DISTURBANCE DECOUPLING IN LINEAR TIME INVARIANT SYSTEMS

by

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ABSTRACT

Disturbance decoupling and disturbance decoupled estimation problems in linear, time invariant, dynamical system are studied in a common framework using geometric approach.

The basic solvability question is investigated for disturbance docoupling problem by static and dynamic state and measurement feedback.

Estimation of the state vector or a function of the state vector of a system in the presence of disturbances is considered. The concepts of observable, unobservable subspaces and observability of a system are generalized for unknown input systems.

In the second part of the thesis, solvability of the above problems in a special kind of system which consists of separated dynamic and algebraic parts is considered. In general, the solvability range of the problems is improved by system decomposition.

All results are fully constructive and an appendix is included for the numerical computation of the developed design methods.

ÖZET

Doğrusal, zamanla değişmeyen, dinamik sistemlerde bozucu bastırma ve bozucu bastırılmış kestirim problemleri ortak bir çerçeve içinde geometrik yaklaşımla incelenmektedir.

Bozucu bastırma problemi için temel çözülebilirlik sorunu, statik ve dinamik durum ve ölçüm geri besleme hallerinde cevaplandırılmaktadır.

Durum vektörünün veya durum vektörünün bir fonksiyonunun kestirimi bozucuların varlığında ele alınmakta; gözlenebilir, gözlenemez alt uzay ve gözlenebilirlik kavramları bilinmeyen girdiği sistemler için genelleştirilmektedir.

Tezin ikinci kısmında yukarıda sözü edilen problemler dinamik ve cebirsel parçalardan oluşmuş özel bir sistem için incelenmekte; çözülebilirlik sınırlarının genişlediği gösterilmektedir.

Varılan bütün sonuçlar yapıcı olup, geliştirilen tasarım yöntemlerinin sayısal hesabı için bir ek verilmiştir.

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I. INTRODUCTION

Control systems usually operate under the presence of unmeasurable, unknown inputs. Various analysis and synthesis techniques have been proposed in literature to deal with such inputs which may be called disturbances. These may include:

1) To treat disturbances as random signals with known statistics and to apply the theory of stochastic processes to analyze systems subject to such inputs.

2) To assume that the disturbances satisfy differential equations with known coefficients, their inifial conditions being unknown or they can be approximated by polynomial inputs of sufficiently high order.

One common method applied in these cases in to augment the system equations with the assumed model for the disturbances so that existing theory can be applied to the augmented system.

It is clear that none of these approaches is best suited to all types of disturbances encountered in practice or to systems excited by different disturbance sources.

In this work we tried to give a unified technique for the regulation and estimation of control systems by considering disturbances as completely unknown signals which take their values from a specified function class F and no existing a priori information about their nature can be used to aid the synthesis problem. The choice of the function class F is not crucial, the space of continuous valued functions, $\{f: R^+ \to R^r\}$, can be adopted for instance.

This selection of the disturbance modal has several advantages as well as disadvantages. Its generality and applicability in different situations, like decentralized control systems; systems subject to man made interference or noise governed by nature is the basic advantage. Besides this a lot of modeling labour is eliminated and systems designed using this approach have simpler structure compared with the methods stated above.

As a disadvantage we must admit that treating disturbance inputs as totally unknown signals considerably reduces the solvability range of the problems studied.

Our treatment of the problems is mainly in time domain. State space representation of control systems is used as a convenient tool for doing this. The problems are formulated and solved in a geometric style as developed by Wonham and his coworkers. The required mathematical background such as controllable and observable subspaces of the state space and certain other invariant subspaces are reviewed in Ch.2. This chapter provides the self containment of the work.

The simplest form of regulation: The disturbance decoupling problem is studied in Ch.3. Disturbance decoupling by state feedback, constant measurement feedback and dynamic measurement feedback are treated in a common framework in sections (3.1), (3.2) and (3.3) respectively. The chapter closes by some interesting remarks.

Ch.4 is devoted to estimation problems. The close connection between disturbance decoupling problem and disturbance decoupled estimation problem is pointed out in section (4.1). A completely new problem: The known initial state observer design problem is also solved in this section. The classical observer design problem is discussed by geometric methods in section (4.2) in its most general setting. The well known concepts of observable and unobservable subspaces of state space are generalized for unknown input systems, observability of an unknown input system is defined based on these subspaces in section (4.3). Finally in section (4.4) it is shown how parameter variations can be considered as unknown inputs which are produced internally and additive to the system. This observation allows us to solve the zero sensitive observer design problem.

To overcome the difficulty stated as a disadvantage of the assumed disturbance model a special kind of system structure is introduced in Ch.5 which consists of the interconnection of a fixed algebraic subsystem and a variable dynamic subsystem. Some properties like controllability, observability and transfer function inveriance of decomposed system are summarized in section (5.1) following the works in [23], [24] and [25]. Disturbance decoupling problem in decomposed systems is studied in section (5.2) in all of its variations. Then in section (5.3) disturbance decoupled estimation problem is investigated for decomposed systems.

Generic solvability conditions are obtained for both problems.

The work is completely original by its way of presentation and some results of chapters 3,4 and 5.

Each problem discussed in the work is precisely defined and formulated first. Necessary and sufficient conditions for the solvability of the problem are given as theorems and corollaries. A constructive synthesis procedure (sometimes an algorithm) is given if a solution exists.

The examples and the computational matrix algorithms collected in the appendix are also an integral part of the thesis which bridge the gap between theory and practice, and which make the study easier for future researchers in this field.

II. MATHEMATICAL PRELIMINARIES AND SYSTEM THEORETIC INTERPRETATIONS

Some definitions and theorems which will be needed in the thesis are collected in this chapter. The aim is to provide ease of reference and to introduce relevant notation. Detailed proofs and extensions can be found in the references cited.

Throughout the thesis uppercase is used for linear mappings and their matrix representations and lowercase for vectors. Vector spaces and subspaces are denoted by bold face capitals. The dimension of a subspace **S** is denoted by dim **S**. $\mathbf{S}_1+\mathbf{S}_2$ and $\mathbf{S}_1\oplus\mathbf{S}_2$ are the sum and direct sum, respectively, of the subspaces $\mathbf{S}_1$ and $\mathbf{S}_2$. $\mathbf{S}^{\perp}$ stands for the orthogonal complement of **S**. The image (kernel) of a map A is written imA (kerA). The symbol $\sigma(A)$ denotes the spectrum (the set of eigenvalues) of A. A' is used to denote the transpose of A.

Consider the linear time invariant system described by

$$\mathring{x}=Ax+Bu \quad , \quad y=Cx \quad , \quad z=Dx \tag{2.1}$$

As usual, x and u are the state and control vectors, y is the vector of measured output variables, and z is the vector of output variables to be regulated. Although the model (2.1) corresponds to a continuous time system, all results stated in this chapter apply equally well to the corresponding discrete time system model, as they are essentially algebraic properties of the 4. tuple (A,B,C,D).

Certain invariant subspaces of the state space **X** are fundemental for a geometric approach to system theory and these are presented below.

Definition 2.1: let $A: X \to X$ be a linear map, a subspace $S \subset X$ is said to be A-invariant if and only if

$$AS \subset S \tag{2.2}$$

The class of A-invariant subspaces of X is denoted by $\underline{I}(A)$. Clearly the 0 subspace, the state space X itself and subspaces spaned by the eigenvectors of A are A-invariant subspaces.

Let S be any A-invariant subspace and let R be such that $S \oplus R = X$. In a basis $\{s_1 \ldots s_k, r_1 \ldots r_m\}$ adopted to this decomposition the map A has the matrix representation:

$$\text{Mat } A = \begin{bmatrix} A_1^{k \times k} & A_3^{k \times m} \\ 0^{m \times k} & A_2^{m \times m} \end{bmatrix} \tag{2.3}$$

The restriction of A to S, $(A|S)$, is characterized by the matrix $A_1^{k \times k}$ in this representation. It satisfies the relation:

$$AS = SA_1 \tag{2.4}$$

where S is the basis matrix for the subspace S whose columns are the basis vectors $(s_1 \ldots s_k)$.

And $A_2^{m \times m}$ is the matrix of the map, $(A|X/S)$ induced by A in the factor space X/S.

The block triangular structure of A in (2.3) implies, via the characteristic polynomial

$$\sigma(A) = \sigma(A|S) \,\dot{\cup}\, \sigma(A|X/S) \tag{2.5}$$

where $\dot{\cup}$ denotes union with any common elements repeated.

Let $\underline{S}$ denote a nonempty set of subspaces, the largest or supremal element S^* of $\underline{S}$, is defined to be that unique member of $\underline{S}$ which contains every member of $\underline{S}$. Thus $S^* \in \underline{S}$, and if $S \in \underline{S}$ then $S \subset S^*$. Similarly the smallest or infimal element S_* of a set of subspaces $\underline{S}$, is defined as the unique subspace which is contained in every member of $\underline{S}$. Hence $S_* \in \underline{S}$, and if $S \in \underline{S}$ then $S_* \subset S$.

These are summarized by writing:

$$S^* = \sup\{S:\ S \in \underline{S}\} = \sup \underline{S} \text{ and } S_* = \inf\{S:\ S \in \underline{S}\} = \inf \underline{S}$$

The following lemmas reveal under what conditions supremal or infimal elements of a set of subspaces exists.

Lemma 2.1a: Let $\underline{S}$ be a nonempty class of subspaces of **X**, closed under subspace addition. Then $\underline{S}$ posseses a supremal element S^*

Lemma 2.1b: Let $\underline{S}$ be a nonempty class of subspaces of **X**, closed under subspace intersection. Then $\underline{S}$ contains an infimal element S_*

Proof These lemmas can be proven by constructing a nondecreasing sequence of subspaces S_1, S_1+S_2, $S_1+S_2+S_3, \ldots$ in the first case and a nonincreasing sequence S_1, $S_1 \cap S_2$, $S_1 \cap S_2 \cap S_3, \ldots$ in the second. As the subspaces are finite dimensional the chains can not be continued beyond, say, k terms then we can set $S^* = S_1 + \ldots + S_k$ and $S_* = S_1 \cap \ldots \cap S_k$. Clearly S^* $(S_*) \in \underline{S}$, and contains (is contained in) every $S \in \underline{S}$. ■

The family of A-invariant subspaces $\underline{I}(A)$ is closed under subspace addition and subspace intersection. Therefore supremal and infimal elements of a set of A-invariant subpaces exist according to the above lemmas.

For example the controllable subspace:

$$\langle A|\mathrm{im}B\rangle \triangleq \sum_{i=1}^{n} \mathrm{im}\,(A^{i-1}B) = \inf\{S:\ S\in \underline{I}(A),\ \mathrm{im}B\subset S\} \qquad (2.6)$$

is the least invariant under the matrix A containing imB and the unobservable subspace:

$$\langle A'|\mathrm{im}C'\rangle^{\perp} = \bigcap_{i=1}^{n} \ker\,(CA^{i-1}) = \sup\{S:\ S\in \underline{I}(A),\ S\subset \ker C\} \qquad (2.7)$$

is the greatest invariant under A contained in kerC.

When $\langle A|\mathrm{im}B\rangle = X$ the pair (A,B) is called contrallable; when $\langle A'|\mathrm{im}C'\rangle^{\perp} = 0$ the pair (C,A) is called observable.

The generalization of simple invariance is the concept of (A,B)-invariance

Definition 2.2: A subspace $V\subset X$ is (A,B)-invariant if and only if

$$AV \subset V + \mathrm{im}B$$

The set of all (A,B)-invariant subspaces in a given subspace $K\subset X$ is denoted by $\underline{V}(A,B:K)$ or simply by $V(K)$ when the matrices under consideration are fixed. It follows from Def.(2.2) that any A-invariant subspace is automatically (A,B)-invariant.

The essential fact about an (A,B)-invariant subspace is that it can be made (A+BF)-invariant by a suitable choice of the matrix F.

Lemma 2.2: Let $V\subset X$. There exists a linear state feedback map $F: X \rightarrow U$ such that

$$(A+BF)\,V \subset V \qquad (2.8)$$

if and only if $V\in\underline{V}(A,B,X)$

Proof: "Only if" Let $\{v_1 \ldots v_k\}$ be a basis for V. (2.8) implies $(A+BF)v_i=w_i$ for some $w_i \in V$ and for $i=1,\ldots,k$ or
$A\ v_i=w_i-BF\ v_i \in V+\mathrm{im}B$. Hence $V\in \underline{V}(A,B)$
"if" Let $V\in\underline{V}(A,B)$. By definition there exists $w_i\in V$ and $u_i\in U$ such that

$A\ v_i=w_i-B\ u_i$ for $i=1,\ldots,k$. define F on V by $F\ v_i=u_i$. Then F will have the required property (2.8) ■

If a subspace V is (A,B)-invariant the class of $F:X\to U$ such that $(A+BF)V\subset V$ is written as $\underline{F}(V)$.

An (A,B)-invariant subspace V is characterized by the property that, for every point x_o in V a control function can be found such that the resulting state trajectory (with x_o as initial point) remains in V for all positive t as shown in Fig.2.1a below. Thus if $F:X\to U$ is chosen according to Lemma 2.2 and u=Fx is set in (2.1), for the autonomous system $\dot{x}=(A+BF)x$, $x(o)\in V$ implies $x(t)\in V$ $(t\geq 0)$; so if x(.) starts in V, it stays in V. V has been made invariant by suitable state feedback.

Fig.2.1 (a) An (A,B)-invariant subspace (b) A controllability subspace

Another generalization of invariance which is, in a sense, the dual of (A,B)-invariance is given by the following definition.

Definition 2.3: A subspace $Q\subset X$ is (C,A)-invariant if and only if

$$A(Q\cap \ker C)\subset Q$$

The set of (C,A)-invariant subspaces containing a given subspace K is denoted by the symbol $\underline{Q}(C,A;K)$.

The duality between (A,B)-invariant subspaces and (C,A)-invariant subspaces is expressed in theorem 2.1 below.

Theorem 2.1: Let $Q\in\underline{Q}(C,A;K)$ then $Q^{\perp}\in\underline{V}(A',C';K^{\perp})$ and conversely let $V\in\underline{V}(A,B;K)$ then $V^{\perp}\in\underline{Q}(B',A';K^{\perp})$

Proof: It suffices to prove only the first part of the theorem, the converse follows by a simple change of symbols. Let M, $N\subset X$ with $AM\subset N$ then $(AM)^{\perp}\supset N^{\perp}$. Thus $(n,Am)=(A'n,m)=0$ for every $n\in N$ and $m\in M$ so that $A'N^{\perp}\subset M^{\perp}$ repeating the same argument for subspaces $N^{\perp}$, $M^{\perp}\subset X'$ and map A' satisfying $A'N^{\perp}\subset M^{\perp}$ it can be concluded that

$$AM\subset N \text{ if and only if } A'N^{\perp}\subset M^{\perp}.$$

Applying this result to the situation at hand.

$$A(\ker C\cap Q)\subset Q \text{ if and only if } A'Q^{\perp}\subset(\ker C\cap Q)^{\perp}=Q^{\perp}+\operatorname{im}C' \quad \blacksquare$$

Thus according to theorem 2.1, the ortogonal complement of a (C,A)-invariant subspace of the system Σ given by (2.1) plays the role of an (A,B)-invariant subspace for the dual system Σ' and vice versa.

The dual of Lemma 2.2 for (C.A)-invariant subspaces is as follows:

Lemma 2.3: Let $Q\subset X$. There exists on output injection map $G:Y\rightarrow X$ such that

$$(A+GC)Q\subset Q \tag{2.9}$$

if and only if $Q\in\underline{Q}(C,A)$

Proof: "if" Suppose $(A+GC)\mathbf{Q}\subset\mathbf{Q}$. Take any $x\in\ker C\cap\mathbf{Q}$. By hypotheses there exists $q\in\mathbf{Q}$ such that $(A+GC)x=Ax=q\in\mathbf{Q}$ thus $A(\ker C\cap\mathbf{Q})\subset\mathbf{Q}$. "Only if" Assume $\mathbf{Q}\in\mathbf{Q}(C,A)$. Let $\{q_1\ldots q_j\ldots q_k\}$ be a basis for $\mathbf{Q}$ such that $\{q_1\ldots q_j\}$ is a basis for $\mathbf{Q}\cap\ker C$. Then

$$Aq_i=s_i \quad \text{for } i=1\ldots j \text{ and } Aq_i=r_i \quad \text{for } i=j+1\ldots k$$

where $s_i\in\mathbf{Q}$ by assumption. Noting that the vectors $Cq_{j+1}\ldots Cq_k$ are linearly independent the map $G:\mathbf{Y}\to\mathbf{X}$ can be defined such that

$$GC\,q_i=-r_i \quad \text{for} \quad i=j+1\ldots k$$

Then $(A+GC)q_i=Aq_i=s_i\in\mathbf{Q}$ for $i=1\ldots j$ and $(A+GC)q_i=0\in\mathbf{Q}$ for $i=j+1\ldots k$ as desired ∎

To discuss stability properties related to subspaces the complex plane is divided into two self conjugate parts $\mathbb{C}_g$ and $\mathbb{C}_b$ which will indicate stability and instability respectively. For continuous time systems the usual choice for $\mathbb{C}_g$ is

$$\mathbb{C}_g=\{s\in\mathbb{C}:\ Re(s)<0\}$$

and for discrete time systems

$$\mathbb{C}_g=\{z\in\mathbb{C}:\ |z|<1\}$$

But any other choice will not change the theory. Thus for example one can define $\mathbb{C}_g=\{s\in\mathbb{C}:\ Re(s)<-\alpha \text{ for some } \alpha>0\}$ if one wants to obtain a prescribed degree of stability.

Given the system (2.1) consider the closed loop system formed by means of state feedback F and the connection of a gain matrix G at the system input as in figure (2.2)

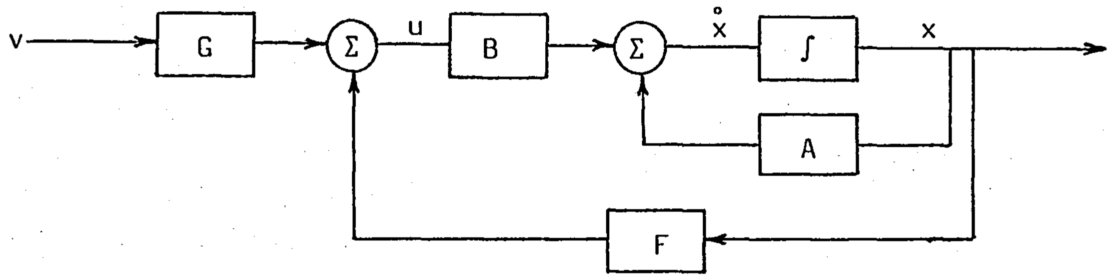


Fig.2.2

The controllability subspaces are defined as follows

Definition 2.4: A subspace $R\subset X$ is an (A,B)-controllability subspace if there exists maps $F:X \to U$ and $G:U \to U$ such that

$$R=\langle A+BF|\,\mathrm{im}(BG)\rangle \qquad (2.10)$$

Thus **R** is the controllable subspace of the pair (A+BF, BG) which is obtained as described in fig.(2.2) The notation $\underline{R}(A,B;K)$ is used to denote the class of (A,B)-controllability subspaces contained in a given subspace **K**. Several facts follow easily from definition 2.4. First, if **R** is an (A,B)-controllability subspace then **R** is (A,B)-invariant moreover ewery state $x_1\in R$ can be reached from the initial state $x_0\in R$ along a controlled state trajectory that is wholly contained in **R**. See fig.2.1b. Trivially **0** is a controllability subspace and so is $\langle A|\mathrm{im}B\rangle$, the controllable subspace of the pair (A,B). If the system is single input these are the only controllability subspaces but in the multi-input case there are lots of controllability subspaces of various dimensions.

By exploiting the equivalence between controllability and spectral assignability the class of controllability subspaces can be characterized by the following basic property:

Theorem 2.2: Let $R\subset X$ be a subspace with $\dim R=\rho\geq 1$. For every symmetric set Λ of ρ complex numbers there exists a map $F:X \to U$ such that

$$(A+BF)R\subset R \text{ and } \sigma[(A+BF)|R]=\Lambda$$

if and only if $R\in\underline{R}(A,B;X)$ ∎

It is this design property of controllability subspaces that makes the concept central in applications. As a special case of theorem (2.2), if the pair (A,B) is controllable $R=\langle A|\text{im}B\rangle=X$ is a controllability subspace so that $\sigma[(A+BF)|X]=\sigma(A+BF)$ can be assigned arbitrarily. The well known pole placement property of controllable pairs is recovered.

Although the theory of controllability subspaces allows to place the closed loop poles at the desired locations, in most practical synthesis problems it is only required that the closed loop system map, A+BF be stable. For this reason stabilizability subspaces are introduced in the definition below.

<u>Definition 2.5</u>: $V\subset X$ is called a stabilizability subspace if there exists $F:X\to U$ such that $(A+BF)V\subset V$ and $\sigma[(A+BF)|V]\subset\mathbb{C}_g$. Where $\mathbb{C}_g$ is understood in the general sense discussed above. It is a direct consequence of definition 2.5 and Lemma 2.2 that stabilizability subspaces are (A,B)-invariant. The familiy of (A,B)-stabilizability subspaces contained in $K\subset X$ is shown by the symbol $\underline{V}_g(K)$ Thus

$$\underline{V}_g=\{V\in\underline{V}(A,B;K):\ \exists F\in\underline{F}(V),\ \sigma[(A+BF)|V]\subset\mathbb{C}_g\} \tag{2.11}$$

If a subspace V belongs to the family $\underline{V}_g$ then for all $x_0\in V$ it is possible to find a feedback map $F:X\to U$ such that the response $e^{(A+BF)t}.x_0\to 0$ as $t\to\infty$. That is, the subspace V of the state space X is stabilized by state feedback. If the state space X is itself a stabilizability subspace then the pair (A,B) is said to be stabilizable. (ie., $\exists F:X\to U$ such that $\sigma(A+BF)\subset\mathbb{C}_g$)

It follows from the above discussion that stabilizability is a weaker properly than controllability. A pair (A,B) may be uncontrollable although it is stabilizable.

The classes $\underline{V}$, $\underline{R}$, and $\underline{V}_g$ are closed under subspace addition and the class $\underline{Q}$ is closed under subspace intersection. Therefore according to Lemma 2.1a and Lemma 2.1b there exists supremal elements V^*, R^*, V_g^* of $\underline{V}$, $\underline{R}$ and $\underline{V}_g$ respectively and infimal element Q_* of Q which can be computed by linear algorithms in finite number of iterations. Before presenting these algorithms a final theorem is given related to (A,B)-invariant and controllability subspaces, which is proved in [1]

Theorem 2.3: Let $V\in\underline{V}(A,B;X)$ and let $R^*=\sup\underline{R}(A,B;V)$. For $F\in\underline{F}(V)$ write $A_F=A+BF$ and $\bar{A}_F$ for the map induced in V/R^* by A_F. Then $\bar{A}_F$ is independent of $F\in\underline{F}(V)$ furthermore

$$\sigma\left[(A+BF)\,|\,V\right]=\sigma_F\dot{\cup}\sigma_o \quad \text{where}$$

$\sigma_F\overset{\Delta}{=}\sigma\left[(A+BF)\,|\,R^*\right]$ is freely assignable by suitable choice of $F\in\underline{F}(V)$ and $\sigma_o\overset{\Delta}{=}\sigma(\bar{A}_F)$ is fixed for all $F\in\underline{F}(V)$ ■

The above theorem is again a generalization of the well known controllability canonical form The controllable poles of the system matrix can be arbitrarily located by state feedback whereas the uncontrollable poles are invariant under state feedback.

ALGORITHMS:

I) Supremal (A,B)-invariant subspace contained in K: $V^*(A,B;K)$

Define the sequence of subspaces V^k according to

$$V^o=K,\quad V^k=K\cap\bar{A}^1(\text{im}B+V^{k-1})$$

The sequence V^k is nonincreasing that is, $V^k\subset V^{k-1}$. For some $k<\dim K$. $V_k=V_{k+1}=\sup\underline{V}(A,B;K)$ can be usedas a stopping rule.

II) Supremal (A,B)-controllability subspace contained in K: $R^*(K)$

Let $V^* = \sup \underline{V}(A,B;K)$

Define the sequence S^k according to

$S^o = 0 \quad , \quad S^k = V^* \cap (A\ S^{k-1} + \operatorname{im} B)$

Induction shows that S^k is nondecreasing and so

$$S^{k+1} = S^k = \sup \underline{R}(A,B;K) \text{ for } k > \dim V^*$$

III) Supremal (A,B)-stabilizability subspace contained in K: $V_g^*(K)$

Let $V^* = \sup\ \underline{V}(A,B;K)$ and $R^* = \sup\ \underline{R}(A,B;K)$

Choose $F_o \in \underline{F}(V^*)$, write $A_o = A + BF_o$, let $P: X \to X/R^*$ be the canonical projection and let $\bar{A}_o$ be the map induced in X/R^* by A_o. Let $\alpha(\lambda)$ be the minimal polynomial of $\bar{A}_o | V^*/R^*$. Factor $\alpha(\lambda) = \alpha_g(\lambda)\alpha_b(\lambda)$ where the zeros of α_g (resp. α_b) belong to $\mathbb{C}_g$ (resp $\mathbb{C}_b$) and write

$\bar{X}_g^* = (V^*/R^*) \cap \ker \alpha_g(\bar{A}_o)$ Then

$V_g^*(K) = P^{-1} \bar{X}_g^*$.

IV) Infimal (C,A)-invariant subspace containing K: $Q_*(C,A;K)$

$Q_*(K)$ can be calcuated by dualizing the supremal (A,B)-invariant subspace algorithm. Define the sequence Q_k as:

$$Q_o = K, \qquad Q_k = K + A(Q_{k-1} \cap \ker C)$$

The sequence Q_k is nondecreasing, $Q_{k-1} \subset Q_k$ and for some $k \leq n - \dim K$

$$Q_k = Q_{k+1} = \inf\ \underline{Q}(C,A;K)$$

The foregoing algorithms are very convenient for the purpose of formulating solvability conditions of the problems considered in subsequent chapters. For example an inclusion relation such as:

$$S \subset V \tag{2.12}$$

where S is a fixed subspace and V is an element of the family $\underline{V}(A,B;K)$ can hold if and only if $S \subset V^*(A,B;K)$.

Thus it is sufficient to check whether the maximal (A,B)-invariant subspace of K contains S or not in order to find out if there exists $V \in \underline{V}(A,B;K)$ satisfying $S \subset V$. Because if (2.12) does not hold for the largest element $V^*(K)$ of the set $V(A,B;K)$ then it cannot hold for any other $V \in \underline{V}(A,B;K)$

On the other hand there may be situations where it is important to know the existence of $V \in \underline{V}(A,B;K)$ other than the maximal satisfying (2.12). For example one may be interested in the minimal element of the set $\underline{V}(A,B;K)$ if the space so constructed somehow equals the state space of a system in a design problem.

Unfortunately there does not exist any algorithms in the literature to calculate the elements of the sets $\underline{V}$, $\underline{R}$ or $\underline{Q}$ other than the ones given above. Any research to this aim seems to be a rewarding study for it opens the doors of solution of many longstanding system problems like minimal order compensator design, minimal order observer design etc. as shown in the following chapters.

III. DISTURBANCE DECOUPLING IN LINEAR TIME INVARIANT SYSTEMS

Consider the dynamical system

$$\dot{x}=Ax+Bu+Eq \tag{3.1a}$$
$$y=Cx \tag{3.1b}$$
$$z=Dx \tag{3.1c}$$

with $x\in\mathbb{R}^n=\mathbf{X}$ the state, $u\in\mathbb{R}^m=\mathbf{U}$ the control, $q\in\mathbb{R}^r=\mathbf{Q}$ the disturbance $y=\mathbb{R}^p=\mathbf{Y}$ the measurement and $z\in\mathbb{R}^l=\mathbf{Z}$ the controlled output.

Definition 3.1: The system (3.1) is disturbance decoupled (relative to the pair q, z) if and only if the forced response

$$z(t)=D\int_0^t e^{A(t-\tau)}E\ q(\tau)d\tau=0$$

for all $q\in\mathbf{Q}$.

Thus, according to Def.3.1 the transfer function: $D(sI-A)^{-1}E$ from q to z is identically zero for a disturbance decoupled system.

Let $\varphi(t;q)$ be the solution of (3.1) with $x(0)=0$, $u(t)=0$. A state $x\in\mathbf{X}$ is reachable from $x(0)=0$ if there exist t and q, with $0<t<\infty$ and $q\in\mathbf{Q}$ such that $\varphi(t;q)=x$. It is easily proven that the set of reachable states of (3.1) under the action of all possible $q\in\mathbf{Q}$ is given by the controllable subspace of the pair (A,E). That is

$$\{x:x=\varphi(t;q) \text{ for some } q\in\mathbf{Q}\}=\langle A|\mathrm{im}E\rangle=\mathrm{im}E+\mathrm{im}(AE)+\ldots+\mathrm{im}(A^{n-1}E) \tag{3.2}$$

The following proposition is fundamental for the disturbance decoupling problem and gives the geometric condition for a system of the form (3.1) to be disturbance decoupled.

Proposition 3.1: The system (3.1) is disturbance decoupled if and only i

$$\exists V \in \underline{I}(A) \quad \text{such that} \quad imE \subset V \subset kerD$$

Proof: "if" Assume the system is disturbance decoupled. From Def.3.1 and formula (3.2) it follows that $<A|imE> \subset kerD$. By Cayley-Hamilton theorem it is easy to show that $<A|imE> \in \underline{I}(A)$ and clearly $imE \subset <A|imE>$ "Only if" Let V be such that $AV \subset V$ and $imE \subset V \subset kerD$ then

$$<A|imE> \subset <A|V> = V \subset kerD$$

Thus the condition of Def 3.1 is satisfied and the system is disturbance decoupled ■

The disturbance decoupling problem applies exactly when Def.3.1 or Prop 3.1 is not satisfied by the system (3.1) and is the problem of finding a feedback control law from the measurements y to the control u such that in the closedloop system the controlled outputs are not affected by the disturbances ie., the forced response

$$z(t) = D\int_0^t e^{A_c(t-\tau)} Eq(\tau) = 0 \qquad (3.3)$$

for all $q \in \mathbf{Q}$, where A_c=Closed loop system matrix.

Hence the problem can be visualized as in the following block diagram.

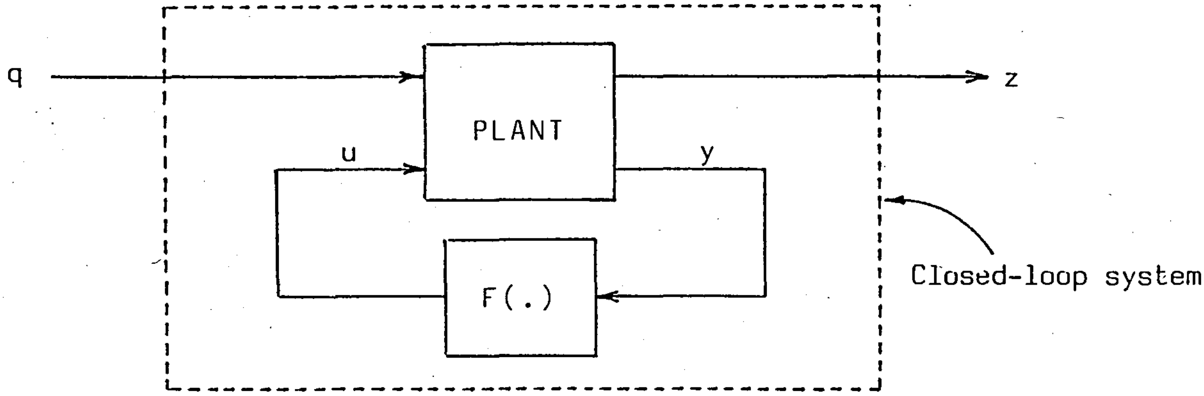


Fig.3.1 DISTURBANCE DECOUPLING PROBLEM (DDP)

1. DISTURBANCE DECOUPLING BY STATE FEEDBACK

The above problem was originally solved by Wonham and Morse assuming that the whole state vector is available (ie., C=I in Fig.3.1). The problem can be defined as follows: Find a feedback control law u=Fx such that the closed loop system

$$\begin{aligned} \mathring{x} &= (A+BF)x+Eq \\ y &= Cx \quad , \quad z=Dx \end{aligned} \tag{3.4}$$

is disturbance decoupled.

From Prop.(3.1) the conditions for the existence of such a feedback matrix are quite obvious.

<u>Theorem 3.1.1.</u>: Disturbance decoupling by state feedback is solvable if and only if

$$\exists V \in \underline{V}(A,B) \quad \text{such that } \operatorname{im}E \subset V \subset \ker D \tag{3.5}$$

<u>Proof</u>: "if" Assume that (3.5) is satisfied. By lemma (2.2) $\exists F:\mathbf{X} \to \mathbf{U}$ such that $V \in \underline{I}(A+BF)$. Hence by Prop.(3.1) the system (3.4) is disturbance decoupled. "Only if" Let $F:\mathbf{X} \to \mathbf{U}$ be a solution of disturbance decoupling by state feedback. That means $\exists V \in \underline{I}(A+BF)$ such that $\operatorname{im}E \subset V \subset \ker D$. Again by lemma (2.2) such a subspace belongs to $\underline{V}(A,B)$ ∎

Condition (3.5) of Theorem (3.1.1) would be useless if we did not have a constructive way of checking it. Fortunately among the (A,B)-invariant subspaces contained in kerD there is a largest one denoted by $V^*(A,B;\ker D)$ which is computable by Algorithm I of Ch.2. So we have the corollary for the solutidn of disturbance decoupling problem by state feedback:

Corollary 3.1.1: Disturbance decoupling by state feedback is solvable if and only if

$$\text{im}E \subset V^*(A,B;\ker D) \qquad (3.6)\blacksquare$$

Theorem 3.1.1 and Cor.(3.1.1) provide a constructive solution to disturbance decoupling problem by state feedback

1. Compute $V^*(A,B;\ker D)$ by Algorithm I of Ch.2 or (A2) of Appendix and check if the condition (3.6) of Cor.(3.1.1) is satisfied

2. Choose any $F \in \underline{F}(V^*)$ as in the proof of lemma (2.2) or by algorithm (A4) of Appendix.

3. $u=Fx$ is the desired control law.

2. DISTURBANCE DECOUPLING BY MEASUREMENT FEEDBACK

Although the theory developed in the preceeding section gives an easy solution to disturbance decoupling problem, it requires that the whole state vector is accessible to direct measurement.This assumption is rather restrictive in practical applications and hence controllers should be of measurement feedback type.

Thus the problem that we pose is to find a feedback map $K:Y \rightarrow U$ such that in the closed loop system

$$\begin{aligned} &\mathring{x}=(A+BKC)x+Eq \\ &y=Cx \quad , \quad z=Dx \end{aligned} \qquad (3.7)$$

resulting from the control law $u=Ky$, the disturbance actions are localized in $\ker D$.

This problem was first solved by Hamano and Furuta in [4]. Theorem (3.2.1) below gives their main result. We first give a definition and a lemma which is motivated by the form of the system matrix in (3.7)

Definition 3.2.1: A subspace $V \subset X$ is said to be an (A,B,C)-invariant subspace if there exists K such that $(A+BKC)V \subset V$.

The set of (A,B,C)-invariant subspaces will be denoted by $\underline{L}(A,B,C)$

Lemma 3.2.1: $\underline{L}(A,B,C)=\underline{V}(A,B)\cap\underline{Q}(C,A)$

Proof: Take $V \in \underline{L}(A,B,C)$. By Def.(3.2.1) there exists a matrix K such that $V \in \underline{I}(A+BKC)$. Setting G=BK and F=KC it is seen that $V \in \underline{I}(A+GC)\cap\underline{I}(A+BF)$ Thus by Lemma (2.2) and Lemma (2.3) $V \in \underline{V}(A,B)\cap\underline{Q}(C,A)$.

For the reverse inclusion take $V \in \underline{V}(A,B)\cap\underline{Q}(C,A)$. Let $\{x_1 \ldots x_j, x_{j+1} \ldots x_k\}$ be a basis for V such that $\{x_1 \ldots x_j\}$ is a basis for $V \cap \ker C$. Because $V \in \underline{V}(A,B)$ there exists $u_i \in U$ and $v_i \in V$ such that

$$Ax_i = v_i + Bu_i \qquad (i=1 \ldots k)$$

Noting that the vectors $\{Cx_{j+1} \ldots Cx_k\}$ are linearly independent we can define $K: Y \to U$ such that

$$KCx_i = -u_i \qquad (i=j+1 \ldots k)$$

Then we have $(A+BKC)x_i = v_i \in V$ for $i=j+1 \ldots k$ but also $(A+BKC)x_i = Ax_i \in V$ for $i=i \ldots j$ because $A(V \cap \ker C) \subset V$. Thus $(A+BKC)V \subset V$ and by Def.(3.2.1) $V \in \underline{L}(A,B,C)$ ■

Theorem 3.2.1 The problem of disturbance decoupling by measurement feedback is solvable if and only if

$$\exists V \in \underline{V}(A,B)\cap\underline{Q}(C,A) \quad \text{such that} \quad \mathrm{im}E \subset V \subset \ker D \qquad (3.8)$$

Proof: The proof is again based on Prop.(3.1) which is fundemental for disturbance decoupling.

"if" Let the closed loop system (3.7) obtained by the control law $u=Ky$ be disturbance decoupled. It follows from Prop.(3.1) that $\exists V \in \underline{I}(A+BKC)$ such that $\mathrm{im}E \subset V \subset \ker D$, The result now follows from Lemma (3.2.1)

"Only if" Assume (3.8) is satisfied. **V** can be made (A+BKC)-invariant by suitable choice of $K:\mathbf{Y} \to \mathbf{U}$ as in the proof of Lemma (3.2.1). Hence the system (3.7) is disturbance decoupled by Prop.(3.1) ■

Once a subspace **V** satisfying (3.8) is given computing a corresponding feedback matrix K such that $(A+BKC)\mathbf{V} \subset \mathbf{V}$ is very easy. Just replace (A+BF) by (A+BKC) in algorithm (A4) of Appendix. The key problem therefore is to check the existence of an (A,B,C) invariant subspace in between imE and kerD and to construct if one exists. This problem is not solved in literature and is open for future research.

Another disadvantage of disturbance decoupling by direct measurement feedback is that the condition (3.8) is stronger than its state feedback counterport (3.5) To overcome these dissiculties disturbance decoupling by dynamic measurement feedback has been propesed in [5], [6] and [7] which is the topic of the next section.

3. DISTURBANCE DECOUPLING BY DYNAMIC MEASUREMENT FEEDBACK

So far the feedback structures used for disturbance decoupling were all linear, time invariant and memoryless. Here we will allow dynamic processing of the measurements before feedback.

We consider the some model (3.1) for the plant and assume that the control input is synthesized by means of the dynamic compensator:

$$\mathring{w}=Nw+My \tag{3.9a}$$
$$u=Lw+Ky \tag{3.9b}$$

where $w \in W$ is the state of the compensator. The order of the compensator is dimW. Combining equations (3.1) and (3.9) gives rise to the closed-loop system with state space $X \oplus W$.

$$\begin{bmatrix} \mathring{x} \\ \mathring{w} \end{bmatrix} = \begin{bmatrix} A+BKC & BL \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} q \tag{3.10a}$$

$$z = \begin{bmatrix} D & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{3.10b}$$

the problem of disturbance decoupling bydynamic measurement feedback can now be formulated as follows: Find the compensator matrices (N,M,K,L) appearing in (3.9) such that the closed loop system (3.10) has zero transfer function from q to z.

The solution to this problem is given in the references cited. We give below an alternative proof which yields the same result.

First we define two mappings between extended state space $X^e = X \oplus W$ and X. The projection $P: X^e \rightarrow X$ is defined by

$$P \begin{bmatrix} x \\ w \end{bmatrix} = x \tag{3.11}$$

and the embedding $S: X \rightarrow X^e$ is defined by

$$Sx = \begin{bmatrix} x \\ 0 \end{bmatrix} \tag{3.12}$$

For a subspace V^e of X^e, we have

$$V_p \overset{\Delta}{=} PV^e = \{x \in X \mid \exists w \in W: \begin{bmatrix} x \\ w \end{bmatrix} \in V^e\} \tag{3.13}$$

$$V_i \overset{\Delta}{=} S^{-1} V^e = \{x \in X \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in V^e\} \tag{3.14}$$

Less precisely, V_p may be viewed as the projection of V^e on the X space, and V_i may be viewed as the intersection of V^e with X space.

Lemma 3.3.1: The closed loop system (3.10) is disturbance decoupled if and only if

$$\exists V \in \underline{I}(A_c) \quad \text{such that} \quad \text{im}E \oplus \mathbf{0} \subset V \subset W \oplus \ker D$$

where A_c is the system matrix of (3.10)

Proof: By noting $\text{im}\begin{bmatrix} E \\ 0 \end{bmatrix} = S(\text{im}E) = \text{im}E \oplus \mathbf{0}$ and $\ker\begin{bmatrix} D & 0 \end{bmatrix} = P^{-1}(\ker D) = W \oplus \ker D$. The lemma is a direct consequence of Prop. (3.1) ■

Lemma 3.3.2: Let $V \subset X^e$ be A_c-invariant then

$$V_i = S^{-1}V \in \underline{Q}(C,A) \text{ and } V_p = PV \in \underline{V}(A,B) \text{ with } V_i \subset V_p$$

Proof: The fact $V_i \subset V_p$ follows directly from (3.13) and (3.14)

To prove that $A(V_i \cap \ker C) \subset V_i$ take $x \in V_i \cap \ker C$: we have to show that $Ax \in V_i$. This follows upon noting that

$$\begin{bmatrix} Ax \\ 0 \end{bmatrix} = \begin{bmatrix} A+BKC & BL \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \in V$$

Next, take $x \in V_p$; we have to prove that $Ax+Bu \in V_p$ for some $u \in U$. Take $w \in W$ such that $\begin{bmatrix} x \\ w \end{bmatrix} \in V$; then

$$\begin{bmatrix} Ax+B(KCx+Lw) \\ MCx+Nw \end{bmatrix} = \begin{bmatrix} A+BKC & BL \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \in V$$

so that $u = KCx+Lw$ suits our purposes. ■

Now the concept of dynamic extension will be introduced:

Definition 3.3.1: Let Σ be the system given by (3.1)
The system Σ^e with input (u,v), state space $X^e = X \oplus W$, measurement (y_1, y_2) and defined by

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} q$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

is called on extension of Σ. The system matrices for the extended system will be respectively denoted by A^e, B^e, E^e, C^e.

According to this definition the extended system is obtained by incorporating q=dimW integrators to the system (3.1) and taking the outputs of the integrators as additional measurements.

Lemma 3.3.3: Any dynamic compensator of the form (3.9) around the system (3.1) is equivalent to a static measurement feedback control applied to the extended system Σ^e.

Proof: The control law $u^e=K^ey^e$ with K^e partitioned as:

$K^e=\begin{bmatrix} K & L \\ M & N \end{bmatrix}$, yields exactly the compensated system (3.10) ■

The following lemma gets us very close to the solution by showing how one can produce (A,B,C)-invariant subspaces by extension.

Lemma 3.3.4: Let $V_1 \in \underline{Q}(C,A)$ and $V_2 \in \underline{V}(A,B)$ with $V_1 \subset V_2$. Then there exists an extension space W of dimension: $\dim W=\dim V_2-\dim V_1$ and an (A^e, B^e, C^e)-invariant subspace V of $X\oplus W$, such that $V_1=S^{-1}V$ and $V_2=PV$.

Proof: Let W be a linear space of dimension: $\dim V_2-\dim V_1$, and let R be a mapping of V_2 onto W such that $\ker R=V_1$. Introduce

$$V=\left\{\begin{bmatrix} x \\ Rx \end{bmatrix}: x\in V_2\right\}\subset X\oplus W$$

Then it is clear that $V_1=S^{-1}V$, $V_2=PV$ and we need only to show that $V\in\underline{L}(A^e,B^e,C^e)$. But this follows by noting that $A^eV\subset V+\mathrm{im}B^e$ which means that $V\in\underline{V}(A^e,B^e)$ and $V\cap\ker C^e=(V_1\cap\ker C)\oplus 0$ so that $A^e(V\cap\ker C^e)\subset V$. Thus $V\in\underline{V}(A^e,B^e)\cap\underline{Q}(C^e,A^e)=\underline{L}(A^e,B^e,C^e)$ by lemma (3.2.1) ■

Lemmas (3.3.2) and (3.3.4) are converses of each other in the sense that Lemma (3.3.2) shows for each $V\in\underline{I}(A_c)$ $PV\in\underline{V}(A,B)$, $S^{-1}V\in\underline{Q}(C,A)$ and (3.3.4) shows that to any pair of subspaces $V_1\in\underline{Q}(C,A)$ and $V_2\in\underline{V}(A,B)$ with $V_1\subset V_2$ there corresponds $K^e:Y^e\rightarrow U^e$ and $V\in\underline{I}(A_c)$.

The following theorem for the solvability of disturbance decoupling problem by dynamic measurement feedback should now be obvious.

Theorem 3.3.1: The problem of disturbance decoupling by dynamic compensation is solvable if and only if there exists a subspace pair (V_1,V_2) such that

$$V_1\in\underline{Q}(C,A),\quad V_2\in\underline{V}(A,B) \text{ and } \mathrm{im}E\subset V_1\subset V_2\subset\ker D \tag{3.15}$$

Proof: Necessity of (3.15) follows from Lemmas (3.3.1) and (3.3.2) Sufficiency can be proved by constructing a subspace $V\in\underline{L}(A^e,B^e,C^e)$ using Lemma (3.3.4) then a map $K^e:Y^e\rightarrow U^e$ as in the proof of Lemma (3.2.1) such that $V\in\underline{I}(A^e+B^eK^eC^e)=\underline{I}(A_c)$ and satisfies Lemma (3.3.1) ■

Of course, one wishes that condition (3.15) of Theorem (3.3.1) can be checked constructively (that is, by an algorithm) if the system parameters are known. This can be achieved by computing the smallest (C,A)-invariant subspace containing imE by algorithm IV of Ch.2 and the largest (A,B)-invariant subspace in kerD by algorithm I. So we arrive at the following constructive criterion.

<u>Corollary (3.3.1)</u>: Disturbance decoupling by dynamic observation feedback is solvable iff

$$Q_*(C,A;\mathrm{im}E) \subset V^*(A,B;\ker D) \qquad (3.16)$$ ■

Several remarks are in order here:

<u>Remark 1</u>: A pair of subspaces (V_1, V_2) having the properties stated in Lemma (3.3.4) is called a (C,A,B) pair by Schumacher and is used effectively in DDP [5] DDEP [11] and Regulator synthesis problems [9].

<u>Remark 2</u>: Condition (3.15) for the solvability of DDP by dynamic compensation is seen to be stronger than (3.5) for the solvability of DDP by state feedback but weaker than (3.8) for the solvability by direct measurement feedback. In short, assuming that all state variables are accessible the following implications hold for the solvability of DDP. DDP by direct measurement feedback ⇒ DDP by dynamic measurement feedback ⇒ DDP by state feedback
In other words (3.8) ⇒ (3.15) ⇒ (3.5)

<u>Remark 3</u>: The order of the feedback compensator that can be designed using the subspace pair in Cor.(3.3.1) is $\dim W = q = \dim V^*(\ker D) - \dim Q_*(\mathrm{im}E)$. This gives an upper bound for the compensator order. The minimal extension order which is necessary for the solution of the problem is given by

$$q^* = \min\{\dim V_2 - \dim V_1 \mid V_1 \in \underline{Q}(C,A),\ V_2 \in \underline{V}(A,B),\ \mathrm{im}E \subset V_1 \subset V_2 \subset \ker D\}$$

A lower bound for q is obviously zero in which case V_1 coincides with V_2 so that disturbance decoupling by direct measurement feedback is possible. Thus static measurement feedback can be viewed as a compensator of order zero. The minimal compensator order q^* is not known and is an interesting research problem.

Remark 4: Suppose that all state variables are available for measurement. In this case one wishes to know if dynamic state feedback in the spirit of Sec.(3.3) brings any improvement for the solvability of disturbance decoupling problem. This can be investigated by setting C=I or kerC=0 in (3.16) so that

$$\mathbf{Q}_*(C,A:\mathrm{im}E)=\inf\{\mathbf{Q}:\ A(\mathbf{Q}\cap\ker C)\subset\mathbf{Q},\ \mathrm{im}E\subset\mathbf{Q}\}=\mathrm{im}E$$

Thus (3.16) reduces to $\mathrm{im}E\subset V^*(A,B:\ker D)$ which is identical to (3.6).

We arrive at the conclusion: There is no difference between static state feedback and dynamic state feedback as for as the solvability of DDP is concerned. Of course we prefer static state feedback because of its simplicitiy.

Remark 5: As a final comment we prove that disturbance decoupling by dynamic measurement feedback is not solvable if $\ker D\subset\ker C$. This can be justified as follows if $\ker D\subset\ker C$, condition (3.16) becomes

$$\mathbf{Q}_*(C\ ,A;\mathrm{im}E)=\langle A|\,\mathrm{im}E\rangle\subset\mathbf{V}^*(A,B;\ker D)\subset\ker D\subset\ker C$$

which means that the original system is already disturbance decoupled contrary to our basic assumption. The result just proved can be stated as: A system that is not disturbance decoupled can not be made so by any form of observation feedback in case the observation is a function of the variable to be regulated.(See also Lemma (4.1.3)

A simple example will now clarify the theory developed

EXAMPE: Let the system (3.1) be given as

$$A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\quad B=\begin{bmatrix}1\\0\\1\end{bmatrix}\quad E=\begin{bmatrix}0\\0\\1\end{bmatrix}\quad C=\begin{bmatrix}1&1&1\end{bmatrix}\ D=\begin{bmatrix}1&0&0\end{bmatrix}$$

It is readily checked that kerD is (A,B)-invariant so that

$$V^*(\ker D) = \mathrm{im}\begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix} \quad \mathrm{im}E \subset V^*(\ker D)$$ thus disturbance decoupling by state feedback is solvable.

But we assume only the measurements y are available for feedback.

$$Q_*(C,A;\mathrm{im}E) = \mathrm{im}\begin{bmatrix}0\\0\\1\end{bmatrix} .$$

Condition (3.16) for disturbance decoupling by dynamic compensation is satisfied.

$V \in \underline{L}(A^e, B^e, C^e)$ is constructed as in the proof of Lemma (3.3.4) For this choose $R: \mathbf{V}_2 \to \mathbf{W}$ to be $R = [0\ 1\ 0]$. Notice that $\ker R = \mathbf{Q}_*(\mathrm{im}E)$.

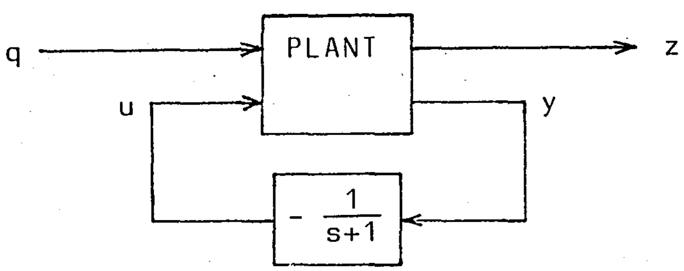
Then $\mathbf{V} = \left\{\begin{bmatrix}x\\Rx\end{bmatrix} : x \in \mathbf{V}_2\right\} = \mathrm{im}\begin{bmatrix}0 & 0\\1 & 0\\0 & 1\\1 & 0\end{bmatrix}$ is (A^e, B^e, C^e)-invariant and can be made $A_c = A^e + B^e K^e C^e$ invariant by suitable $K^e: \mathbf{Y}^e \to \mathbf{U}^e$ (Defining a dynamic feedback law) Such a K^e can be calculated as in Lemma 3.2.1 or by the procedure described in algorithm 4 of Appendix

$$K^e = \begin{bmatrix}K & L\\M & N\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & -1\end{bmatrix}$$

The compensator (3.9) is now given by: $\dot{w} = -w + y$, $u = -w$

F(s) in Fig.(3.1) is: $F(s) = \dfrac{U(s)}{Y(s)} = \dfrac{-1}{s+1}$

The closed loop system is shown in the figure below



IV) DISTURBANCE DECOUPLED ESTIMATION PROBLEM (DDEP)

Consider a linear, time invariant dynamical system represented by

$$\overset{\circ}{x}=Ax+Bu+Eq$$
$$y=Cx \quad , \quad z=Dx \tag{4.1}$$

where $x\in X=\mathbb{R}^n$, $u\in U=\mathbb{R}^m$, $y\in Y=\mathbb{R}^p$ are the state, the input and the observation respectively, $q\in Q=\mathbb{R}^r$ represents the unmeasurable disturbances, $z\in Z=\mathbb{R}^l$ denotes the to be estimated outputs.

The disturbance decoupled estimation problem is the problem of constructing a related system, an observer, driven by u(t) and y(t) of system (4.1) and giving the output $\hat{z}(t)$

$$\overset{\circ}{w}=Nw+My+Gu$$
$$\hat{z}=Lw+Ky \tag{4.2}$$

such that the resulting estimation error

$$e(t)\overset{\Delta}{=}\hat{z}(t)-z(t) \tag{4.3}$$

depends only on the initial conditions x(0), w(0) and not on the disturbance q or on the input u.

Thus the problem can be visualized as in the following block diagram.

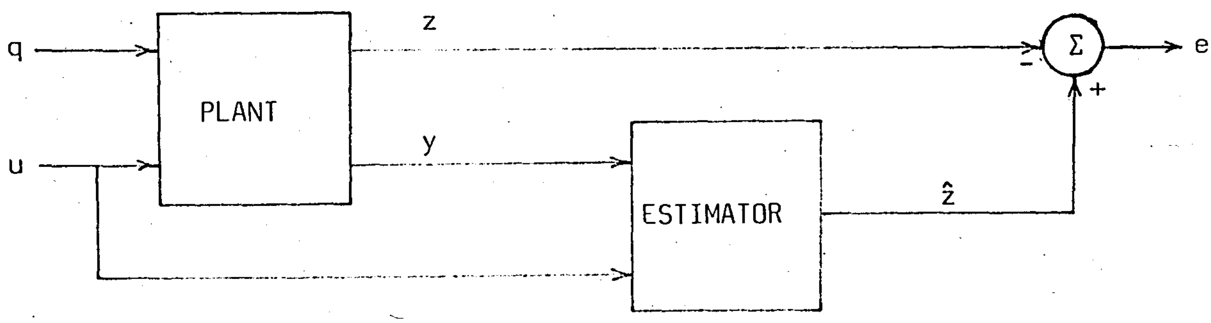


Fig.4.1

In sec.(4.1) it will be shown that this problem formulation allows us to design an estimator of the form (4.2) that estimates the function z=Dx when the initial state x(0) of (4.1) is given.

1. ESTIMATION WITH GIVEN INITIAL STATES

Combining equations (4.1), (4.2) and (4.3) the composite system corresponding Fig.4.1 can be described by

$$\begin{bmatrix} \mathring{x} \\ \mathring{w} \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B & E \\ G & 0 \end{bmatrix} \begin{bmatrix} u \\ q \end{bmatrix} \qquad (4.1.1)$$

$$e=\hat{z}-z= \begin{bmatrix} KC-D & L \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

It follows from the above problem definition that the composite system (4.1.1) must be disturbance decoupled relative to the pair $[u,\ q]$,e if (4.2) is to be an estimator for the plant (4.1). If the composite system matrices are denoted by A_c, B_c and D_c respectively this requirement can be stated formally as:

<u>Lemma 4.1.1</u>: The system (4.2) serves as an estimator for the system (4.1) if and only if there exists on A_c-invariant subspace **V** of the extended state space X^e=X⊕W such that

$$\text{im}B_c \subset V \subset \ker D_c \qquad (4.1.2)$$

<u>ProofX</u> The lemma is a direct consequence of proposition (3.1) and the problem statement ■

For the proof of the main theorem on disturbance decoupied estimator design two more lemmas are needed. The first one is the analog of Lemma (3.3.2). The projection P and the embedding S between X^e and X are defined as in (3.11) and (3.12)

<u>Lemma 4.1.2</u>: Let **V** be an A_c-invariant subspace of **X+W** then

$$V_i \overset{\Delta}{=} S^{-1}V \in \underline{\mathbf{Q}}(C,A) \text{ and } V_p \overset{\Delta}{=} PV \in \underline{I}(A)$$

<u>Proof</u>: Take $x \in V_i \cap \ker C$; then $\begin{bmatrix} x \\ 0 \end{bmatrix} \in V$ and

$$\begin{bmatrix} A & 0 \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Ax \\ 0 \end{bmatrix} \in \mathbf{V}$$ since **V** is A_c-invariant by assumption.

So $Ax \in \mathbf{V}_i$ which proves that $A(\mathbf{V}_i \cap \ker C) \subset \mathbf{V}_i$. hence $\mathbf{V}_i$ is an (C,A)-invariant subspace of $\mathbf{X}$ by definition.

To prove that PV is A-invariant, take $x \in \mathbf{V}_p$ then there exists $w \in \mathbf{W}$ such that $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathbf{V}$. and so

$$\begin{bmatrix} A & 0 \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} Ax \\ MCx+Nw \end{bmatrix} \in \mathbf{V}.$$ Thus $Ax \in \mathbf{V}_p$ ■

The next lemma is a standard result on linear matrix equations.

Lemma 4.1.3: Let M and N be given matrices of arbitrary dimensions. There exists a matrix X of appropriate dimension such that XM=N if and only if $\ker M \subset \ker N$

Proof: "if" Let X be such that XM=N, $u \in \ker M$ implies that $XMu=Nu=0$ thus $u \in \ker N$ proving that $\ker M \subset \ker N$
"Only if" Assume that $\ker M \subset \ker N$. Choose a basis $\{u_1 \ldots u_m, u_{m+1} \ldots u_n\}$ for **X** such that $\{u_{m+1} \ldots u_n\}$ is a basis for kerM. Then

$$Mu_i = Nu_i = 0 \quad \text{for } i=m+1 \ldots n$$

Define the matrix $X_o : \text{im}M \to \mathbf{X}$ by its action on the linearly independent vectors Mu_i $(i=1 \ldots m)$ such that

$$X_o(Mu_i) = Nu_i \quad \text{for } i=1 \ldots m$$

and let X be any extension of X_o from imM to **X** then X satisfies the equation XM=N ■

Theorem 4.1.1: There exists a solution to the disturbance decoupled estimation problem
if and only if

$$\exists \mathbf{Q} \in \mathbf{Q}(C,A:\text{im}E) \text{ such that } \mathbf{Q} \cap \ker C \subset \ker D \qquad (4.1.3)$$

Proof: "Necessity" Operating on both sides of (4.1.2) with S^{-1} one gets:

$$S^{-1}(imB_c) \subset S^{-1}V \subset S^{-1}(kerD_c) \qquad (4.1.4)$$

Setting $V_i = S^{-1}V$ and using the relations $imE \subset S^{-1}(imB_c)$, $S^{-1}(kerD_c) = ker(KC-D)$ (4.1.4) can be simplified to

$$imE \subset V_i \subset ker(KC-D) \qquad (4.1.5)$$

The fact that $V_i \in \underline{Q}(C,A)$ has been proven in Lemma (4.1.2) thus it suffices to show that $V_i \cap kerC \subset kerD$ to complete the proof of the necessity part. For this take any $x \in V_i \cap kerC$ then

$$Dx = (D-KC)x = 0$$

where the first equality is a result of $x \in kerC$ and the second follow from $x \in V_i$, $V_i \subset ker(D-KC)$ by (4.1.5) Thus $x \in kerD$ which implies that $V_i \cap kerC \subset kerD$

"Sufficiency" What needs to be shown is that: given a subspace $Q \in \underline{Q}(C,A;\ imE)$ with $Q \cap kerC \subset kerD$ the observer parameters (N,M,G,L,K) can be chosen such that there exists an A_c-invariant subspace in between imB_c and $kerD_c$ as required by Lemma (4.1.1).

Let V_2 be an A-invariant subspace containing $(Q+imB)$. Such a subspace can always be found because the state space X is A-invariant and $(Q+imB) \subset X$. Define the observer state space W to be a linear space of dimension:

$$dimW = dimV_2 - dimQ$$

Let R be a linear mapping from V_2 onto W with $kerR = Q$ and write

$$V = \left\{ \begin{bmatrix} x \\ Rx \end{bmatrix} : x \in V_2 \right\} \qquad (4.1.6)$$

As **Q** is (C,A)-invariant $G_o: Y \to X$ can be picked such that $(A+G_oC)\mathbf{Q} \subset \mathbf{Q}$. Now lemma (4.1.3.) allows us to define N such that:

$$NRx = R(A+G_oC)x \quad , \text{ for al } x \in V_2 \qquad (4.1.7)$$

because $x \in \mathbf{Q} = \ker R$ implies $(A+G_oC)x \in \mathbf{Q}$. Thus $\ker R \subset \ker R(A+G_oC)$. Set

$$M = -RG_o \qquad (4.1.8)$$

For K we do the following construction.

Let $\{v_i \ldots v_j, v_{j+1} \ldots v_k\}$ be a basis for **Q** such that $\{v_i \ldots v_j\}$ is a basis for $\mathbf{Q} \cap \ker C$. K is defined by its action an the linearly independent vectors Cv_i $(i=j+1 \ldots k)$ as

$$KCv_i = Dv_i \text{ for } i=j+1 \ldots k \qquad (4.1.9)$$

Note that for $i=1 \ldots j$ this relation is automatically satisfied since $KCv_i = Dv_i = 0$ for all $v_i \in \mathbf{Q} \cap \ker C \subset \ker D$ by assumption. Thus $(D-KC)v_i = 0$ for $i=1 \ldots k$ or in other words $\mathbf{Q} \subset \ker(D-KC)$ has been achieved by this construction of K.

Again employing lemma (4.1.3) L can be computed from

$$LR = D-KC \qquad (4.1.10)$$

because $\ker R = \mathbf{Q} \subset \ker(D-KC)$. Finally set

$$G = RB \qquad (.4.1.11)$$

All parameters of the system (4.2) have been specified. Now it remains to check that the so constructed system is indeed an estimator for the plant (4.1). It follows from (4.1.7) and (4.1.8) that

$$(MC+NR)x=RAx \quad \text{for all } x\in V_2$$

so that

$$\begin{bmatrix} A & 0 \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ Rx \end{bmatrix} = \begin{bmatrix} Ax \\ RAx \end{bmatrix} \in \mathbf{V} \text{ for all } x\in V_2$$

so it is seen that the subspace **V** defined by (4.1.6) is A_c-invariant. Moreover it follows from (4.1.10) that $\mathbf{V}\subset\ker[KC-D\ L]=\ker D_c$ and from the assumptions $\mathrm{im}E\subset\mathbf{Q}$, $\mathrm{im}B\subset V_2$ and (4.1.11) that $\mathrm{im}B_c\subset\mathbf{V}$ ■

Remark:1A distinctive property of the estimator designed by the method of Thm.(4.1.1) is its "dead-beat" character. The observer output $\hat{z}(t)$ tracks the to be estimated output $z(t)$ from $t=0$ on if the proper initial condition $w(o)$ is put on the system (4.2), that is the error $e(t)$ defined by (4.3) is identically zero for all $t\geq 0$.

The largest A_c-invariant subspace of $\ker D_c$ which is defined in (2.7) as the unobservable subspace of the composite system (4.1.1), contains the initial values $[x(0),\ w(0)]$ from which the composite system will move in such a way that the error is zero for all time. It was shown in the proof of Thm(4.1.1) that the space **V** defined by (4.1.6) is A_c-invariant and contained in $\ker D_c$ therefore it is a subspace of the unobservable subspace of (4.1.1). Hence if the initial state $x(0)$ of the plant is known the desired dead-beat response can be obtained by starting the estimator system with the initial condition

$$w(0)=Rx(0) \tag{4.1.12}$$

On the other hand if the initial condition $w(0)$ is not properl y chosen the convergence of $\hat{z}(t)$ to $z(t)$ can not be guaranteed because in the problem statement it is only required that the error be independent of the input u and the disturbance q. nothing has been said about the asymptotic stability of the composite system (4.1.1). Thus the proposed method may be viewed as: estimation in the presence of unknown inputs but known initial conditions. In the next section we will allow both the initial state and the input to be unknown.

Theorem (4.1.1) and the remark 1 above, provide a constructive procedure for designing estimators when the initial state x(0) of (4.1) is known. Given a subspace $\mathbf{Q}\subset\mathbf{X}$ satisfiying condition (4.1.3) of Thm.(4.1.1) an estimator can be designed quite easily following the proof of sufficiency. On the other hand there is a constructive way of checking whether (4.1.3) holds or not for a given problem. Recall from Ch.2 that among the (C.A)-invariant subspaces containing a given subspace there is a smallest one which can be computed by algorithm 4 of Ch.2 so (4.1.3) can be checked by computing the subspace $\mathbf{Q}_*(C,A;\mathrm{im}E)$. If (4.1.3) does not hold for this choice of $\mathbf{Q}\in\underline{\mathbf{Q}}(C,A;\mathrm{im}E)$ then none of the members in $\underline{\mathbf{Q}}(C,A;\mathrm{im}E)$ can satisfy (4.1.3) because $\mathbf{Q}_*(\mathrm{im}E)\subset\mathbf{Q}$ and $\mathbf{Q}_*(\mathrm{im}E)\cap\ker C\subset\mathbf{Q}\cap\ker C$ for all $\mathbf{Q}\in\underline{\mathbf{Q}}(C,A;\mathrm{im}E)$. This result is stated as a corollary below.

<u>Corollary 4.1.1</u>: Given the initialstate of (4.1), there exists an observer estimating the function z=Dx, if and only if

$$\ker C\cap\mathbf{Q}_*(C,A;\mathrm{im}E)\subset\ker D \qquad (4.1.13)\ \blacksquare$$

Before closing this section we would like to point out two important aspects of our approach that differs from classical observer theory which will also be discussed by geometric methods in the next section.

<u>Remark 2</u>: The system (4.2) is the most general linear, time invariant, dynamical system that can be thought as an estimator. No assumptions have been made about the structure of the plant or of the estimator. This may be compared with the usual approach to observer design where it is implicitely assumed that the plant (4.1) is controllable and the observer (4.2) is observable.

<u>Remark 3</u>: The common practice in observer design is to take the subspace $\mathbf{V}_2$ introduced in the proof of Thm(4.1.1) to be the state space $\mathbf{X}$. In fact this is the only possible choice for $\mathbf{V}_2$ if the system (4.1) is controllable. In this case one can design an observer of order $(n-\dim\mathbf{Q}_*(\mathrm{im}E))$. Of course, this choice of $\mathbf{V}_2$

need not be optimal. For instance once the subspace $Q_*(\mathrm{im}E)$ has been computed V_2 can be calculated by formula (2.6) to be the minimal A-invariant subspace containing $\mathrm{im}B+Q_*(\mathrm{im}E)$ to reduce the order of the designed observer. This fact is especially clear when imB and/or imE coincides with the zero subspace. No extra dynamics is required to estimate the states of a stable undisturbed system. This is in contrast with the classical approach to observer design which always predicts (n-p) for the order of a minimal order observer

EXAMPLE: Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

and $x(0)=x_0$ is given

$Q_*(C,A;\mathrm{im}E)=\mathrm{im}E=\mathrm{im}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ it is easily verified that (4.1.13) holds

Let $V_2=X=\mathbb{R}^3$ then it is possible to design on estimator or order two.

$R: X \rightarrow W=\mathbb{R}^2$ is given by $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Notice that $\ker R=Q_*(\mathrm{im}E)$

$\underline{G}(Q_*) \triangleq \{G: Y \rightarrow X \mid (A+GC)Q_* \subseteq Q_*\}$ can be calculated by using Lemma (2.3) or algorithm 4 of Appendix as:

$$\underline{G}(Q_*) = \left\{ \begin{bmatrix} 0 \\ -1 \\ g \end{bmatrix} \text{ where } g \in \mathbb{R} \right\}$$

$G_0 \in \underline{G}(Q_*)$ is chosen as $G_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ then $A+G_0C = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Calculate N using (4.1.7): $N = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and M from (4.1.8):

$M = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ K is found from (4.1.9) as: K=0. (4.1.10) gives the value $L = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Finally G is calculated by (4.1.11) to be $G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now the estimator system is given by

$$\overset{\circ}{w} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad w(0)=Rx_0$$

$$\hat{z} = \begin{bmatrix} 1 & 0 \end{bmatrix} w$$

It can be immediately shown that $\mathbf{V} = \left\{ \begin{bmatrix} x \\ Rx \end{bmatrix} , \; x \in \mathbf{X} \right\} = \operatorname{im} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 \end{bmatrix} \quad \text{invariant, contains } \operatorname{im}B_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and is contained in $\ker D_c = \ker \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \end{bmatrix}$

2. OBSERVER DESIGN FOR LTI SYSTEMS WITH UNKNOWN INPUTS AND MEASUREMENT ERRORS

A linear time invariant system of the form (4.2) serves as an observer for the plant

$$\overset{\circ}{x} = Ax + Bu + Eq$$
$$y = Cx + Fd \quad , \quad z = Dx \qquad (4.2.1)$$

if and only if

$$\lim_{t\to\infty} \left[\hat{z}(t) - z(t)\right] = 0 \text{ for al } x_0, w_0, q, d, u \qquad (4.2.2)$$

The new term $d \in \mathbf{D} = \mathbb{R}^s$ in the model (4.2.1) represents the unknown measurement errors which must be taken into account for a more accurate description of the physical problem. Though the observer problem has been studied extensively since the original work of Luenberger [9] it was not solved in literature by taking the measurement errors into consideration so the results below are new with this respect.

Two types of observer can be defined depending on the convergence in (4.2.2) I) STABLE OBSERVERS: The convergence in (4.2.2) takes place with exponents belonging to a "good part" $\mathbb{C}_g$ of the complex plane. The error poles lie in $\mathbb{C}g$.

II) FIXED POLE OBSERVERS: The exponents of convergence in (4.2.2) belong to a specified symmetric set, Λ, of complex numbers. The error poles can be assigned arbitrarily.

The choice of error dynamics is of practical importance in observer problem because the response of the estimator system must be rapid compared with the time constants of the plant if it will be used to implement a feedback control law.

Defining the estimation error as in (4.3) and combining equations (4.2.1) and (4.2) give rise to the following composite system (observer+plant).

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B & E & 0 \\ G & 0 & MF \end{bmatrix} \begin{bmatrix} u \\ q \\ d \end{bmatrix} \tag{4.2.3}$$

$$e = \begin{bmatrix} KC-D & L \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + KFd$$

The following assumptions are made for the plant and the observer:

A1. The pair (A,B) is assumed to be controllable.
A2. The pair (L,N) is assumed to be observable.

Assumption (A2) is by no means restrictive. If the observer is not observable a lower order observer can be designed satisfying the same stability requirement (4.2.2) as the original one by taking the observable part. Thus (A2) is a necessary condition to design a minimal order observer. Assumption (A1) can also be justified because the unwanted signals are modeled separately as the disturbance term q hence it is reasonable to assume that the plant is completely controllable by the control vector u which is at the controller's disposal.

Necessary and sufficient conditions for (4.2) to be an observer for (4.2.1) are given in the theorem below following the works in [10], [11] and [12] and making the required change to accomodate the presence of measurement errors d.

Theorem 4.2.1: The necessary and sufficient conditions for (4.2) to be a stable observer for (4.2.1) are:

$$\sigma(N)\subset\mathbb{C}g \qquad (4.2.4a)$$

and there exists a matrix V such that

$$NV-VA+MC=0 \qquad (4.2.4b) \qquad VE=0 \qquad (4.2.4e)$$
$$D-LV-KC=0 \qquad (4.2.4c) \qquad MF=0 \qquad (4.2.4f)$$
$$G-VB=0 \qquad (4.2.4d) \qquad KF=0 \qquad (4.2.4h)$$

Proof: Sufficiency is proved by defining the estimation errors: $e_1 \overset{\Delta}{=} w-Vx$, $e \overset{\Delta}{=} \hat{z}-z$ and by noting that e_1 is governed by the differential equation:

$$\mathring{e}_1=Ne_1+(NV-VA+MC)x+(G-VB)u-VEq+MFd$$

and $$e=Le_1+(LV+KC-D)\ x+KFd$$

It is clear from these relations that if (4a)-(4g) hold $e(t)\to 0$ as $t\to\infty$ with exponents lying in $\mathbb{C}g$.

Because of the references given if suffices to prove the necessity of only (4f) and (4g) which however follows by writing the transfer function from d to e of (4.2.3) as:

$$KF+L(sI-N)^{-1}MF=0$$

Expanding $(sI-N)^{-1}$ into powers of s^{-1} according to:

$$(sI-N)^{-1} = Is^{-1}+Ns^{-2}+N^2s^{-3}+\ldots..$$

and equating the coefficient of s^{-k} to zero yields

$KF=0$ and $\begin{bmatrix} L \\ LN \\ LN^2 \\ \vdots \end{bmatrix} MF=0$ which implies $MF=0$ since

$\ker \begin{bmatrix} L \\ LN \\ LN^2 \\ \vdots \end{bmatrix} =0$ because of the assumption that the pair (L,N) is observable. ■

The conditions (4a)-(4g) are given in matrix terms in the above theorem. These will be translated into subspace relations in the next theorem. This theorem also shows the power and economy of the geometric approach in a striking way by shrinking the seven conditions (4) to a single subspace inclusion relation.

<u>Theorem 4.2.2</u>: There exists a stable observer for the system (4.2.1) if and only if

$$\exists V_g \in \underline{V}_g(A',\bar{C}';\ker E') \quad \text{such that} \quad \mathrm{im}D' \subset V_g+\mathrm{im}\bar{C}' \qquad (4.2.5)$$

where $\bar{C}=TC$, T being a matrix such that $\ker T=\mathrm{im}F$

<u>Proof</u>: "Only if" if the system (4.2) serves as an observer for (4.2.1) there exists a matrix V satisfying (4a)-(4g). Let $\mathbf{V}$ be a subspace spaned by the columns of V' ie., $\mathbf{V}=\mathrm{im}V'$ and let $T:Y \to Y$ be such that $\ker T=\mathrm{im}F$ then (4f) and (4g) hold if and only if there exists matrices $\bar{M}$, $\bar{K}$ of appropriate dimensions such that

$$M=\bar{M}T \qquad (4.2.6) \qquad \text{and} \qquad K=\bar{K}T \qquad (4.2.7)$$

this follows from Lemma (4.1.3). Writin $\bar{C}=TC$, substituting (6) and (7) into (4b) and (4c) and taking transposes it is seen that $V\in\underline{V}(A',\bar{C}')$. Then by lemma (2.2) at least a map $\tilde{M}'\in\underline{F}(V)$ exists such that

$$(A'-\bar{C}'\tilde{M}')V'=V'N' \qquad (4.2.8)$$

The reason is evident: assuming that (8) does not hold for all $M'\in\underline{F}(V)$ contradicts with the fact that $V\in\underline{V}(A',\bar{C}')$. Now comparing (8) with (2.4) we can conclude that

$$N'=(A'-\bar{C}'\tilde{M}')|V \qquad (4.2.9)$$

In view of (4a), (4e), (8) and the fact that $\sigma(N')=\sigma(N)$ we have from definition (2.5): $V\in\underline{V}g(A',\bar{C}';\ker E')$ and (4c) implies

$$\mathrm{im}D'\subset\mathrm{im}\bar{C}'+V$$

"if" Given a subspace $Vg\in\underline{V}g(A',\bar{C}',\ker E')$ with $\mathrm{im}D'\subset\mathrm{im}\bar{C}'+Vg$ start by computing a suitable $\tilde{M}'\in\underline{F}(Vg)$ satisfying

$$\sigma[(A'-\bar{C}'\tilde{M}')|Vg]\subset\mathbb{C}g \qquad (4.2.10)$$

Let V be a matrix such that

$$\mathrm{im}V'=Vg \qquad (4.2.11)$$

Calculate $\bar{M}$ from: $\bar{M}=V\tilde{M}$ and M from

$$M=\bar{M}T$$

where T is defined as earlier: $\ker T=\mathrm{im}F$. Set

$$N'=(A'-\bar{C}'\tilde{M}')|Vg$$

Calculate L and $\bar{K}$ satisfying

$$LV+\bar{K}\bar{C}=D \tag{4.2.12}$$

and K from

$$K=\bar{K}T$$

Set

$$G=VB \tag{4.2.13}$$

Then (N,M,G,L,K) are the desired observer parameters satisfying (4a)-(4g)

The above theorem gives a design procedure when a subspace Vg is given having the properties required in the theorem statement. On the other hand the existence of such a subspace can be checked contructively by computing the supremal (A',C')-stabilizability subspace in kerE' by algorithm 3 of Ch.2 or (A5) of the appendix so we arrive at the following constructive corollary for the observer design problem for the system (4.2.1)

Corollary 4.2.1: There exists a stable observer for the system (4.2.1) if and only if

$$\mathrm{im}D'\subset \mathrm{im}\bar{C}'+V_g^*(A',\bar{C}';\ker E') \tag{4.2.14}$$

where $\bar{C}$ is defined as in theorem 2. ■

It is seen from the above theorems that the measurement errors d essentially cause the measurements which are corrupted by the noise to be completely discarded. An analogous situation occurs for the disturbance input q as well this will be shown in theorem (4.3.3)

Next we consider the kind of error poles of the observer which is constructed for the subspace Vg satisfying the conditions of theorem 4.2.2 Theorem (2.3) plays a key role here the result is stated as a theorem below.

Theorem 4.2.3: Let $V_g \in \underline{V}_g(A',\bar{C}';\ker E')$ be a stabilizability subspace for which an observer is designed as in theorem 4.2.2 Let $n_o \overset{\Delta}{=} \dim V_g$ and $n_1 \overset{\Delta}{=} \dim R^*(A',\bar{C}';V_g)$. Then among the n_o poles of the observer n_1 can be freely assignable by suitable choice of $F \in \underline{F}(V_g)$ and (n_o-n_1) are fixed but guaranteed to be in $\mathbb{C}_g$.

Proof: Because of the observability of the pair (L,N) the error poles are identical with the observer poles and are given by the eigenvalues of the matrix N. From (4.2.9) and theorem (2.3) we can write

$$\sigma(N')=\sigma(N)=\sigma\left[(A'-\bar{C}'\tilde{M}')\,|\,V_g\right]=\sigma_F\cup\sigma_o \quad \text{where}$$

$$\sigma_F \overset{\Delta}{=} \sigma\left[(A'+\bar{C}'F)\,|\,R^*\right] \text{ is freely assignable and}$$

$$\sigma_o \overset{\Delta}{=} \sigma\left[(A'+\bar{C}'F)\,|\,V_g/R^*\right] \text{ is fixed for all } F \in \underline{F}(V_g)$$

Now the conclusion follows since there are $n_1 \overset{\Delta}{=} \dim R^*$ elements in σ_F and $\dim(\frac{V_g}{R^*})=\dim V_g-\dim R^*=n_o-n_1$ elements in σ_o ■

From the spectral assignability property of controllability subspaces (theorem (2.2)) and theorem 4.2.3 above, we can immediately obtain the solution of the fixed pole observer problem posed at the beginning of the section.

Theorem 4.2.4: There exists a fixed pole observer for the system (4.2.1) If and only if

$$\exists R \in \underline{R}(A',\bar{C}',\ker E') \text{ such that } \operatorname{im} D' \subset R+\operatorname{im}\bar{C}' \qquad (4.2.15)$$ ■

The constructive form of theorem 4 can be given as:

Corollary 4.2.2: There exists a fixed pole observer for the system (4.2.1) if and only if

$$\operatorname{im} D' \subset \operatorname{im}\bar{C}'+R^*(A',\bar{C}';\ker E') \qquad (4.2.16)$$ ■

The above theorems can be proven by exactly the some method used to prove theorem 4.2.2 and Cor.(4.2.1)

Remark 1: It is obvious that theorems 2 and 4, and their corollaries are also applicable to the observer design problem for the special cases when no unknown inputs and/or measurement errors are present. The solution for these cases can be obtained by setting $E=0$ and/or $\bar{C}=C$ respectively. Similarly the results can be specialized to full state observers by taking $D=I$. Thus the problem was considered in its most general formulation except from the assumption (A1) and (A2). It is worth investigating the constraints imposed by these assumptions on the solution of the problem.

Remark 2: The order of the observer which is constructed by the method of Theorem 2 or Theorem 4 is at most equal to

$$\dim(\ker E')=n-r$$

where $r=\operatorname{rank}E=$Number of disturbances. This gives an upper bound for the observer order if a solution of the problem exists.

The order of the minimal order stable observer that estimates the function $z=Dx$ of the system (4.2.1) is given by $\dim V_o$ where

$$V_o \triangleq \inf\{V : V\in \underline{V}g(A',\bar{C}';\ker E'),\ \operatorname{im}D'\subset V+\operatorname{im}\bar{C}'\}$$

A subspace V satisfying the above condition is called a generalized stable cover for imD' (cf.[28], [29]). Thus the equivalance of the minimal stable dynamic cover problem to the minimal order observer design problem is seen. Though there have been a number of papers on the former problem its complete solution is not known yet.

Remark 3: Sometimes the direct feedthrough term K may be constrained to zero in the observer system (4.2) in order to prevent additive measurement noise from passing unfiltered into the estimate of $z=Dx$. An observer for which $K=0$ is called a Kalman Observer and the general case $K\neq 0$ is refered to as a Luenberger Observer.

The necessary and sufficient conditions for the solution of Kalman Observer problem can be obtained by setting K=0 in Theorem (4.2.1) Theorem (4.2.2) Theorem (4.2.4) The results are:

There exists a stable Kalman Observer for the system (4.2.1) if and only if

$$\exists V_g \in \underline{V}_g(A',\bar{C}';\ker E') \quad \text{such that} \quad \operatorname{im} D' \subset V_g \tag{4.2.17}$$

There exists a fixed pole Kalman Observer for the system (4.2.1) if and only if

$$\exists R \in \underline{R}(A',\bar{C}';\ker E') \quad \text{such that} \quad \operatorname{im} D' \subset R \tag{4.2.18}$$

From (4.2.5) and (4.2.15) it is clear that solvability of Kalman Observer problem implies solvability of Luenberger Observer problem but not vice versa.

EXAMPLE: Let the system (4.2.1) be given as:

$$A = \begin{bmatrix} -1 & +1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ D = I, \ F = 0$$

To determine the type of observer that can be designed for this system $R^*(A',C';\ker E')$ is computed by (A3) of Appendix I as:

$$R^*(A',C';\ker E') = \operatorname{im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

. Since the condition of Cor. 4.2.2. is not satisfied a fixed pole observer can not be designed. Next we try a stable observer by defining the stability region as usual $\mathbb{C}_g=\{s\in\mathbb{C}: \operatorname{Re}(s)<0\}$ and computing $V_g^*(A',C':\ker E')$ by (A5) of Appendix I as:

$$V_g^*\ (A',C';\ker E')=\operatorname{im}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&1\end{bmatrix}$$ Cor 4.2.1 is satisfied by this subspace

hence a stable observer can be constructed with one fixed pole and two assignable poles. Following the proof of sufficiency of Theorem 4.2.2.

We compute $\underline{F}(V_g^*)=\{F:(A'-C'F)V_g^*\subset V_g^*\}$ by the method described in (A4) of Appendix I. This gives

$$F=\begin{bmatrix}0&1&f_{13}&1\\f_{21}&f_{22}&f_{23}&f_{24}\end{bmatrix}$$ where $f_{13},f_{21},f_{22},f_{23},f_{24}\in\mathbb{R}$

are arbitrary

and $$A'-C'F=\begin{bmatrix}-1&0&2&0\\1&0&-1&1\\-f_{11}&1-f_{12}&-f_{13}&1-f_{14}\\2-f_{21}&-f_{22}&1-f_{23}&-f_{24}\end{bmatrix}$$ The restriction of this matrix to the

subspace V_g^* is calculated to be:

$$N'=(A'-C'F)|V_g^*=\begin{bmatrix}-1&0&0\\1&0&1\\2-f_{21}&-f_{22}&-f_{24}\end{bmatrix}$$

The characteristic polynomial p(s) of N' is

$$p(s)=\det(sI-N')=(s+1)(s^2+f_{24}s+f_{22})$$

It is seen that one of the error poles is fixed at $s_1=1$ and the remaining poles can be assigned by suitable choice of f_{24}, f_{22}. Letting $f_{21}=f_{13}=0$ and $f_{24}=f_{22}=4$ gives $s_1=s_3=-2$. Thus

$$N'=\begin{bmatrix}-1&0&0\\1&0&1\\2&-4&-4\end{bmatrix}$$ and $\tilde{M}'$ of (4.2.10) is selected to be

$\tilde{M}'=F = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 4 \end{bmatrix}$ Continuing the cnostruction given in theorem (4.2.2)

We have from (4.2.11) $V' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $M=V\tilde{M} = \begin{bmatrix} 0 & 0 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$

L and K are calculated from (4.2.12) as:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$ where the free parameters

are taken as zeros for simplicity. Finally from (4.2.13) we have

$G = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, Thus the observer system (4.2) is given by

$$\mathring{w} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & -4 \\ 0 & 1 & -4 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ 1 & 4 \\ 1 & 4 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$\hat{z}=\hat{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} y$$

Notice that the observer dynamics is used to recover the missing components of the state vector x.

3. OBSERVABLE AND UNOBSERVABLE SUBSPACES FOR SYSTEMS SUBJECT TO UNKNOWN INPUTS

We turn back to the system description given by (4.1) Our aim is to generalize the concepts of observable and unobservable subspaces which are well defined for systems having only the control inputs u; to those systems which are subject to disturbance inputs q as well. This study will lead to a new definition of observability of the system (4.1) and also some interesting results which do not exist in literature.

Definition 4.3.1: The largest subspace of the state space $\mathbf{X}$ on which the orthogonal projection of the state of (4.1) can be estimated is called the observable subspace of (4.1) and denoted by $\mathbf{0}$. The orthogonal complement of $\mathbf{0}$ is defined as the unobservable subspace and denoted by the symbol $\bar{\mathbf{0}}$.

The term "largest" is justified below by showing that the set of subspaces on which the projection of the state vector can be estimated are ordered by subspace inclusion ($\subseteq$)

The estimation method is not specified in Def.(4.3.1). If a fixed pole observer of the form (4.2) is used for estimation purposes then it can be easily verified that the observable subspace which will be denoted by $\mathbf{0}_I$ is given by

$$\mathbf{0}_I = \text{im}C' + \mathbf{R}^*(A',C';\ker E') \tag{4.3.1}$$

To see this, let $D:\mathbf{X} \to \mathbf{X}$ be the ortogonal projection operator on $\mathbf{0}_I$, Now, as $\text{im}D' = \text{im}D = \text{im}C' + \mathbf{R}^*(A',C';\ker E')$ the function $z=Dx$ can be estimated by a fixed pole observer according to Cor.(4.2.2). Furthermore $\mathbf{0}_I$ is the greatest subspace having the stated property because $\mathbf{R}^*(A',C';\ker E')$ is the largest member of the family $\underline{R}(A',C';\ker E')$.

The expression (4.3.1) given for the observable subspace of (4.1) is completely in agreement with the usual definition of observable subspace for systems which are not subject to unknown inputs. If we set $E=0$, $\ker E'=X$ in (4.3.1) and $R^*(A',C';X)$ is calculated by the algorithm 2 of Ch.2, O_I reduces to

$$O_I=\mathrm{im}C'+R^*(A',C';X)=\langle A'|\mathrm{im}C'\rangle$$

which was defined in (2.7) as the observable subspace of (2.1).

By similar reasoning it can be proved that given the initial condition x(0), the unobservable subspace of (4.1) which will be denoted by $\bar{O}_K$ is given as:

$$\bar{O}_K=\ker C\cap Q_*(C,A;\mathrm{im}E) \tag{4.3.2}$$

This follows from Cor.(4.1.1) and Def.(4.3.1)

It is appropriate to identify O_I and O_K as observable subspaces of (4.1) forobservers based on integrators, because the observer system (4.2) used to estimate the ortogonal projection of the state vector on these subspaces can always be realized by $n_1=\dim R^*(A',C';\ker E')$ and $n_2=\dim V^*(A',C';\ker E')$ integrators respectively.

In [15], Basile and Marro showed that by taking successive derivatives of the measurements y(t), it is possible to estimate the ortogonal projectino of the state on the least (E',A')-invariant subspace containing imC'. and it is not possible to find any greater subspace where the projection of the state can be observed because this would mean that a trajectory on the greatest (A,E)-invariant subspace contained in kerC could affect the output y(t), which is clearly a contradiction.

The estimation procedure proposed in [15] was as follows: Start from the observations

$$y(t)=Cx(t) \tag{4.3.3}$$

which in order to emphasize the iterative character of the argument, can be written as

$$y_o(t)=Q_o x(t) \tag{4.3.4}$$

where $y_o(t)=y(t)$ is a vector of known functions of the time and $Q_c=C$ is a known constant matrix. Using solely the vector equation (4.3.4), that is pseudo-inverting the matrix Q_o, the ortogonal projection of the vector x(t) on the image of the transpose of the coefficient matrix can be determined. We denote this subspace by the symbol $\mathbf{Q}_o=\text{im}Q_o'=\text{im}C'$.

In general, more knowledge of the state can be gained by using also the state equation (4.1): in fact, taking the first derivatives of (4.3.4) and using (4.1), we have

$$\mathring{y}_o=(t)=Q_oAx(t)+Q_oBu(t)+Q_oEq(t) \tag{4.3.5}$$

Since the disturbance vector function q(t) is unknown, in order to deduce some information on the state from equation (4.3.5) we must employ its projection on the subspace $[\text{im}Q_oE]^{\perp}=\ker(E'Q_o')$. Letting P_1 denote the orthogonal projection operator on this subspace, we obtain

$$P_1\mathring{y}_o(t)=P_1Q_oAx(t)+P_1Q_o(Bu(t) \tag{4.3.6}$$

A twofold advantage is obtained by this projection; the unknown input is dropped and a vector equation is obtained, both sides of which are again differentiable. In more compact notation (4.3.4) and (4.3.6) can be written together as:

$$y_1=Q_1x(t) \tag{4.3.7}$$

where

$$\mathbf{Q}_1 = \begin{bmatrix} Q_o \\ P_1 Q_o A \end{bmatrix} , \qquad y_1 = \begin{bmatrix} y_o \\ P_1 \dot{y}_o - P_1 Q_o Bu \end{bmatrix}$$

In order to deduce information about the state, it is convenient to employ (4.3.7) instead of (4.3.4), because $\mathbf{Q}_o \subset \mathbf{Q}_1 = \text{im}Q_1'$.
In fact, since

$$\text{im}P_1 = \text{im}P_1' = \{y: E'Q_o'y = 0\} \qquad (4.3.8)$$

is the set of vectors which are mapped by Q_o' into $\ker E'$, the subspace $\text{im}(Q_o'P_1') = Q_o'\text{im}P_1'$ is equal to $\text{im}Q_o' \cap \ker E' = \mathbf{Q}_o \cap \ker E'$ so that the image of the transpose of the coefficient matrix of (4.3.7) is

$$\mathbf{Q}_1 = \text{im}Q_1' = \mathbf{Q}_o + A'(\mathbf{Q}_o \cap \ker E') \qquad (4.3.9)$$

and therefore, $\mathbf{Q}_o \subset \mathbf{Q}$,

Now starting from equation (4.3.7) by means of the some procedure one can derive the equation

$$y_2 = Q_2 x(t) \qquad (4.3.10)$$

which make it possible to determine the projection of the state on the subspace

$$\mathbf{Q}_2 = \text{im}Q_2' = \mathbf{Q}_o + A'(\mathbf{Q}_1 \cap \ker E') \qquad (4.3.11)$$

(4.3.11) can be proved by the same arguments used in the proof of (4.3.9) noting that $A'(\mathbf{Q}_o \cap \ker E') \subset A'(\mathbf{Q}_1 \cap \ker E')$, so that we can have $\mathbf{Q}_o$ instead of $\mathbf{Q}_1$ in the right side of (4.3.11)

Iterating (n-1) times, one finally obtains

$$y_{n-1} = Q_{n-1} x(t) \qquad (4.3.12)$$

where y_{n-1} is a known function of the observations y, inputs u and of their derivatives, and Q_{n-1} is a known matrix such that

$$Q_{n-1}=\text{im}Q'_{n-1}=Q_o+A'(Q_{n-2}\cap\ker E') \qquad (4.3.13)$$

The sequence of subspaces Q_k is nondecreasing and converges to the minimal (E',A')-invariant subspace containing imC' which can be seen by comparing the sequence with the algorithm 4 given in Ch.2 Thus the largest subspace on which the projection of the state can be estimated ie., the observable subspace of (4.1) for observers based on differentiators is

$$O_D=Q_*(E',A';\text{im}C') \qquad (4.3.14)$$

By taking orthogonal complements of subspaces, it follows from Def.(4.3.1) and Theorem (2.1) that the unobservable subspace is given by

$$\bar{O}_D=V^*(A,E;\ker C) \qquad (4.3.15)$$

As for the subspace O_I, O_D reduces to the usual definition of observable subspace if E=0 ie., if there are no unknown inputs. That is for systems which are subject to only control inputs u, the subspaces O_I and O_D coincide and given by

$$O_I=O_D=\langle A'|\text{im}C'\rangle \quad \text{if } q=0, \text{ equivalently if } E=0 \qquad (4.3.16)$$

The precise relation between O_I and O_D will be given after the following theorem

<u>Theorem 4.3.1</u>: Let R*(A,B; kerC)=Largest (A,B)-controllability subspace contained in kerC

V*(A,B;kerC)=Largest (A,B)-invariant subspace contained in kerC
Q_*(C,A; imB)=Minimal (C,A)-invariant subspace containing imB

then

$$R^*(A,B;\ker C)=V^*(A,B;\ker C)\cap Q_*(C,A;\operatorname{im}B) \tag{4.3.17}$$

<u>Proof</u>: Define subspace sequences V_k, Q_k and R_k according to:

$$V_k=\ker C\cap A^{-1}(V_{k-1}+\operatorname{im}B) \quad , \quad V_o=X \tag{4.3.18}$$
$$Q_k=\operatorname{im}B+A(Q_{k-1}\cap\ker C) \quad , \quad Q_o=0 \tag{4.3.19}$$
$$R_k=V^*\cap(AR_{k-1}+\operatorname{im}B) \quad , \quad R_o=0 \tag{4.3.20}$$

It is known that the sequences V_k, Q_k and R_k converge respectively to $V^*(A,B;\ker C)$, $Q_*(C,A;\operatorname{im}B)$ and $R^*(A,B;\ker C)$ (c.f algorithms 1,2.4 of Ch.2). Thus it is enough to show that

$$R_k=V^*\cap Q_k \qquad k=1,2,\dots,n \tag{4.3.21}$$

Since $R_1=V^*\cap\operatorname{im}B$ and $Q_1=\operatorname{im}B$, (4.3.21) is true for k=1. Assuming it holds at k=i there follows

$$\begin{aligned} R_{i+1}&=V^*\cap(AR_i+\operatorname{im}B)=V^*\cap(A(V^*\cap Q_i)+\operatorname{im}B)\\ &\subset V^*\cap(A(\ker C\cap Q_i)+\operatorname{im}B)=V^*\cap Q_{i+1}\end{aligned} \tag{4.3.22}$$

where the induction assumption $R_i=V^*\cap Q_i$ and the fact $V^*\subset\ker C$ is made use of.

For the reverse inclusion, let $x\in V^*\cap Q_{i+1}$ that means $x=At+b$ for some $t\in Q_i\cap\ker C$ and $b\in\operatorname{im}B$. Since $x\in V^*$, $At+b\in V^*$ which implies $At\in V^*+\operatorname{im}B$ or $t\in A^{-1}(V^*+\operatorname{im}B)$. Therefore

$$t\in Q_i\cap\ker C\cap A^{-1}(V^*+\operatorname{im}B)=Q_i\cap V^*=R_i$$

where the induction assumption is used in the last step. Hence $x\in AR_i+\operatorname{im}B$ furthermore $x\in V^*\cap(AR_i+\operatorname{im}B)=R_{i+1}$ which establishes

$$V^*\cap Q_{i+1}\subset R_{i+1} \tag{4.3.23}$$

From (22) and (23), (21) and (17) follow. ∎

<u>Corollary 4.3.1</u>: Let

$\bar{O}_I \triangleq$ Unobservable subspace of (4.1) for observers based on integrators
$\bar{O}_D \triangleq$ Unobservable subspace of (4.1) for observers based on differentiators
$\bar{O}_K \triangleq$ Unobservable subspace of (4.1) given the initial condition x(0).

then

$$\bar{O}_I=\bar{O}_D+\bar{O}_K \quad \text{and} \quad O_I=O_D\cap O_K \qquad (4.3.24a.b)$$

<u>Proof</u>: It suffices to prove only one of expressions (24) the other one follows by taking orthogonal complements of subspaces and recalling the identity:

$$(S_1+S_2)^{\perp}=S_1^{\perp}\cap S_2^{\perp} \qquad (4.3.25)$$

From (4.3.1)

$$Q_I=\text{im}C'+R^*(A',C';\ker E')$$

Applying theorem 1 to write the equivalent expression for $R^*(A',C';\ker E')$ with A',C',E' playing the roles of A,B,C respectively, we have

$$O_I=\text{im}C'+(V^*(A',C';\ker E')\cap Q_*(E',A';\text{im}C')) \qquad (4.3.26)$$

Taking orthogonal complements of subspaces in (26), employing identity (26) and theorem (2.1) repeatedly

$$\bar{O}_I=\ker C\cap(V^*(A,E;\ker C)+Q_*(C,A;\text{im}E)) \qquad (4.3.27)$$

Now the result (24a) follows by applying the distributive rule in (27) which holds since $V^*(A,E;\ker C)\subset\ker C$.

$$\bar{O}_I=V^*(A,E;\ker C)+[\ker C\cap Q_*(C,A;\text{im}E)]$$ ■

The two types of estimation method described in this chapter have been devoloped seperately and independently in literature. The close relationship between the two methods, given by Cor(4.3.1) does not seem to be appreciated before.

Each estimation method has its own advantages and disadvantages. The differentiator based observer of this section provides a one step estimation procedure and has the largest observable subspace of the two types of observers but it has some practical difficulties in its implementation. First, the use of differentiators amplifies noise which is inherent in any kind of measurement scheme; secondly the method has some numerical problems which stems from its one step character: It is very probable that the matrix seen in (4.3.12) is illconditioned. Thus the method is not suitable for on line operation.

The integrator based observer of section (4.2) has no such problems. It can provide an asymptotic; on-line identification of the to be estimated outputs at any desired rate. But these desirable properties are purchased at the price of increasing the unobservable subspace in fact, this price can be seen exactly in Cor.(4.3.1)

We can also find out from Cor(4.3.1) why $\mathbf{O}_I$ and $\mathbf{O}_D$ coincides for a system that has only the control imputs. This is because we have $\bar{O}_K=0$ for a known input system, which can be seen by setting $E=0$ in eqn.(4.3.2) The state x(t) of a system whose initial state x(0) and inputs u(t) are known, can always be estimated for all time t. This is not the case if some of the inputs, q(t) are unknown.

By an alternative line of thought the result of Cor 4.3.1 may be interpreted interms of separation. If $\bar{O}_I=0$ in a given problem, the initial state x(0) can be estimated by an observer based on differentiators, then a dead-beat observer can be designed as in theorem (4.1.1) to estimate the state x(t) for all. time t. Thus the condition $\bar{O}_I=0$ expresses two seperate phases of an estimation problem: 1) Estimation of the initial state x(0) 2) Estimation of x(t), t>0; given the initial state x(0)

Estimators that employ differentiators and integrators simultaneously can also be tried. In this case there may be a compromise between the number of differentiators used, and the dimension of the observable subspace of (4.1) This possibility is clear from the nondecreasing sequence of subspaces $\{Q_k\}$.

The above discussion naturally suggest the following definitions of observability for the system (4.1) Two kinds of observability are defined depending on whether the initial state x(0) is known or not. The first definition on known initial state observability is as follows.

Definition 4.3.2: The system (4.2) is known initial state observable
If and only if

$$O_K \stackrel{\Delta}{=} \ker C \cap Q_*(C,A;\mathrm{im}E) = \mathbf{0} \quad (4.3.28)$$

or equivalently

$$O_K \stackrel{\Delta}{=} \mathrm{im}C' + \mathbf{V}^*(A',C';\ker E') = \mathbf{X} \quad (4.3.29)$$

Unknown initial state observability is simply refered to as observability and defined as:

Definition 4.3.3: The system (4.1) is observable
If and only if

$$O_I \stackrel{\Delta}{=} \mathrm{im}C' + \mathbf{R}^*(A',C';\ker E') = \mathbf{X} \quad (4.3.30)$$

or equivalently

$$\bar{O}_I = \mathbf{V}^*(A,E;\ker C) + [\ker C \cap Q_*(C,A;\mathrm{im}E)] = \mathbf{0} \quad (4.3.31)$$

The above definitions are nothing but specializations of Cor.(4.1.1) and Cor.(4.2.2) to full state observers. Thus a system is called observable whenever there exists a linear time invariant observer of the form (4.2) estimating the states of the system. This result is a well-known theorem in (q=0) case and is the main reason of prefering $O_I=X$ to $\bar{O}_D=0$ as the definition of observability. The latter condition is known as "extended observability" or [16], [17] "strong observability" but does not imply the existence of a full state observer of the form (4.2) whereas Definition 3 implies both Def.2 and the condition $\bar{O}_D=0$ as seen from Cor.(4.3.1)

Theorems 2 and 3 below greatly simplify definition 2 and 3.

<u>Theorem 4.3.2:</u> The system (4.1) is known initial state observable if and only if

$$\ker C \cap \operatorname{im} E = 0 \qquad (4.3.32)$$

<u>Proof</u>: The necessity of (32) follows (29) Since $Q_*(C,A;\operatorname{im}E)$ is a subspace containing imE, its intersection with kerC can not be equal to the zero subspace unless $\operatorname{im}E \cap \ker C=0$
(32) is also sufficient for known initial state observability because subspaces having zero intersection with kerC are (C,A)-invariant thus (28) is satisfied by the subspace $Q_*(C,A;\operatorname{im}E)=\operatorname{im}E$ ■

The next lemma preapares our final theorem an observability of (4.1)

<u>Lemma 4.3.1</u>: Let S be a subspace such that

$$\ker C \subset S \quad \text{and} \quad S \oplus \operatorname{im}E = X \qquad (4.3.33)$$

Let $P: X \to X$ be a projection on S along imE. Then

$$\underline{V}(A,E;\ker C)=\underline{I}(PA;\ker C) \qquad (4.3.34)$$

where $\underline{I}(PA;\ker C)$ denotes the set of PA-invariant subspaces contained in kerC.

Proof: Let $V\in\underline{V}(A,E;\ker C)$. By definition (2.2)

$$AV\subset V+\mathrm{im}E \quad , \quad V\subset\ker C \tag{4.3.35}$$

Operating on both sides of (35) with P.

$$PAV\subset PV=V\subset\ker C$$

which means $V\subset\underline{I}(PA;\ \ker C)$. Hence

$$\underline{V}(A,E;\ker C)\subset\underline{I}(PA;\ker C) \tag{4.3.36}$$

For the reverse inclusion, let $V\in\underline{I}(PA;\ \ker C)$. By definiton (2.1)

$$PAV\subset V \quad , \quad V\subset\ker C \tag{4.3.37}$$

Operating on (37) with P^{-1} (the functional inverse of P)

$$P^{-1}(PAV)=\mathrm{im}E+AV\subset P^{-1}V=V+\mathrm{im}E$$

or $AV\subset V+\mathrm{im}E$. Thus

$$\underline{I}(PA;\ \ker C)\subset\underline{V}(A,E;\ker C) \tag{4.3.38}$$

From (36) and (38) it follows that $\underline{V}(A,E;\ker C)=\underline{I}(PA;\ker C)$ ■

The only assumption required for Lemma (4.3.1) is that

$$\ker C\cap\mathrm{im}E=0 \tag{4.3.39}$$

that is, the system must be known initial state observatable. This, in turn impies that

$$\mathrm{rank}E=r\leqslant\mathrm{rank}C=P \qquad \text{or} \tag{4.3.40}$$

No. of disturbances$\leq$No. of observations

Lemma 4.3.1 has also significance for the easy computation of supremal (A.E)-invariant subspace of kerC. There have been papers of Wonham and Morse [30] and Bhattacharyya [31] on this subject but none of them is applicable when rank E<rank C. Therefore the result of Lemma 1 is complementary to those of [30] and [31]. Formula (2.7) can be used to compute V*(A,E;kerC) as:

$$V^*(A,E;\ker C)=\bigcap_{i=1}^{n} \ker C(PA)^{i-1} \qquad (4.3.41)$$

when (4.3.39) is satisfied which is generically true if $r\leq p$.

<u>Theorem 4.3.3</u> The system (4.1) is observable if and only if $\ker C\cap \mathrm{im}E=0$ and the pair (C,PA) is observable. where P is defined as in Lemma 4.3.1

<u>Proof:</u> The theorem is an immediate consequence of Def. 4.3.3 Thm 4.3.2 and Lemma 4.3.1 ∎

4. ZERO SENSITIVITY OBSERVER DESIGN PROBLEM

In the observer design problems considered so far it has been assumed that the plant parameters (A,B,C,D) are exactly known. But this assumption is somehow unrealistic from a practical point of view. In practice the design is carried out for a nominal parameter set (A_o,B_o,C_o). The actual system matrices are related to the nominal ones by

$$A=A_o+\delta A \quad , \quad B=B_o+\delta B \quad , \quad C=C_o+\delta C \qquad (4.4.1)$$

where $(\delta A, \delta B, \delta C)$ represents a perturbation around nominal values.

If the parameters of the observer are not chosen carefully a nonzero perturbation causes loss of identification of the to be estimated output z and may induce steady state errors as shown in [18].

Efforts have been made in the past for the design of so called "robust observers" which have low sensitivity to parameter to parameter variations [19], [20]. Other authors, [21] [22], treated the same problem with the objective of obtaining complete insensitivity (zero sensivitiy.)

The theory developed in previous sections allows us to give an easy solution to this latter problem at the same time generalizing some of the results of [21], [22]. The problem that we are to solve can be formulated as follows:

Zero Sensitivity Observer Design Problem: Given the system

$$\begin{aligned} \dot{x} &= (A_o+\delta A)x+(B_o+\delta B)u \\ y &= (C_o+\delta C)x \quad , \quad z=Dx \end{aligned} \tag{4.4.2}$$

with nominal parameters (A_o, B_o, C_o) determine, if they exist, the parameters (N,M,G,L,K) of the observer system

$$\begin{aligned} \dot{w} &= Nw+My+Gu \\ \hat{z} &= Lw+Ky \end{aligned} \tag{4.4.3}$$

such that

$\lim_{t\to\infty} [\hat{z}(t)-z(t)]=0$ for all $x(0)$, $w(0)$, u and every perturbation $(\delta A, \delta B, \delta C)$ which is assumed to be completely arbitrary.

The perturbations are assumed to be generated by the variation of real, scalor unknown parameters a_i, b_i, c_i in

$$\delta A=a_1A_1+a_2A_2+\ldots+a_pA_p \tag{4.4.4a}$$

$$\delta B=b_1B_1+b_2B_2+\ldots+b_kB_k \tag{4.4.4b}$$

$$\delta C=c_1C_1+c_2C_2+\ldots+c_sC_s \tag{4.4.4c}$$

with A_i, B_i, C_i being known matrices. This assumption doesnot restrict generality, since by taking the standard basis for the spaces of (nxn), (nxm) and (pxn) matrices any perturbation can be written in the form (4.4.4)

Substituting equations (4.4.4) into (4.4.2) and rearranging result in:

$$\mathring{x}=A_ox+B_ou+[A_1|A_2|\cdots|A_p|B_1|B_2|\ldots|B_k]\begin{bmatrix}a_1x\\a_2x\\\vdots\\a_px\\b_1u\\b_2u\\\vdots\\b_ku\end{bmatrix} \tag{4.4.5a}$$

$$y=C_ox+[C_1|C_2|\ldots|C_s]\begin{bmatrix}c_1x\\c_2x\\\vdots\\c_sx\end{bmatrix} \tag{4.4.5b}$$

As the parameters a_i, b_i, $c_i \in \mathbb{R}$ are arbitrary and may assume any real values the vectors $[a_1x', a_2x' \ldots a_px', b_1u', b_2u' \ldots b_ku']'$ and $[c_1x', c_2x', \ldots, c_sx']'$ effectively act as unknown inputs which are produced internally through variation of system parameters.

Eliminating redundant inputs in (4.4.5) and defining new disturbance matrices E and F such that

$$imE=im\delta A+im\delta B=\sum_{i=1}^{p} imA_i+\sum_{i=1}^{k} imB_i \qquad (4.4.6)$$

$$imF=im\delta C=\sum_{i=1}^{s} imC_i \qquad (4.4.7)$$

equations (4.4.5) take the form

$$\mathring{x}=A_o x+B_o u+Eq$$

$$y=C_o x+Fd \quad , \quad z=Dx$$

where q and d are unknown, unmeasurable signals and E, F are matrices defined as in (4.4.6) and (4.4.7)

In this setting, the problem is seen to be equivalent to the one, just solved in section (4.2). So we have the following theorem for the solution of the zero sensitvitiy observer design problem.

Theorem 4.4.1: There exists a zero sensitive stable observer for the system (4.4.2)
if and only if

$$\exists V_g \in \underline{V}_g(A_o', \bar{C}_o', \ker E') \quad \text{such that} \quad imD' \subset V_g + im\bar{C}_o'$$

where $\bar{C}_o=TC_o$. T being a matrix such that $\ker T=imF=im\delta C$ ■

Equations (4.4.6) and (4.4.7) and theorem 4.4.1 provide a constructive and conceptually clear solution of the problem posed in this section. The existence of a subspace satisyfing the condition of theorem 1 can be checked constructively as in Cor. (4.2.1) then a zero sensitive observer can be designed following the proof of sufficiency of theorem (4.2.2). A fixed pole zero sensitive observer can be designed similarly.

The solvability of the problem might have been improved if there had been bounds on the magnitudes of the variations $\|\delta A\|$, $\|\delta B\|$, $\|\delta C\|$. Although our result is also applicable to this situation it is an open area for future research.

The proposed method differs from the existing results in literature in several ways: 1) More insight is gained by showing the equivalance of the problem to the unknown input observer design problem 2) The presence of measurement errors d in the model (4.2.1) allows us to consider variations in the observation matrix C, which is not treated in [22] 3) Estimation of z=Dx has been studied rather than the special case z=x.

This chapter is closed with the following table summarizing some important results of section (4.3).

		OBSERVABLE SUBSPACE	UNOBSERVABLE SUBSPACE	THE SYSTEM IS OBSERVABLE IFF
x(0) UNKNOWN $q=0$		$O = \langle A' \mid imC' \rangle$	$\bar{O} = \bigcap_{i=1}^{n} \ker(CA^{i-1})$	The pair (C,A) is observable
x(0) KNOWN $q \neq 0$		$O_K = imC' + V^*(A',C';\ker E')$	$\bar{O}_K = \ker C \cap Q_*(C,A;imE)$	$\ker C \cap imE = 0$
x(0) UNKNOWN, $q \neq 0$	For observers based on integrators	$O_I = imC' + R^*(A',C';\ker E')$	$\bar{O}_I = \bar{O}_K + \bar{O}_D$	i) $\ker C \cap imE = 0$
	For observers based on differentiators	$O_D = Q_*(E',A';imC')$	$\bar{O}_D = V^*(A,E;\ker C)$	ii) The pair (C,PA) is observable

Table 4.4.1 OBSERVABLE, UNOBSERVABLE SUBSPACES AND OBSERVABILITY OF THE SYSTEM (4.1)

V. USE OF SYSTEM DECOMPOSITION TO IMPROVE SOLVABILITY OF DDP, DDEP

A common drawback of the disturbance decoupling problems and some of disturbance decoupled estimation problems studied in previous chapters is that, they are generically unsolvable.[1] Certain structural constraints must be satisfied by the system matrices (A,B,C,D,E) for the problems to have a solution. This unpleasant situation is partially a consequence of the model chosen for the disturbances as mentioned earlier.

One possibility to circumvent this difficulty is to design systems having variable structure. A problem which is not solvable for one realization of the system may be solved by adjusting the variable parameters of the system properly.

Linear m-port systems composed of separated lossless and algebraic parts have been used for this purpose in DDP by state feedback before. [24], [25], [26], [27]. The results in these works which will be needed later are summarized below and application of system decomposition to disturbance decoupling by measurement feedback (both static and dynamic) and to disturbance decoupled estimation problems are discussed in subsequent sections.

1. SYSTEMS CONSISTING OF DYNAMIC AND ALGEBRAIC PARTS, TRANSFER FUNCTION INVARIANCE

We consider linear time invariant systems composed of dynamic and algebraic parts as shown in Fig.(5.1.1)

---

1 If the elements of the matrices A,B,C,D,E are listed in arbitrary order and regarded as a data point $p\in\mathbb{R}^N$, $N=n^2+nm+pn+ln+nr$, then nontrivial algebraic equations are satisfied by the elements p_i of p for which the problem is solvable. Thus the points p for which the problem is solvable lie on a hypersurface in $\mathbb{R}^N$ in a small neighborhood of which the problem is unsolvable

A random assignment of the entries of A,B,C etc. will almost surely result in a problem with no solution.

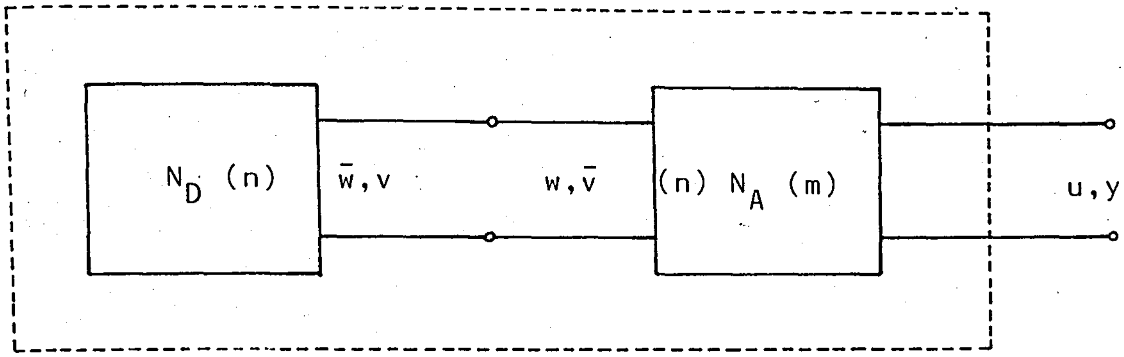


Fig.5.1.1

The algebraic subsystem N_A is defined as

$$w=-F_1\bar{v}-F_2u-E_1q \qquad y=C\bar{v} \;,\quad z=D\bar{v} \tag{5.1.1}$$

and is assumed to be fixed by construction. The dynamic subsystem N_D is described with

$$v=Hx \quad , \quad \mathring{x}=\bar{w} \tag{5.1.2}$$

and can be made to vary by suitable choice of nonsingular H matrices. When the subsystems N_D and N_A are interconnected subject to the constraints:

$$\bar{w}=-w \qquad \text{and} \qquad \bar{v}=v \tag{5.1.3}$$

they give rise to the state space description

$$\mathring{x}=HF_1x+F_2u+E_1q \qquad y=CHx \;,\quad z=DHx \tag{5.1.4}$$

or to the equivalent description

$$\mathring{v}=HF_1v+HF_2u+HE_1q \qquad y=Cv \;,\quad z=Dv \tag{5.1.5}$$

Since the matrices H are nonsigngular the state equations (5.1.4) and (5.1.5) are equivalent under the transformation of variables $v=Hx$.

This kind of system representation arises naturally in electrical networks where for example N_A consists of all algebraic components such as resistors, controlled sources and N_D consists of inductors, capacitors etc. In this case the variables $(\bar{v},w)$ and (u,y) denote hybrid pairs of port voltages and port currents and the elements of the state vector x are flux linkage or charge depending on $\bar{w}$.

In our formulation $u\in\mathbb{R}^m=U$, $y\in\mathbb{R}^p=Y$, $z\in\mathbb{R}^l=Z$ and $q\in\mathbb{R}^r=Q$ denote the input, observation, the controlled output and the unmeasurable disturbance which enters through the algebraic subsystem N_A respectively. We also assume that F_2, E_1, C and D are full rank matrices

In [23] controllability and observability of m-port systems consisting of algebraic and dynamic subsystems and the change of these properties with the selection of the dynamic part N_D have been investigated for the case of no disturbance is present. These results are given without proof in the tables below.

rank F_2	rank $[F_1, F_2]$	CONCLUSION: The m-port system
$=0$	arbitrary	can not be made controllable
$\neq 0$	$<n$	can not be made controllable
$\neq 0$	$=n$	$\exists H$ such that the m-port is controllable
$=n$	$=n$	controllable for all possible choices of H.

Table 5.1.1.A CONTROLLABILITY OF m-PORT DEPENDING ON THE SELECTION OF N_D

rank C	rank $\begin{bmatrix} F_1 \\ C \end{bmatrix}$	CONCLUSION: The m-port system
=0	arbitrary	can not be made observable
≠0	<n	can not be made observable
≠0	=n	∃H such that the m-port is observable
=n	=n	observable for all possible choices of H.

Table 5.1.1B OBSERVABILITY OF m-PORT DEPENDING ON THE SELECTION OF N_D.

It is often desired that a system be both controllable and observable at the same time. For simultaneous controllability and observability of m-port systems composed of fixed algebraic and variable dynamic parts the following theorem is given in [23].

Theorem 5.1.1: The triple (F_1, F_2, C) can be made both controllable and observable with the same matrix H if and only if

$$\text{(i)} F_1 \neq 0,\ \text{rank}[F_1, F_2] = n \quad \text{and} \quad \text{(ii)} C \neq 0\ ,\ \text{rank}\begin{bmatrix} F_1 \\ C \end{bmatrix} = n. \qquad \blacksquare$$

In the some work, algorithms to construct a matrix H that defines the dynamic n-port N_D, which will make a given algebraic (m+n)-port controllable and/or observable are developed.

The advantage of considering the system decomposition is in the additional degree of freedom gained by the variable structure of the subsystem N_D. Among the class of H matrices which realizes a given transfer function matrix one can choose these which makes the solution of a given problem -say DDP or DDEP- possible. The problem of finding the class of n-ports N_D when connected to N_A yield a given transfer function matrix has been studied and solved in [24], [25], [26] and [27] with the objective of improving the solvability of DDP by state feedback. Theorems (5.1.3) and (5.1.4) below summarize some of their results.

We start with the obvious definition of a system matrix.

Definition 5.1.1: Let the state equations of a linear time invariant system be given as:

$$\begin{aligned} \mathring{x} &= Ax+Bu \\ y &= Cx+Du \end{aligned} \qquad (5.1.6)$$

After taking Laplace transforms with zero initial conditions equation (5.1.6) can be written in the form

$$\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Y(s) \end{bmatrix} \qquad (5.1.7)$$

The matrix $P(s) = \begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix}$ which appears in (7) is callad a system matrix or more precisely a polynomial system matrix in state space form.

System matrices are useful in control theory because they contain all the mathematical information about the system furthermore all transformations of the system equations can be expressed as operations on P(s). We are particularly interested in transformations which leave unchanged the transfer function matrix.

Theorem 5.1.2: Consider two completely controllable, completely observable systems described by the system matrices in state space form $P_1(s)$ and $P_2(s)$. Then $P_1(s)$ and $P_2(s)$ give rise to the same transfer function matrix
if and only if they are system similar ie. there exists a constant nonsingular matrix T such that

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & 1 \end{bmatrix} P_1(s) \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} = P_2(s)$$

■

Sufficiency of theorem 2 is obsvious and the necessity follows from the assumption of nimimality. The complete proof can be found for example in [24]. The above theorem will now be used to get the desired result.

Theorem 5.1.3: Let Σ_1 and Σ_2 be two controllable and observable m-ports wiith the same algebraic part (F_1, F_2, D) but possibly different dynamic parts characterized by nonsingular matrices H_1 and H_2 respectively. Σ_1 and Σ_2 have identical $(u \to z)$ transfer functions if and only if there exists constant and nonsingular matrices M and N such that the following equations are satisfied:

$$MF_1N = F_1 \tag{5.1.8a}$$

$$MF_2 = F_2 \tag{5.1.8b}$$

$$DN = D \tag{5.1.8c}$$

$$H_2 = NH_1M \tag{5.1.8d}$$

Proof: According to theorem (5.1.2) Σ_1 and Σ_2 must be similar in order to yield the same transfer function matrix from state spare description (5.1.5) this requirement can be expressed in terms of system matrices as:

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sI-H_2F_1 & H_2F_2 \\ -D & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI-H_1F_1 & H_1F_2 \\ -D & 0 \end{bmatrix}$$

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_2^{-1}s-F_1 & F_2 \\ -D & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_1^{-1}s-F_1 & F_2 \\ -D & 0 \end{bmatrix}$$

$$\begin{bmatrix} H_1^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_2^{-1}s-F_1 & F_2 \\ -D & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} H_1^{-1}s-F_1 & F_2 \\ -D & 0 \end{bmatrix}$$

$$\begin{bmatrix} H_1^{-1}T^{-1}H_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_2^{-1}s-F_1 & F_2 \\ -D & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} H_1^{-1}s-F_1 & F_2 \\ -D & 0 \end{bmatrix}$$

Defining $M \triangleq H_1^{-1} T^{-1} H_2$ and $N \triangleq T$ and equating both sides of the above equation the theorem follows. ■

Given an m-port Σ with an algebraic part (F_1, F_2, D) and a dynamic part characterized by H_1 theorem 3 provides a means of generating all other dynamic subsystems which give rise to the same transfer function matrix as Σ when connected to the algebraic subsystem (F_1, F_2, D). Just compute the M,N couples satisfying (8a), (8b) and (8c) parametrically, the H matrices defining the dynamic subsystems that we looked for, canbe calculated by substituting these values into equation (8d). Notice that equations (8a,b,c) have always the trivial solution M=I, N=I which gives $H_2 = H_1$. The following theorem taken from [24] gives necessary and sufficient conditions for equations (8) to have notrivial solutions.

Theorem 5.1.4.: There exists nontrivial matrix solutions to equations (5.1.8)
if and only if rank $F_2 < n$ and rank $D < n$ ■

Since most systems have input and output spaces of smaller dimension than their state space the conditions of theorem 5.1.4 are usually satisfied and the degree of freedom gained in a certain problem depends on the free parameters obtained in the solution of 5.1.8).

2. SOLVABILITY OF DDP IN A DECOMPOSED m-PORT SYSTEM

Disturbance decoupling problem have been discussed in detail in Chapter 3. Before presenting the application of system-decomposition to DDP the necessary and sufficient conditions for the solvability of DDP by state feedback, static measurement feed-back- dynamic measurement feedback are summarized in the following table for convenience. The model (3.1) is assumed for the plant as usual.

DDP	Necssary and Sufficient conditions for solvability
State feedback	$\exists V \in \underline{V}(A,B;\ker D)$ such that $\operatorname{im} E \subset V$
Static Meas. feedback	$\exists V \in \underline{V}(A,B;\ker D) \cap \underline{Q}(C,A;\operatorname{im} E)$
Dynamic Meas. feedback	$\exists\, Q \in \underline{Q}(C,A;\operatorname{im} E)$ and $\exists V \in \underline{V}(A,B;\ker D)$ st. $Q \subset V$

Table 5.2.1

Remark 1: It is seen from Table (5.2.1) that the necessary condition

$$DE=0 \tag{5.2.1}$$

must be satisfied for the solution of disturbance decoupling problem in all three cases. Moreover it is known that [7] for the subclass of systems with $DE=0$ we have generic solvability for DDP if and only if

$$\text{rank } B \geq \text{rank } D \quad \text{and} \quad \text{rank } C \geq \text{rank } E \tag{5.2.2}$$

Since F_2, E_1, C and D are assumed to be full rank matrices (5.2.2) is equivalent to:

$$\begin{aligned}&\text{No. of controls} \geq \text{No. of controlled outputs}\\&\text{No. of measurements} \geq \text{No. of disturbances}\end{aligned} \tag{5.2.3}$$

for a decomposed system.

With these preliminary results available we are now ready to state the problem.

Problem statement: Consider the decomposed system of Fig.(5.1.1) described by the state equations

$$\begin{aligned}&\dot{x}=H_1F_1x+H_1F_2u+H_1E_1q\\&y=Cx \quad , \quad z=Dx\end{aligned} \tag{5.2.4}$$

Assume that disturbance decoupling problem is not solvable for this realization of the system. The objective is to vary the dynamic subsystem N_D in such a way that the input-output transfer function matrix of (5.2.4) is kept unchanged but the noise component E_1q entering through N_A can be decoupled from the output after a feedback is applied.

Combining theorem (5.1.3) and remark 1 above the following theorem is given as a necessary condition for the solution of this problem.

Theorem 5.2.1: Let N_D be the dynamic subsystem which solves the above problem then the matrix H defining N_D satisfies

$$H = N\ H_1 M$$

where the nonsingular matrices M, N are the solutions of

$$\text{(i) } MF_1N = F_1 \qquad \text{(ii) } MF_2 = F_2 \qquad \text{(iii) } DN = D \qquad \text{(iv) } DH_1ME_1 = 0$$

Proof: The first three condnitions follow from the requirement that H should yield the same transfer function as H_1 does, and the last one is a consequence of (5.2.1) ■

In view of remark 1 the conditions (i)-(iv) of theorem (5.2.1) are almost sufficient for the solution of DDP if conditions (5.2.3) are satisfied. The type of feedback is specified neither in the problem statement nor in theorem (5,2.1), the theorem is applicable to all forms of decoupling listed in Table (5.2.1) provided equations (i)-(iv) have nonsingular solutions M and N.

If transfer function invariance is not required (ie., the input-output transfer function of (5.2.4) is permitted to change with changing dynamic part) then a more powerful result can be given by

Theorem 5.2.2: Given the algebraic subsystem $N_A(F_1,F_2,E_1,C,D)$ there exists a dynamic subsystem N_D (a nonsingular matrix H) such that disturbance decoupling problem is generically solvable in (F_1,F_2 and C) for the system resulting from the interconnection of N_A with N_D
if and only if

$$\text{rank } E_1+\text{rank}<D\ n \qquad (5.2.5a)$$

$$\text{rank } E_1\leqslant \text{rank } C \qquad (5.2.5b)$$

$$\text{rank } D\leqslant \text{rank } F_2 \qquad (5.2.5c)$$

are simultaneously satisfied by N_A.

Proof: When (5.2.1) is satisfied (5.2.5b,c) are necessary ond sufficient for the generic solvability of DDP [7]. Thus, assume that $DHE_1=0$ for some nonsingular matrix H, that means, $\text{im}(HE_1)\subset\ker D$ or $\text{rank } E_1\leqslant \text{null } D=n-\text{rank } D$ from which the necessity of (5.2.5a) follows. Next, assume that $\text{rank } E_1=r\leqslant \text{null } D=n-l$.

Let $\{e_1\ldots e_r\}$ be a basis for $\text{im}E_1$ and $\{d_1\ldots d_{n-l}\}$ be a basis for kerD. Complete $\{e_1\ldots e_r\}$ to a basis for X as $\{e_1\ldots e_r,e_{r+1}\ldots e_n\}$ and $\{d_1\ldots d_{n-l}\}$ to a basis for X as $\{d_1\ldots d_{n-l}, d_{n-l+1}\ldots d_n\}$. Define the nonsingular matrix H characterizing N_D by

$$H\ e_i=di \qquad \text{for} \quad i=1\ldots n$$

This selection of H satisfies (5.2.1) ∎

Notice that (5.2.5 b,c) are not necessary for the solution of a particular problem they are only required for generic solvability. Thus if one or both of (5.2.5b,c) fail to hold one can still construct a dynamic n-port by the procedure described in Thm (5.2.1) or Thm(5.2.2 then if one is lucky enough DDP is solvable for the resulting interconnected system.

Disturbance decoupling by state feedback in decomposed m-ports has been studied extensively in [24], [25], [26] and [27] but the assumption that the whole state vector is accessible to direct measurement is rather restrictive in practical applications so the problem must be solved by measurement feedback ultimately. Towards this aim a condition is given in the following theorem for disturbance decoupling by direct measurement feedback to be equivalent to disturbance decoupling by state feedback.

Theorem 5.2.3: Let $\ker C \cap \ker D$ be A-invariant in the system (3.1) then DDP by static measurement feedback is solvable iff DDP by state feedback is solvable.

Proof: That the disturbance can be decoupled by state feedback if it can be decoupled by measurement feedback follows from Remark(3.3.2). To prove the converse we need the assumption that $\ker C \cap \ker D \in \underline{I}(A)$. Hence $\ker C \cap \ker D$ is an (A,B)-invariant subspace of kerD. Thus

$$\ker C \cap \ker D \subset V^*(A,B;\ker D)$$

on the other hand it is clear that $\ker C \cap \ker D \subset \ker C$. And so

$$\ker C \cap \ker D \subset \ker C \cap V^*(A,B;\ker D) \quad (5.2.6)$$

Since by definition it holds that $V^*(A,B;\ker D) \subset \ker D$ it follows that

$$\ker C \cap V^*(A,B;\ker D) \subset \ker C \cap \ker D \quad (5.2.7)$$

From (5.2.6) and (5.2.7) we have

$$\ker C \cap \ker D = \ker C \cap V^*(A,B;\ker D) \quad (5.2.8)$$

Therefore

$$A(\ker C \cap V^*(A,B;\ker D)) = A(\ker C \cap \ker D) \subset \ker C \cap \ker D = \ker C \cap V^*(A,B;\ker D)$$

or

$$A(\ker C \cap V^*(A,B;\ker D)) \subset V^*(A,B;\ker D)$$

Thus $V^*(A,B;\ker D) \in \underline{Q}(C,A)$ by Def.(2.3). Since DDP by state feedback is solvable $\text{im}E \subset V^*(A,B;\ker D)$. the conclusion now follows from Theorem (3.2.1) because $V^*(A,B;\ker D) \in \underline{V}(A,B;\ker D) \cap \underline{Q}(C,A;\text{im}E)$ ■

The use of Theorem (5.2.3) is several folded. First, it provides a constructive procedure for checking the condition of Theorem (3.2.1) for disturbance decoupling by static measurement feedback, Though the condition

$$\ker C \cap \ker D \in \underline{I}(A) \tag{5.2.9}$$

is not necessary for the solution of DDP. the solvability of DDP by static measurement feedback can be determined as in Cor.(3.1.1) by computing the largest (A,B)-invariant subspace in kerD and checking the condition $\text{im}E \subset V^*(A,B;\ker)$ if (5.2.9) is satisfied in a given system.

In a design problem one usually has a certain degree of freedom in carrying out measurements. In this case the C matrix can be selected to satisfy (5.2.9). A similar situation may arise when constructing the subsystem N_D according to Theorem (5.2.1). After conditions (i)-(iv) have been satisfied, if there still remains some free parameters in the matrix H then these parameters can be chosen such that (5.2.9) holds true.

<u>EXAMPLE</u>: Consider the decomposeed system

$$\begin{aligned} &\mathring{x} = H_oF_1x + H_oF_2u + H_oE_1q \\ &y = Cx \quad , \quad z = Dx \end{aligned} \tag{5.2.10}$$

where

$$H_o = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

From $DH_oE_1=0$ it is readily found that DDP is solvable only if $e_2=e_3=0$. From Theorem (5.1.3) the M, N couples and the class of dynamic n-ports which leave the transfer function of (5.2.10) invariant is found to be:

$$M = \begin{bmatrix} 1/\alpha & 0 & 0 \\ \frac{1-\alpha}{\alpha} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} \alpha & 0 & \alpha-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 2-\alpha & \alpha & -1 \\ \frac{1-\alpha}{\alpha} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\alpha \in \mathbb{R}-\{0\}$ is arbitrary. Substituting these values into (iv) of Theorem (5.2.1) one obtains:

$$e_3=0 \quad , \quad \alpha=\frac{e_1}{e_1-e_2} \quad .$$

Thus any disturbance component of the form $E_1 = \begin{bmatrix} e_1 \\ e_2 \\ 0 \end{bmatrix}$ with with $e_1 \neq 0$ and $e_1 \neq e_2$ can be generically decoupled by suitable choice of α.

As a numarical example let $e_1=1$, $e_2=-1$ then $\alpha=1/2$

$$H = \begin{bmatrix} 3/2 & 1/2 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A=HF_1 = \begin{bmatrix} 2 & 1/2 & 3/2 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B=HF_2 = \begin{bmatrix} -1/2 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and}$$

$$E \stackrel{\Delta}{=} HE_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

It is immediately verified that kerD is (A,B)-invariant and

$$\text{im}E \subset V^*(A,B;\ker D) = \text{im}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ thus DDP by state feedback is solvable.

On the other hand since $\ker D \cap \ker C = 0$ is A-invariant thm (5.2.3) predicts that DDP by static measurement feedback is also solvable.

In fact the subspace $\text{im}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is (A+BKC)-invariant for $K=\begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Hence $u=\begin{bmatrix} -2 \\ 2 \end{bmatrix} y$ is the desired control law.

3. SOLVABILITIY OF DDEP IN A DECOMPOSED m-PORT SYSTEM

As in disturbance decoupling problem some important results of Chapter 4 are summarized in the folowing table. The conclusions are valid for the system (4.1)

DDEP		Necessary and Sufficient conditions for solvability
x(0) KNOWN Dead-beat observer		$\exists Q \in \underline{Q}(C,A;\text{im}E)$ such that $\ker C \cap Q \subset \ker D$
x(0) UNKNOWN	Fixed pole Observer	$\exists V \in \underline{R}(A',C';\ker E')$ such that $\text{im}D' \subset V + \text{im}C'$
	Stable Observer	$\exists V \in \underline{V}g(A',C';\ker E')$ such that $\text{im}D' \subset V + \text{im}C'$

Table 5.3.1

The results in Table (5.3.1) are given to design a Luenberger observer. If a Kalman observer is to be designed (K=0 in (4.2.)) then the conditions should be modified as in (5.2.17) and (5.2.18). In this case the system matrices must satisfy DE=0 if a solution of the problem exists. In the previous section it was shown how this condition can be met without changing the transfer function matrix, by suitable selection of the dynamic n-port N_D in a decomposed system. These results will not be repeated here. We will concentrate on the existence of a Luenberger observer estimating the state vector of a decomposed system described by

$$\mathring{x}=HF_1x+HF_2u+HE_1q \qquad (5.3.1)$$
$$y=Cx \quad , \quad z=x$$

with (F_1,F_2,E_1,C) representing the fixed algebraic subsystem and H defining the variable dynamic subsystem.

As the dynamic part N_D is variable in a decomposed system the interconnected system given by (5.3.1) may not be observable for some choices of the nonsingular matrix H. So the following definition of [23] applies.

<u>Definition 5.3.1</u>: The algebroic (m+n)-port N_A (F_1,F_2,E_1,C) <u>can be made observable</u> if and only if there exists a dynamic n-port N_D (a nonsingular matrix H) such that the m-port resulting from the interconnection of N_A with N_D as depicted by Fig.(5.1.1) is observable; N_A is <u>absolutely observable</u> if and only if (5.3.1) is observable for all nonsingular matrices H.

As in (q=0) case absolute observability of N_A requires that rankC=n. In what follows it will be assumed that rankC<n, that is the system is not absolutely observable, which is the only interesting and nontrivial case.

The main result on "known initial state observability" of decomposed systems is given in the theorem below.

Theorem 5.3.1: Suppose that the initial state x(0) of (5.3.1) is given then the following statements are equivalent

(i) The algebroic (m+n)-port (F_1,F_2,E_1,C) can be made observable
(ii) There exists a nonsingular matrix H such that

$$\ker C\cap \operatorname{im}(HE_1)=0 \quad \text{and } \operatorname{rank}(CHE_1)=\operatorname{rank}E_1=r \qquad (5.3.2a,b)$$

(iii) $\operatorname{rank}E_1=r\leqslant \operatorname{rank}C=p$ (5.3.3)

Proof: According to theorem (4.3.2) the system (5.3.1) is observable with known initial state if and only if $\ker C\cap \operatorname{im}(HE_1)=0$ which is exactly the restatement of (5.3.2a) Next, let $x\in\ker C\cap \operatorname{im}(HE_1)$ and $x\neq 0$ then $Cx=0$ and $x=HE_1q$ for some $q\in Q=\mathbb{R}^r$ hence $CHE_1q=0$ implying $\ker(CHE_1\neq 0$ which contradicts $\operatorname{rank}(CHE_1)=r$. Conversely if $CHE_1q=0$ then x can be defined such that $x=HE_1q$ and $Cx=0$, therefore $x=0$ by (2a) which implies $q=0$ because E_1 is assumed to be full column rank and H is nonsingular. This establishes the equivalence of (i) and (ii)

To prove the equivalence of (ii) to (iii) recall that $r=\operatorname{rank}(CHE_1)\leqslant\min(p,r)$ which shows that $r\leqslant p$. And if $r\leqslant p$ is given we can always wind r linearly indeendent vectors $\{x_1\ldots x_r\}$ in the complement of kerC. Form a matrix M whose columns are the vectors $\{x_1\ldots x_r\}$ Since E_1 and M have the some rank r, E_1 can be transformed to M by elemantary row operations. Let H be the nonsingular matrix representing these operations. This construction of H satisfies (5.3.2a) and (5.3.2b) ■

It is seen from (ii) of the above theorem that the problem of disturbance decoupled estimator design with given initial state is generically solvable in a decomposed system if and only if

$$\text{No. of disturbance}\leqslant\text{No. of measurements} \qquad (5.3.4)$$

because (5.3.2b) fails only if all rxr minors of CHE_1 vanish which obviously represents a hypersurface in $\mathbb{R}^N$, $N=n^2+pn+rn$. (See the footnote on pp. 64)

Because of this large freedom in the choice of H matrices, satisfying theorem 5.3.1 one suspects that the matrix, which will make the (m+n)-port observable may even be found among the class of matrices which leaves the transfer function of (5.3.1) invariant. This possibility is shown by an example below.

The following lemma is given as a necessary condition for the observability of (5.3.1) with unknown initial state.

<u>Lemma 5.3.1</u>: The algebraic subsystem (F_1, F_2, E_1, C) can be made observable only if

$$\text{(i) } \operatorname{rank}E_1 = r < \operatorname{rank}C = p \quad \text{and} \quad \text{(ii) } \operatorname{rank}\begin{bmatrix} F_1 & E_1 \\ C & 0 \end{bmatrix} = n+r$$

<u>Proof</u>: Assume that the algebraic subsystem can be made observable with the matrix H. then the condition

$$\ker C + \operatorname{im}(HE_1) \subseteq X \tag{5.3.5}$$

must be satisfied, otherwise there exists a nontrivial (HF_1, HE_1)-invariant subspace in kerC which contradicts the condition given in Def.(4.3.3) From (5) it follows that

$$\dim(\ker C + \operatorname{im}(HE_1)) < \dim X = n$$

by noting that $\ker C \cap \operatorname{im}(HE_1) = 0$ there follows: $\operatorname{null}C + \operatorname{rank}E_1 < n$ or $r < p$.

On the other hand the condition $V^*(HF_1, HE_1; \ker C) = \mathbf{0}$ implies that

$$\sup\{V: (HF_1 + HE_1 L)V \subset V \subset \ker C\} = \mathbf{0} \text{ for all } L: X \rightarrow \mathbf{Q}$$

Thus it is seen that the pair (C,F_1+E_1L) can be made observable by the matrix H for all $L:\mathbf{X}\to\mathbf{Q}$. Now in view of the condition given in Table (5.1.1B) we can conclude that

$$\text{rank}\begin{bmatrix} F_1+E_1L \\ C \end{bmatrix}=n \quad \text{or}\ \ker\begin{bmatrix} F_1+E_1L \\ C \end{bmatrix}=0 \quad \forall L:\mathbf{X}\to\mathbf{Q}$$

that is, $\begin{bmatrix} F_1+E_1L \\ C \end{bmatrix}$ $x=0$ implies $x=0$. Writing $q=Lx$, it follows that

$$x=0,\ q=0 \text{ is required for } \begin{bmatrix} F_1+E_1L \\ C \end{bmatrix} x=\begin{bmatrix} F_1 & E_1 \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ q \end{bmatrix}=0.$$

Therefore

$$\ker\begin{bmatrix} F_1 & E_1 \\ C & 0 \end{bmatrix}=0 \quad \text{or} \quad \text{rank}\begin{bmatrix} F_1 & E_1 \\ C & 0 \end{bmatrix}=n+r$$ ■

As expected, the conditions of Lemma 5.3.1 strenghten those given in Thm. 5.3.1 for known initial state observability and those of Table (5.1.1B) for known input observability. Formulation of necessary and sufficient conditions for unknown initial state, unknown input observability of decomposed systems is under investigation. Neverthless when Lemma 5.3.1 is satisfied, the algorithm of [23] can be applied to the pair (C,F_1+E_1L) by writing the matrix L in terms of its elements $\{l_{ij}\}$ and each time checking condition (5.3.2) after finding a matrix H. But this essentially becomes a trial and error procedure.

<u>EXAMPLE</u>: The decomposed system of example (5.2.1) is considered which is rewritten for convenience:

$$H_o=\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad F_1=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad F_2=\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},\quad C=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad E_1=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

One more measurement is added to satisfy (i) of Lemma 5.3.1

$$\ker C \cap \operatorname{im}(H_o E_1) = \operatorname{im}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq \mathbf{0}.$$

Hence the system is not observable with this selection of the dynamic part even if the initial state is given.

Taking $H=\begin{bmatrix} 3/2 & 1/2 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ it is easily verified that (5.3.2) is satisfied thus the composite system is known initial state observable. Moreover it was shown in example (5.2.1) that this choice of the matrix H does n ot change the u to z transfer function of (5.3.1)

The dynamic subsystem defined by $H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, makes the system observable.

APPENDIX I

SUBSPACE ALGORITHMS AND THEIR MATRIX EQUIVALENTS

It is prefered to delay computational algorithms and to collect them in an appendix in order not to interrupt the theoretical developement of the work. Matrix algorithms will be given below for the basic operations involving subspaces such as subspace addition, subspace intersection etc. and for the computation of supremal (A,B)-invariant, controllability, stabilizability subspaces and infimal (C,A)-invariant subspace.

Some of these algorithms may be found in the exercises of [1], some have been collected by a literature survey and a few of them like minimal (C,A)-invariant subspace algorithm, supremal stabilizability subspace algorithm are believed to be new. The algorithms are not claimed to be optimal with respect to numerical behaviour, neverthless they provide mechanized procedures for the solution of synthesis problems discussed in previous chapters. Much work still remains to be done in order to implement the algorithms on a digital computer.

A1. BASIC OPERATIONS ON SUBSPACES:

A subspace S is represented by a matrix S whose columns span the subspace, that is S=imS. It is assumed that the matrix S has linearly independent columns and is named to be the <u>basis matrix</u> of S. It is clear that the basis matrix of a subspace is not unique and depends on the particular coordination chosen for the subspace. Conversely from any matrix M a basis matrix can be obtained by eliminating redundant columns of M.

If M,X,Y are matrices, with M given, a <u>maximal solution</u> of the equation MX=0 (resp. YM=0) is a solution X (resp. Y) of maximal rank, having linearly independent columns (resp. rows) and written as $X=M^{\vdash}$ (res $Y=M^{\perp}$). Thus X is a basis matrix for kerM and Y' is a basis matrix for kerM'.

Basic operations on subspaces and their matrix counterparts are summarized in Table A1 below. R and S denote subspaces of X, A is an abritrary map and A^{-1} denotes the functional inverse of A defined by

$$A^{-1}S=\{x\in X:\ Ax\in S\} \qquad (1)$$

	Subspace Operation	Matrix Equivalent
1	S or R	S or R
2	$S^{\perp}$	$S^{\perp}$
3	dim S	rank S
4	S+R	$[S,\ R]$
5	$S\cap R$	$\begin{bmatrix} S^{\perp} \\ R^{\perp} \end{bmatrix}^{\vdash}$
6	$R\subset S$	rank $S=$rank$[R,S]$
7	$R=S$	rank $R=$rank$[R,S]=$rank$S$
8	AS	AS
9	$A^{-1}S$	$(S^{\perp}A)^{\vdash}$

TABLE A1

In the above table the matrix equivalents of the operations are given in terms of the image of the matrix on the right. The symbol im(-) is not written explicitely, the matrix is understood to be a basis matrix of the subspace considered after the elimination of linearly dependent columns. The only exception of this rule is for the orthogonal complement of a subspace in which case the equivalence is given as ($S^{\perp}$=row space $S^{\perp}$) since $S^{\perp}\subset X'$. The formula for $S\cap R$ follows from the well known identity:

$$(S\cap R)^{\perp}=S^{\perp}+R^{\perp} \qquad (2)$$

and the matrix equivalent of $A^{-1}S$ is a consequence of the relationship:

$$(A^{-1}S)^{\perp}=A'S^{\perp} \qquad (3)$$

The basic relations in Table A1 can be used to convert more complicated subspace algorithms to their matrix equivalents.

A2. COMPUTATION OF $\mathbf{V}^*(A,B;\mathbf{K})=\sup\{\mathbf{V}:A\mathbf{V}\subset\mathbf{V}+\mathrm{im}B,\ \mathbf{V}\subset\mathbf{K}\}$

Writing algorithm (I) of Ch.2 in matrices one has:

<u>step 1</u>: Set $k=0$

<u>step 2</u>: Let V_k be a basis matrix for $\mathbf{K}$. (ie $\mathrm{im}V_o=\mathbf{K}$)

<u>step 3</u>: Calculate W_{k+1} from: $W_{k+1}=[B,\ V_k]^{\perp}$

<u>step 4</u>: Let $V_{k+1}=\begin{bmatrix} V_o^{\perp} \\ W_{k+1}A \end{bmatrix}^{\vdash}$

<u>step 5</u>: If rank V_{k+1}=rank V_k set $\mathbf{V}^*(\mathbf{K})=\mathrm{im}V_k$ and stop. Else

<u>step 6</u>: Increment k by one and return to step 3

The sequence $\mathrm{im}V_k$ converges to the maximal (A,B)-invariant subspace of $\mathbf{K}$ in at most $k^*=\dim\mathbf{K}$ iterations.

A3. COMPUTATION OF $R^*(A,B;K)=\sup\underline{R}(A,B;K)$

(Algorithm II in Ch.2)

<u>step 1</u>: Calculate $\mathbf{V}^*(A,B;\mathbf{K})\triangleq\mathrm{im}V^*$ using algorithm (2) above.

<u>step 2</u>: Let W^* be a maximal solution of $W^*V^*=0$. That is $W^*=V^{*\perp}$

<u>step 3</u>: Set $k=1$ and $S_{k-1}=0$

<u>step 4</u>: Calculate T_k as $T_k=[AS_{k-1},\ B]^{\perp}$

<u>step 5</u>: Let $S_k=\begin{bmatrix} W^* \\ T_k \end{bmatrix}^{\vdash}$

<u>step 6</u>: If rank S_k=rank S_{k-1} set $\mathbf{R}^*(K)=\mathrm{im}S_k$ and stop. Else

<u>step 7:</u> Increment k by one and return to step 4.

The sequence $\mathrm{im}S_k$ converges to the maximal controllability subspace of K in at most $n=\dim\mathbf{X}$ iterations.

A4. COMPUTATION OF $\underline{F}(V^*)=\{F:X \to U \mid (A+BF)V^* \subset V^*\}$

<u>step 1</u> Let V^* be a basis matrix of the subspace V^*
<u>step 2</u>: Calculate the maximal solution W^* of $W^*V^*=0$
<u>step 3:</u> Write the matrix F in terms of its elements $[f_{ij}]$
<u>step 4</u>: Determine $[f_{ij}]$ $i=1,\dots,m$ $j=i,\dots,n$ from the equation:

$$W^*(A+BF)V^*=0 \quad \text{or} \quad W^*BFV^*=-W^*AV^*$$

A5. COMPUTATION OF $V_g^*(K)=\sup\{V\in\underline{V}(A,B;K),\ \exists F\in\underline{F}(V),\sigma[(A+BF)\mid V]\subset\mathbb{C}_g\}$
(Algorithm III of Ch.2)

<u>step 1</u>: Compute $V^*(K)$, $R^*(K)$ and their basis matrices V^*,R^* using algorithms (2) and (3).
<u>step 2</u>: Let S_0 be a complement of R^* in X and S_1 be a complement of R^* in V^* with basis matrices S_0 and S_1 respectively. In short

$$X=R^*+S_0 \qquad S_0=\text{im}S_0$$
$$V^*=R^*+S_1 \qquad S_1=\text{im}S_1$$

<u>step 3</u>: Let P_0 and P_1 be such that

$$P_0[R^*,S_\sigma] = [0,I_\sigma] \qquad \text{where } \sigma=\dim S_0$$
$$P_1[R^*,S_1] = [0,I_\rho] \qquad \text{where } \rho=\dim S_1$$

<u>step 4</u>: Choose any map F_0 from the class $\underline{F}(V^*)$ and write $A_0\overset{\Delta}{=}A+BF_0$
<u>step 5</u>: Calculate the map induced by A_0 in X/R^* and its restriction to V^*/R^* as:

$$A_0\mid X/R^*=P_0A_0S_0$$
$$A_0\mid V^*/R^*=P_1A_0S_1$$

<u>step 6</u>: Let $\alpha(\lambda)$ be the minimal polynomial of $A_0\mid V^*/R^*=P_1A_0S$.
Factor $\alpha(\lambda)=\alpha_g(\lambda)\alpha_b(\lambda)$ where the zeros of $\alpha_g(\lambda)$ (resp. $\alpha_b(\lambda)$) belong to $\mathbb{C}_g$ (resp. $\mathbb{C}_b$)
<u>step 7:</u> Calculate the basis matrix (P_0V^*) of V^*/R^* and form the matrix

$$M = \begin{bmatrix} \alpha_g(P_oV^*)^{\perp} \\ \alpha_g(P_oA_oS_o) \end{bmatrix}$$

step 8: The supremal stabilizability subspace $V_g^*(K)$ in K is given by:

$$V_g^*(K)=\mathrm{im}V_g^* \qquad \text{where} \qquad V_g^*=(MP_o)^{\vdash}$$

A6. COMPUTATION OF $Q_*(C,A;K)=\inf\{Q:A(\ker C\cap Q)\subset Q,\ K\subset Q\}$
(Algorithm IV in Ch.2)

step 1: Let K be a basis matrix for the subspace K.
step 2: Set k=0 and $Q_k=K$

step 3: Calculate T_k from $T_k=\begin{bmatrix} Q_{k\perp} \\ C \end{bmatrix}^{\vdash}$

step 4: Let $Q_{k+1}=[K,AT_k]$
step 5: If rank Q_k=rank Q_{k+1} set $Q_*(K)=\mathrm{im}Q_k$ and stop. Else
step 6: Increment k by one and return to step 3.

The sequence of subspaces $\mathrm{im}Q_k$ convergees to the minimal C,A)-invariant subspace containing K. in at most n-dim K steps.

IV. CONCLUSION

Disturbance decoupling and disturbance decoupled estimation problems in linear time invariant dynamical systems have been studied in a common framework using geometric approach. When concluding we would like to emphasize some contributions of the work and suggest some open areas for future research.

It has been assumed throughout the thesis that the disturbance signals are totally unknown and may take any real values. One of the extensions of the results is to examine the same problems and others under the condition that the disturbance are again arbitrary but bounded signals. This point is a practical and largely unexplored field.

Another general comment is that the problems have beentreated in time domain using state space representations whereas it may be true that a reformulation of the results in frequency domain through polynomial system matrices will be more effective in some cases.

The disturbance decoupling problem has been solved in Ch.3 starting from the simplest form of decoupling. Disturbance decoupling by state feedback followed by more advanced forms of decoupling, by static and dynamic measurement feedback. The problem of disturbance decoupling by dynamic measurement feedback has been solved by showing that any dynamic compensation around a system is equivalent to a direct output feedback applied to a properly augmented system. This observation may found application in other problems where dynamic compensation is employed. Constructive solvability criteria have been given for both disturbance decoupling by state feedback and dynamic measurement feedback. Though theorem 3 of sec.(5.2) gives a verifiable condition for the solvability of disturbance decoupling by static measurement feed-back which can be used in some cases the general case remains unsolved. It has been also proven in sec.(3.3) that disturbance decoupling by dynamic state feedback is equivalent to disturbance decoupling by constant state feedback. Thus, dynamic state feedback brings no improvement on the disturbance decoupling problem as far as solvability is concerned.

In Ch.4 the disturbance decoupled estimation problem or what amounts to the same thing, the unknown input observer design problem has been solved in its up to date most general form by taking the measurement errors into consideration as well. The problem has been studied for two different observer structures, one based on integrators and the other based or differentiators. Advantages and drawbacks of each approach have been discussed in sec.(4.3) and the close relationship between the two methods has been pointed out in Cor.(4.3.1). This result which has important consequences is totally new and ties two theories together which have been developed separately and independently in literature. Then a new condition has been given for the observability of an unknown input system based on the new definitions of observable and unobservable subspaces.

The result of Lemma (4.3.1) can be used in conjunction with those of [30] and [31] to characterize (A,B)-invariant subspaces and for the easy computation of supremal (A,B)-invariant subspace of kerC.

Minimal order observer and minimal order compensator synthesis problems have not been solved in the thesis but their equivalance to generalized dynnamic cover problem [28] [29] has been shown and some hints have been given at the end of Ch.2 for the solution of this problem.

Solvability of disturbance decoupling problem and disturbance decoupled estimation problem in a decomposed system which consists of algebraic and dynamic subsystems has been investigated in Ch.5. The use of decomposed systems to improve the solvability of disturbance decoupling problem by state feedback has been originally.

proposed in [24], [25], and [26]. The results of these works have been extended to disturbance decoupling by static and dynamic measurement feedback in sec.(5.2). Necessary and generically sufficient conditions have been given for the existence of dynamic subsystems which, when connected to the algebraic subsystem yield a given transfer function matrix and at the some time making it possible to decouple a noise component after some form of feedback is applied.

Of course, there is room for future research here. Our results have been given in terms of genericity. One may go about to give sufficient conditions for disturbance decoupling to be exactly possible by suitable selection of the dynamic subsystem N_D.

Known initial state observability of decomposed systems has been completely solved by Theorem 1 of sec.(5.3). The result of this theorem clearly reveals the advantage of considering system decomposition. Some necessary conditions have been obtained for unknown input, unknown initial state observability of decomposed systems which strenghtenes those given in [23] for the case where all inputs are known.

Finally some computational algorithms have been collected in the appendix to translate the relatively abstract synthesis methods developed in the work to everyday matrix arithmetic suitable for computer implementation. Some of these algorithms are believed to be new but much work remains to be done in order to obtain good, numerically stable algorithms.

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