INTEGRABILITY OF NON-LINEAR DIFFERENTIAL EQUATIONS; LAX FORMULATION AND BI-HAMILTONIAN STRUCTURES

by

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ABSTRACT

The integrability of non-linear differential equations are studied on the basis of Lax and bi-Hamiltonian formulations. The relations between the Lax formalism and bi-Hamiltonian structures are analysed and illustrated with well known examples such as the KdV system. Various methods resulting from this analysis are then applied to multicomponent KdV equations.

KISA ÖZET

Lineer olmayan diferansiyel denklemlerin integre edilebilirliği Lax ve bi-Hamiltonyen formalizmi çerçevesinde çalışılmıştır. Lax formalizmi ve bi-Hamiltonyen yapılar arasındaki ilişkiler incelenmiş ve KdV sistemi gibi iyi bilinen örnekler ile açıklanmıştır. Bu incelemeden elde edilen yöntemler çok bileşenli KdV denklemlerine uygulanmıştır.

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LIST OF SYMBOLS

A	second operator of the Lax pair
$d_F S[u]$	Fréchet derivative of $S[u]$
H	Hamiltonian density
$\mathcal{H}$	Hamiltonian integral
L	Lax operator
$\mathcal{L}_{\mathcal{P}\theta}(S[u])$	Lie derivative of $S[u]$ with respect to $(\mathcal{P}\theta)$
L	Lagrangian density
$\mathcal{P}$	first Hamiltonian operator
$\mathcal{Q}$	second Hamiltonian operator
$\mathcal{R}$	recursion operator
R	R-matrix operator
$\mathrm{Res}(A)$	residue of the operator A
$S[u]$	functional of u
$u(x,t)$	dynamical variable
u_i	ith velocity potential
x	spatial coordinate
t	time
$\mathrm{Tr}(A)$	trace of A
λ	spectral parameter
$\wedge$	exterior product
$\phi_i\ , i = 1, \ldots, N$	multicomponent KdV field, N being a positive integer
$\pi_\pm$	projection operators
Π_i	canonical momentum conjugate to ϕ_i
Θ	functional bi-vector
θ	a uni-vector

1. INTRODUCTION

In the year 1895 D.J. Korteweg and his student G. deVries derived the equation

$$u_t = u_{xxx} + uu_x \tag{1.1}$$

in their study of shallow water waves [1]. This is a (1+1) dimensional nonlinear partial differential equation admitting a travelling wave solution $u(x,t) = 3c sech^2 \frac{\sqrt{c}}{2}(x+ct)$, with the constant $c > 0$ being the speed of the wave. These solutions are called "solitons" and their properties were discovered numerically in the early sixtees. A formal method for solving the KdV equation was introduced by Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura in 1967 [2]. Later this method became a standart tool for solving nonlinear systems and formed the basis of inverse scattering theory.

In 1968 Peter D. Lax developed a beautiful formulation of integrable systems and showed that (1.1) is just one member of an infinite hierarchy of equations [3]. More interest was shown to KdV and other nonlinear models when their integrability conditions were seen to lead to new mathematical properties such as the bi-Hamiltonian structure [4].

A general review of the integrability of the KdV equation is given in Chapter 2. For this purpose the concept of infinite number of conserved quantities is presented in section 2.1. A motivation of the Lax formalism concerning the linear eigenvalue problem connection is the subject of section 2.2. In section 2.3 the Lax representation is analysed on the basis of Lax's original approach to the KdV equation [3]. As a second example, nonlinear Schrödinger equation with matrix Lax pair is discussed. Section 2.4 is on the bi-Hamiltonian formalism and its application to the proof of integrability of the KdV equation. This section also completes the concept of infinite conservation laws.

The third chapter is devoted to the relationship between the Lax formalism and the bi-Hamiltonian formalism. The first attempt is to find a Lax representation of a system that admits bi-Hamiltonian structure. For this purpose the recursion operator technique is introduced in section 3.1. The next section is about the Gelfand-Dickii bracket method which can be used as a tool for calculating the Hamiltonian operators starting from a Lax operator. The r-matrix bracket connection is given in section 3.3. A method of finding the second Hamiltonian operator using the Lax-Nijenhuis equation is also mentioned in section 3.4. These methods are illustrated with various well known systems such as the Boussinesq and Kaup-Broer equations.

The last part of this work is on the multicomponent KdV equations which is already known to be bi-Hamiltonian [5] [6]. A Lax operator is constructed for this system and it is shown that it can be used to calculate the second Hamiltonian operator. An alternative Lax representation which is achieved through the use of the recursion operator is given in section 4.5.

2. INTEGRABILITY OF KdV EQUATION

2.1. Infinite Number of Conservation Laws

Let us consider the equation (1.1) assuming that u is zero at infinity. A conserved quantity (also called a constant of motion) of this system will satisfy

$$\frac{dQ}{dt} = 0 \tag{2.1}$$

where Q is an integral of a *density*, $Q = \int_{-\infty}^{+\infty} T dx$ such that

$$\frac{d}{dt}\int T dx = \int T_t dx = 0. \tag{2.2}$$

This implies a conservation law of the form (so called the continuity equation)

$$T_t + X_x = 0 \tag{2.3}$$

where $-X[u]$ is called the *flux.* If both X and T are polynomials in u and its x derivatives and not dependent explicitly on x and t, this is called a *polynomial conservation law.*

A conservation law for the KdV equation (1.1) is easily obtained if one writes it as

$$u_t = (u_{xx} + \frac{1}{2}u^2)_x. \tag{2.4}$$

So the conserved density is u and the flux is $-u_{xx} - \frac{1}{2}u^2$.

The second constant of motion is given by $\int \frac{1}{2}u^2 dx$ with the flux $-\frac{1}{3}u^3 + \frac{1}{2}u_x^2 - uu_{xx}$. Three more polynomial conserved densities were found by Whitham, Kruskal and Zabusky just by brute force and five additional ones were computed by Miura *et al* ([7] and references therein). Finally it was understood that in fact there were an infinite number of them [1].

To construct the conserved quantities let us introduce the Miura transformation [8]

$$u(x,t) = v^2(x,t) + i\sqrt{6}v_x \tag{2.5}$$

which maps a solution of the modified KdV (mKdV) equation

$$v_t = v_{xxx} + v^2 v_x \tag{2.6}$$

into a solution of the KdV equation. However mKdV equation is not Galilean invariant, under the Galilean transformation $t \to t$, $x \to x + \frac{3}{2\epsilon^2}t$, $u \to u + \frac{3}{2\epsilon^2}$, and $v \to \frac{\epsilon}{\sqrt{6}}v + \frac{\sqrt{6}}{2\epsilon}$ mKdV goes to

$$v_t = v_{xxx} + vv_x + \frac{\epsilon^2}{6}v^2 v_x \tag{2.7}$$

which is called the Gardner equation. Here ϵ is a real parameter. A solution of this equation is mapped into a solution of the KdV equation by the Gardner transformation

$$u = \frac{\epsilon^2}{6}v^2 + v + i\epsilon v_x. \tag{2.8}$$

(2.5) and (2.8) are related to each other by the Galilean transformation given above. The Gardner equation (2.7) is a generalized one, *i.e.*, it reduces to the KdV equation when $\epsilon = 0$, and if $\epsilon \to \infty$, under a rescaling $v \to \frac{\epsilon}{\sqrt{6}}v$ it becomes the mKdV equation.

We can write (2.7) in the form of a continuity equation

$$v_t = (\frac{\epsilon^2}{18}v^3 + \frac{1}{2}v^2 + v_{xx})_x \tag{2.9}$$

which means that $v(x,t)$ can be regarded as a density and its integral is a conserved quantity of the Gardner equation.

$$\frac{d}{dt}\int v dx = 0. \tag{2.10}$$

We can invert the map (2.8) and expand v in terms of u as

$$v = \sum_{n=0}^{\infty} \epsilon^n h_n[u] \tag{2.11}$$

then

$$\int v_t dx = 0 = \sum_{n=0}^{\infty} \epsilon^n \int \frac{\partial}{\partial t} h_n[u] dx \tag{2.12}$$

leads to an infinite number of conservation laws for the KdV equation. But if some $h_n[u]$'s are themselves total derivatives the corresponding conservation laws will be trivial. In fact the following analysis shows that only the terms with even n in the expansion (2.11) provide nontrivial conservation laws.

First let us note that pure polynomial terms in u cannot be written as total derivatives. And it is obvious that ignoring $i\epsilon v_x$ in (2.8) will not affect the existence of such terms in the inversion of (2.8). Then $u = v + \frac{\epsilon^2}{6}v^2$ leads to

$$\begin{aligned} v &= \frac{3}{\epsilon^2}[(1 + \frac{2}{3}\epsilon^2 u)^{\frac{1}{2}} - 1] \\ &= u - \frac{1}{4}\epsilon^2 u^2 + \frac{1}{8}\epsilon^4 u^3 - \ldots \end{aligned} \tag{2.13}$$

It is seen that the monomials in u are the coefficients of ϵ^{2n} terms.

We do not know whether the odd powers of ϵ are total derivatives yet. But if we set $v = y + iz$, $u(x,t)$ becomes

$$u = (y - \epsilon\frac{\partial z}{\partial x} + \frac{\epsilon^2}{6}(y^2 - z^2)) + i(z + \epsilon\frac{\partial y}{\partial x} + \frac{\epsilon^2}{3}yz) \tag{2.14}$$

through the relation (2.8). The fact that u is real leads to vanishing of imaginary part of u. Then we obtain

$$\begin{aligned} z &= -\frac{3}{\epsilon}\frac{\partial}{\partial x}ln(1 + \frac{\epsilon^2}{3}y) \\ &= \frac{\partial}{\partial x}(-\epsilon y + \frac{1}{6}\epsilon^3 y^2 - \frac{1}{9}\epsilon^5 y^3 + \ldots) \end{aligned} \tag{2.15}$$

which is a total derivative and involves odd power ϵ terms. To show that there are no odd power terms in the real part of v we should analyse the scaling properties of the Gardner transformation (2.8). The scaling dimensions of u, v, x, and ϵ can be obtained from

$$\begin{aligned} [u] &= [\frac{\epsilon^2}{6}v^2 + v + i\epsilon v_x] \\ &= [\frac{\epsilon^2}{6}v^2] = [v] = [i\epsilon v_x] \\ &= 2[\epsilon] + 2[v] = [v] = [\epsilon] + [v] - [x] \end{aligned} \tag{2.16}$$

which gives $[u] \equiv 1$, $[v] = 1$, $[x] = -\frac{1}{2}$, $[\epsilon] = -\frac{1}{2}$. Then one can show that (2.8) is invariant under the transformations

$$u \rightarrow au$$

$$
\begin{aligned}
v &\rightarrow av \\
x &\rightarrow a^{-\frac{1}{2}}x \\
\epsilon &\rightarrow a^{-\frac{1}{2}}\epsilon
\end{aligned}
$$

where a is a scaling parameter. So an odd power ϵ term should involve an odd number of derivatives and since such a term must be imaginary due to the factor i in the derivative term in (2.8) we conclude that the real terms are all of even powers of ϵ.

Now let us construct the conserved quantities. If we substitute the expansion (2.11) into the Gardner transformation, we obtain a recursion relation

$$
h_n + i\frac{\partial h_{n-1}}{\partial x} + \frac{1}{6}\sum_{m=0}^{n-2} h_{n-m-2}h_m = 0 \tag{2.17}
$$

where $n > 0$ and $h_{-1} = h_{-2} \equiv 0$, $h_0 = u$. Using this relation we find

$$
\begin{aligned}
h_1 &= -iu_x && (2.18)\\
h_2 &= -\frac{1}{6}u^2 - u_{xx} && (2.19)\\
h_3 &= i(\frac{1}{3}u^2 + u_{xx})_x && (2.20)\\
h_4 &= \frac{1}{3}(\frac{1}{6}u^3 - \frac{1}{2}u_x^2) + (\frac{1}{2}u^2 + u_{xx})_{xx} && (2.21)\\
&\vdots
\end{aligned}
$$

We see that odd n terms are total derivatives and they are imaginary. Let us define the nth conserved density as

$$
T_n = 3(-1)^n h_{2n} \tag{2.22}
$$

then the nth conserved quantity is

$$H_n = 3(-1)^n \int h_{2n} dx. \tag{2.23}$$

Thus the first three conserved quantities are

$$\begin{aligned} H_0 &= 3\int u dx & (2.24)\\ H_1 &= \int (\frac{1}{2}u^2) dx & (2.25)\\ H_2 &= \int (\frac{1}{6}u^3 - \frac{1}{2}u_x^2) dx & (2.26)\\ &\vdots \end{aligned}$$

such that H_1 and H_2 are the first and the second Hamiltonian of the KdV system respectively.

2.2. The Associated Eigenvalue Problem

In 1967 Gardner, Greene, Kruskal and Miura found a method for solving the KdV equation which we now call the inverse scattering method [1] [2]. The most important part of this discovery is that KdV is associated with a linear eigenvalue equation whose scattering problem determines the required data for solving $u(x,t)$ of (1.1). We will not go through the details of this computation but give just the mentioned relation between the two equations and it will be a motivation for the Lax formalism.

Consider the Sturm-Liouville equation

$$\psi_{xx} + \frac{1}{6}u\psi = -\lambda\psi \tag{2.27}$$

with the eigenvalue λ. This can also be wieved as a time independent Schrödinger equation with a potential u which is a function of x and a parameter t [9]. $\lambda(t)$ is then the energy eigenvalue and $\psi(x,t)$ is the corresponding eigenfunction [1]. Now let us make an assumption that $u(x,t)$ evolves according to the KdV equation and write

$$u = -6(\lambda + \frac{\psi_{xx}}{\psi}) \tag{2.28}$$

using (2.27). Then the KdV equation (1.1) becomes

$$\lambda_t \psi^2 - [\psi^2(\frac{\psi_{xxx} - \psi_t + (\frac{1}{2}u - 3\lambda)\psi_x}{\psi})_x]_x = 0 \tag{2.29}$$

in terms of the eigenfunction of (2.27). Assuming ψ is zero at $x = \pm\infty$ one obtains

$$\lambda_t = 0 \tag{2.30}$$

by integrating eq.(2.29) over x, *i.e.*, the eigenvalues are constant when u is a solution of the KdV equation. Then it is said that the Schrödinger equation (2.27) is *isospectral.*

A linearization of the Miura transformation (2.5) leads to the eigenvalue problem. This can be achieved by means of the Cole-Hopf transformation [7] [10]

$$v = i\sqrt{6}\frac{\partial}{\partial x}(ln\psi) = i\sqrt{6}\frac{\psi_x}{\psi}. \tag{2.31}$$

Then eq.(2.5) becomes

$$\psi_{xx} + \frac{1}{6}u\psi = 0. \tag{2.32}$$

One can rederive eq.(2.27) by using the Galilean invariance of KdV with the transformation $u \to u + 6\lambda$.

2.3. The Lax Representation

A general principle for associating nonlinear evolution equations with linear operators was presented by Peter D. Lax [3]. Let u change with t according to the equation

$$u_t = K[u] \tag{2.33}$$

where K is a function of u and its x derivatives. We want to find self-adjoint operators $L(t)^\dagger = L(t)$ which remain unitarily equivalent when (2.33) holds. In other words

$$U(t)^{-1}L(t)U(t) = L(0). \tag{2.34}$$

$U^{-1} = U^\dagger$ is a unitary operator and satisfies a differential equation of the form

$$U_t = AU \tag{2.35}$$

with $A^\dagger = -A$. Differentiating both sides of (2.34) with respect to t one obtains

$$-U^{-1}U_tU^{-1}LU + U^{-1}L_tU + U^{-1}LU_t = 0. \tag{2.36}$$

Substitution of (2.35) into this and multiplication by U on the left and by U^{-1} on the right gives

$$L_t = AL - LA = [A, L]. \tag{2.37}$$

One expects this operator equation to be related to the evolution equation (2.33) since the problem was to find u dependent operators which remain unitarily equivalent as u evolves according to equation (2.33).

Now let us write the eigenvalue problem for the operator L namely

$$L\psi = -\lambda(t)\psi. \tag{2.38}$$

Here t appears only as a parameter. Remembering the discussion of the previous section one can differentiate this with respect to t

$$L_t\psi + L\psi_t = -\lambda_t\psi - \lambda\psi_t \tag{2.39}$$

and find

$$(AL - LA)\psi + L\psi_t = -\lambda_t\psi - \lambda\psi_t \tag{2.40}$$

using (2.37). Since $\psi(t)$ is unitarily related to its value at $t = 0$,*i.e.* , $\psi(t) = U(t)\psi(0)$,

$$\begin{aligned} \frac{\partial\psi(t)}{\partial t} &= \frac{\partial U(t)}{\partial t}\psi(0) \\ &= A(t)U(t)\psi(0) \\ &= A(t)\psi(t). \end{aligned} \tag{2.41}$$

Substituting this into (2.40)

$$(AL - LA)\psi + LA\psi = -\lambda_t\psi - \lambda A\psi \tag{2.42}$$

and using (2.38) to cancel the $AL\psi$ term, it is seen that

$$\lambda_t = 0. \tag{2.43}$$

As we have seen in the previous section the eigenvalues of the Schrödinger equation

$$(\partial^2 + \frac{1}{6}u)\psi = -\lambda\psi \tag{2.44}$$

are invariant if u is a solution to the KdV equation. Here ∂ is a shorthand for the derivative operator $\frac{\partial}{\partial x}$. So let us choose

$$L = \partial^2 + \frac{1}{6}u. \tag{2.45}$$

Then the left hand side of the "Lax equation" (2.37) is simply a multiplication which is $L_t = \frac{1}{6}u_t$. In order to have a multiplication on the right hand side, the operator $A^\dagger = -A$ should involve ∂ terms. Furthermore they should be in odd powers of ∂ since $\partial^\dagger = -\partial$. The first choice is obviously

$$A = \partial. \tag{2.46}$$

The commutator is

$$\begin{aligned} [A, L] &= \partial(\partial^2 + \frac{1}{6}u) - (\partial^2 + \frac{1}{6}u)\partial \\ &= \partial^3 + \frac{1}{6}u_x + \frac{1}{6}u\partial - \partial^3 - \frac{1}{6}u\partial \\ &= \frac{1}{6}u_x \end{aligned} \tag{2.47}$$

where we have used the fact that $\partial u = (\partial u) + u\partial = u_x + u\partial$. So (2.37) reads

$$u_t = u_x \tag{2.48}$$

Secondly one can try

$$A = \partial^3 + a(x,t)\partial + \partial a(x,t) \tag{2.49}$$

where $a(x,t)$ is a coefficient to be determined. This form is adequate for A to be anti-Hermitian since $(a\partial)^\dagger = -\partial a$ [11]. The commutator is calculated as

$$\begin{aligned} [A, L] &= (\partial^3 + a\partial + \partial a)(\partial^2 + \frac{1}{6}u) - (\partial^2 + \frac{1}{6}u)(\partial^3 + a\partial + \partial a) \\ &= (-4a_x + \frac{1}{2}u_x)\partial^2 + (\frac{1}{2}u_{xx} - 4a_{xx})\partial + \frac{1}{6}u_{xxx} + \frac{1}{3}au_x - a_{xxx}. \end{aligned} \tag{2.50}$$

Since $[A, L]$ should be simply a multiplication, one eliminates the ∂ terms by choosing $a = \frac{1}{8}u$. Then the calculation of the commutator leads to a familiar result

$$[A, L] = \frac{1}{24}(u_{xxx} + uu_x). \tag{2.51}$$

Indeed after a rescaling $A \to 4A$ one obtains the KdV equation (1.1). But this is not the whole story. A can be generalized to

$$A_q = \partial^{2q+1} + \sum_1^q [a_j(x,t)\partial^{2j-1} + \partial^{2j-1}a_j(x,t)] \tag{2.52}$$

in order to get higher order equations of the form

$$u_t = K_q[u] \tag{2.53}$$

which is called the KdV hierarchy. So (2.48) is the first member of this hierarchy with $q = 0$ and (1.1) is the second with $q = 1$.

An alternative method of deriving the operators A_q uses formal operator techniques. First we introduce the inverse of ∂ as

$$\partial^{-1}\partial = \partial\partial^{-1} = 1. \tag{2.54}$$

The square root of $L = \partial^2 + \frac{1}{6}u$ is defined to be an infinite series in powers of ∂^{-1}, *i.e.*,

$$L^{1/2} = \partial + a_0 + a_1\partial^{-1} + a_2\partial^{-2} + \dots \tag{2.55}$$

where a_i are functionals of u. These coefficients can be determined up to any order by squaring the series and requiring it to be equal to L up to that order. A_q is identified with $(L^{\frac{2q+1}{2}})_+$ where the subscript $+$ stands for the non-negative powers of ∂. Then the calculation of

$$[(L^{\frac{2q+1}{2}})_+, L] = [L, (L^{\frac{2q+1}{2}})_-] \tag{2.56}$$

gives us $K_q[u]$ up to a multiplicative constant.

Suppose that one attempts to determine the series (2.55) up to the third order. Then

$$L^{1/2} = \partial + a_0 + a_1\partial^{-1} + a_2\partial^{-2} + a_3\partial^{-3} \tag{2.57}$$

and

$$\begin{aligned} L &= (\partial + a_0 + a_1\partial^{-1} + a_2\partial^{-2} + a_3\partial^{-3})(\partial + a_0 + a_1\partial^{-1} + a_2\partial^{-2} + a_3\partial^{-3}) \\ &= \partial^2 + 2a_0\partial + a_{0,x} + 2a_1 + a_0^2 + (a_{1,x} + 2a_2 + 2a_0a_1)\partial^{-1} \\ &\quad + (a_{2,x} + 2a_3 + a_1^2 + 2a_0a_2 - a_1a_{0,x})\partial^{-2} + O(\partial^{-3}). \end{aligned} \tag{2.58}$$

Comparing this with (2.45) a set of equations is obtained:

$$
\begin{aligned}
2a_0 &= 0 && (2.59)\\
a_{0,x} + 2a_1 + a_0^2 &= \frac{1}{6}u && (2.60)\\
a_{1,x} + 2a_2 + 2a_0a_1 &= 0 && (2.61)\\
a_{2,x} + 2a_3 + a_1^2 + 2a_0a_2 - a_1a_{0,x} &= 0. && (2.62)
\end{aligned}
$$

Solving these equations for a_0, a_1, a_2, and a_3 and substituting them into (2.55) one gets

$$L^{1/2} = \partial + \frac{1}{12}u\partial^{-1} - \frac{1}{24}u_x\partial^{-2} + \frac{1}{48}(u_{xx} - \frac{1}{6}u^2)\partial^{-3} + O(\partial^{-4}). \qquad (2.63)$$

According to our identification, A_3 is $(L^{3/2})_+$ and can be calculated as

$$
\begin{aligned}
(L^{2/3})_+ &= (LL^{1/2})_+\\
&= ((\partial^2 + \frac{1}{6}u)(\partial + \frac{1}{12}u\partial^{-1} - \frac{1}{24}u_x\partial^{-2} + \ldots))_+\\
&= (\partial^3 + \frac{1}{4}u\partial + \frac{1}{8}u_x + \frac{1}{12}u_{xx}\partial^{-1} + \ldots)_+\\
&= \partial^3 + \frac{1}{8}(\partial u + u\partial) \qquad (2.64)
\end{aligned}
$$

which is the same as the previous result.

In 1972, Zakharov and Shabat obtained the Lax-pair operators L and A for the nonlinear Schrödinger equation (NSE) [12] [13]

$$iq_t = -q_{xx} + 2k(q^*q)q, \quad -iq_t^* = -q_{xx}^* + 2k(q^*q)q^* \qquad (2.65)$$

described in terms of the complex variable $q(x,t)$. These are matrix valued operators

$$L = \begin{pmatrix} i(1+\beta)\partial & q^* \\ q & i(1-\beta)\partial \end{pmatrix} \tag{2.66}$$

and

$$A = \begin{pmatrix} i\beta\partial^2 - \frac{iq^*q}{1+\beta} & q_x^* \\ -q_x & ik\partial^2 + \frac{iq^*q}{1-\beta} \end{pmatrix} \tag{2.67}$$

where $k = 2/(1-\beta^2)$. Then $L_t = [A, L]$ implies (2.65).

Historically the NSE was the second example of an integrable nonlinear partial differential equation after the discovery of the KdV equation [12]. The Lax process became a powerful method in the search for other integrable nonlinear systems.

2.4. Bi-Hamiltonian Structure

So far we have seen that the KdV equation possesses infinitely many conservation laws and there exists a KdV hierarchy described by the Lax equation. Now Hamiltonian structure of the KdV system will be analysed to explain its integrability from another point of view.

The Hamiltonian formulation of mechanics can be found in many texts on classical mechanics (see [14] for example). This description uses N generalized coordinates and N generalized momenta as dynamical variables which parametrize a $2N$ dimensional phase space. A theorem by Liouville states that the volume in the phase space remains unchanged during the time evolution of the system. The change in the coordinates and momenta during the motion may be regarded as canonical transformations. Now assume that there exist N functionally independent conserved quantities for this system. These conserved quantities can be thought as new momenta obtained as a result of a canonical transformation, if their Poisson brackets with one another vanish (*i.e.*, they are in involution). These are called the action variables. This transformation gives us the coordinates called the angle variables. The Hamilton's equations in terms of the action-angle variables can be integrated to obtain solutions for the angle variables. Then the inversion of the solutions will lead to the evolution equations. Although the determination of these canonical transformations may be practically impossible, our assumption at the beginning asserts the existence of solutions. So we phrase the following [1]: A Hamiltonian system whose phase space is $2N$ dimensional is integrable by the method of quadratures if and only if there exist exactly N functionally independent conserved quantities which are in involution.

A covariant description can be achieved using $2N$ generalized coordinates y^μ half of which are the generalized coordinates and half of which are the generalized momenta. Then the Hamilton's equations can be written covariantly as

$$y_t^\mu = \{y^\mu, H\} = \{y^\mu, y^\nu\}\frac{\partial H}{\partial y^\nu} \tag{2.68}$$

where H is the Hamiltonian and curly brackets represent Poisson brackets. In the case of the continuum systems such as the KdV equation, there is an infinite number of degrees of freedom. So one should replace y^μ with the dynamical variable $u(x,t)$ and the partial derivatives with functional derivatives. Then (2.68) reads

$$u_t = \{u, \mathcal{H}\} = \int dy \{u(x), u(y)\}\frac{\delta \mathcal{H}}{\delta u(y)}. \tag{2.69}$$

The Hamiltonian is an integral $\mathcal{H} = \int dx H[u]$ where $H[u]$ is a function of $u, u_x, \dots$, and the functional derivative of $\mathcal{H}$ is computed using the formula

$$\frac{\delta \mathcal{H}}{\delta u} = (\frac{\partial}{\partial u} - \frac{\partial}{\partial x}\frac{\partial}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial u_{xx}} - \dots)H[u]. \tag{2.70}$$

The operator acting on the density H is called the "Euler" operator [11].

Eq. (2.69) may also be written as $u_t = \mathcal{P}\delta\mathcal{H}$ where δ is a shorthand for the functional derivative (2.70). $\mathcal{P}$ is called the Hamiltonian operator. Then the fundamental Poisson bracket is

$$\{u(x), u(y)\} = \mathcal{P}\delta(x-y). \tag{2.71}$$

We will use the terms Poisson bracket structure and Hamiltonian operator interchangeably. Before going on further, as an example let us try to formulate the KdV equation this way. Let $\mathcal{P} = \partial$ and $\mathcal{H} = H_2 = \int dx(\frac{1}{6}u^3 - \frac{1}{2}u_x^2)$. Then

$$\frac{\delta\mathcal{H}}{\delta u} = \frac{1}{2}u^2 - \frac{\partial}{\partial x}(-u_x) \tag{2.72}$$

and

$$\frac{\partial}{\partial x}\frac{\delta\mathcal{H}}{\delta u} = uu_x + u_{xxx}. \tag{2.73}$$

Consequently $\{u(x), u(y)\} = \partial\delta(x-y)$.

A skew-symmetric operator $\mathcal{P}^\dagger = -\mathcal{P}$ is a Hamiltonian operator if and only if its Lie derivative acting on the associated functional bi-vector $\Theta_{\mathcal{P}} = \frac{1}{2}\int \theta \wedge (\mathcal{P}\theta)dx$ is zero (Jacobi identity) [11]

$$\mathcal{L}_{\mathcal{P}\theta}(\Theta_{\mathcal{P}}) = 0. \tag{2.74}$$

Here $\mathcal{L}_{\mathcal{P}\theta}$ is the Lie derivative with respect to $(\mathcal{P}\theta)$, $\theta \neq \theta[u]$ being a uni-vector, given by the formula

$$\mathcal{L}_{\mathcal{P}\theta}(R[u]) = \frac{\partial R}{\partial u}(\mathcal{P}\theta) + \frac{\partial R}{\partial u_x}(\mathcal{P}\theta)_x + \dots \tag{2.75}$$

where $R[u]$ is any function of u, u_x, ...etc. The property (2.74) is also known as the Jacobi identity of the Hamiltonian operators. So for the KdV equation $\Theta_{\mathcal{P}} = \frac{1}{2}\int \theta \wedge \theta_x dx$ and its Lie derivative is zero since there is no u dependence.

It is well known that the KdV equation admits bi-Hamiltonian structure [4] [1], that is, it can be written in Hamiltonian form in two different ways:

$$u_t = \mathcal{P}\delta\mathcal{H} = \mathcal{Q}\delta\mathcal{H}' \tag{2.76}$$

where the second Hamiltonian operator $\mathcal{Q}$ is

$$\begin{aligned} \mathcal{Q} &= \partial^3 + \frac{1}{3}(\partial u + u\partial) \\ &= \partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x \end{aligned} \tag{2.77}$$

and the corresponding Hamiltonian is $\mathcal{H}' = H_1 = \int(\frac{1}{2}u^2)dx$. Then $\frac{\delta\mathcal{H}'}{\delta u} = u$ and $(\mathcal{Q}u) = u_{xxx} + uu_x$. The proof that $\mathcal{Q}$ is a Hamiltonian operator is as follows: The functional bi-vector is

$$\begin{aligned} \Theta_{\mathcal{Q}} &= \frac{1}{2}\int \theta \wedge (\mathcal{Q}\theta)dx \\ &= \frac{1}{2}\int \theta \wedge (\theta_{xxx} + \frac{2}{3}u\theta_x + \frac{1}{3}u_x\theta)dx \\ &= \frac{1}{2}\int (\theta \wedge \theta_{xxx} + \frac{2}{3}u\theta \wedge \theta_x)dx. \end{aligned} \tag{2.78}$$

Applying the Lie derivative (2.75) we get

$$\begin{aligned} \mathcal{L}_{\mathcal{Q}\theta}(\Theta_{\mathcal{Q}}) &= \frac{1}{3}\int \mathcal{L}_{\mathcal{Q}\theta}(u) \wedge \theta \wedge \theta_x dx \\ &= \frac{1}{3}\int (\theta_{xxx} + \frac{2}{3}u\theta_x + \frac{1}{3}u_x\theta) \wedge \theta \wedge \theta_x dx \\ &= \frac{1}{3}\int \theta_{xxx} \wedge \theta \wedge \theta_x dx \end{aligned} \tag{2.79}$$

since $\theta \wedge \theta$ terms are zero. Integrating by parts

$$\int \theta_{xxx} \wedge \theta \wedge \theta_x dx = -\int \theta_{xx} \wedge \theta_x \wedge \theta_x dx - \int \theta_{xx} \wedge \theta \wedge \theta_{xx} dx \tag{2.80}$$

The operators $\mathcal{P}$ and $\mathcal{Q}$ form a Hamiltonian pair if and only if they satisfy

$$\mathcal{L}_{\mathcal{P}\theta}(\Theta_{\mathcal{Q}}) + \mathcal{L}_{\mathcal{Q}\theta}(\Theta_{\mathcal{P}}) = 0 \tag{2.81}$$

[11]. It is easy to see that this condition holds for the Hamiltonian operators of KdV since $\Theta_{\mathcal{P}} = \frac{1}{2}\int \theta \wedge \theta_x dx$ has no u dependence and $\Theta_{\mathcal{Q}}$ is just $\frac{1}{3}\int u\theta \wedge \theta_x dx$ which results in

$$\begin{aligned} \mathcal{L}_{\mathcal{P}\theta} &= \frac{1}{3}\int (\mathcal{P}\theta) \wedge \theta \wedge \theta_x dx \\ &= \frac{1}{3}\int \theta_x \wedge \theta \wedge \theta_x dx \\ &= 0. \end{aligned} \tag{2.82}$$

The two Poisson brackets corresponding to $\mathcal{P}$ and $\mathcal{Q}$ are respectively

$$\{u(x), u(y)\}_1 = \partial\, \delta(x-y) \tag{2.83}$$

$$\{u(x), u(y)\}_2 = (\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x)\, \delta(x-y). \tag{2.84}$$

Now let us turn back to the question of integrability of KdV equation. Since it is a continuum system, according to the discussion we made in the beginning of this section, there should exist an infinite number of conserved quantities. Moreover they should be independent and in involution with one another. We have seen in section 2.2 that the KdV system indeed possesses infinitely many conservation laws but we do not know whether the conserved quantities are independent or not. In order to exhibit the independence of these quantities we may investigate the scaling behaviour of the recursion relation of equation (2.17)

$$\begin{aligned} [h_n] &= [i\frac{\partial h_{n-1}}{\partial x}] \\ &= [h_{n-1}] - [x] \\ &= [h_{n-1}] + \frac{1}{2} \\ &= \frac{n}{2} + 1 \end{aligned} \tag{2.85}$$

where we have used $[h_0] = [u] = 1$ at the last step. Then

$$[\rho_n] = [h_{2n}] = n + 1. \tag{2.86}$$

so each of the conserved quantities scales with a different integer power law which clearly indicates that they are independent.

The last requirement still needed to be shown is that the conserved quantities are all in involution.

Let us recall that by (2.76) there are two Hamiltonian operators for the KdV equation satisfying

$$u_t = \mathcal{P}\delta H_2 = \mathcal{Q}\delta H_1. \tag{2.87}$$

If we act $\mathcal{Q}$ on $\delta H_0 = \frac{\delta}{\delta u}(3\int u dx) = 3$ we get

$$(\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x)\delta H_0 = u_x \tag{2.88}$$

which is exactly $\mathcal{P}\delta H_1 = (\partial u)$. So we suspect that we should have a recursion relation in the form

$$\mathcal{P}\delta H_n = \mathcal{Q}\delta H_{n-1} \tag{2.89}$$

for all n [11] . Therefore

$$\begin{aligned}
\{H_n, H_m\}_1 &= \int \delta H_n \mathcal{P}\delta H_m dx \\
&= -\int \delta H_m \mathcal{P}\delta H_n dx \\
&= -\int \delta H_m \mathcal{Q}\delta H_{n-1} dx \\
&= \int \delta H_{n-1} \mathcal{Q}\delta H_m dx \\
&= \int \delta H_{n-1} \mathcal{P}\delta H_{m+1} dx \\
&= \{H_{n-1}, H_{m+1}\}_1 \\
&= \dots \\
&= \{H_0, H_{m+n}\}_1 \\
&= \{H_{-1}, H_{m+n+1}\}_1 \\
&= 0
\end{aligned} \tag{2.90}$$

since H_{-1} is zero. Similarly it can be shown that

$$\{H_n, H_m\}_2 = 0. \tag{2.91}$$

So H_n which is conserved through the relation

$$\frac{dH_n}{dt} = \{H_n, H_2\}_1 = \{H_n, H_1\}_2 = 0 \tag{2.92}$$

is also in involution with all other H_m.

Since each of the conserved quantities can be thought of as a Hamiltonian we have a hierarchy of evolution equations

$$u_t = K_n[u] = \{u, H_n\}_2 = \{u, H_{n+1}\}_1. \tag{2.93}$$

The first three equations in the hierarchy are

$$\begin{aligned} u_t &= u_x && (2.94) \\ u_t &= u_{xxx} + uu_x && (2.95) \\ u_t &= u_{xxxxx} + \frac{5}{3}uu_{xxx} + \frac{10}{3}u_xu_{xx} + \frac{5}{6}u^2u_x && (2.96) \\ &\vdots \end{aligned}$$

Eq. (2.93) provides the evolution equations of so-called the KdV hierarchy.

3. RELATIONS BETWEEN FORMULATIONS

3.1. The Recursion Operator

We have seen that the Lax equation (2.37) results in a hierarchy of equations

$$u_t = K_q[u] \quad , \qquad q = 1, 2, 3, \ldots \tag{3.1}$$

The recursion operator ($\mathcal{R}$) is used to obtain the $q+1$ th equation in this hieararchy if K_q is known, *i.e.*,

$$K_{q+1} = (\mathcal{R}K_q). \tag{3.2}$$

In the case of bi-Hamiltonian systems the recursion operator is given by [11] [15]

$$\mathcal{R} = \mathcal{Q}\mathcal{P}^{-1} \tag{3.3}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are the first and second Hamiltonian operators respectively. According to this formula the recursion operator for the KdV equation is

$$\begin{aligned} \mathcal{R} &= (\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x)(\partial^{-1}) \\ &= \partial^2 + \frac{2}{3}u + \frac{1}{3}u_x\partial^{-1}. \end{aligned} \tag{3.4}$$

Now action of this operator on the right hand side of the first equation of the KdV hierarchy (2.94) gives

$$\begin{aligned} (\mathcal{R}K_1) &= (\partial^2 u_x) + \frac{2}{3}uu_x + \frac{1}{3}u_x(\partial^{-1}u_x) \\ &= u_{xxx} + uu_x \end{aligned} \tag{3.5}$$

where we have used the the fact that $(\partial^{-1}u_x) = u$. Similarly we can calculate $K_3[u]$ of (2.94) as

$$\begin{aligned}(\mathcal{R}K_2) &= ((\partial^2 + \frac{2}{3}u + \frac{1}{3}u_x\partial^{-1})(u_{xxx} + uu_x)) \\ &= u_{xxxxx} + \frac{5}{3}uu_{xxx} + \frac{10}{3}u_xu_{xx} + \frac{5}{6}u^2u_x. \end{aligned} \tag{3.6}$$

Clearly we have used that $(\partial^{-1}uu_x) = \frac{1}{2}u^2$. Eq.(3.2) can also be expressed in terms of the Hamiltonians as

$$\delta H_{n+1} = (\mathcal{R}^\dagger \delta H_n). \tag{3.7}$$

Now if the bi-Hamiltonian structure of an equation of the form

$$u_t = K[u] \tag{3.8}$$

is known, then the operators

$$\mathcal{R} = \mathcal{Q}\mathcal{P}^{-1} \quad , \qquad B = d_F K[u] \tag{3.9}$$

form a Lax pair [16] [17] ,where $d_F K[u]$ is called the Fréchet derivative of $K[u]$ and defined by

$$d_F K[u](v) = \Big(\frac{d}{d\epsilon}K[u + \epsilon v]\Big)_{\epsilon=0}. \tag{3.10}$$

The operators B and $\mathcal{R}$ satisfy the equation

$$\mathcal{R}_t = [B, \mathcal{R}] \tag{3.11}$$

which implies eq. (3.8). In fact (3.10) is the action of the operator

$$d_F K[u] = \frac{\partial K}{\partial u} + \frac{\partial K}{\partial u_x}\partial + \frac{\partial K}{\partial u_{xx}}\partial^2 + \dots \tag{3.12}$$

on v. To illustrate this let us take the KdV equation where $K = u_{xxx} + uu_x$ as an example. Then

$$\begin{aligned} d_F K(v) &= \Big(\frac{d}{d\epsilon}[u_{xxx} + \epsilon v_{xxx} + (u+\epsilon v)(u_x + \epsilon v_x)]\Big)_{\epsilon=0} \\ &= v_{xxx} + uv_x + vu_x \end{aligned} \tag{3.13}$$

or

$$d_F K = \partial^3 + u\partial + u_x \tag{3.14}$$

which can also be directly calculated using (3.12). A straightforward calculation of eq.(3.11) leads to

$$\frac{2}{3}u_t + \frac{1}{3}u_{xt}\partial^{-1} = \frac{2}{3}(u_{xxx} + uu_x) + \frac{1}{3}(u_{xxxx} + uu_{xx} + u_x^2)\partial^{-1} \tag{3.15}$$

from which we get the KdV equation after comparing the coefficients of ∂^0 and ∂^{-1} terms.

Secondly we study the Kaup-Broer system [18] which is given by

$$L_t = [(L^2)_{\geq 1}, L] \tag{3.16}$$

with the Lax operator $L = \partial + v + \partial^{-1}\Psi$. ($\geq 1$ indicates the ∂^n terms with $n \geq 1$).

Then

$$
\begin{aligned}
(L^2)_{\geq 1} &= (\partial^2 + v_x + 2v\partial + \Psi + v^2 + v\partial^{-1}\Psi \\
&\quad + \partial^{-1}\Psi\partial + \partial^{-1}\Psi v + \partial^{-1}\Psi\partial^{-1}\Psi)_{\geq 1} \\
&= \partial^2 + 2v\partial
\end{aligned} \tag{3.17}
$$

and

$$
L_t = v_t + \partial^{-1}\Psi_t \tag{3.18}
$$

$$
[(L^2)_{\geq 1}, L] = v_{xx} + 2vv_x + 2\Psi_x + 2\partial^{-1}\Psi_x v + 2\partial^{-1}\Psi v_x - \partial^{-1}\Psi_{xx}. \tag{3.19}
$$

So after equating ∂^0 and ∂^{-1} terms we obtain the coupled system

$$
\begin{aligned}
v_t &= v_{xx} + 2\Psi_x + 2vv_x \\
\Psi_t &= -\Psi_{xx} + 2(v\Psi)_x.
\end{aligned} \tag{3.20}
$$

which can be written in Hamiltonian form in two different ways:

$$
U_t = K[U] = \mathcal{P}\delta H_2 = \mathcal{Q}\delta H_1 \tag{3.21}
$$

where $U = \begin{pmatrix} v \\ \Psi \end{pmatrix}$. Now the Hamiltonian operators are 2×2 matrix diferential operators so (3.21) is a matrix equation of the form

$$
\begin{aligned}
\begin{pmatrix} v_t \\ \Psi_t \end{pmatrix} &= \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \delta H_2/\delta v \\ \delta H_2/\delta\Psi \end{pmatrix} \\
&= \begin{pmatrix} 2\partial & \partial^2 + \partial v \\ -\partial^2 + v\partial & \partial\Psi + \Psi\partial \end{pmatrix} \begin{pmatrix} \delta H_1/\delta v \\ \delta H_1/\delta\Psi \end{pmatrix}
\end{aligned} \tag{3.22}
$$

with the Hamiltonian functions

$$H_1 = \int dx(v\Psi) \tag{3.23}$$

$$H_2 = \int dx(v_x\Psi + v^2\Psi + \Psi^2). \tag{3.24}$$

The recursion operator is then

$$\begin{aligned}\mathcal{R} &= \mathcal{Q}\mathcal{P}^{-1} \\ &= \begin{pmatrix} 2\partial & \partial^2 + \partial v \\ -\partial^2 + v\partial & \partial\Psi + \Psi\partial \end{pmatrix} \begin{pmatrix} 0 & \partial^{-1} \\ \partial^{-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \partial + v_x\partial^{-1} + v & 2 \\ \Psi_x\partial^{-1} + 2\Psi & -\partial + v \end{pmatrix}\end{aligned} \tag{3.25}$$

In this case $K[U]$ is a vector

$$(K_i) = \begin{pmatrix} v_{xx} + 2\Psi_x + 2vv_x \\ -\Psi_{xx} + 2(v\Psi)_x \end{pmatrix} \tag{3.26}$$

so we calculate its Fréchet derivative according to the formula

$$d_F K_{ij} = \frac{\partial K_i}{\partial U^j} + \frac{\partial K_i}{\partial U^j_x}\partial + \frac{\partial K_i}{\partial U^j_{xx}}\partial^2 + \ldots \tag{3.27}$$

to obtain

$$B = (d_F K_{ij}) = \begin{pmatrix} \partial^2 + 2v_x + 2v\partial & 2\partial \\ 2\Psi\partial + 2\Psi_x & -\partial^2 + 2v_x + 2v\partial \end{pmatrix}. \tag{3.28}$$

Then eq.(3.11) leads to eq.(3.20) which shows that $\mathcal{R}$ and B form a matrix Lax pair for the Kaup-Broer system besides L and $(L^2)_{\geq 1}$.

3.2. Gelfand-Dikii Brackets

Having seen that it is possible to obtain a Lax description through the use of the recursion operator when one knows the bi-Hamiltonian structure, it is natural to ask the following question: Is there any solution for the reverse problem, or how can the knowledge of Lax operator formalism help to investigate the bi-Hamiltonian structure? The present section is devoted to this question. For this aim the Gelfand-Dikii brackets will be employed.

Let us introduce the generalized Lax operator

$$L = \partial^n + u_{-1}\partial^{n-1} + u_0\partial^{n-2} + \ldots + u_{n-2} \tag{3.29}$$

where the coefficients u_{-1}, u_0, ... are functions of x. We also introduce two operators Q and V with negative powers of ∂ as

$$\begin{aligned} Q &= \partial^{-n}q_{-1} + \partial^{-n+1}q_0 + \partial^{-n+2}q_1 + \ldots + \partial^{-1}q_{n-2} \\ V &= \partial^{-n}v_{-1} + \partial^{-n+1}v_0 + \partial^{-n+2}v_1 + \ldots + \partial^{-1}v_{n-2} \end{aligned} \tag{3.30}$$

These are auxillary operators which will be used during the construction of the Hamiltonian structures.

Before going on further, let us define the residue and the trace of differential operators. For a general (pseudo-)differential operator of the form

$$A = \sum a_i\partial^i \tag{3.31}$$

the residue is defined as the coefficient of the ∂^{-1} term

$$\mathrm{Res}(A) = a_{-1}. \tag{3.32}$$

Then the trace is defined as

$$\mathrm{Tr}(A) = \int dx \mathrm{Res}(A) \tag{3.33}$$

with the property that for any two pseudo-differential operators A and B

$$\mathrm{Tr}[A, B] = 0 \tag{3.34}$$

where $[A, B] = AB - BA$.

The two Hamiltonian structures associated with the Lax operator L of (3.29) are identified from the Gelfand-Dikii brackets [19]:

$$\{F_Q(L), F_V(L)\}_1 = \mathrm{Tr}(L[V, Q]) \tag{3.35}$$

$$\{F_Q(L), F_V(L)\}_2 = \mathrm{Tr}\{LQ(LV)_+ - QL(VL)_+\} \tag{3.36}$$

where

$$F_Q(L) = \mathrm{Tr}(LQ) \tag{3.37}$$

and

$$F_V(L) = \mathrm{Tr}(LV). \tag{3.38}$$

For the KdV equation the Lax operator is given in the eq.(2.45) so according to our convention $n = 2$, $u_{-1} = 0$, and $u_0 = \frac{1}{6}u$. Then the operators given in eqs.(3.30) become

$$
\begin{aligned}
Q &= \partial^{-2}q_{-1} + \partial^{-1}q_0, \\
V &= \partial^{-2}v_{-1} + \partial^{-1}v_0
\end{aligned}
\tag{3.39}
$$

and the calculation of (3.37) and (3.38) gives

$$
F_Q(L) = \int dx(\frac{1}{6}uq_0), \tag{3.40}
$$

$$
F_V(L) = \int dx(\frac{1}{6}uv_0). \tag{3.41}
$$

One sees that there is no q_1 or v_1 dependence. Now let's calculate the first bracket according to the formulae (3.35)

$$
\begin{aligned}
\mathrm{Tr}(L[V,Q]) &= \int dx \mathrm{Res}\Big((\partial^2 + \frac{1}{6}u)(\partial^{-2}v_{-1}\partial^{-2}q_{-1} \\
&\quad +\partial^{-2}v_{-1}\partial^{-1}q_0 + \partial^{-1}v_0\partial^{-2}q_{-1} + \partial^{-1}v_0\partial^{-1}q_0 \\
&\quad -\partial^{-2}q_{-1}\partial^{-2}v_{-1} - \partial^{-2}q_{-1}\partial^{-1}v_0 - \partial^{-1}q_0\partial^{-2}v_{-1} \\
&\quad -\partial^{-1}q_0\partial^{-1}v_0)\Big) \\
&= \int dx \mathrm{Res}(v_{-1}\partial^{-1}q_0 + \partial v_0\partial^{-2}q_{-1} + \partial v_0\partial^{-1}q_0 \\
&\quad -q_{-1}\partial^{-1}v_0 - \partial q_0\partial^{-2}v_{-1} - \partial q_0\partial^{-1}v_0) \\
&= \int dx(v_{0,x}q_0 - q_{0,x}v_0)
\end{aligned}
\tag{3.42}
$$

It is once more seen that the absence or presence of a ∂^{-2} term in (3.39) has no effect. The first bracket is only dependent on q_0 and v_0. In order to find the Hamiltonian operator from the information above one should remember the formula

$$\{F_Q(L), F_V(L)\} = \int dxdy \frac{\delta F_Q(L)}{\delta u(x)} \{u(x), u(y)\} \frac{\delta F_V(L)}{\delta u(y)} \tag{3.43}$$

and its antisymmetry property

$$\begin{aligned} \{F_Q(L), F_V(L)\} &= \frac{1}{2}\int dxdy \Big[\frac{\delta F_Q(L)}{\delta u(x)} \{u(x), u(y)\} \frac{\delta F_V(L)}{\delta u(y)} \\ &\quad - \frac{\delta F_V(L)}{\delta u(x)} \{u(x), u(y)\} \frac{\delta F_Q(L)}{\delta u(y)}\Big]. \end{aligned} \tag{3.44}$$

Then it is easy to see that

$$\{F_Q(L), F_V(L)\}_1 = \frac{1}{72}\int dxdy \Big[q_0\{u(x), u(y)\}_1 v_0 - v_0\{u(x), u(y)\}_1 q_0\Big]. \tag{3.45}$$

If this is compared with (3.42), the first fundamental Poisson bracket can be identified as

$$\{u(x), u(y)\}_1 = 72\, \partial\, \delta(x-y) \tag{3.46}$$

which is equal to (2.83) up to a multiplicative constant.

Now using the property of trace given in (3.34), one can rewrite the second bracket as

$$\{F_Q(L), F_V(L)\}_2 = \mathrm{Tr}[((LV)_+ L - L(VL)_+)Q]. \tag{3.47}$$

A straightforward calculation yields the integral

$$\begin{aligned} \{F_Q(L), F_V(L)\}_2 &= \int dx \Big(\frac{1}{3} v_{0,x} u q_0 + \frac{1}{6} v_0 u_x q_0 - v_{-1,xx} q_0 + v_{0,xxx} q_0 \\ &\quad - 2 v_{-1,x} q_{-1} + v_{0,xx} q_{-1}\Big). \end{aligned} \tag{3.48}$$

Now besides q_0 and v_0, the result contains the functions v_{-1} and q_{-1}. Since for KdV $u_{-1}=0$, neither $F_Q(L)$ nor $F_V(L)$ contain these functions therefore their bracket also should not be dependent on them. Remember that the presence or absence of a $\partial^{-2}q_{-1}$ or $\partial^{-2}v_{-1}$ term in (3.39) had no effect in the first bracket. In order to obtain such a situation in the second bracket, one imposes the following constraint

$$\mathrm{Tr}[((LV)_+L - L(VL)_+)\partial^{-2}] = 0. \tag{3.49}$$

This guarantees that there will be no ∂ term in $(LV)_+L - L(VL)_+$ which is equivalent to saying that there will be no q_{-1} in the second bracket.

Eq.(3.49) reads

$$-2v_{-1,x} + v_{0,xx} = 0 \tag{3.50}$$

or

$$v_{-1} = \frac{1}{2}v_{0,x}. \tag{3.51}$$

The use of this relation in the integral (3.48) leads to

$$\begin{aligned}\{F_Q(L), F_V(L)\}_2 &= \int dx(\frac{1}{3}v_{0,x}uq_0 + \frac{1}{6}v_0u_xq_0 + \frac{1}{2}v_{0,xxx}q_0) \\ &= \int dxdy\delta(x-y)q_0(\frac{1}{2}v_{0,xxx} + \frac{1}{3}uv_{0,x} + \frac{1}{6}u_xv_0).\end{aligned} \tag{3.52}$$

If this is compared with

$$\begin{aligned}\{F_Q(L), F_V(L)\}_2 &= \int dxdy\frac{\delta F_Q(L)}{\delta u(x)}\{u(x), u(y)\}_2\frac{\delta F_V(L)}{\delta u(y)} \\ &= \int dxdy(\frac{1}{6}q_0)\{u(x), u(y)\}_2(\frac{1}{6}v_0)\end{aligned} \tag{3.53}$$

the second fundamental Poisson bracket is identified as

$$\{u(x), u(y)\}_2 = 18\ (\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x)\ \delta(x-y). \tag{3.54}$$

The brackets (3.35) are often written in a shorter form called the Adler-Gelfand-Dickey brackets [20]. Let α be a pseudo-differential operator

$$\alpha = \partial^{-1}\alpha_0 + \partial^{-2}\alpha_1 + \ldots + \partial^{-n}\alpha_{n-1}. \tag{3.55}$$

Then the two Poisson structures corresponding to the first and second Hamiltonian operators associated with the Lax operator (3.29) are given respectively as

$$P_L(\alpha) = [\alpha, L]_+ \tag{3.56}$$

$$Q_L(\alpha) = L(\alpha L)_+ - (L\alpha)_+ L \tag{3.57}$$

These are called the first and the second Adler-Gelfand-Dickey brackets and they are simply the action of the Hamiltonian operators on α.

For the KdV example $P_L(\alpha)$ is

$$[\alpha, L]_+ = -2\alpha_{0,x} \tag{3.58}$$

so the first Hamiltonian operator is equal to ∂ up to a constant. The calculation of the second structure leads to

$$Q_L(\alpha) = (-\alpha_{0,xx} + 2\alpha_{1,x})\partial - \alpha_{0,xxx} + \alpha_{1,xx} - \frac{1}{3}u\alpha_{0,x} - \frac{1}{6}\alpha_0 u_x \tag{3.59}$$

Since this should be just a multiplication, the coefficient of ∂ term has to be set equal to zero. This condition gives

$$\alpha_1 = \frac{1}{2}\alpha_{0,x} \tag{3.60}$$

then

$$Q_L(\alpha) = -\frac{1}{2}(\alpha_{0,xxx} + \frac{2}{3}u\alpha_{0,x} + \frac{1}{3}u_x\alpha_0) \tag{3.61}$$

which is the action of the second Hamiltonian operator $\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x$ on α multiplied by a constant.

3.3. The R-matrix Bracket

An algebraic approach to integrable systems is the R-matrix bracket [21] which enables one to find the fundamental Poisson brackets if the Lax operator is known. These brackets are the generalized forms of the Adler-Gelfand-Dickey brackets.

Let us introduce the modified Lie bracket

$$[X, Y]_R = [RX, Y] + [X, RY]. \tag{3.62}$$

The linear operator R is called a classical R-matrix if the bracket satisfies the Jacobi identity,

$$[Z, [X, Y]_R]_R + \text{cyclic permutation} = 0, \tag{3.63}$$

and anti-symmetry property

$$[X, Y]_R = -[Y, X]_R \tag{3.64}$$

It is also possible to express (3.63) as

$$[Z, [RX, RY] - R([X, Y]_R)] + c.p. = 0 \tag{3.65}$$

This is obvious when the second entry is just a scalar multiple of the original Lie bracket

$$[RX, RY] - R([X, Y]_R) = -\beta[X, Y]. \tag{3.66}$$

The case $\beta = 1$ is called the modified classical Yang-Baxter equation [20] [21] [22].

The two Poisson strutures associated with the Lax operator (3.29) are defined by

$$P_L(\alpha) = R([L, \alpha]) - [L, R\alpha] \tag{3.67}$$

$$Q_L(\alpha) = LR(\alpha L) - R(L\alpha)L \tag{3.68}$$

where α is the pseudo-differential operator given in (3.55).

The connection between (3.67) and (3.35) is as follows. Let $X = X_+ + X_-$ be any pseudo-differential operator. Define the projection operators π_+ and π_- as

$$\pi_\pm X = X_\pm \tag{3.69}$$

and set

$$R = \frac{1}{2}(\pi_+ - \pi_-). \tag{3.70}$$

It is straightforward to check that this choice for the R-matrix operator satisfies the modified classical Yang-Baxter equation. Now using (3.70), the first bracket (3.67) becomes

$$\begin{aligned} R([L, \alpha]) - [L, R\alpha] &= \frac{1}{2}([L, \alpha]_+ - [L, \alpha]_-) - \frac{1}{2}[L, \alpha_+ - \alpha_-] \\ &= \frac{1}{2}([L, \alpha]_+ - [L, \alpha]_- + [L, \alpha]_+ + [L, \alpha]_-) \\ &= [L, \alpha]_+ \end{aligned} \tag{3.71}$$

where we have used the fact that $\alpha_+ = 0$ and therefore $\alpha = \alpha_-$.

Similarly the second R-matrix bracket reduces to

$$
\begin{aligned}
Q_L(\alpha) &= LR(\alpha L) - R(L\alpha)L \\
&= \frac{1}{2}\Big((L\alpha)_+L - (L\alpha)_-L - L(\alpha L)_+ + L(\alpha L)_-\Big) \\
&= \frac{1}{2}\Big((L\alpha)_+L - [L\alpha - (L\alpha)_+]L - L(\alpha L)_+ \\
&\qquad + L[\alpha L - (\alpha L)_+]\Big) \\
&= L(\alpha L)_+ - (L\alpha)_+L. \qquad (3.72)
\end{aligned}
$$

It is seen that the Adler-Gelfand-Dickey brackets are in fact a particular case of the R-matrix brackets (3.67).

All these brackets are associated with standart Lax operators of the form (3.29). There also exist nonstandart Lax operators which include ∂^{-1} terms [23] . A generalization of the Gelfand-Dickii brackets for the case of nonstandart Lax equations is discussed in [24] .

3.4. The Lax-Nijenhuis Equation

In order to investigate the relationship between the Lax formulation of an integrable system and the bi-Hamiltonian structure, Magri and Kosmann-Schwarzbach introduced the so-called "Lax-Nijenhuis equation" [20]:

$$\mathcal{L}_{\mathcal{Q}\alpha}(L) = \frac{1}{2}\mathcal{L}_{\mathcal{P}\alpha}(L^2) + [L, \hat{L}] \qquad (3.73)$$

where $\mathcal{L}_{\mathcal{Q}\alpha}(L)$ is the Lie derivative (2.75) of the Lax operator L with respect to $(\mathcal{Q}\alpha)$. $\mathcal{P}$ and $\mathcal{Q}$ are the first and the second (higher order) Hamiltonian operators, respectively.

If L is a differential operator of order ∂^n then $\hat{L}$ is

$$\hat{L} = \lambda_{n-1}\partial^{n-1} + \lambda_{n-2}\partial^{n-2} + \ldots + \lambda_0 \tag{3.74}$$

where the coefficients λ_i are functions of $\alpha(x)$.

This equation enables one to find the second Hamiltonian operator $\mathcal{Q}$ if $\mathcal{P}$ and L are known. $\hat{L}$ is an auxiliary operator and its coefficients will be determined during the calculation.

The Lax operator L and the first Hamiltonian operator $\mathcal{P}$ of the KdV equation are

$$L = \partial^2 + \frac{1}{6}u \tag{3.75}$$

$$\mathcal{P} = \partial. \tag{3.76}$$

Then the left hand side of eq.(3.73) is

$$\mathcal{L}_{\mathcal{Q}\alpha}(L) = \mathcal{L}_{\mathcal{Q}\alpha}(\frac{1}{6}u) = \frac{1}{6}(\mathcal{Q}\alpha) \tag{3.77}$$

where $\mathcal{Q}$ is the second Hamiltonian operator to be determined. The squaring of the Lax operator yields

$$L^2 = \partial^4 + \frac{1}{3}u\partial^2 + \frac{1}{3}u_x\partial + \frac{1}{36}u^2 + \frac{1}{6}u_{xx}. \tag{3.78}$$

The Lie derivative of (3.78) is

$$\mathcal{L}_{\mathcal{P}\alpha}(L^2) = \frac{1}{3}\alpha_x\partial^2 + \frac{1}{3}\alpha_{xx}\partial + \frac{1}{18}u\alpha_x + \frac{1}{6}\alpha_{xxx}. \tag{3.79}$$

Since L is a second order differential operator we choose $\hat{L}$ as

$$\hat{L} = \lambda_1 \partial + \lambda_0 \tag{3.80}$$

whose commutator with L gives

$$[L, \hat{L}] = 2\lambda_{1,x}\partial^2 + (\lambda_{1,xx} + 2\lambda_{0,x})\partial + \lambda_{0,xx} - \frac{1}{6}\lambda_1 u_x. \tag{3.81}$$

Now by inserting (3.77), (3.79), and (3.81) into (3.73), one can find the coefficients λ_1 and λ_0. By matching the coefficients of ∂^2 terms, one gets

$$\partial^2 : \quad 0 = \frac{1}{6}\alpha_x + 2\lambda_{1,x} \tag{3.82}$$

which has the solution

$$\lambda_1 = -\frac{1}{12}\alpha. \tag{3.83}$$

λ_0 can be found by matching the ∂ terms,

$$\begin{aligned} \partial : \quad 0 &= \frac{1}{6}\alpha_{xx} + \lambda_{1,xx} + 2\lambda_{0,x} \\ &= \frac{1}{12}\alpha_{xx} + 2\lambda_{0,x} \end{aligned} \tag{3.84}$$

so

$$\lambda_0 = -\frac{1}{24}\alpha_x. \tag{3.85}$$

Finally equating the coefficients of ∂^0 terms, one gets

$$\partial^0 : \quad \frac{1}{6}(\mathcal{Q}\alpha) = \frac{1}{2}(\frac{1}{18}u\alpha_x + \frac{1}{6}\alpha_{xxx}) + \lambda_{0,xx} - \frac{1}{6}\lambda_1 u_x$$

$$= \frac{1}{36}u\alpha_x + \frac{1}{24}\alpha_{xxx} + \frac{1}{72}\alpha u_x \tag{3.86}$$

or

$$(\mathcal{Q}\alpha) = \frac{1}{4}((\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x)\alpha). \tag{3.87}$$

This is the familiar form of the second Hamiltonian structure of the KdV equation.

As a second example let us consider the Boussinesq equation:

$$u_{tt} = -\frac{1}{3}(u_{xxx} + 8uu_x)_x. \tag{3.88}$$

Introducing the variable $v(x,t)$, this equation may also be written in the following form

$$\begin{aligned} u_t &= v_x \\ v_t &= -\frac{1}{3}u_{xxx} - \frac{8}{3}uu_x \end{aligned} \tag{3.89}$$

then Eq.(3.88) is just a result of the consistency relation

$$v_{tx} = v_{xt}. \tag{3.90}$$

The Boussinesq system can be expressed in Hamiltonian form as

$$U_t = \mathcal{P}\delta H_1 \tag{3.91}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \mathcal{P} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{3.92}$$

and

$$H_1 = \int (\frac{1}{6}u_x^2 - \frac{4}{9}u^3 + \frac{1}{2}v^2)dx. \tag{3.93}$$

Then (3.91) gives

$$\begin{aligned}
\begin{pmatrix} u_t \\ v_t \end{pmatrix} &= \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \delta H_1/\delta u \\ \delta H_1/\delta v \end{pmatrix} \\
&= \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{3}u_{xx} - \frac{4}{3}u^2 \\ v \end{pmatrix} \\
&= \begin{pmatrix} v_x \\ -\frac{1}{3}u_{xxx} - \frac{8}{3}uu_x \end{pmatrix}.
\end{aligned} \tag{3.94}$$

The Boussinesq equation is known to have a Lax formulation

$$L_t = [A, L] \tag{3.95}$$

with the pair of operators

$$L = \partial^3 + 2u\partial + u_x + v \tag{3.96}$$

$$A = \partial^2 + \frac{4}{3}u. \tag{3.97}$$

Then Eq.(3.95) reads

$$2u_t\partial + u_{xt} + v_t = -\frac{1}{3}u_{xxx} + v_{xx} + 2v_x\partial - \frac{8}{3}uu_x. \tag{3.98}$$

The comparison of the coefficients of ∂^1 and ∂^0 terms gives Eq.(3.88).

Now suppose that, having the Lax operator (3.96) and the Hamiltonian operator (3.92) in hand, one wishes to calculate the second Hamiltonian operator -if it exists- through the use of Lax-Nijenhuis equation

$$\mathcal{L}_{\mathcal{Q}\theta}(L) = \frac{1}{2}\mathcal{L}_{\mathcal{P}\theta}(L^2) + [L, \hat{L}]. \tag{3.99}$$

The squaring of the Lax operator leads to

$$\begin{aligned} L^2 &= \partial^6 + 4u\partial^4 + (8u_x + 2v)\partial^3 + (9u_{xx} + 3v_x + 4u^2)\partial^2 + \\ &+(5u_{xxx} + 8uu_x + 3v_{xx} + 4uv)\partial + u_{xxxx} + v_{xxx} + 2uu_{xx} \\ &+2uv_x + 2u_x v + u_x^2 + v^2. \end{aligned} \tag{3.100}$$

Since L is a third order differential operator, $\hat{L}$ should be chosen as a second order differential operator with arbitrary coefficients:

$$\hat{L} = \mu(x)\partial^2 + \eta(x)\partial + \lambda(x). \tag{3.101}$$

The commutator on the right hand side of eq.(3.99) can be calculated as

$$\begin{aligned} [L, \hat{L}] &= 3\mu_x\partial^4 + (3\mu_{xx} + 3\eta_x)\partial^3 + (3\eta_{xx} + \mu_{xxx} + 3\lambda_x + 2u\mu_x - 4\mu u_x)\partial^2 \\ &+(\eta_{xxx} + 3\lambda_{xx} + 2u\eta_x - 4\mu u_{xx} - 2\mu v_x - 2\eta u_x)\partial \\ &\lambda_{xxx} + 2u\lambda_x - \mu u_{xxx} - \mu v_{xx} - \eta u_{xx} - \eta v_x \end{aligned} \tag{3.102}$$

The Hamiltonian operators are now matrices so the definition of the Lie derivative given by Eq.(2.75) should be generalized to

$$\mathcal{L}_{\mathcal{P}\theta}(R[U]) = \sum_{i,j} \frac{\partial R}{\partial U_i}(\mathcal{P}_{ij}\theta_j) + \frac{\partial R}{\partial U_{i,x}}(\mathcal{P}_{ij}\theta_j)_x + \dots \tag{3.103}$$

If $\theta_1 = \alpha$ and $\theta_2 = \beta$ then

$$(\mathcal{P}\theta) = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta_x \\ \alpha_x \end{pmatrix}. \tag{3.104}$$

so

$$\mathcal{L}_{\mathcal{P}\theta}(R[u,v]) = \frac{\partial R}{\partial u}\beta_x + \frac{\partial R}{\partial v}\alpha_x + \frac{\partial R}{\partial u_x}\beta_{xx} + \frac{\partial R}{\partial v_x}\alpha_{xx} + \dots \tag{3.105}$$

e.g.

$$\begin{aligned} \mathcal{L}_{\mathcal{P}\theta}(u) &= \beta_x \\ \mathcal{L}_{\mathcal{P}\theta}(v_x) &= \alpha_{xx} \\ \mathcal{L}_{\mathcal{P}\theta}(uv) &= v\beta_x + u\alpha_x \qquad \text{etc}\dots \end{aligned} \tag{3.106}$$

Now using (3.100) and (3.102) , one may rewrite the Lax-Nijenhuis equation and match the coefficients of ∂^n terms on both sides. Since the coefficient of the ∂^6 term in L^2 is one, its Lie derivative is zero, so the leading terms are of fourth degree:

$$\begin{aligned} \partial^4: \qquad 0 &= \frac{1}{2}\mathcal{L}_{\mathcal{P}\theta}(4u) + 3\mu_x \\ 0 &= \frac{1}{2}(4\beta_x) + 3\mu_x \end{aligned} \tag{3.107}$$

whose solution for μ is

$$\mu = -\frac{2}{3}\beta. \tag{3.108}$$

Similarly

$$\begin{aligned} \partial^3: \qquad 0 &= \frac{1}{2}\mathcal{L}_{\mathcal{P}\theta}(8u_x+2v)+3\mu_{xx}+3\eta_x \\ 0 &= \frac{1}{2}(8\beta_{xx}+2\alpha_x)+3(-\frac{2}{3}\beta_{xx})+3\eta_x \end{aligned} \tag{3.109}$$

Then

$$\eta = -\frac{2}{3}\beta_x - \frac{1}{3}\alpha \tag{3.110}$$

In order to obtain a solution for λ, one needs to match the ∂^2 terms,

$$\begin{aligned} \partial^2: \qquad 0 &= \frac{1}{2}\mathcal{L}_{\mathcal{P}\theta}(9u_{xx}+3v_x+4u^2)+3\eta_{xx}+\mu_{xxx}+3\lambda_x+2u\mu_x-4\mu u_x \\ 0 &= \frac{1}{2}(9\beta_{xxx}+3\alpha_{xx}+8u\beta_x)+3(-\frac{2}{3}\beta_{xxx}-\frac{1}{3}\alpha_{xx}) \\ &\quad -\frac{2}{3}\beta_{xxx}+3\lambda_x+2u(-\frac{2}{3}\beta_x)-4u_x(-\frac{2}{3}\beta) \\ 0 &= \frac{11}{6}\beta_{xxx}+\frac{1}{2}\alpha_{xx}+\frac{8}{3}(u\beta)_x+3\lambda_x \end{aligned} \tag{3.111}$$

and therefore

$$\lambda = -\frac{11}{18}\beta_{xx} - \frac{1}{6}\alpha_x - \frac{8}{9}u\beta. \tag{3.112}$$

So the three coefficients of the operator $\hat{L}$ are found. If we go one step further and work with the ∂ term, we find

$$\begin{aligned} \partial: \qquad \mathcal{L}_{\mathcal{Q}\theta}(2u) &= \frac{1}{2}\mathcal{L}_{\mathcal{P}\theta}(5u_{xxx}+8uu_x+3v_{xx}+4uv) \\ &\quad +\eta_{xxx}+3\lambda_{xx}+2u\eta_x-4\mu u_{xx}-2\mu v_x-2\eta u_x. \end{aligned} \tag{3.113}$$

Since μ, η, and λ are known in terms of α and β, a straightforward calculation yields

$$\mathcal{L}_{\mathcal{Q}\theta}(u) = \frac{1}{3}\alpha_{xxx}+\frac{2}{3}u\alpha_x+\frac{1}{3}u_x\alpha+v\beta_x+\frac{2}{3}v_x\beta. \tag{3.114}$$

Now remembering the formula (3.103) one can write

$$\begin{aligned}\mathcal{L}_{Q\theta}(u) &= (\mathcal{Q}_{11}\theta_1) + (\mathcal{Q}_{12}\theta_2) \\ &= (\mathcal{Q}_{11}\alpha) + (\mathcal{Q}_{12}\beta). \end{aligned} \tag{3.115}$$

A comparison of eq.(3.114) with eq.(3.115) shows that

$$\begin{aligned}\mathcal{Q}_{11} &= \frac{1}{3}(\partial^3 + 2u\partial + u_x) \\ \mathcal{Q}_{12} &= \frac{1}{3}(3v\partial + 2v_x). \end{aligned} \tag{3.116}$$

The last part of this task is to match the ∂^0 terms which will give

$$\begin{aligned}\partial^0 : \quad \mathcal{L}_{Q\theta}(u_x + v) &= \frac{1}{2}\mathcal{L}_{P\theta}(u_{xxxx} + v_{xxx} + 2uu_{xx} + 2uv_x + 2u_x v + u_x^2 + v^2) \\ &+\lambda_{xxx} + 2u\lambda_x - \mu u_{xxx} - \mu v_{xx} - \eta u_{xx} - \eta v_x. \end{aligned} \tag{3.117}$$

Now performing the algebra in a similar fashion of the previous case and using

$$\begin{aligned}\mathcal{L}_{Q\theta}(u_x) &= \sum_j (\mathcal{Q}_{1j}\theta_j)_x \\ &= (\mathcal{L}_{Q\theta}(u))_x \end{aligned} \tag{3.118}$$

one obtains

$$\begin{aligned}\mathcal{L}_{Q\theta}(v) &= v\alpha_x + \frac{1}{3}v_x\alpha - \frac{1}{9}\beta_{xxxxx} - u_{xx}\beta_x - \frac{10}{9}u\beta_{xxx} \\ &-\frac{5}{3}u_x\beta_{xx} - \frac{2}{9}u_{xxx}\beta - \frac{16}{9}uu_x\beta - \frac{16}{9}u^2\beta_x. \end{aligned} \tag{3.119}$$

Then

$$\mathcal{Q}_{21} = \frac{1}{3}(3v\partial + v_x) \tag{3.120}$$

$$\mathcal{Q}_{22} = -\frac{1}{9}[\partial^5 + 5(u\partial^3 + \partial^3 u) - 3(u_{xx}\partial + \partial u_{xx}) + 16u\partial u]. \tag{3.121}$$

In order to check that this is correct, let us calculate the second Hamiltonian structure by means of the Adler-Gelfand-Dickey bracket of section 3.2. The Lax operator is

$$L = \partial^3 + u_0\partial + u_1 \tag{3.122}$$

where $u_0 = 2u$ and $u_1 = u_x + v$. The Hamiltonian operator can be extracted from L by the relation [25]

$$L_t = (L\alpha)_+ L - L(\alpha L)_+ \tag{3.123}$$

where

$$\alpha = \partial^{-3}\alpha_{-1} + \partial^{-2}\alpha_0 + \partial^{-1}\alpha_1. \tag{3.124}$$

Equation (3.123) will lead to

$$u_{0,t} = (A_{00}\alpha_0) + (A_{01}\alpha_1) \tag{3.125}$$

$$u_{1,t} = (A_{10}\alpha_0) + (A_{11}\alpha_1) \tag{3.126}$$

and the Poisson brackets are defined in terms of A_{ij} by

$$\{u_i(x), u_j(y)\} = A_{ij}\delta(x-y) \tag{3.127}$$

where $i, j = 1, 2$.

After a straightforward calculation eq.(3.123) becomes

$$\begin{aligned} 2u_t\partial &+ u_{xt} + v_t \\ &= (-\alpha_{1,xxx} + 3\alpha_{0,xx} - 3\alpha_{-1,x} - 2u\alpha_{1,x} - 2\alpha_1 u_x)\partial^2 \\ &+(2\alpha_1 v_x - 3\alpha_{-1,xx} - 2\alpha_{1,xxxx} + 5\alpha_{0,xxx} - 2\alpha_1 u_{xx} - 4\alpha_{1,xx}u \\ &-5\alpha_{1,x}u_x + 2u_x\alpha_0 + 3v\alpha_{1,x} + 4u\alpha_{0,x})\partial \\ &+\alpha_1 v_{xx} + \alpha_0 u_{xx} + \alpha_0 v_x + 2\alpha_{1,x}v_x - \alpha_{1,xxxxx} - 6\alpha_{1,xx}u_x \\ &+2\alpha_{0,xxxx} - \alpha_{-1,xxx} - 4uu_x\alpha_1 - 4u^2\alpha_{1,x} + 4u\alpha_{0,xx} - 2u\alpha_{-1,x} \\ &-\alpha_1 u_{xxx} - 4u_{xx}\alpha_{1,x} + 3u_x\alpha_{0,x} + 3v\alpha_{0,x} - 4u\alpha_{1,xxx} \end{aligned} \tag{3.128}$$

Obviously the coefficient of ∂^2 term is zero,

$$0 = -\alpha_{1,xxx} + 3\alpha_{0,xx} - 3\alpha_{-1,x} - 2u\alpha_{1,x} - 2\alpha_1 u_x \tag{3.129}$$

and its solution for α_{-1} is

$$\alpha_{-1} = \alpha_{0,x} - \frac{1}{3}\alpha_{1,xx} - \frac{2}{3}u\alpha_1. \tag{3.130}$$

The next equation obtained from matching the ∂ terms is, with α_{-1} equal to (3.130),

$$\begin{aligned} 2u_t &= -\alpha_{1,xxxx} + 2\alpha_{0,xxx} - 2u\alpha_{1,xx} - u_x\alpha_{1,x} \\ &+2u_x\alpha_0 + 4u\alpha_{0,x} + 2v_x\alpha_1 + 3v\alpha_{1,x}. \end{aligned} \tag{3.131}$$

Then according to (3.125),

$$\begin{aligned} A_{00} &= 2\partial^3 + 2u_x + 4u\partial \\ A_{01} &= -\partial^4 - 2u\partial^2 - u_x\partial + 2v_x + 3v\partial. \end{aligned} \tag{3.132}$$

Similarly ∂^0 terms give

$$\begin{aligned} u_{xt} + v_t &= -\frac{8}{3}u\alpha_{1,xxx} + 2u\alpha_{0,xx} - \frac{8}{3}uu_x\alpha_1 - \frac{8}{3}u^2\alpha_{1,x} \\ &-\frac{1}{3}u_{xxx}\alpha_1 + \alpha_{0,xxxx} - \frac{2}{3}\alpha_{1,xxxxx} - 4u_x\alpha_{1,xx} \\ &-2u_{xx}\alpha_{1,x} + v_{xx}\alpha_1 + u_{xx}\alpha_0 + v_x\alpha_0 \\ &+2v_x\alpha_{1,x} + 3u_x\alpha_{0,x} + 3v\alpha_{0,x}. \end{aligned} \tag{3.133}$$

Then

$$\begin{aligned} A_{10} &= 2u\partial^2 + \partial^4 + 3u_x\partial + 3v\partial + u_x x \\ A_{11} &= -\frac{8}{3}u\partial^3 - \frac{8}{3}uu_x - \frac{8}{3}u^2\partial - \frac{1}{3}u_{xxx} \\ &-\frac{2}{3}\partial^5 - 4u_x\partial^2 - 2u_{xx}\partial + v_{xx} + 2v_x\partial. \end{aligned} \tag{3.134}$$

Now according to the definition given in (3.127)

$$\{u_0(x), u_0(y)\} = A_{00}\delta(x-y) \tag{3.135}$$

and since $\{u_0(x), u_0(y)\} = 4\{u(x), u(y)\}$ one obtains

$$\{u(x), u(y)\} = \frac{1}{2}(\partial^3 + 2u\partial + u_x)\delta(x-y) \tag{3.136}$$

The next one is

$$\{u_0(x), u_1(y)\} = A_{01}\delta(x-y) \tag{3.137}$$

where

$$\begin{aligned}\{u_0(x), u_1(y)\} &= \{2u, u'+v\} \\ &= 2\{u, u'\} + 2\{u, v\} \\ &= -2\{u, u\}\partial + 2\{u, v\}.\end{aligned} \tag{3.138}$$

Here the primes denote the first derivatives. Comparing this with eq.(3.137) and using (3.136) and (3.132) one obtains

$$\{u(x), v(y)\} = \frac{1}{2}(3v\partial + 2v_x)\delta(x-y) \tag{3.139}$$

Similarly

$$\{u_1(x), u_0(y)\} = A_{10}\delta(x-y) \tag{3.140}$$

where

$$\begin{aligned}\{u_1(x), u_0(y)\} &= \{u'+v, 2u\} \\ &= 2\{u', u\} + 2\{v, u\} \\ &= 2\partial\{u, u\} + 2\{v, u\}.\end{aligned} \tag{3.141}$$

Then

$$\{v(x), u(y)\} = \frac{1}{2}(3v\partial + v_x)\delta(x-y). \tag{3.142}$$

Note that this may also be obtained by calculating the adjoint of $\{u(x), v(y)\}$, *i.e.*,

$$\{u, v\} = \{v, u\}^{\dagger} \tag{3.143}$$

The last bracket is

$$\{u_1(x), u_1(y)\} = A_{11}\delta(x-y) \tag{3.144}$$

which can be decomposed to

$$\begin{aligned} \{u_1(x), u_1(y)\} &= \{u' + v, u' + v\} \\ &= \{u', u'\} + \{u', v\} + \{v, u'\} + \{v, v\} \\ &= -\partial\{u, u\}\partial + \partial\{u, v\} - \{v, u\}\partial + \{v, v\} \end{aligned} \tag{3.145}$$

If this is compared with eq.(3.144), using (3.134), (3.136), (3.139) and (3.142) one obtains

$$\begin{aligned} \{v(x), v(y)\} &= -\frac{1}{6}(\partial^5 + 10u\partial^3 + 15u_x\partial^2 + 16u^2\partial + 9u_{xx}\partial \\ &\quad +2u_{xxx} + 16uu_x)\delta(x-y) \end{aligned} \tag{3.146}$$

or

$$\begin{aligned} \{v(x), v(y)\} &= -\frac{1}{6}[\partial^5 + 5(u\partial^3 + \partial^3 u) - 3(u_{xx}\partial + \partial u_{xx}) \\ &\quad +16u\partial u]\delta(x-y). \end{aligned} \tag{3.147}$$

So the second Hamiltonian operator is a matrix of the form

$$Q = \frac{1}{2}\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \tag{3.148}$$

where

$$\begin{aligned} Q_{11} &= \partial^3 + 2u\partial + u_x \\ Q_{12} &= 3v\partial + 2v_x \\ Q_{21} &= 3v\partial + v_x \\ Q_{22} &= -\frac{1}{3}[\partial^5 + 5(u\partial^3 + \partial^3 u) - 3(u_{xx}\partial + \partial u_{xx}) + 16u\partial u] \end{aligned} \tag{3.149}$$

If this is rescaled with $\frac{2}{3}$, eqs.(3.116) and (3.120) are obtained.

The Hamiltonian of the Boussinesq equation associated with the second Hamiltonian operator $\mathcal{Q} = (\mathcal{Q}_{ij})$ is rather simple. An easy calculation shows that

$$U_t = \mathcal{Q}\delta H_2 \tag{3.150}$$

implies (3.89) where

$$H_2 = \int \frac{3}{2} v dx. \tag{3.151}$$

It is seen that the Lax-Nijenhuis equation is a suitable tool in the calculation of the second Hamiltonian operator of an integrable system if the Lax operator and the first Hamiltonian operator are known. But the reverse problem, *i.e.*, the investigation of a Lax operator by using the Lax-Nijenhuis equation with the bi-Hamiltonian structure in hand, is an open question.

4. MULTICOMPONENT KdV EQUATIONS

So far the relations between the Lax formulation of an integrable system and its bi-Hamiltonian structure are analysed. Two different Lax representations are directly related to the same Hamiltonian operator pair. These are the recursion operator representation and the formulation with the Lax operator which gives the Hamiltonian operators through Gelfand-Dickii brackets or Lax-Nijenhuis equation. So in this point of view they are especially of interest among the other possible Lax representations, and as a final example we will analyse the multicomponent KdV equations in the light of this fact.

4.1. The Lagrangian and Equations of Motion

Now consider the Lagrangian density written in terms of an N-component field, $\phi_i = \phi_i(x,t)$,

$$\mathsf{L} = \frac{1}{2}\delta_{ij}\phi_{i,x}\phi_{j,t} + \frac{1}{2}b_{ij}\phi_{i,xx}\phi_{j,xx} - \frac{1}{3}c_{ijk}\phi_{i,x}\phi_{j,x}\phi_{k,x}. \tag{4.1}$$

Here b_{ij} and c_{ijk} are completely symmetric constant coefficients with subindices $i, j, k = 1 \ldots N$. The multicomponent KdV action is defined as the integral of (4.1),

$$S = \int \mathsf{L}(\phi_{i,x}, \phi_{i,xx}, \phi_{i,t})\,dx\,dt \tag{4.2}$$

The principle of least action $\delta S = 0$ leads to the Lagrange equation of motion

$$\frac{\partial}{\partial t}\frac{\partial \mathsf{L}}{\partial \phi_{i,t}} + \frac{\partial}{\partial x}\frac{\partial \mathsf{L}}{\partial \phi_{i,x}} - \frac{\partial^2}{\partial x^2}\frac{\partial \mathsf{L}}{\partial \phi_{i,xx}} = 0 \tag{4.3}$$

if the variations $\delta\phi_i$ are zero at the boundaries. Using the symmetry properties of the coefficients b_{ij} and c_{ijk} one can easily calculate that

$$\frac{\partial \mathsf{L}}{\partial \phi_{i,t}} = \frac{1}{2}\phi_{i,x} \tag{4.4}$$

$$\frac{\partial \mathsf{L}}{\partial \phi_{i,x}} = \frac{1}{2}\phi_{i,t} - c_{ijk}\phi_{j,x}\phi_{k,x} \tag{4.5}$$

$$\frac{\partial \mathsf{L}}{\partial \phi_{i,xx}} = b_{ij}\phi_{j,xx}. \tag{4.6}$$

Then the equation of motion (4.3) can be written as

$$\phi_{i,xt} = b_{ij}\phi_{j,xxxx} + c_{ijk}(\phi_{j,x}\phi_{k,x})_x. \tag{4.7}$$

Defining the velocity potentials $u_i = \phi_{i,x}$ one can rewrite eq.(4.7) as

$$u_{i,t} = b_{ij}u_{j,xxx} + c_{ijk}(u_j u_k)_x \tag{4.8}$$

which will be referred as the multicomponent KdV equation [5].

4.2. The Lax Pair

A matrix valued Lax operator for the multicomponent KdV equation is given as [26]

$$L_{ij} = b_{ij}\partial^2 + \frac{1}{3}c_{ijk}u_k. \tag{4.9}$$

It is proper to choose the second operator of the Lax pair as

$$A_{ij} = 4b_{ij}\partial^3 + c_{ijk}(u_k\partial + \partial u_k). \tag{4.10}$$

When b_{ij} and c_{ijk} satisfy the following constraints

$$\begin{aligned} b_{ij}c_{ikl} - b_{ik}c_{ijl} &= 0 \\ c_{ijk}c_{imn} - c_{imk}c_{ijn} &= 0 \end{aligned} \tag{4.11}$$

the Lax equation

$$L_t = [A, L] \tag{4.12}$$

reads

$$c_{ijk}u_{k,t} = b_{im}c_{mjl}u_{l,xxx} + 2c_{imk}c_{mjl}u_k u_{l,x}. \tag{4.13}$$

Again by using eqs.(4.11) the multicomponent KdV equations (4.8) are obtained.

4.3. The Hamiltonian Formulation

Using the Lagrangian given in eq.(4.1) one can obtain the canonical momentum variable conjugate to ϕ_i formally as

$$\Pi_i = \frac{\partial \mathsf{L}}{\partial \phi_{i,t}} = \frac{1}{2}\phi_{i,x}. \tag{4.14}$$

Then the Hamiltonian is

$$\begin{aligned} H_1 &= \int (\delta_{ij}\Pi_i\phi_{j,t} - \mathsf{L})dx \\ &= \int (-\frac{1}{2}b_{ij}\phi_{i,xx}\phi_{j,xx} + \frac{1}{3}c_{ijk}\phi_{i,x}\phi_{j,x}\phi_{k,x})dx \\ &= \int (-\frac{1}{2}b_{ij}u_{i,x}u_{j,x} + \frac{1}{3}c_{ijk}u_i u_j u_k)dx. \end{aligned} \tag{4.15}$$

The functional derivative of H_1 is

$$\begin{aligned} \frac{\delta H_1}{\delta u_m} &= \frac{\partial H_1}{\partial u_m} - \frac{\partial}{\partial x}\frac{\partial H_1}{\partial u_{m,x}} + \frac{\partial^2}{\partial x^2}\frac{\partial H_1}{\partial u_{m,xx}} - \dots \\ &= c_{mjk}u_j u_k + b_{mj}u_{j,xx}. \end{aligned} \tag{4.16}$$

The Hamiltonian formulation of the system is achieved if one finds a matrix operator P_{ij} such that

$$u_{i,t} = P_{ij}\frac{\delta H_1}{\delta u_j}. \tag{4.17}$$

It is obvious that the proper choice is

$$P_{ij} = \delta_{ij}\partial \tag{4.18}$$

then (4.17) reads

$$u_{i,t} = (b_{ij}u_{j,xx} + c_{ijk}u_j u_k)_x \tag{4.19}$$

which is the desired result.

Having the Hamiltonian operator in hand the fundamental Poisson bracket can be written as

$$\{u_i(x), u_j(y)\} = \partial\delta(x-y)\delta_{ij}. \tag{4.20}$$

4.4. The Second Hamiltonian Structure

The second Hamiltonian operator will be calculated using the Lax-Nijenhuis equation

$$\mathcal{L}_{Q\alpha}(L_{ij}) = \frac{1}{2}\mathcal{L}_{P\alpha}(L^2_{ij}) + [L, \hat{L}]_{ij}. \tag{4.21}$$

The square of the Lax operator given in (4.9) is

$$\begin{aligned} L^2_{ij} &= L_{im}L_{mj} \\ &= b_{im}b_{mj}\partial^4 + \frac{2}{3}b_{im}c_{mjk}u_k\partial^2 + \frac{2}{3}b_{im}c_{mjq}u_{q,x}\partial \\ &+\frac{1}{3}b_{im}c_{mjk}u_{k,xx} + \frac{1}{9}c_{imk}c_{mjq}u_k u_q. \end{aligned} \tag{4.22}$$

Let the operator $\hat{L}_{ij}$ be

$$\hat{L}_{ij} = \lambda_{ij}(x) + \mu_{ij}(x)\partial \tag{4.23}$$

where λ_{ij} and μ_{ij} are symmetric coefficients. Then the commutator in the Lax-Nijenhuis equation can be calculated as

$$\begin{aligned}
[L,\hat{L}]_{ij} &= L_{im}\hat{L}_{mj} - \hat{L}_{im}L_{mj} \\
&= (b_{im}\mu_{mj} - \mu_{im}b_{mj})\partial^3 \\
&\quad +(b_{im}\lambda_{mj} - \lambda_{im}b_{mj} + 2b_{im}\mu_{mj,x})\partial^2 \\
&\quad +(2b_{im}\lambda_{mj,x} + b_{im}\mu_{mj,xx} + \frac{1}{3}c_{imk}\mu_{mj}u_k - \frac{1}{3}c_{mjk}\mu_{im}u_k)\partial \\
&\quad +b_{im}\lambda_{mj,xx} + \frac{1}{3}c_{imk}\lambda_{mj}u_k - \frac{1}{3}\lambda_{im}c_{mjk}u_k - \frac{1}{3}\mu_{im}c_{mjk}u_{k,x}.
\end{aligned} \tag{4.24}$$

The Lie derivatives are calculated according to the generalized formula given in eq.(3.103), e.g.,

$$\begin{aligned}
\mathcal{L}_{P\alpha}(u_k) &= \frac{\partial u_k}{\partial u_i}(P_{ij}\alpha_j) = \delta_{ki}\delta_{ij}\alpha_{j,x} = \alpha_{k,x} \\
\mathcal{L}_{P\alpha}(u_{k,x}) &= \alpha_{k,xx} \\
\mathcal{L}_{P\alpha}(u_k u_m) &= \alpha_{k,x}u_m + u_k\alpha_{m,x} \quad \text{etc}\ldots
\end{aligned} \tag{4.25}$$

Now let us rewrite Lax-Nijenhuis equation using equations (4.22) and (4.24), and match the ∂^n terms. The first one will be

$$\partial^3 : \qquad 0 = b_{im}\mu_{mj} - b_{mj}\mu_{im}. \tag{4.26}$$

The next equation is

$$\partial^2 : \qquad 0 = \frac{1}{3} b_{im} c_{mjk} \alpha_{k,x} + b_{im} \lambda_{mj} + 2 b_{im} \mu_{mj,x} - \lambda_{im} b_{mj}. \tag{4.27}$$

Assuming that

$$b_{im} \lambda_{mj} - b_{mj} \lambda_{im} = 0 \tag{4.28}$$

eq.(4.27) can be solved for μ_{mj},

$$\mu_{mj} = -\frac{1}{6} c_{mjk} \alpha_k. \tag{4.29}$$

With this solution (4.26) is nothing but the first constraint of equations (4.11). The matching of ∂ terms leads to

$$\begin{aligned} \partial : \qquad 0 &= \frac{1}{3} b_{im} c_{mjk} \alpha_{k,xx} + 2 b_{im} \lambda_{mj,x} + b_{im} \mu_{mj,xx} \\ &\quad + \frac{1}{3} c_{imk} \mu_{mj} u_k - \frac{1}{3} \mu_{im} c_{mjk} u_k \\ &= \frac{1}{3} b_{im} c_{mjk} \alpha_{k,xx} + 2 b_{im} \lambda_{mj,x} - \frac{1}{6} b_{im} c_{mjk} \alpha_{k,xx} \\ &\quad - \frac{1}{18} c_{imk} c_{mjq} \alpha_q u_k + \frac{1}{18} c_{mjk} c_{imq} u_k \alpha_q \\ &= \frac{1}{6} b_{im} c_{mjk} \alpha_{k,xx} + 2 b_{im} \lambda_{mj,x} \end{aligned} \tag{4.30}$$

whose solution for λ_{mj} is

$$\lambda_{mj} = -\frac{1}{12} c_{mjk} \alpha_{k,x}. \tag{4.31}$$

Then the assumption (4.28) is just a reexpression of the second constraint given in eq.(4.11).

The final step is to equate the ∂^0 terms which gives

$$\begin{aligned}
\partial^0: \quad \frac{1}{3}c_{ijk}\mathcal{L}_{Q\alpha}(u_k) &= \frac{1}{2}(\frac{1}{3}b_{im}c_{mjk}\alpha_{k,xxx} + \frac{1}{9}c_{imk}c_{mjq}\alpha_{k,x}u_q + \frac{1}{9}c_{imk}c_{mjq}u_k\alpha_{q,x}) \\
&+ b_{im}(-\frac{1}{12}c_{mjk}\alpha_{k,xxx}) + \frac{1}{3}c_{imk}(-\frac{1}{12}c_{mjq}\alpha_{q,x})u_k \\
&- \frac{1}{3}(-\frac{1}{12}c_{imq}\alpha_{q,x})c_{mjk}u_k - \frac{1}{3}(-\frac{1}{6}c_{imq}\alpha_q)c_{mjk}u_{k,x} \\
&= \frac{1}{12}b_{ik}c_{kjm}\alpha_{m,xxx} + \frac{1}{9}c_{ikm}c_{kjq}u_q\alpha_{m,x} + \frac{1}{18}c_{ikq}c_{kjm}u_{m,x}\alpha_q \\
&= \frac{1}{12}b_{mk}c_{kji}\alpha_{m,xxx} + \frac{1}{9}c_{ikj}c_{kmq}u_q\alpha_{m,x} \\
&+ \frac{1}{18}c_{ikj}c_{kqm}u_{m,x}\alpha_q \qquad (4.32)
\end{aligned}$$

or

$$\mathcal{L}_{Q\alpha}(u_k) = \frac{1}{4}b_{km}\alpha_{m,xxx} + \frac{1}{3}c_{kmq}u_q\alpha_{m,x} + \frac{1}{6}c_{kmq}u_{m,x}\alpha_q \qquad (4.33)$$

By definition $\mathcal{L}_{Q\alpha}(u_k)$ is

$$\frac{\partial u_k}{\partial u_i}(Q\alpha)_i = \delta_{ki}Q_{im}\alpha_m = Q_{km}\alpha_m \qquad (4.34)$$

so after comparing equations (4.33) and (4.34) the second Hamiltonian operator of the multicomponent KdV equations can be identified as

$$Q_{km} = \frac{1}{4}b_{km}\partial^3 + \frac{1}{3}c_{kmq}u_q\partial + \frac{1}{6}c_{kmq}u_{q,x}. \qquad (4.35)$$

Scaling Q to $4Q$ one can find the following operator given in [5],

$$Q_{km} = b_{km}\partial^3 + \frac{2}{3}c_{kmq}(\partial u_q + u_q\partial). \qquad (4.36)$$

The corresponding Hamiltonian is

$$H_0 = \int (\frac{1}{2}\delta_{ij}u_iu_j)dx \tag{4.37}$$

In order to prove that Q_{ij} is a Hamiltonian operator one should check the two criteria stated in section 2.5. The first one is skew-symmetry and this is obvious since

$$\begin{aligned} Q^{\dagger}_{ij} &= b_{ij}(-\partial^3) + \frac{2}{3}c_{ijk}(-u_k\partial - \partial u_k) \\ &= -Q_{ij} \end{aligned} \tag{4.38}$$

where we have used the fact that $(\partial u)^{\dagger} = -u\partial$. Secondly, Q_{ij} should satisfy the Jacobi identity

$$\mathcal{L}_{Q\theta}(\Theta_Q) = 0. \tag{4.39}$$

The functional bivector Θ_Q is

$$\begin{aligned} \Theta_Q &= \frac{1}{2}\int \theta_i \wedge (Q_{ij}\theta_j)dx \\ &= \int (\frac{1}{2}b_{ij}\theta_i \wedge \theta_{j,xxx} + \frac{1}{3}c_{ijk}u_{k,x}\theta_i \wedge \theta_j + \frac{2}{3}c_{ijk}u_k\theta_i \wedge \theta_{j,x})dx \end{aligned} \tag{4.40}$$

The second term is zero since it is a contraction of two objects, c_{ijk} and $\theta_i \wedge \theta_j$ which are symmetric and anti-symmetric in the indices i and j, respectively. That is

$$\begin{aligned} c_{ijk}\theta_i \wedge \theta_j &= \frac{1}{2}(c_{ijk}\theta_i \wedge \theta_j + c_{jik}\theta_j \wedge \theta_i) \\ &= \frac{1}{2}(c_{ijk}\theta_i \wedge \theta_j - c_{jik}\theta_i \wedge \theta_j) \\ &= 0 \end{aligned} \tag{4.41}$$

then

$$\Theta_Q = \int (\frac{1}{2} b_{ij}\theta_i \wedge \theta_{j,xxx} + \frac{2}{3} c_{ijk} u_k \theta_i \wedge \theta_{j,x}) dx. \tag{4.42}$$

The Lie derivative in eq.(4.39) can be calculated as

$$\begin{aligned} \mathcal{L}_{Q\theta}(\Theta_Q) &= \int \frac{2}{3} c_{ijk} \mathcal{L}_{Q\theta}(u_k) \wedge \theta_i \wedge \theta_{j,x} \\ &= \int (\frac{2}{3} b_{kn} c_{ijk} \theta_{n,xxx} \wedge \theta_i \wedge \theta_{j,x} + \frac{4}{3} c_{ijk} c_{knl} u_{l,x} \theta_n \wedge \theta_i \wedge \theta_{j,x} \\ &\quad + \frac{8}{9} c_{ijk} c_{knl} u_l \theta_{n,x} \wedge \theta_i \wedge \theta_{j,x}) dx \end{aligned} \tag{4.43}$$

The last two terms are zero because $c_{ijk}c_{knl}$ is symmetric in indices n and i by eq.(4.11). And one can write

$$\begin{aligned} \mathcal{L}_{Q\theta}(\Theta_Q) &= \int [\frac{2}{3} b_{kn} c_{ijk} (\theta_{n,xx} \wedge \theta_i \wedge \theta_{j,x})_x - \frac{2}{3} b_{kn} c_{ijk} \theta_{n,xx} \wedge \theta_{i,x} \wedge \theta_{j,x} \\ &\quad - \frac{2}{3} b_{kn} c_{ijk} \theta_{n,xx} \wedge \theta_i \wedge \theta_{j,xx}] dx \end{aligned} \tag{4.44}$$

which vanishes by the fact that the first term is a total derivative and in the second term c_{ijk} is symmetric and $b_{kn}c_{ijk} = b_{kj}c_{ink}$. So the second Hamiltonian operator satisfies the Jacobi identity. But one should also check if the operators P_{ij} and Q_{ij} form a Hamiltonian pair, so that the multicomponent KdV equations can be written in bi-Hamiltonian form:

$$u_{k,t} = P_{ks} \frac{\delta H_1}{\delta u_s} = Q_{ks} \frac{\delta H_0}{\delta u_s} \tag{4.45}$$

The operators P_{ij} and Q_{ij} form a Hamiltonian pair if and only if they satisfy

$$\mathcal{L}_{P\theta}(\Theta_Q) + \mathcal{L}_{Q\theta}(\Theta_P) = 0 \tag{4.46}$$

[11]. The first term vanishes by symmetry considerations

$$\begin{aligned}\mathcal{L}_{P\theta}(\Theta_Q) &= \int \frac{2}{3}c_{ijk}\mathcal{L}_{P\theta}(u_k)\wedge\theta_i\wedge\theta_{j,x}dx \\ &= \int \frac{2}{3}c_{ijk}\theta_{k,x}\wedge\theta_i\wedge\theta_{j,x}dx \\ &= 0. \end{aligned} \tag{4.47}$$

The vanishing of the second term is obvious since the bivector

$$\begin{aligned}\Theta_P &= \frac{1}{2}\int \theta_i\wedge(P_{ij}\theta_j)dx \\ &= \frac{1}{2}\int \theta_i\wedge\theta_{i,x}dx \end{aligned} \tag{4.48}$$

has no functional u_k dependence and therefore its Lie derivative is zero.

It is seen that the multicomponent KdV equations admit a Bi-Hamiltonian formulation with the operators given in equations (4.18) and (4.36). The two corresponding Poisson brackets are

$$\begin{aligned}\{u_i(x),u_j(y)\}_1 &= \delta_{ij}\partial\delta(x-y) \\ \{u_i(x),u_j(y)\}_2 &= [b_{ij}\partial^3+\frac{2}{3}c_{ijk}(\partial u_k+u_k\partial)]\delta(x-y). \end{aligned} \tag{4.49}$$

4.5. An Alternative Lax Representation

In this section the recursion operator method will be used to obtain a second Lax pair for the multicomponent KdV equations

$$u_{i,t}=S_i(u)=b_{ij}u_{j,xxx}+c_{ijk}(u_ju_k)_x. \tag{4.50}$$

The recursion operator for this system is

$$\begin{aligned} R_{ij} &= Q_{im}P^{-1}_{mj} \\ &= (b_{im}\partial^3 + \frac{4}{3}c_{imk}u_k\partial + \frac{2}{3}c_{imk}u_{k,x})(\delta_{mj}\partial^{-1}) \\ &= b_{ij}\partial^2 + \frac{4}{3}c_{ijk}u_k + \frac{2}{3}c_{ijk}u_{k,x}\partial^{-1} \end{aligned} \quad (4.51)$$

Using this operator one may find higher order evolution equations and related conserved quantities of the multicomponent KdV hierarchy [5]. Also a Lax pair representation is possible by means of the recursion operator. The Lax equation is

$$R_{ij,t} = [d_F S, R]_{ij} \quad (4.52)$$

where $d_F S$ is the Fréchet derivative of $S_i(u)$,

$$\begin{aligned} (d_F S)_{ij} &= \frac{\partial S_i}{\partial u_j} + \frac{\partial S_i}{\partial u_{j,x}}\partial + \frac{\partial S_i}{\partial u_{j,xx}}\partial^2 + \dots \\ &= b_{ij}\partial^3 + 2c_{ijk}u_k\partial + 2c_{ijk}u_{k,x}. \end{aligned} \quad (4.53)$$

The commutator in eq.(4.52) can be calculated as

$$\begin{aligned} [d_F S, R]_{ij} &= (d_F S)_{im}R_{mj} - R_{im}(d_F S)_{mj} \\ &= \frac{4}{3}b_{im}c_{mjl}u_{l,xxx} + \frac{8}{3}c_{imk}c_{mjl}u_l u_{k,x} \\ &\quad -\frac{4}{3}c_{imk}c_{mjl}u_{k,x}(u_l\partial^{-1} - u_{l,x}\partial^{-2} + u_{l,xx}\partial^{-3} - \dots)\partial \\ &\quad +\frac{2}{3}b_{im}c_{mjl}u_{l,xxxx}\partial^{-1} + \frac{4}{3}c_{imk}c_{mjl}u_k u_{l,xx}\partial^{-1} \\ &\quad -\frac{4}{3}c_{imk}c_{mjl}u_{k,x}(u_{l,x}\partial^{-1} - u_{l,xx}\partial^{-2} + \dots) \end{aligned}$$

$$= \frac{4}{3}b_{im}c_{mjl}u_{l,xxx} + \frac{8}{3}c_{imk}c_{mjl}u_lu_{k,x}$$
$$+(\frac{2}{3}b_{im}c_{mjl}u_{l,xxx} + \frac{4}{3}c_{imk}c_{mjl}u_ku_{l,x})_x\partial^{-1}. \quad (4.54)$$

This is equal to

$$R_{ij,t} = \frac{4}{3}c_{ijk}u_{k,t} + \frac{2}{3}c_{ijk}u_{k,xt}\partial^{-1}. \quad (4.55)$$

The ∂^0 and ∂^{-1} terms of (4.54) and (4.55) can be equated to give

$$c_{ijk}u_{k,t} = b_{im}c_{mjl}u_{l,xxx} + c_{imk}c_{mjl}(u_lu_k)_x \quad (4.56)$$

which gives after using eqs.(4.11)

$$u_{k,t} = b_{kl}u_{l,xxx} + c_{klm}(u_lu_m)_x \quad (4.57)$$

So the operators (4.51) and (4.53) form a Lax pair for the multicomponent KdV system.

5. CONCLUSION

In this thesis we have studied various connections between the two basic properties of integrable nonlinear partial differential equations, namely the Lax representation and the bi-Hamiltonian structure. The analyses of these connections are mainly based on practical purposes such as calculating the second Hamiltonian structure , rather than a rigorous study involving differential geometry and group theory.

The Lax operator description of integrable systems is a worthy tool for many purposes. As in the case of KdV equation, a hierarchy of evolution equations are best understood within the Lax representation. Furthermore the linear problem associated to the nonlinear equation is achieved through the use of the Lax operator. The linearization scheme is a crucial step in solving the system via the so called inverse scattering method. An interesting property of this representation is the non-uniqueness of the Lax pair, as seen in the KdV and Kaup-Broer examples. The alternative Lax descriptions often involve non-standart (pseudo-differential) Lax operators like the recursion operator.

The existence of a bi-Hamiltonian structure is a perfect proof of the integrability of a system. Although the first Hamiltonian operator is usually obvious, the search of a second compatible structure may not be so easy. The Lax-Nijenhuis equation is quite helpful at this point, since we have the Lax operator in hand we can calculate the second Hamiltonian operator. In this thesis this method worked succesfully in the cases of the KdV, Boussinesq and multicomponent KdV equations. More examples including the Toda system and n-dimensional rigid body can be found in [20].

The second technique of constructing the bi-Hamiltonian structure is Gelfand-Dickii brackets. The advantage of this method is the simultaneous calculation of the two Hamiltonian operators. But although the Lax operator used in this method is a generalized one, it has some restrictions. For example an operator with a leading order n of the form $L = f[u]\partial^n + \sum_{i=0}^{i=n-1} g_i[u]\partial^i$ is not suitable because of the coefficient

$f[u]$. Another restriction comes from the fact that this generalized Lax operator is a "scalar" operator. The application of this method to matrix valued Lax operators is an open question. The Gelfand-Dickii brackets or the similar Adler-Gelfand-Dickey brackets are in fact particular cases of the R-matrix brackets and these brackets satisfy the classical modified Yang-Baxter equation.

The multicomponent KdV equations are the last examples studied in this thesis. These equations can be derived starting from a Lagrangian and the Hamiltonian formulation is achieved using this Lagrangian. In order to formulate the multicomponent KdV system in the Lax description, a matrix valued operator is introduced. Using this operator and the Lax-Nijenhuis equation a second Hamiltonian operator is obtained. It is shown that the new Hamiltonian operator satisfies the required properties like the Jacobi identity. So the multicomponent KdV equations are both bi-Hamiltonian and Lax representable. An alternative description concerning the recursion operator is also presented.

APPENDIX A

The properties of ∂ :

The operator ∂ is defined as

$$\partial u = (\partial u) + u\partial \tag{A.1}$$

where

$$(\partial u) = u_x. \tag{A.2}$$

The multiple action of ∂ is

$$\underbrace{\partial \ldots \partial}_{p} = \partial^p \tag{A.3}$$

and it is obvious that

$$\partial^p \partial^q = \partial^{p+q}. \tag{A.4}$$

Using eqs.(A.1) and (A.3) one may easily find

$$\partial^2 u = u_{xx} + 2u_x\partial + u\partial^2 \tag{A.5}$$

and

$$\partial^3 u = u_{xxx} + 3u_{xx}\partial + 3u_x\partial^2 + u\partial^3 \tag{A.6}$$

etc. So the general rule is

$$\partial^p u = \sum_{k=0}^{\infty} C(p,k) u_{\underbrace{xx\ldots x}_{k}} \partial^{p-k} \tag{A.7}$$

where

$$C(p,k) = \frac{p!}{(p-k)!k!}. \tag{A.8}$$

The operator ∂^{-1} is defined as

$$\partial^{-1}\partial = \partial\partial^{-1} = I. \tag{A.9}$$

The analog of eq.(A.2) is

$$(\partial^{-1}u) = \int dxu \tag{A.10}$$

but this is not equal to $\partial^{-1}u$ in fact. In order to obtain an expression for the latter one, first write eq.(A.1) as

$$u\partial = \partial u - u_x \tag{A.11}$$

then multiply this with ∂^{-1} on the right to get

$$u = \partial u\partial^{-1} - u_x\partial^{-1}. \tag{A.12}$$

The action of ∂^{-1} on (A.12) gives

$$\partial^{-1}u = u\partial^{-1} - \partial^{-1}u_x\partial^{-1}. \tag{A.13}$$

Although this is a correct expression for $\partial^{-1}u$ it is not good for practical purposes just because of the second term. But since

$$\partial^{-1}u_x = u_x\partial^{-1} - \partial^{-1}u_{xx}\partial^{-1} \tag{A.14}$$

one may iterate (A.13) and get

$$\partial^{-1}u \;=\; u\partial^{-1} - (u_x\partial^{-1} - \partial^{-1}u_{xx}\partial^{-1})\partial^{-1}$$

$$= u\partial^{-1} - u_x\partial^{-2} + \partial^{-1}u_{xx}\partial^{-2}. \tag{A.15}$$

The next iteration gives

$$\partial^{-1}u = u\partial^{-1} - u_x\partial^{-2} + u_{xx}\partial^{-3} - \partial^{-1}u_{xxx}\partial^{-4}. \tag{A.16}$$

Continuing this way one can always expand this up to a desired order

$$\partial^{-1}u = u\partial^{-1} - u_x\partial^{-2} + u_{xx}\partial^{-3} - \ldots + (-1)^n u_{\underbrace{xx\ldots x}_{n}}\partial^{-n-1}. \tag{A.17}$$

A similar analysis may be applied to $\partial^{-2}u$ to get an expansion of the form

$$\partial^{-2}u = u\partial^{-2} - 2u_x\partial^{-3} + 3u_{xx}\partial^{-4} - 4u_{xxx}\partial^{-5} + - \ldots \quad . \tag{A.18}$$

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