THE PROBLEM OF 'SPACE' IN FREGE'S ONTOLOGY

A TRANSCENDENTAL INQUIRY

by

Mehmet Türker Armaner

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ABSTRACT

This thesis is an attempt to evaluate Frege's ontology of numbers in a critical way in view of Kant's transcendental philosophy. Despite the significant divergences between Kant and Frege, these two philosophers will be taken in continuity. It is shown that these divergences originate mostly from Frege's way of misevaluating Kant's system. The main problem of the thesis is the "space" of the objective thoughts in Frege's system. The most important conceptual similarities between Kant and Frege are the relations between object and concept, and concept and judgement. For both, arithmetical objects are neither empirical nor sensible and these objects do not exist by themselves. Numbers, as arithmetical objects, are found in Number concepts. Objects and concepts are found in judgements. Both Kant and Frege attempt to secure an objective status for natural numbers. In Frege's system, neither the formation nor the space of objects is mentioned. He does not consider 'synthesis' or 'unity' which are crucial for the formation of objects in Kant's system and to the objective thoughts, Frege does not assign a space. Frege's ontology does not provide a medium for connecting the objective thoughts either. Thus Frege's system is ontologically incomplete.

ÖZET

Bu tezde, Frege'nin 'sayı ontolojisi'ni Kant'ın transandantal felsefesi açısından eleştirel bir biçimde değerlendirmeyi deneyeceğiz. Kant ile Frege, aralarındaki önemli farklılıklara rağmen, bir süreklilik içinde ele alınacaktır. Tezde, Frege'nin Kant'a yönelttiği eleştirilerin, çoğunlukla, Kant'ın dizgesinin Frege tarafından yanlış bir biçimde değerlendirilmiş olmasından kaynaklandığı gösterilmeye çalışılacaktır. Kant ile Frege arasındaki en önemli kavramsal benzerlikler, nesne ile kavramı ve kavram ile yargı arasındaki ilişkilerdedir. Her iki filozof için, aritmetiksel nesneler empirik ya da duyulara ait değildir. Bu nesneler, kendi başlarına da varolamazlar. Aritmetiksel nesneler olarak sayılar, sayı kavramlarının altında bulunur. Nesneler ile kavramlar da, yargıların içinde bulunur. Frege'nin dizgesinde, Kant'ın dizgesindeki en önemli noktalardan olan nesnelerin oluşumundan ya da mekanından sözedilmez. Frege'nin dizgesinde 'nesnel düşünceler' için de bir mekan bulunmaz. Bu 'nesnel düşünceler'i birbirine bağlayacak 'ontolojik aracılar' da yoktur. Bu nedenlerle, Frege'nin dizgesi, ontolojik olarak eksiktir.

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I- Introduction

The ontology of numbers has constituted one of the most significant problems in the history of philosophy. With the beginning of ancient philosophy, there have been different philosophical approaches to this subject. Some philosophers claimed that arithmetical objects are empirical and exist in the external world, some others claimed that these objects are subjective and psychological; some argued that numbers really exist in a sphere peculiar to mathematical objects and some others claimed that numbers are nothing but names.

Pythagoras, his followers and Plato claimed that the world of the numbers are quite different from that of the sensible things. Pythagoreans claimed that things exist by imitating numbers and number is the substance of all things. Plato, on the other hand, introduced his theory of ideas and instead of 'imitation', he used the concept of 'participation'. Plato argued that the objects of mathematics belong to a realm other than those of sensible things and of ideas.

Aristotle held a view which is different from Pythagoreans and Plato. He claimed that it is impossible for mathematical objects to exist within sensible things and secondly, it is not possible for mathematical objects to exist separately on their own.

In the present thesis, we are concerned with a critical evaluation of Frege's ontology of numbers in view of Kant's transcendental philosophy. In The Foundations of Arithmetic, Frege considers the views of some philosophers and mathematicians concerning the theory of numbers and comments on these. Among them, it seems, Frege has been inspired mostly from Leibniz and Kant- and especially from the latter one. Frege severly critisizes the philosophers who hold the view that numbers are the properties of external things, or are subjective, and laws of arithmetic are inductive truths. Frege's comments on these views are summarized in the beginning of chapter III. In the present thesis, Frege's system, despite the divergences, will be taken in continuity with that of Kant. Kant and Frege are the two remarkable philosophers who have influenced contemporary philosophy and philosophy of mathematics.

The most important conceptual similarities between Kant Frege are the relations between object and concept, and concept and judgement. For both, arithmetical objects are neither empirical nor sensible. In both systems, objects do not exist by themselves. Numbers, as arithmetical objects, are found in number concepts. Objects and concepts are found in judgements. Both Kant and Frege attempt to secure an objective status for natural numbers.

In the second chapter, Kant's transcendental ontology is exposed. In Kant's transcendental philosophy, objects are not given entities; they are acquired as a result of a certain procedure, namely, the three-fold synthesis. Objects, as we mentioned above, cannot be found by themselves either. An object, in Kant's system, is found together with a concept. Objects and concepts are in unity and both are found in a judgement. Thus a judgement is the ultimate space for an object. For arithmetical objects, that is numbers, this space is the synthetic a priori judgement. In sections II-1. and in II-2., respectively, Transcendental Aesthetic as the objective grounds of sensibility, and Transcendental Logic as the objective grounds of thought are discussed. Sensibility and thought, in Kant's system, are the two capacities of human soul. In the former section, the metaphysical and transcendental expositions of space and time are considered. The metaphysical exposition is required to discover the origin and transcendental exposition to demonstrate the objective validity of these concepts. In section II-3., Understanding, as another capacity of human soul, and its elements are exposed. This faculty, according to Kant, enables us to think the representations that are received by the faculty of sensibility. In section II-4., it is shown that the material for natural numbers is obtained from the a priori manifold in time. This a priori manifold is synthesized subjectively; however, this synthesis is subject to an objective rule and in the case of the natural numbers, this rule is subordination.

In Frege's system, numbers, as arithmetical objects, are located in number concepts. These number concepts are found in 'objective thoughts'. Objective thoughts do not necessitate a thinking being in order to exist. Objective thoughts are propositions which do not belong to any language. They exist without being thought or uttered. These thoughts, according to Frege, are logical entities and may carry truth values. Thus, Frege claims, these thoughts are not produced, but only grasped by us.

In Frege's system, neither the formation nor the space of objects is mentioned. He does not consider 'synthesis' or 'unity' which are crucial for the formation of objects in Kant's system. To the objective thoughts, Frege does not assign a space either. Although he mentiones that thoughts are logical entities, he does not mention their grounds. Frege's ontology also lacks the mediums for connecting the elements of separate realms. On the other hand, in Kant's transcendental ontology, categories and schemata serve as mediums of this sort. According to Kant, Pure Reason is the space for judgements in which we find objects in unity with concepts.

In the Conclusion, Frege's ontology is critisized in view of transcendental ontology. It is shown that the divergences between Kant's ontology and that of Frege originate mostly from Frege's way of misevaluating Kant's system. Frege critisizes Kant for, according to him, Kant's system involves subjectivism. On the other hand, it is Kant who declares judgements to be objective. The point that Frege misses in Kant's system is that, judgement, as an act of understanding, has a logical side which can only be captured by transcendental considerations. Frege, following Kant, divides judgements as analytic and synthetic. While, in Kant's ontology, we find numbers in synthetic judgements, according to Frege, they are found in the analytical ones. However, according to transcendental ontology, it is not possible to find any kind of object in the analytical judgement, that is, these judgements are empty in terms of objects; they only contain representations of objects. It is mentioned in section II-2. that the most important division in Kant's divisions of logic is between general logic and transcendental logic. In Frege's system, on the other hand, transcendental logic is not considered.

We should also note that the term 'Understanding', it is explained in a footnote, is not a proper translation of 'Verstand'[1]. What Kant intends by the term 'Verstehen' is not the act of 'understanding'; it is rather grasping or capturing.

The present thesis inquires the ontologies of Kant and Frege. And, the main problem of the thesis is the 'space' of the objective thoughts in Frege's system. This problem, as far as we could see, is not mentioned in the works of Kant and Frege.

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[1] See thesis page 18, footnote 56

II. Kant's Ontology of Natural Numbers

The ground of Kant's transcendental philosophy is the 'human soul'.[2] 'Soul', according to Kant, is the transcendental self which is 'I'. This is a status of 'inner thinking' itself: " 'I', as thinking, am an object of inner sense, and am called 'soul'. That which is an object of the outer senses is called 'body'. Accordingly, the expression 'I', as a thinking being, signifies the object of that psychology which may be entitled the 'rational doctrine of the soul"[3]. 'Soul', considered as the 'thinking being', contains the appearances. 'I', as the transcendental object of the inner being[4], connects these appearances. In 'thought', which is one of the capacities of the 'soul', the main function of 'I' is to convert representations into 'thoughts'.

This function of 'I' is necessary for combining different representations into one unity. In defining the concept 'soul', Kant asserts that it is a 'substance'. Thoughts in 'I' inhere only as determinations. But 'I', cannot be the determination of another thing. This is due to the 'substantiality' of the soul. Substance cannot be the predicate of another thing.[5] Hence, being a substance, 'I' also cannot be a predicate. 'I', in itself has a logical identity. Its existence is 'permanent': "The 'I', is indeed in all thoughts but there is not in these representations the least trace of intuition, distinguishing the 'I' from other objects of intuition. Thus we can indeed perceive that this representation is invariably present in thought, but not that it is an abiding and continuing intuition, wherein the thoughts, as being transitory, give place to one another." [6]

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[2]Kant uses two words for denoting 'soul': /Gemut/ and /Gesele/. While the former is being used interchangeably with 'mind', the latter is used in the sense of the basic element mentioned above. The Turkish word for 'mind' would be 'zihin' and for 'soul', 'ruh'.
[3] Kant Immanuel, *Critique of Pure Reason*, tr. Norman Kemp Smith, (1950; rpt. London:Macmillan,1990) A342/B400
[4]ibid., A367
[5]ibid.,A348/B406
[6]ibid.,A350

We, on the other hand, do not have the knowledge of the transcendental self 'I'. What we barely know is that it is consciousness which makes representations to be thoughts and in it, perceptions are found.We do not have access to the knowledge of the soul since it is an internal 'thing in itself': "All that is here required is that we follow the guidance of the categories, with this difference only, that since our starting point is a given thing, 'I' as thinking being, we begin with the category of substance, whereby a thing in itself is represented..." [7]

The other three features of the Soul are: "as regards to its quality it is simple, as regards the different times in which it exists, it is numerically identical, that is unity (not plurality), it is in relation to possible objects in space."[8] 'Unity of representations' is required in Kant's transcendental philosophy. 'I', which combines different representations as a manifold into one unity, in itself should have a unity. 'Substantiality' of the soul is the first condition to realize this requirement.

'Simplicity' is another condition of forming a unity. According to Kant, only as a simple substance it is possible to make up a thought out of a variety of representations. This unity, according to him, is not derived from experience: "If a multiplicity of representations are to form a single representation, they must be contained in the absolute unity of the thinking subject...the unity of the thought ... may relate just as well to the collective unity of different substances acting together."[9] Since soul is considered as a 'thing in itself', the statement, 'I am a substance', to Kant, signifies only the pure category.[10] For it belongs to the inner sense; soul cannot be found among outer appearances and outer appearances do not contain thoughts: "That the mode in which our outer sense is thereby

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[7]ibid., A344/B402
[8]ibid.
[9]ibid., A352-353
[10]ibid., A352-353

affected gives us no intuition of representations, will, etc., but only of space and its determinations, proves nothing to the contrary... the predicates of inner sense, representations and thought, are not inconsistent with its nature."[11]

The two capacities of the soul which are primarily significant to Kant's transcendental philosophy are 'sensibility' and 'thought'.[12] It is in Transcendental Aesthetic and Transcendental Logic that Kant respectively investigates 'sensibility' and 'thought', by clearly separating the 'objective' from the 'subjective' and thus clarifies the necessary and *a priori* grounds of these two capacities.

II. 1 Transcendental Aesthetic: The Objective Grounds of Sensibility

In the Critique of Pure Reason, Kant undertakes two basic tasks: inquiry of the 'pure elements' in the subjective act, and secondly, discovery of the 'origin' of these pure elements and demonstration of their 'objective validity'. In "Transcendental Aesthetic", which Kant entitles as a science of all principles of *a priori* sensibility [13] Kant investigates the pure elements in 'sensibility'. These elements are 'space' and 'time'. In this section, Kant provides a *metaphysical* and a *transcendental* exposition of these elements. The former exposition is required to discover the origin and the latter to demonstrate the objective validity of these concepts.

By "exposition", Kant means the clear representation of that which belongs to a concept. If the exposition contains the source of the concept and exhibits it as given *a priori*, it is a *metaphysical exposition*..[14] *Transcendental exposition* of a concept, according to Kant, is the discovery of a principle by means of which *a*

[11]ibid., A356
[12]ibid., A358-359
[13]Ibid., A21/B35
[14]ibid.,A23/B38

priori synthetic knowledge can be obtained from that concept. [15]

Space and time, being the pure elements in sensibility, are also its forms. Space is the form of the 'outer sense' and time is the form of the 'inner sense.' The origin of space and time, according to Kant's inquiry, is the human soul itself. The origin of these pure elements is not the 'thing in itself' nor are they derived from experience: "By means of outer sense...we represent to ourselves objects as outside us, and all without exception in space...Inner sense, by means of which the mind intuits itself of its inner state, yields indeed no intuition of the soul itself as an object; but there is nevertheless a determinate form, namely time, in which alone the intuition of the inner states is possible, and everything which belongs to inner determinations is therefore represented in relations of time. Time cannot be outwardly intuited, any more than space can be intuited as something in us."[16]

Sensibility, as a capacity of the human soul, enables the soul to receive representations. These representations, though having a necessary connection among themselves, are not in the sphere of the 'thing in itself': "The effect of an object upon the faculty of representation, so far as we are affected by it, is *sensation*."[17] On the form and matter of the sensation Kant says: "...that in the appearance which corresponds to sensation I term its 'matter'; but that which so determines the manifold of appearances that it allows of being ordered in certain relations, I term the 'form' of appearance...That intuition which is in relation to the object through sensation, is empirical. The undetermined object of an empirical intuition is entitled appearance."[18] Intuition, according to Kant, is that through which it is in immediate relation to the objects, and to which all thought as a means is directed. "I term all representations *pure*...in which there is nothing that

[15] ibid.,A25/B41
[16] ibid.,A23/B37
[17] ibid.,A20/B34
[18] ibid.

belongs to sensation...The pure form of sensible intuitions in general, must be found in the mind *a priori.* This pure form of sensibility may also itself be called *pure intuition*.. Intuitions are in a necessary relation with 'thoughts' and which have their seat in the 'understanding'."[19] In Transcendental Aesthetic, *metaphysical exposition* of the two forms of sensibility is provided by revealing the *a priori* nature of space and time.[20] Since experience is possible only through space and time, these obviously cannot have their origin in experience. Thus, space and time are not empirical concepts. "The representation of space cannot, therefore, be empirically obtained from the relations of outer appearance. On the contrary, this outer experience is itself possible at all only through that representation."[21] "...neither coexistence nor succession would ever come within our perception, if the representation of time were not presupposed as underlying them *a priori.*"[22]

Space and time, according to Kant, are the necessary *a priori* representations which underly all intuitions. Space underlies outer intuitions and time, inner intuitions: "The apodeictic certainty of all geometrical propositions, and the possibility of their *a priori* construction, is grounded on this *a priori* necessity of space. Were this representation of space a concept acquired *a posteriori*... the first principle of mathematical determination would be nothing but perceptions."[23]

Space and time are not discursive (general concepts) but pure intuitions. An *a priori* intuition underlies all concepts of space.[24] Different times are parts of one and the same time; and the representation which can be given only through

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[19]ibid.,A19/B30
[20]ibid.,A22-25/B37-40
[21]ibid.,A23/B38
[22]ibid.,A30/B46
[23]ibid.,A24/B39
[24]ibid.,A25/B39

a single object is intuition. [25] Time has only one dimension; different times are not simultaneous but successive. On the other hand, different spaces are not successive but simultaneous.[26]

Kant considers space and time as infinitely given magnitudes. "All parts of space exist *ad infinitum*". [27] In view of 'time', as the inner form of sensibility, a 'unity' is presupposed in Kant's system. Time is infinite and "every determinate magnitude of time is possible only through limitations of one single time that underlies it. The original representation, *time* , must therefore be given as unlimited."[28]

By *transcendentally exposing* the concepts of 'space' and 'time', Kant seeks for their objective validity. From this exposition a principle which concerns space is derived: things stand side by side in space. This is the principle of "coordination". It is by means of this principle that things are constructed in geometry. Geometry determines the properties of space synthetically. Kant asserts that "...our representation of space...must in its origin be intuition...this intuition must be *a priori* , that is, it must be found in us prior to any perception of an object, and must therefore be pure, not empirical intuition...it obtains *immediate representation,* that is, *intuition* of them."[29]

Space, according to Kant, does not represent any property of things in themselves, and it does not represent them in their relation to one another. It is the form of all appearances of outer sense. Space is the subjective condition of sensibility, under which alone outer intuition is possible for us. This subjective (not psychological) condition is subject to an objective condition, that is; 'things stand side by side in space'. This proposition is valid universally and without

[25]ibid.,A32/B47
[26]ibid.,A31/B47
[27]ibid.,A23-25/B38-39
[28]ibid.,A32/B48
[29]ibid.,A25/B41

limitation.[30] Consequently, space is the form of the representations (of outer objects) of our sensibility.[31]

Time, according to Kant, is the form of our inner sense. It can be represented prior to objects. It is by means of time that representations in our inner state are related and connected. It is the formal *a priori* condition of all appearances. Time is the immediate condition of inner appearances of our souls, and thereby the mediate condition of our appearances. All objects of the senses (appearances) are in time and necessarily stand in time-relations.[32]

Two rules arise from the transcendental exposition of space and time: i) Objects are side by side in space ii) Objects are one after another in time. With respect to logical relations, the first rule turns out to be the rule of 'coordination' and the second, 'subordination'. "The proposition, that all things are side by side in space, is valid under the limitation that these things are viewed as objects of our sensible intuition."[33] "Only in time can two contradictorily opposed predicates meet in one and the same object, namely, *one after the other*."[34]

The transcendental exposition of space gives rise to the possibility of 'geometry' and enables it to be thought as a body of *a priori* synthetic knowledge.[35] From the transcendental exposition of time, the possibility of 'mechanics' becomes possible as a body of *a priori* synthetic knowledge. Kant relates this to the concept of 'change'. In his system, the form of psychological states is time and these states are always one after the other. He asserts that "the concept of alteration, and with it the concept of motion, as alteration of place, is possible only through and in the representation of time.[36]

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[30]ibid.,A27/B43
[31]ibid.,A26-30/B42-46
[32]ibid.,A33-36/B49-53
[33]ibid.,A27/B43
[34]ibid.,A32/B49
[35]ibid.,A25/B41
[36]ibid.,A32/B48

From this exposition, Kant concludes that space is 'real' in the empirical sense and 'ideal' in the transcendental sense: "We assert... the *empirical reality* of space, as regards all possible outer experience; and yet at the same time we assert its transcendental ideality-...its limitation to possible experience, and so look upon it as something that underlies things in themselves."[37] Time, as well as space, is therefore described as empirically real and transcendentally ideal. Real, in the sense that it is the mode of representation of oneself as object. Ideal, in the sense that it does not belong to things as a property of them. [38] Apart from being forms of sensibility, time and space are also two sources of knowledge. From these sources, '*a priori* synthetic knowledge' is obtained. But their objective validity is possible only if the objects that they are applied are viewed as appearances.[39] "As appearances, they cannot exist in themselves, but only in us. What objects may be in themselves, and apart from all this receptivity of our sensibility, remains completely unknown to us."[40]

II.2 Transcendental Logic: The Objective Grounds of Thought

In Kant's exposition of logic, we find three main divisions: the first one concerns the distincion between 'pure general logic' and 'applied general logic'. The second concerns the division of general logic into two 'pure general logic' and 'special general logic'. The third one concerns the division between 'general logic' and 'transcendental logic'.

With respect to the first division, Kant states the point of difference as follows: "General logic is called applied, when it is directed to the rules of employment of understanding under the subjective empirical conditions dealt with by psychology. Applied logic has therefore empirical

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[37]ibid.,A28/B44
[38]ibid.,A36-37/B53-54
[39]ibid.,A39/B56
[40]ibid.,A42/B59

principles...Consequently it is neither a canon of understanding in general nor an organon of special sciences..."[41]

Applied general logic, according to Kant, "is a representation of the understanding and of the rules of its necessary employment *in concreto*, that is, under the accidental subjective conditions which may hinder or help its application, and which are all given only empirically."[42] General logic, on the other hand, is abstracted from all empirical conditions which our understanding is subject to.[43] General logic is called pure when it is abstracted from psychological subjective conditions under which general logic is used. Thus pure general logic is objective; thims logic does not differ from person to person.

In Logic, Kant states the difference as follows: "In pure logic we segregate the understanding from the other powers of the mind and contemplate what it does by itself. Applied logic contemplates the understanding so far as it is intermingled with other powers of the mind that influence its acts and slant in some direction, so that it does not proceed according to the laws which its own insight knows are the right ones. Applied logic actually should not be called logic. It is a psychology in which we contemplate how things usually work in our thinking, not how they are to work."[44] From this division Kant infers two rules: "1. As general logic, it abstracts from all content of knowledge of understanding from all differences in its objects, and deals with nothing but the mere form of thought. 2. As pure logic, it has nothing to do with empirical principles...and has no influence whatever on the canon of understanding. Pure logic is a body of demonstrated doctrine, and everything in it must be certain entirely *a priori*." [45]

The second division concerns the distinction between 'pure general logic'

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[41]ibid.,A53/B77-78
[42]ibid.,A54/B78
[43]ibid.,A53/B77
[44]Kant Immanuel, *Logic*, ed. Gottlob Benjamin Jache p.21
[45]Op.Cit., A54/B78

and 'special general logic'. On this distinction, Kant states the following: "The former contains the absolutely necessary rules of thought...It therefore treats of understanding without any regard to difference in the objects to which the understanding may be directed. The logic of the special employment of the understanding contains the rules of correct thinking as regards a certain kind of objects. The former may be called the logic of elements, the latter the organon of this or that science."[46] General logic cannot be used as an organon; a special application of it only coulo serve such a purpose.

For the third division, Kant asserts that "general logic...considers only the logical form in the relation of any knowledge to other knowledge; that is, it treats of the form of thought in general."[47] Transcendental logic, on the other hand, has the *a priori* forms of intuition and of thought as its content and this content cannot be captured by general logic.

Transcendental logic deals with the *a priori* elements in the understanding and investigates how these elements are related to objects *a priori* . Transcendental logic, according to Kant, "should contain solely the rules of the pure thought of an object, would exclude only those modes of knowledge which have empirical content."[48] In other words, transcendental logic investigates the *a priori* form of the acts of thought: "It concerns itself with the laws of understanding and of reason solely inso far as they relate *a priori* to objects."[49] General logic, since it considers only the logical form, does not give us the origin of the object (whether it is empirical or *a priori*). Consequently, transcendental logic is the investigation of application of the *a priori* concepts of the understanding to objects.

Knowledge, according to Kant, is not a proposition but a judgement. The

[46]ibid.,A52/B76
[47]ibid.,A55/B80
[48]ibid.
[49]ibid.,A57/B82

logical criterion of truth is the negative condition of truth. This criterion tells us under which conditions the proposition will be false. These conditions are the canons of the understanding. Any proposition which does not agree with the canons of the understanding is false for the sake of logic. General logic cannot give a general criterion of truth because the investigation of truth requires the investigation of the object and general logic is empty in view of objects; it only deals with the form of judgement. Transcendental logic, on the other hand, is the source of truth. What makes the object possible is also what makes the truth possible. The grounds of judgement and of truth, according to Kant, is transcendental.

Kant also divides 'pure general logic' and 'transcendental logic' into 'analytic' and 'dialectic'. According to Kant, the elements of a whole is the 'analytic' of that whole. In pure general logic, analytic is empty of content and distinctions between objects are not considered. Transcendental analytic, similarly, is the determination of the elements in transcendental logic together with the rules they are subject to.

Logic, according to Kant, must provide the criteria of truth according to the universal rules of understanding. Everything that contradicts these rules is considered to be false. Kant calls this "purely logical criterion of truth" as the "negative condition of all truth".[50] He concludes that 'analytic' yields this "negative touchstone of truth"[51]. The treatment of general logic (which he considers as a 'canon') as an 'organon' is called 'dialectic' by him.[52] 'Dialectic' is the process of acquiring new thoughts from the given ones.

Kant defines 'transcendental analytic' as that "...part of transcendental logic which deals with the elements of the pure knowledge yielded by understanding,

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[50]ibid.,A59/B84
[51]ibid.,A60/B84
[52]ibid.,A61/B85

and the principles without which no object can be thought."[53] If transcendental logic is used beyond its limits, it is called 'transcendental dialectic'. Its limits are restricted to being a canon for passing judgement upon the empirical employment of the understanding.[54]

II.3 Understanding[55] (Verstand): the seat of judgements, concepts and objects

It is by means of sensibility that we receive representations. The faculty of understanding enables us to *think* these representations: "The understanding is a faculty of thought. Thought is knowledge by means of concepts. But concepts, as predicates of possible judgements, relate to some representation of a not *yet* determined object."[56]

By means of the faculty of understanding (verstand), different representations are brought together . For this process, an act is required. This act is the 'judgement'. A judgement is not a linguistic entity. A pure judgement is a *pure* act of the understanding and it is not subjective.

In Kant's transcendental philosophy, it is only in the context of a judgement that a 'concept' can be found. Judgements, as acts of the understanding, are the grounds of the concepts. The knowledge acquired by these concepts is called 'discursive' by Kant[57]: "The knowledge yielded by understanding, or at least by the human understanding, must therefore be by means of concepts, and so is not intuitive, but discursive. Whereas all intuitions, as sensible, rest on affections, concepts rest on functions. By 'function' I mean the unity of the act of bringing various representations under one common representation. Concepts are based

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[53]ibid.,A62/B87
[54]ibid.,A63/B88
[55]In terms of their dictionary meanings, 'understanding' and 'Verstand' appear to be equal. However, in the context of transcendental philosophy, 'Verstand' has a completely different sense from that of the 'understanding'. By means of the functions of Verstand, what is significant is the generation of the concept and object within the judgement. The Turkish word for 'Verstand' would be 'Müdrike'.
[56]ibid.,A69/B93
[57]ibid.,A68/B93

on the spontaneity of thought, sensible intuitions on the receptivity of impressions... Since no representation, save when it is an intuition, is in immediate relation to an object, no concept is ever related to an object immediately, but to some other representation of it, be that other representation an intuition, or itself a concept. Judgement is therefore the mediate knowledge of an object, that is, the representation of a representation of it. In every judgement there is a concept which holds of many representations, and among them of a given representation that is immediately related to an object."[58]

In transcendental philosophy, the constituents of a judgement are the concept and the object. These two elements are in a unity within the judgement and neither of them has an existence independent from or prior to judgement itself. A judgement is an act of the understanding, and concepts and objects are found within these acts.

A judgement can be either empirical or pure. With respect to general logic, the content of a judgement is not relevant. In transcendental logic, however, one considers an *a priori* content. On the content and division of judgements Kant states the following: "If we abstract all content of a judgement, and consider only the mere form of understanding, we find that the function of thought in judgement can be brought under four heads, each of which contain three moments."[59] The well-known division is as follows:

Quantity of judgements: universal, particular, singular

Quality of judgements: affirmative, negative, infinitive

Relation of judgements: categorical, hypothetical, disjunctive

Modality of judgements: problematic, assertoric, apodeictic

As we mentioned before, transcendental logic provides us with the *laws* of pure thought. Transcendental analytic, in which we find the definitions of

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[58]ibid.
[59]ibid.,A70/B95

'judgement' and 'category', provides us with the *elements* of pure thought. 'Pure thoughts' are 'pure judgements' and the elements of these are the 'pure concepts of the understanding'. A judgement, according to Kant, let us repeat, is an act of the faculty of understanding; it is a unitary act of thought.

The rules of the (synthetical) unity of the judgement are contained in transcendental consciousness (which is not empirical). This synthesis starts spontaneously in time and proceeds within 'imagination'. Here, consciousness provides a unity to an image (which is received by sensibility) in order to avoid its being lost in the passing away time: "Space and time contain a manifold of pure *a priori* intuition... But if this manifold is to be known, the spontaneity of our thought requires that it be gone through in a certain way, taken up, and connected. This act I name *synthesis*... By *synthesis,* in its most general sense, I understand the act of putting different representations together, and of grasping what is manifold in them in one act of knowledge. Such a synthesis is pure."[60]

The pure elements of transcendental aesthetic provides the material for pure synthesis. And we acquire the pure concepts of the understanding that is, the categories by pure synthesis.

In order to realize pure synthesis, the synthetic unity of space and time is required. It is only by such a way that synthesis is possible and *a priori.* Transcendental logic inquires the possibility of bringing the pure synthesis of representations under concepts. After pure synthesis is realized, the form of judgement has to be found out. Therefore, it is only in these forms that the categories, as pure concepts of understanding, can be acquired. For this acquisition, the representations that are brought together should have a content but not an empirical one. This content, which is a priori, consists of the manifolds in space and time. The process of gathering the manifold in space under a

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[60]ibid.,A77/B103

concept is objective. At this point, Kant attempts to derive the "objective" from "subjective". Transcendental content becomes possible only if there exists a synthetical unity in the manifold. Judgements of pure mathematics, according to Kant, have transcendental content.

A concept is the unitary 'consciousness' of the picture that is synthesized in imagination. Such consciousness is universal with regard to its form and it provides a rule which explains how an object is formed in accordance with the concept. Categories are pure concepts. Therefore, there must be a pure consciousness in their origin. This is the 'transcendental consciousness' which yields the unchanging 'self'. Without this 'transcendental consciousness' the categories would not have been possible:

"The same function which gives unity to the various representations *in a judgement* also gives unity to the mere synthesis of various representations *in an intuition* ; and this unity, in its most general expression, we entitle the pure concept of the understanding."[61]

The number of these pure concepts is as same as that of the division of judgements. Kant, with reference to Aristotle, calls these pure concepts categories. Kant's well-known division is as follows:

Of Quantity: unity, plurality, totality

Of Quality: reality, negation, limitation

Of Relation: of inherence and subsistence

of causality and dependence

of community

Of Modality: possibility --- impossibility

existence --- non-existence

necessity --- contingency [62]

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[61]ibid.,A79/B105
[62]ibid.,A80/B106

Kant then entitles these pure concepts 'predicaments' and distinguishes them from 'predicables' which are the pure *derivative* concepts of categories.

Kant brings these four divisions under two groups: "...those in the first group being concerned with objects of intuition, pure as well as empirical, those in the second group with the existence of these objects, in their relation either to each other or to the understanding... The categories in the first group I would entitle the *mathematical*, those in the second group the *dynamical*... In view of the fact that all *a priori* division of concepts must be by dichotomy, it is significant that in each class the number of the categories is always the same, namely, three. Further, it may be observed that the third category in each class always arises from the combination of the second category with the first."[63]

Kant introduces three concepts, namely, 'unity', 'truth', and 'perfection' in order to bring categories under general logical rules. "In all knowledge of an object there is *unity* of concept, which may be entitled as *qualitative unity* ... Secondly, there is *truth*, in respect of its consequences. The greater the number of true consequences that follow from a given concept, the more criteria are there for its objective reality. This might be entitled the *qualitative plurality* of characters... Thirdly, and lastly, there is *perfection*...This may be entitled the *qualitative completeness* (totality)... Thus the criterion of the possibility of a concept (not of an object) is the definition of it, in which the *unity* of the concept, the *truth* of all that may be immediately deduced from it, and finally, the *completeness* of what has been thus deduced from it, yield all that is required for the construction of the whole concept."[64]

In the section on the "analytic of concepts", Kant states his view about "the deduction of the pure concepts of understanding". In this "transcendental deduction", the possibility of relating categories, as pure concepts of the

63ibid.,B110
64ibid.,B114-115

understanding, to objects in an *a priori* manner is investigated. In other words, the question which Kant discusses is the following: how the categories make experience possible. Kant distinguishes the transcendental deduction from empirical deduction "which shows the manner in which a concept is acquired through experience and through reflection upon experience.."[65] The aim for this task is to demonstrate the objective validity of categories. Their objective validity rests, according to Kant, upon the condition that they serve as the *a priori* conditions of possible experience.[66]

In Kant's system, experience starts with sensation and appearance, and ends with judgement. In order to acquire the judgement, a "three-fold synthesis" is required. The constituents of this synthesis are apprehension, reproduction, and recognition.

As a consequence of the modification of sensibility, which is a capacity of the soul, we receive appearances. The two forms of sensibility are space and time. Space is the form of outer sense and time of the inner sense. In the former, nothing changes and in the latter nothing remains as it is. Representations, according to Kant, regardless of their origin (I.e. either empirical or *a priori*) are the modifications of the mind and belong to the inner sense. Every intuition of outer sense leaves in the inner sense a series of imprints which are in a flux . These intuitions have an absolute unity in themselves. "Synthesis of apprehension" is the act of bringing these imprints together into one unity.

If, in the synthesis of apprehension, everything that belongs to the sensibility is isolated, there remains something *a priori*. This is what Kant calls the *pure* synthesis of apprehension: "This synthesis of apprehension must also be exercised *a priori*, that is, in respect of representations which are not empirical. For without it we should never have *a priori* the representations either

[65]ibid.,A85/B117
[66]ibid.,A96

of space or of time."[67]

The faculty of imagination of the soul reproduces the impressions that are apprehended in the inner sense, and, in a way, generates a picture. In other words, this faculty "re-synthesizes" these impressions. These impressions are appearances and not things in themselves. Appearances, according to Kant, can be reduced to the determinations of the inner sense. Kant seeks for an *a priori* ground of a necessary synthetic unity of appearances that makes their reproduction (or, picturing) possible. Without this process of reproduction, our intuitions cannot yield knowledge.[68]

If we abstract from an image within imagination that which corresponds to the sensation, there remains a 'pure image' which is *a priori.*. It is these pure images that underlie all knowledge. The transcendental synthesis of imagination is realized by pure images; it is the ground of all experience: "For experience as such necessarily presupposes the reproducibility of appearances."[69] A particular natural number, for instance, can never be obtained unless it undergoes this process of reproduction. The representations of space and time cannot arise either.[70]

In the first step of the three-fold synthesis, that is in apprehension, impressions are received by the inner sense and then synthesized to yield a unity in respect of the objects. In the second step, that is in reproduction, these impressions are reproduced and synthesized again. In other words, they are pictured by means of the faculty of imagination. In the third step of the synthesis, by means of the consciousness of the picture in imagination, concepts arise: "If we were not conscious that what we think is the same as what we thought a

[67]ibid.,A99-100
[68]A101 (Kant entitles the faculty of imagination and apprehension as transcendental in the sense that they provide the grounds for experience and all sorts of knowledge - either empirical or pure *a priori)*
[69]ibid.,A102
[70]ibid.

moment before, all reproduction in the series of representations would be useless."[71] In the lack of consciousness, the manifold of representations will also lack unity and it will not be possible for them to form a whole and it would not be possible for concepts to arise.[72] In counting, units are added one after each other in a successive order. But if the units are forgotten it would not be possible to acquire the totality of the units hence a representation of a natural number would not have been produced: "For the concept of number is nothing but the consciousness of this unity of synthesis."[73]

The manifold of what is successively intuited is combined by the 'unitary consciousness' in one representation. Whenever one is conscious of this representation, a concept arises and so does the knowledge of an object.[74] Appearances, according to Kant, are sensible representations that exist in space. Objects, on the other hand, exist in judgements. Thus the object that corresponds to knowledge is an entity within an act of thought. The relation of all knowledge to its object is *a priori* and necessary. Our modes of knowledge possesses the unity of the object and this unity also constitutes the concept of that object. [75]

The unitary consciousness serves as a rule and reproduction in imagination functions according to it: "...the unity which the object makes necessary can be nothing else than the formal unity of consciousness in the synthesis of the manifold of the representations...But this unity is impossible if the intuition cannot be generated in accordance with a rule by means of such a function of synthesis as makes the reproduction of the manifold *a priori* necessary, and renders possible a concept in which it is united"[76]

In Kant's system, it is the categories, as pure concepts of the

[71]ibid.,A103
[72]ibid.
[73]ibid.
[74]ibid.,A104
[75]ibid.
[76]ibid.,A105

understanding, that make experience and objects possible. Concepts are *homogeneous* with the representations of the objects that fall under them. This is not the case for categories: "Pure concepts of the understanding being quite heterogeneous from empirical intuitions, and indeed from all sensible intuitions, can never be met in any intuition."[77] Kant concludes that in both relations, that is concept-object and category-appearance, a mediating representation is necessary. This representation has to be *pure, intellectual, and sensible.* Kant calls it 'the transcendental schema'. [78]

Time, being the formal condition of the connection of all representations contains an *a priori* manifold in pure intuition. A transcendental determination in time is homogeneous both with the category and the appearance. The schema is universal and rests upon an *a priori* rule. This determination constitutes the unity of the category[79]: "Thus an application of the category to appearances becomes possible by means of the transcendental determination in time, which, as the schema of the concepts of understanding, mediates the subsumption of the appearances under the category." [80]

Category, in itself, carries the possibility of all the synthetical unity of time. Thus in the absence of the category, objects and concepts would not be possible. Time is the form of everything that belongs to the inner sense. Transcendental determination in time enables us to form the image of an object that falls under a concept. So what imagination carries in itself is not the object itself. This is something subjective but has an *a priori* ground: "It is simply the pure synthesis...to which the category gives expression. It is a transcendental product of imagination, a product which concerns the determination of inner sense in general according to conditions of its form (time), in respect of all

[77]ibid.,A138/B117
[78]ibid.
[79]ibid.,A139/B178
[80]ibid.

representations, so far as these representations are to be connected *a priori* in one concept in conformity with the unity of apperception."[81]

In Kant's system, thinking, which is one act of thought, is realized through concepts and concepts are meaningless if no object is given to them.[82] Thinking goes together with schematism. A schema is a 'plan' that enables us to picture the object that falls under a concept. It is a product of the faculty of imagination which is a spontaneous act[83]: "Pure *a priori* concepts...must contain *a priori* certain formal conditions of sensibility, namely those of inner sense. These conditions of sensibility constitute the universal condition under which alone the category can be applied to any object. This formal and pure condition of sensibility to which the employment of the concept of understanding is restricted, we shall entitle the *schema* of the concept."[84]

Kant distinguishes the image from the schema. A schema is, in a sense, the representation of a representation within imagination. It is in between the image and the concept and provides the image of an object: "Indeed it is schemata, not images of objects, which underlie our pure sensible concepts."[85]

II.4. Objects and Natural Numbers in Kant's Transcendental Ontology

All objects, both empirical and *a priori*, are synthetical according to Kant. Objects, in Kant's transcendental philosophy, are not given entities as, for example, appearances are given; but acquired as a consequence of a certain procedure, namely, the three-fold synthesis. Something that is to be determined as an object acquires synthetical unity as a consequence of this procedure.

Synthesis is a spontaneous act of the understanding which provides a

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[81]ibid.,A142/B181
[82]ibid.
[83]ibid.,A140/B179
[84]ibid.
[85]ibid.,A141/B180

unity that is necessary for the formation of the representations of objects, that is, concepts. Empirical objects have their source in sensation. We receive representations by the modifications of sensibility which is one of the capacities of the human soul. These representations are given to us as appearances in intuition. As appearances, they do not have a synthetic unity; these appearances need to be brought together by a function of understanding to be determined as an object. This synthetical unity is provided by a concept. A concept is universal due to its form and it is a rule which regulates the unity of the representations as an object.[86]

As we previously mentioned, in Kant's system, objects are not found by themselves. It is the judgement, that is the unitary act of thought, that provides the space for objects. Objects exist in judgements together (in unity) with concepts. Concepts and objects are the constituents of a judgement and they do not have a priority over the whole. Judgement comes into being as an act of the understanding once the three-fold synthesis is completed; in the judgement, we find the object and its concept. It is not meaningful, in Kant's transcendental philosophy, to consider one of these three elements independently from the others. Objects exist in unity with their concepts in the judgements.

Empirical objects have their source in sensation and *a priori* objects, for example the objects of geometry and arithmetic, have their source in the *a priori* manifolds of space and time. A geometrical object arises in a synthetic judgement by giving synthetic unity to the elements of the *a priori* manifold of space. For arithmetical objects, for example the natural numbers, the material for the synthesis consists of the undetermined durations in the *a priori* manifold of time. In order to obtain a certain duration in the *a priori* manifold of time, the transcendental determination of time is required. It is by means of transcendental

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[86] A105

determination that we picture the object that falls under a concept within the faculty of imagination.

The space of *a priori* objects are synthetic *a priori* judgements. The fundemental question of the first Critique is: how are synthetic judgements possible *a priori* ? In Kant's system, objects are formed only in synthetic judgements. It is the synthetic judgement in which the object and concept are found. A synthetic judgement itself is a unity and it also provides a synthetic unity to the object by means of the concept. It is this unity that makes the object possible. In analytic judgements, only the representations of the objects are found. Thus an analytic judgement always needs a synthetic judgement as its grounds if it is to be meaningful. [87]

We mentioned above that in order to obtain number as an object, the synthesis of the *a priori* manifold in time is required. In the first step of the three-fold synthesis, that is in the apprehension in the inner intuition, the moments of time are received by the inner sense. Then, a series of imprints are produced in the imagination. These are spontaneously synthesized to yield a unity. Since the representations that are synthesized to form the image of a number are *a priori*, a pure synthesis is required for realizing this. In order to acquire this pure synthesis, as it was mentioned in the previous section, everything that belongs to sensibility is isolated and abstracted.

In the second step of the synthesis, that is the reproduction in imagination, the image of the apprehended moments is formed. That is to say, the imprints of the apprehended moments are 're-synthesized' in order to form an image of the number. But this image is not the image of the number yet. For, number as an object is formed after the synthesis is completed. This is only the image of the object that is to be found in the judgement. The image itself is subjective and

[87] These points have been explained in the lectures of the following courses: phil 471 (philosophy of mathematics), phil 530 (Kant), phil 565 (cosmology).

psychological. This is the image of a transcendental determination in time and the transcendental determination requires a homogeneous intuition in time. After the synthesis is completed, there exists in the judgement an object which corresponds to this image. But number five, as an arithmetical object for example, has nothing in common neither with its image in the imagination nor with the symbol '5' in the intuition. While the former exists in thought (that is, in the judgement), the latter, being an appearance, exists in intuition.

In the third step of the synthesis, that is synthesis by recognition, by means of the consciousness of the picture of the number in imagination, that is, of the schema of the category of quantity, the concept of number arises. The number as an object falls under the concept of number. On one side, the concept of number is universal and on the other side it is a rule for the unity of the number as an object: "The concept of number is nothing but the consciousness of this unity of synthesis."[88] Consciousness enables us to hold the concept in a unity.

We should note that the space that the concept and the object share, that is the judgement, is different from the space of the source of the material of arithmetical objects, that is time. Judgement, as a unitary act of thought, is the transcendental medium for connecting an object to its concept. The definition of judgement in <u>Logic</u> is as follows: "A judgement is the presentation of the unity of the consciousness of several presentations, or the presentation of their relation so far as they make up one concept."[89] Here Kant considers object and concept respectively as the matter and the form of the judgement: "To every judgement belong, as its essential components, matter and form. The matter of judgement consists in given cognitions that are joined in judgement into unity of consciousness; in the determination of their manner in which various presentations as such belong to one consciousness consists the form of

[88] ibid.,A103
[89] Op.Cit., p.106

judgement."[90]

In order to *think* a concept in unity with its object, the pure concepts of understanding, that is the categories are required. Categories secure the function of giving unity to various representations in a judgement.

In the beginning of the section on 'schematism', Kant asserts that the representation of an object that falls under a concept must be *homogeneous* with the concept:[91] "...the concept must contain something which is represented in the object that is to be subsumed under it."[92] It is objects that concepts are in immediate relation within the judgement.

Categories, according to Kant, being pure concepts of the understanding, are *heterogeneous* with all sensible intuitions.[93] Here the following question arises: how would it be possible to apply a category to appearances? Kant suggests a third entity as a solution, namely the transcendental schema which is pure, intellectual, and sensible. By means of these properties, the transcendental schema is homogeneous with the category as well as with the appearances and thus makes the application of the former to the latter possible.[94]

Schema is the form of the image of the object. Schema enables the faculty of imagination to provide an image to the concept in the second step of the synthesis and schematism is the name of this process.[95]

It is the categories that provide the synthetic unity to an object. Categories are the forms of this synthetic unity which has its source is transcendental apperception. For, *a priori* objects, particularly the objects of arithmetic, the category of <u>unity</u> enables the number as an object to be subsumed under the concept of that number. The application of a category to the *a priori*

[90]Op.Cit., Ibid.
[91]Ibid.,A137/B176
[92]Ibid.
[93]Ibid.
[94]Ibid.,A138/B177
[95]Ibid.,A140/B179

representations in time is possible only if there is a medium. This is nothing but the transcendental schema of the category. For, without schemata, categories are nothing more than mere forms. The schema, as it is mentioned in the previous section, is the product of the faculty of imagination. The product of the empirical faculty of understanding is the image and the product of the pure *a priori* imagination is the pure image. Schema of the pure concept of quantity provides a pure image of the number subsumed under the concept of number. Schema exists in thought and it is a rule of synthesis of the imagination in respect of the a priori successive units in time.

Magnitude (*quantitatis*) is a concept of the understanding.[96] Pure schema of magnitude is number. It is the pure schema, as a representation, that "comprises the successive addition of homogeneous units." Number is "the unity of the synthesis of the manifold of a homogeneous intuition."[97] It is a "unity due to generating time itself in the apprehension of the intuition."[98] This is a spontaneous act of the understanding. 'Unity', as one of the pure concepts of the category of quantity, has a transcendental schema in the understanding. Magnitude is the concept of this transcendental schema: "The concept of magnitude is the determination of a thing whereby we are enabled to think how many times a unit is posited in it."[99] When the schema of the category of quantity, which provides 'unity', is found within the unitary act of thought, that is within the judgement as a concept, number as an object arises. In this schema, a pure and homogeneous intuition is reproduced. The schema of unity is an a priori form that enables us to form the image of the number as an object by means of the schema of magnitude. This is possible only if the schema of the category of unity is a concept.

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[96]Ibid.,A142/B182
[97]Ibid.,A143
[98]Ibid.
[99]Ibid.,A242/B300

In the case of arithmetical objects, the three-fold synthesis, as a spontaneous act of the understanding, is *a priori* and functions according to an objective rule, that is subordination. This rule is acquired by transcendentally exposing the concept of time. What this rule tells us is that *a priori* moments are one after the other in time. Thus the image of number that is the product of a priori imagination is subjective but number as an object which exists in judgement and brought into a synthetic unity according to a rule, is objective. At this point, one can state that the order of the numbers - since they are formed as a consequence of the transcendental determination in time, which is the form of inner sense, that is one of the two forms of sensibility which is one of the capacities of the human soul and which belongs to a person and hence subjective - is subject to an objective rule, that is subordination.

In Kant's transcendental ontology, all objects are grounded in a well-determined space. Objects which have the same ontological status share the same space and are connected within that space. Objects which have different ontological status and thus not sharing the same space, on the other hand, are not connected without a mediating representation. It is only in this way that the unity of representations becomes possible. Kant considers this approach as a requirement for acquiring a certain kind of knowledge. Mathematical knowledge, as we mentioned above, contains synthetic *a priori* judgements. A judgement, having concept and object as its constituents, is a mediating representation between the concept and the object. An image is the representation of the object within the faculty of imagination. And it is the schema which provides this image to the concept. The space of an image is imagination and the space of a concept is the judgement which is an act of the understanding. Here the schema is the mediating representation between these two spaces. Imagination is also the space of a schema. But images and schemata are the products of two distinct functions of the imagination.

Natural numbers are *a priori* objects in Kant's transcendental ontology and Kant secures an objective status to natural numbers in his system. This objectivity is derived from a subjective act. However, what belongs to me and what is subjective is the representation of the number, not the number itself. Number, as an object, on the other hand, is objective and independent from my psychological states; number is synthetic and has a unity. The space of a number as an object is the synthetic *a priori* judgement.[100]

[100]It is synthetic, because, 7, in itself has a unity, so does 5. But 12 has a different synthetic unity in itself, different from those of 7 and 5. Here there are two concepts: the concept of "7+5" and the concept of "12" which necessitate two different synthese. The judgement is *a priori*, because it cannot be derived from experience. A concept of number is bound to its object with a necessity in a synthetic *a priori* judgement. Time is the source of arithmetical knowledge which is synthetic *a priori* .

III. Frege's Ontology of Natural Numbers

In The Foundations of Arithmetic, Frege criticizes certain views on the concept of number and elucidates his view on 'unit' with regards to their 'distinguishability' and 'identity'. His criticism consists of three main sections: (i) nature of arithmetical propositions, (ii) concept of Number and (iii) unity and one. Frege first asks three questions : a) Are numerical formulae provable?, b) Are the laws of arithmetic inductive truths?, and c) Are the laws of arithmetic synthetic a priori or analytic?.

In view of the first question above, Frege distinguishes between numerical formulae and general laws.[101] According to Hankel, Frege states, numerical formulae, being infinitely numerous, are unprovable. Hankel finds the conception 'infinitely numerous unprovable primitive truths' incongruous and paradoxical if it is to be used for numerical formulae.[102] Frege rejects Hankel's view and gives the following reasons: such a conception, that is, 'infinitely numerous unprovable primitive truths', "conflicts with one of the requirements of reason which must be able to embrace all first principles in a survey." [103] Secondly, numerical formulae for large numbers are not self-evident and it is not, according to Frege, convenient to distinguish between small and large numbers. Furthermore, Frege claims that such a distinction is not possible at all. [104]

Leibniz on the other hand, Frege states, holds the opposite view that numerical formulae are provable; Leibniz gives the following proof: "It is not an immediate truth that 2 and 2 are 4; provided it be granted that 4 signifies 3 and 1. It can be proved, as follows:

Definitions: (1) 2 is 1 and 1

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[101]Gottlob Frege, *The Foundations of Arithmetic*, tr. J. L. Austin (1950; rpt. Illinois: Blackwell,1968) p. 5
[102] Ibid., p.6
[103]Ibid.
[104]Ibid.

(2) 3 is 2 and 1

(3) 4 is 3 and 1

Axiom: If equals be substituted for equals, the equality remains.

Proof: 2+2 = 2+1+1 (by Def.1) = 3+1 (by Def.2) = 4 (by Def.3).

Therefore: 2+2 = 4 (by the Axiom)."[105]

Frege finds the following gap in the proof above: The transformation of the right hand part of the first equation, that is 2+1+1, into 2+(1+1) by adding brackets is not allowed by the present rules. Similarly, the second equation, that is 3+1=4 is transformed into (2+1)+1=3+1. Thus, Frege claims, the proof by Leibniz implicitly uses the rule of associativity.

According to Frege, "a similar proof can be given for every formula of addition. Every number, that means, is to be defined in terms of its predecessor... Through such definitions we reduce the whole infinite set of numbers to number one and increase by one, and every one of the infinitely many numerical formulae can be proved from a few general propositions."[106]

Another mathematician, Grassmann, Frege states, shares this view and attempts to obtain the law a+(b+1)=(a+b)+1 by means of the following rule: "If a and b are any arbitrary members of the basic series, then by the sum a+b is to be understood that member of the basic series for which the formula a+(b+ e) = a+b+ e is valid."[107]

Frege criticizes Grassmann's definition above in two ways: "First, sum is defined in terms of itself. If we do not understand the meaning of a+b, we do not understand the expression a+(b+ e) either." [108] And, Frege interprets Grassmann's sum as addition: "In that case, the criticism could still be brought that a+b would be an empty symbol if there were either no member or several

[105]Ibid., p.7
[106]Ibid., pp.7-8
[107]Ibid., p.8
[108]Ibid.

members of the basic series which satisfied the prescribed condition." [109]

Frege furthermore charges a severe criticism towards Mill's empirical approach. Mill explains numbers in terms of observed or, physical facts. Like Leibniz, Mill also tries to base arithmetic on definitions, but in a quite different way. According to Mill: "these definitions are not definitions in the logical sense; not only do they fix the meaning of a term, but they also assert along with it an observed matter of fact." [110] Frege quotes from Mill: "The calculations do not follow from the definition itself but from the observed matter of fact." [111] Frege however points out that this sense that is to be attached to '2+1' cannot be acquired from observation. Besides, Frege argues, Mill does not give any example other than number 3, which Mill separates as 'oo o'.

According to Frege, Mill's view that "the definition of each individual number did really assert a special physical fact" gives rise to some difficulties. [112] Frege claims that this principle can imply that "it would be enough if we had derived through induction a general law in which they were all included together." [113] Frege formulates this law as follows: "There exist large collections of things which can be split up." [114] According to Frege, however, this is not sufficient. "For this does not state that there exist collections of such a size and of such a sort as are required for, say, the number 1,000,000, nor is the manner in which they are to be divided up specified any more precisely." [115]

Secondly, Frege argues, Mill's theory is not sufficient for explaining the numbers '0' and '1', since no physical fact underlies these numbers. [116]

Thirdly, Frege argues that Mill has supposed that "the physical facts would

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[109]Ibid.
[110]Ibid., p.9
[111]Ibid., p.10
[112]Ibid.
[113]Ibid.
[114]Ibid.
[115]Ibid.
[116]Ibid., p.9

be used only for the smaller numbers, say up to '10', while the remaining numbers could be constructed out of these." [117] In this case, then, some numbers will not be constructed out of their own characteristics. For example '11', if constructed like '10+1' while the construction of some numbers, for example '2', if constructed like '1+1' "must depend on the observation of a particular collection, separated in its own peculiar way." [118]

Fourthly, since numbers owe their existence to physical phenomena according to Mill, then, Frege argues, it would not be correct to talk about, for example, a clock striking three times [119] because it does not impress our senses in the way Mill describes three as 'oo o'. Accordingly, Frege shows that the definitions of individual numbers "neither assert observed facts nor presuppose them for their legitimacy."[120]

In view of the second question, that is, 'are the laws of arithmetic inductive truths?', Frege considers the controversial views by Mill and Leibniz and gives his criticism on these. Here, Frege states that his task is to "ascertain the nature of the laws involved." [121]

In his system, Mill has introduced the following principle: "whatever is made up of parts, is made up of parts of those parts" as an attempt to define inductive truth. He claims such truths to be "a law of nature of the highest order". [122] Frege reformulates this principle as "the sums of equals are equals". [123] According to Frege, Mill misuses the conception of 'arithmetical truth' : "In order to be able to call arithmetical truths laws of nature, Mill attributes them a sense which they do not bear."[124] That is, such truths cannot be based upon physical

[117]Ibid., p.11
[118]Ibid., pp.11-12
[119]Ibid., p.9
[120]Ibid., p.10
[121]Ibid.
[122]Ibid., p.12
[123]Ibid.
[124]Ibid., p.13

facts.

Another misconception, Mill uses, according to Frege, is 'addition'. Mill considers 'addition' as a process that expresses "the relation between the parts of a physical body or of a heap and the whole body or heap." [125] According to Frege, however, when two physical entities are brought together, we cannot acquire the meaning of an arithmetical formula. This process of 'bringing together' can only be thought as the application of the related arithmetical formula, and not the arithmetic formula itself: "Mill always confuses the applications that can be made of an arithmetical proposition, which often are physical and do presuppose observed facts, with the pure mathematical proposition itself." [126]

Frege's argument against Mill's point is as follows: "We can't speak even here of parts; but then we are using the word not in physical or geometrical sense, but in its logical sense... This is a matter of logical subordination... It follows that the general laws of addition cannot, for their part, be laws of nature." [127]

Frege rejects the view which asserts that the general laws of 'addition' are inductive truths. His main reason for such a claim is that there is no uniformity between the numerical formulae and the numbers. [128] At this point, Frege agrees with Leibniz who believes that numbers and geometrical figures are dissimilar. [129]

Frege makes a distinction between the spatio-temporal and non- spatio-temporal and asserts that numbers are of the latter sort: "In ordinary inductions we often make good use of the proposition that every position in space and every

[125] Ibid.
[126] Ibid.
[127] Ibid., pp.13-14
[128] Ibid., p.14
[129] Ibid.

movement in time is as good in itself as every other... In the case of numbers this does not apply, since they are not in space or time. Position in the number series is not a matter of indifference like position in space." [130]

Numbers, according to Frege have their own unique peculiarities, especially in the cases of '0','1' and '2'. Arithmetic, Frege claims, is prior to induction and induction, since it is based on probability, must presuppose arithmetic. [131] Leibniz opposes this point and introduces his conception of 'innate'. Frege quotes: "The truths of number are in us and yet we still learn them." [132]

The last question is: 'are the laws of arithmetic synthetic a priori or analytic?' In the Foundations of Arithmetic, Frege gives these distinctions and claims that they concern the justification for making the judgement, but not the content of it: [133] "If it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one. For a truth to be a posteriori it must be impossible to construct a proof of it without including an appeal to facts... But if, on the contrary, its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof, then the truth is a priori." [134]

Frege immediately eliminates the possibility that arithmetical judgements are analytic a posteriori. So, two possibilities remain: synthetic a priori and analytic. According to Frege, philosophers and mathematicians such as Baumann and Hankel belive in the former: "a pure intuition as the ultimate ground of our knowledge of such judgements... Hankel bases the theory of real

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[130] Ibid., p.15
[131] Ibid., pp.16-17
[132] Ibid., p.17
[133] Ibid., p.3
[134] Ibid., p.4

numbers on three fundamental propositions, to which he ascribes the character of common notions."[135]

Frege quotes from Hankel: "Once expounded they are perfectly self-evident; they are valid for magnitudes in every field, as vouched for by our pure intuition of magnitude."[136] Frege hesitates to use magnitude for the concept of number. Moreover, according to him, it is not appropriate to use the terms 'intuition of magnitude' and 'pure intuition of magnitude'.

Frege makes a distinction between the natures of geometrical and arithmetical entities. In agreement with Leibniz, he asserts that "in geometry, it is quite intelligible that general propositions should be derived from intuition... But with the numbers it is different; each number has its own peculiarities." [137]

In relation to the above distinction between geometry and arithmetic, Frege claims that while the axioms of the former, together with the primitive laws of logic, are synthetic, the fundamental propositions of the latter are analytic. Frege considers the basis of arithmetic as prior to geometry and to any other empirical science. He concludes that the laws of number should be connected to the laws of thought for arithmetic which is the "widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable." [138] In Thoughts, Frege argues the 'actuality' of thoughts and claims that "A thought...is not the sort of thing to which it is usual to apply the term actual...We will hardly admit what is timeless and unchangeable to be actual." [139]

We mentioned above that Leibniz holds the view that arithmetical formulae, which are analytic a priori, are provable. According to Leibniz, algebra has its basis in logic. Moreover, he claims that "in the case of necessary truths a

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[135]Ibid., p.18
[136]Ibid.
[137]Ibid., p.19
[138]Ibid., p.21
[139]Gottlob Frege, "Thoughts", *Collected Papers*, ed. Brian McGuiness, 1st. ed. (rpt.Glasgow:Blackwell,; 1984)p. 370

proof or reduction to identities is possible." [140] Frege quotes from Leibniz: "every truth has its proof a priori derived from the concept of the terms, notwithstanding it does not always lie in our power to achieve this analysis." [141] Here Frege mentions Jevons' view which coincides with that of Leibniz: "I hold that algebra is a highly developed logic, and number but logical discrimination." [142] Frege criticizes this view by Leibniz on the 'reduction to identities'. Frege asserts that 'empty forms of logic' are insufficient to fill in the 'rich content' of arithmetic. [143]

Frege finds the method of induction as insufficient for arriving at arithmetical propositions. In order to show that it is not possible to base arithmetic on induction, he gives an example from geology: The example is based on the supposition that there is a borehole inserted to earth where we find a wide variety of rock strata and we observe that in the borehole temperature increases regularly with the depth. Frege claims that we cannot infer anything about the nature of the strata at deeper levels although it is true that under the concept of 'whatever you come to by going on boring', both the strata so far observed and those at lower levels alike fall. In relation to his theory of numbers, Frege claims that "it will be no help to us to learn in the case of the numbers that these all fall together under the concept of 'whatever you get by going on increasing by one'." [144]

Frege does not prefer to use deduction as applied to facts. Rather, he suggests the following: "Instead of linking our chain of deductions direct to any matter of fact, we can leave the fact where it is, while adopting its content in the form of a condition. By substituting in this way conditions for facts throughout the whole of a train of reasoning, we shall finally reduce it to a form in which a certain

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[140] Op.Cit., p.21
[141] Ibid.
[142] Ibid., p.22
[143] Ibid.
[144] Ibid., pp.15-16

result is made dependent on a certain series of conditions. This truth would be established by thought alone.“.[145] According to Frege, by this method, “the truths of arithmetic would then be related to those of logic in much the same way as the theorems of geometry to the axioms.”[146]

In second main section on ‘the concept of number’, Frege asks the following questions: ‘is number a property of external things?’, ‘is number something subjective?’ and ‘is number a set?’ In the beginning of this section, Frege distinguishes between the concept of Number and individual numbers. According to Frege, arithmetical laws concern the general concept of Number, and not the definitions of individual numbers.

In view of the first question, Frege considers Cantor’s and Schröder’s replies which are positive. For Cantor, mathematics is an empirical science and “It begins with the consideration of things in the external world.”[147] Cantor agrees with Schröder in acquiring the numbers by abstraction: “For E. Schröder number is modeled on actuality, derived from it by a process of copying the actual units with ones, which he calls the abstraction of number. In this copying, the units are only represented in point of their frequency...Here frequency is only another name for number.”[148] Baumann, on the other hand, holds the opposite view; Frege quotes: “The reason being that external things do not present us with any strict units; they present us with isolated groups or sensible points, but we are at liberty to treat each of these itself again as a many.”[149]

According to Frege, individual numbers cannot be ascribed to the objects in the way that some properties are given to them as colour, shape, etc. Frege asserts that these properties do not depend on our choice, while, on the other

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[145]Ibid., p.23
[146]Ibid., p.24
[147]Ibid., p.27
[148]Ibid., pp.27-28
[149]Ibid., p.28

hand, different numbers can be ascribed to an object "with equal right". [150] Even though it can, in some sense, be said that a Number 'belongs to' an object, this Number belongs to the object "in view of the way in which we have chosen to regard it; and even then not in such a way that we can simply assign the Number to it as a predicate." [151]

Frege furthermore criticizes Mill's view that the number is a property of the agglomeration of things. On this point, Mill states the following: "The name of a number connotes...some property belonging to the agglomeration of things which we call by the name; and that property is the characteristic manner in which the agglomeration is made up of, and may be separated into, parts."[152] In the first place, Frege criticizes the phrase 'the characteristic manner': "There are very various manners in which an agglomeration can be separated into parts, and we cannot say that one alone would be characteristic."[153] Besides, it is more difficult to consider the number '0' as such a property.[154]

On the 'applicability' of number, Frege states Leibniz's view that considers number as "a kind of metaphysical figure". [155] According to Leibniz, number is applicable also to immaterial things and, in Frege's view, Leibniz "holds that number is of supreme universality and belongs to metaphysics".[156]

According to Frege, it is not sensible to presuppose the occurrence of a sensible thing in something non-sensible. He inquires the relation between the sensible and non-sensible and asserts that the knowledge of the sensible things, that is the symbols, do not reveal anything about the non-sensible things, that is the numbers. [157]

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[150]Ibid., p.29
[151]Ibid.
[152]Ibid., pp.29-30
[153]Ibid., p.30
[154]Ibid.
[155]Ibid., p.31
[156]Ibid.
[157]Ibid., p.32

According to Locke Frege states, the individual number exists only as a notion. Mill, we mentioned above, holds the view that the number is something physical. Frege's critisizes Mill in view of this point that Mill does not distinguish between the physical things and their concepts; Mill also considers the latter as something physical. [158]

Of the second question "is number something physical?" is answered positively, according to Frege, then it follows that numbers come about as a result of a mental process. Lipschitz holds this view. Frege quotes from Lipschitz: "Anyone who propose to make a survey of a number of things, will begin with some one particular thing and proceed by continually adding a new one to those previously selected." [159]

Frege rejects this view and claims that the numbers are objective, i.e., independent from our ideas. [160] On the other hand, Frege distinguishes 'objective' from 'spatial' or 'actual'. [161] What is objective, according to Frege, "is not a creature of thought, the product of a psychological process, but is only recognized or apprehended by thought."[162] Frege then claims that 'being recognized' and 'being created' are quite different and they do not imply each other. Frege defines 'objectivity' as follows: "What is objective in it is what is subject to laws, what can be conceived and judged, what is expressible in words...I understand objective to mean what is independent of our sensation, intuition and imagination, and of all construction of mental pictures out of memories of earlier sensations, but not what is independent of our reason." [163]

Schloemilch claims, Frege states, the subjectivity of numbers in a different manner. According to him, "number is the idea of the position of an item in a

[158]Ibid., p.33
[159]Ibid.
[160]Ibid., p.34
[161]Ibid., p.35
[162]Ibid.
[163]Ibid., pp.35-36

series". [164] According to Frege, the implication of this proposition leads to the view that arithmetic is nothing more than psychology. Such a view obviously cannot be defended by Frege; if numbers were ideas, then they would belong to us and it would not be possible to talk about a very big number which hardly anyone has an idea. [165] In Thoughts, Frege claims that "if everything is idea, then there is no owner of ideas...If there is no owner of ideas then there are also no ideas, for ideas need an owner and without one they cannot exist." [166] Ideas become possible if there is a subject which thinks them. This is not the case for thoughts. Hence, for Frege, the space of ideas is different from the space of thoughts.

According to Frege, the theory that attempts to define number as a set, namely 'set theory', fails mostly because it is insufficient to explain numbers '0' and '1'. Secondly, the process of 'bringing together' is left obscure in this theory: "Thomae requires for the formation of number that item-sets which differ be given different names. By this he evidently means to refer to a process of bringing out more sharply the characteristics of the sets in question, of which the given of names is only the external sign. The question is, just what is this process like?"[167] Frege considers two different views:"Some call number a set of things or objects; others, following Euclid, define it as a set of units." [168]

According to Frege, Dedekind uses the word 'system' for 'set'. Frege quotes from Dedekind: "Such a system S... is completely determined if for every thing it is determined whether it is an element of S or not. Hence a system S is the same as a system T (in symbols, S=T) if every element of S is also an element of T and every element of T is also an element of S."[169] "It very

[164]Ibid., p.37
[165]Ibid., p.38
[166]Op.Cit., p.366
[167] *The Foundations of Arithmetic*, p.38
[168]Ibid., p.39
[169]Gottlob Frege, *Basic Laws*, ed./tr. Montgomery Furth (1964; rpt. Berkeley and Los Angeles: Univ. of

frequently occurs that different things *a,b,c,...*, regarded for some reason from a common point of view, are put together in the mind; and we say then that they form *a system S.*" [170]

Frege rejects this view and claims that "this 'regarding', this 'putting together in the mind', is not an objective characteristic. I ask in whose mind?" [171] And he quotes the following passage from Dedekind: "For uniformity of expression it is advantageous to admit also the special case in which a system S consists of a *single* (one and only one) element *a*, i.e., in which the thing *a* is an element of S but every thing different from *a* is not element of S."[172] Frege's comment on this view is that, according to Dedekind, it is the elements that really make up the system.[173]

Frege investigates the latter view in a separate section, namely 'Unity and One'. This section consists of the following questions: "Does the word 'one' stand for a property of objects?", "Are the units identical with one another?", "Are attempts to overcome the difficulty sufficient?" and lastly Frege gives his own solution.

Frege has already rejected the view that number is a property of external things. Under the first question, that is, "Does the number word 'one' stand for a property of objects?", he inquires whether 'one' and 'unit' can be used interchangeably. Euclid, according to Frege, has used the word 'monas' in a vague way: "sometimes an object to be numbered, sometimes a property of such an object, and sometimes the number one."[174] In a footnote[175], Frege states that the Greek word 'monas' is translated into German as 'Einheit' (in english:

California Press, 1967) pp.29-30
[170]Ibid., 30
[171]Ibid.
[172]Ibid.
[173]Ibid.
[174]*The Foundations of Arithmetic*, p.39
[175]Ibid.

'unity', in turkish:'birlik') which is derived from 'Ein' (One). In english there is no such derivation. 'Unity' is not derived from 'one'.

Frege considers Schröder's view on this point: "Each of the things to be numbered is called a unit." [176] Frege rejects this view for he does not see the necessity of bringing "the things under the concept of unity, instead of simply defining number right away."[177] In Schröder's definition, according to Frege, when the things that are brought together are called units, "we are supposed to be adding our description of them; under the influence of the grammatical form, we are adding 'one' as a word for a property." [178]

What Frege objects is the usage of 'one' as a predicate. In the plural case, it is easier to see that it is not appropriate to use 'one' as a predicate for a property of things. If 'x' is the property of both A and B, then it can be said that 'A and B are x' but when 'one' is used as a property, we must be able to assert the proposition 'A and B are one', which we cannot. [179]

According to Frege, there is an affinity between the views of Schröder and those of Dedekind with respect to 'set'. Frege claims that, when Dedekind calls a system part of a system, he means "the subordination of a concept under a concept or an object's falling under a concept: cases that he distinguishes no better than Schröder, owing to an error of conception shared by them both; for Schröder too at bottom regards the elements as what constitutes his *class*. With him an empty class may really no more occur than may an empty system with Dedekind."[180] Frege quotes from Dedekind: "It very frequently occurs that different things *a, b, c,...*, regarded for some reason from a common point of view, are put together in the mind; and we say then that they form a *system S.* "[181]

[176]Ibid.
[177]Ibid.
[178]Ibid., pp.39-40
[179]Ibid., p.40
[180]*The Basic Laws of Arithmetic*, p.30
[181]Ibid.

Frege critisizes this view of Dedekind for according to him, "this putting together in the mind is not an objective characteristic."[182]

According to Frege, both Schröder and Dedekind define a concept by means of its characteristic marks.Frege comments on this view in the following manner: "If it is the characteristic marks that make up the concept, and not the objects falling under it, then there are no difficulties or objections to an empty set. Of course then an object can never be at the same time a concept; and a concept under which falls only one object must not be confused with it. In this way, then, it will finally be acknowledged that a statement of number contains an assertion about a concept. I have reduced Number to the relation of equinumeracy, and reduced the latter to many-one correspondence." [183]

Frege rejects both Leibniz's and Baumann's views on this: "Leibniz indeed says that 'by *one* is meant whatever we grasp in one act of the understanding,' but this is to define 'one' in terms of itself" [184] Baumann's view, according to Frege, leads us to the errors of subjectivism again. Frege quotes from Baumann: "Whatever we take as a point, or refuse to take as further subdivided into parts, that we regard as one." [185] In the result of this view, according to Frege, "everything is made dependent on our way of regarding them." [186]

Baumann assigns the property of 'indivisibility' to units. [187] Frege does not accept this as a criterion since it does not contain 'consciousness'. "Such properties of things as being undivided or being isolated, which animals perceive quite as well as we do, cannot be what is essential in our concept." [188]

Köpp assigns the property of 'indivisibility' to units besides

[182]Ibid.
[183]Ibid., p.31
[184]Op. Cit., p.41
[185]Ibid.
[186]Ibid.
[187]Ibid.
[188]Ibid., p.42

'undividibility'.[189] Baumann's criterion is close to that of Köpp's, that is 'strictness in units'. [190] Frege rejects both of these views since, according to him, these arguments do not lead us to draw any conclusions. Apart from this, he asserts that, there are cases where we cannot avoid of thinking possibilities of dissection, "where a conclusion is based on the way in which a unit is made up of parts". [191]

The second question Frege considers is: "Are units identical with one another?" According to Schröder, Frege states, the necessity for calling things 'unit' is for to ascribe the necessary identity to the items that are to be numbered. [192] According to Frege, if the things that are to be numbered are objects, then he claims that, "no two objects are ever completely identical". [193] The term, "the identity of units", according to Frege, is used without qualification. [194] "Hobbes states that 'Number in the absolute sense in mathematics presupposes units identical one with another out of which it is formed.' Hume holds the component parts of quantity and number to be entirely similar. Thomae calls the individual member of his set a unit, and says in so many words that units are identical with each other." [195]

According to Frege, these views do not help to acquire the concept of number: "The properties which serve to distinguish things from one another are, when we are considering their Number, immaterial and beside the point." [196] Thus, Frege claims, the concept of Number cannot be reached by abstracting the differences between things.

If the units that are brought together are identical, then it is difficult to

[189]Ibid., p.43
[190]Ibid.
[191]Ibid.
[192]Ibid.
[193]Ibid.
[194]Ibid., p.45
[195]Ibid.
[196]Ibid.

obtain plurality. If the units are completely different from each other, then the things that are brought together do not form a unity. Frege declares this problem by stating Descartes', Schröder's and Jevons' views; they all agree on the view that numbers originate from diversity: "Descartes says, the number (or better, the plurality) in things arises from their diversity. And as E. Schröder justly observes: 'That things should be numbered is a reasonable demand only where the objects submitted appear clearly distinguishable from one another (for example, spatially and temporally separated) and isolated in contrast with one another.' " 197

Jevons elaborates more on this: "Number is but another name for diversity. Exact identity is unity, and with difference arises plurality." (ibid.) Frege gives Jevons' definition of unit : "Any object of thought which can be discriminated from every other object treated as a unit in the same problem."[198] Jevons defines number '5' as '1+1+1+1+1' or, '1'+1''+1'''+1''''+1'''''. [199] And for the arithmetical operations, Frege argues, Jevons would claim the following: "Instead of 3-2=1, (1'+1''+1''')-(1''+1''')=1' can be used." [200]

Frege's objection to Jevons is as follows: "If, however, we adopt the alternative plan, of assigning different symbols to different things, it is hard to see why we still retain in our symbols a common element; why not write, instead of 1'+1''+1'''+1''''+1''''', simply a+b+c+d+e ? ... We must have identity - hence the 1; but we must have difference - hence the strokes; only unfortunately, the latter undo the work of the former." [201]

Frege faces Jevons' probable way of formulating arithmetical operations as follows: If '3-2=1' can be written as '(1'+1''+1''')-(1''+1''')=1', "what would be the

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197Ibid., p.46
198Ibid.
199Ibid., p.47
200Ibid., p.49
201Ibid., pp. 47-48

remainder of (1'+1"+1"')-(1""+1""') ? Certainly not 1'. It follows, therefore, that on his view there would be not only distinct ones but also different two's and so on; for 1""+1""' could not be substituted for 1"+1"'. This puts us in a position to see quite clearly that number is not an agglomeration of things... It is therefore impossible to regard 1 as a symbol for different distinct objects." [202]

On the relation between unit and one, Frege considers Locke's, Leibniz's and Hesse's views: "Leibniz understands by *unitas* a concept under which this one and that one and the other one fall...Locke and Hesse seem to use unit and one to mean the same. Indeed Leibniz, in the last analysis, does so too; for when he calls each individual object falling under his concept of *unitas* a *unum*, this word is being used to signify not the individual object but the concept under which they all fall."[203]

In order to take a step to overcome the confusion between 'one' and 'unit', Frege suggests to draw a strict distinction between these two terms: "When we speak of 'the number one', we indicate by means of the definite article a definite and unique object of scientific study. There are not diverse numbers one, but only one."[204] At this point, Frege distinguishes between the individual numbers and number concept: "Only concept words can form a plural. If, therefore, we speak of 'units', we must be using the word not as equivalent to the proper name 'one', but as a concept word. If this term 'unit' means 'object to be numbered', then number cannot be defined as units."[205]

Frege claims that to use '1' does not either help to overcome this difficulty that is stated above: "If we use 1 to stand for each of the objects to be numbered, we make the mistake of assigning the same symbol to different things. But if we

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[202]Ibid., p.49
[203]Ibid., p.48
[204]Ibid., p.49
[205]Ibid., p.50

provide the 1 with differentiating strokes, it becomes unusable for arithmetic."[206]

Frege investigates the views of certain philosophers and mathematicians on this problem under the question: "Are attempts to overcome the difficulty sufficient?"

According to Frege, the view that numbers are spatio-temporal entities cannot overcome this difficulty. Hobbes and Thomae hold such a view. Frege quotes from Thomae: "If we consider a set of individuals or units in space and number them one after the other, for which time is necessary, then, abstract as we will, there remain always as discriminating marks of the units their different positions in space and in the order of succession in time."[207]

Leibniz, Baumann and Jevons reject the view above. According to Leibniz, "number results from the mere division of the continuum and cannot be applied to immaterial things."[208]

On the 'atemporality' of number, Frege claims that "if the objects numbered do not follow one after another in actual fact, but it is only that they are numbered one after another, then time cannot be the ground of discrimination between them...Time is only a psychological necessity or numbering, it has nothing to do with the concept of number."[209] Taking points in time for numbers, according to Frege, gives rise to another difficulty which we previously mentioned. Since there can be different intervals between these points, there can be different two's, different three's and so on.[210] In order to evade this difficulty, "more generalized point of series" can be suggested but Frege criticizes this point as well: "their positions in the series cannot be the basis on which we distinguish the objects, since they must already have been able to arrange them in a series. Any such

[206]Ibid.
[207]Ibid., p.52
[208]Ibid.
[209]Ibid., pp.52-53
[210]Ibid., p.53

arrangement always presupposes relation between the objects...which serve to lead us on from one object to the next and which are necessarily bound up with distinguishing between them."[211]

Frege criticizes some other significant suggestions to overcome this difficulty before stating his own view. One of the suggestions is by Schröder who allows the number "only to copy the object".[212] By this way of copying, Schröder claims, a natural number can be defined as a 'sum of ones'. According to Frege, this definition by Schröder gives the numerals, but not the number.[213] Moreover, Frege claims, Schröder assigns no content to the process of addition: "The symbol + is introduced solely to serve as a visible mark, without any content of its own, for linking up the other symbols; only later does he define addition."[214]

According to Frege, "It is probably in order to avoid the difficulties which Jevons runs into through making each symbol 1 mean one of the objects numbered, that Schröder allows it only to copy the object."[215] He suggests abstraction of the distinguishing properties in order "to avoid carrying over into number the distinguishing marks of the things numbered."[216] Frege quotes from Jevons: "Abstract number, then, is *the empty form of difference*."[217]

Frege's critisizes the view above in the following manner: "By it we should never ...arrive at a number like 10,000 for it is beyond our powers to grasp so many differences at once and retain the fact of their existence; while to go through them one after another is not enough, for number would never be complete...Moreover, to tell us how to abstract is not, in any case, to define for us what abstraction is."[218]

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[211] Ibid., p.54
[212] Ibid.
[213] Ibid.
[214] Ibid., p.55
[215] Ibid., p.54
[216] Ibid., p.55
[217] Ibid., p.56
[218] Ibid.

As a solution to the difficulty above, Frege first considers some properties of numbers: "Number is not abstracted from things...nor is it a property of things in the sense that they are...Number is not anything physical, but nor is it anything subjective (an idea)...Number does not result from the annexing of things to thing...The terms 'multitude', 'set' and 'plurality' are unsuitable...for use in defining number."[219]

In *Thoughts,* Frege discusses whether thoughts are ideas or not. As a basis of this discussion, he distinguishes ideas from the things of the outer world in four ways: "First: ideas cannot be seen, or touched, or smelled, or tasted, or heard...Secondly: ideas are something we have...An idea that someone has belongs to the content of his consciousness...Thirdly: ideas need an owner. Things of the outer world are on the contrary independent...Fourthly: every idea has only one owner; no two men have the same idea."[220] Frege puts the question as follows: "Being isolated, being undivided, being incapable of dissection -none of these can serve as a criterion for what we express by the word 'one'."[221]

Frege distinguishes 'one' and 'unit' in order to overcome this difficulty: "The word 'one', as the proper name of an object of mathematical study, does not admit of a plural. Consequently, it is nonsense to make number result from the putting together of ones. The plus symbol in 1+1=2 cannot mean such a thing together."[222]

Frege considers Number "in the context of a judgement which brings out its basic use."[223] According to Frege, "the content of a statement of number is an assertion about a concept. This is perhaps clearest with the number 0."[224] By

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[219]Ibid., p.58
[220]*Collected Papers*, p.361
[221]Ibid.
[222]Op. Cit. pp.58-59
[223]Ibid., p.59
[224]Ibid.

'sentence', Frege means "sentences in which we communicate or assert something"[225] In the assertoric sentence, Frege distinguishes between the content which is the thought or at least contains the thought.[226] In order to avoid a confusion, in a footnote[227] Frege asserts that he uses the word 'thought' "more or less in the same sense 'judgement' has in the writings of logicians."

Frege rejects the view that takes numbers as subjective. If, according to him, "the concepts themselves were subjective, then the subordination of one to the other, being a relation between them, would be subjective too, just as a relation between ideas is."[228]

A thought, being used for 'judgement' as explained above, contains a number-concept in which we find number as an object. Frege basicly distinguishes three things for a thought: "1- The grasp of a thought - thinking, 2- the acknowledgement of the truth of a thought - the act of judgement, 3- the manifestation of this judgement."[229] Frege claims that "the grasp of a thought presupposes someone who grasps it, who thinks. He is the owner of the thinking, not of the thought."[230] Thoughts, according to Frege, are just grasped, not produced by us.[231] And since Frege claims that thoughts stand in the closest connection to truths, and since truths are timeless, as laws of mathematics,[232] thoughts cannot change from time to time nor from person to person. On the other hand, Frege claims that a thought acts by being grasped and taken to be true. But, according to Frege, "when a thought is grasped, it at first only brings about changes in the inner world of the one who grasps it; yet it remains untouched in the core of its essence, for the changes it undergoes affect only

[225] Op. Cit. p.354
[226] Ibid.
[227] Ibid., p.355
[228] *The Foundations of Arithmetic*, p.60
[229] Op. Cit. p.355
[230] Ibid., p.369
[231] Ibid., p.368
[232] Ibid., p.358

inessential properties"[233]

Frege distinguishes between 'being true' from 'being taken to be true' and he considers laws of logic as laws of truth, not laws of takings-to-be-true.[234] Laws of truth, according to Frege, are not psychological laws and "they do not bear the relation to thought that the laws of grammar bear to language; they do not make explicit the nature of our human thinking and change as it changes."[235] Frege claims that this can be explained only by reducing a law of logic to another law of logic.[236]

Since Frege considers thoughts to be timeless, and since actuality necessitates change in time, he considers thoughts as being unactual. But, according to Frege, the term 'actual' has different senses when used for thoughts. So Frege intends to state that thoughts are -in a sense- actual but this actuality is quite different from that of things. It was mentioned above that, for thoughts, Frege claims that their truth does not depend on a thinker and furtherly asserts that "they are not wholly unactual even then, at least if they *could* be grasped and so brought into action."[237] In The Basic Laws of Arithmetic, Frege claims that, "All determinations of the place, the time, and the like, belong to the thought whose truth is in point; its truth itself is independent of place or time."[238]

According to Frege, numbers are mathematical objects, which are not independent from concepts: "If a concept is something objective, an assertion about a concept can have for its part a factual content."[239] Frege explains the function of a concept as follows: "The concept has a power of collecting together far superior to the unifying power of synthetic apperception."[240] Numbers,

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[233]Ibid., p.371
[234]*The Basic Laws of Arithmetic*, p.13
[235]Ibid.
[236]Ibid., p.15
[237]Op. Cit. p.371
[238]Ibid., p.14
[239]*The Foundations of Arithmetic*, p.61
[240]Ibid.

according to Frege, "are assigned only to the concepts, under which are brought both the physical and mental alike, both the spatial and temporal and the non-spatial and non-temporal."[241] At this point, there appears an interesting agreement between Frege and Spinoza; Frege quotes Spinoza: "I answer that a thing is called one or single simply with respect to its existence, and not with respect to its essence; for we only think of things in terms of number after they have first been reduced to a common genus."[242] Frege also mentions that abstraction is not the only way to arrive at a concept: "We can, on the contrary, arrive at a concept equally well by starting from defining characteristics."[243]

On concepts, Frege considers Schröder who claims that "if we are able to speak of the frequency of a thing, the name of the thing concerned must always be a *generic name*, a general concept word or *notio communis*." Frege criticizes this view as follows: "it will not do to call a general concept word the name of a thing. That leads straight to the illusion that the number is a property of a thing. The business of a general concept word is precisely to signify a concept."[244]

On a peculiarity of german language that leads one to confusion about the assertion of numbers, Frege claims that "ordinary language does assign number not to concepts but to objects: we speak of 'the number of the bales' just as we do of 'the weight of the bales'. Thus on the face of it we are talking about objects, whereas really we are intending to assert something of a concept."[245]

By distinguishing the 'number as an object' and 'concept-number', Frege seems to solve the problem of properties asserted of a concept: "By properties which are asserted of a concept I naturally do not mean the characteristics which make up the concept. These latter are properties of the things which fall under

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[241]Ibid., pp.61-62
[242]Ibid., p.62
[243]Ibid.
[244]Ibid., p.63
[245]Ibid., p.64

the concept, not of the concept."[246]

Frege relates this to the controversy of 'existence' and 'number nought': "Existence is analogous to number. Affirmation of existence is in fact nothing but denial of the number nought...It would also be wrong to deny that existence and oneness can ever themselves be component characteristics of a concept...If, for example, we collect under a single concept all concepts under which there falls only one object, then oneness is a component characteristic of this new concept."[247] At this point, Frege distinguishes between a concept's falling under a higher concept and a concept's being subordinate to another concept.

Finally, Frege arrives at the problem of 'unit'. Frege adopts Schröder's definition of unit and quotes from him: "This generic name or concept will be called the denomination of the number formed by the method given, and constitute, in effect, what is meant by its unit."[248] Frege's view on this is that, "for it is the case that the concept, to which the number is assigned, does in general isolate in a definite manner what falls under it."[249] Here Frege thus reconciles the properties of 'identity' and 'distinguishability' for units: "Only a concept which isolates what falls under it in a definite manner, and which does not permit any arbitrary division of it into parts, can be a unit relative to a finite number."[250] Frege argues that the term 'unit' is used in a double sense. The identity of the units is acquired by the meaning of the word 'unit' that is explained above. On the other hand, the 'distinguishability' of units is acquired whenever we mean that the things numbered are distinguishable.[251]

In the article *Function and Concept*, Frege defines concepts in terms of functions

[246]Ibid.
[247]Ibid., p.65
[248]Ibid., pp.65-66
[249]Ibid., p.66
[250]Ibid.
[251]Ibid., pp.66-67

and argues that functions are his starting-point.[252] Frege critizes mathematical works, especially formal theories of his time for their inadequacy in ontological background. These theories, according to Frege, do not distinguish between form and content, sign and the thing signified. Difference of sign, Frege claims, cannot be a sufficient ground for difference of the thing signified.[253] Frege makes a distinction between 'numerals' and 'numbers' in the same way: "The characters we call numerals...have physical and chemical properties depending on the writing material."[254] Numbers, on the other hand, do not have these properties.

Frege considers equality as a relation that holds between signs of objects, and not between the objects themselves. According to Frege, a=a and a=b are two different thoughts but they designate the same thing, i.e. their meanings are the same: "If we were to regard equality as a relation between that which the names 'a' and 'b' designate, it would seem that a=b could not differ from a=a (i.e. provided a=b is true.)"[255]

By considering the difficulties between the sign and the thing signified (or what the sign designates), Frege entitles the latter as "the *sense* of the sign, wherein the mode of presentation is contained."[256] According to Frege, the designation of a single object can consist of several words. He calls every such designation a proper name.[257]

According to Frege, there exists a relation between the sign, its sense, and what it means; he explains this relation as follows: "To the sign there corresponds a definite sense and to that in turn a definite thing meant, while to a given thing meant (an object) there does not belong only one sign. The same sense has

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[252]Gottlob Frege, "Function and Concept", *Collected Papers*, p.137
[253]Ibid., p.138
[254]Ibid., p.139
[255]Gottlob Frege, "On Sense and Meaning", *Collected Papers*, p.157
[256]Ibid., pp.158
[257]Ibid.

different expressions in different languages or even in the same language."[258]

In Begriffsschrift, Frege distinguishes two kinds of signs: "those by which we may understand different objects and those that have a completely determinate meaning."[259] Letters like a,b,c are of the first kind and +,-,0,1,2 are of the second.

Frege distinguishes between function and a mathematical expression. A mathematical expression, according to him, *indicates* a number; not that itself is a number: "If a function were really the meaning of a mathematical expression, it would just be a number; and nothing new would have been gained for arithmetic by speaking of functions."[260]

Frege introduces his concept of 'argument' and distinguishes between functions and numbers. He asserts that "I am concerned to show that the argument does not belong with a function, but goes together with the function to make up a complete whole; for a function by itself must be called incomplete, in need of supplementation, or 'unsaturated'. And in this respect functions differ fundementally from numbers."[261] Frege considers 'the sign of the argument' and 'the expression of the function' as the two parts into which a mathematical expression is split up.[262]

Frege defines 'the value of a function for an argument' as the result of completing the function with the argument.[263] According to Frege, the equality between the values of functions must be considered as the equality between the value ranges of functions; a rule, that must be taken as a fundemental law of logic.[264]

Frege replaces the traditional dichotomy of *subject-predicate* by his *function-*

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[258]Ibid., p.159
[259]Gottlob Frege, *Begriffsschrift*, p.11
[260]Op. Cit. p.140
[261]Ibid.
[262]Ibid., p.141
[263]Ibid.
[264]Ibid., p.142

argument. His reason for this is mainly to avoid the ambiguities arising from the ordinary language. For example, Frege converts the proposition 'Archimedes perished at the capture of Syracuse' into 'The violent death of Archimedes at the capture of Syracuse is a fact.'[265] Here, Frege claims, "the subject contains the whole content, and the predicate serves only to turn the content into a judgement. Such a language would have only a single predicate for all judgements, namely, 'is a fact'."[266]

According to Frege, an expression can be divided into two as a stable component, representing the totality of relations, and the sign, regarded as replaceable by others, that denotes the object standing in these relations. Frege calls the former 'function' and the latter 'argument' of the expression[267]:"If an expression, whose content need not be capable of becoming a judgement, a simple or a compound sign has one or more occurences and if we regard that sign as replaceable in all or some of these occurences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a function, and the replaceable part the argument of the function."[268]

Another basic distinction that Frege introduces is between 'thoughts' and 'meanings'. According to him, the meaning of two mathematical expressions can be the same but from this it does not follow that they express the same thought.[269]

We mentioned above that Frege defines concepts by means of functions. He asserts that "a concept is a function whose value is always a truth-value...We can designate as an extension the value-range of a function whose value for every argument is a truth value."[270]

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[265]Op. Cit. p.12
[266]Ibid.
[267]Ibid., p.22
[268]Ibid.
[269]Gottlob Frege, "Function and Concept", *Collected papers*, p.145
[270]Ibid., p.146

Frege holds the view that 'identity' is a relation between extensions of the concepts, and not between the concepts themselves. "The linguistic form of equations is a statement. A statement contains a thought as its sense; and this thought is in general true or false; i.e. it has in general a truth-value, which must be regarded as what the sense means."[271] Frege, as he does for expressions, divides statements into two parts: a complete (or saturated) and an incomplete (or unsaturated) part.

According to Frege, not only the numbers, but all the objects can be considered as arguments: "An object is anything that is not a function, so that an expression for it does not contain any empty place...What a statement means is a truth-value. Thus the two truth-values are objects."[272] Frege considers the value-ranges of functions, but not the functions themselves as objects.[273] Likewise, he considers the extensions of concepts as objects, not the concepts themselves.[274]

In defining function, Frege introduces 'variable' as a concept that must be questioned. If variables are objects, then the variables of arithmetic must be numbers. According to Frege, the attempt to define a variable as a magnitude fails. For magnitudes are applied to physical objects and arithmetic has nothing to do with physical objects either.[275]

Frege hence investigates whether the variables of Analysis are *variable* numbers or not. Frege asserts that any variation takes place in time and Analysis has nothing to do with temporal processes.[276] Moreover, Frege claims that nothing would remain the same if a number varies. Thus, according to him, a number does not vary and we have no proper names for variable numbers. Frege does not consider letters like 'x','y','z' as proper names for variable

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[271]Ibid.
[272]Ibid., p.147
[273]Ibid.
[274]Ibid., p.148
[275]Gottlob Frege, "What is a Function", *Collected Works*, p.285
[276]Ibid., p.286

numbers in the way that '2' and '3' are proper names for constant numbers. The difference between the numbers that are designated by 'x' and 'y' cannot be seen. Thus, according to Frege, "since we cannot conceive of each variable as an individual, we cannot attach any proper names to variables."[277]

Frege considers two of E. Czuber's arguments which are attempts to solve the difficulties mentioned above and then states his own criticisms of them. Czuber's first point is about eliminating time by defining a variable as an indefinite number.[278] Frege rejects this view by claiming that there are no indefinite numbers. Taking 'n' as an example, Frege claims that " 'n' is not the proper name of any number, definite or indefinite."[279] When the expression 'number n' is used, according to Frege, it is used in a context. He also claims that the letter 'n' indicates generality, i.e. it must be replaced by the name of a number. This generality can be considered as 'indefiniteness'. But, according to Frege, "here the word 'indefinite' is not an adjective of 'number', but 'indefinitely' as an adverb, e.g. of the verb 'to indicate'. We cannot say that 'n' designates an indefinite number, but we *can* say that it indicates numbers indefinitely." By this point, Frege concludes that there are no indefinite numbers and thus Czuber's attempt fails.

Czuber's second point is to avoid the difficulty that "we cannot conceive of any variable so as to distinguish it from others." His suggestion is to call the "totality of the values that a variable may assume, the range of the variable."[280] Frege quotes from Czuber: "The variable x counts as having been defined when it can be determined as regards any assigned real number whether it belongs to the range or not."[281] Frege refutes this argument by claiming that "since there are

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[277]Ibid., p.287
[278]Ibid.
[279]Ibid.
[280]Ibid., p.288
[281]Ibid.

no indefinite numbers, it is impossible to define any indefinite number."[282] Moreover, Frege claims that "the expression 'a variable assumes a value' is completely obscure."[283] Frege asserts that in this expression the number plays two roles: an object, as it is called a variable, and a property, as it is called a value.[284]

From variables to arrive at functions, Frege states Czuber's view and attempts to refute it; he quotes from Czuber: "If every value of the real variable x that belongs to its range has correlated with it a definite number y, then in general y also is defined as a variable, and is called a function of the real variable x. This relation is expressed by an equation of the form y=f(x)"[285] Frege claims that the delimitation of a function does not tell what it is and claims that "y is called a definite number, whereas on the other hand, being a variable, it would have to be an indefinite number. y is neither definite nor an indefinite number; but the sign 'y' is attached incorrectly to a plurality of numbers, and then afterwards he talks as if there were only a single number."[286]

Frege further claims that without mentioning the law of 'correlation', from variables one cannot arrive at functions. Frege asserts that the "distinctions between laws of correlation will go along with distinctions between functions; and these cannot any longer be regarded as quantitive...Our general way of expressing such a law of correlation is equation...Functions have indeed be defined as being such mathematical expressions."[287]

Frege distinguishes functions from numbers by using one of his fundemental dichotomies: saturatedness/unsaturatedness: "If a function is completed by a number so as to yield a number, the second is called the value of the function for

[282]Ibid.
[283]Ibid.
[284]Ibid.
[285]Ibid., pp.288-289
[286]Ibid., p.289
[287]Ibid., pp.289-290

the first as argument."[288]

Frege investigates the concept of number in view of two properties: 'self-subsistence' and 'numerical identity'. By considering the first one, Frege claims that "every individual number is a self-subsistent object."[289]

Along this line, Frege attempts to complete Leibniz's definition of numbers beginning with '0','1', and 'n+1'. Then he argues that these suggested definitions are not sufficient. For numbers '1' and '0' Frege suggests the following: "the number 0 belongs to a concept, if the proposition that a does not fall under that concept is true universally, whatever a may be. Similarly we could say: the number 1 belongs to a concept F, if the proposition that a does not fall under F is not true universally, whatever a may be, and if from the propositions 'a falls under F' and 'b falls under F' it follows universally that a and b are the same."[290]

For the succession of numbers, Frege suggests the following: "the number (n+1) belongs to a concept F, if there is an object a falling under F and such that the number n belongs to the concept 'falling under F, but not a'."[291]

Frege gives two main reasons to show the insufficiency of these suggestions. First, he asserts that "strictly speaking we do not know the sense of the expression 'the number n belongs to the concept G' any more than we do that of the expression 'the number (n+1) belongs to the concept F'."[292] The second one concerns the 'unprovability' of the expression "the number which belongs to the concept F"[293] It follows, according to Frege, that it is not possible to prove a numerical identity for these definitions do not enable us to achieve a determinate number.[294]

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[288] Ibid., p.292
[289] *The Foundations of Arithmetic*, p.67
[290] Ibid.
[291] Ibid.
[292] Ibid., p.68
[293] Ibid.
[294] Ibid.

According to Frege, "the content of a statement of number is an assertion about a concept."[295] Frege does not consider numbers as the properties of the concepts to which they belong. "Precisely because it forms only an element in what is asserted, the individual number shows itself for what it is, a self-subsistent object."[296]

Frege relates this to his suggestion of converting the propositions. As an example, he gives the following: " 'Jupiter has four moons' can be converted into 'the number of Jupiter's moons is four'...Here 'is' has the sense of 'is identical with' or 'is the same as'...Identities are, of all forms of proposition, the most typical of arithmetic."[297]

According to Frege, one might object to this view by arguing that the idea of a number cannot be formed. Frege admits this criticism but he asserts that "that is not the fault of the self-subsistence we have ascribed to the number."[298] Frege claims that "we can form no idea of the number either as a self-subsistent object or as a property in an external thing, because it is not in fact either anything sensible or a property of an external thing. But the point is clearest in the case of the number 0."[299]

Frege argues against the view that thoughts are ideas. He relates ideas to the content of consciousness' and claims that "the words 'true' and 'false'... also be applicable only in the realm of my consciousness, if they were not meant to apply to something of which I was not the owner, but to characterize in some way the content of my consciousness. Truth then would be confined to this content and it would remain doubtful whether anything at all similar occured in the consciousness of others...If someone takes thoughts to be ideas, what he then

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[295]Ibid.
[296]Ibid.
[297]Ibid., p.69
[298]Ibid.
[299]Ibid., p.70

accepts as true is, on his view, the content of consciousness, and does not properly concern other people at all...Thoughts are neither things in the external world nor ideas."[300]

According to Frege, the meanings of the words should be considered in the context of propositions. This can help us to avoid subjectivism. "Only in a proposition have the words really a meaning. It may be that mental pictures float before us all the while, but these need not correspond to the logical elements in the judgement. It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts also their content...The self-subsistence which I am claiming for number is not to be taken to mean that a number word signifies something when removed from the context of a proposition, but only to preclude the use of such words as predicates or attributes, which appreciably alters their meaning"[301]

Frege makes a distinction between 'spatial' and 'non-spatial' and claims that we do not have to locate all the objects to one part of this dichotomy. Numbers, according to Frege, as arithmetical objects, are "neither outside us nor within us."[302] Frege asserts that, being spatial is not an indispensable condition of being an object. "4 is not a spatial object, not that it is not an object at all...Not every objective object has a place."[303]

The second concept, besides 'self-subsistence' is 'numerical identity' Frege starts by attempting to find a criterion for this concept. We mentioned above that according to Frege, the sense of a word should be inquired in the context of a proposition only. Frege relates this view to numerical identity by posing his problem as "to define the sense of a proposition in which a number word

[300]Op. Cit. p.363
[301]Op. Cit. pp.71-72
[302]Ibid., p.72
[303]Ibid.

occurs."[304]

Frege explains the necessity of introducing his concept of "identity" as to recognize a number as the same again: "If we are to use the symbol a to signify an object, we must have a criterion for deciding in all cases whether b is the same as a, even if it is not always in our power to apply this criterion."[305] According to Frege, in order to realize this, the sense of the following proposition has to be defined: "the number which belongs to the concept F is the same as that which belongs to the concept G."[306]

At this point, Frege introduces three possible objections which he calls "doubts". The first one considers the view which takes numerical identity as one-one correlation. Frege claims that "It is not only among numbers that the relationship of identity is found."[307] Thus, according to him, the content of a judgement has to be constructed which can be taken as an identity such that each side of it is a number. For this, Frege claims, the concept of identity has to be fixed initially.

The second doubt is concerned with Frege's suggestion for fixing the concept of identity. Here, Frege introduces one of his most important concepts: 'carving up the content'. Frege uses two analogies from geometry for defining the concept of identity. The first one considers the concept of direction. To obtain the concept of direction he takes the judgement "line a is parallel to line b" which can be symbolized as a//b. Then it can be said that "the direction of line a is identical with the direction of line b".[308] Here, the symbol '// ' is replaced by =, "through removing what is specific in the content of the former and dividing it between a and b. We carve up the content in a way different from the original way, and this yields us a new concept."[309] Frege distinguishes between two concepts, those

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[304]Ibid., p.73
[305]Ibid.
[306]Ibid.
[307]Ibid., p.74
[308]Ibid.
[309]Ibid., p.75

of a 'straight line' and its 'direction' and asserts that "the concept of direction is only discovered at all as a result of a process of intellectual activity which takes its start from the intuition. On the other hand, we do have an idea of parallel straight lines."[310]

The second analogy that Frege borrows from geometry is the similarity of triangles: "From geometrical similarity is derived he concept of shape, so that instead of 'the two triangles are similar' we say 'the two triangles are of identical shape' or 'the shape of the one is identical with that of the other'." [311] In both examples, as the result of carving up the content of an initial concept, a new concept is introduced. In the case of lines, 'direction', and in the case of the triangles 'shape'.

Frege's second doubt concerns the possibility of substituting a concept for another in an identity; in view of such a substitution, he states Leibniz's definition, that is "things are the same as each other, of which one can be substituted for the other without loss of truth" and proposes to use 'identical' for the term 'same'.[312] According to Frege's definition, it must be possible to substitute "the direction of b" everywhere for "the direction of a" if line a is parallel to line b.[313]

In Begriffsschrift, Frege distinguishes identity of content from conditionality and negation for it applies to names and not to contents: "Whereas in other contexts signs are merely representatives of their content, so that every combination into which they enter expresses only a relation between their respective contents, they suddenly display their own selves when they are combined by means of the sign for identity of content; for it expresses the circumstance that two names have the same content. Hence the introduction of a sign for identity of content necessary produces a bifurcation in the meaning of all

[310] Ibid.
[311] Ibid.
[312] Ibid., p.76
[313] Ibid., p.77

signs: they stand at times for their content, at times for themselves...The need for a sign for identity of content rests upon the following consideration: the same content can be completely determined in different ways; but that in a particular case *two ways of determining it* really yield the *same result* is the content of the *judgement*."[314]

Frege we mentioned above, distinguishes between the meaning and the sense of a sign. He also distinguishes these entities from the 'idea': "The same sense is not always connected, even in the same man, with the same idea. The idea is subjective: one man's idea is not that of another."[315] On the other hand, the meaning is objective, i.e. it does not differ from one person to another.

Against a subjectivist approach, Frege secures the existence of the 'self ' and claims that "I am not my own idea; and when I assert something about myself, e.g. that I am not feeling any pain at the moment, then my judgement concerns something which is not a content of my consciousness, is not my idea, namely myself."[316] Frege states a possible objection by considering that the word 'I' is something in the content of one's consciousness. Frege's view against this objection, however is that: "I have an idea of myself, but I am not identical with this idea. What is a content of my consciousness, my idea, should be sharply distinguished from what is an object of my thought. Therefore the thesis that only what belongs to the content of my consciousness can be the object of my awareness, of my thought, is false."[317]

Frege's third doubt concerns the possibility that the "criterion of identity" may fail to cover all the cases. This problem, according to Frege, originates from not specifying -in the case of the lines- the concept of 'direction'. Frege asserts that, "in the proposition 'the direction of a is identical with the direction of b' the

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[314] *Begriffsschrift*, pp.20-21
[315] *Collected Papers*, p.160
[316] Ibid., p.366
[317] *The Foundations of Arithmetic*, p.77

direction of a plays the part of an object"[318] This, according to Frege, tells nothing about the proposition " 'the direction of a is identical with q' should be affirmed or denied, except for the one case where q is given in the form of 'the direction of b'. What we lack is the concept of direction."[319] To overcome this difficulty, Frege makes a possible suggestion and shows a gap: "the temptation is to give as our definition: q is a direction, if there is a line b whose direction is q. But then we have obviously come round a circle. For in order to make use of this definition, we should have to know already in every case whether the proposition 'q is identical with the direction of b' was to be affirmed or denied."[320]

According to Frege, the concept of identity cannot be supplemented by taking, as a defining characteristic of a concept, the way in which an object is introduced. His reason for this as follows: "The definition of an object does not, as such, really assert anything about the object, but only lays down the meaning of a symbol. After this has been done, the definition transforms itself into a judgement, which does assert about the object; but now it no longer introduces the object, it is exactly on a level with other assertions made about it."[321]

Frege suggests to define number as the extension of a concept. According to him, identity can be asserted for the extensions of the concepts, and not for the concepts themselves. What can be asserted for the concepts is their equality. In view of the analogies taken from geometry, Frege suggests the following: "the direction of line a is the extension of the concept 'parallel to line a'; the shape of triangle t is the extension of the concept 'similar to triangle t' "[322] And for the case of numbers Frege's definition is as follows: "the Number which belongs to the concept F is the extension of the concept 'equal to the concept F'."[323]

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[318]Ibid.
[319]Ibid., p.78
[320]Ibid.
[321]Ibid.
[322]Ibid., p.79
[323]Ibid., pp.79-80

Frege makes a distinction between numbers and their extensions; he asserts that: "the extension of the concept 'equal to the concept F' is identical with the extension of the concept 'equal to the concept G' is true if and only if the proposition 'the same number belongs to the concept F as to the concept G' is also true...Certainly we do not say that one number is wider than another, in the sense in which the extension of one concept is wider than that of another; but then it is also quite impossible for a case to occur where the extension of the concept 'equal to the concept F' would be wider than the extension of the concept 'equal to the concept G'. For on the contrary, when all concepts equal to G are also equal to F, then conversely also all concepts equal to F are equal to G. 'Wider' as used here must not, of course, be confused with 'greater' as used of numbers."[324]

In order to prove his definition of number, Frege attempts to derive the properties of individual numbers from his concept of Number. For this, in the first place, Frege gives a more precise account of the term 'equality' which he defines in terms of 'one-one correlation'.[325] One-one correlation, according to Frege, is a relation concept. Frege's suggestion for acquiring this relation concept is to abstract the objects from the judgement-content: "If from a judgement-content which deals with an object a and an object b we subtract a and b, we obtain as remainder a relation-concept which is, accordingly, incomplete at two points."[326] Frege defines two types of relations: one holds between an object and a concept and the other holds between the correlating objects and the relation concept.

Frege restricts the use of the term 'concept' to the sphere of pure logic and claims that there is a clear-cut distinction between the concept and the object: "A concept is predicative. On the other hand, name of an object, a proper name, is

[324]Ibid., pp.80-81
[325]Ibid., p.81
[326]Ibid., p.82

quite incapable of being used as a grammatical predicate."[327] We mentioned above that Frege defines the relation between concept and object by means of the term 'to fall under', which, according to him, is an irreversible relation. And a grammatical predicate, according to him, *means* this concept.[328] According to Frege, a concept cannot be a grammatical subject because of its predicative nature. In order to play this role, this concept must first be converted to an object.[329]

In The Foundations of Arithmetic, Frege states the principle that we should "never to lose sight of the distinction between concept and object."[330] And related to this principle, he asserts the following: "As to the third point, /the mentioned principle/ it is a mere illusion to suppose that a concept can be made an object without altering it."[331] According to Frege, the fundamental difference between an object and a concept is that "an object can never occur predicatively or unsaturatedly."[332] Hence, it appears, Frege's concept/object distinction, in view of this analysis, is a logical distinction; one cannot substitute an object and a concept into each other's logical place without altering them.[333]

Frege calls the 'correlated objects' as the 'subject' of the relation concept. At this point, Frege argues in terms of pure logic: "the doctrine of relation concepts is thus, like that of simple concepts, a part of pure logic. What is of concern to logic is not the special content of any particular relation, but only the logical form. And whatever can be asserted of this, is true analytically and known a priori."[334]

Frege defines the concept of 'correlation' as follows: "If now every object which falls under the concept F stands in the relation ϕ to an object falling under the

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[327] Op. Cit. p.183
[328] Ibid.
[329] Ibid., p.186
[330] Op. Cit. p.x
[331] Ibid.
[332] *Collected Papers*, p.281
[333] Ibid., p.283
[334] Op. Cit. p.83

concept G, and if to every object which falls under G there stands in the relation ϕ an object falling under F, then the objects falling under F and under G are correlated with each other by the relation ϕ "[335] By using the definition of 'correlation', Frege attempts to explain the meaning of the expression " 'every object which falls under F stands in the relation ϕ to an object falling under G' in the case where no object at all falls under F."[336] For this, in the first place, Frege considers two propositions: "*a* falls under F" and "*a* does not stand in the relation ϕ to any object falling under G"; p and q respectively. According to Frege, (p and q) involves a contradiction, whatever signified by *a* .[337] From this, he infers that "the proposition 'every object which falls under F stands in the relation ϕ to an object falling under G' is, in the case where there is no object falling under F, true; for in that case the first proposition '*a* falls under F' is always false, whatever *a* may be. In the same way the proposition 'to every object which falls under G there stands in the relation ϕ an object falling under F' means that the two propositions '*a* falls under G' and 'no object falling under F stands to *a* in the relation ϕ' cannot, whatever *a* may be, both be true together."[338]

After defining 'correlation' and specifying ϕ as the relation which correlates the concepts F and G, Frege defines the concept of Number through one-one relations. Frege first considers the following two propositions:

"1. If d stands in the relation ϕ to a, and if d stands in the relation ϕ to e, then generally, whatever d, a, and e may be, a is the same as e.

2. If d stands in the relation ϕ to a, and if b stands in the relation ϕ to a, then generally, whatever d, b and a may be, d is the same as b."[339]

Frege claims that if the two propositions above both hold true, then "this reduces

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[335]Ibid.
[336]Ibid.
[337]Ibid., p.84
[338]Ibid.
[339]Ibid.

one-one correlation to purely logical relationships, and enables us to give the following definition: the expression 'the concept F is equal to the concept G' is to mean the same as the expression 'there exists a relation ϕ which correlates one to one the objects falling under the concept F with the objects falling under the concept G' ."[340]

By using these definitions, Frege defines the relation between the number as object and the concept of Number by claiming that "the expression '*n* is a Number' is to mean the same as the expression 'there exists a concept such that *n* is the Number which belongs to it' as an addition to his original definition that "the Number which belongs to the concept F is the extension of the concept 'equal to the concept F'. "[341]

Frege argues that 'identity' holds between numbers and 'equality' holds between concepts: "the Number which belongs to the concept F is identical with the Number which belongs to the concept G if the concept F is equal to the concept G."[342] According to Frege, in the above definition of 'the Number which belongs to the concept F' it has to be shown that " the extension of the concept 'equal to the concept F' is the same as the extension of the concept 'equal to the concept G', if the concept F is equal to the concept G."[343] In order to show this, Frege asserts that "it is to be proved that, for F equal to G, the following two propositions hold good universally: 'if the concept H is equal to the concept F, then it is also equal to the concept G' and 'if the concept H is equal to the concept G, then it is also equal to the concept F."[344]

Frege's proof for the first proposition is as follows: "there exists a relation which corelates one to one the objects falling under the concept H with those faling

340Ibid., pp.84-85
341Ibid., p.85
342Ibid.
343Ibid.
344Ibid., pp.85-86

under the concept G, if there exists a relation ϕ which correlates one to one the objects falling under the concept F with those falling under the concept G and if there exists also a relation φ which correlates one to one the objects falling under the concept H with those falling under the concept F."[345] This, according to Frege, can be symbolized as follows: H φ F ϕ G. Frege claims that this relation can be found in the judgement content "there exists an object to which c stands in the relation φ and which stands to b in the relation ϕ".[346] As mentioned above, if c and b are subtracted, we acquire the relation concept. Frege claims that the proof of the second one can be given in the same way.

By using the definitions above, Frege gives the definitions of the individual number nought and the number 1 which, in his view, are indispensable for giving the definition of the concept of Number.

Frege defines number nought as follows: "0 is the Number which belongs to the concept 'not identical with itself '." He explains his reasons for using the concept of 'not identical with itself ' as follows: "On my use of the word 'concept', 'a falls under the concept F' is the general form of a judgement-content which deals with an object a and permits of the insertion for a of anything whatever. And in this sense 'a falls under the concept 'not identical with itself ' ' has the same meaning as 'a is not identical with itself ' or 'a is not identical with a'. "[347]

According to Frege, nothing falls under the concept which the Nought falls. In terms of 'equality', Frege explains this as follows: "Every concept under which no object falls is equal to every other concept under which no object falls, and to them alone; from which it follows that 0 is the Number which belongs to any such concept, and that no object falls under any concept if the number which belongs to that concept is 0."[348]

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[345] Ibid., p.86
[346] Ibid.
[347] Ibid., pp.87-88
[348] Ibid., p.88

Frege proves this proposition through his definition of relation ϕ: "Every object which falls under F stands in the relation ϕ to an object which falls under G; and to every object which falls under G there stands in the relation ϕ an object falling under F."[349]

Frege also considers two concepts such that some object falls under one of them while nothing falls under the other: " 'a falls under G' and 'no object falling under F stands to a in the relation ϕ ' are both true together for every relation ϕ ...If, that is, there exists no object falling under F, then a fortiori there exists no object falling under F which stands to a in any relation whatsoever. There exists, therefore, no relation by which the objects falling under F can be correlated with those falling under G so as to satisfy our definition, and accordingly the concepts F and G are unequal."[350]

Frege defines the expression "n follows in the series of natural numbers directly after m" by the following proposition: "there exists a concept F, and an object falling under it *x*, such that the Number which belongs to the concept F is *n* and the Number which belongs to the concept 'falling under F but not identical with *x*' is *m*"[351] According to Frege, the meaning of this proposition is the same as that of the following: "*n* follows in the series of natural numbers directly after *m*."[352]

After proving the definition of the number Nought and defining 'n follows in the series of natural numbers directly after m', Frege attempts to define number 1. We should note that Frege formerly defined the concept 'identical with 0' as a concept that number 0 - number as an object - falls under. But no object falls under the concept 'identical with 0 but not identical with 0'. This concept, according to Frege, is the concept that the Number 0 - number as a concept - belongs to. Therefore, Frege claims, the following propositions hold true: "the

349Ibid.
350Ibid.
351Ibid.
352Ibid.

Number which belongs to the concept 'identical with 0' is identical with the Number which belongs to the concept 'identical with 0' " and "the Number which belongs to the concept 'identical with 0 but not identical with 0' is 0. "[353] Frege concludes the proof as follows: "Therefore, on our definition /of succesivity/, the Number which belongs to the concept 'identical with 0' follows in the series of natural numbers directly after 0 ... 1 is the Number which belongs to the concept 'identical with 0' ... Thus 1 follows in the series of natural numbers directly after 0."[354]

Frege furthermore states the folowing six propositions, which, according to him, can be proved by means of the definitions mentioned above:

" 1. If a follows in the series of natural numbers directly after 0, then a is = 1.

2. If 1 is the Number which belongs to a concept, then there exists an object which falls under that concept.

3. If 1 is the Number which belongs to a concept F; then, if the object x falls under the concept F and if y falls under the concept F, x=y; that is, x is the same as y.

4. If an object falls under the concept F, and if it can be inferred generally from the propositions that x falls under the concept F and that y falls under the concept F that x = y, then 1 is the Number which belongs to the concept F.

5. The relation of m to n which is established by the proposition: 'n follows in the series of natural numbers directly after m' is a one-one relation.

6. Every number except 0 follows in the series of natural numbers directly after a Number."[355]

In order to prove the last proposition, by which the concept of 'succession' is defined, Frege claims that, a concept must be produced to which this latter

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[353]Ibid., p.90
[354]Ibid.
[355]Ibid., pp.91-92

Number belongs. For this, he chooses the following concept: "member of the series of natural numbers ending with n."[356] The proof is as follows: "The proposition 'if every object to which x stands in the relation ϕ falls under the concept F, and if from the proposition that d falls under F the concept F it follows universally, whatever d may be, that every object to which d stands in the relation ϕ falls under the concept F, then y falls under the concept F, whatever concept F may be' is to mean the same as 'y follows in the ϕ- series after x' and again the same as 'x comes in the ϕ-series before y.' [357]

Frege claims that the 'following of numbers' in the series of natural numbers is objective. Thus, according to him, "the series is not necessarily to be conceived in the form of a spatial and temporal arrangement".[358] Frege does not admit the view that makes such a following dependent to someone's attention that is transferred from one object to another to which it stands in the relation ϕ since it involves subjectivity. According to Frege, this view "describes a way of discovering that y follows, it does not define what is meant by y's following... Frege furthermore claims that "one proposition follows from certain others is something objective, something independent of the laws that govern the movements of our attentions."[359] And: "we have no need always to run through all the members of a series intervening between the first member and some given object, in order to ascertain that the latter does follow after the former. Given, for example, that in the f-series b follows after a and c after b, then we can deduce from our definition that c follows after a, without even knowing the intervening members of the series."[360]

Frege explains the ϕ relation by the expression "n follows in the series of natural

[356] Ibid., p.92
[357] Ibid.
[358] Ibid.
[359] Ibid., p.93
[360] Ibid.

numbers directly after m".[361] This expression, according to Frege, enables us to substitute φ-series with "series of natural numbers".[362] Frege considers a natural number as a 'serie' in the φ-series and adds the following propositions to his definition of 'following in the series of natural numbers': " The proposition 'y follows in the f-series after x or y is the same as x' is to mean the same as 'y is a member of the φ-series beginning with x' and again the same as 'x is a member of the f-series ending with y'. It follows that a is a member of the series of natural numbers ending with n, if n either folows in the series of natural numbers after a or is identical with a."[363]

After giving the definition of 'following in the series of natural numbers', Frege claims that he has also proved that there exists no last member of this series.[364] Frege states the outline of his proof that there is no last member of the series of natural numbers as follows: "If a follows in the series of natural numbers directly after d, and if it is true of d that the number which belongs to the concept 'member of the series of natural numbers ending with d' follows in the series of natural numbers directly after d, then it is also true of a that: the Number which belongs to the concept 'member of the series of natural numbers ending with a' follows in the series of natural numbers directly after a. It is then to be proved, secondly, that what is asserted of d and of a in the propositions just stated holds for the number 0. And finally it is to be deduced that it also holds for n if n is a member of the series of natural numbers beginning with 0"[365]

Frege concludes by giving the definition of finite number and the proof of the proposition that 'no finite Number follows in the series of natural numbers after itself'.

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[361]Ibid., p.94
[362]Ibid.
[363]Ibid.
[364]Ibid.
[365]Ibid.

In *Thoughts*, Frege asserts that "what is improperly called the truth of pictures and ideas is reduced to the truth of sentences."[366] Here Frege introduces his concept of 'sense' and assigns 'truth' to the sense of the sentences: "when we call a sentence true we really mean that its sense is true." [367] According to Frege, the question of 'truth' arises in the context of a thought only. He furthermore claims that "thoughts are sense of sentences, without wishing to assert that the sense of every sentence is a thought...We say a sentence *expresses* a thought...A thought is something imperceptible: anything the senses can perceive is excluded from the realm of things for which the question of truth arises. Truth is not a quality that answers to a particular kind of sense-impressions."[368]

According to Frege, the need for meaning originates mostly because we are concerned with the truth-values.[369] By the truth-value of a sentence Frege means "the circumstance that it is true or false...Every assertoric sentence concerned with what its words mean is therefore to be regarded as a proper name, or its meaning."[370]

Judgement, Frege claims, is a medium in order to advance from sense to meaning. Frege rejects the view which asserts that the relation of the thought to the True must be regarded as that of subject to predicate instead of that of sense to meaning. Accordingly, "subject and predicate are just elements of thought; they stand on the same level for knowledge. By combining subject and predicate, one reaches only a thought, never passes from sense to meaning, never from a thought to its truth-value."[371] On the other hand, it is not possible only to concern with the meaning of the sentence. But a thought, by itself, does not yield any

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[366]Gottlob Frege, "On Sense and Meaning", *Collected Papers*, p.353
[367]Ibid.
[368]Ibid., p.354
[369]Ibid., p.163
[370]Ibid.
[371]Ibid., p.164

knowledge.[372]

Frege, on the other hand, asserts that "the being of a thought does not consist in its being true."[373] Frege argues against the view that false thoughts are thoughts that has no being and distinguishes between the senselessness of a sentence and the falsity of the thought. Frege claims that "A false thought must be admitted, not indeed as true, but as sometimes indispensable: first, as the sense of an interrogative sentence; secondly, as part of a hypothetical thought complex; thirdly, in negation."[374]

Frege considers axioms as thoughts. The truth of axioms, according to him, is provable by a chain of logical inferences like the laws of logic.[375] Definitions, according to Frege, are mathematical propositions which have no meaning. Axioms, on the other hand, like theorems, "must contain no proper name, no concept-word, no relation-word, no function-sign whose meaning has not previously been established."[376] Definitions can be used in constructing proofs by forming self-evident propositions if we give a word meaning by their means. Principles are also used in this way. Frege, however, does not count definitions among principles.[377] Definitions, according to Frege, "are arbitrary stipulations and thus differ from all assertoric propositions...By defining, no knowledge is engendered; and thus one can only say that definitions that have been altered into assertoric propositions formally play the role of principles but really are not principles at all...Never may something be represented as a definition if it requires proof or intuition to establish its truth. On the other hand, one can never expect principles or theorems to settle the reference of a word or sign...Axioms do not contradict one another, since they are true; this does not stand in need of

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[372]Ibid.
[373]Ibid., p.145
[374]Ibid., p.147
[375]Gottlob Frege, "On the Foundations of Geometry: First Series", *Collected Papers*, p.273
[376]Ibid., p.274
[377]Ibid.

proof. Definitions must not contradict each other."[378]

Frege admits the existence of the primitive elements. But, according to him, definitions cannot be counted among them although they are constructed out of these primitive elements.[379] Frege claims that a definition itself is a constituent of the system of a science. [380] "The mental activities leading to the formulation of a definition may be of two kinds: analytic or synthetic...But the mental work preceding the formulation of a definition does not appear in the systematic structure of mathematics; only its result, the definition does. Thus it is all the same for the system of mathematics, whether the preceding activity was of an analytic or a synthetic kind...Therefore so far as the system is concerned, every definition is the giving of a name, regardless of the manner in which we arrived at it. It is self-evident that what is given a name (sign) must be determined by the definition. A word without a determinate meaning has no meaning so far as mathematics is concerned."[381]

According to Frege, even though it can be accepted that the mere definition of an object is not sufficient to tell whether that object falls under the defined concept or not, the complete knowledge of the object together with the definition must suffice.[382] This knowledge, according to Frege, is expressed by means of propositions.[383]

As an example of this in the number theory, Frege gives the 'Gaussian definition of the number-congruence'. Here, the word 'congruent', according to him, is reduced to the expressions 'different' and 'a number divides evenly into number'.[384] "If we posit the Gaussian definition of the number-congruence, then

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[378]Ibid., pp.274-275
[379]Gottlob Frege, "On the Foundations of Geometry: Second Series", *Collected Papers*, p.302
[380]Ibid.
[381]Ibid., p.302-303
[382]Ibid., p.304
[383]Ibid., p.305
[384]Ibid., p.304

in order to recognize that 2 is congruent to 8 modulo 3, we need only the propositions '8-2=3+3' and '3 goes evenly into 3+3' which neither contain the sign for congruence nor presuppose knowledge of it. We saw that a definition which is to assign a meaning to a word must determine this meaning."[385]

On the 'ambigious' signs (like a,b,c) that occur in the propositions, Frege claims that it is not right to think that two propositions like '(a+b) x c=axc+ bxc' and '(2+3) x 7=2x7+3x7'. The difference only lies in the fact that the former proposition is a general one.[386] If, according to Frege, the proper names and the concept-words are sharply distinguished, then the ambiguity does not arise at all.[387] In a proposition like 'x > 0', the letter 'x' is designated here as a concept, that is, positive number.[388] Frege rejects the view that introduces the concept of 'interpretation'. According to him, "a thought leaves no room for different interpretations".[389]

A proposition, according to Frege, expresses a thought. After claiming this, he attempts to distinguish the real propositions from the pseudo ones. This latter kind, Frege claims, has only the grammatical form of a proposition.[390] Pseudo-propositions are even incapable of expressing the false thoughts. They are, according to Frege, are neither valid nor invalid.[391] In some cases, according to Frege, it happens that the combination of two pseudo-propositions, occuring as an antecedent and a consequent, can form a real proposition. In order to avoid this, Frege suggests not to take an antecedent proposition as the explanation of a letter occuring in it. He gives the following example: If the counterpositive of the proposition 'If a is a whole number, (ax(a-1)) is an even number' , that is, 'If (ax(a-

[385]Ibid., p.305
[386]Ibid., pp.306-307
[387]Ibid., p.307
[388]Ibid.
[389]Ibid., p.315
[390]Ibid., p.308
[391]Ibid., p.310

1)) is not an even number, then a is not a whole number' is treated as the same, then we must consider 'Let (ax(a-1)) not be an even number' as an explanation of the letter 'a', which involves contradiction.[392]

Frege claims that if a system of general theorems coincide in their antecedent pseudo-propositions, we can obtain a theory. And the consequent pseudo-propositions can be combined in one composite one by conjunction. Thus the whole theory can be reduced to a set of pseudo-propositions as antecedent and a composite pseudo-proposition as consequent.[393]

According to Frege, "the linguistic expression of a group of thoughts consists of real propositions connected by 'and'. We can think of a group of thoughts as one thought constituted out of other thoughts."[394]

Frege does not admit that the negation of a thought can dissolve the thought into its component parts. Because, we mentioned above, the truth (or falsity) and the make-up of a thought, according to Frege, is not dependent on our act of negating. What is dissolved, Frege claims, is indeed the 'model' of a thought in the world of sentences, expressions, words, signs.[395]

Moreover, Frege claims that "consideration of the law *duplex negatio affirmat* makes it specially plain to see that negation has no separating or dissolving effect."[396] Since we cannot affect a thought by our acts, according to Frege, "no non thought is turned into a thought by negation, just as no thought is turned into a non-thought by negation."[397]

With regards to negation, Frege refutes the following two views: In the first one, it is supposed that, "negation extends to the whole thought when 'not' is attached to the verb of the predicate." In this view, according to Frege, the distinction

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[392]Ibid., p.312
[393]Ibid.
[394]Ibid., p.334
[395]Gottlob Frege, "Negation", *Collected Papers*, p.377
[396]Ibid.
[397]Ibid., p.379

between the negative and affirmative judgements is not clear. Frege suggests to drop the difference between the two kinds of judgements or thoughts until we find a criterion that enables us to distinguish them with certainty. [398]

The second view asserts that "the judging subject sets up the connection or order of the parts in the act of judging and thereby brings the judgement into existence."[399] Frege rejects this point by claiming that in this view 'the act of grasping a thought' and 'the acknowledgement of its truth' is confused. According to Frege, "even the act of grasping a thought is not a production of the thought, is not an act of setting its parts in order; for the thought was already true, and so was already there with its parts in order, before it was grasped."[400]

Frege distinguishes 'judging' and 'negating' by their being dependent or independent on a subject. According to Frege, judging, as an act, is a physical process and needs a judging subject. Negation, on the other hand, being a part of a thought, is independent from a subject and hence is not a content of a consciousness.[401]

Frege argues that it is generally held that there are two ways of judging: affirmative and negative; And these require three items: "1- assertoric force for affirmatives; 2- assertoric force for negatives, e.g. inseparably attached to the word 'false' ; 3- a negating word like 'not' in sentence uttered non-assertorically."[402] Frege rejects this view for the sake of 'economy' and claims that two items are sufficient, that are "1- assertoric force; 2- a negating word."[403]

From here, Frege concludes that "for every thought there is a contradictory thought; we acknowledge the falsity of a thought by admitting the truth of its contradictory. The sentence that expresses the contradictory thought is formed

[398]Ibid., p.380
[399]Ibid., p.381
[400]Ibid., p.382
[401]Ibid.
[402]Ibid., p.385
[403]Ibid.

from the expression of the original thought by means of a negative word."[404]

According to Frege, a thought and its negation are quite different with respect to self-sufficiency. While the former does not necessitate a completion, the latter needs to be completed by the former.[405] Frege claims that "the negation of a thought is itself a thought, and can again be used to complete the negation."[406] Frege introduces his concept of 'wrapping up a thought' which he uses to explain the 'double negation', that is, 'the negation of a negation.' This, according to him, does not change the truth value of a thought.

In a footnote, Frege clarifies his concept of 'sentence' as follows: "I am not using the word 'sentence' here in quite the same sense as grammar does, which also includes subordinate clauses. An isolated subordinate clause does not always have a sense about which the question of truth can arise, whereas the complex sentence to which it belongs has such a sense."[407]

We mentioned above that, according to Frege, sentences are the models of thoughts. He also claims that we must be able to distinguish the parts of a thought which correspond to the parts of a sentence. Frege asserts that there can be thoughts which can have thoughts as parts of it. He calls these kind of thoughts as 'compound thoughts'. Frege considers six kinds of compound thoughts:

"1- A and B; 2-not(A and B); 3-(not A) and (not B);

4-not ((not A) and (not B)); 5- (not A) and B;

6-not ((not A) and B)." [408]

For the first kind of compound thought, Frege distinguishes between 'the thought expressed' and 'the assertion'. According to him, the sentences conjoined by

[404]Ibid.
[405]Ibid., p.386
[406]Ibid., p.387
[407]Ibid., p.355
[408]Gottlob Frege, "Compound Thoughts", *Collected Papers*, p.403

'and' are to be uttered without assertoric force. Since non of the two sentences conjoined are uttered by assertive force, Frege discusses whether the whole is a thought or not and claims that the whole must be distinguished from the component parts. For the whole contains "that which combines them together; and this corresponds in language to the word 'and'. This word is used here in a particular way; we are concerned with its use as a conjunction between two sentences proper. I call any sentence proper if it expresses a thought. But a thought is something which must be either true or false, *tertium non datur.*"[409]

From this, Frege infers the following:

A is true,

B is true; therefore

(A and B) is true. [410]

The second kind of compound thought is the negation of the first kind: "Whenever a compound thought of the first kind out of two thoughts is false, the compound of the second kind out of them is true, and conversely. A compound of the second kind is false only if each compounded thought is true, and a compound of the second kind is true whenever at least one of the compounded thoughts is false."[411] From these, Frege infers the following:

not (A and B) is true;

A is true; therefore

B is false.[412]

In view of the third kind of compound thought, Frege asserts the following: "A compound of the first kind, formed from the negation of one thought conjoined with the negation of another thought, is also a compound of these thoughts themselves. I call it a compound of the third kind out of the first thought and the

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[409]Ibid., p.392
[410]Ibid., p.393
[411]Ibid., p.394
[412]Ibid., p.395

second."[413] Frege's inference for this kind is as follows:

A is false;

B is false; therefore

(neither A nor B) is true.[414]

Frege states that "The negation of a compound of the third kind between two thoughts is likewise a compound of these two thoughts: it may be called a compound thought of the fourth kind. A compound of the fourth kind out of two thoughts is a compound of the second kind out of the negations of these thoughts...

Here, too, follows another inference:

(A or B) is true;

A is false; therefore

B is true."[415]

Frege explains the fifth kind of compound thought as follows: "By forming a compound of the first kind out of the negation of one thought and a second thought, we get a compound of the fifth kind out of these two thoughts. Given that 'A' expresses the first thought and 'B' expresses the second, the sense of '(not A) and B' is such a compound thought."[416]

The last kind of compound thought is the negation of the fifth kind; In view of this, Frege infers that "A compound of the fifth kind is true if and only if its first component thought is false, but the second is true. From this it follows that a compound of the sixth kind out of two thoughts is false if and only if its first component is false, but the second is true. Such a compound thought is therefore true given only the truth of its first component thought, regardless of whether the second is true or false. It is also true given only the falsehood of itssecond

[413]Ibid.,
[414]Ibid., p.396
[415]Ibid., pp.398-399
[416]Ibid., p.398

component thought, regardless of whether the first is true or false."[417]

Frege entitles the compound thoughts formed with the aid of negation from compounds of the first kind as 'mathematical compound thoughts' and claims that "if one component of a mathematical compound thought is replaced by another thought having the same truth-value, then the resultant compound thought has the same truth-value as the original."[418]

Frege above showed that "the Number which belongs to the concept 'member of the series of natural numbers ending with a' follows in the series of natural numbers directly after a. In order to prove this proposition, Frege asserts, it must be shown that a is the Number which belongs to the concept 'member of the series of natural numbers ending with a, but not identical with a'. And, according to Frege, it must also be shown that, the extension of this concept is identical with the extension of the concept 'member of the series of natural numbers ending with d'. Frege furtherly adds that "for this we need the proposition that no object which is a member of the series of natural numbers beginning with 0 can follow in the series of natural numbers after itself. "[419]

In order to conclude the proof, Frege claims that the condition, that n must be a member of the series of natural numbers beginning with 0, must be added to the proposition: "the Number which belongs to the concept 'member of the series of natural numbers ending with n' follows in the series of natural numbers directly after n"[420] Frege abbreviates this condition by the proposition "n is a finite Number"[421] Thus, Frege belives, the following proposition is proved: "no finite Number follows in the series of natural numbers after itself "[422]

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[417]Ibid., p.399
[418]Ibid., p.406
[419]*The Foundations of Arithmetic*, p.95
[420]Ibid., p.96
[421]Ibid.
[422]Ibid.

IV- Conclusion: Object and Space - A Criticism of Frege's Ontology In View of Transcendental Philosophy

In Kant's transcendental ontology, objects are not given entities; they arise as a consequence of a certain procedure, namely the three-fold synthesis[423] . Kant argues that something that is to be determined as an object acquires its synthetical unity only as the consequence of this procedure. One should note that, in Kant's ontology, all objects, both empirical and a priori, are synthetic.

The space of an object in Kant's ontology is the judgement. Objects exist in judgements together and in unity with concepts. In transcendental ontology, concepts and objects are the constituents of a judgement and these constituents do not have a priority over the whole. A judgement is formed as an act of the understanding once the three-fold synthesis is completed. In the judgement, we find the object and its concept.

Since objects and concepts are formed only within a synthetic unity, in Kant's ontology, the space of all objects are synthetic judgements. It is the synthetic judgement in which the object and its concept are found. It is within the synthetic judgement that synthetic unity is given to the entity which is determined as the object. It is this synthetic unity that constitutes the entity as an object. Thus, in view of Kant's Transcendental ontology, in analytical judgements, only the representations of the objects are found. An analytical judgement therefore always needs a synthetical judgement as its grounds if it is to be meaningful. One should note that no analytical judgement is a space of some object in transcendental ontology.

According to Frege, however, the distinction between the synthetic and analytic judgements concern the justification for making the judgement, and not its

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[423] See thesis p. 25

content. Let us recall that general logic, according to Kant's distinction[424], deals only with the forms of the judgements and not with their contents. Hence, in the Kantian sense, Frege considers general logic as the sphere in which one observes the distinction between the synthetic and analytic judgements. Obviously, in view of transcendental philosophy, the analytic/synthetic distinction in the domain of general logic is not meaningful. In his criticism of Kant's ontology, Frege does not consider the synthetical unity which is required for the formation of the object and its concept. The transcendental grounds of an object and the a priori forms of all objects are beyond the domain of general logic and Kant develops transcendental logic to deal with such issues. According to Frege: "...what is of concern to logic is not the special content of any particular relation, but only the logical form. And whatever can be asserted of this, is true analytically and known a priori."[425]

It appears that, Frege considers (general) logic as a space for judgements, that is objective thoughts to which truth is attributed. At this point, the following question immediately arises: Is logic itself a space? Similarly, one can ask whether physics is a space or chemistry is a space. Frege himself makes such comparisons; in "Thoughts", he claims that: "logic has much the same relation to truth as physics has to weight or heat".[426] Obviously physics, as a science, is not a space. Physical space, on the other hand, is obviously different from physics itself. This is also true for logical objects, that is the objective thoughts. Logic, as a science, is not the space of logical objects. Thus, for objective thoughts, a space must be assigned.

Logic, as Frege states, discovers the laws of truth[427] but logic, as a science itself, cannot be a space for the entities to which truth is attributed. Frege

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424 See thesis p. 15

425 Ibid., p.83

426 *Collected Papers*, p.351

427 Ibid., p.352

does not clarify the space of objective thoughts. It is therefore not clear how these logical objects are connected and how they may be subject to reason.

Frege critisizes Kant in terms of intuition and claims that this element involves subjective, empirical and psychological aspects. However, intuition in Kant's ontology has a logical side as well as a subjective one. Intuition has two objective forms giving rise to two logical principles, namely, coordination and subordination.

We mentioned in section II-2. that transcendental logic provides us with the a priori elements and the *laws* of pure thought. A judgement, according to Kant, is an act of the faculty of understanding; it is a unitary act of thought. Transcendental analytic, in which we find the definitions of 'judgement' and 'category', provides us with the *elements* of pure thought. 'Pure thoughts' are 'pure judgements' and categories are pure concepts which are found only in 'pure thoughts'. [428]

Frege's distinction of synthetic/analytic excludes the objective logical forms of 'intuition' in Kant's sense. Frege claims that Kant's attempt to introduce the notion of 'intuition' is to ground the synthetic a priori judgements. According to Frege, however, Kant does not at all clarify 'intuition': "...whether it is spatial or temporal, or whatever else it may be."[429]

In Logic, Kant distinguishes 'intuition' and 'concept' as follows: "All cognitions, that is, all presentations consciously referred to an object, are either *intuitions* or *concepts*. Intuition is a singular presentation (*repraesentatio singularis*), the concept is a general (*repraesentatio per notas communes*) or reflected presentation (*repraesentatio discursiva*)...Concept is opposed to intuition, for it is a general presentation or a presentation of what is common to several

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[428] See thesis p. 43
[429] Op. Cit., p.18

objects, a presentation, *therefore, so far as it may be contained in different objects."* [430]

These two different uses of the term 'intuition', that is, in the sense of Logic and in the sense of *Transcendental Aesthetic,* however, cannot be compared by means of being 'narrower' (or, wider). On the basis of this distinction Frege claims that "the sense of the word 'intuition' is wider in the Logic than in the *Transcendental Aesthetic*" [431] Frege, however, ignores that the term 'intuition' is used in the general logical sense in the former and in terms of sensibility and in the 'transcendental logical' sense in the latter. In Frege's works, we almost nowhere find a discussion and an explicit criticism of transcendental logic.

In *Transcendental Aesthetic,* Kant defines intuition in terms of the faculty of sensibility as follows: "In whatever manner and by whatever means a mode of knowledge may relate to objects, *intuition* is that through which it is in immediate relation to them, and to which all thought as a means is directed. But intuition takes place only in so far as the object is given to us. This again is only possible, to man at least, in so far as the mind is affected in a certain way. The capacity (receptivity) for receiving representations through the mode in which we are affected by objects, is entitled *sensibility.* Objects are *given* to us by means of sensibility, and it alone yield us *intuitions*; they are *thought* through the understanding, and from the understanding arise *concepts.* But all thought must, directly or indirectly, by way of certain characters, relate ultimately to intuitions, and therefore, with us, to sensibility, because in no other way can an object be given to us."[432]

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430 *Logic,* p.96

431 Op. Cit., p.19

432 *The Critique of pure Reason,* A19/B34

Frege critisizes Kant's view of sensibility as well: "Without sensibility no object would be given to us"[433] For, according to Frege, "nought and one are objects which cannot be given to us in sensation"[434]

In *Transcendental Aesthetic,* Kant defines sensation as "the effect of an object upon the faculty of representation, so far as we are affected by it[435] Frege furthermore asserts that "perhaps Kant used the word 'object' in a rather different sense; but in that case he omits altogether to allow for nought and one...- for these are not concepts either, and even of a concept Kant requires that we should attach its object to it in intuition."[436]

Frege, however, by confusing sensation with sensibility, ignores the distinction between the sources of empirical and non-empirical objects in Kant's ontology. Although both types of objects have a priori grounds, the former has its source in sensation and the latter in the a priori manifolds of space and time.

Frege also questiones the term 'self-evidence' in Kant's system. He claims that to use the term 'intuition' to provide grounds for 'self-evidence' gives rise to philosophical difficulties. For an arithmetical proposition like '135664 + 37863 = 173527', according to Frege, calling on our intuition of fingers or points for support runs the risk of "making these propositions appear to be empirical...Moreover, the term 'intuition' seems hardly appropriate, since even 10 fingers can, in different arrangements, give rise to very different intuitions. And have we, in fact, an intuition of 135664 fingers or points at all?"[437]

Frege, unfortunately, confuses here the a priori object and its representation in thought. Intuitions, be them fingers or points are merely tools for representing non-empirical objects.

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433 Ibid., A51/B75
434 *The Foundations of Arithmetic*, p.101
435 Op.Cit., A34/B20
436 Ibid., p.101
437 Ibid., p.6

Kant's transcendental philosophy secures the objectivity of a priori judgements and system provides a means for their investigation. Arithmetical judgements are a priori. According to Frege, Kant's transcendental logic involves subjectivity and he claims that mathematical objects, that is natural numbers appear as psychological elements in Kant's ontology.

On the other hand, Kant himself declares the objectivity of mathematical judgements. Natural numbers are *a priori* objects in Kant's ontology and Kant secures an objective status to natural numbers in his system. This objectivity is transcendentally derived from a subjective act, namely the three-fold synthesis.[438] In Kant's ontology, what belongs to me as an image is subjective; this subjective entity, however, is the representation of the number and not the number itself. Number is an a priori object and as an a priori object it has no psychological aspects; it is synthetic and has a unity, transcendentally derived from pure consciousness. In transcendental ontology, the space of a number as an object is the synthetic *a priori* judgement.

In Frege's ontology, numbers as arithmetical objects are found in number concepts. Frege claims that "the content of a statement of number is an assertion about the concept."[439] Concepts, in his system, are confined to purely logical use and they should be distinguished from the objects; these concepts are found in objective thoughts together with objects. In view of the relation between an object and its concept, Frege uses the term 'to fall under' and, for first and second level concepts, 'subordination'. Frege claims that "each individual pair of correlated objects stands to the relation- concept much as an individual object stands to the concept under which it falls."[440] In Frege's system, however, a space for objective thoughts is not determined. Nor does he specify how the relation between the

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[438] See thesis p. 21

[439] Ibid., p.59

[440] Ibid., p.82

objective thoughts and the subordination between the first and second level concepts is formed. In view of transcendental philosophy, for a complete ontology, the space of an object should also be determined because an entity without a space cannot be an object.

In Kant's transcendental philosophy, objects have a synthetical unity but appearances do not. The rules of this unity have their source in transcendental consciousness which is the consciousness of the unchanging 'self'. The ultimate grounds of objects, according to Kant, is the transcendental or pure consciousness. The second step of the three-fold synthesis, let us recall, is realized within the faculty of imagination. According to Kant, transcendental or pure consciousness within imagination is the form of all possible knowledge.[441] Categories, as pure concepts of the faculty of understanding, have a transcendental content which can only be captured by transcendental logic. This transcendental content has its source a priori in the manifolds in space and in time. In Kant's ontology, for the formation of an a priori object, 'pure intuition', containing the a priori manifold, must be provided, this manifold must be synthesized in imagination and a concept should arise as a consequence of the representation of this manifold in the pure synthesis. The form of an object (be the object empirical or a priori) is a priori and this synthesis is made possible by means of transcendental consciousness. Transcendental consciousness is the a priori source of all objects. Frege's consideration of transcendental ontology confuses the image (or the representation) of a number in the Kantian sense with the number itself as an a priori object.

The representation itself is psychological and belongs as an image to the mind whereas the number, as an a priori object, is within the pure thought. On the other hand, Frege, in his system, does not clarify how a natural number as a

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[441] See thesis p. 25

mathematical object is formed and where exactly this formation takes place, although he considers mathematical objects as a priori entities within analytical judgements. We mentioned in section III that Frege distinguishes between 'being true' from 'being taken to be true' and he considers laws of logic as laws of truth, not laws of being taken to be true and this can be explained only by reducing a law of logic to another law of logic. Hence, according to Frege, objective thoughts are the elements of logic and their truth is timeless. But, by claiming these, Frege does not at all clarify, in view of transcendental philosophy, the grounds of these objective thoughts.

In Kant's system, we mentioned in section II.4.; two separate ontological spaces are connected by means of a medium only. The schema serves as such a medium between the categories and appearances. The schema provides an image to the concept of number, that is, a representation of number and as a representation it involves psychological aspects. The space of natural numbers, as non-empirical objects are synthetic a priori judgements. And the space of such judgements, according to Kant, is pure reason.

Frege's ontology, on the other hand, is incomplete with respect to Kant's transcendental philosophy. In Frege's system, like that of Kant, numbers are found under number concepts and number as object is found in unity with the concept of number in judgements. Frege, however, considers analytical judgements as the space of mathematical objects. An analytical judgement rests on the principle of identity. In view of transcendental ontology, we should note, analytical judgements are empty in terms of objects.

We mentioned in section III that Frege introduces the concept of 'identity' to recognize a number as the same again and he tries to find a criterion for this principle. For this task, he attempts to define the following proposition: "the number which belongs to the concept F is the same as that which belongs to the concept G." In order to fix the sense of identity statements, Frege introduces his concept of 'carving up the content'. Frege uses two analogies from geometry in order to explain this. The first one concerns the concept of 'direction'. According to Frege, the following judgement is an analytical one: "the proposition 'line a is parallel to line b' (let us call it A) can be replaced by 'the direction of line a is identical with the direction of line b' (let us call it B). " Here, the content of the concept in A is carved up and it yields a new concept in B, namely, 'direction'. The objects of these proposition, Frege claims, are 'the direction of line a' and 'the direction of line b'. Frege claims that we can acquire numbers in the same way. For the case of numbers, let us repeat, Frege's definition is as follows: "the Number which belongs to the concept F is the extension of the concept 'equal to the concept F'. "[442]

At this point, the judgement that Frege claims to be analytical, should be questioned. According to Frege, it is analytical to claim that 'proposition A is identical with the proposition B'. But does it follow that the contents of the propositions A and B are analytical in themselves?[443] That is to say, it seems doubtful to claim that the proposition 'line A is parallel to line B', if true, is an analytical one. What helps us to distinguish the two lines from each other is the space that these lines are supposed to be in. But Frege's definition of identity does not clarify how parallel lines themselves are formed as objects and where this relation of 'being parallel' happens to be. By the same way, Frege's definition of the identity of numbers does not clarify how arithmetical objects are formed and to exactly where this formation belongs. The identity statement, by itself, is not sufficient to generate a mathematical object *ex nihilo*.

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442 Ibid., pp.79-80

443 I owe this point to my supervisor; Prof. Dr. Yalçın Koç.

Consequently, the significant points that Frege misevaluates in Kant's ontology, are in the first place the distinction between the analytic and synthetic judgements. Frege treats analytic judgements as if natural numbers inhere in them while for Kant, the space of natural numbers are synthetic *a priori* judgements. But, as we mentioned above, analytic judgements can contain no objects- either empirical or a priori. Secondly, Kant's use of 'intuition' does not involve subjectivism as Frege considers it to be. Kant uses this term objectively in connection with the faculty of sensibility in order to clarify the source of empirical objects. Thirdly, Frege totally ignores that in Kant's transcendental philosophy, let us repeat, all objects are synthetical.

Furthermore, it is not certain where Frege's objective thoughts can be found. Frege's ontology does not provide a medium for connecting the objective thoughts. Thus Frege's system is ontologically incomplete with respect to the ultimate space of such objects.

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