# SCHUBERT VARIETIES OF GRASSMANNIANS

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# ABSTRACT

## SCHUBERT VARIETIES OF GRASSMANNIANS

In this endeavor, studying Grassmannians in affine and projective space is essential. We firstly present Grassmannians as varieties then divide Grassmannians into disjoint Schubert cells and apply Plücker embedding to realize Grassmannian as a subvariety of projective space. We also briefly introduce the similar concepts for Flag manifolds. In this thesis, one of our aim is to expand the underlying ideas behind the Schubert calculus. For this purpose, we introduce some enumerative problems. We also review the Schubert polynomials and Schur functions. Finally, we associate the ring of symmetric polynomials with the cohomology ring of Grassmannians by indexing the Schubert class of cohomology ring of Grassmannians with Schur functions. To understand the product of Schur functions Pieri's formula is given. Moreover, we associate the products of Schur functions with the intersection of Schubert varieties of Grassmannians. Similarly, to understand the product of Schubert polynomials Monk's rule is given. Then the Schubert class of cohomology ring of Flag manifold is indexed by Schubert polynomials and we associate the products of Schubert polynomials with the intersection of Schubert varieties of Flag manifold.

# ÖZET

## GRASSMANN ÇOKKATLISININ SCHUBERT VARYETELERİ

Bu çalışmada, afin ve izdüşümsel uzaylarda Grassmann çokkatlısını çalışmak esas alınmıştır. Öncelikle Grassmann'ın bir varyete olduğunu fark edip sonra onu ayrışık Schubert hücrelerine böldük ve Plücker gömmesini Grassmann çokkatlısını izdüşümsel uzayın altvaryetesi olarak fark etmek için uyguladık. Benzer konseptleri kısaca Flag çokkatlısı için de tanıttık. Bu tezde, amaçlarımızdan biri de Schubert kalkülüsün ana fikirlerini açmak oldu. Bu amaçla, bazı enumerative problemler tanıttık. Aynı zamanda Schubert polinomları ve Schur fonksiyonları için bir tekrar yaptık. Son olarak da Grassmann çokkatlısının eşbenzeti halkasının Schubert sınıflarını Schur fonksiyonlarıyla indeksleyerek simetrik polinomlar halkası Grassmann çokkatlısının eşbenzeti halkasıyla ilişkilendirildi. Schur fonksiyonlarının çarpmasını anlamak için Pieri formülü verildi. Dahası Schur fonksiyonlarının çarpmasıyla Grassmann çok katlısının eşbenzeti halkasının Schubert sınıflarının kesişimi arasında bir ilişki kuruldu. Benzer şekilde, Schubert polinomlarının çarpmasını anlamak için Monk's kuralı verildi. Daha sonra Flag çokkatlısının eşbenzeti halkasının Schubert sınıfları Schubert polinomları ile indekslendi ve Schubert polinomlarının çarpması Flag çokkatlısının Schubert varyetelerinin kesişimiyle ilşkilendirldi.

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# LIST OF SYMBOLS

| | |
|---|---|
| $:=$ | Equality that includes definition |
| $\subset$ | Subset |
| $\mathbb{C}^n$ | n-dimensional complex coordinate space |
| dim(V) | The dimension of V |
| GL(n,$\mathbb{C}$) | The set of $n \times n$ invertible matrices whose entries in $\mathbb{C}$ |
| $Hom_{\mathbb{Z}}(A, \mathbb{Z})$ | The set of all $\mathbb{Z}$-module homomorphisms from A to $\mathbb{Z}$ |
| $k_0, ..., \check{k_\lambda}, ..., k_n$ | Ignore the $\lambda^{th}$ term of the sequence |
| (p(0),...,p(n)) | The point of $\mathbb{P}^n$ |
| $p_i(j)$ | The entry of a matrix lying in the $i^{th}$ row and $j^{th}$ column |
| $[p_i(j)]$ | Matrix with the entries $p_i(j)$ |
| $p(k_0...k_n)$ | The determinant of (n+1)×(n+1) matrix whose columns are $k_0, ..., k_n$ |
| $\mathbb{P}^n$ | n-dimensional complex projective space |
| $S_n$ | Symmetric group on n-elements |
| $S_\infty$ | The union of $S_n$'s over all $n \in \mathbb{N}$ |
| $S^n$ | n-dimensional unit sphere |
| $T^2$ | Ring torus |
| $\mathbb{Z}$ | The set of integers |
| $\delta_{m,n}$ | Kronocker delta function |
| $\wp_n$ | The ring of polynomials with integral coefficients in n variables |
| $\Lambda_n$ | The ring of symmetric polynomials with integral coefficients and n variables |

# 1. INTRODUCTION

In 1874, Schubert coped with finding the number of geometric objects such as lines and planes which satisfy the certain conditions. Let's try to understand first ideas by examples in the three dimensional spaces. For instance, 'How many lines exist in three dimensional space which intersect with four lines?' is one of the enumerative question. Schubert tries to solve this question by adding some specifications to the four lines as following: 'We are given four lines such that first two intersects and the remainig two intersect with each other. Under these conditions how many lines exist which intersect these four lines?' First of all, intersecting lines can be seen on the same planes. The line which joins the intersection point of first two lines and the intersection point of the third and fourth lines will cross the four lines simultaneously. Moreover, there is a line which lies in the intersection of these two planes. This line also crosses the four given lines. According to Schubert, we can generalize the number of solutions as 2 by 'Principle of Conservation of Number'. However, we encounter with some problems in affine case. What if these two planes are parallel? Then the answer becomes 1. Or what if third and fourth lines are parallel and planes lying 3rd and 4th lines are parallel to the planes lying first and second lines? Then there is no line which intersects all four lines at the same time. As a result, to solve such enumerative geometric problems we prefer projective space, $\mathbf{P}^n$ because all parallel objects have an intersecting point at infinity.

In the preliminaries chapter, the Schubert polynomials and Schur functions are mentioned. Some properties are given. Pieri's Formula for Schur functions and Monk's Rule for Schubert polynomials are introduced.

In the third chapter, first of all we introduce the Grassmannians $\mathbb{G}_{m,n}$ as a set of m-dimenisonal subspaces of $\mathbb{C}^{m+n}$ and realize them as geometric objects in $\mathbb{P}(\Lambda^m\mathbb{C}^{m+n})$, that is to say, as a root of certain polynomials with $\binom{m+n}{m}$-variables thanks to Plücker embedding.

Then, we examined the set of d-dimensional subspaces of $\mathbb{P}^n$ which is denoted as $\mathbb{G}_{d,n}$. The Plücker embedding also defined for this set and $\mathbb{G}_{d,n}$ is realized as geometric object in $P^{\binom{n+1}{d+1}-1}$.

In the fourth chapter, we start to develop algebraic methods to solve enumerative problems. First of all, the intersection of lines in projective space is discussed. Then, to understand Grassmannian manifold in a better way we divide it to disjoint pieces called as Schubert cells. Moreover, the closure of a Schubert cell is defined as Schubert variety and their intersection problems is conveyed to the combinatoric problems. Moreover, we construct a relation between the ring of symmetric polynomials and the cohomology ring of Grassmannian manifold thanks to Pieri formula.

In the final chapter, the Grassmannian is generalized as Flag manifold and the similar concepts are reviewed for Flag manifold. The cup product of Schubert cycles in the cohomology ring of Falg manifold is given by Monk's formula. Moreover, one can define a relation between the polynomial ring and the cohomology ring of Flag manifold thanks to Monk's formula.

I hope that the thesis provides a good overview to understand how algebraic geometry, combinatoric and algebraic topology cooperate to solve enumerative problems.

# 2. PRELIMINARIES

## 2.1. Schur Functions

We can define the Schur functions in a various way. One of the oldest version is made by Jacobi in 1850. First of all, we define an antisymmetric polynomial in the following way:

$$a_\mu = \sum_{w \in S_n} \epsilon(w) x^{w(\mu)}. \tag{2.1}$$

For $w \in S_n$, consider the shortest string such that $s_{q_1} s_{q_2} \ldots s_{q_k}$=w where $s_{q_i}$'s are transpositions. Then we call $q_1 q_2 \ldots q_k$ as a reduced word for w and k is the length of w denoted with $\ell(w)$. Remember that $p(x_1, \ldots, x_n)$ is called as antisymmetric polynomial if $p(x_1, \ldots, x_n)$ equals to $(-1)^{\ell(w)} p(x_{w_1}, \ldots, x_{w_n})$.

In this formula, $\epsilon(w)$ denotes the signature of the permutation and $x^{w(\mu)} = x_1^{\mu_{w(1)}} ... x_n^{\mu_{w(n)}}$ for an n-tuple $\mu = (\mu_1, ..., \mu_n)$ where $\mu_i$'s are natural numbers. To understand better, let's make a few observations on example.

**Example 2.1.** *Let* $\mu = (5, 4, 2)$ *n=3 then*

$$\begin{aligned} a_{(5,4,2)} &= \sum_{w \in S_3} \epsilon(w) x^{w(\mu_1 \mu_2 \mu_3)} \\ &= x_1^5 x_2^4 x_3^2 - x_1^4 x_2^5 x_3^2 - x_1^2 x_2^4 x_3^5 - x_1^5 x_2^2 x_3^4 + x_1^2 x_2^5 x_3^4 + x_1^4 x_2^2 x_3^5 \\ &= x_1^2 x_2^2 x_3^2 (x_1^3 x_2^2 + x_2^3 x_3^2 + x_1^2 x_3^3 - x_1^3 x_3^2 - x_2^2 x_3^3 - x_1^2 x_2^3). \end{aligned}$$

*Note that for* $\mu' = (5, 4, 4)$ $a_{\mu'} = 0$. *In general, for the partitions which has equal parts, the antisymmetric polynomial becomes zero.*
*Also note that for* $\mu'' = (5, 2, 4)$*:*
$a_{(5,2,4)} = x_1^2 x_2^2 x_3^2 (-x_1^3 x_2^2 - x_2^3 x_3^2 - x_1^2 x_3^3 + x_1^3 x_3^2 + x_2^2 x_3^3 + x_1^2 x_2^3) = -a_{(5,4,2)}$.

As we observed in the example, in general $a_\mu$ polynomial remains same under $S_n$ action on partition, up to sign. Therefore, we can restrict our analysis on polynomials which can be indexed by strictly decreasing partitions. Any strictly decreasing partition can be decomposed as largest strictly decreasing part $\delta = (n-1, n-2, ..., 1, 0)$ and the remaining partition. For example, $\nu = (8,4,3,2) = (5,2,2,2) + (3,2,1,0)$.

Moreover, by using these antisymmetric polynomials, we can obtain symmetric functions which is firstly introduced by Jacobi, called Schur functions.

**Definition 2.2.** *(Jacobi's Schur function) For a partition* $\lambda$*,*

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta} = \frac{det(x_i^{\lambda_j+n-j})}{det(x_i^{n-j})} \tag{2.2}$$

*for* $1 \leq i, j \leq n$*.*

Let's make some observations on $a_\mu$. Since we can obtain Schur functions by using these polynomials, we can use these informations to understand Schur functions. For instance, $a_{\lambda+\delta}$ is zero whenever $\lambda + \delta$ have equal components imply that $s_\lambda$ equals to zero. Moreover, if there exists a partiton $\lambda$ and a permutation w which satisfies $\alpha + \delta = w(\lambda + \delta)$ then $s_\lambda = \epsilon(w) s_\alpha$. So, when $\alpha$ is not a partition we obtain same Schur function up to sign. As a result, we can restrict our Schur function definition just for partition.

**Remark 2.3.**

- *$a_\delta$ is a homogenous polynomial with degree n(n-1)/2 and degree of $a_{\lambda+\delta}$ equals to $\lambda_1 + \ldots + \lambda_n + n(n-1) - n(n-1)/2 = \lambda_1 + \ldots + \lambda_n + n(n-1)/2$. Then degree of Schur function is $\lambda_1 + \ldots + \lambda_n + n(n-1)/2 - n(n-1)/2 = \lambda_1 + \ldots + \lambda_n$.*
- $s_\delta = \prod_{1 \leq i < j \leq n} \frac{x_i^2 - x_j^2}{x_i - x_j} = \prod_{1 \leq i < j \leq n} (x_i + x_j)$.
- $s_{(1,\ldots,1)} = e_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \ldots x_{i_k}$ *where* $(1, \ldots, 1)$ *is a partition consists of* $k$ *components and* $e_k$ *is called as elementary symmetric functions.*

- $s_{(k)}=h_k=\sum_{1\leq i_1\leq \ldots \leq i_k\leq n} x_{i_1}\ldots x_{i_k}$ *where* $h_k$ *is called as complete symmetric functions.*

**Theorem 2.4.** *(The Fundamental Theorem of Symmetric Funtions) Schur functions form a basis for symmetric functions in n variable as* $\lambda$ *varies through the partitions having at most n-length.*

In the following theorem, we are going to see a general formula to understand the product of Schur functions with the complete and elementary symmetric polynomials, which is called as Pieri's formula.

**Theorem 2.5.** *With the preceding notation;*

$$s_\lambda e_h = \sum_{\nu\in\lambda\otimes 1^h} s_\nu \quad s_\lambda h_g = \sum_{\nu\in\lambda\otimes g} s_\nu \tag{2.3}$$

*where* $\lambda\otimes 1^h$ *denotes the set of partitions obtained by adding h many boxes at most one per row and* $\lambda\otimes g$ *denotes the set of partitions obtained by adding g many boxes at most one per column.*

*Proof.* To understand the product of $s_\lambda$ and $e_h$, let's check $a_{\lambda+\delta}e_h$.

$$\begin{aligned}
a_{\lambda+\delta}e_h &= \sum_{w\in S_n}\epsilon(w)x^{w(\lambda+\delta)}\sum_{1\leq i_1<\ldots<i_h\leq n} x_{i_1}\ldots x_{i_h}\\
&= \sum_{w\in S_n}\sum_{1\leq i_1<\ldots<i_h\leq n}\epsilon(w)x^{w(\lambda+\delta)}x_{w(i_1)}\ldots x_{w(i_h)}\\
&= \sum_{\alpha\in\{0,1\}^n} a_{\lambda+\alpha+\delta}.
\end{aligned}$$

Then,

$$s_\lambda e_h = \frac{a_{\lambda+\delta}e_h}{a_\delta} = \sum_{\alpha\in\{0,1\}^n} \frac{a_{\lambda+\alpha+\delta}}{a_\delta} = \sum_{\nu\in\lambda\otimes 1^h} s_\nu.$$

For the second formula we obtain the similar identity;

$$a_{\lambda+\delta}h_g = \sum_{|\alpha|=g} a_{\lambda+\alpha+\delta}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\alpha_i$'s are not necessarily decreasing. We need to show that in the summation we just end up with partitions whose result in adding one box per column. In one of the summand, let $\alpha_{i+1}$ such that $\alpha_{i+1} > \lambda_i - \lambda_{i+1}$ for some partiton which satisfies $|\alpha| = g$. Define a sequence $\gamma$ such that $\gamma_i = \alpha_{i+1} - (\lambda_i - \lambda_{i+1} + 1)$ and $\gamma_{i+1} = \alpha_i + (\lambda_i - \lambda_{i+1} + 1)$ also $\gamma_j = \alpha_j$ for all $i \neq j$. Note that the summation of $\gamma_i$'s is again g since $\gamma_{i+1} + \gamma_i = \alpha_i + (\lambda_i - \lambda_{i+1} + 1) + \alpha_{i+1} - (\lambda_i - \lambda_{i+1} + 1) = \alpha_i + \alpha_{i+1}$. Then, by properties of antisymmetric polynomial a, we have $a_{\lambda+\alpha+\delta} = -a_{\lambda+\gamma+\delta}$. Therefore, cancellation occurs between such summands in the summation and the ones stem from addition one box per-column survive. $\square$

## 2.2. Schubert Polynomials

Let p be a polynomial in the polynomial ring with infinite set of variables. Let's define a simple reflection operator $r_i$ by the following action:

$$(r_i.p)(x_1, ..., x_i, x_{i+1}, ...) := p(x_1, ..., x_{i+1}, x_i, ...). \tag{2.4}$$

So, p is symmetric in $x_i, x_{i+1}$ if $p = r_i.p$ i.e., under the change of $x_i, x_{i+1}$ variables polynomial remains same, or equivalently $(1 - r_i).p = 0$.

Let $p - r_i.p = q$ then $r_i.q = r_i(p - r_i.p) = r_i.p - p = -(p - r_i.p) = -q$. Then,

$$\begin{aligned} r_i.q &= -q \\ q(x_1, ..., x_{i+1}, x_i, ...) &= -q(x_1, ..., x_i, x_{i+1}, ...) \\ q(x_1, ..., x_i, x_i, ...) &= -q(x_1, ..., x_i, x_i, ...) \\ q(x_1, ..., x_i, x_i, ...) &= 0. \end{aligned}$$

So, q is an antisymmetric polynomial. By using the long division algorithm for polynomials, $q(x_1..., x_i, ...) = (x_i - x_{i+1})q' + k$ where $deg(k) < deg(x_i - x_{i+1})$ . Since $deg(x_i - x_{i+1}) = 1$ , $deg(k) = 0$.i.e. $k \in Z$. For $(x_i = x_{i+1})$, we will obtain 0=0+k which implies that $k = 0$.

Therefore, $q(x_1..., x_i, ...) = (x_i - x_{i+1})q'$ and the following actions on polynomials results in a polynomial again:

$$\partial_i p := \frac{p - r_i.p}{x_i - x_{i+1}}. \tag{2.5}$$

Also we call "$\partial$" as divided difference operator. We are going to introduce some properties of divided difference operators.

**Proposition 2.6.** *1)* $\partial_i(pq) = (\partial_i p)q + (r_i p)\partial_i q$.
*2) If* $\partial_i p = 0$, *then* $\partial_i(pq) = p\partial_i q$.
*3)* $\partial_i^2 = 0$.
*4)* $\partial_i\partial_j = \partial_j\partial_i$ *if* $| i - j | > 1$.
*5)* $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$.

Let $s_i$ denotes the transposition that replace the $i^{th}$ and $(i + 1)^{th}$ entries when acts on a permutation from right.

Remember that $s_1, \ldots, s_n$ generate the symmetric group on n-elements $S_n$ with the following braid relations:

$$\begin{aligned} {s_i}^2 &= 1 \\ s_i s_j &= s_j s_i \ for \ |i-j| > 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}. \end{aligned} \tag{2.6}$$

Since $r_i$'s satisfies the braid relations, $\partial$ operator satisfies the above properties and the following proposition.

**Proposition 2.7.** *Let $\pi \in S_n$ and $q_1 \ldots q_{\ell(\pi)}$ is the reduced word of $\pi$ then*

$$\partial_\pi = \partial_{q_1} \ldots \partial_{q_{\ell(\pi)}}. \tag{2.7}$$

*Moreover, this equation is independent of reduced word and $\partial_\pi$ is well-defined.*

**Definition 2.8.** *(Schubert polynomials with divided difference operators)*
*Let $w_0 = [n, n-1, n-2, ...]$ be the longest element in $S_n$.*
*Then we define the Schubert polynomial which is indexed by w, $\vartheta_w$ as following:*

$$\vartheta_w = \partial_{w^{-1}w_0} \vartheta_{w_0} \tag{2.8}$$

*where $\vartheta_{w_0} = x_1^{n-1} x_2^{n-2} ... x_{n-1}^1$.*

We can discuss whether this is an appropriate definiton or not since definition depends on n. If w is the element of $S_n$ then it is also element of $S_{n+1}$. Then we may start with $\vartheta_{w_0} = x_1^n x_2^{n-1} ... x_n^1$ or $\vartheta_{w_0} = x_1^{n-1} x_2^{n-2} ... x_{n-1}^1$ to compute Schubert polynomial for the same w. Does it make any difference in the resulting polynomial $\vartheta_w$? Or even if we can obtain a Schubert polynomial after this operation, each Schubert polynomial can be obtainable by this operation? Answers are given by Theorem 2.9.

**Theorem 2.9.** *There exists a unique way of indexing Schubert polynomials with* $\pi \in S_\infty$ *such that* $\vartheta_{id} = 1$ *and*

$$\partial_i \vartheta_\pi = \begin{cases} \vartheta_{\pi \circ s_i} & if \pi(i) > \pi(i+1) \\ 0 & if \pi(i) < \pi(i+1). \end{cases}$$

**Example 2.10.**

- $\vartheta_{s_i} = x_1 + x_2 + \ldots + x_i$
- *Let's show that* $\vartheta_{id} = 1$.

  $\vartheta_{id} = \partial_{w_0^n} \vartheta_{w_0}$ *for* $w_0^n = [n, n-1, ..., 1]$ *with one line notation.*

  $= \partial_{w_0^{n-1}} \partial_{n-1} \partial_{n-2} ... \partial_1 (x_1^{n-1} x_2^{n-2} ... x_{n-1}^1)$ *for* $w_0^{n-1} = [n-1, n-2, ..., 1]$

  $= \partial_{w_0^{n-1}} \partial_{n-1} \partial_{n-2} ... \partial_2 (x_1^{n-2} x_2^{n-2} ... x_{n-1}^1)$

  $= ...$

  $= \partial_{w_0^{n-1}} (x_1^{n-2} x_2^{n-3} ... x_{n-1}^0)$.

  *Observe that this is the Schubert polynomial in* $S_{n-1}$ *indexed by identity. By induction,* $\vartheta_{id} = \partial_{w_0^{(3-1)}} (x_1^{3-2} x_2^{3-3}) = \partial_{w_0^{(2)}} (x_1^1 x_2^0) = \partial_1 (x_1) = (x_1 - x_2)/(x_1 - x_2) = 1.$

Schubert polynomials may be constructed from the diagrams by using some algorithms. Fomin and Kirillov [1993] introduced a set of diagrams that encode the Schubert polynomials which is called as RC-graph(reduced word compatible sequence graph). Besides, Coşkun and Taşkın [2013] introduced the Tower tableaux to encode Schubert polynomials, see [4]. There are other diagrams and algorithms to obtain Schubert polynomials.

**Theorem 2.11.** *The Schubert polynomials form an integral basis for* $Z[x_1, x_2, ...]$.

For the detailed proof of Theorem 2.11, see [3].

Now, a generalization for Pieri's formula will be given, which is Monk's Rule. The following form of this rule is used to understand the product of Schubert polynomials.

**Theorem 2.12.** *(Monk's Rule) For all permutations $w \in S_\infty$, and for all natural numbers $n$,*

$$\vartheta_w \times \vartheta_{s_n} = \sum_{j \leq n < k,\ l(wt_{jk}) = l(w)+1} \vartheta_{wt_{jk}}. \tag{2.9}$$

One of the nicest proof of Monk's Rule which uses RC-Graphs can be seen in [2].

# 3. GRASSMANNIANS AS ALGEBRAIC VARIETIES

Complex Grassmannian is a set of linear subspaces of dimension m in $\mathbb{C}^{m+n}$, denoted as $\mathbb{G}_{m,n}$. We can define a right group action of GL(m+n,$\mathbb{C}$) on $\mathbb{G}_{m,n}$. Since for all x, y $\in \mathbb{G}_{m,n}$ one can obtain g $\in$ GL(m+n,$\mathbb{C}$) such that xg=y, right action is transitive and $\mathbb{G}_{m,n}$ consists of one orbit under this action. Also, let $F_{m,n}$ be the subgroup of GL(m+n,$\mathbb{C}$) which stabilizes the first m vectors of the canonical basis of $\mathbb{C}^{m+n}$. Then,

$$\mathbb{G}_{m,n} \cong GL(m+n,\mathbb{C})/F_{m,n}.$$

We are going to analyze the relations between the elements in the same Grassmannian. Let V $\in \mathbb{G}_{m,n}$ and $v_1, ..., v_m$ be a basis of this space. Extend this basis to a basis of $\mathbb{C}^{m+n}$ with vectors $v_{m+1}, ..., v_{m+n}$ and denote the space spanned by these vectors by $V^{\perp}$. Let W $\in \mathbb{G}_{m,n}$ such that $W \cap V^{\perp} = \{0\}$. Then the basis of W can be written as the following form:

$$w_i = v_i + \sum_{j=1}^{n} x_{ij} v_{m+j} \qquad 1 \leq i \leq m.$$

We can see W as a space which is spanneed by the rows of the following matrix:

$$\begin{bmatrix} 1 & 0 & ... & ... & 0 & x_{11} & ... & x_{1n} \\ 0 & 1 & ... & ... & 0 & x_{21} & ... & x_{2n} \\ ... & ... & ... & ... & ... & ... & .... & ... \\ 0 & 0 & ... & .... & 1 & x_{m1} & ... & x_{mn} \end{bmatrix}.$$

Then the set of all possible W's is isomorphic to $\mathbb{C}^{mn}$. Take another m-plane ($\subset \mathbb{C}^{m+n}$) in the neighborhood of W, say V. Since the change of basis equations between W and V are affine maps, the Grassmannian has the structure of complex algebraic variety.

In the next section, we will see the ideal of Grassmannian variety is defined by quadratic relations stem from Plücker embedding. For more detailed explanation about ideal of a variety, see Appendix A.

### 3.1. Plücker Embedding

In this part, we will see how one can realize Grassmannian $\mathbb{G}_{m,n}$ as a projective subvariety. First of all, let's define Plücker embedding $\phi$ in the following way:

$$\phi : \mathbb{G}_{m,n} \to \mathbb{P}(\Lambda^m \mathbb{C}^{m+n})$$

where $\Lambda^m \mathbb{C}^{m+n}$ denotes the $m^{th}$ exterior power of $\mathbb{C}^{m+n}$. Let W be an arbitrary element in Grassmannian, in other words, an m dimensional subspace in $\mathbb{C}^{m+n}$ which can be represented as:

$$W = \begin{bmatrix} x_{11} & x_{12} & \dots & \dots & x_{1,m+n} \\ x_{11} & x_{11} & \dots & \dots & x_{2,m+n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & \dots. & x_{m,m+n} \end{bmatrix}.$$

Then, the homogenous coordinates of the image of W under $\phi$ is defined as the minors of order m in W and denoted as:

$$P_{i_1,..,i_m} = det(x_{p,i_q})_{1 \leq p,q \leq m} \ , 1 \leq i_1 < ... < i_m \leq m+n.$$

In other words, m columns from m×(m+n) matrix in Grassmannians are choosen and the determinant of this m×m matrix is taken as one coordinate of the image point. Then, we order these determinants lexicographically to find $\phi(W)$. Let $P_{i_1,..,i_m}$ and $P_{i'_1,..,i'_m}$ be two Plücker coordinates of W. In lexicographic order $P_{i_1,..,i_m}$ comes before $P_{i'_1,..,i'_m}$ if $i_1 < i'_1$, and vice versa. If $i_1 = i'_1$ then we go on until finding first unequal ones and compare them.

At that point one question arises: “Do the Plücker coordinates remain same if we use different basis for W?”. To answer it let's fix two basis of W and compare the corresponding Plücker coordinates. Let $P_i = (p_i(1), ..., p_i(m+n))$ and $Q_i = (q_i(1), ..., q_i(m+n))$ be two basis for i=1,...,m.

Then there is an $A \in GL(m, \mathbb{C})$ such that $[q_i(j)] = A[p_i(j)]$. Then $Q_{i_1,..,i_m} = det(A)P_{i_1,..,i_m}$ holds. As a result, lexicographic order of these two determinants define points (...,$P_{i_1,..,i_m}$,...) and (...,$Q_{i_1,..,i_m}$,...)=(...,det(A)$P_{i_1,..,i_m}$,...) which is same point in $\mathbb{P}^{\binom{m+n}{m}}$. Therefore, the Plücker coordinates do not change even if we use the different basis for m-dimensional subspaces in $\mathbb{G}_{m,n}$.

We know that $\mathbb{G}_{m,n}$ is a compact, connected manifold whose points correspond to m-dimensional subspaces of $\mathbb{C}^{m+n}$. Let us consider how one can obtain the dimension of manifold by using this fact. Each m-dimensional subspaces of $\mathbb{C}^{m+n}$ can be reduced to the folowing form by using the linear algebra tools:

$$\begin{bmatrix} 1 & 0 & ... & ... & 0 & x_{11} & ... & x_{1n} \\ 0 & 1 & ... & ... & 0 & x_{21} & ... & x_{2n} \\ ... & ... & ... & ... & ... & ... & .... & ... \\ 0 & 0 & ... & .... & 1 & x_{m1} & ... & x_{mn} \end{bmatrix}$$

where each $x_{ij} \in \mathbb{C}$. So, the question about the dimension of $\mathbb{G}_{m,n}$ turns into that “In how many ways the matrix in the above form can be constructed?”. The answer will be given by the number of free coordinates which is mn. Therefore, the dimension of $\mathbb{G}_{m,n}$ is mn, that is to say, $\mathbb{G}_{m,n}$ is isomorphic to $\mathbb{C}^{mn}$.

**Proposition 3.1.** *$\phi$ is an embedding, which is called Plücker embedding.*

*Proof.* To prove that the map $\phi$ is an embedding, we need to show that the map itself and its differential at each point is one-to-one. Let V $\in \mathbb{G}_{m,n}$ and $v_1, ..., v_m$ be a basis of this space. Let $v_{m+1}, \ldots, v_{m+n}$ be the basis of an orthogonal complement of V.

And suppose that W $\in \mathbb{G}_{m,n}$ represented with a matrix with respect to the previous basis such that;

$$\begin{bmatrix} 1 & 0 & \ldots & \ldots & 0 & x_{11} & \ldots & x_{1n} \\ 0 & 1 & \ldots & \ldots & 0 & x_{21} & \ldots & x_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots. & \ldots \\ 0 & 0 & \ldots & \ldots. & 1 & x_{m1} & \ldots & x_{mn} \end{bmatrix}$$

where $x_{ij} \in \mathbb{C}$. Then we get $P_{1,\ldots,i-1,m+j,i+1,\ldots,m} = x_{ij}$ by choosing the $(1,\ldots,i-1,m+j,i+1,\ldots,m)^{th}$ coordinates of $\phi(W)$. If $\phi(W) = \phi(V)$ then all $x_{ij}$'s must be the nonzero multiple of the $(1,\ldots,i-1,m+j,i+1,\ldots,m)^{th}$ coordinate of $\phi(V)$. In the matrix representation of V with the given basis, all the coordinates in the position of $x_{ij}$'s are zero. Therefore, $\phi(W) \neq \phi(V)$ if all $x_{ij}$ are not same with the coordinates of $\phi(V)$ in the position of $x_{ij}$. Then, $\phi$ is injective. Moreover, the differential of $\phi$ is a $\binom{m+n}{m} \times (mn)$ matrix and $P_{1,\ldots,i-1,m+j,i+1,\ldots,m} = x_{ij}$ implies also that in the $(1,\ldots,i-1,m+j,i+1,\ldots,m)^{th}$ position 1 occurs for each $x_{ij}^{th}$ row. Then the rank of this matrix mn since the position of 1 is different from each other which result in the injectivity of differential of $\phi$. □

The Plücker embedding pave the way for the polynomials which are used to construct the Grassmannian as an algebraic subvariety and these polynomials are given by the following quadratic relations.

**Theorem 3.2.** *Let $j_1,\ldots,j_m$ and $k_1,\ldots,k_m$ be two sets of integers between 1 and m+n and take a fix integer i between 1 and m. Then on Grassmannian $\mathbb{G}_{m,n}$ we have the quadratic relation:*

$$\sum_{w \in S/S' \times S''} \epsilon(w) P_{j_1,\ldots,j_{i-1},w(j_i),\ldots,w(j_m)} P_{w(k_1),\ldots,w(k_i),k_{i+1},\ldots,k_m} = 0 \tag{3.1}$$

*where $S''$ is the group of permutations of the symbols $k_1,\ldots,k_i$, $S'$ that of $j_1,\ldots,j_m$ and S that of $k_1,\ldots,k_i,j_1,\ldots,j_m$. These quadratic relations are called as Plücker relations.*

*Moreover, the Grassmannian is completely determined by the Plücker relations and these relations generate the ideal of Grassmannian variety.*

After this point we will focus on the set of d-dimensional linear subspaces in n-dimensional projective space which will be denoted as $\mathbb{G}_{d,n}$. As we mentioned in the introduction, counting intersection points in projective space has several adventages such as parallel linear objects have intersection point at infinity. Besides, we will give the proof of Theorem 3.2 for the projective correspondent.

In the projective space, d-planes are spanned by d+1 points $A_i = (a_i(0), \dots, a_i(n))$ with $i = 0, \dots, d$. So, for the projective case the matrix representation of d-plane in $\mathbb{P}^n$ has dimension $(d+1) \times (n+1)$. For this matrix Plücker coordinates are obtained similarly as in the affine case and satisfies the quadratic relations for projective case.

**Definition 3.3.** *Quadratic relations are defined on Plücker coordinates as following:*

$$\sum_{\lambda=0}^{d+1} (-1)^{\lambda} p(j_0 ... j_{d-1} k_{\lambda}) p(k_0 ... \check{k}_{\lambda} ... k_{d+1}) = 0 \tag{3.2}$$

$j_0, ..., j_{d-1}$ *and* $k_0, ..., k_{d+1}$ *are any sequences of integers for* $0 \leq j_{\mu}, k_{\nu} \leq n$ *where* $p(j_0, \dots, j_d)$ *denotes the determinant of the* $(d+1) \times (d+1)$ *matrix with* $0 \leq j_0 < \dots < j_d \leq n$.

We can move one more step and by restricting the codomain of Plücker embedding we can obtain a bijective map.

**Theorem 3.4.** *There is a bijective correspondence between the d-planes in* $P^n$ *and the points of* $P^{\binom{n+1}{d+1}-1}$ *whose coordinates satisfy the quadratic relations.*

*Proof.* First of all we need to show that Plücker coordinates of d planes should satisfy the quadratic relations (3.2).

More precisely, we want to put the relations among the determinants.

Consider the following polynomial in Plücker coordinates:

$$\sum_{\lambda=0}^{d+1}(-1)^{\lambda}\begin{vmatrix} \vdots & & \vdots & \vdots \\ p_i(j_0) & \cdots & p_i(j_{d-1}) & p_i(k_\lambda) \\ \vdots & & \vdots & \vdots \end{vmatrix}\begin{vmatrix} \cdots & \check{p}_0(k_\lambda) & \cdots \\ & \vdots & \\ \cdots & \check{p}_d(k_\lambda) & \cdots \end{vmatrix}.$$

If we expand the first determinants along the last column then we get,

$$\sum_{\lambda=0}^{d+1}(-1)^{\lambda}\left\{\textstyle\sum_{i=0}^{d}(-1)^{d+i}\begin{vmatrix} \vdots & & \vdots \\ \check{p}_i(j_0) & \cdots & \check{p}_i(j_{d-1}) \\ \vdots & & \vdots \end{vmatrix} p_i(k_\lambda)\right\}\begin{vmatrix} \cdots & \check{p}_0(k_\lambda) & \cdots & p_0(k_{d+1}) \\ & \vdots & & \vdots \\ \cdots & \check{p}_d(k_\lambda) & \cdots & p_d(k_{d+1}) \end{vmatrix}.$$

Then, if we group the terms related with $\lambda$ in one summation and the terms containing 'i' in another one;

$$\sum_{i=0}^{d}\left\{(-1)^{d+i}\begin{vmatrix} \vdots & & \vdots \\ \check{p}_i(j_0) & \cdots & \check{p}_i(j_{d-1}) \\ \vdots & & \vdots \end{vmatrix}\textstyle\sum_{\lambda=0}^{d+1}(-1)^{\lambda}p_i(k_\lambda)\begin{vmatrix} \cdots & \check{p}_0(k_\lambda) & \cdots & p_0(k_{d+1}) \\ & \vdots & & \vdots \\ \cdots & \check{p}_d(k_\lambda) & \cdots & p_d(k_{d+1}) \end{vmatrix}\right\}.$$

Now, one can realize that first summation comes from the determinantal expansion of the matrix, $\begin{bmatrix} \cdots & p_i(k_\lambda) & \cdots \\ \cdots & p_0(k_\lambda) & \cdots \\ & \vdots & \\ \cdots & p_d(k_\lambda) & \cdots \end{bmatrix}$. The determinant of this matrix equals to zero since two rows are equal.

As a result, Plücker coordinates of a d-plane in $\mathbb{P}^n$ satisfy the quadratic relations.

Conversely, to prove that any point in $\mathbb{P}^{\binom{n+1}{d+1}-1}$ whose coordinates satisfying the quadratic relations match with a unique d-plane in $\mathbb{P}^n$, we are going to solve the quadratic relations.

First of all, assume that $p(k_0...k_d)$ is not zero. Under this assumption, we want to show that $\binom{n+1}{d+1}$ many Plücker coordinates determined by the [(d+1)(n-d)+1] many coordinates of the form $p(k_0...\check{k}_\lambda...k_d j_\alpha)$, in other words by the coordinates $p(i_0...i_d)$ with at most one of $i_0, ..., i_d$ not among $k_0, ..., k_d$. Observe that $(k_0...\check{k}_\lambda...k_d j_\alpha)$ sequence can be constructed (d+1)[(n+1)-(d+1)]+1 many ways since $\check{k}_\lambda$ can be choosen d+1 many ways and if $j_\alpha$ will be different from $k_i$'s it can be choosen (n+1)-(d+1) many. There is also one more way to obtain nonzero determinant which is omitting $k_d$ and add to sequence $k_d$ again in place of $j_\alpha$ (+1 in the formula).

Let's examine the quadratic relations which arise from such coordinates. Let $\ell_0...\ell_d$ and $k_0...k_d$ be two sequences of integers where $0 \leq \ell_\nu, k_\mu \leq n$. They might have uncommon terms, say t many and also let $\ell_\gamma$ be one of these terms. The quadratic relation for the sequences $\ell_0...\check{\ell}_\gamma...\ell_d$ and $k_0...k_d\ell_\gamma$;

$$\begin{aligned} 0 &= \sum_{\lambda=0}^{d+1}(-1)^\lambda p(\ell_0...\check{\ell}_\gamma...\ell_d k_\lambda)p(k_0...\check{k}_\lambda...k_d\ell_\gamma) \\ \mp p(\ell_0...\check{\ell}_\gamma...\ell_d\ell_\gamma)p(k_0...k_d) &= \sum_{\lambda=0}^{d}(-1)^\lambda p(\ell_0...\check{\ell}_\gamma...\ell_d k_\lambda)p(k_0...\check{k}_\lambda...k_d\ell_\gamma). \end{aligned} \tag{3.3}$$

Then if $k_\lambda$ is among $\ell_i$ sequence $p(\ell_0 \cdots \check{\ell}_\gamma \cdots \ell_d k_\lambda) = 0$. If $k_\lambda$ is not one of the $\ell_i$'s, in the $(\ell_0...\check{\ell}_\gamma...\ell_d k_\lambda)$ sequence there are exactly t-1 many terms which are not among $k_0, \ldots, k_d$. If t=1, the summation in the quadratic relation just have one nonzero summand and quadratic equation becomes:

$$\mp p(\ell_0...\check{\ell}_\gamma...\ell_d\ell_\gamma)p(k_0...k_d) = p(\ell_0...\check{\ell}_\gamma...\ell_d k_\lambda)p(k_0...\check{k}_\lambda...k_d\ell_\gamma)$$

where $k_\lambda$ is the sole noncommon term of sequence. So, one can obtain other Plücker coordinates thanks to $p(k_0...k_d)$, $p(k_0...\check{k}_\lambda...k_d\ell_\gamma)$ and the quadratic relations.
The left-hand side of (3.3) becomes $p(\ell_0...\ell_d)p(k_0...k_d)$ for $\gamma = d$. If $t \geq 2$, this product can be expressed in terms of the coordinates $p(i_0, \ldots, i_d)$ with at most (t-1) of $i_0, \ldots, i_d$ not among $k_0, \ldots, k_d$.

Now multiply the equation with $p(k_0, \ldots, k_d)$ and obtain;

$$\mp p(\ell_0 \ldots \ell_d) p(k_0 \ldots k_d) p(k_0 \ldots k_d) = \sum_{\lambda=0}^{d} (-1)^{\lambda} \{p(\ell_0 \ldots \check{\ell}_{\gamma} \ldots \ell_d k_{\lambda}) p(k_0 \ldots k_d)\} p(k_0 \ldots \check{k}_{\lambda} \ldots k_d \ell_{\gamma}).$$

Apply the quadratic relation for $\{p(\ell_0 \ldots \check{\ell}_{\gamma} \ldots \ell_d k_{\lambda}) p(k_0 \ldots k_d)\}$. We will see that we can express $\mp p(\ell_0 \ldots \ell_d) p(k_0 \ldots k_d) p(k_0 \ldots k_d)$ in terms of coordinates $p(i_0, \ldots, i_d)$ with at most t-2 many $i_0, \ldots, i_d$ not among $k_0, \ldots, k_d$. If we go on multiplying with $p(k_0 \ldots k_d)$ and expressing the terms in the summation with quadratic relations, at last step we see that $p(\ell_0 \ldots \ell_d) p(k_0 \ldots k_d)^{t-1}$ can be seen as a polynomial in the coordinates $p(i_0, \ldots, i_d)$ with at most one of $i_0, \ldots, i_d$ not among $k_0, \ldots, k_d$. As a result, we have proved that, when $k_0, \ldots, k_d$ is nonzero [(d+1)(n-d)+1] many coordinates determine the other Plücker coordinates thanks to quadratic relations.

Secondly, without loss of generality assume that $p(k_0 \cdots k_d) = 1$. To show the existence of a d-plane in $\mathbb{P}^n$, we are going to construct a d-plane whose Plücker coordinates corresponds to a point in $\mathbb{P}^{\binom{n+1}{d+1}-1}$. Let $p_i(j) = p(k_0 \cdots k_{i-1} j k_{i+1} \cdots k_d)$ with $i = 0, \cdots, d$ and $j = 0, \cdots, n$. More precisely, $p_i(j)$ equals to the determinant of a matrix whose columns are $k_p$'s for $p = 0, \cdots, i-1, i+1, \cdots, d$ and has $j^{th}$ column in place of $i^{th}$ column. If $\alpha \neq$ i then $p_i(k_{\alpha}) = 0$ since we have the same columns in the square matrix, determinant equals to zero. By the assumption $p(k_0 \cdots k_d) = 1$, $p_i(k_i) = 1$. Therefore, $(p_i(0), \cdots, p_i(n))$ with $i = 0, \cdots, d$ are linearly independent set of vectors. Observe that, $(p_i(0), \cdots, p_i(n))$ is a vector such that in the $k_i^{th}$ position there is '1', in the $k_{\gamma}^{th}$ positions for $\gamma \neq i$ there are zeroes and in the other positions there might be any real number. So, this set of vectors span a d-plane in $\mathbb{P}^n$, say K. Up to this point we constructed a d-plane K just considering one coordinate of $\mathbb{P}^{\binom{n+1}{d+1}-1}$, so also we need to show that this plane determines the rest of the coordinates satisfying quadratic relations.

Now consider the Plücker coordinates of K and observe that they coincide with the coordinates satisfying quadratic relations. Let $p'(\ell_0 \ldots \ell_d)$ be the Plücker coordinate of K.

This Plücker coordinate is defined as the determinant of matrix $[p_i(\ell_\beta)]$ for $i, \beta = 0, \ldots, d$. Therefore, if we have $\ell_\beta = k_\beta$ for $\beta \neq \lambda$ this $(d+1) \times (d+1)$ square matrix coincides with the identity matrix except the $\lambda^{th}$ column. In other words,

$$p'(\ell_0 \ldots \ell_d) = p'(k_0 \ldots \ell_\lambda \ldots \kappa_d) = \begin{vmatrix} p_0(k_0) & \ldots & p_0(\ell_\lambda) & \ldots & p_0(k_d) \\ p_1(k_0) & \ldots & p_1(\ell_\lambda) & \ldots & p_1(k_d) \\ \vdots & & \vdots & & \vdots \\ p_d(k_0) & \ldots & p_d(\ell_\lambda) & \ldots & p_d(k_d) \end{vmatrix} \tag{3.4}$$

$$= \begin{vmatrix} 1 & \ldots & p_0(\ell_\lambda) & \ldots & 0 \\ 0 & \ldots & p_1(\ell_\lambda) & \ldots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \ldots & p_d(\ell_\lambda) & \ldots & 1 \end{vmatrix} = p_\lambda(\ell_\lambda).$$

Since $p(k_0 \ldots k_d) = 1$, for the sequence $\ell_0 \ldots \ell_d$ where the at most one of $\ell_i's$ not among $k_0 \ldots k_d$, $p_\lambda(\ell_\lambda) = p(\ell_0 \ldots \ell_d)$ holds. As a result, one of the Plücker coordinate of the d-plane K equals to the coordinate satifying quadratic relation in the $\mathbb{P}^{\binom{n+1}{d+1}-1}$. Previously, we proved that these coordinates determine the rest ones. So, $p'(\ell_0 \ldots \ell_d)$ equals to $p(\ell_0 \ldots \ell_d)$ for all sequences. Thus, the point $(\ldots, p(\ell_0 \ldots \ell_d), \ldots)$ in $\mathbb{P}^{\binom{n+1}{d+1}-1}$ arises from d-plane K in $\mathbb{P}^n$.

Finally, we will show that there is a unique d-plane in $\mathbb{P}^n$ whose Plücker coordinates corresponds to given point in $\mathbb{P}^{\binom{n+1}{d+1}-1}$. Let's assume that $K'$ another d-plane matching with the same point and $P_i' = (p_i'(0), \cdots, p_i'(n))$ for i=0,$\cdots$,d spans $K'$. Then, the (d+1)×(d+1)-matrix $[p_i'(k_\beta)]$ is invertible for $i, \beta = 0, \cdots, d$ because its determinant is a nonzero multiple of $p(k_0, \cdots, k_d) = 1$. Multiplying $P_i'$ with the inverse matrix of $P_i'$, we get $[p_i'(k_\alpha)]$ is the identity matrix.

Then for any sequence $\ell_0 \ldots \ell_d$ the determinant $\det[p_i'(\ell_\beta)]$ equals to $p(\ell_0, \ldots \ell_d)$. Now fix $\gamma$ and $\ell$ for $0 \leq \gamma \leq d$ and $0 \leq \ell \leq n$, and put $\ell_\beta = k_\beta$ for $\beta \neq \gamma$ and $\ell_\gamma = \ell$.

Then similar with the previous case $[p'_i(\ell_\beta)]$ coincides with the identity matrix outside the $\gamma$-th column.

As a result we have;

$$p'_\gamma(j) = det[p'_i(j_\beta)] = p(\ell_0 \dots \ell_d) = p_\gamma(j).$$

Since we reached that K' and K have same coordinates in their matrix representation, they are spanned with same d+1 points. In other words, $P_\gamma = P'_\gamma$ and so K'=K. □

In this proof, the assumption of $p(k_0 \dots k_d) = 1$ can be changed with $p(k_0 \dots k_d)$ is a nonzero real number. In this case, we will choose the entries of d-plane K as $p_i(j) = p(k_0 \cdots k_{i-1} j k_{i+1} \cdots k_d)/p(k_0 \dots k_d)$ and revise the proof.

In the previous parts, we discussed the dimension of Grassmannians for the affine case and asserts that for the Grassmannian consists of m-dimensional subspaces in $\mathbb{C}^{m+n}$, the dimension is m((m+n)-m)=mn. Similarly, in the projective case, for the Grassmannian consists of d-planes in $\mathbb{P}^n$ the dimension equals to (d+1)((n+1)-(d+1))=(d+1)(n-d). Particularly, one can say that d-planes in $\mathbb{P}^n$ are represented by the points of the (d+1)(n-d)-dimensional Grassmann manifold. For instance, the lines in $\mathbb{P}^3$ are represented by the points of 4-dimensional Grassmannian $\mathbb{G}_{1,3}$. By Theorem 3.4, the points of $\mathbb{G}_{1,3}$ can be seen as points in $\mathbb{P}^{\binom{3+1}{1+1}-1} = \mathbb{P}^5$ whose coordinates $p(i_0 i_1)$ satisfy the quadratic relations.

**Example 3.5.** Let's find the quadratic relations which is satisfied by the points of $\mathbb{G}_{1,3}$ using equation (3.2). For n=3 and d=1 two sequences $j_0$ and $k_0k_1k_2$ with $0 \leq j_\beta, k_\gamma \leq 3$ can be choosen $4\binom{4}{3} = 16$ many different ways. So, we may construct 16 summation formulas but some of them may generate same quadratic relations and some of them may not give new informations about coordinates in projective space.

For $j_0 = 0$ and $k_0k_1k_2 = 012$ the quadratic relation is;

$$p(00)p(12) - p(01)p(02) + p(02)p(01) = 0. \tag{3.5}$$

Since p is an alternating function(precisely determinant) p(00)=0, equation becomes;

$$-p(01)p(02) + p(02)p(01) = 0. \tag{3.6}$$

So, this relation gives no more information about the Plücker coordinates. Note that the main reason for this result is common 0 term in the sequences. So, to check whether we can generalize the situation for the other sequences, consider the sequences $j_0 = 0$ and $k_0k_1k_2 = 023$ then quadratic relation is;

$$p(00)p(23) - p(02)p(03) + p(03)p(02) = 0. \tag{3.7}$$

Then equivalently equation becomes;

$$-p(02)p(03) + p(03)p(02) = 0 \ for \ p(00) = 0. \tag{3.8}$$

We obtained a similar result since $j_0$ completely lies in the $k_0k_1k_2$ sequence. So, to get meaningful equations try non-intersect sequences. For $j_0 = 0$ and $k_0k_1k_2 = 123$ quadratic relation is;

$$p(01)p(23) - p(02)p(13) + p(03)p(12) = 0. \tag{3.9}$$

Let's try to obtain different quadratic relations by choosing non-intersect sequences.

For $j_0 = 1$ and $k_0k_1k_2 = 023$; p(10)p(23)-p(12)p(03)+p(13)p(02)=0.
For $j_0 = 2$ and $k_0k_1k_2 = 013$; p(20)p(13)-p(21)p(03)+p(23)p(01)=0.
For $j_0 = 3$ and $k_0k_1k_2 = 012$; p(30)p(12)-p(31)p(02)+p(32)p(01)=0.

One can easily observe that last four possible cases generate the same equation. Since the other possible sequences causes meaningless equations the sole quadratic relation is;

$$p(01)p(23) - p(02)p(13) + p(03)p(12) = 0. \tag{3.10}$$

Therefore; one point of $\mathbb{G}_{1,3}$ can be represented as a point in $\mathbb{P}^5$ which satisfies the above quadratic relation.

In this example, we tried other possible sequences after finding one equation. However; if we can guess the number of quadratic relations, we can stop before trying all possible cases. To guess it, we will use some tools of linear algebra and Theorem 3.4.

Since we know that there is a bijective correspondence between the points of $\mathbb{G}_{d,n}$ and $P^{\binom{n+1}{d+1}-1}$ whose coordinates satisfy the quadratic relations, we should obtain dim($P^{\binom{n+1}{d+1}-1}$)-dim($\mathbb{G}_{d,n}$)=$[\binom{n+1}{d+1} - 1] - [(d+1)(n-d)]$ many quadratic relations.

For our example n=3 and d=1, the number of quadratic relation equals to $[\binom{3+1}{1+1} - 1] - [(1+1)(3-1)] = [\binom{4}{2} - 1] - [(2)(2)]$=6-1-4=1. So, we should not look for any other quadratic relation after finding one equation.

# 4. SCHUBERT CALCULUS

## 4.1. Schubert Condition

Now, we will work on necessary and sufficient condition for a d-plane in $\mathbb{P}^n$ to intersect a given sequence of linear spaces in $\mathbb{P}^n$.

**Definition 4.1.** *Let* $A_0 \subset A_1 \subset \ldots \subset A_d$ *be a strictly increasing sequence of (d+1) linear spaces in* $\mathbb{P}^n$*. A d-plane K satisfies Schubert condition if* $dim(A_i \cap K) \geq i$ *for all* $i \in \{0, \ldots, d\}$*. The set of all such d-planes K can be seen as a subset of* $\mathbb{G}_{d,n}$ *denoted as* $\Omega(A_0, \ldots, A_d)$*.*

For instance, take an arbitrary line as $A_0 \subset \mathbb{P}^3$ and $A_1 = \mathbb{P}^3$. For this case d=1 and n=3 then $\Omega(A_0, A_1)$ is the subset of $\mathbb{G}_{1,3}$. The 1-dimensional elements K in $\Omega(A_0, A_1)$ should satisfy $dim(A_0 \cap K) \geq 0$ and $dim(A_1 \cap K) \geq 1$. All 1-dimensional linear subspaces directly satisfies the second condition. By the first condition, we derive that K's in $\Omega(A_0, A_1)$ intersect with $A_0$ at one point or infinitely many points ($K = A_0$).

In the previous theorem, we have proved that d-planes of $\mathbb{P}^n$ can be embedded into $\mathbb{P}^{\binom{n+1}{d+1}-1}$. Remember that mainly our objective is to understand the intersection of d-planes. To understand better the intersections of elements in $\mathbb{G}_{d,n}$ it is better to see them as the points of $\mathbb{P}^{\binom{n+1}{d+1}-1}$.

**Proposition 4.2.** *Let* $0 \leq a_0 < \ldots < a_d \leq n$ *be a strictly increasing sequence of integers and* $A_i$ *be the* $a_i - dimensional$ *linear space in* $\mathbb{P}^n$ *consisting of points of the form* $(p(0), \ldots, p(a_i), 0 \ldots, 0)$*. Then,* $\Omega(A_0, \ldots, A_d)$ *includes exactly the points* $(\ldots, p(\ell_0 \ldots \ell_d), \ldots)$ *in* $\mathbb{G}_{d,n}$ *satisfying* $p(\ell_0 \ldots \ell_d) = 0$ *when* $\ell_i > a_i$ *for some i.*

*Proof.* Let $K \in \Omega(A_0, \ldots, A_d)$.

By definition, K should satisfy $dim(A_i \cap K) \geq i$ for $i \in \{0, \dots, d\}$. Choose a point $P_i = (p_i(0), \dots, p_i(n))$ in $A_i \cap K$ for i=1,...,d so that $P_i$'s are linearly independent. So, $\{P_0, \dots, P_d\}$ construct a set of basis for K. Since $P_i \in A_i \cap K$ it also lies in $A_i$ and it should be in the same form of points in $A_i$. By assumption of this form, $p_i(j)$'s are zero for $j = (a_i + 1), \dots, n$. Therefore the matrix representation of K is:

$$\begin{bmatrix} * & \dots & * & 0_{0,a_0+1} & \dots & 0 & \dots & 0 \\ * & \dots & \dots & \dots & * & 0_{1,a_1+1} & \dots & 0 \\ & & \vdots & & \vdots & & \vdots & \\ * & \dots & \dots & * & \dots & * & 0_{d+1,a_d+1} & 0 \end{bmatrix}_{(d+1)\times(n+1)}$$

where * might be zero also. d-plane K can be seen as a point of $\mathbb{G}_{d,n}$ and its points are represented in $\mathbb{P}^{\binom{n+1}{d+1}-1}$ with coordinates $p(\ell_0 \dots \ell_d) = det[p_i(\ell_\alpha)]$ for $i \in \{0, \dots, d\}$ and $\ell_\alpha \in \{0, \dots, n\}$. Specifically, let's choose d+1 columns of this matrix such that one of these columns $\ell_i$, which is indexed with $\gamma$, greater than $a_i$ . Then, these d+1 columns construct the square block matrix:

$$[p_i(\ell_\alpha)] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}_{(d+1)\times(d+1)}$$

where $P_{12}$ is $(\gamma + 1) \times (d + 1 - (\gamma + 1)) = (\gamma + 1) \times (d - \gamma)$ zero matrix. If one expand the determinant of this matrix along the last columns, it can be easily seen that $p(\ell_0 \dots \ell_d) = det[p_i(\ell_\alpha)] = 0$.

Conversely, pick a point on $\mathbb{G}_{d,n}$ which is represented with $(\dots, p(\ell_0 \dots \ell_d), \dots)$ satisfying $p(\ell_0 \dots \ell_d) = 0$ when $\ell_i > a_i$ for some i. Choose a right-most non-zero coordinate $p(k_0, \dots, k_d)$. Since we use lexicographic order in this representation, $\sum_{\alpha=0}^{d} k_\alpha$ should be maximum among the nonzero coordinates. Since $(\dots, p(\ell_0 \dots \ell_d), \dots)$ is a point in the projective space, we may replace $p(\ell_0 \dots \ell_d)$ with $p(\ell_0 \dots \ell_d)/p(k_0, \dots, k_d)$ and also we may assume $p(k_0, \dots, k_d) = 1$.

We know from the previous results, $(\ldots, p(\ell_0 \ldots \ell_d), \ldots)$ represents the d-plane K spanned by the points $P_i = (p_i(0), \ldots, p_i(n))$ where $p_i(j) = p(\ldots \ell_{i-1} j \ell_{i+1})$. Fix a $j \in \{0, \ldots, n\}$ such that $j > a_i$.

We need to show that $p_i(j) = 0$ to complete the proof. Since $p(k_0, \ldots, k_d)$ is nonzero, $k_i \leq a_i$ for all $i \in \{0, \ldots, d\}$ and so $k_i < j$. As a result, $\sum_{\alpha=0}^{d} k_\alpha < j + \sum_{\alpha \neq i} k_\alpha$. Since $\sum_{\alpha=0}^{d} k_\alpha$ is the maximum among nonzero terms, $p_i(j) = p(\ldots k_{i-1} j k_{i+1} \ldots)$ should be zero. Hence $P_i$ lies in $A_i$. $P_0$ lies in $A_0$ , $P_1$ lies in $A_1$ and $A_0 \subset A_1$ implies that $\{P_0, P_1\} \subset A_1$ and so $dim(K \cap A_1) \geq 1$. Similarly, $dim(K \cap A_i) \geq i$ for all $i \in \{0, ..., d\}$. Therefore, K satisfies the Schubert condition and $(\ldots, p(\ell_0 \ldots \ell_d), \ldots)$ satisfying $p(\ell_0 \ldots \ell_d) = 0$ when $\ell_i > a_i$ lies in $\Omega(A_0, \ldots, A_d)$. ☐

Now we will see that if there is a relation between two different set of linear subspaces of $\mathbb{P}^n$ then it causes a relation among corresponding $\Omega(A_0, \ldots, A_d)$.

**Proposition 4.3.** *Let $A_0 \subset \ldots \subset A_d$, $B_0 \subset \ldots \subset B_d$ be two strictly increasing sequences of linear spaces in $\mathbb{P}^n$ and assume $dim(A_i) = dim(B_i)$ for all $i \in \{0, \ldots, d\}$. Then there is an invertible linear transformation which carries $\mathbb{G}_{d,n} \subset P^{\binom{n+1}{d+1}-1}$ into itself such that $\Omega(A_0, \ldots, A_d)$ is mapped to $\Omega(B_0, \ldots, B_d)$.*

*Proof.* Equality of $dim(A_i)$ and $dim(B_i)$ causes the existence of an invertible $(n+1) \times (n+1)$ matrix $[t_{ij}]$ which carries $B_i$ onto $A_i$ for all i. Right multiplication of a point in $\mathbb{P}^n$ with $[t_{ij}] = T$ map to a point into itsef as follows:

$$[(p(0), \ldots, p(n)] \begin{bmatrix} t_{00} & t_{01} & \ldots & t_{0n} \\ t_{10} & t_{11} & \ldots & t_{1n} \\ \vdots & & & \\ t_{n0} & t_{n1} & \ldots & t_{nn} \end{bmatrix} = \Big( \sum_{i=0}^{n} p(i) t_{i0}, \ldots, \sum_{i=0}^{n} p(i) t_{in} \Big).$$

Note that, T carries a d-plane K in $\mathbb{P}^n$ into another d-plane T(K). Moreover, if K$\in \Omega(A_0, \ldots, A_d)$ K should satisfy Schubert condition and $dim(A_i \cap K) \geq i$ for all i. Since $T(A_i) = B_i$ , $dim(B_i \cap T(K)) \geq i$ for all i.

Let's pick d+1 many linearly independent points $P_i(p_i(0),\dots,p_i(n))$ spanning K for i=0,...,d. Then the image of these points under T , $T(P_i)=(q_i(0),\dots,q_i(n))$'s span T(K) and each $q_i(j)$ is of the form:

$$q_i(j)=\sum_{\gamma=0}^{n} p(\gamma)t_{\gamma j} \quad for \quad j=0,\dots,n.$$

This relation result in that Plücker coordinates of T(K) for the sequence $j_0\dots j_d$ is a certain linear combination of Plücker coordinates of K for the same sequence.

Therefore, there is a linear transformation $\tau[t_{ij}]$ of $P^{\binom{n+1}{d+1}-1}$ into itself by mapping $\Omega(A_0,\dots,A_d)$ to $\Omega(B_0,\dots,B_d)$. Besides, invertibility of $[t_{ij}]$ implies the invertibility of $\tau[t_{ij}]$. □

**Corollary 4.4.** *Let $A_0\subset\dots\subset A_d$ be a strictly increasing sequence of linear spaces in $\mathbb{P}^n$. Then $\Omega(A_0,\dots,A_d)$ includes the points in $\mathbb{G}_{d,n}$ whose coordinates $p(\ell_0\dots\ell_d)$ satisfy certain linear equations; that is to say, $\Omega(A_0,\dots,A_d)$ is the intersection of $\mathbb{G}_{d,n}$ and a certain linear spaces in $P^{\binom{n+1}{d+1}-1}$. Besides, the linear space is a hyperplane if and only if $dim(A_0)=(n-d-1)$ and $dim(A_i)=(n-d+(i+1)-1)=(n-d+i)$ where i=1,...,d.*

*Proof.* Let $A_i$'s be $a_i$-dimensional linear spaces and $B_i$'s are also linear spaces with $dim(A_i)=dim(B_i)=a_i$ for all $i\in\{0,\dots,d\}$. Pick $B_i$'s such that they consists of points of the form $(p(0),\dots,p(a_i),0,\dots,0)$. By Proposition 4.3 , there is a linear transformation $\Gamma$ of $P^{\binom{n+1}{d+1}-1}$ into itself such that $\Omega(A_0,\dots,A_d)$ is mapped to $\Omega(B_0,\dots,B_d)$ by $\Gamma$. By proposition 4.2, the image of a point K in $\Omega(A_0,\dots,A_d)$ lies in $\Omega(B_0,\dots,B_d)$ if and only if each coordinates of $\Gamma(K)$, $q(j_0,\dots,j_d)$ equals to zero when $j_i>a_i$ for some $i\in\{0,\dots,d\}$. As we observed in the previous proof, $q(j_0,\dots,j_d)$'s are certain linear combination of coordinates of K's, $p(j_0,\dots,j_d)$'s. Then, certain linear combination of coordinates of K's equal to zero whenever $j_i>a_i$.

Since $\Omega(B_0, \ldots, B_d)$ exactly consists of d-planes whose coordinates satisfying $q(j_0, \ldots, j_d) = 0$ for some $j_i > a_i$, K lies in $\Omega(A_0, \ldots, A_d)$ if and only if certain linear combinations of $p(j_0, \ldots, j_d)$'s equal to zero; in other words, $\Omega(A_0, \ldots, A_d)$ is the intersection of $\mathbb{G}_{d,n}$ and a certain linear space. Then the number of such equations are equal to the number of sequences such that $j_i > a_i$ for some i and also these linear equations are linearly independent because $\Gamma$ is invertible.

Moreover, if our strictly increasing sequence of dimensions are $a_0 = n - d - 1$, $a_1 = n - d + 1$, ... , $a_d = n$ then strictly increasing sequence $j_0, \ldots, j_d$ for $q(j_0, \ldots, j_d) = 0$ can be constructed just one way as $j_0 = n - d$, $j_1 = n - d + 1$, ... , $j_d = n$ where $j_0 > a_0$. Then, there is just one linearly independent eqution defining linear space implies that linear space is a hyperplane. □

Let's turn back to the first question in the introduction: "How many lines in three dimensional space intersect with four given lines?". In one of the previous examples, we noted that the lines in $\mathbb{G}_{1,3}$ can be seen as points of the form (p(01),p(02),p(03),p(12),p(13),p(23)) in $\mathbb{P}^5$ which satisfy solely the quadratic relation:

$$p(01)p(23) - p(02)p(13) + p(03)p(12) = 0.$$

We also observed that the set of lines which intersects a line A in $\mathbb{P}^3$ is $\Omega(A, \mathbb{P}^3)$. Therefore, the set of lines which intersects four given lines $L_1$, $L_2$, $L_3$, $L_4$ can be represented as:

$$M = \bigcap_{i=1}^{4} \Omega(L_i, \mathbb{P}^3).$$

By Corollary 4.4, the linear spaces which are used to obtain $\Omega(L_i, \mathbb{P}^3)$ are hyperplanes since dim($L_i$)=(3-1-1)=1 and $dim(\mathbb{P}^3) = (3 - 1 + 1) = 3$ for n=3 and d=1.

So, for each i we have $\Omega(L_i, \mathbb{P}^3)$=$\mathbb{G}_{1,3} \cap H_i$ for a proper hyperplane in $P^{\binom{3+1}{1+1}-1} = \mathbb{P}^5$. Then;

$$M = \bigcap_{i=1}^{4} (\mathbb{G}_{1,3} \cap H_i) = \mathbb{G}_{1,3} \cap \Big( \bigcap_{i=1}^{4} H_i \Big).$$

If $H_i$'s are linearly independent hyperplanes then then their intersection is a line $\mathbb{P}^5$, say S. Then by using $p(01)p(23) - p(02)p(13) + p(03)p(12)$ equals to zero for lines in $\mathbb{G}_{1,3}$ and express M as the zeroes of a certain quadratic polynomial in parameter S. It can be easily seen that M consists of 2 point which can be coincided when S is tangent to $\mathbb{G}_{1,3}$. If the $H_i$'s are linearly dependent then the dimension of $\cap H_i$ can be higher that one which causes the increasing in dimension of M. In that case dim(M) must be infinite. As a result, the number of lines which intersect four given lines can be either infinite or two or one.

As an example, take three lines $L_1$, $L_2$, $L_3$ such that they do not intersect each other and do not parallel i.e. skew lines. Fix a point $P_1$ on the first line $L_1$. Let $g_1$ be the plane spanned by $L_2$ and $P_1$ also let $g_2$ be the plane spanned by $L_3$ and $P_1$. Surely, $g_1$ and $g_2$ are different planes since $L_2$ and $L_3$ are skew lines. Take $L_4$ as a line in the intersection of $g_1$ and $g_2$ so $L_4$ also passes through $P_1$. It also intersects $L_2$ in a point $P_2$ and $L_3$ in a point $P_3$. Since $L_1$, $L_2$ and $L_3$ are skew lines, $P_1$, $P_2$ and $P_3$ are distinct points and any two of them can define $L_4$. Let L be any line intersecting four of them simultaneously. To understand the conditions L should satisfy to intersect four of them let's make case analysis.

*Case 1:* L conicides with $L_4$ if L passes through $P_2$ and $P_3$. Since $L_4$ intersect $L_1$, $L_2$, $L_3$ we are done.

*Case 2:* If L does not pass through $P_2$, L must lie in $g_1$ plane to intersect both $L_2$ and $L_4$. So L should pass through the intersection point of $g_1$ and $L_1$, which is $P_1$ to intersect $L_1$. To intersect $L_3$ also, L should pass through intersection point of $g_1$ and $L_3$, which is $P_3$. Then, L pass through $P_1$ and $P_3$ implies that L coincides with $L_4$.

*Case 3:* If L does not pass through $P_3$, L must lie in the $g_2$ plane to intersect both $L_3$ and $L_4$. So, to intersect $L_1$ L should pass through the intersection point of $g_2$ and $L_1$, which is $P_1$. Since L should intersect $L_2$ also L includes intersection point of $L_2$ and $g_2$ which is $P_2$. Then L passes through $P_1$ and $P_2$ implies that L is actually $L_4$.

*Case 4*: If L do not pass through $P_2$ and $P_3$, L cannot intersect all four lines at the same time.

Therefore, $L_4$ is the sole line intersecting four given lines when $L_1$, $L_2$, $L_3$ are skew lines. Precisely, if one choose $L_4$ as $L_1$ itself, there will be infinite number of lines intersecting four given lines. We can pick an L corresponding to each point of $L_1$. One can consider examples such that if four given lines pass through one point then there are infinitely many lines intersecting all of them simultaneously or if all lie in the same plane without considering they intersect each other or not one can assert that there are infinitely many lines intersecting all of them.

In some special cases, as we see in the examples M can be infinite so we should revise the 'principle of conservation of numbers' in the following fashion: 'If the number of solutions is finite in any special case of problem then the number of solutions is the same in the general case also, up to multiplicities'. Schubert realized that something must be conserved under specialization of the general problem which is exactly the cohomology classes.

## 4.2. Schubert Varieties of Grassmannians

In projective space, linear subspaces can be seen as a celluler decomposition and define the cohomology groups of the projective spaces. In Grassmannians, Schubert varieties which can be encoded by certain partitions play a similar role. In this section we will decompose Grassmannians by using Schubert cells and varieties then examine their meaning in algebraic topology.

**Definition 4.5.** *Fix a complete flag,* $\mathbb{F}$.

$$0 = F_0 \subset F_1 \subset \cdots \subset F_i \subset \cdots \subset F_{n+m} = \mathbb{C}^{n+m}$$

*namely, strictly increasing sequence of vector subspaces of* $\mathbb{C}^{n+m}$*, where each* $F_i$ *is of dimension i. Let* $\lambda$ *be a partition contained in an* $m \times n$ *rectangle, that is, a nonincreasing sequence of integers* $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$*. For the fix flag* $\mathbb{F}$ *and a partition* $\lambda$*, Schubert cell is defined as;*

$$X_\lambda = \{H \in \mathbb{G}_{m,n},\ dim(H \cap F_j) = i \ \ if \ \ n + i - \lambda_i \leq j \leq n + i - \lambda_{i+1}\}. \tag{4.1}$$

*Also Schubert variety based on the same fix flag and partition defined as;*

$$\Omega_\lambda = \{H \in \mathbb{G}_{m,n}, dim(H \cap F_{n+i-\lambda_i}) \geq i, 1 \leq i \leq m\}. \tag{4.2}$$

Note that in the previous section we defined $\Omega(A_0, \ldots, A_d)$ in a similar manner with $\Omega_\lambda$. However, to define $\Omega(A_0, \ldots, A_d)$ we use (d+1)-many spaces and to define $\Omega_\lambda$ we have used complete flag of $\mathbb{C}^{m+n}$. To better understand the definition let's examine a few examples.

**Example 4.6.** *(i) Let* $\lambda = \varnothing$*. Then;*

$\Omega_\varnothing = \{H \in \mathbb{G}_{m,n},\ dim(H \cap F_{n+i}) \geq i, 1 \leq i \leq m\}$*. That means* $\Omega_\varnothing$ *consists of all possible m dimensional subspaces of* $\mathbb{C}^{m+n}$*, which is* $\mathbb{G}_{m,n}$.

*(ii) If* $\lambda_1 = \cdots = \lambda_m = n$ *then*

$\Omega_{(n,\cdots,n)} = \{H \in \mathbb{G}_{m,n}, dim(H \cap F_i) \geq i, 1 \leq i \leq m\} = F_m$.

*(iii) For a partition with just one nonzero part, say* $\lambda = (k)$

$\Omega_{(k)} = \{H \in \mathbb{G}_{m,n},\ dim(H \cap F_{n+1-k}) \neq 0 \ and \ dim(H \cap F_{n+i}) \geq i \ \ for \ \ 2 \leq i \leq m\}$.

*(iv) Let* $\lambda(r,t)$ *be the partition which is the complement of* $r \times t$ *rectangle in* $m \times n$ *rectangle. In other words,* $\lambda_1(r,t) = \cdots = \lambda_{m-r}(r,t) = n \quad \lambda_{m-r+1}(r,t) = \cdots = \lambda_m(r,t) = n - t$.

$\Omega_{\lambda(r,t)} = \{H \in \mathbb{G}_{m,n}, dim(H \cap F_{n+i-(n)}) \geq i,\ 1 \leq i \leq m-r \ \ and$
$\dim(H \cap F_{n+i-(n-t)}) \geq i, \quad m-r+1 \leq i \leq m\}.$

*Since for i=m-r,* $dim(H \cap F_{n+i-(n)}) = dim(H \cap F_{m-r}) \geq m-r$ *and also for i=m,* $dim(H \cap F_{n+m-(n-t)}) = dim(H \cap F_{m+t}) \geq m$ *implies that;*

$$\Omega_{\lambda(r,t)} = \{H \in \mathbb{G}_{m,n},\ F_{m-r} \subset H \subset F_{m+t}\}.$$

*Moreover,* $\Omega_{\lambda(r,t)}$ *is isomorphic to* $\mathbb{G}_{r,t}$*.*

*(v) Let* $\lambda$ *be a partition contained in* $6 \times 7$ *rectangle such that* $\lambda_1 = 7$*,* $\lambda_2 = 5$*,* $\lambda_3 = 4$*,* $\lambda_4 = 3$*,* $\lambda_5 = 2$*,* $\lambda_6 = 2$*. Then;*

$$X_{(7,5,4,3,2,2)} = \{W \in \mathbb{G}_{6,7},\ dim(W \cap F_j) = i \ \ if \ \ n+i-\lambda_i \leq j \leq n+i-\lambda_{i+1}\}.$$

$i=1 \quad dim(W \cap F_j) = 1$ *if* $1 \leq j \leq 3$*,*
$i=2 \quad dim(W \cap F_j) = 2$ *if* $4 \leq j \leq 5$*,*
$i=3 \quad dim(W \cap F_j) = 3$ *if* $6 \leq j \leq 7$*,*
$i=4 \quad dim(W \cap F_j) = 4$ *if* $8 \leq j \leq 9$*,*
$i=5 \quad dim(W \cap F_j) = 5$ *if* $10 \leq j \leq 10$*,*
$i=6 \quad dim(W \cap F_j) = 6$ *if* $11 \leq j \leq 13$*.*
*In other words,*
$dim(W \cap F_1) = 1$, $dim(W \cap F_2) = 1$, $dim(W \cap F_3) = 1$,
$dim(W \cap F_4) = 2$, $dim(W \cap F_5) = 2$,
$dim(W \cap F_6) = 3$, $dim(W \cap F_7) = 3$,
$dim(W \cap F_8) = 4$, $dim(W \cap F_9) = 4$,
$dim(W \cap F_{10}) = 5$,
$dim(W \cap F_{11}) = 6$, $dim(W \cap F_{12}) = 6$, $dim(W \cap F_{13}) = 6$.
*Therefore, if* $W \in X_{(7,5,4,3,2,2)}$ *there are jumps occuring at* $F_1,\ F_4,\ F_6,\ F_8,\ F_{10},\ F_{11}$ *which is exactly* $F_{7+i-\lambda_i}$ *for each* $i \in \{1, \ldots, 6\}$*. In general, these dimensional jumps occur at* $F_{n+i-\lambda_i}$ *for each* $i \in \{1, \ldots, m\}$*.*

*As a result matrix representation of Schubert cell is in the following form:*

$$X_{(7,5,4,3,2,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & * & 0 & * & 0 & * & 1 & 0 & 0 & 0 \\ 0 & * & * & 0 & * & 0 & * & 0 & * & 0 & 1 & 0 & 0 \end{pmatrix}_{6\times(13)}$$

*for* $* \in \mathbb{C}$. *Also, observe that cell is isomorphic to* $\mathbb{C}^{19}$ *where 19=(6×7)-(7+5+4+3+2+2) which is not just coincidence!*

**Definition 4.7.** *Let* $\lambda = (\lambda_1, \cdots, \lambda_k), \mu = (\mu_1, \cdots, \mu_n)$ *be partitions and t=max(k,n). We say* $\lambda \subset \mu$ *if and only if* $\lambda_i \leq \mu_i$ *for* $1 \leq i \leq t$.

**Remark 4.8.** *To understand in a better way why Schubert variety and cells of Grassmannians can be indexed with partitions* $\lambda \subset m \times n$ *one can also check the following set up to define Schubert cells:*

*Let* $\mathbb{G}(t,k)$ *denotes the set of k-planes in* $\mathbb{C}^t$. *Then* $X_j$ *is defined as the following set;*

$$\left\{ \begin{bmatrix} * & \ldots & * & 1_{1j_1} & 0 & \ldots & 0 & \ldots & & \ldots & 0 & 0 & \ldots & 0 \\ * & \ldots & * & 0 & * & \ldots & 1_{2j_2} & 0 & & \ldots & 0 & 0 & \ldots & 0 \\ & \vdots & & & & \vdots & & \vdots & & & & & \vdots & \\ * & \ldots & * & 0 & * & \ldots & 0 & * & \ldots & & 1_{kj_k} & 0 & \ldots & 0 \end{bmatrix} : j = (j_1, \ldots, j_k), * \in \mathbb{C} \right\}.$$

*In other words, we are indexing the cells with the position of 1's. In comparison with our orjinal notation;*

*n=t-k , m=k, m+n=t,* $j_i = n + i - \lambda_i$. *Let's define* $a_\ell := j_\ell - \ell$ *and* $\phi_\ell := t - k - a_\ell$ *for* $\ell = 1, \ldots, k$. *Note that* $\ell$ *stands for the row number. So,* $a_\ell$ *counts the number of stars in each row and constitutes an increasing seuence means* $\phi_\ell$ *will be decreasing one so that maximum value of* $\phi_\ell$ *can be t-k=n. As a result,* $\phi_\ell$ *has to be partition of* $k \times t - k$.

*Similarly, in our case* $\phi_\ell = t - k - a_\ell = n - a_\ell = n - (n + i - \lambda_i - i) = \lambda_i$ *for* $i = 1, \ldots, m$. *Therefore,* $\lambda$ *is a partition of* $m \times n$ *in our definition.*

**Proposition 4.9.** *For all partitions* $\lambda \subset m \times n$,

- *(i) The Schubert variety* $\Omega_\lambda$ *is an algebraic subvariety of* $\mathbb{G}_{m,n}$
- *(ii)* $X_\lambda \simeq \mathbb{C}^{mn-|\lambda|}$
- *(iii)* $\mathbb{G}_{m,n} = \bigsqcup_{\lambda \subset m \times n} X_\lambda$
- *(iv)* $\Omega_\lambda = \bigsqcup_{\lambda \subset \mu} X_\mu$
- *(v)* $\Omega_\mu \subset \Omega_\lambda$ *if and only if* $\lambda \subset \mu$
- *(vi)* $\Omega_\lambda = \overline{X_\lambda} = \bigsqcup_{\lambda \subset \mu} X_\mu$.

*Proof.* (i) To prove that Schubert variety is an algebraic subvariety of $\mathbb{G}_{m,n}$ we need to find the polynomial equations which is satisfied by $\Omega_\lambda$. Let $W \in \mathbb{G}_{m,n}$. Then, $dim(W \cap F_i) \geq j$ if and only if the rank of map $W \subset \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}/F_i$ is less than or equal to m-j. If this map is expressed as a matrix then minors of order m-j+1 are always vanishing and W is the root of these polynomials. Since the Schubert variety is defined with such incidence conditions, it can be seen as a subvariety of $\mathbb{G}_{m,n}$.

(ii) Let $G \in X_\lambda$. Then this space admits a unique basis consisting of vectors such that

$$g_i = f_{n+i-\lambda_i} + \sum_{1 \leq j \leq n+i-\lambda_i, j \neq n+k-\lambda_k for k \leq i} x_{ij} f_j$$

for $1 \leq i \leq m$. Here we choose $f_1, \ldots, f_{m+n}$ as a basis set of $\mathbb{C}^{m+n}$ where $F_i = \langle f_1, \ldots f_i \rangle$ for all $i \in \{1, \ldots, m\}$. Then, the existence of mn-$|\lambda|$ possible $x_{ij}$ implies the existence of an isomorphism defined by $x_{ij}$ from $X_\lambda$ to $\mathbb{C}^{mn-|\lambda|}$ where $|\lambda| = \lambda_1 + \ldots + \lambda_m$. So, $X_\lambda \simeq \mathbb{C}^{mn-|\lambda|}$.

(iii) If W is an element of Grassmanian then the sequence of intersections $W \cap F_i$ where i runs from 0 to m, increasing at most one dimension at each step.

We can mark the jumping point as $n+i-\mu_i$ for the partition $\mu \subset m \times n$. This implies that

$$\mathbb{G}_{m,n} = \coprod_{\mu \subset m \times n} X_\mu .$$

(iv) If the dimension of $W \cap F_{n+i-\lambda_i}$ is greater than or equal to i then the first i increasing in dimension occurs before $n+i-\lambda_i$. Therefore, $n+i-\lambda_i \geq n+i-\mu_i$ implies that $\lambda_i \leq \mu_i$. As a result,

$$\Omega_\lambda = \coprod_{\lambda \subset \mu} X_\mu .$$

(v) If $\lambda \subset \mu$ then by (iv), it immediately follows that;

$$\Omega_\mu = \coprod_{\mu \subset \nu} X_\nu \subset \coprod_{\lambda \subset \nu} X_\nu = \Omega_\lambda$$

and vice versa.

(vi) By (iv) or by definition, we may note that $X_\lambda \subseteq \Omega_\lambda$ then $\overline{X}_\lambda \subseteq \overline{\Omega}_\lambda$. Since $\Omega_\lambda$ is a variety, it is Zariski closed which means $\Omega_\lambda = \overline{\Omega}_\lambda$. Then $\overline{X}_\lambda \subseteq \Omega_\lambda$.

By (ii), one can realize that $X_\lambda$ consists of m dimensional subspaces which can be represented in the following form:

$$\begin{bmatrix} * & \dots & * & 0 & \dots & 0 & \dots & \dots & \dots & 0 \\ * & \dots & * & \dots & * & 0 & \dots & \dots & \dots & 0 \\ & & \vdots & & \vdots & & \vdots & & & \\ * & \dots & * & \dots & \dots & \dots & * & 0 & \dots & 0 \end{bmatrix}_{m \times (m+n)}$$

where each rightmost asterisk is nonzero and lies in $(n+i-\lambda_i)^{th}$ position in the $i^{th}$ row and each $* \in \mathbb{C}$. Therefore, if $\mu \supset \lambda$ then $X_\mu \subset X_\lambda \subset \overline{X}_\lambda$. Then, $\Omega_\lambda = \bigcup_{\mu \supset \lambda} X_\mu \subset \bigcup_{\mu \supset \lambda} \overline{X}_\lambda = \overline{X}_\lambda$.

As a result, $\Omega_\lambda = \overline{X}_\lambda$. Moreover, $\coprod_{\lambda \subset \nu} X_\nu = \Omega_\lambda$ implies that $\coprod_{\lambda \subset \nu} X_\nu = \overline{X}_\lambda$.

□

The Schubert varieties of Grassmannian $\mathbb{G}_{m,n}$ are projective varieties which can be indexed with partitions of $m \times n$ rectangle. This nice property provides a combinatoric way to determine geometric properties of Schubert variety.

One can also wonder that if we consider the (m+n)-dimensional subspace in $\mathbb{C}^{(m+n)}$ instead of m-dimensional subspaces whether we will obtain a combinatoric property of the set of full-ranked matrices to study geometric properties of object. The answer will be given in the next chapter which is permutation.

### 4.3. Intersection of Schubert Varieties of Grassmannians

Up to now, we defined Schubert varieties as a family of subvarieties of Grassmannians which can be indexed by partitions of certain rectangle. The intersection properties of these variables are very similar with the product properties of Schur functions. The standard monomials defined for Schur functions pave the way for describing the ideal of Schubert variety of Grassmannian.

By proposition 4.9. we know that Schubert variety consists of certain Schubert cells. So, their intersection problem turns into whether Schubert cells they include intersects or not. In this section, we will focus on the intersection of Schubert cells whose reference flag different.

Let's fix the flag of reference as $\{0\} = F_0 \subset \cdots \subset F_{m+n} = \mathbb{C}^{m+n}$ and a basis $f_1, \cdots, f_{m+n}$ which respects this flag, that is, $F_i = \langle f_1, \cdots, f_i \rangle$.

Now, we will construct the new flag with the same basis which is called dual flag as follows:

$$F'_i = \langle f_{m+n-(i-1)}, \cdots, f_{m+n-(i-i)} \rangle = \langle f_{m+n-i+1}, \cdots, f_{m+n} \rangle.$$

Schubert cells and Schubert varieties corresponding to dual flag are denoted as $X'_\lambda$ and $\Omega'_\lambda$, respectively. We know that each element in Schubert cell $X_\lambda$ can be reduced to the following form:

$$\begin{pmatrix}
* & \dots & * & 1 & 0 & \dots & 0 & \dots & \dots & 0 & 0 & \dots \\
* & \dots & * & 0 & * & \dots & 1 & 0 & \dots & 0 & 0 & \dots \\
 & & \vdots & & & \vdots & & \vdots & \vdots & & & \\
* & \dots & * & 0 & * & \dots & 0 & * & \dots & 1 & 0 & \dots
\end{pmatrix}_{m\times(m+n)}$$

where the position of 1 in the $i^{th}$ row is $n+i-\lambda_i$. Similarly; an m-dimensional subspace in Schubert cell $X'_\mu$ should be in the following form:

$$\begin{pmatrix}
0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots & 0 & * & \dots \\
 & & \vdots & & & \vdots & & & \vdots & \vdots & & & \\
0 & \dots & \dots & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & * & \dots \\
0 & \dots & 0 & 0 & 0 & \dots & & 0 & 0 & \dots & 1 & * & \dots
\end{pmatrix}_{m\times(m+n)}$$

where the position of 1 in the $i^{th}$ row is $\mu_{m+1-i}+i$.

If we take $L \in X'_\mu \cap X_\lambda$ then L should satisfy both form of matrices in $X'_\mu$ and $X_\lambda$. So, first row of m-plane in $X_\lambda$ should be a linear combination of rows of m-plane in $X'_\mu$. This implies that the position of 1 in the first row of m-plane in $X_\lambda$ seen in the larger index than the position of 1 in the first row of m-plane in $X'_\mu$, that is to say, $\mu_{m+1-1}+1 \leq n+1-\lambda_1$ then $\lambda_1+\mu_m \leq n$.

To generalize, if we take $i^{th}$ row of the m-plane in $X_\lambda$ it should be the combination of the last m-(i-1)=m+1-i rows of m-plane in $X'_\mu$. The position of 1 condition for the $i^{th}$ row holds if the inequality $\mu_{m+1-i}+i \leq n+i-\lambda_i$ then $\lambda_i+\mu_{m+1-i} \leq n$ is satisfied. Therefore, we may conclude that if $X'_\mu \cap X_\lambda$ is nonempty then $\mu \subset \hat{\lambda}$ for $\hat{\lambda}$ denotes the complementary parition of $\lambda$ in an $m \times n$ rectangle.

## 4.4. Pieri's Formula

Schubert varieties of Grassmannian manifold can be indexed with partitions. The cohomology ring of Grassmannian can be decomposed into Schubert classes which is denoted as $\sigma$ and indexed with partitions (or equivalently Schur functions) and the cup product of two Schubert classes is related with the product of Schur functions. In section 2.1, the product of Schur functions is understood thanks to Pieri's formula. To prove the other form of Pieri's formula we need following proposition in this set-up.

**Proposition 4.10.** *Let $\gamma$ and $\alpha$ be two partitions lying in $m \times n$ rectangle, and suppose that $|\gamma| + |\alpha| = mn$. Then,*

$$\sigma_{\gamma} \smile \sigma_{\alpha} = \delta_{\hat{\gamma},\alpha}. \tag{4.3}$$

So, an element of cohomology ring of Grassmannian can be seen as following summation:

$$x = \sum_{\gamma \in m \times n} (x \smile \sigma_{\hat{\gamma}}) \sigma_{\gamma}. \tag{4.4}$$

As we did in Section 4.3 , if $X'_{\mu} \cap X_{\nu}$ is nontrivial then $\mu \subset \hat{\nu}$. By Proposition 4.10., we may derive that for $|\nu| + |\mu| = mn$ if $\nu \neq \hat{\mu}$ then $\Omega_{\mu} \cap \Omega'_{\nu}$ is empty. Hence, the cup product of Schubert classes indexed by $\mu$ and $\nu$ equals to zero. If $\nu = \hat{\mu}$ then $\Omega_{\mu} \cap \Omega'_{\nu}$ is nonempty and $\sigma_{\mu} \smile \sigma_{\nu}$ equals to 1.

**Theorem 4.11.** *(Pieri's Formula) If $\lambda \subset m \times n$ is a partition, and $k$ is an integer between 1 and n, then*

$$\sigma_{\lambda} \smile \sigma_{k} = \sum_{\nu \subset m \times n,\ \nu \in \lambda \otimes k} \sigma_{\nu}.$$

*Proof.* By using the above facts, Proposition 4.10 and Equation (4.4); it is enough to show that if $|\lambda| + |\mu| = mn - k$ then $\sigma_\lambda \smile \sigma_\mu \smile \sigma_k = 1$ when the following condition is satisfied;

$$n - \lambda_m \geq \mu_1 \geq n - \lambda_{m-1} \geq \mu_2 \geq \ldots \geq n - \lambda_1 \geq \mu_m$$

and the cup product is zero otherwise.

Remember that the nonempty intersection of $X_\lambda$ and $X'_\mu$ occurs when $\lambda_i + \mu_{m+1-i} \leq n$. So, for i=m this condition turns out that $n - \lambda_m \geq \mu_1$ and for i=m-1 it becomes $n - \lambda_{m-1} \geq \mu_2$ and so on. Since the non-empty intersection of Schubert cells implies non-empty intersection of varieties and it ends-up with non-zero cup product of corresponding Schubert cycles, from now on we will assume that $\lambda_i + \mu_{m+1-i} \leq n$ for all i, to obtain nonzero $\sigma_\lambda \smile \sigma_\mu$. Let's construct the following vector spaces;

$$\begin{aligned}
K_i &= \langle v_1, \ldots, v_{n+i-\lambda_i} \rangle && = V_{n+i-\lambda_i} \\
L_i &= \langle v_{\mu_{m+1-i}+i}, \ldots, v_{m+n} \rangle && = V'_{n+m+1-i-\mu_{m+1-i}} \\
M_i &= \langle v_{\mu_{m+1-i}+i}, \ldots, v_{n+i-\lambda_i} \rangle && = K_i \cap L_i \\
M_{i+1} &= \langle v_{\mu_{m-i}+i+1}, \ldots, v_{n+i+1-\lambda_{i+1}} \rangle && = K_{i+1} \cap L_{i+1}.
\end{aligned}$$

These vector spaces are choosen in these forms since we will use them to define the incidence conditions of $\Omega_\lambda$ and $\Omega'_\mu$. Let's compare $M_i$ and $M_{i+1}$ to observe their intersection conditions. If they are non intersect vector spaces then,

$$n + i - \lambda_i < \mu_{m-i} + i + 1 \quad \sim \quad n - \lambda_i \leq \mu_{m-i}$$

holds, which is exactly the condition we want to prove.

So, the above condition is satisfied if and only if $M_i$'s are direct summands, that is, if and only if the sum of their dimensions is,

$$\sum_{i=1}^{m}(n+i-\lambda_i)-(\mu_{m+1-i}+i)=mn-|\lambda|-|\mu|=k.$$

Then if $D\in\Omega_\lambda\cap\Omega'_\mu$, by definition of Schubert variety we should have $dim(D\cap K_i)\geq i$ and $dim(D\cap L_i)\geq m-i+1\sim dim(D\cap L_{i+1})\geq m-i$. Since D is also an m-dimensional subspace in $\mathbb{C}^{m+n}$, $D\subset K_i\cap L_{i+1}$ for all i. Then, $D\in\bigcap(K_i+L_{i+1})$.

Now let E be a subspace of dimension n+1-k of $\mathbb{C}^{m+n}$, as we did in the previous sections Schubert variety corresponding to partition k is;

$$\Omega_k(E)=\{D\in\mathbb{G}_{m,n},D\cap E\neq 0\}.$$

If $\Omega_\lambda\cap\Omega'_\mu\cap\Omega_k(E)=\varnothing$ then we know that $\sigma_\lambda\smile\sigma_k\smile\sigma_\mu=0$. If the above condition is not satisfied then $dim(\bigcap K_i+L_{i+1})\leq m+k-1$ hence E can be choosen such that $(\bigcap K_i+L_{i+1})\cap E$ intersects trivially which results in $\sigma_\lambda\smile\sigma_k\smile\sigma_\mu=0$.

On the other hand, if $dim(\bigcap K_i+L_{i+1})=m+k$ then $dim(\bigcap K_i+L_{i+1}\cap E)$ has to be at least one since (n+1-k)+(m+k)=m+n+1. Then, the triple intersection $\Omega_\lambda\cap\Omega'_\mu\cap\Omega_k(E)=W$. Since their intersection is transverse $\sigma_\lambda\smile\sigma_k\smile\sigma_\mu=1$. □

To understand the relation between being transverse and cup product operation, see [1] page 158-159. In the preliminary chapter we have defined the ring of symmetric polynomials with integral coefficients and denote it by $\Lambda_n$ if it contains symmetric polynomials with n variables. Now, we are going to give a ring morphism between $\Lambda_n$ and the cohomology ring of $\mathbb{G}_{m,n}$, $H^*(\mathbb{G}_{m,n})$.

**Corollary 4.12.** *There is a surjective ring morphism,*

$$\theta_{m,n} : \Lambda_n \to H^*(\mathbb{G}_{m,n})$$

*such that each Schur functions* $s_\lambda$ *in* $\Lambda_n$ *is mapped to the Schubert class* $\sigma_\lambda$ *for* $\lambda \subset m \times n$ *and zero otherwise.*

This corollary is of importance since it transports the problems in combinatorics to algebraic toplogy.

# 5. FLAG MANIFOLD

Let $F_\bullet$ be a complete flag generated by $(F_1, F_2, \ldots, F_n)$ where $dim(F_i) = i$ for each $i \in \{1, \ldots, n\}$. For instance, $\mathbb{F}_\bullet = \langle 4e_1+3e_2, 2e_1+3e_3, 8e_1+e_3+e_4, e_2\rangle$ is a complete flag with standard basis of $\mathbb{C}^4$. And let $\mathbb{F}_n$ be the set of complete flags of vector subspaces of $\mathbb{C}^n$. Then we call $\mathbb{F}_n$ as a flag manifold and each point of flag manifold corresponds to a complete flag. Since there is an invertible matrix in GL(n,$\mathbb{C}$) which transforms a full-ranked matrix to a full-ranked matrix, the complex linear group GL(n,$\mathbb{C}$) has a transitive action on $\mathbb{F}_n$. Therefore we have,

$$\mathbb{F}_n \simeq GL(n, \mathbb{C})/L$$

where L denotes the set of invertible lower triangular matrices. Since the flag is full-ranked when an arbitray flag in $\mathbb{C}^n$ written in the canonical form then the position of the leading 1's correspond to a permutation in $S_n$. So, we may use the permutations to index the elements of flag varieties thanks to the uniqueness of canonical forms. As in the Grassmannian variety, we may use the combinatorics on permutations to understand the geometry of flag manifold and also we may define the Schubert varieties and Schubert cell of flag manifold.

## 5.1. Schubert Varieties of a Flag Manifold

Let's fix a complete flag $G_\bullet$ spanned by vectors $g_1, \ldots, g_n$ in $\mathbb{C}^n$ such that $g_1, \ldots, g_i$ generates $G_i$. If $G'_i$ is spanned with $g_{n+1-i}, \ldots, g_n$ and if $H_\bullet \in \mathbb{F}_n$ satisfies $H_i \cap G'_{n-i} = 0$ for all i then there exists a unique basis $h_1, \ldots, h_n$ of $\mathbb{C}^n$ such that;

$$h_i = g_i + \sum_{j>i} x_{ij} e_j, \quad 1 \leq i \leq n$$

where aech $H_i$ is spanned with $h_1, \ldots, h_i$.

Then $H_\bullet$ can be represented with the following matrix:

$$\begin{bmatrix} 1 & x_{12} & \dots & \dots & \dots & x_{1n} \\ 0 & 1 & \dots & \dots & \dots & x_{2n} \\ & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & \dots & 1 & x_{n-1,n} \\ 0 & 0 & \dots & \dots & 0 & 1 \end{bmatrix}$$

for $x_{ij} \in \mathbb{C}^n$. The number of free coordinates of matrix is $(n-1)+(n-2)+\ldots+1 = (n-1+1)(n-1)/2 = n(n-1)/2$. Then the set of all possible such matrices $\mathbb{F}_n$ is isomorphic to $\mathbb{C}^{n(n-1)/2}$.

As in the Grassmannian variety case, we may define Plücker embedding on flag variety. Note that all possible choices for the first row of $F_\bullet$ is isomorphic to $\mathbb{G}_{1,n-1}$ and all possible choices for the first two rows of $F_\bullet$ lie in $\mathbb{G}_{2,n-2}$. Then in the last case, we may see the flag variety as a closed subvariety of $\mathbb{G}_{1,n-1} \times \mathbb{G}_{2,n-2} \times \ldots \times \mathbb{G}_{n-1,1}$ and on each $\mathbb{G}_{i,n-i}$ Plücker embedding can be seen as $\phi_i : \mathbb{G}_{i,n-i} \to \mathbb{P}^{\binom{n}{i}-1}$ for each $i \in \{1,\ldots,n-1\}$.

**Definition 5.1.** *Let $w \in S_n$, then the rank function is defined as;*

$$r_w(p,q) = |\{(i,w(i)) : i \leq p, w(i) \leq q\}|. \tag{5.1}$$

In other words; rank function $r_w$ associates the point (p,q) with the number of 1 in the $p \times q$ upper-left minor of matrix representation of $F_\bullet$.

Now, let's define the Schubert cell and variety of the Flag manifold. Schubert cell is defined as similar with Grassmannians.

**Definition 5.2.** *The Schubert cell of Flag manifold is defined as*

$$X_w = \{F_\bullet \in \mathbb{F}_n, dim(F_p \cap G_q) = r_w(p,q), 1 \leq p,q \leq n\}. \tag{5.2}$$

One may observe that the dimension of Schubert cell equals to the length of permutation indexing Schubert cell. The Zariski closure of Schubert cell is Schubert variety and defined as follows.

**Definition 5.3.**

$$\Omega_w = \{F_\bullet \in \mathbb{F}_n, dim(F_p \cap G_q) \geq r_w(p,q), 1 \leq p,q \leq n\}. \tag{5.3}$$

Schubert variety of a Flag manifold can be seen as a disjoint union of Schubert cells and also as an algebraic subvariety of Flag manifold. More precisely, $\Omega_w$ consists of Schubert cells indexed with $v \leq w$ where $' \leq'$ represents the Bruhat order.

## 5.2. Monk's Rule

Schubert varieties of Flag manifold can be indexed with permutations. Since Schubert polynomials are also indexed with permutations these polynomials can be seen as representatives of Schubert classes in the cohomology of a flag manifold. Moreover, the product of Schubert polynomials are associated with the intersection of these Schubert classes. In the section 2.2, we discussed the Monk's Rule for the product of Schubert polynomials. Now, we are going to see Monk's Rule for the cup product of Schubert cycles of Flag manifold where cycles denoted with $\sigma$. Since Schubert cycles of Flag manifold can be represented with Schubert polynomials, it is not surprise to observe the similar formulas.

**Theorem 5.4.** *(Monk's Rule) For all permutations $w \in S_n$, and all integers $i < n$, we get*

$$\sigma_w \smile \sigma_i = \sum_{j \leq i < k,\ l(wt_{jk}) = l(w)+1} \sigma_{wt_{jk}}. \tag{5.4}$$

Let $\wp_n$ be the ring of polynomials with integral coefficients in n variables. By Theorem 2.11, we know that its basis are Schubert polynomials which causes the existense of followig ring morphism between the ring of polynomials and cohomology ring of Flag manifold.

**Corollary 5.5.** *The map* $\Im_n : \wp_n \to H^*(\mathbb{F}_n)$*, defined by*

$$\Im_n(\wp_n) = \begin{cases} \sigma_w & if w \in S_n \\ 0 & otherwise \end{cases}$$

*is a surjective ring morphism.*

This corollary is important because it transports the questions in combinatorics to algebraic topology.

# 6. CONCLUSION

In this brief overview, we have started from combinatoric notions related with Schubert polynomials and Schur functions. Then we realized Grassmannians as algebraic projective subvarieties thanks to Plücker embedding and sketch the Schubert calculus. Our main objective was to see the connection between symmetric function ring and cohomology ring of Grassmannians since this connection provides tools coming from algebraic topology for problems in algebraic geometry.

To sum up, we have tried to enhance the explanations and proofs about the topics and provide a good way to learn Schubert calculus for the students in the level of master degree in mathematics. Next step may be working on well-written proofs in Schubert calculus from algebraic topologic perspective.

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# APPENDIX A: BASIC NOTIONS IN ALGEBRAIC GEOMETRY

In this part, mostly we aim to introduce the meaning of the affine and projective variety and Zariski closure of variety.

**Definition A.1.** *Let $k$ be a field, and let $p_1, \ldots, p_s$ be polynomials in $k[x_1, \ldots, x_n]$. Then;*

$$V(p_1, \ldots, p_t) = \{(k_1, \ldots, k_n) \in k^n : p_i(k_1, \ldots, k_n) = 0 \ for\ all\ 1 \leq i \leq t\}. \tag{A.1}$$

*We call $V(p_1, \ldots, p_t)$ the affine variety defined by $p_1, \ldots, p_t$. A subset $C$ of $k^n$ of the form $V(p_1, \ldots, p_t)$ is said to be Zariski closed in $k^n$.*

In other words, $V(p_1, \ldots, p_t) \subset k^n$ is the set of all solutions of equations $p_1(x_1, \ldots, x_n) = \ldots = p_t(x_1, \ldots, x_n)$=0. For instance, in $\mathbb{R}^2$ $V(x^2 + y^2 - 1)$ is a circle centered at the origin with radius 1.

**Lemma A.2.** *If $V, W \subset k^n$ are affine varieties, then so are $V \cup W$ and $V \cap W$.*

As a natural consequence of this lemma one can deduce that finite intersections and unions of affine varieties are again affine varieties.

**Definition A.3.** *The Zariski topology on $k^n$ is the topology whose closed sets are the Zariski closed.*

It can be defined as topology since the followings are satisfied for two sets of polynomials $F, G \subset k[x_1, \ldots, x_n]$ ;

(i) $V(\varnothing) = k^n$

(ii) $V(k[x_1, \ldots, x_n]) = \varnothing$

(iii) $V(F \cup G) = V(F) \cap V(G)$

(iv) $V(FG) = V(F) \cup V(G)$.

**Definition A.4.** *Let $W \subset k^n$ be an affine variety. Then;*

$$I(W) = \{p \in k[x_1, \ldots, x_n] : \ p(k_1, \ldots, k_n) = 0 \ for \ all \ (k_1, \ldots, k_n) \in W\}. \qquad \text{(A.2)}$$

*We call $I(W)$ as the ideal of W.*

**Proposition A.5.** *Let V and W be affine varieties in $k^n$. Then;*
*(i) $V \subset W$ if and only if $I(V) \supset I(W)$*
*(ii) V=W if and only if $I(V) = I(W)$.*

**Definition A.6.** *Let $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n_{\geq 0}$. "v" is called as lexicographically smaller than "w" if in the vector difference $v - w \in \mathbb{Z}^n$ the left-most nonzero entry is negative.*

For instance, $(5, 4, 8, 7, 2)$ is lexicographically smaller than $(5, 5, 1, 1, 1)$ since $(5, 4, 8, 7, 2) - (5, 5, 1, 1, 1) = (0, -1, 7, 6, 1)$ and first left-most nonzero entry is -1.

**Theorem A.7.** *If $I = \langle p_1, \ldots, p_r \rangle, J = \langle q_1, \ldots, q_s \rangle$ are ideals in $k[x_1, \ldots, x_n]$, then $V(I.J) = V(I) \cup V(J)$ where $I.J = \langle p_i q_j : 1 \leq i \leq r, 1 \leq j \leq s \rangle$.*

**Definition A.8.** . *The Zariski closure of a subset of affine space $S \subset k^n$ is the smallest affine algebraic variety containing this set. The Zariski closure of S denoted as $\bar{S}$ and also equals to V(I(S)).*

**Definition A.9.** *Let k be a field and let $p_1, \ldots, p_t \in k[x_0, \ldots, x_n]$ be homogenous polynomials. Then;*

$$V(p_1, \ldots, p_t) = \{(k_0, \ldots, k_n) \in \mathbb{P}^n(k) : p_i(k_0, \ldots, k_n) = 0 \ for \ all \ 1 \leq i \leq t\}. \qquad \text{(A.3)}$$

*$V(p_1, \ldots, p_t)$ is called as projective variety defined by $p_1, \ldots, p_t$.*

# APPENDIX B: BASIC CONCEPTS IN ALGEBRAIC TOPOLOGY

**Definition B.1.** *A cell complex is a toplogical space built inductively as follows:*

- *Pick a discrete set of points $X^0$ (or 0-skeleton), whose elements called as 0-cells or vertices.*
- *Construct n-skeleton $X^n$ from $X^{n-1}$ by attaching n-cells $e^n_\alpha$ by the following map:*

  $$\phi_\alpha : \partial D^n = S^{n-1} \to X^{n-1}.$$

  *Hence,*

  $$X^n = X^{n-1} \coprod_\alpha D^n_\alpha / x \sim \phi_\alpha(x) \tag{B.1}$$

  *where $x \in \partial D^n_\alpha$ and $\coprod$ denotes the disjoint union and defined as $\coprod\limits_\alpha D_\alpha = \bigcup\limits_\alpha \{(x, i) : x \in D_\alpha\}$. One can easily observe that this map is not necessarily one-to-one.*
- *One can stop at some stage n to adding process or go on indefinitely then,*

  $$X = \bigcup_{n=0}^{\infty} X^n. \tag{B.2}$$

  *In this case, a subset of X is open if and only if $A \cap X^n$ is open for all n. This is called Weak topology on X. Such a complex is called as CW-complex where C comes from closure finiteness and W comes from weak topology.*

CW-structure is not unique. Let's observe this fact on examples.

**Example B.2.** *(i) CW-structure for a circle is* $S^1 = e^0 \cup e^1$*.*

*(ii) One may built two different CW-structure for* $S^2$*:*

$$S^2 = e^0_1 \cup e^0_2 \cup e^1_1 \cup e^1_2 \cup e^2_1 \cup e^2_2 = e^0_1 \cup e^2_1.$$

*For the first CW-structure of* $S^2$ *one can consider that* $e^1_1$*,* $e^1_2$*,* $e^0_1$ *and* $e^0_2$ *construct the equator of the* $S^2$ *and* $e^2_1$ *canstruct the upper hemisphere and* $e^2_2$ *construct the lower hemisphere.*

*The attaching map for the second CW-structure defined as*

$\phi_1 : \partial D^2 = S^1 \to X^1 = \{e^0_1\}$.

*(iii) Let's consider the CW-structure of torus,* $T^2$*. It is exactly* $e^0_1 \cup e^1_1 \cup e^1_2 \cup e^2_1$*.*

$\phi^1_1 : \partial D^1 = \partial\{e^1_1\} \to X^0 = \{e^0_1\}$

$\phi^1_2 : \partial D^1 = \partial\{e^1_2\} \to X^0 = \{e^0_1\}$

$\phi^2_1 : \partial D^2 = S^1 \to X^1$

*(iv) Schubert cells constructs a CW-structure for the Grassmannian manifold.*

$$\psi_\alpha : D^n_\alpha \to X^n = X^{n-1} \coprod_\alpha D^n_\alpha / x \sim \phi_\alpha(x)$$

$\psi_\alpha$ *is called as characteristic function of the n-cell. Moreover, if this characteristic function is an embedding then the cell complex is regular and regular simlexes are called as simplicial complexes.*

**Definition B.3.** $\triangle^n = \left\{ \sum_{i=1}^{n+1} t_i e_i = (t_0, \ldots, t_n) | \ t_i \geq 0, \ \sum_{i=1}^{n+1} t_i = 1 \right\}$ *where* $e_i$ *denotes the standard n-simplex for* $\mathbb{R}^{n+1}$*.*

**Definition B.4.** *Let X be a* $\triangle$*-complex.* $\triangle_r(X)$ *is a free abelian group with the open n-simplices* $e^r_\alpha$ *of X. Elements of* $\triangle_r(X)$ *are called as r-chains.*

**Example B.5.** *(i)* $\triangle^0 = \{t_1 e_1 | \ t_1 \geq 0, \ t_1 = 1\} = \{e_1\}$*.*

*(ii)* $\triangle^1$ *is a line segment whose end points are (1,0) and (0,1) in* $\mathbb{R}^2$*.*

*(iii)* *$\triangle_r(X)$ and n-simplices decomposition for torus are:*

$T^2 = e_1^0 \cup e_1^1 \cup e_2^1 \cup e_3^1 \cup e_1^2 \cup e_2^2$

$\triangle_0(T^2) = \mathbb{Z}\langle v_0\rangle = \{nv_0 | n \in \mathbb{Z}\}$

$\triangle_1(T^2) = \mathbb{Z}\langle e_1^1, e_2^1, e_3^1\rangle = \Big\{\sum_{i=1}^{3} n_i e_i^1 | n_i \in \mathbb{Z}\Big\}$

$\triangle_2(T^2) = \mathbb{Z}\langle e_1^2, e_2^2\rangle = \{n_1 e_1^2 + n_2 e_2^2 | n \in \mathbb{Z}\}.$

**Definition B.6.** *Boundary of simplex is defined as:*

$$\partial([v_0, v_1, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]. \tag{B.3}$$

*Moreover, naturally one can extend the boundary operator $\partial$ to $\triangle_n(X)$.*

**Definition B.7.** *Take the following chain complex of X:*

$$\dots \to \triangle_{n+1}(X) \to \triangle_n(X) \to \triangle_{n-1}(X) \to \dots \to \triangle_1(X) \to \triangle_0(X) \to 0$$

*where each $\to$ between $\triangle_{i+1}(X)$ and $\triangle_i(X)$ is given by $\partial_{i+1}$. The $i^{th}$ simplicial homology of X is defined as quotient group;*

$$H_i^{\triangle}(X) = ker\partial_i / Im\partial_{i+1}. \tag{B.4}$$

*Besides, elements of $ker\partial_i$ are called i-cycles on X and elements of $Im\partial_{n+1}$ are called i-boundaries on X.*

**Remark B.8.** *(i) As in the case of divided difference operator, $\partial$ used to define Schubert polynomial, $\partial^2$ equals to zero for the boundary operator on n-simplices.*

*(ii) $\partial_n \circ \partial_{n+1} = 0$ if and only if $Im\partial_{n+1} \subset ker\partial_n$.*

*(iii) Different cell structures for the same geometric object gives the same homology groups.*

*(iv) If geometric object X is path-connected space then $H_0(X) = \mathbb{Z}$. So, $H_0(\mathbb{G}_{m,n}) = \mathbb{Z}$.*

**Example B.9.** *r-chain for torus occurs as follows;*

$$0 \to \triangle_2(T^2) \to \triangle_1(T^2) \to \triangle_0(T^2) \to 0.$$

*Equivalently;*

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

*More precisely;* $\partial_2$ *maps* $e_1^2$ *to* $e_3^1 - e_1^1 - e_2^1$ *and* $e_2^2$ *to* $e_1^1 + e_2^1 - e_3^1$. $\partial_1$ *maps each* $e_1^1, e_2^1, e_3^1$ *to* $e_0 - e_0 = 0$. *Then;*
$H_0^{\triangle}(T^2) = \dfrac{ker\partial_0}{Im\partial_1} = \mathbb{Z}/\{0\} = \mathbb{Z}$
$H_1^{\triangle}(T^2) = \dfrac{ker\partial_1}{Im\partial_2} = \dfrac{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}{<(-1,-1,1),(1,1,-1)>} = \dfrac{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}{<(1,1,-1)>} = \mathbb{Z} \oplus \mathbb{Z}$
$H_2^{\triangle}(T^2) = \dfrac{ker\partial_2}{Im\partial_3} =< e_1^2 + e_2^2 > /\{0\} = \mathbb{Z}.$

**Definition B.10.** *A continuous map* $f : S^n \to S^n$ *gives rise to corresponding map* $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ *where* $\tilde{H}_n(S^n)$ *denotes the reduced homology of* $S^n$ *which is equivalent to* $\mathbb{Z}$. *Hence,* $f_*$ *is a map from* $\mathbb{Z}$ *to* $\mathbb{Z}$ *which is always multiplication by an integer. This integer is called as degree of* $f$ *and denoted with deg(f).*

For instance, degree of identity function on $S^n$ is 1 and degree of antipodal map on $S^n$ which sends x to -x is $(-1)^{n+1}$.

To obtain cohomology groups, we will dualize the chain of $\triangle_r(X)'s$ and $\partial$ by replacing them with $Hom(\triangle_r(X), \mathbb{Z})$ and $\delta$ operator, respectively. Let's denote the $\triangle_r(X)'s$ with $C_r$'s and $Hom(\triangle_r(X), \mathbb{Z})$ with $C^r$'s. Then $C^r$'s construct cochain complex as follows:

$$\ldots \leftarrow Hom(C_{r+1}, \mathbb{Z}) \leftarrow Hom(C_r, \mathbb{Z}) \leftarrow Hom(C_{r-1}, \mathbb{Z}) \leftarrow \ldots$$

where each $\leftarrow$ between $Hom(C_{r+1}, \mathbb{Z})$ and $Hom(C_r, \mathbb{Z})$ is given by $\delta^r$ in other words $\delta^r : C^r \to C^{r+1}$ and take $\psi \in Hom(C_r, \mathbb{Z})$, $\sigma \in C_{r+1}$. Then, $\delta^r(\psi)(\sigma) := \psi\partial_{r+1}(\delta)$. Besides this chain complex is denoted as $C_*$.

**Definition B.11.** *The* $n^{th}$ *cohomology group* $H^n(C_*, \mathbb{Z})$ *is defined by,*

$$H^n(C_*, \mathbb{Z}) := H_n(C^*) := \frac{ker(\delta^r : C^r \to C^{r+1})}{Im(\delta^{r-1} : C^{r-1} \to C^r)}. \tag{B.5}$$

One of the most important adventage of cohomology over homology is that it is possible to define direct sum of cohomology groups:

$$H^*(X) = \bigoplus H^r(X) \tag{B.6}$$

and obtain the graded ring structure. By defining another operation on direct sum of cohomology groups, we may extend the structure as ring. This operation is

$$\smile : H^k(X) \otimes H^l(X) \to H^{k+l}(X)$$

where $\otimes$ denotes the cross product operation. This $\smile$ operation will be cup product and formally defined as follows.

**Definition B.12.** *Let X be a topological space and fix a coefficient ring as* $\mathbb{Z}$. *Let* $\phi \in C^k(X, \mathbb{Z})$ *and* $\psi \in C^l(X, \mathbb{Z})$. *The cup product of* $\phi$ *and* $\psi$ *lies in* $C^{k+l}(X, \mathbb{Z})$ *and let* $\sigma$ *be k+l-simplex of X. Then cup product operation defined as,*

$$(\phi \smile \psi) = \phi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+l}]}) \tag{B.7}$$

*where the right-hand side is the product in* $\mathbb{Z}$.

An element of $H_n(X, \mathbb{Z})$ whose image in $H_n(X|x, \mathbb{Z})$ is a generator for all x is called a fundamental class for X with coefficients in $\mathbb{Z}$. This is the fundamental class definition for homology group and it can be extended for the cohomology ring case.

In this appendix section to make easier the concepts, $\mathbb{Z}$ is used to define $Hom(C_r, \mathbb{Z})$ and cohomology. However, one can use an arbitrary abelian group in place of $\mathbb{Z}$.