frac{L'}{L}(s, \overline{\psi_1}) \\
&= \frac{\overline{\psi_1}(q_2)\log q_2}{q_2^s - \overline{\psi_1}(q_2)} + \frac{L'}{L}(s, \overline{\psi_1}) \\
&\ll_{q_2} ((\log q_1\tau)^{2-2\sigma} + 1)\min\left(\frac{1}{|\sigma - 1|}, \log\log q_1\tau\right) \\
&\ll_{q_2} ((\log q_1(x+4))^{2-2\sigma} + 1)\log\log q_1(x+4) \\
&\ll_{q_2} ((\log q_1(x+4))^{2-2(1-1/\log\log q_1(x+4))} + 1)\log\log q_1(x+4) \\
&\ll_{q_2} \log\log q_1(x+4).
\end{aligned}$$

By Cauchy's theorem, for $r := 1/\log\log q_1(x+4)$,

$$\begin{aligned}
\frac{d}{ds}\frac{L'}{L}(s, \overline{\psi}\psi_{0,q_2}) = \frac{1}{2\pi i}\int_{|w-s|=r} \frac{\frac{L'}{L}(s, \overline{\psi_1}\psi_{0,q_2})}{(w-s)^2}\,dw &\ll_{q_2} \frac{\log\log q_1(x+4)}{r^2} r \\
&= (\log\log q_1(x+4))^2,
\end{aligned}$$

and

$$\begin{aligned}
\frac{L''}{L}(s, \overline{\psi_1}\psi_{0,q_2}) &= \frac{d}{ds}\frac{L'}{L}(s, \overline{\psi_1}\psi_{0,q_2}) + \left(\frac{L'}{L}(s, \overline{\psi_1}\psi_{0,q_2})\right)^2 \\
&\ll_{q_2} (\log\log q_1(x+4))^2 + (\log\log q_1(x+4))^2 \\
&\ll_{q_2} (\log\log q_1(x+4))^2.
\end{aligned}$$

Now by using the identity $\frac{L^{(w+1)}}{L} = (\frac{L^{(w)}}{L})' + \frac{L^{(w)}}{L}\frac{L'}{L}$ and induction on w, we obtain

$$\frac{L^{(w)}}{L}(s, \overline{\psi}\psi_{0,q_2}) \ll_{w,q_2} (\log\log q_1(x+4))^w, \tag{2.46}$$

along the line of integration in I_1.

Since $|\frac{x^s}{s}| \ll 1$ along this line, we have

$$I_1 \ll_{q_2} (A(k) \log\log q_1(x+4))^{K+1}.$$

By symmetry, we also have

$$I_3 \ll_{q_2} (A(k) \log\log q_1(x+4))^{K+1}.$$

Now, we will find an upper bound for I_2. Along the upper part of the line of integration in I_2, we have $s = 1 - \frac{1}{\log\log q_1(x+4)} + it$ with $0 \leq t \leq x$. Note that for large x, we have $\frac{1}{2} + \frac{1}{3} < 1 - \frac{1}{\log\log q_1(x+4)}$ and clearly, the inequality $\frac{1}{2} + \frac{1}{\log\log q_1(t+4)} \leq \frac{1}{2} + \frac{1}{3}$ is equivalent to $t \geq \frac{\exp(\exp 3)}{q_1} - 4$. Thus, for $t \geq \left|\frac{\exp(\exp 3)}{q_1} - 4\right| =: t_0$, we can use Lemma 2.10. We split the upper part of the integral in I_2 as $\int_0^{t_0} + \int_{t_0}^x$. Note that for $t \geq t_0$, we have

$$\begin{aligned}
\frac{L'}{L}(s, \overline{\psi_1}\psi_{0,q_2}) &\ll_{q_2} ((\log q_1\tau)^{2-2(1-\frac{1}{\log\log q_1(x+4)})} + 1) \min\left(\frac{1}{|1 - \frac{1}{\log\log q_1(x+4)} - 1|}, \log\log q_1\tau\right) \\
&\ll_{q_2} ((\log q_1(x+4))^{\frac{2}{\log\log q_1(x+4)}} + 1)(\log\log q_1(x+4)) \\
&\ll_{q_2} \log\log q_1(x+4),
\end{aligned}$$

by Lemma 2.10 and the same argument as in (2.46) shows that

$$\frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi_{0,q_2}) \ll_{w,q_2} (\log\log q_1(x+4))^w,$$

for $t \geq t_0$. Thus, the part $\int_{t_0}^x$ in I_2 is

$$\begin{aligned}
&\ll_{q_2} (A(k) \log\log q_1(x+4))^{K+1} x^{1-\frac{1}{\log\log q_1(x+4)}} \int_{\frac{\exp(\exp 3)}{q_1}-4}^{x} \frac{dt}{\sqrt{(\frac{1}{\log\log q_1(x+4)})^2 + t^2}} \\
&\ll_{q_2} (A(k) \log\log q_1(x+4))^{K+1} x^{1-\frac{1}{\log\log q_1(x+4)}} \left(\log\left(\sqrt{\frac{1}{\log\log q_1(x+4)})^2 + x^2} + x\right)\right) \\
&\ll_{q_2} x^{1-\frac{1}{\log\log q_1(x+4)}} (\log x)(A(k) \log\log q_1(x+4))^{K+1}.
\end{aligned}$$

On the part $\int_0^{t_0}$ in I_2, we have $s = 1 - \frac{1}{\log\log q_1(x+4)} + it$, $0 \leq t \leq t_0$.

$$\frac{L^{(v+1)}}{L}(s, \overline{\psi_1}\psi_{0,q_2}) \prod_{w=1}^{k} \frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi_{0,q_2}) \frac{x^s}{s} \ll x^{1-\frac{1}{\log\log q_1(x+4)}}.$$

Thus, the part $\int_0^{t_0}$ in I_2 is $\ll x^{1-\frac{1}{\log\log q_1(x+4)}}$. Combining the results for $\int_0^{t_0}$ and $\int_{t_0}^{x}$ and using symmetry for the lower part of I_2, we obtain

$$I_2 \ll_{q_2} x^{1-\frac{1}{\log\log q_1(x+4)}}(\log x)(A(k)\log\log q_1(x+4))^{K+1}.$$

Combining the results for I_1, I_2, and I_3 finishes the proof. □

Lemma 2.27. *With the notations in Lemma 2.26, we have*

$$\sum_{\frac{x}{2} \leq n \leq x} \frac{b_n(i_1, ..., i_k; v; \overline{\psi_1})\psi_{0,q_2}(n)}{\left(\log \frac{q_1 n}{q_2}\right)^K} = O_{q_1,q_2}\left(\frac{x^{1-\frac{1}{\log\log q_1(x+4)}}(A(k)\log\log q_1(x+4))^{K+1}}{(\log x)^{K-1}}\right)$$

Proof. We apply partial summation to the result of Lemma 2.26. □

3. PROOF OF THEOREM 1.1

As mentioned in [11], the work of Titchmarsh (Theorem 11.5C in [3]) (for $k = 1$) and the work of Spira [15] (for $k \geqslant 2$) give the existence of zero-free half-planes $\sigma \geqslant \sigma_k$ for $\zeta^{(k)}(s)$. Hence, we can find $\sigma_k' \geqslant \sigma_k$ such that $\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(\sigma_k' + it) \ll_k 1$. By the work of Levinson and Montgomery [16] (Theorem 9), there are no non-real zeros of $\zeta'(s)$ in the left half plane. Also Spira [15] showed that for each $\delta > 0$ there exists an r_k such that $\zeta^{(k)}(s) \neq 0$ in the region defined by $\sigma < -\delta$, $|t| > \delta$, $|s| > r_k$.

We will use the same contour as in the proof of Theorem 1 in [11]. Let R be the rectangle with vertices $\sigma_k' + iA$, $\sigma_k' + iB$, $-\delta + iB$ and $-\delta + iA$, with a fixed δ such that $o < \delta < \frac{1}{8}$ and let C be the positively oriented contour on the boundary of R. By the residue theorem, for large A, $A < B \leq 2A$, we have

$$S_k(A,B) := \sum_{A<\gamma_{\zeta,k}\leqslant B} \chi_\psi(\rho_{\zeta,k}) = \frac{1}{2\pi i}\int_C \chi_\psi(s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s)\,ds.$$

We want that the horizontal parts of the contour to be in a distance $\gg \frac{1}{\log A}$ from any zero of $\zeta^{(k)}(s)$, which we can have by adjusting A and B by an amount of $\leqslant 1$. For such suitably chosen, large A, $A < B \leqslant 2A$, we have, on the horizontal parts of the contour C, $\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s) \ll_k (\log A)^2$ and $\chi_\psi(s) \ll_q A^{\frac{1}{2}+\delta}$. Moreover, on the right side of the contour C, (the line from $\sigma_k' + iA$ to $\sigma_k' + iB$), we have $\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s) \ll_k 1$ and from Lemma 2.4 we see that $\chi_\psi(s) \ll_q A^{\frac{1}{2}-\sigma_k'}$.

We put

$$I_1 := \frac{1}{2\pi i}\int_{-\delta+iA}^{\sigma_k'+iA} \chi_\psi(s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s)\,ds \ll_{k,q} A^{\frac{1}{2}+\delta}(\log A)^2,$$

$$I_2 := \frac{1}{2\pi i}\int_{\sigma_k'+iA}^{\sigma_k'+iB} \chi_\psi(s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s)\,ds \ll_{k,q} A^{\frac{3}{2}-\sigma_k'} \ll_{k,q} A^{\frac{1}{2}},$$

$$I_3 := \frac{1}{2\pi i} \int_{\sigma_k'+iB}^{-\delta+iB} \chi_\psi(s) \frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s)\, ds \ll_{k,q} A^{\frac{1}{2}+\delta} (\log A)^2,$$

$$I_4 := \frac{1}{2\pi i} \int_{-\delta+iB}^{-\delta+iA} \chi_\psi(s) \frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s)\, ds.$$

Now,

$$\begin{aligned} I_4 &= -\frac{1}{2\pi} \int_A^B \chi_\psi(-\delta+it) \frac{\zeta^{(k+1)}}{\zeta^{(k)}}(-\delta+it)\, dt \\ &= \overline{-\frac{1}{2\pi} \int_A^B \chi_{\overline{\psi}}(-\delta-it) \frac{\zeta^{(k+1)}}{\zeta^{(k)}}(-\delta-it)\, dt} \\ &= \overline{-\frac{1}{2\pi} \int_A^B \chi_{\overline{\psi}}(1-(1+\delta+it)) \frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-(1+\delta+it))\, dt} \\ &= \overline{-\frac{1}{2\pi i} \int_{1+\delta+iA}^{1+\delta+iB} \chi_{\overline{\psi}}(1-s) \frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-s)\, ds}. \end{aligned}$$

Thus we have

$$\begin{aligned} \overline{S_k(A,B)} &= \sum_{i=1}^{4} \overline{I_i} = \overline{I_4} + O_{k,q}(A^{\frac{1}{2}+\delta}(\log A)^2) \\ &= -\frac{1}{2\pi i} \int_{1+\delta+iA}^{1+\delta+iB} \chi_{\overline{\psi}}(1-s) \frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-s)\, ds + O_{k,q}(A^{\frac{1}{2}+\delta}(\log A)^2). \end{aligned}$$

From the functional equation of the ζ-function, and from Lemma 2.4 we get for $\sigma \leqslant \frac{1}{2}$

$$\begin{aligned} \zeta^{(k)}(s) &= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \chi_\zeta^{(i)} \zeta^{(k-i)}(1-s) \\ &= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \chi_\zeta(s) \left(-\ell + O\left(\frac{1}{|t|}\right)\right)^i \zeta^{(k-i)}(1-s) \end{aligned}$$

$$=\chi_\zeta(s)\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(-\ell+O\left(\frac{1}{|t|}\right)\right)^{i}\zeta^{(k-i)}(1-s)$$

$$=\chi_\zeta(s)\left(1+O\left(\frac{1}{|t|}\right)\right)\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(-\ell)^{i}\zeta^{(k-i)}(1-s)$$

$$=\chi_\zeta(s)\left(1+O\left(\frac{1}{|t|}\right)\right)\left(-\ell+\frac{d}{ds}\right)^{k}\zeta(1-s), \tag{3.1}$$

where $\ell=\log\frac{|t|}{2\pi}$. If we put

$$G_k(s,z):=\left(z+\frac{d}{ds}\right)^{k}\zeta(s)=z^k\zeta(s)+kz^{k-1}\zeta'(s)+...+\zeta^{(k)}(s),$$

where the differentiation in G' is with respect to s, then by differentiating both sides of (3.1) we see that

$$\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-s)=-\left(\ell+\frac{G_k'}{G_k}(s,\ell)\right)\left(1+O\left(\frac{1}{|t|}\right)\right).$$

We have

$$\begin{aligned}\frac{G_k'}{G_k}(s,z)=&\frac{z^k\zeta'(s)+kz^{k-1}\zeta''(s)+...+\zeta^{(k+1)}(s)}{z^k\zeta(s)+kz^{k-1}\zeta'(s)+...+\zeta^{(k)}(s)}\\ =&\frac{z^k(\zeta'(s)+kz^{-1}\zeta''(s)+...+z^{-k}\zeta^{(k+1)}(s))}{z^k(\zeta(s)+kz^{-1}\zeta'(s)+...+z^{-k}\zeta^{(k)}(s))}\\ =&\frac{\sum_{v=0}^{k}\binom{k}{v}\frac{1}{z^v}\frac{\zeta^{(v+1)}}{\zeta}(s)}{1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{z^w}\frac{\zeta^{(w)}}{\zeta}(s)}.\end{aligned} \tag{3.2}$$

Since $\frac{\zeta^{(w)}}{\zeta}(s)\ll_w 1$ for $\sigma\geqslant 1+\delta$, we have $\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{\zeta^{(w)}}{\zeta}(s)\ll_k\frac{1}{\log A}$, which gives $\frac{G_k'}{G_k}(s,\ell)\ll_k 1$ for $\sigma\geqslant 1+\delta$ and $A\leqslant t\leqslant B$. So we get

$$\overline{S_k(A,B)}=\frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB}\chi_{\overline{\psi}}(1-s)\frac{G_k'}{G_k}(s,\ell)\,ds+O_k(A^{\frac{1}{2}+\delta}\log^2 A), \tag{3.3}$$

by Lemma 2.21.

As in [11], for $\sigma \geqslant 1+\delta, A \leqslant t \leqslant B$ for large A, we can write

$$\begin{aligned}\left(1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{\zeta^{(w)}}{\zeta}(s)\right)^{-1} &= \sum_{u=0}^{\infty} (-1)^u \left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{\zeta^{(w)}}{\zeta}(s)\right)^{u} \\ &= \sum_{u\leq \frac{\log A}{\log\log A}} (-1)^u \left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{\zeta^{(w)}}{\zeta}(s)\right)^{u} + O\left(\frac{1}{A}\right).\end{aligned} \tag{3.4}$$

We first combine (3.2) and (3.4), and put the result in the equation (3.3), then we get

$$\begin{aligned}\overline{S_k(A,B)} = &\sum_{u\leqslant \frac{\log A}{\log\log A}} \sum_{v=0}^{k}(-1)^u\binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\geqslant 0}} \binom{u}{i_1,\ldots i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w} \\ &\times \frac{1}{2\pi}\int_A^B \frac{\chi_{\overline{\psi}}(-\delta-it)}{\ell^K}\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(1+\delta+it)\right)^{i_w}\frac{\zeta^{(v+1)}}{\zeta}(1+\delta+it)\,dt \\ &+ O_k(A^{\frac{1}{2}+\delta}\log^2 A),\end{aligned}$$

where $K = i_1+2i_2+\ldots+ki_k+v$. Define $\sum_{n=1}^{\infty}\frac{b_n(i_1,\ldots,i_k;v)}{n^s} := \frac{\zeta^{(v+1)}}{\zeta}(s)\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}$ for $\sigma>1$. We put $A=\frac{T}{2}$, $B=T$, and by Lemma 2.19, we get

$$\begin{aligned}\overline{S_k\left(\frac{T}{2},T\right)} = &\sum_{u\leqslant \frac{\log T}{\log\log T}} \sum_{v=0}^{k}(-1)^u\binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\geqslant 0}} \binom{u}{i_1,\ldots i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w} \\ &\times \frac{1}{2\pi}\int_{\frac{T}{2}}^{T} \frac{\chi_{\overline{\psi}}(-\delta-it)}{(\log\frac{t}{2\pi})^K}\left(\sum_{n=1}^{\infty}\frac{b_n(i_1,\ldots,i_k;v)}{n^{1+\delta+it}}\right)dt \\ &+ O_{k,q}(T^{\frac{1}{2}+\delta}\log^2 T) \\ = &\sum_{u\leqslant \frac{\log T}{\log\log T}} \sum_{v=0}^{k}(-1)^u\binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\geqslant 0}} \binom{u}{i_1,\ldots i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w} \\ &\times \frac{\tau(\overline{\psi})}{q}\left(\sum_{\frac{qT}{4\pi}<n\leqslant\frac{qT}{2\pi}}\frac{b_n(i_1,\ldots,i_k;v)}{\left(\log\frac{n}{q}\right)^K}e\left(\frac{-n}{q}\right)\right) \\ &+ O_{k,q}(T^{\frac{1}{2}+\delta}\log^2 T).\end{aligned} \tag{3.5}$$

Note that every natural number n with $\frac{qT}{4\pi} < n \leq \frac{qT}{2\pi}$ is congruent a number $a \pmod q$. Moreover if $n \equiv a \pmod q$, then $e(-\frac{n}{q}) = e(-\frac{a}{q})$.

Using the orthogonality property of Dirichlet characters for n with $(n, q) = 1$ and the fact that $e(-\frac{n}{q}) = 1$ for $(n, q) > 1$, we get

$$\begin{aligned}
&\sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}} \frac{b_n(i_1,...,i_k,v)e^{-\frac{2\pi in}{q}}}{(\log\frac{n}{q})^K} \\
&= \sum_{a \pmod q} \sum_{\substack{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi} \\ n\equiv a \pmod q}} \frac{b_n(i_1,...,i_k,v)e^{-\frac{2\pi in}{q}}}{(\log\frac{n}{q})^K} \\
&= \sum_{\substack{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi} \\ q|n}} \frac{b_n(i_1,...,i_k,v)}{(\log\frac{n}{q})^K} \\
&\quad + \frac{1}{\varphi(q)} \sum_{a \pmod q} e^{-\frac{2\pi ia}{q}} \sum_{\psi' \pmod q} \overline{\psi'}(a) \sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}} \frac{b_n(i_1,...,i_k,v)\psi'(n)}{\left(\log\frac{n}{q}\right)^K} \\
&= \sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}} \frac{b_n(i_1,...,i_k,v)}{(\log\frac{n}{q})^K} - \sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}} \frac{b_n(i_1,...,i_k,v)\psi_0(n)}{(\log\frac{n}{q})^K} \\
&\quad + \frac{1}{\varphi(q)} \sum_{a \pmod q} e^{-\frac{2\pi ia}{q}} \sum_{\psi' \pmod q} \overline{\psi'}(a) \sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}} \frac{b_n(i_1,...,i_k,v)\psi'(n)}{\left(\log\frac{n}{q}\right)^K} \\
&= \sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}} \frac{b_n(i_1,...,i_k,v)}{(\log\frac{n}{q})^K} - \sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}} \frac{b_n(i_1,...,i_k,v)\psi_0(n)}{(\log\frac{n}{q})^K} \\
&\quad + \frac{1}{\varphi(q)} \sum_{a \pmod q} e^{-\frac{2\pi ia}{q}} \overline{\psi_0}(a) \sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}} \frac{b_n(i_1,...,i_k,v)\psi_0(n)}{\left(\log\frac{n}{q}\right)^K} \\
&\quad + \frac{1}{\varphi(q)} \sum_{a \pmod q} e^{-\frac{2\pi ia}{q}} \sum_{\substack{\psi' \pmod q \\ \psi'\neq\psi_0}} \overline{\psi'}(a) \sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}} \frac{b_n(i_1,...,i_k,v)\psi'(n)}{\left(\log\frac{n}{q}\right)^K}, \qquad (3.6)
\end{aligned}$$

where in the last line we split the sum over characters modulo q into two parts: The term with $\psi' = \psi_0$ and the term with $\psi' \neq \psi_0$.

Now we use (3.6) in (3.5),

$$\overline{S_k(T/2,T)} = \sum_{u \le \frac{\log T}{\log\log T}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w}$$
$$\times \frac{\tau\left(\overline{\psi}\right)}{q} \left(\sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)}{(\log\frac{n}{q})^K} - \sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)\psi_0(n)}{(\log\frac{n}{q})^K} \right.$$
$$\left. + \frac{1}{\varphi(q)} \sum_{a \pmod q} e^{-\frac{2\pi i a}{q}} \overline{\psi_0}(a) \sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)\psi_0(n)}{\left(\log\frac{n}{q}\right)^K} \right)$$
$$+ E_{\psi'\ne\psi_0} + O_{k,q}(T^{\frac{1}{2}+\delta}\log^2 T), \tag{3.7}$$

where

$$E_{\psi'\ne\psi_0} := \sum_{u \le \frac{\log T}{\log\log T}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w}$$
$$\times \frac{\tau\left(\overline{\psi}\right)}{q\varphi(q)} \sum_{a \pmod q} e^{-\frac{2\pi i a}{q}} \sum_{\psi'\ne\psi_0 \pmod q} \overline{\psi'}(a) \sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)\psi'(n)}{\left(\log\frac{n}{q}\right)^K}.$$

By Lemma 2.25 with $x = \frac{qT}{2\pi}$

$$E_{\psi'\ne\psi_0} \ll_{k,q} \sum_{u \le \frac{\log T}{\log\log q(T+4)}} \sum_{v=0}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w}$$
$$\times \frac{T^{1-\frac{1}{\log\log q(T+4)}} (A(k)\log\log q(T+4))^{K+1}}{(\log T)^{K-1}}$$
$$\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}} (\log\log q(T+4))^2$$
$$\times \sum_{u \le \frac{\log T}{\log\log T}} \sum_{v=0}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k} \left(\frac{A(k)\log\log q(T+4)}{\log T}\right)^{K-1}$$
$$\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}} (\log\log q(T+4))^2$$
$$\times \sum_{u \le \frac{\log T}{\log\log q(T+4)}} \left(\frac{A(k)\log\log q(T+4)}{\log T}\right)^{u-1} \sum_{v=0}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k}$$

$$
\begin{aligned}
&\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}} (\log\log q(T+4))^2 \\
&\quad \times \sum_{u \le \frac{\log T}{\log\log T}} \left(\frac{A(k)\log\log q(T+4)}{\log T} \right)^{u-1} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k} \\
&\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}} (\log\log q(T+4))^2 \\
&\quad \times \sum_{u \le \frac{\log T}{\log\log T}} \left(\frac{A(k)\log\log q(T+4)}{\log T} \right)^{u-1} k^u \\
&\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}} (\log\log q(T+4))^2 \sum_{u \le \frac{\log T}{\log\log T}} \left(\frac{A(k)\log\log q(T+4)}{\log T} \right)^{u-1} \\
&\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}} (\log\log q(T+4))^2 \frac{\log T}{\log\log q(T+4)} \frac{1}{1-\frac{A(k)\log\log q(T+4)}{\log T}} \\
&\ll_{k,q} \frac{T}{\log T}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\overline{S_k(T/2,T)} = &\sum_{u \le \frac{\log T}{\log\log T}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w} \\
&\times \frac{\tau\left(\overline{\psi}\right)}{q} \left(\sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)}{(\log \frac{n}{q})^K} - \sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)\psi_0(n)}{(\log \frac{n}{q})^K} \right. \\
&\left. + \frac{1}{\varphi(q)} \sum_{a \pmod q} e^{-\frac{2\pi i a}{q}} \overline{\psi_0}(a) \sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)\psi_0(n)}{\left(\log \frac{n}{q}\right)^K} \right) \\
&+ O_{k,q}\left(\frac{T}{\log T} \right) \\
= &\sum_{u \le \frac{\log T}{\log\log T}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w} \\
&\times \frac{\tau\left(\overline{\psi}\right)}{q} \left(\sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)}{(\log \frac{n}{q})^K} - \sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)\psi_0(n)}{(\log \frac{n}{q})^K} \right. \\
&\left. + \frac{\tau\left(\overline{\psi_0}\right)}{\varphi(q)} \sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k,v)\psi_0(n)}{\left(\log \frac{n}{q}\right)^K} \right) \\
&+ O_{k,q}\left(\frac{T}{\log T} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{u \leq \frac{\log T}{\log\log T}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1, \ldots, i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w} \\
&\quad \times \frac{\tau\left(\overline{\psi}\right)}{q} \left(\sum_{\frac{qT}{4\pi} < n \leq \frac{qT}{2\pi}} \frac{b_n(i_1, \ldots, i_k, v)}{(\log \frac{n}{q})^K} - \sum_{\frac{qT}{4\pi} < n \leq \frac{qT}{2\pi}} \frac{b_n(i_1, \ldots, i_k, v)\psi_0(n)}{(\log \frac{n}{q})^K} \right. \\
&\quad \left. + \frac{\mu(q)}{\varphi(q)} \sum_{\frac{qT}{4\pi} < n \leq \frac{qT}{2\pi}} \frac{b_n(i_1, \ldots, i_k, v)\psi_0(n)}{\left(\log \frac{n}{q}\right)^K} \right) \\
&\quad + O_{k,q}\left(\frac{T}{\log T}\right).
\end{aligned}
$$

We use Lemma 2.5 in [11], and Lemma 2.25 with $x = \frac{qT}{2\pi}$, and we get

$$
\begin{aligned}
\overline{S_k(T/2, T)} = &- \frac{\tau\left(\overline{\psi}\right)\mu(q)}{\varphi(q)} \frac{T}{4\pi} \left(\sum_{u=0}^{\infty} - \sum_{u > \frac{\log T}{\log\log T}} \right) \left[(-1)^u \sum_{v=0}^{k} \binom{k}{v} \right. \\
&\left. \times \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1, \ldots, i_k} \prod_{w=1}^{k} \left\{ (-1)^w w! \binom{k}{w} \right\}^{i_w} \frac{(-1)^v (v+1)!}{K!} \right] \\
&+ E_1 + E_2 + E_3 + O_{k,q}\left(\frac{T}{\log T}\right), \quad \text{say.}
\end{aligned} \tag{3.8}
$$

Using Lemma 2.24, Lemma 2.25, and (2.43) we have

$$
\begin{aligned}
E_1 &\ll_{k,q} \log^2 T \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{v=0}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1, \ldots, i_k} (A(k))^K \\
&\ll_{k,q} (A(k))^{\frac{\log T}{\log\log T}} \log^2 T \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1, \ldots, i_k} \\
&\ll_{k,q} (A(k))^{\frac{\log T}{\log\log T}} \log^2 T \sum_{u \leqslant \frac{\log T}{\log\log T}} k^u \\
&\ll_{k,q} (A(k))^{\frac{\log T}{\log\log T}} \log^2 T \\
&\ll_{k,q} T^{\epsilon},
\end{aligned}
$$

and

$$\begin{aligned}
E_2 &\ll_{k,q} \frac{T}{\log T} \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{v=0}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1, \ldots, i_k} \frac{(A(k))^K}{(K-1)!} \\
&\ll_{k,q} \frac{T}{\log T} \sum_{u \leqslant \frac{\log T}{\log\log T}} \frac{(A(k))^u}{(u-1)!} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1, \ldots, i_k} \\
&\ll_{k,q} \frac{T}{\log T} \left(1 + O\left(e^{A(k)}\right)\right) \ll_{k,q} \frac{T}{\log T},
\end{aligned}$$

and

$$\begin{aligned}
E_3 &\ll_{k,q} \frac{T(\log T)^{2+3\epsilon}}{e^{\delta_1(k)(\log T)^{\frac{1}{3}-\epsilon}}} \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{v=0}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1, \ldots, i_k} \left(\frac{A(k)}{(\log T)^{\frac{1}{3}-\epsilon}}\right)^K \\
&\ll_{k,q} \frac{T(\log T)^{2+3\epsilon}}{e^{\delta_1(k)(\log T)^{\frac{1}{3}-\epsilon}}} \sum_{u \leqslant \frac{\log T}{\log\log T}} \left(\frac{A(k)}{(\log T)^{\frac{1}{3}-\epsilon}}\right)^u \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1, \ldots, i_k} \\
&\ll_{k,q} \frac{T(\log T)^{2+3\epsilon}}{e^{\delta_1(k)(\log T)^{\frac{1}{3}-\epsilon}}} \frac{1}{1 - A(k)(\log T)^{-\frac{1}{3}+\epsilon}}.
\end{aligned}$$

The sum over $u > \log T / \log\log T$ in (3.8) is

$$\ll_{k,q} T \sum_{u > \frac{\log T}{\log\log T}} \frac{(A(k))^u}{u!} \ll_{k,q} T \frac{(A(k))^{\lceil \log T/\log\log T \rceil}}{\lceil \log T/\log\log T \rceil !} \ll_{k,q} T^{\frac{2\log\log\log T}{\log\log T}},$$

by Stirling's formula. Thus we get

$$\begin{aligned}
\overline{S_k(T/2,T)} =& \frac{\tau\left(\overline{\psi}\right)\mu(q)}{\varphi(q)} \left[-\sum_{u=0}^{\infty} (-1)^u \sum_{v=0}^{k} \binom{k}{v} \right. \\
& \left. \times \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1, \ldots, i_k} \prod_{w=1}^{k} \left\{ (-1)^w w! \binom{k}{w} \right\}^{i_w} \frac{(-1)^v (v+1)!}{K!} \right] \frac{T}{4\pi} \\
& + O_{k,q}\left(\frac{T}{\log T}\right) \\
=& \frac{\tau\left(\overline{\psi}\right)\mu(q)}{\varphi(q)} \frac{\mathcal{A}_k}{2} \frac{T}{2\pi} + O_{k,q}\left(\frac{T}{\log T}\right). \qquad (3.9)
\end{aligned}$$

Note that $\mu(q) = -1$ and $\varphi(q) = q - 1$, since q is assumed to be prime.

Observe that from (1.3) and (2.5) we have the trival estimate

$$S_k(0, \sqrt{T}) \ll_q T^{\frac{3}{4}+\frac{\delta}{2}} \log T. \tag{3.10}$$

Now, using the result (3.9) for $S_k(T/2, T)$ with T replaced by $\frac{T}{2}, \frac{T}{4}, \ldots$ down to almost $\sqrt{T}$, then adding up, and also using the trivial estimate (3.10), we obtain

$$\sum_{0<\gamma_{\zeta,k}<T} \chi_\psi(\rho_{\zeta,k}) = -\frac{\overline{\tau(\overline{\psi})}}{q-1} \mathcal{A}_k \frac{T}{2\pi} + O_{k,q}\left(\frac{T}{\log T}\right), \quad (T \to \infty), \text{ upon GRH.}$$

4. PROOF OF THEOREM 1.2

Let q_1 and q_2 be two distinct fixed odd prime numbers, $k \geqslant 1$ be an integer, ψ_1 be a non-principal Dirichlet character $\pmod{q_1}$ and ψ_2 be a nonprincipal Dirichlet character $\pmod{q_2}$. Note that since q_1 and q_2 are odd prime numbers, ψ_1 and ψ_2 are primitive Dirichlet characters modulo q_1, q_2, respectively. By the residue theorem, for large A, $A < B \leqslant 2A$,

$$S_{k,q_1,q_2}(A,B) := \sum_{A<\gamma_{\psi_1,k}<B} \chi_{\psi_2}(\rho_{\psi_1,k}) = \frac{1}{2\pi i}\int_C \chi_{\psi_2}(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi_1)\,ds, \tag{4.1}$$

for a suitable contour C.

By Lemma 2.8, for $\sigma > \sigma_{k,q_1} := 1+2\left(1+\sqrt{1+\frac{2k^2}{2\log 2}}\right) > 1$, we have $|L^{(k)}(s,\psi)| > \frac{(\log 2)^k}{2^{\sigma+1}}$. In [9](Theorem 3), Yıldırım showed that for any $\delta > 0$ there exists $M = M(k,\delta,\mathfrak{a})$ such that $L^{(k)}(s,\psi_1) \neq 0$ in the region $\sigma < -\delta, |t| > \delta, |s| > q^M$. So let C be the positively oriented rectangle with vertices $\sigma'_{k,q_1} + iA, \sigma'_{k,q_1} + iB, -\delta + iB, -\delta + iA$ for a fixed real number δ with $0 < \delta < \frac{1}{8}$ and for a fixed $\sigma'_{k,q_1} > \sigma_{k,q_1}$.

Following the argument in the proof of Theorem 1 in [11], we note that by (1.6), we can take horizontal sides of the rectangle to be a distance of $\gg \frac{1}{\log A}$ from any zero of $L^{(k)}(s,\psi_1)$ by adjusting A and B by an amount of ≤ 1. In view of (2.4), this means adding or deleting $O(\log A)$ number of terms each of size $\ll_{q_1,q_2} A^{\frac{1}{2}+\delta}$. Such an adjustment ensures $\frac{L^{(k+1)}}{L^{(k)}}(s,\psi_1) \ll (\log A)^2$ on the horizontal parts of C. On the right vertical line of C, we have $\frac{L^{(k+1)}}{L^{(k)}}(s,\psi_1) \ll 1$, also by (2.4), the contribution from this line is $\ll_{q_1,q_2} A^{\frac{1}{2}}$. So the contributions from the horizontal lines and from the right vertical line are $\ll_{q_1,q_2} A^{\frac{1}{2}+\delta}\log^2 A$. We have

$$\frac{1}{2\pi i}\int\limits_{-\delta+iB}^{-\delta+iA} \chi_{\psi_2}(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi_1)\,ds = -\frac{1}{2\pi}\int\limits_{A}^{B} \chi_{\psi_2}(-\delta+it)\frac{L^{(k+1)}}{L^{(k)}}(-\delta+it,\psi_1)\,dt$$

$$
\begin{aligned}
&= \overline{-\frac{1}{2\pi}\int_A^B \chi_{\overline{\psi_2}}(-\delta - it)\frac{L^{(k+1)}}{L^{(k)}}(-\delta - it, \overline{\psi_1})\, dt} \\
&= \overline{-\frac{1}{2\pi}\int_A^B \chi_{\overline{\psi_2}}(1-(1+\delta+it))\frac{L^{(k+1)}}{L^{(k)}}(1-(1+\delta+it), \overline{\psi_1})\, dt} \\
&= \overline{-\frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB} \chi_{\overline{\psi_2}}(1-s)\frac{L^{(k+1)}}{L^{(k)}}(1-s, \overline{\psi_1})\, ds.}
\end{aligned}
$$

So,

$$
\overline{S_{k,q_1,q_2}(A,B)} = -\frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB} \chi_{\overline{\psi_2}}(1-s)\frac{L^{(k+1)}}{L^{(k)}}(1-s, \overline{\psi_1})\, ds + O_{k,q_1,q_2}\left(A^{\frac{1}{2}+\delta}\log^2 A\right).
$$

From the functional equation of the L-function, and from Lemma 2.8 we get for $\sigma \leqslant \frac{1}{2}$

$$
\begin{aligned}
L^{(k)}(s,\psi_1) =& \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\chi_{\psi_1}^{(i)}(s)L^{(k-i)}(1-s,\overline{\psi}) \\
=& \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\chi_{\psi_1}(s)\left(-\ell + O\left(\frac{1}{|t|}\right)\right)^i L^{(k-i)}(1-s,\overline{\psi_1}) \\
=& \chi_{\psi_1}(s)\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(-\ell + O\left(\frac{1}{|t|}\right)\right)^i L^{(k-i)}(1-s,\overline{\psi_1}) \\
=& \chi_{\psi_1}(s)\left(1 + O\left(\frac{1}{|t|}\right)\right)\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(-\ell)^i L^{(k-i)}(1-s,\overline{\psi_1}) \\
=& \chi_{\psi_1}(s)\left(1 + O\left(\frac{1}{|t|}\right)\right)\left(-\ell + \frac{d}{ds}\right)^k L(1-s,\overline{\psi_1}), \qquad (4.2)
\end{aligned}
$$

where $\ell = \log\frac{q_1|t|}{2\pi}$. We put

$$
\begin{aligned}
\frac{G_k'}{G_k}(s,z,\psi) &= \left(z + \frac{d}{dz}\right)^k L(s,\psi) \\
&= z^k L(s,\psi) + kz^k L'(s,\psi) + \dots + L^{(k)}(s,\psi),
\end{aligned}
$$

where the differentiation in $G^{'}$ is with respect to s.

Then by differentiating both sides of (4.2), we see that

$$\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi_1}) = -\left(\ell + \frac{G_k'}{G_k}(s,\ell,\overline{\psi_1})\right)\left(1+O\left(\frac{1}{|t|}\right)\right).$$

Now, we have

$$\frac{G_k'}{G_k}(s,z,\overline{\psi_1}) = \frac{\sum_{v=0}^{k}\binom{k}{v}\frac{1}{z^v}\frac{L^{(v+1)}}{L}(s,\overline{\psi_1})}{1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{z^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1})}. \tag{4.3}$$

Since $\frac{L^{(w)}}{L}(s,\overline{\psi_1}) \ll_w 1$, when $\sigma \geq 1+\delta$ we get

$$\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1}) \ll_k \frac{1}{\log A} \tag{4.4}$$

and

$$\frac{G_k'}{G_k}(s,\ell,\overline{\psi_1}) \ll_k 1, \quad (\sigma \geq 1+\delta, A \leq t \leq B). \tag{4.5}$$

Thus we get,

$$\begin{aligned}
&\int_{1+\delta+iA}^{1+\delta+iB} \chi_{\overline{\psi_2}}(1-s)\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi_1})\,ds \\
&= \int_{A}^{B} \chi_{\overline{\psi_2}}(-\delta-it)\left(-\left(\ell+\frac{G_k'}{G_k}(1+\delta+it,\ell,\overline{\psi_1})\right)\left(1+O\left(\frac{1}{t}\right)\right)\right)dt.
\end{aligned}$$

Now using (4.5), together with Lemma 2.21, we get

$$\overline{S_{k,q_1,q_2}(A,B)} = \frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB} \chi_{\overline{\psi_2}}(1-s)\frac{G_k'}{G_k}(s,\ell,\overline{\psi_1})\,ds + O_{k,q_1,q_2}(A^{\frac{1}{2}+\delta}\log^2 A). \tag{4.6}$$

In order to approximate $\frac{G_k'}{G_k}(s,\ell,\overline{\psi_1})$ by a Dirichlet series, we expand the denominator of (4.3) as a power series in the region $\sigma \geqslant 1$, $A \leqslant t \leqslant B$, for large A where (4.4) holds.

Thus,

$$\left(1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{z^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{-1}=\sum_{u=0}^{\infty}(-1)^u\left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^u$$
$$=\sum_{u\le\frac{\log A}{\log\log A}}\left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^u$$
$$+O\left(\frac{1}{A}\right). \tag{4.7}$$

Now we combine (4.3), (4.6) and (4.7), and for $K:=i_1+2i_2+...+ki_k+v$, we get

$$\overline{S_{k,q_1,q_2}(A,B)}=\sum_{u\le\frac{\log A}{\log\log A}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+...+i_k=u\\ i_1,...,i_k\ge 0}}\binom{u}{i_1,...,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}$$
$$\times\frac{1}{2\pi}\int_A^B\frac{\chi_{\overline{\psi_2}}(-\delta-it)}{\ell^K}\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(1+\delta+it,\overline{\psi_1})\right)^{i_w}\frac{L^{(v+1)}}{L}(1+\delta+it,\overline{\psi_1})\,dt$$
$$+O_{k,q_1,q_2}(A^{\frac{1}{2}+\delta}\log^2 A).$$

We put $A=T/2, B=T$ and use Lemma 2.20 to get

$$\overline{S_{k,q_1,q_2}(T/2,T)}=\sum_{u\le\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+...+i_k=u\\ i_1,...,i_k\ge 0}}\binom{u}{i_1,...,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}$$
$$\times\frac{\tau\left(\overline{\psi_2}\right)}{q_2}\sum_{\frac{q_2T}{4\pi}<n\le\frac{q_2T}{2\pi}}\frac{b_n(i_1,...,i_k;v;\overline{\psi_1})e^{-\frac{2\pi in}{q_2}}}{(\log\frac{q_1n}{q_2})^K}+O_{k,q_1,q_2}(T^{\frac{1}{2}+\delta}\log^2 T), \tag{4.8}$$

where

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v;\overline{\psi_1})}{n^s}:=\frac{L^{(v+1)}}{L}(s,\overline{\psi_1})\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{i_w}.$$

Now we split the sum over n in 4.8 into two parts: the terms with $q_2|n$ and the terms with $q_2\nmid n$.

The first part is equal to

$$\sum_{\substack{\frac{q_2T}{4\pi}<n\leq\frac{q_2T}{2\pi} \\ q_2|n}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})e^{-\frac{2\pi in}{q_2}}}{(\log\frac{q_1n}{q_2})^K} = \sum_{\substack{\frac{q_2T}{4\pi}<n\leq\frac{q_2T}{2\pi} \\ q_2|n}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})}{(\log\frac{q_1n}{q_2})^K}$$

$$= \sum_{\frac{q_2T}{4\pi}<n\leq\frac{q_2T}{2\pi}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})}{(\log\frac{q_1n}{q_2})^K} - \sum_{\substack{\frac{q_2T}{4\pi}<n\leq\frac{q_2T}{2\pi} \\ q\nmid n}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})}{(\log\frac{q_1n}{q_2})^K}$$

$$= \sum_{\frac{q_2T}{4\pi}<n\leq\frac{q_2T}{2\pi}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})}{(\log\frac{q_1n}{q_2})^K} - \sum_{\frac{q_2T}{4\pi}<n\leq\frac{q_2T}{2\pi}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_{0,q_2}(n)}{(\log\frac{q_1n}{q_2})^K}, \tag{4.9}$$

where we used the fact that the condition $q_2 \nmid n$ is equivalent to $(n,q_2)=1$ since q_2 is a prime number.

In the the second part, i.e., in the part with $q_2 \nmid n$, every natural number n with $\frac{q_2T}{4\pi} < n \leq \frac{q_2T}{2\pi}$ and $(n,q_2)=1$ is congruent a number $a \pmod{q_2}$ with $(a,q_2)=1$. Noting that $e^{\frac{-2\pi n}{q_2}} = e^{\frac{-2\pi a}{q_2}}$ for such n, we can can use the cancellation property of Dirichlet characters and write

$$\sum_{\substack{\frac{q_2T}{4\pi}<n\leq\frac{q_2T}{2\pi} \\ (n,q_2)=1}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})e^{-\frac{2\pi in}{q_2}}}{(\log\frac{q_1n}{q_2})^K}$$

$$= \sum_{\substack{a \pmod{q} \\ (a,q_2)=1}} \sum_{\substack{\frac{q_2T}{4\pi}<n\leq\frac{q_2T}{2\pi} \\ n\equiv a \pmod{q_2}}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})e^{-\frac{2\pi in}{q}}}{(\log\frac{q_1n}{q_2})^K}$$

$$= \frac{1}{\varphi(q_2)} \sum_{\substack{a \pmod{q_2} \\ (a,q_2)=1}} e^{-\frac{2\pi ia}{q_2}} \sum_{\psi \pmod{q_2}} \overline{\psi}(a) \sum_{\frac{q_2T}{4\pi}<n\leq\frac{q_2T}{2\pi}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})\psi(n)}{\left(\log\frac{q_1n}{q_2}\right)^K}. \tag{4.10}$$

For the sum over $\psi \pmod{q_2}$ in (4.10), we will consider the principal character modulo q_2 separately. We will denote the principal character modulo q_2 as ψ_{0,q_2}. Combining (4.9) and (4.10), we obtain that the sum over n in (4.8) is equal to

$$\sum_{\frac{q_2 T}{4\pi}<n\le\frac{q_2 T}{2\pi}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})}{(\log\frac{q_1 n}{q_2})^K} + \left(-1+\frac{\mu(q_2)}{\varphi(q_2)}\right)\sum_{\frac{q_2 T}{4\pi}<n\le\frac{q_2 T}{2\pi}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_{0,q_2}(n)}{(\log\frac{q_1 n}{q_2})^K}$$
$$+\frac{1}{\varphi(q_2)}\sum_{\substack{a \pmod{q_2}\\(a,q_2)=1}} e^{-\frac{2\pi i a}{q_2}} \sum_{\substack{\psi \pmod{q_2}\\ \psi\neq\psi_{0,q_2}}} \overline{\psi}(a) \sum_{\frac{q_2 T}{4\pi}<n\le\frac{q_2 T}{2\pi}} \frac{b_n(i_1,...,i_k;v;\overline{\psi_1})\psi(n)}{\left(\log\frac{q_1 n}{q_2}\right)^K}, \tag{4.11}$$

where $\mu(q_2)$ comes from considering ψ_{0,q_2} separately in (4.10) and from the fact that $\sum_{\substack{a \pmod{q_2}\\(a,q_2)=1}} e^{-\frac{2\pi i a}{q_2}} = \mu(q_2)$. Note that $\mu(q_2)=-1$ since q_2 is a prime number.

Now, the first term in (4.11) can be calculated by Lemma 2.23 and the second term in (4.11) can be calculated by Lemma 2.27. In the third term of (4.11), since $\psi\neq\psi_{,0,q_2}$ and $\overline{\psi_1}$ and ψ are primitive characters modulo q_1 and q_2, respectively, the character $\psi_1\psi$ is a primitive character modulo q_1q_2 (see Lemma 9.3 in [5]). In view of Lemma 2.13, the third term in (4.11) can be calculated by Lemma 2.23 by taking $q=q_1q_2$. Combining these all, for large T, we obtain

$$\begin{aligned}
\overline{S_{k,q_1,q_2}(T/2,T)} \ll_{k,q_1,q_2} & \sum_{u\le\frac{\log T}{\log\log(T+4)}} \sum_{v=0}^{k}\binom{k}{v} \sum_{\substack{i_1+...+i_k=u\\ i_1,...,i_k\ge 0}} \binom{u}{i_1,...,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w} \\
& \times \frac{T^{1-\frac{1}{\log\log(T+4)}}(A(k)\log\log(T+4))^{K+1}}{(\log T)^{K-1}} + T^{\frac{1}{2}+\delta}\log^2 T \\
\ll_{k,q_1,q_2} & T^{1-\frac{1}{\log\log(T+4)}}\log T(\log\log(T+4)) \\
& \times \sum_{u\le\frac{\log T}{\log\log T}} \sum_{v=0}^{k}\binom{k}{v} \sum_{\substack{i_1+...+i_k=u\\ i_1,...,i_k\ge 0}} \binom{u}{i_1,...,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w} \\
& \times \left(\frac{A(k)\log\log(T+4)}{\log T}\right)^K + T^{\frac{1}{2}+\delta}\log^2 T \\
\ll_{k,q_1,q_2} & T^{1-\frac{1}{\log\log(T+4)}}\log T(\log\log(T+4)) \\
& \times \sum_{K\le k\frac{\log T}{\log\log(T)}+k} \left(\frac{A(k)\log\log(T+4)}{\log T}\right)^K + T^{\frac{1}{2}+\delta}\log^2 T
\end{aligned}$$

$$\ll_{k,q_1,q_2} T^{1-\frac{1}{\log\log(T+4)}} \log T \log\log(T+4). \tag{4.12}$$

Observe that from (1.6) and (2.4) we have the trival estimate

$$S_{k,q_1,q_2}(0,\sqrt{T}) \ll_{q_1,q_2} T^{\frac{3}{4}+\frac{\delta}{2}} \log T. \tag{4.13}$$

Now, using the result (4.12) for $S_{k,q_1,q_2}(T/2,T)$ with T replaced by $\frac{T}{2}, \frac{T}{4}, ...$ down to almost $\sqrt{T}$, then adding them up, and also using the trivial estimate (4.13), we obtain

$$\sum_{0<\gamma_{\psi_1,k}<T} \chi_{\psi_2}(\rho_{\psi_1,k}) \ll_{k,q_1,q_2} T^{1-\frac{1}{\log\log T}} \log T \log\log T, \quad (T\to\infty),$$

upon GRH.

5. CONCLUSION

For the sum $\sum_{0<\gamma_{\zeta,k}<T} \chi_\psi(\rho_{\zeta,k})$, we obtained a main term. On the other hand, in view of Theorem 1.2, we could not obtain a main term for the average value $\sum_{0<\gamma_{\psi_1,k}<T} \chi_{\psi_2}(\rho_{\psi_1,k})$ of $\chi_{\psi_1}(s)$ at the zeros of the k^{th} derivative of $L(s,\psi_1)$, where the involved Dirichlet characters are of distinct moduli.

One quick observation is that when the Dirichlet characters are of distinct moduli, χ_{ψ_2} does not know much about the zeros $\rho_{\psi_1,k}$ of $L^{(k)}(s,\psi_1)$, however χ_ψ is aware of the zeros $\rho_{\zeta,k}$ of $\zeta^{(k)}(s)$.

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