A COMPILATION OF DUALITIES BETWEEN 4D $\mathcal{N} = 2$ SUPERSYMMETRIC GAUGE THEORIES AND 2D CONFORMAL FIELD THEORIES

by

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I dedicate this thesis to my parents and professors.

ABSTRACT

A COMPILATION OF DUALITIES BETWEEN $4D$ $\mathcal{N}=2$ SUPERSYMMETRIC GAUGE THEORIES AND $2D$ CONFORMAL FIELD THEORIES

In this thesis AGT-W conjecture is reviewed. AGT conjecture is a phenomenon which is shown by Luis F. Alday, Davide Gaiotto and Yuji Tachikawa in their paper published in 2009. It relates the partition functions of both $\mathcal{N}=2$ supersymmetric $U(2)$, $SU(2)$ gauge theories and Liouville field theory which is a type of conformal field theory. One month later, Niclas Wyllard extended the conjecture and proved the connection between $\mathcal{N}=2$ supersymmetric $SU(N)$ gauge theories and A_{N-1} Toda field theories. In addition to AGT-W conjecture, gauge/Liouville Triality, spectral duality and $2D/4D$ duality through Nekrasov-Shatashvili limit are summarized. $\mathcal{N}=2$ supersymmetric gauge theories are also studied in type IIA, IIB and topological string theory background.

ÖZET

$4B$ $\mathcal{N} = 2$ SÜPERSİMETRİK AYAR KURAMLARI VE $2B$ AÇIKORUR ALAN KURAMLARI ARASINDAKİ İKİLİKLERİN BİR DERLEMESİ

Bu tezde AGT-W konjektürü incelenmektedir. AGT konjektürü, Luis F. Alday, Davide Gaiotto ve Yuji Tachikawa'nın 2009'da yayınlanan makalelerinde gösterilmiş bir olgudur. $\mathcal{N} = 2$ süpersimetrik $U(2)$ ve $SU(2)$ ayar kuramları ile bir çeşit açıkorur alan kuramı olan Liouville alan kuramını ilişkilendirmektedir. Bir ay sonrasında, Niclas Wyllard konjektürü genelleştirerek $\mathcal{N} = 2$ süpersimetrik $SU(N)$ ayar kuramlarının A_{N-1} Toda alan kuramları ile bağlantısını kanıtlamıştır. AGT-W konjektürüne ek olarak ayar/Liouville üçlüğü, spektral ikilik ve Nekrasov-Shatashvili limiti yardımı ile $2B/4B$ ikiliği özetlenmiştir. $\mathcal{N} = 2$ süpersimetrik alan kuramları ayrıca Tip IIA, IIB sicim ve topolojik sicim kuramları zemininde incelenmiştir.

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LIST OF SYMBOLS

$\mathcal{N}$	Number of Supersymmetries
$U(1)$	Unitary Group of Degree One
$U(2)$	Unitary Group of Degree Two
$U(N)$	Unitary Group of Degree N
$SU(2)$	Special Unitary Group of Degree Two
$SU(N)$	Special Unitary Group of Degree N
A_n	A-Type Quiver with n Nodes
D_n	D-Type Quiver with n Nodes
$\widehat{A}_1$	Affine A_1 Quiver
$\widehat{D}_4$	Affine D_4 Quiver
Tr	Trace of a Matrix

LIST OF ACRONYMS/ABBREVIATIONS

$2D$	Two Dimensional
$3D$	Three Dimensional
$4D$	Four Dimensional
$5D$	Five Dimensional
$6D$	Six Dimensional
QED	Quantum Electrodynamics
QCD	Quantum Chromoodynamics
SW Theory	Seiberg-Witten Theory
CFT	Conformal Field Theory
LFT	Liouville Field Theory
TFT	Toda Field Theory
DF Integral	Dotsenko-Fateev Integral

1. INTRODUCTION

QCD was introduced as an extension of QED in order to express electroweak and strong interactions mathematically. Although it is quite useful in many practical applications, our knowledge of it is limited. We are not able to some solve problems such as quark confinement analytically. Methods like lattice QCD can be used to understand the behavior of quarks and gluons, however supersymmetry gives birth to more general form of solutions. Supersymmetric versions of QCD are its "sibling" theories. In $4D$ spacetime, supersymmetric theories are the only way to reach such extent of generality.

"Number of supersymmetries" which will be explained in the following sections is denoted by $\mathcal{N}$. The number of generators in a theory with $\mathcal{N}$ supersymmetries have $4\mathcal{N}$ supercharges, which which is a concept that will also become clear later. $\mathcal{N} = 1$ supersymmetic theories with 4 supercharges are not sufficient to introduce analyticality. The more supersymmetries a theory contains, more strictly it is constrained. There comes a point that the theory only consists of gauge fields and matter fields become redundant because of insufficient number of degrees of freedom. Since supergravity is a whole separate subject, we do not want spin-2 particles. Therefore $\mathcal{N} = 4$ supersymmetric theories with 16 supercharges are the theories with largest number of supersymmetries. $4D$ $\mathcal{N} = 2$ supersymmetric theories are exceptional not only because they are both analytically solvable and have enough degrees of freedom, but also have applications in many fields. Under certain conditions, they are related are related to classical and quantized integrable systems, geometric invariants like Gromov-Witten invariants and Gopakumar-Vafa invariants, various types of conformal field theories and string theory.

It is pretty challenging to study $\mathcal{N} = 2$ supersymmetric gauge theories alone. Fortunately, string theory brings powerful mathematical tools that can be used to understand Seiberg-Witten theory. Nathan Seiberg and Edward Witten are the ones who predicted the instanton countribution to the prepotential of the low energy effective action which restrict the theory.

The solution proposed by Seiberg and Witten is self-consistent, in other words, it is not derived using any other theory. A detailed computation is possible through construction of the moduli space of instantons, called ADHM space. Then, contributions due to separate instanton configurations are summed up. This method was orchestrated by Andrey S. Losev, Gregory Moore, Nikita Nekrasov and Samson L. Shatashvili.

2. THEORY

2.1. A Short Review

At the beginning of the thesis, there will be a review of Supersymmetric Yang-Mills theories, Liouville and Toda Field theories and AGT-W conjecture.

2.1.1. Seiberg-Witten Theory

We will reach the well-established Seiberg-Witten theory step-by-step from the mere beginning.

2.1.1.1. Quantum Electrodynamics. One starts with the simplest case scenario; quantum electrodynamics which is the non-supersymmetric and abelian Yang-Mills theory. QED Lagrangian density with spin-0, spin-1 and spin-1/2 fields reads

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\psi}\gamma^{\mu}D_{\mu}\psi - m_{\psi}\overline{\psi}\psi + \frac{1}{2}D^{\mu}\phi D_{\mu}\phi - \frac{1}{2}m_{\phi}^{2}\phi^{2}. \tag{2.1}$$

This Lagrangian density provides one of the most precise theories in the history of physics which is both classically solvable and quantizable.

2.1.1.2. Yang-Mills Theory. If the spin-1 field exhibits $SU(N)$ gauge symmetry, the relevant theory is Yang-Mills Theory. The corresponding Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}Tr(F^{\mu\nu}F_{\mu\nu}) + i\overline{\psi}\gamma^{\mu}D_{\mu}\psi - m_{\psi}\overline{\psi}\psi + \frac{1}{2}(D^{\mu}\phi)^{\dagger}D_{\mu}\phi - \frac{1}{2}m_{\phi}^{2}|\phi|^{2}. \tag{2.2}$$

Yang-Mills theory does not only have non-linear equations of motion, but it also contains anomalies. Although we are able to calculate quark-quark scattering amplitudes and cross sections in QCD, which is a Yang-Mills theory, analytically; we fail to find the general solution of confinement problem. Lattice QCD is one resolution of the issue. Another one is introducing supersymmetry(SUSY) which gives rise to a new family of quantum field theories.

2.1.1.3. SUSY. The concept of SUSY is almost as old as quantum mechanics. Once, Erwin Schödinger solved the radial part of the hydrogen atom problem in a similar fashion with quantum harmonic oscillator. He introduced creation and annihilation operators of the Hamiltonian. The energy eigenstates occured in pairs with the same energies. However, no one regarded these states as “fermionic” and “bosonic”’ back then. No one named the phenomenon “SUSY”. It was later understood that any basic quantum mechanical system is also solvable via the same method.

In 1967, Sidney Coleman and Jeffrey Mandula proposed a larger symmetry of particle physics containing the Poincaré symmetry in their well-known paper. The total symmetry group is the direct product of Poincaré group and an internal symmetry group such that $G = G_{\text{Poincaré}} \times G_{\text{Internal}}$. This internal symmetry group corresponds to the group of transformations which preserve the equations of motion in superstring theory. Unfortunately, Coleman-Mandula theorem is a no-go theorem, that means, it is not possible to physically observe the proposal. On the other hand; it is reasonable to try to build SUSY version of the Standard Model, because there exists “fermionic” degrees of freedom on the worldsheet of string theory and we want our theory about elementary particles to be compatible with string theory. Supersymmetry is also particularly useful because it resolves Higgs mass hierarchy problem, unifies the coupling constants in grand unified theories and introduces the superpartners which can be thought as possible candidates for Dark Matter.

Although the above mentioned approaches in quantum mechanics and quantum field theory seem to be distinct, they are strictly related. In supersymmetric quantum mechanics, there are twin energy eigenstates with the same energy for each state. The generator of transformation between the bosonic and fermionic degrees of freedom is called *supercharge*. Hamiltonian can be then expressed in terms the supercharges. In quantum field theory, number of fields is doubled after adding SUSY. For any bosonic field there is also a fermonic field with the same mass and vice versa. Since Lagrangian is used rather than Hamiltonian in quantum field theory, it is more convenient to build a Lagrangian formulation of the theory which is a challenge. Let us define the fermion number operator F and the operator $(-1)^F$. $\phi(x)$ and $\psi(x)$ denote the bosonic and the fermionic field, respectively. The action of the operator $(-1)^F$ on the fields $\phi(x)$ and $\psi(x)$ is

$$\begin{aligned}(-1)^F\phi &= \phi,\\ (-1)^F\psi &= -\psi.\end{aligned} \tag{2.3}$$

The index $I = 1, 2, ..., \mathcal{N}$ is called *number of supersymmetries*, that means there are $\mathcal{N}$ spin for each field [1]. Then it is easy to see that it anticommutes with the supercharge Q^I of the theory for both quantum mechanics and quantum field theory with any number of supersymmetries:

$$\{Q^I, (-1)^F\} = 0. \tag{2.4}$$

SUSY in Quantum Mechanics. Let us stick with supersymmetric quantum mechanics to introduce the algebra of SUSY. In supersymmetric quantum mechanics with $\mathcal{N}$=2, in other words with 2 states having the same energy, $A^\dagger$ and and A are creation and annihilation operators, respecively. Their relation to the *superpotential* $W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'}{\psi_0}$ is

$$\begin{aligned}A &= \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x),\\ A^\dagger &= -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x).\end{aligned} \tag{2.5}$$

Let us also define twin Hamiltonians H_1 and H_2:

$$\begin{aligned} H_1 &= A^\dagger A, \\ H_2 &= AA^\dagger. \end{aligned} \tag{2.6}$$

Then the total Hamiltonian has the form

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}. \tag{2.7}$$

Supercharges are defined as follows:

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad \overline{Q} = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \tag{2.8}$$

The Hamiltonian in terms of supercharges is

$$H = \{Q, \overline{Q}\}. \tag{2.9}$$

SUSY in Quantum Field Theory. We start with the simplest case: $4D$ $\mathcal{N} = 1$ case with 4 generators. $\mathcal{N}$ means only transformations between spin-0, spin-1/2 and spin-1 fields are allowed. For a complex scalar field ϕ with its conjugate scalar field $\phi^*(x)$ and Dirac spinor ψ with its adjoint spinor $\overline{\psi}(x)$ equation of motions are

$$\begin{aligned} \partial^\mu \partial_\mu \phi(x) - m^2 \phi(x) &= 0, \\ i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) &= 0, \\ \partial^\mu \partial_\mu \phi^*(x) - m^2 \phi^*(x) &= 0, \\ i\partial^\mu \overline{\psi}(x)\gamma_\mu - m\overline{\psi}(x) &= 0. \end{aligned} \tag{2.10}$$

We can introduce transformations $\phi(x) \mapsto \phi(x)+\delta\phi(x)$ and $\psi(x) \mapsto \psi(x)+\delta\psi(x)$ that leave the equations of motion invariant, where θ is a Grassmann spinor which acts like a Dirac spinor, also $\theta^\dagger$ and $\overline{\theta}$ are conjugate transpose and adjoint spinors respectively:

$$\begin{aligned} \delta\phi(x) &= \theta\psi(x), \\ \delta\psi(x) &= -i\sigma^\mu\theta^\dagger\partial_\mu\phi(x). \end{aligned} \tag{2.11}$$

The spinor indices α and $\dot{\alpha}$ of ψ_α and $\psi^\dagger_{\dot{\alpha}}$ are suppressed for the sake of simplicity. Those transformations indicate their dual transformations:

$$\begin{aligned} \delta\phi^*(x) &= \overline{\theta}\,\overline{\psi}(x), \\ \delta\psi^\dagger(x) &= i\theta\sigma^\mu\partial_\mu\phi^*(x). \end{aligned} \tag{2.12}$$

Using Noether's theorem, we then introduce the SUSY currents J^μ_α, $\overline{J}^\mu_{\dot{\alpha}}$ and generators Q_α and $\overline{Q}_{\dot{\alpha}}$ along with momentum operator $P^\mu = i\partial_\mu$:

$$\begin{aligned} J^\mu_\alpha &= \sqrt{2}(\sigma^\nu\overline{\sigma}^\mu\psi)_\alpha\partial_{nu}\phi^*, \\ \overline{J}^\mu_{\dot{\alpha}} &= \sqrt{2}(\overline{\psi}\overline{\sigma}^\mu\sigma_\nu)_{\dot{\alpha}},\partial_{nu}\phi. \end{aligned} \tag{2.13}$$

However, there is a slight problem. SUSY transformations leave the equations of motions invariant on-shell, but for a closed algebra, the equations must be invariant off-shell as well. In order to achieve that goal, we introduce auxiliary fields $\mathcal{F}(x)$ and $\mathcal{F}^*(x)$. The fields $\mathcal{F}(x)$ and $\mathcal{F}^*(x)$ have no dynamics, and therefore no physical meaning. The extended set of transformations is

$$\begin{aligned}
\delta\phi(x) &= \theta\psi(x), \\
\delta\psi(x) &= -i\sigma^{\mu}\theta^{\dagger}\partial_{\mu}\phi(x) + \theta\mathcal{F}(x), \\
\delta\mathcal{F}(x) &= -i\theta^{\dagger}\overline{\gamma}^{\mu}\partial_{\mu}\psi(x), \\
\delta\phi^{*}(x) &= \overline{\theta}\,\overline{\psi}(x), \\
\delta\overline{\psi}(x) &= i\gamma^{0}\theta\sigma^{\mu}\partial_{\mu}\phi^{*}(x) + \overline{\theta}\mathcal{F}^{*}(x), \\
\delta\mathcal{F}^{*}(x) &= i\partial_{\mu}\psi^{\dagger}(x)\overline{\gamma}^{\mu}\theta.
\end{aligned} \tag{2.14}$$

We now introduce $M_{\mu\nu}$, generator of Lorent transformations expressed as boost K_i's and rotation J_k's:

$$\begin{aligned}
M_{\mu\nu} &= -M_{\nu\mu}, \\
M_{0i} &= K_i, \\
M_{ij} &= \epsilon_{ijk}J_k
\end{aligned} \tag{2.15}$$

and Z^{IJ} terms are central charges which satisfying the anti-symmetry condition $Z^{IJ} = -Z^{JI}$. Finally, SUSY algebra which points out a larger symmetry than Poincaré algebra is [2]

$$\begin{aligned}
[P_{\mu}, Q^{I}_{\alpha}] &= 0, & [M_{\mu\nu}, \overline{Q}^{I\dot{\alpha}}] &= i(\overline{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\overline{Q}^{I\dot{\beta}}, \\
[P_{\mu}, \overline{Q}^{I}_{\alpha}] &= 0, & \{Q^{I}_{\alpha}, \overline{Q}^{J}_{\dot{\beta}}\} &= 2\sigma^{\mu}_{\alpha\dot{\beta}}P_{\mu}\delta^{IJ}, \\
[P_{\mu}, \overline{Q}^{I}_{\alpha}] &= 0, & \{Q^{I}_{\alpha}, Q^{J}_{\beta}\} &= \epsilon_{\alpha\beta}Z^{IJ}, \\
[M_{\mu\nu}, Q^{I}_{\alpha}] &= i(\sigma_{\mu\nu})_{\alpha}{}^{\beta}Q^{I}_{\beta}, & \{\overline{Q}^{I}_{\dot{\alpha}}, \overline{Q}^{J}_{\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}}(Z^{IJ})^{*}.
\end{aligned} \tag{2.16}$$

SUSY algebra has three essential consequences. First, it implies that the number of fermionic and bosonic states in each multiplet is equal. Second, the vacuum state of the multiplet, which is the name given to the mathematical expression of fermionic and bosonic fields as a single field, has positive energy eigenvalue and that value can always be adjusted to 0. And lastly, if $\mathcal{N}$=2 theory is massive the mass m must satisfy the limit $m \geq \sqrt{2}|Z|$ where Z is the central charge. This is called *BPS limit.* Note that only one central charge remain because of the antisymmetry condition in $\mathcal{N}$=2 SUSY [3].

Superspace. Supersymmetry is particularly interesting because not only the fields, but also the coordinates have “fermionic” counterparts that do not have meaning in classical mechanics. They are nothing but θ’s, the Grassmann spinors. Grassmann spinors look like Dirac spinors, they consist of components θ_α and $\overline{\theta}_{\dot{\alpha}}$ where $\alpha, \dot{\alpha} = 1, 2$. Because of Grassmannian nature of spinors, the anticommutation relations hold: $\theta_\alpha \theta_\beta = -\theta_\beta \theta_\alpha$, $\overline{\theta}_{\dot{\alpha}} \overline{\theta}_{\dot{\beta}} = -\overline{\theta}_{\dot{\beta}} \overline{\theta}_{\dot{\alpha}}$. For any supersymmetric quantum field theory, number of Grassmann spinor θ’s is equal to $\mathcal{N}$. We introduce the scalar functions $f(x)$, $g(x)$, $h(x)$ and auxillary spinors μ, ν, ξ. In the case of $4D$, the most general form of a function $F(x, \theta, \overline{\theta})$ which depends on the spacetime coordinate x and superspace coordinates θ, $\overline{\theta}$ is of the form

$$\begin{gathered} F(x,\theta,\overline{\theta}) = f(x) + \theta\mu(x) + \overline{\theta}\overline{\nu} + \theta\theta g(x) + \overline{\theta\theta}h(x) + \theta\sigma^{\mu}\overline{\theta}v_{\mu}(x) + \theta\theta\overline{\theta}\overline{\xi}(x) + \\ \theta\overline{\theta\theta}\overline{\eta}(x) + \theta\theta\overline{\theta\theta}l(x). \end{gathered} \tag{2.17}$$

The notation $\mu\nu$ is shorthand for $\mu^\alpha \nu_\alpha = \epsilon_{\alpha\beta}\mu^\alpha \nu^\beta$, where $\epsilon_{\alpha\beta}$ is the antisymmetric SUSY metric

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.18}$$

We define the SUSY covariant derivative operators which commute with the supercharges

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\overline{\theta}^{\alpha}\frac{\partial}{\partial x^\mu} \quad \overline{\mathcal{D}}_{\dot{\alpha}} = \frac{\partial}{\partial\theta^{\dot{\alpha}}} + i\sigma^{\mu}_{\alpha\alpha}\overline{\theta}^{\alpha}\frac{\partial}{\partial x^\mu}. \tag{2.19}$$

Using the notation $\theta^2 = \theta_1\theta_2$ and $\overline{\theta}^2 = \overline{\theta}_1\overline{\theta}_2$, we also normalize our Berezin integral such that

$$\int d^2\theta\,\theta^2 = \int d^2\overline{\theta}\,\overline{\theta}^2 = 1. \tag{2.20}$$

Chiral Superfield. We need the concept of *chiral superfield* in order to include the information about spin-0 and spin-1/2 fields in a single field. It is also a field the field that solves the differential equation $\overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$. There is also a dual field which is called *antichiral superfield* solves the partner equation $\mathcal{D}_\alpha\overline{\Phi} = 0$. In order to most general solution of the differential equation be obtained, components of the chiral superfield must satisfy certain constraints. Using the notations $y^\mu = x^\mu + i\theta\sigma\overline{\theta}$, $\overline{y}^\mu = x^\mu - i\theta\sigma\overline{\theta}$ and the auxillary fields $\mathcal{F}(y)$, $\mathcal{F}^*(y)$ we introduced earlier; we obtain the simplified form of both $\Phi(y,\theta)$ and $\overline{\Phi}(\overline{y},\overline{\theta})$:

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\,\theta\,\psi(y) + \theta\,\theta\mathcal{F}(y). \tag{2.21}$$

$$\overline{\Phi}(\overline{y},\overline{\theta}) = \phi(\overline{y}) + \sqrt{2}\,\overline{\theta}\,\overline{\psi}(\overline{y}) + \overline{\theta}\,\overline{\theta}\mathcal{F}^*(\overline{y}). \tag{2.22}$$

Wess-Zumino Model. Wess-Zumino model is the simplest type of supersymmetric interacting quantum field theory. Its Lagrangian is given in terms of the fields $\phi_i(x)$, $\psi_i(x)$ and $f_i(x)$:

$$\mathcal{L} = \partial^\mu\phi^i\partial_\mu\phi_i + i\overline{\psi}^i\gamma^\mu\partial_{mu}\psi_i - f^{i*}f_i - \frac{1}{2}M^{ij}\psi^i\psi^j - \frac{1}{2}y^{ijk}\psi_i\psi_j\phi_k + h.c., \tag{2.23}$$

where M_{ij} and y_{ijk} are tensors.

Vector Superfield. *Vector Superfield* is type of superfield which is of the form

$$V(x,\theta,\overline{\theta}) = C + i\theta\lambda - i\overline{\theta}\lambda + i\theta\sigma^{\mu}\overline{\theta}v^{\mu}, \tag{2.24}$$

where $\lambda(x)$ is just another spinor like ψ. For a $U(1)$ gauge field which is included in the superfield, the field strength tensor W_{α} is

$$W_{\alpha} = -\frac{1}{4}\overline{\mathcal{D}}^{2}\mathcal{D}_{\alpha}V. \tag{2.25}$$

For a general $SU(N)$ theory it becomes

$$W_{\alpha} = \frac{1}{8}\overline{\mathcal{D}}^{2}e^{2V}\mathcal{D}_{\alpha}e^{-2V}. \tag{2.26}$$

4D $\mathcal{N}$=1 Supersymmetric Yang-Mills Theory. Lagrangian density of the $\mathcal{N}$=1 supersymmetric Yang-Mills theory reads

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{matter} + \big(\mathcal{L}_{FI}\big). \tag{2.27}$$

The last term on the right-hand side is called *Fayet-Iliopoulos term* which and appears if there exists a $U(1)$ subgroup in the theory. Before we delve into the parts of the Lagrangian, we need to define the complexified coupling constant τ which is equal to $\frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$ and SUSY equivalent of the field strength W^{α}.

Three separate parts mentioned are as follows:

$$\mathcal{L}_{FI} = \frac{1}{2}\xi \int d^2\theta d^2\overline{\theta}\, V. \tag{2.28}$$

$$\mathcal{L}_{gauge} = \frac{1}{32\pi}\text{Im}\Big(\tau \int d^2\theta \, Tr(W^\alpha W_\alpha)\Big), \tag{2.29}$$

$$\mathcal{L}_{matter} = \int d^2 d^2\overline{\theta} \, \Phi^\dagger e^V \Phi + \int d^2\theta \, W + \int d^2\overline{\theta} \, W^\dagger, \tag{2.30}$$

Φ is chiral superfield containing information of spin-0 and spin-1/2 fields in non-supersymmetric quantum field theory and conventionally named as "matter field". V is vector vector superfield, contains information of spin-1/2 and spin-1 fields and named as "gauge field". Finally, W is called superpotential like the one in supersymmetric quantum mechanics [2]. $\mathcal{N}$=1 SUSY is a rather familiar subject because form of the action is well-undertood. However, formulation of the $\mathcal{N}$=2 theory requires a little more mathematical background.

4D $\mathcal{N}$=2 Supersymmetric Yang-Mills Theory. $\mathcal{N} = 2$ supersymmetric gauge theories which are denoted by the umbrella term *Seiberg-Witten theory* are popular among supersymmetric gauge theories and the thesis focuses particularly on them. There are two subcases os Seiberg-Witten theory. First case is called *Higgs branch* where the scalar fields contributing to the vector multiplet and their conjugate transposes are set to be 0. In this thesis, Higgs branch will not be reviewed. The second case is called *Coulomb branch*, where the scalar fields contributing to the hypermultiplet are 0. Then the scalar fields ϕ_i belonging to the vector multiplet satisfy the condition $\text{Tr}[\phi_i, \phi_i^\dagger]^2 = 0$ and they have vacuum expectation values a_i. Our condition implies scalar fields are spanned by Cartan root vectors. The name Coulomb comes from the fact that the gauge group breaks into the product of $U(1)$ subgroups which result in theories that look like QED [3, 4].

Coulomb branch of $\mathcal{N} = 2$ supersymmetric $SU(2)$ theory in $4D$ preserves its symmetry both in perturbative and non-perturbative regimes. The symmetry in perturbative regime appears as S-duality transformations, that means the theory is invariant under the transformation $\tau \mapsto 1/\tau$. In non-perturbative regime, it then appears as monodromies. Both transformations are expressed as $SL(2,\mathbb{Z})$ action. $SL(2,\mathbb{Z})$ action can be modelled with an integer lattice on a torus. In fact, couplings of any $\mathcal{N}=2$ supersymmetric $SU(N)$ gauge theory are constrained on a smooth manifold $\mathcal{C}$ called *Seiberg-Witten curve.* Genus of the Seiberg-Witten curve is $N-1$. It is possible to obtain vacuum expectation values a_i and their dual values a_i^D through a set of contour integrals of Seiberg-Witten differential $\mathrm{d}S$ on Seiberg-Witten curve $\mathcal{C}$ along closed loops A and B [3,5]:

$$\begin{aligned} a_i &= \oint_{A_i} \mathrm{d}S, \\ a_i^D &= \oint_{B_i} \mathrm{d}S. \end{aligned} \tag{2.31}$$

Also, the coupling constants T_{ij} satisfy the relations [6]

$$\begin{aligned} a_i^D &= \frac{\partial \mathcal{F}}{\partial a_i}, \\ T_{ij} = \frac{a_j^D}{a_i} &= \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}. \end{aligned} \tag{2.32}$$

Low-energy effective Lagrangian of density Seiberg-Witten theory is simply

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{matter} \tag{2.33}$$

with the gauge term

$$\mathcal{L}_{gauge} = \frac{1}{4\pi}\mathrm{Im}\Big(\int d^4\theta\, \Phi^\dagger \frac{\partial \mathcal{F}}{\partial \Phi} + \frac{1}{2}\int d^2\theta\, \frac{\partial^2 \mathcal{F}}{\partial \Phi^2} W^\alpha W_\alpha\Big). \tag{2.34}$$

$\mathcal{F}(\Lambda, \vec{a})$ is called *prepotential* which also shows holomorphic property, that means it depends on Φ but not on $\Phi^\dagger$. Λ is the gauge coupling parameter which is equal to $e^{2\pi i \tau}$. Prepotential $\mathcal{F}(\Lambda)$ consists of a perturbative part $\mathcal{F}_{Perturbative}(\Lambda)$, which obviously contains perturbative contributions to the solution and also an instanton part $\mathcal{F}_{Instanton}(\Lambda)$, which contains non-perturbative contributons and is a result of Nekrasov's rigorous instanton calculation. Perturbative part can also seperated into the classical part $\mathcal{F}_{Classical}(\Lambda)$ and the 1-loop part $\mathcal{F}_{1-loop}(\Lambda)$, which carry the information about the perturbative quantum corrections. Those quantum corrections disappear in the diagrams with more than 1-loops, that is where the name comes from:

$$\begin{gathered}\mathcal{F}(\Lambda) = \mathcal{F}_{Perturbative}(\Lambda) + \mathcal{F}_{Instanton}(\Lambda) = \\ \mathcal{F}_{Classical}(\Lambda) + \mathcal{F}_{1-Loop}(\Lambda) + \mathcal{F}_{Instanton}(\Lambda).\end{gathered} \tag{2.35}$$

Let us return to the theory with one vector multiplet. Two out of three parts of the prepotential look like as follows:

$$\begin{aligned}\mathcal{F}_{1-Loop} &= \frac{i}{\pi}(\ln \Phi - \ln \Lambda), \\ \mathcal{F}_{Instanton} &= \sum_{0}^{\infty} \mathcal{F}_k \Phi^{4k-2}\, \Phi^{4k}.\end{aligned} \tag{2.36}$$

And $\vec{a}$ terms are the vacuum expectation value vectors spanned in the root space of corresponding gauge group, as mentioned before. For $U(2)$ vector multiplet, $\vec{a} = (a_1, a_2)$ and for $SU(2)$, $\vec{a} = (1, -1)$. This is the case for a single gauge group, of course. For each vector multiplet, we add another $\mathcal{F}(\Lambda_i, \vec{a}_i)$ term to the prepotential. If we determine the true form of $\mathcal{F}(\Lambda, \vec{a})$, we can understand the physics behind 4D $\mathcal{N}$=2 supersymmetric Yang-Mills theory. In order to calculate the prepotential, we must first calculate *Nekrasov partition function* first. Through vigorous instanton counting method found by Nikita Nekrasov, it is possible to express both 1-loop and instanton contributions as functions of root vectors [3,7]. Then, Nekrasov partition function gives us the prepotential. The relation between two will be explored in this thesis.

In the presence of multiple matter fields which is a condition for scale invariance the extra contribution is

$$\mathcal{L}_{matter} = \sum_{i=1}^{N_f} \int d^4\theta \, Q_i^\dagger e^{-2V} Q_i + \int d^2\theta \left(\sqrt{2}\tilde{Q}_i \Phi Q_i + m_i \tilde{Q}_i Q_i\right) + h.c., \tag{2.37}$$

which looks like the Yukawa interaction terms in quantum field theory. Note that Q_i terms are the matter multiplets and N_f is the number of flavors or in other words, the number of matter fields.

If a hypermultiplet is coupled to one single vector multiplet along with another hypermultiplet, it is mathematically expressed in the fundamental representation. If one single hypermultiplet is coupled to two separate vector multiplets, it is in the bifundamental representation. If one hypermultiplet is coupled to a vector multiplet two times, it is in the adjoint representation [3].

Since we want to underline the relation between supersymmetric Yang-Mills theories and a certain family of conformal field theories, we want our gauge theory to be also scale invariant. Hence, only certain types of quiver diagrams are allowed. For instance; if we have one single $SU(N_C)$ vector multiplet, the number of allowed flavors N_F is equal to $2N_C$. For a theory with a $SU(2)$ vector multiplet, 4 $SU(2)$ hypermultiplets are allowed.

In supersymmetric Yang-Mills theories running coupling constant is determined by the β-function with the energy scale Λ like any type of quantum field theory:

$$\beta(g) = \frac{\mathrm{d}g(\Lambda)}{\mathrm{d}\ln(\Lambda)}. \tag{2.38}$$

In general, the form of the β-function is

$$\beta(g) = -\frac{g^2}{2\pi}\left[3T_G - \sum_i T(R_i(1-\gamma_i))\right]\left(1 - \frac{T_G g}{2\pi}\right)^{-1}, \tag{2.39}$$

where $T(R_i)$ is the Dynkin index of the representation R_i except the adjoint ones, T_G is the Dynkin index of the adjoint representation of G and γ_i is the anomalous dimension [5, 8–10]. If we focus our attention to $\mathcal{N} = 2$ theories,

$$\beta(g) = -\frac{g^2}{16\pi^2}\big[3T_G - \sum_i T(R_i)\big]. \tag{2.40}$$

And finally, for $G = SU(2)$, which is relevant to the original form of the AGT conjecture, the β-function takes the form

$$\beta(g) = -\frac{g^2}{16\pi^2}\big[4 - 2T(R_i)\big]. \tag{2.41}$$

We also add that $T(fund) = 1/2$, $T(bifund) = 1$, $T(ad) = 2$. That means if there exists only one $SU(2)$ vector multiplet, then it can only couple to two fundamental and two antifundamental hypermultiplets without violating the scale invariance as seen in Figure 2.1 [3]. It is possible to extend the diagram provided that previously explained conditions are met. As the next example we join n $SU(2)$ vector multiplets linearly as in Figure 2.2. In fact the quivers do not need to be linear. One can easily build exotic quivers like Sicilian quiver, necklace quiver or trifundamental quiver. It is also convenient to classify quivers in an analogy with semisimple Lie algberas. Each vector multiplet corresponds to a node in the Dynkin diagram. The case does not necessarily apply only to simply laced diagrams. More recently, it became possible to calculate the partition functions for B and C type diagrams using the root parameters. Quivers

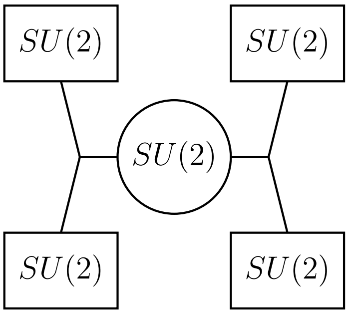


Figure 2.1. Quiver diagram of simplest $SU(2)$ theory.

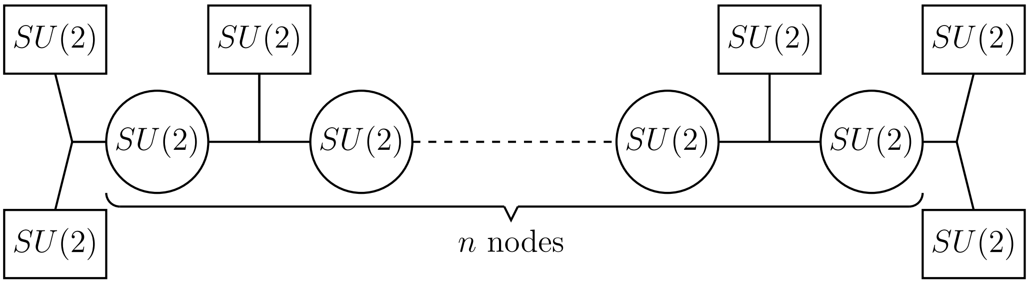


Figure 2.2. Quiver diagram of the theory with n $SU(2)$ vector multiplets.

are discussed in the next section.

2.1.1.4. Nekrasov Partition Function. The next step is to determine the form of prepotential $\mathcal{F}$. The challenge is that Nekrasov partition function is divergent in both IR and UV regimes. We fix the IR divergence by Ω-*deformation* and we fix the UV divergence by *Uhlenbeck compactification.*

Uhlenbeck compactification means the expression of moduli space of the theory with gauge group $\mathcal{G}$ $\mathcal{M}^G$ as a series expansion over instanton number k. Expectation value of identity operator then becomes

$$\langle 1 \rangle = \sum_{k=0}^{\infty} \Lambda^k \oint_{\mathcal{M}_k^G} 1. \tag{2.42}$$

Ω-deformation means deformation of spacetime by parameters ϵ_1 and ϵ_2. In the 0 limit of those parameters the infinite integral $\int \mathrm{d}^4x = \mathrm{d}\tau \mathrm{d}x \mathrm{d}y \mathrm{d}z$ becomes finite.

$$\begin{aligned} (\tau, x) &\mapsto (e^{-\epsilon_1 \tau^2}, e^{-\epsilon_1 x^2}), \\ (y, z) &\mapsto (e^{-\epsilon_2 y^2}, e^{-\epsilon_2 z^2}), \\ \int d^4x &\mapsto \int d^4x e^{-\tau^2} e^{-x^2} e^{-y^2} e^{-z^2}, \end{aligned} \tag{2.43}$$

where τ is Wick rotated time $\tau = it$.

Through ADHM construction and infinite dimensional path integral not on spacetime $\mathbb{R}^{3,1}$ but on moduli space $\mathcal{M}$ of coupling constants with a twist which results in a twisted SUSY algebra and Ω-deformation, we obtain the Nekrasov partition function Z. Then we obtain the prepotential, if we take the logarithm of partition function which is given for a quiver with one vector multiplet,

$$\mathcal{F}(\Lambda, \vec{a})) = \ln \left[\lim_{\epsilon_1, \epsilon_2 \to 0} \epsilon_1 \epsilon_2 Z(\Lambda, \vec{a}; \epsilon_1, \epsilon_2) \right], \tag{2.44}$$

where the ϵ terms are Ω-deformation parameters.

Nekrasov partition function is actually product of three subfunctions:

$$Z = Z_{Classical} \times Z_{1-Loop} \times Z_{Inst}, \tag{2.45}$$

where the classical part $Z_{Classical}(\vec{a}_1, ..., \vec{a}_n; \epsilon_1, \epsilon_2)$ provides us with the classical equations of motion, obviously. In 4D, the 1-loop part $Z_{1-Loop}(\vec{a}, m; \epsilon_1, \epsilon_2)$ contains perturbative quantum corrections, whereas the instanton part $Z_{Inst}(\vec{a}, m, \vec{\mu}; \epsilon_1, \epsilon_2)$ gives us non-perturbative quantum corrections. For a theory with n vector multiplets, the classical part which is the easiest is

$$Z_{Classical}(\vec{a}_1, ..., \vec{a}_n; \epsilon_1, \epsilon_2) = e^{-\frac{2\pi i \sum_{i=1}^{n} \tau_i a_i^2}{\epsilon_1 \epsilon_2}}. \tag{2.46}$$

where τ_i is the coupling constant of ith vector multiplet. The 1-loop part consists of

$$\begin{aligned}
Z^{Fund}_{1-Loop}(\vec{a}, m; \epsilon_1, \epsilon_2) &= \prod_i \Gamma_2(a_i - m + \epsilon_1 + \epsilon_2), \\
Z^{Antifund}_{1-Loop}(\vec{a}, m; \epsilon_1, \epsilon_2) &= \prod_i \Gamma_2(m - a_i), \\
Z^{Bifund}_{1-Loop}(\vec{a}, \vec{b}, m; \epsilon_1, \epsilon_2) &= \prod_{i,j} \Gamma_2(a_i - b_j - m + \epsilon_1 + \epsilon_2), \\
Z^{Vector}_{1-Loop}(\vec{a}, \epsilon_1; \epsilon_2) &= -\prod_{i<j}(\Gamma_2(a_i - a_j + \epsilon_1)\Gamma_2(a_i - a_j + \epsilon_2)).
\end{aligned} \tag{2.47}$$

where

$$ln(\Gamma_2(x|t_1, t_2) = \frac{\partial}{\partial y}\zeta_2(x, y|t_1, t_2)\bigg|_{y=0}, \tag{2.48}$$

and

$$\zeta_2(x, y|t_1, t_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(x + mt_1 + nt_2)^y}. \tag{2.49}$$

$\vec{a}$, $\vec{b}$ are vacuum expectation value vectors of the vector multiplets and m is the mass of hypermultiplet. Also the products are over the components of $\vec{a}$ and $\vec{b}$. In order to write the instanton part, we follow a similar approach except the sum over Young diagrams:

$$
\begin{gathered}
Z_{Inst}^{Fund}(\vec{a}, m, \vec{\mu}; \epsilon_1, \epsilon_2) = \prod_k \prod_{i,j \in \mu_k} (a_k - m + \epsilon_1 i + \epsilon_2 j), \\
Z_{Inst}^{Antifund}(\vec{a}, m, \vec{\mu}; \epsilon_1, \epsilon_2) = Z_{Inst}^{Fund}(\vec{a}, \epsilon_1 + \epsilon_2 - m, \vec{\mu}; \epsilon_1, \epsilon_2), \\
Z_{Inst}^{Bifund}(\vec{a}, \vec{b}, m, \vec{\mu}, \vec{\nu}; \epsilon_1, \epsilon_2) = \\
\prod_{k,l} \Big[\prod_{i,j \in \mu_k} (a_k - b_l - m - (\nu_{l,j}^{\vee} - i)\epsilon_1 + (\mu_{k,i} - j + 1)\epsilon_2) \times \\
\prod_{i,j \in \nu_l} (a_k - b_l - m + (\mu_{k,j}^{\vee} - i + 1)\epsilon_1 - (\nu_{l,i} - j)\epsilon_2) \Big], \\
Z_{Inst}^{Adj}(\vec{a}, m, \vec{\mu}; \epsilon_1, \epsilon_2) = Z_{Inst}^{Bifund}(\vec{a}, \vec{a}, m, \vec{\mu}, \vec{\mu}; \epsilon_1, \epsilon_2), \\
Z_{Inst}^{Vector}(\vec{a}, \vec{\mu}; \epsilon_1, \epsilon_2) = \frac{1}{Z_{Inst}^{Adj}(\vec{a}, 0, \vec{\mu}; \epsilon_1, \epsilon_2)},
\end{gathered}
\tag{2.50}
$$

where $\vec{\mu}$ and $\vec{\nu}$ are vectors consisting of Young diagrams and share the basis of $\vec{a}$ $\vec{b}$. $\nu_k^{\vee}$ is the dual partition of ν_k. The gauge groups in 1-loop and instanton terms are of the form $U(N)$, because considering $SU(N)$ case is more tricky [3, 7].

If we want to write the instanton partition part of the 5D partition function, we must rewrite Z_{Inst}^{Fund}:

$$
Z_{Inst}^{Fund}(\vec{a}, m, \vec{\mu}; \epsilon_1, \epsilon_2) = \prod_k \prod_{i,j \in \mu_k} (1 - e^{-a_k + m - \epsilon_1 i - \epsilon_2 j}). \tag{2.51}
$$

Antifundamental, bifundamental, adjoint and vector terms are calculated using the 5D fundamental term. If $G = SU(2)$, total instanton partition function has the form

$$
Z_{Inst} = \sum_{\mu_1, \mu_2} \Lambda^{|\mu_1| + |\mu_2|} Z_{Inst}^{Fund,1} Z_{Inst}^{Fund,2} Z_{Inst}^{Antifund,1} Z_{Inst}^{Antifund,2} Z_{Inst}^{Vector}. \tag{2.52}
$$

All distinct contributions are expressed in terms of the following fundamental factors:

$$\begin{aligned} n_{\mu_1\mu_2}(a,\epsilon_1,\epsilon_2) = \prod_{i,j\in\mu_1} [a-\epsilon_2(\mu_{1,i}-j)+\epsilon_1(\mu^{\vee}_{2,j}-i)]\times \\ \prod_{i,j\in\mu_2} [a+\epsilon_2(\mu_{2,i}-j)-\epsilon_1(\mu^{\vee}_{1,j}-i)], \end{aligned} \tag{2.53}$$

then their 5D generalization [14]

$$N_{\mu_1\mu_2}(Q;q,t) = \prod_{i,j\in\mu_1}(1-Qq^{\mu_{1,i}-j}t^{\mu^{\vee}_{2,j}-i+1}) \times \prod_{i,j\in\mu_2}(1-Qq^{-\mu_{2,i}+j-1}t^{-\mu^{\vee}_{1,j}+i}), \tag{2.54}$$

where $Q=\frac{e^{-a}}{e^{a}}=e^{-2a}$, $q=\epsilon_2$, $t=\epsilon_1$ and μ, ν are two distinct Young diagrams. 5D case is particularly important in thesis, because 4D case is mere a limit of the latter. For $N_F=4$ A_1 gauge theory theory the total instanton partition function which is impressively algorithmic is

$$\begin{aligned} Z^{A_1}_{Inst} = \sum_{\mu_1,\mu_2}(v\Lambda)^{|\mu_1|+|\mu_2|}\frac{N_{\phi\mu_1}\big(v\frac{f_1^+}{e_1}\big)N_{\phi\mu_1}\big(v\frac{f_2^+}{e_1}\big)N_{\phi\mu_2}\big(v\frac{f_1^+}{e_2}\big)N_{\phi\mu_2}\big(v\frac{f_2^+}{e_2}\big)}{N_{\mu_1\mu_1}(1)N_{\mu_1\mu_2}\big(\frac{e_1}{e_2}\big)N_{\mu_2\mu_1}\big(\frac{e_2}{e_1}\big)N_{\mu_2\mu_2}(1)}\times \\ N_{\mu_1\phi}\Big(v\frac{e_1}{f_1^-}\Big)N_{\mu_1\phi}\Big(v\frac{e_1}{f_2^-}\Big)N_{\mu_2\phi}\Big(v\frac{e_2}{f_1^-}\Big)N_{\mu_2\phi}\Big(v\frac{e_2}{f_2^-}\Big), \end{aligned} \tag{2.55}$$

where $e_i=e^{-a_i}$ and $f_i^{\pm}=e^{-m_i^{\pm}}$. Instanton partition function of a $N_F=6$ A_2 gauge theory is

$$\begin{aligned} Z^{A_2}_{Inst} = \sum_{\mu_1,\mu_2}\sum_{\nu_1,\nu_2}(v\Lambda_1)^{|\mu_1|+|\mu_2|}(v\Lambda_2)^{|\nu_1|+|\nu_2|}\times \\ \frac{N_{\phi\mu_1}\big(v\frac{f_1^+}{e_1}\big)N_{\phi\mu_1}\big(v\frac{f_2^+}{e_1}\big)N_{\phi\mu_2}\big(v\frac{f_1^+}{e_2}\big)N_{\phi\mu_2}\big(v\frac{f_2^+}{e_2}\big)}{N_{\mu_1\mu_1}(1)N_{\mu_1\mu_2}\big(\frac{e_1}{e_2}\big)N_{\mu_2\mu_1}\big(\frac{e_2}{e_1}\big)N_{\mu_2\mu_2}(1)}\times \\ \frac{N_{\phi\nu_1}\big(v\frac{f_1^+}{e'_1}\big)N_{\phi\nu_1}\big(v\frac{f_2^+}{e'_1}\big)N_{\phi\nu_2}\big(v\frac{f_1^+}{e'_2}\big)N_{\phi\nu_2}\big(v\frac{f_2^+}{e'_2}\big)}{N_{\nu_1\nu_1}(1)N_{\nu_1\nu_2}\big(\frac{e'_1}{e'_2}\big)N_{\nu_2\nu_1}\big(\frac{e'_2}{e'_1}\big)N_{\nu_2\nu_2}(1)}\times \\ N_{\mu_1\nu_1}\Big(f^+\frac{e_1}{e'_1}\Big)N_{\mu_1\nu_2}\Big(f^+\frac{e_1}{e'_2}\Big)N_{\mu_2\nu_1}\Big(f^-\frac{e_2}{e'_1}\Big)N_{\mu_2\nu_2}\Big(f^-\frac{e_2}{e'_2}\Big). \end{aligned} \tag{2.56}$$

Diagram of $\widehat{A}_1$ $SU(2)$ theory is shown in the Figure 2.3. For $N_F = 2$ $\widehat{A}_1$ version,

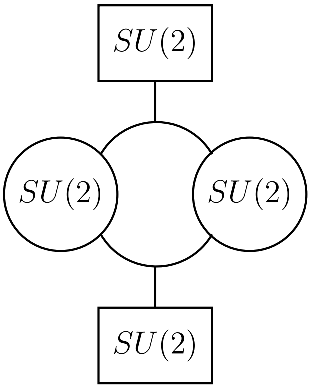


Figure 2.3. $\widehat{A}_1$ gauge theory for $SU(2)$.

the partition function reads

$$
\begin{gathered}
(v\Lambda_2)^{|\nu_1|+|\nu_2|} \times \\
\frac{N_{\mu_1\nu_1}\left(f_1^+\frac{e_1}{e_1'}\right)N_{\mu_1\nu_2}\left(f_1^+\frac{e_1}{e_2'}\right)N_{\mu_2\nu_1}\left(f_1^-\frac{e_2}{e_1'}\right)N_{\mu_2\nu_2}\left(f_1^-\frac{e_2}{e_2'}\right)}{N_{\mu_1\mu_1}(1)N_{\mu_1\mu_2}\left(\frac{e_1}{e_2}\right)N_{\mu_2\mu_1}\left(\frac{e_2}{e_1}\right)N_{\mu_2\mu_2}(1)} \times \\
\frac{N_{\mu_1\nu_1}\left(f_2^+\frac{e_1}{e_1'}\right)N_{\mu_1\nu_2}\left(f_2^+\frac{e_1}{e_2'}\right)N_{\mu_2\nu_1}\left(f_2^-\frac{e_2}{e_1'}\right)N_{\mu_2\nu_2}\left(f_2^-\frac{e_2}{e_2'}\right)}{N_{\nu_1\nu_1}(1)N_{\nu_1\nu_2}\left(\frac{e_1'}{e_2'}\right)N_{\nu_2\nu_1}\left(\frac{e_2'\cdot}{e_1'}\right)N_{\nu_2\nu_2}(1)},
\end{gathered}
\tag{2.57}
$$

where the variables q and t are omitted [11].

2.1.2. Conformal Field Theory, Liouville Field Theory and Toda Field Theory

To explore the other branch of the correspondence we now discuss general form of conformal field theories in $2D$, Liouville field theory and Toda field theory.

2.1.2.1. Conformal Field Theory. Conformal field theories are quantum field theories which are invariant under scale transformations but preserve angles. In string theory, worldsheets with complex geometries are encountered. Using topology, we simplify the worldsheets to nothing but planes or spheres with handles and holes. In order to succeed in that, a scale invariant and angle preserving theory is needed which is the 2D version of conformal field theory. Conformal field theory is not only used in string theory, it is also a tool for fields of physics used in statistical mechanics and condensed matter physics to study phase transitions etc.. 2D conformal field theories, which are studied in this thesis, have great importance because it has infinite number of generators.

Conformal transformations are angle preserving, they can be obtained by multiplying the metric by a scalar function $\Lambda(x)$:

$$g_{\mu\nu} \to \Lambda(x) g_{\mu\nu}. \tag{2.58}$$

General form of coordinate transform is

$$g_{\mu\nu} \to \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g_{\alpha\beta}. \tag{2.59}$$

We make the Ansatz $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$ and $\Lambda(x) = e^{-\omega(x)}$ to get the differential equation

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\nu} = \omega(x) g_{\mu\nu}. \tag{2.60}$$

Although higher dimensional CFTs are particularly significant, we are going to concentrate on 2 dimensional case. In 2 dimensions, the classical equation of motion reduces to

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\nu} = \partial^{\rho}\epsilon_{\rho} g_{\mu\nu}. \tag{2.61}$$

We now consider $\mathbb{R}^2$, 2 dimensional flat space and transform it into $\mathbb{C}$:

$$\begin{aligned} z &= x_1 + ix_2, \\ \overline{z} &= x_1 - ix_2, \\ \epsilon &= \epsilon_1 + i\epsilon_2, \\ \overline{\epsilon} &= \epsilon_1 + i\epsilon_2. \end{aligned} \tag{2.62}$$

Partial derivative operators take the form

$$\begin{aligned} \partial_1 &= \frac{1}{2}(\partial_z + \partial_{\overline{z}}), \\ \partial_2 &= \frac{1}{2i}(\partial_z - \partial_{\overline{z}}). \end{aligned} \tag{2.63}$$

We are left with two simple equations

$$\begin{aligned} \partial z\overline{\epsilon} &= 0, \\ \partial \overline{z}\epsilon &= 0. \end{aligned} \tag{2.64}$$

They imply that ϵ depends only on z and $\overline{\epsilon}$ only on $\overline{z}$. $\epsilon(z)$ and $\overline{\epsilon}(\overline{z})$ are called holomorphic and holomorphic parts, respectively. Most general form of the solution is given in terms of Laurent series

$$\begin{aligned} \epsilon(z) &= \sum_{-\infty}^{\infty} \epsilon_n z^{n+1}, \\ \overline{\epsilon}(\overline{z}) &= \sum_{-\infty}^{\infty} \overline{\epsilon}_n \overline{z}^{n+1}. \end{aligned} \tag{2.65}$$

Generators of conformal transformations $l_n = -z^{n+1}\partial_z$ and $\bar{l}_n = -\overline{z}^{n+1}\partial_{\overline{z}}$ form the *Witt algebra*

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n}, \\ [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n}, \\ [\bar{l}_m, l_n] &= 0. \end{aligned} \tag{2.66}$$

Since we are interested in quantized version of CFT, Witt algebra is not sufficient, we need. We use the *Virasoro algebra* with the generators L_m, the central extension of Witt algebra which will remove quantum anomalies

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}. \tag{2.67}$$

In $2D$ conformal field theories any field $V(z,\overline{z})$ satisfy the operator product expansion of the form, if the fields are dependent on $\overline{z}$ which is the complex conjugate of the complex number z as well,

$$V_i(z_1,\overline{z_1})V_j(z_2,\overline{z_2}) = \frac{C_{ij}^k}{|z_1-z_2|^2}V_k(z_2,\overline{z_2}). \tag{2.68}$$

For transformations of z and $\overline{z}$, as $z' = f(z)$ and $\overline{z}' = \overline{f}(\overline{z})$, a field is called a *primary field*, if it transforms such as

$$\Big(\frac{\partial f}{\partial z}\Big)^{-\Delta}V(z',\overline{z}') = \Big(\frac{\partial f}{\partial z}\Big)^{-\Delta}\Big(\frac{\partial \overline{f}}{\partial \overline{z}}\Big)^{-\overline{\Delta}}V(z,\overline{z}), \tag{2.69}$$

where Δ is *scaling dimension.* The transformation of primary fields are in the form

$$T(z_1)V(z_2) = \frac{\Delta}{(z_1-z_2)^2}V(z_2) + \frac{V(z_2)}{z_1-z_2} + \text{Regular Terms}, \tag{2.70}$$

where $T(z)$ is the energy momentum tensor. It is possible to build the two-point correlation function

$$\langle V_{\alpha_1}(z_1,\overline{z_1}), V_{\alpha_2}(z_2,\overline{z_2})\rangle = \frac{\delta_{\Delta_1\Delta_2}C(\alpha_1,\alpha_2)}{(z_1-z_2)^{2(\Delta_1+\Delta_2)}}, \tag{2.71}$$

where C_{12} is the *structure constant.* It is worth noting that the exponent in the denominator is fixed by conformal symmetry. Three-point correlation function reads

$$\begin{gathered}\langle V_{\alpha_1}(z_1,\overline{z_1}),V_{\alpha_2}(z_2,\overline{z_2}),V_{\alpha_3}(z_3,\overline{z_3})\rangle = \\ \frac{C(\alpha_1,\alpha_2,\alpha_3)}{(z_1-z_2)^{2(\Delta_1+\Delta_2-\Delta_3)}(z_1-z_3)^{2(\Delta_1+\Delta_3-\Delta_2)}(z_2-z_3)^{2(\Delta_2+\Delta_3-\Delta_1)}}.\end{gathered} \tag{2.72}$$

And finally, four-point correlation function is

$$\begin{gathered}\langle V_{\alpha_1}(z_1,\overline{z_1}),V_{\alpha_2}(z_2,\overline{z_2}),V_{\alpha_3}(z_3,\overline{z_3}),V_{\alpha_4}(z_4,\overline{z_4})\rangle = \\ \sum_{\alpha} C(\alpha_1,\alpha_2,\alpha)C(\alpha,\alpha_3,\alpha_4)\mathcal{F}^s(\alpha_1,\alpha_2,\alpha_3,\alpha_4;z_1,z_2,z_3,z_4)\times \\ \overline{\mathcal{F}}^s(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\overline{z_1},\overline{z_2},\overline{z_3},\overline{z_4}).\end{gathered} \tag{2.73}$$

Where $\mathcal{F}^s$ and $\overline{\mathcal{F}}^s$ are called s-channel *conformal blocks* [13]. Any conformal field theory including Liouville and Toda field theories holds the form given above. Distinct conformal field theories only differ in the form of structure constants C and conformal blocks $\mathcal{F}^s$. We are ready to define those terms for Liouville field theory now.

2.1.2.2. Liouville Field Theory. Liouville field theory appears after quantization of classical *Liouville's equation*

$$\nabla^2 ln(\phi) + R\phi^2 = 0 \tag{2.74}$$

Where the equation is $2D$ and R is the constant curvature. That is a special type of conformal field theory which is not free; primary fields interact both with each other via the *Liouville potential* and the background with a curvature. The action in the most general case is:

$$\int d^2\xi\sqrt{g}(\frac{1}{4\pi}g^{ab}\partial_a\phi\partial_b\phi + \mu\, e^{2b\phi} + \frac{Q}{4\pi}R\phi) \tag{2.75}$$

Where g^{ab} is the metric of Riemann surface, R is the curvature, μ and b are coupling constants of the Liouville potential and Q is the coupling constant of the interaction with the background. The theory is conformal if the constraint

$$Q = b + \frac{1}{b} \tag{2.76}$$

is satisfied. The central charge c is

$$c = 1 + 6Q^2. \tag{2.77}$$

Actually, Liouville field theory is a mere member of group of conformal feld theories called *Toda field theories.* Toda field theories have an action of the form [12]

$$\int d^2\xi\sqrt{g}(\frac{1}{8\pi}g^{ab}\langle\partial_a\phi,\partial_b\phi\rangle - \frac{m^2}{\beta^2}\sum_{i=1}^{r} n_i e^{\beta\langle\alpha_i,\phi\rangle}), \tag{2.78}$$

where the operation $< .,. >$ is the inner product defined in the space spanned by the simple roots of the semisimple Lie algebra the theory is built on, m is the mass, β is the coupling constant of the interaction, α_i is the simple root and n_i is the *Coxeter number.*

Action for the A_{N-1} Toda field theory is

$$\int d^2\xi\sqrt{g}(\frac{1}{8\pi}g^{ab}\langle\partial_a\phi,\partial_b\phi\rangle + \mu\sum_{i=1}^{N-1} e^{\langle\alpha_i,\phi\rangle} + \frac{\langle Q,\phi\rangle}{4\pi}R\phi). \tag{2.79}$$

It turns out to be Liouville field theory is nothing but A_1 Toda field theory. Toda field theories have particular importance while applying AGT conjecture to higher order gauge groups and extending it to AGT-W conjecture. Liouville structue constant C is

$$\begin{aligned} C(\alpha_1,\alpha_2,\alpha_3) = [\pi\mu\gamma(b^2)b^{2-2b^2}]^{\frac{Q-\alpha_1-\alpha_2-\alpha_3}{b}} \times \\ \frac{\Upsilon(b)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(-\alpha_1+\alpha_2+\alpha_3)} \times \\ \frac{1}{\Upsilon(\alpha_1-\alpha_2+\alpha_3)\Upsilon(\alpha_1+\alpha_2-\alpha_3)}, \end{aligned} \tag{2.80}$$

where

$$\Upsilon(x) = \frac{1}{\Gamma_2(x|b,\frac{1}{b})\Gamma_2(Q-x|b,\frac{1}{b})}. \tag{2.81}$$

Definition of double gamma function Γ_2 is given in previous section.

For conformal blocks, we simplify the expression fixing z_1 at 0 and z_4 at ∞:

$$\begin{aligned} \mathcal{F}^s(\alpha_1,\alpha_2,\alpha_3,\alpha_4;z_2,z_3) = z_2^{\Delta_1-\Delta 2-\Delta_\alpha} z_3^{\Delta_\alpha-\Delta 3-\Delta_4} \times \\ \sum_{n=0}^{\infty}\left(\frac{z_3}{z_2}\right)^n \mathcal{B}_n(\alpha_1,\alpha_2,\alpha_3,\alpha_4). \end{aligned} \tag{2.82}$$

Projective invariance allows us to fix those two points. We also need to avoid the pole at z=0. We simplify the expression further by setting $z_2 = 1$ and $z_3 = Q$:

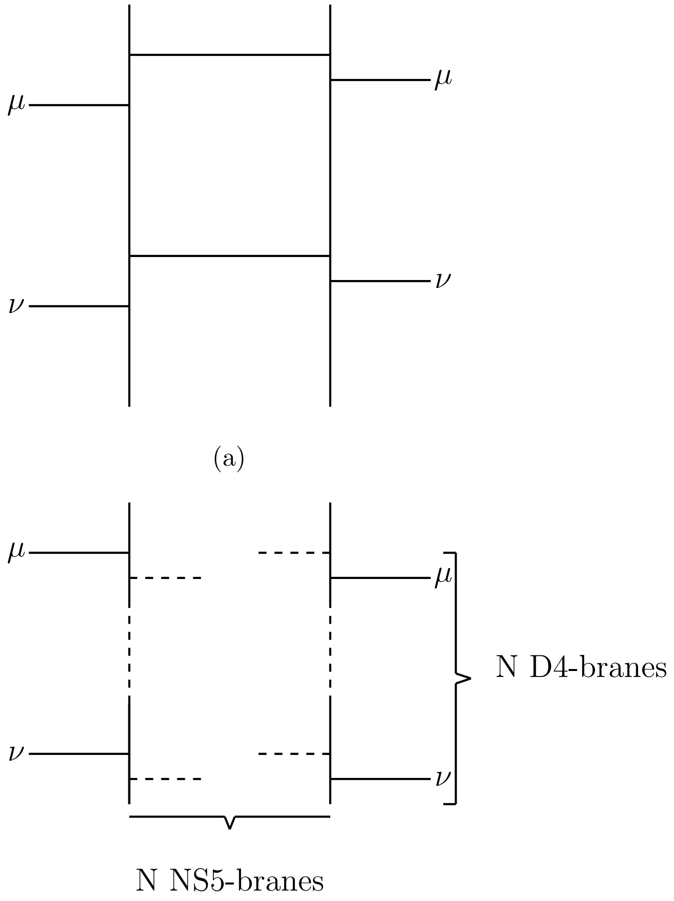
$$\mathcal{F}^s(\alpha_1,\alpha_2,\alpha_3,\alpha_4;Q) = Q^{-\Delta_3+\Delta_4}\sum_{n=0}^{\infty} Q^{\Delta_\alpha+n}\mathcal{B}_n(\alpha_1,\alpha_2,\alpha_3,\alpha_4), \tag{2.83}$$

where α is a complex parameter such that $\alpha = \frac{Q}{2} + i\mathbb{R}^+$, Q is the very same gauge coupling parameter used in Nekrasov partition function and $\mathcal{B}_n$ terms are sums over all Young diagrams with size n.

2.2. Brane and Gaiotto Constructions

2.2.1. Type IIA and IIB Backgrounds

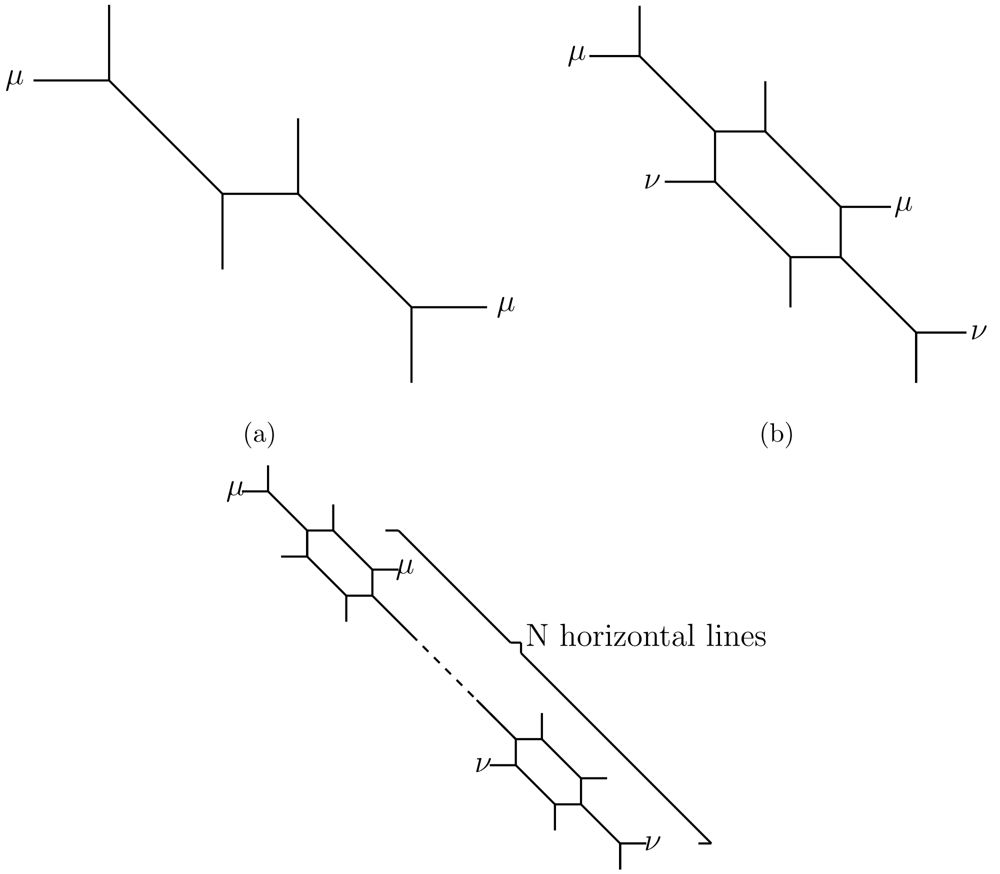
It is also possible to express $\mathcal{N} = 2$ supersymmetric gauge theories in type IIA and IIB string theory backgrounds. In type IIA background, there are N D4 and N and N NS5-branes. D4-branes are placed vertically and NS-5 branes are placed horizontally as shown in the figure 2.4. Position of D4-branes on NS5-branes determine the masses m and vacuum expectation values a in the gauge theory. In the case of type IIB set up a more complex but similar geometry exists as in Figure 2.5. For an affine A-type quivers, the diagram is wrapped around a cylinder, an infinite lattice occurs.



(a)

(b)

Figure 2.4. Type IIA set ups for SU(2) (a) and SU(N) (b) and SU(N) (c) theories.



(a) (b)

(c)

Figure 2.5. Type IIB set ups for U(1) (a), SU(2) (b) and SU(N) (c) theories.

2.2.2. Gaitto Construction

The simplest theory in the case of Liouville CFT lives on a Riemann surface with 1 channel and 4 holes as depicted in Figure 2.6. The surface can be extended as long as it satisfies the conformal invariance condition. It can be described as a "curve" which is called *Gaiotto curve*. Each individual spherical part need to contain a total number of 3 channels and holes combined. It is possible to generalize Gaioto picture. Each vector multiplet corresponds to a channel in the Gaiotto curve, each hypermultiplet corresponds to a hole and finally each loop in the gauge theory corresponds to handle. Number of handles of the Riemann surface is genus g and Euler characteristic $\chi(g)$ of that surface is equal to $2-2g$. For instance, A_n theory which has no loops corresponds to Gaiotto curve in Figure 2.7 [3, 12].

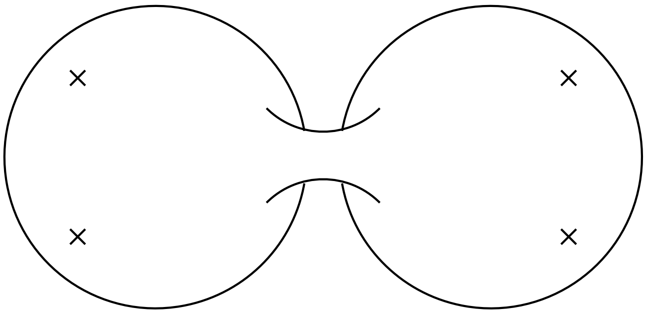


Figure 2.6. Riemann surface with 1 channel and 4 holes.

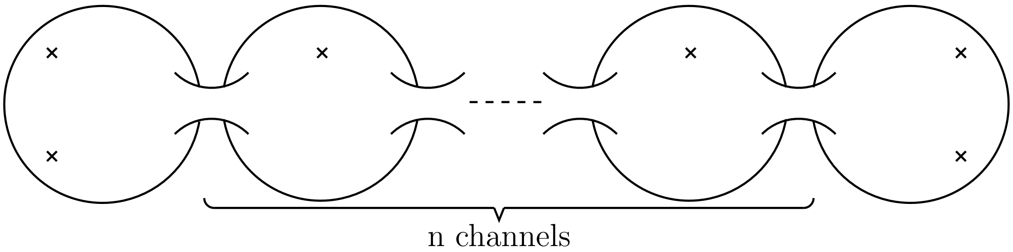


Figure 2.7. Riemann surface with n channels and n+3 holes.

2.3. AGT-W Conjecture

In 2009, Luis F. Alday, Davide Gaiotto and Yuji Tachikawa published the paper called "Liouville Correlation Functions from Four-dimensional Gauge Theories" to study the relationship between partition functions of the theories. Perturbative part of Nekrasov partition function turns out to be nothing but structure constants of Liouville field theory. The same duality exists between instanton part of the first theory and conformal blocks of the latter.

Everything is just a consequence of the attempt to extend the electromagnetic duality not only to Yang-Mills theories but also to SUSY equivalent of them both in weak and strong coupling regions. Those theories exhibit $SL(2,\mathbb{Z})$ symmetry which is also the conformal symmetry group of Liouville field theory. If the theory contains only one SU(2) vector multiplet Nekrasov partition function and four-point Liouville correlation functions are equal to

$$Z(Q,\vec{a};\epsilon_1,\epsilon_2) = \langle V_{\alpha_1}(0), V_{\alpha_2}(1), V_{\alpha_3},(Q), V_{\alpha_4}(\infty)\rangle. \tag{2.84}$$

Only if $\epsilon_1 = b$ and $\epsilon_2 = \frac{1}{b}$.

Just 6 weeks after the paper, Wyllard showed a similar correspondence applies to SU(N) vector multiplets. The conformal field theories corresponding to SU(N) gauge theories are A_{N-1} Toda field theories living on the same Riemann surface.

The correspondence also apply to $5D$ and $6D$ cases. $4D$ Nekrasov partition function a mere limit of $5D$ one. $5D$ spacetime is expressed as $S^1 \times \mathbb{R}^{3,1}$ where S^1 is a 1-sphere with radius R. In the limit $R \to 0$, $S^1 \times \mathbb{R}^{3,1}$ reduces to $\mathbb{R}^{3,1}$.

Unfortunately, it is not possible to relate $5D$ gauge theory to $2D$ conformal field theories using the previous approach. The previous relations do not apply anymore. In fact, lifting the supersymmetric gauge theory to $5D$ corresponds to q-deformation of the

Virasoro algebra in the language of CFT. A Dotseko-Fateev integral with q-deformed version of Green's function must be performed [15]. The commutation relation

$$[a_m, a_n^\dagger] = m\delta_{m,n} \tag{2.85}$$

becomes

$$[a_m, a_n^\dagger] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m,n} \tag{2.86}$$

after q-deformation.

We will not only lift the theory to 5 dimensions, but also do it for the affine quiver. The simplest affine quiver $\tilde{A}_1$ corresponds to a genus-1 Riemann surface, better known as a torus. Therefore, we need a q-deformed Green's function on a torus.

5D Dotsenko-Fateev integral has the general form

$$\begin{gathered} Z_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z_1, z_2, z_3, z_4) = \\ \langle : e^{\alpha_1\phi(z_1)} :: e^{\alpha_2\phi(z_2)} :: e^{\alpha_3\phi(z_3)} :: e^{\alpha_4\phi(z_4)} :: \oint dz e^{\alpha\phi(z)} :\rangle = \\ \prod_{i<j} V(z_i - z_j) \oint dz \prod_i V(z - z_i), \end{gathered} \tag{2.87}$$

where V's are the Green's functions. If CFT is not q-deformed and lives on a genus-0 surface, after fixing the points at z_1=0, $z_2 = Q_1$, $z_3 = Q_2$ and $z_4 = \infty$, it takes the form

$$\begin{gathered} Z_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(Q_1, Q_2) = \\ Q_1^{2\alpha_1\alpha_2} Q_2^{2\alpha_1\alpha_3} (Q_1 - Q_2)^{2\alpha_2\alpha_3} \oint dz z^{2\alpha^2} z^{2\alpha\alpha_1} (z - Q_1)^{2\alpha\alpha_2} (z - Q_2)^{2\alpha\alpha_3}. \end{gathered} \tag{2.88}$$

The dictionary between Seiberg-Witten theory and Liouville field theory is as shown in Table 2.1. Forms of Green's functions needed after q-deformation and handle addition

Table 2.1. Dictionary between Seiberg-Witten theory and Liouville field theory.

SW Theory with a Quiver Γ	*LFT on a Riemann Surface* $\mathcal{C}$
Vector multiplet	Channel
Hypermultiplet	Hole
ϵ_1	b
ϵ_2	$\frac{1}{b}$
External momentum α_i	Mass m_i
Internal momentum α_k	Vacuum expectation value a_k

are as in Figure 2.8 [16, 17].

$$\begin{tikzcd}
\begin{array}{c}\text{q-Deformed Vandermonde Factor}\\ \frac{\varphi(z_i/z_j)}{\varphi(tz_i/z_j)}\end{array} \arrow[r] & \text{Yet to be determined} \\
\begin{array}{c}\text{Deformed Vandermonde Factor:}\\ (z_i-z_j)^{2\alpha_i\alpha_j}\end{array} \arrow[u, "\text{q-Deformation}"] \arrow[r, "g=0\to g=1"'] & \begin{array}{c}\theta_*\ \text{Function}\\ \theta_*(z_i-z_j)\end{array} \arrow[u]
\end{tikzcd}$$

Figure 2.8. Form of V$(z_i - z_j)$ in various cases.

Two recently introduced functions are

$$\theta_*(x) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2+n}{2}} sin\Big(\frac{(2n+1)x}{2}\Big), \tag{2.89}$$

$$\varphi(x) = \prod_{n=0}^{\infty} (1 - q^n x), \qquad |q| < 1. \tag{2.90}$$

The form of Green's function after both q-deformation for affine gauge theories are still worked on. Once a result is found, it will be published as a paper.

2.3.1. Gauge/Liouville Triality

In addition to the duality proposed in AGT conjecture, there is a triality between a $3D$ $\mathcal{N} = 2$ $U(N)$ gauge theory $\mathcal{G}_{3D}$ with $2M$ hypermultiplets, $N = 1$ $5D$ $U(M)$ gauge theory $\mathcal{G}_{5D}$ with $2M$ hypermultiplets and q-deformed Liouville field theory q-LFT [17]. Even Toda field theory can be q-deformed, which gives us q-TFT. The poles of the contour integral in $\mathcal{G}_{5D}$ corresponds to truncation $Q = t^N$ of semistandard Young tableaux in $\mathcal{G}_{5D}$. Additionally, both poles and semistandard Young tableaux correspond to internal points y_i and external points z_a of Dotsenko-Fateev integral in q-LFT. This relation is shown in Figure 2.9. Strangely, $\mathcal{G}_{5D}$ obtained via the triality is not the same theory obained via AGT conjecture, but rather its spectral dual $\tilde{\mathcal{G}}_{5D}$ [17]. Spectral duality will be explained in the next section of this thesis.

$$\begin{tikzcd}
 & \text{q-LFT} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\mathcal{G}_{3D} \arrow[rr, leftrightarrow] & & \mathcal{G}_{5D}
\end{tikzcd}$$

Figure 2.9. Triality between $\mathcal{G}_{3D}$, $\mathcal{G}_{5D}$ and q-LFT.

2.3.2. Spectral Duality, Fiber-Base Duality and S-Duality

Although it is not quite obvious in the partition function, there is a correspondence called *spectral duality* between $5D$ $U(N)$ gauge theory and $5D$ A_{N-1} quiver gauge theory with $U(2)$ vector multiplets. $U(2)^{N-1}$ theory is nothing but $U(N)$ theory 90° rotated web diagram in string theoretical background. Different mathematical methods lead to *fiber base duality* and *S-duality* which indicate the same correspondence a the first one. Expressing gauge theories in type IIA and IIB backgrounds gives us S-duality, that means D4 and NS5-branes are swithced. Using topological string theory results in fiber-base duality, that means base space B and fiber space F are swiched. That means means the product space $E = B \times F$ becomes $F \times B$. The diagrams shown in Figure 2.10 and 2.11 belong to dual theories mentioned.

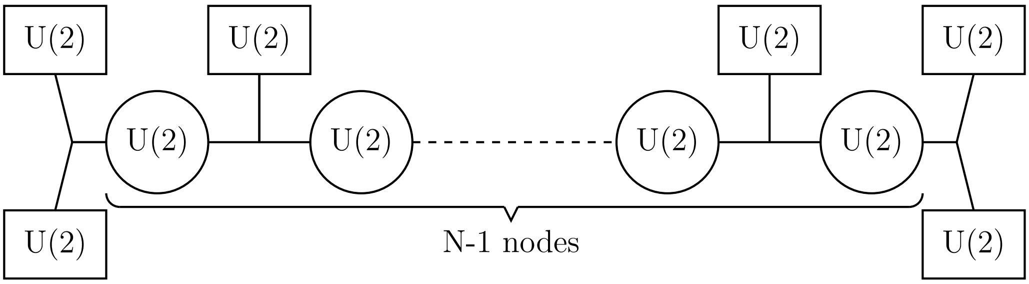


Figure 2.10. Quiver diagram of the A_{N-1} theory with N-1 U(2) vector multiplets.

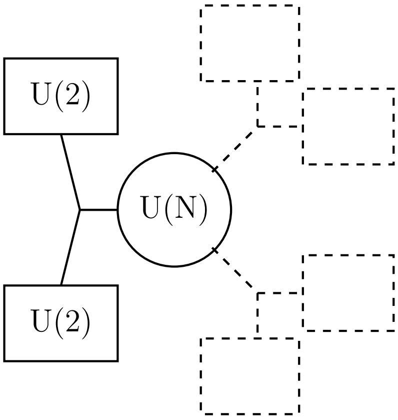


Figure 2.11. Quiver diagram of U(N) gauge theory with 2N fundamental hypermultiplets, dual theory of A_{N-1} theory.

2.3.3. Nekrasov-Shatashvili Limit and Integrable Systems

Supersymmetric Yang-Mills Theories have one more limit called *Nekrasov-Shatashvili Limit*: $\epsilon_1 \to \epsilon$, $\epsilon_2 \to 0$. Ω-deformation of the 4D gauge theory corresponds to not only q-deformation of CFT but also the quantization of Bethe Ansatz equation of Heisenberg spin chain [18,19]. The deformation parameter ϵ corresponds to $\hbar$. The symmetry group of Heiseberg spin chain is $SL(2)$ for a gauge theory with a single node. For a generalized quiver of type A_N, the symmetry group is $SL(N-1)$. Also a 2d gauge theory can be expressed as our integrable system. It turns out to exist another connection between 2D and 4D theories via Bethe Ansatz and Yang-Baxter equations. 4D SU(N) gauge theory with 2N fundamental hypermultiplets with Nekrasov-Shatashvili limt is related to quantized version of SL(2) Heisenberg spin chain model which is related to a 2D U(M) gauge theory with 2N fundamental chiral multiplets and an additional adjoint chiral multiplet. The deormation parameter ϵ of the 4D theory corresponds to the mass of the adjointmultiplet of the 2D theory. Twisted superpotential $\mathcal{W}$ of the gauge theories are obtained from the prepotential $\mathcal{F}$ which look like free energy in statistical mechanics:

$$\mathcal{W}(\vec{a},\epsilon) = \frac{1}{\epsilon}\mathcal{F}(\vec{a},\epsilon) - 2\pi\vec{k}\cdot\vec{a}. \tag{2.91}$$

Where $\vec{k}$ is a vector with integer components. Through their relation tointegrable systems, there is another correspondence between 4D and 2D gauge theories. The vacua of those theories are equal except some extra constant term:

$$\mathcal{W}_{4D} \equiv \mathcal{W}_{4D}. \tag{2.92}$$

3. CONCLUSION

So far, we showed how Nekrasov Partition functions of $4D$ and $5D$ SW theories with simply laced diagrams gauge groups $U(2)$ and $SU(2)$, correlation functions of LFTs and stressed the relation between them. We also pointed out a more generalized relation between SW theories with gauge group $SU(N)$ and A_{N-1} TFTs. That relation is called AGT-W conjecture. We summarized another relation called gauge/Liouville triality between a $3D$ $\mathcal{N} = 2$ supersymmetric Yang-Mills theories, $5D$ $\mathcal{N} = 1$ supersymmetric Yang-Mills theories and q-deformed LFTs. Additionally, we stated another remarkable duality between $5D$ $U(N)$ theories and $5D$ $U(2)^{N-1}$ theories, which is simultaneously called spectral duality, fiber-base duality and S-duality. We also underlined the connection of $2D$ and $4D$ superymmetric Yang-Mills theories with quantizable integrable sysytems through Nekrasov-Shatashvili limit. Finally, we gave insight into type IIA and IIB string theory backgrounds.

Our next goal is to calculate Nekrasov partition function for $\widehat{A}_1$ case and find a closed form of Green's function in DF integral. Then we will extend our calculation to $\widehat{A}_n$ quivers. We are hoping to calculate the Nekrasov partition functions for $\widehat{D}_n$ theories starting with $\widehat{D}_4$. We will check the form of their corresonding DF Green's functions. We will hope to figure out what type of compactification pattern in the string theoretical background is needed in order to obtain affinization of quivers. Finally,we will extend our study to fractional quivers.

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APPENDIX A: QUIVERS

In mathematics, quivers are objects that depict multiple vector spaces and maps between them. However, in physics, the concept quiver has a narrower meaning. Quivers represent the interactions between vector multiplets and hypermultiplets. Squares in the diagram of quiver Γ represent hypermultiplets and circles represent vector multiplets. If two multiplet is interacting, they are connected.

It is phenomenally interesting that it is possible to build gauge theories which correspond to Dynkin diagrams representing semisimple Lie algebras. The simplest case is A_n gauge theories. It is also possible to build D_n, E type gauge theories which are simply laced. Even non-simply laced B_n and C_n theories are possible with the help of advanced mathematical tools. Dynkin diagrams of semisimple Lie algebras are shown in Figure A.1.

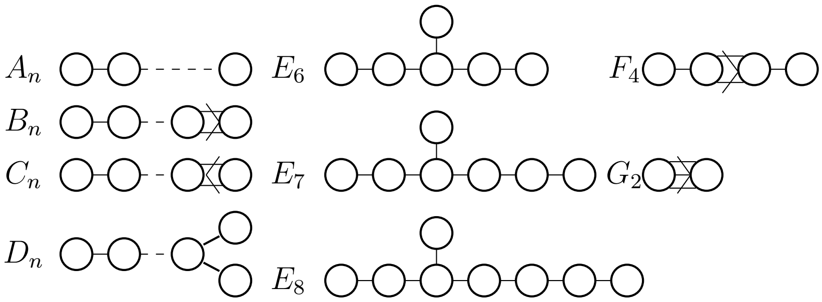

Figure A.1. Dynkin diagrams of semisimple Lie algebras.