RECENT CHARACTERIZATIONS OF LITTLEWOOD-RICHARDSON COEFFICIENTS

by

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ABSTRACT

RECENT CHARACTERIZATIONS OF LITTLEWOOD-RICHARDSON COEFFICIENTS

This thesis discusses the latest methods for calculating Littlewood-Richardson coefficients along with their role in the theory of group representations, symmetric functions, and Schubert varieties of Grassmanians. The methods prescribed here are constructed through honeycombs, Berenstein Zelevinky triangles, and hives. The classical approach, which uses Littlewood-Richardson tableaux, is also covered briefly since it allows us to introduce a new algorithm that efficiently calculates a lower bound for non-zero Littlewood-Richardson coefficients.

ÖZET

LITTLEWOOD-RICHARDSON KATSAYILARININ GÜNCEL KARAKTERİZASYONU

Bu tezde Littlewod-Richardson katsayılarının grup representasyonları, simetrik fonksiyonlar ve Schubert döngüsü teorilerindeki yeriyle birlikte bu katsayıları hesaplamak için kullanılan honeycombs, Berenstein Zelevinky üçgenleri ve hive gibi güncel metodlar incelenecektir. Bunlarla birlikte klasik bir method olan Littlewood-Richardson tabloları kısaca tanıtılacak ve bu tablolar üzerinde Littlewood-Richardson katsayılarının sıfır olmadığı durumlar için dominans sıralmaya göre bir alt sınır bulan yeni bir algoritma sunulacaktır.

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LIST OF SYMBOLS

$BZ_n(\lambda, \mu, \nu)$	Berenstein-Zelevinsky triangle of type (λ, μ, ν)
$\mathbb{C}$	Complex numbers
$\mathbb{C}[x_1, x_2, \ldots]$	Ring of formal power series
$c_{\lambda\mu}^{\nu}$	Littlewood-Richardson coeffecient
$\mathbb{C}[G]$	Group algebra of G
$\mathbb{C}S$	Vectors space generated by set S
$\mathbb{G}_{k,n}$	Grassmanian variety
$GL(V)$	General linear group of vector space V
$H_n(\lambda, \mu, \nu)$	Hive of type (λ, μ, ν)
k_λ	Size of conjugacy class of μ
$K_{\lambda\mu}$	Kostka number
$\ell(r_i)$	Number of cells in the ith row
$LR_n(\lambda, \mu, \nu)$	Littlewood-RİchardsonTriangle of type (λ, μ, ν)
M^λ	Permutation module associated to partition λ
$\mathbb{N}$	Natural numbers
$\mathrm{NE}_{\gamma/\lambda}(c)$	North-East of the Cell c in the diagram γ/λ
$\mathrm{NE}\#_T(c, k)$	Number of label k at the North-East of c in tableau T
p_n	nth power sum symmetric functions
$\mathbb{P}^n$	Projective space of dimension n
p_λ	Power sum symmetric functions associated to partition λ
$\mathbb{R}$	Real Numbers
$R(G)$	Space of class functions on group G
r_i	ith row of tableau
S_n	The symmetric group on n elements
S_λ	Young subgroup of Partition λ
s_λ	Schur function of λ
S^λ	Specht module associated to partition λ
$Tr(M)$	Trace of matrix M

X_{reg}	Regular representation
X^T	Monomial of tableau T
X_λ	Schubert variety
Λ	Ring of symmetric functions
$\rho \uparrow_H^G$	Induced representation of subgroup H
$\rho \downarrow_H^G$	Reduced representation of G to subgroup H
σ_λ	Schubert cycle
Ω_λ	Schubert cell
$\#_T(k)$	Number of label k in tableau T
$\partial_{(i)}(\gamma/\lambda)$	i-th lower standardization of γ/λ
$\uparrow_\leftarrow (\gamma/\lambda)$	Left-top standardization of γ/λ
$\leftarrow\uparrow (\gamma/\lambda)$	Top-left standardization of γ/λ

LIST OF ACRONYMS/ABBREVIATIONS

BZT	Berenstein Zelevinsky Triangle
LRC	Littlewood Richardson Coeffecient
LRT	Littlewood Richardson Triangle
NE	North-East

1. INTRODUCTION

Littlewood-Richardson(LR) coefficients, $c^{\lambda}_{\mu\nu}$, are nonnegative integers appearing in the following decomposition

$$(S^{\mu} \bigotimes S^{\nu}) \uparrow^{S_n} = \bigoplus_{\lambda} c^{\lambda}_{\mu\nu} S^{\lambda}$$

where S^{λ}, S^{μ} and S^{ν} are Specht modules indexed by three positive integer partitions λ, μ and ν respectively. As we see later in section 2.1, Specht modules constitute all of the irreducible representations of symmetric groups, and $(S^{\mu} \bigotimes S^{\nu}) \uparrow^{S_n}$ represents certain induced representations of their tensor products.

Littlewood-Richardson rule (LR rule), a first combinatorial method to calculate $c^{\lambda}_{\mu\nu}$ is introduced in 1934 by D. E. Littlewood and A. R. Richardson [1]. This method is based on Young tableaux, combinatorial objects named after Albert Young, and the first complete proof of the LR rule was given by Schützenberger in 1977 [2], four-decade later after it's stated by Littlewood with the contributions of C. Schensted and Knuth [3].

Later it was discovered that LR coefficients describe the relationships between many other mathematical objects in various fields, such as the characters of finite-dimensional irreducible representations of general linear groups in representation theory [4], Schur functions that constitute a basis for symmetric polynomials [5], and the cohomology classes of Schubert varieties in algebraic geometry [6]. Recently its study gained traction due to its role in the proof of Horn's conjecture related to the eigenvalues of Hermitian matrices. In Chapter 2, we define some of these objects and introduce LR coefficients in the related context.

Appearances of LR coefficients in different areas also proposed various other combinatorial rules for calculating LR coefficients. In Chapter 3, these new models will be introduced along with the classical approach that uses the LR tableau to calculate LR coefficients, and their equivalences will be proved. As we will discuss in section 3.4, Berenstein and Zelevinsky proposed calculating LR coefficients by counting the integer points of certain polytopes and generalizing the LR coefficients in the representations of groups other than the general linear group [7]. Later Knutson and Tao propose honeycombs [8], which we will introduce in Section 3.5, based on the Berenstein Zelevinsky cones to prove the Saturation conjecture. The main advantage of this new model was the "overlay" operation. Tamsen Whitehead stated an equivalent operation on the LR tableaux with the help of a new construction named flows on honeycombs [9]. In the honeycomb article, Knutson and Tao also introduced the hive model, which saturation conjecture also proved separately from honeycombs [10]. We will discuss hives and their connection to honeycombs in Section 3.3.

The computational complexity of calculating LR coefficients was also studied in the more recent literature [11–13]. Although calculating the LR coefficients' exact value is an #P problem, it is shown that their positivity can be decided in polynomial time [14]. Various algorithms were proposed on different combinatorial tools to decide whether the LR coefficient is non-zero for given partitions. In Chapter 4, a new method for deciding the positivity of LR coefficients will be introduced on the LR tableau.

2. APPEARANCES OF THE LITTLEWOOD-RICHARDSON COEFFECIENT

This chapter briefly introduces some of the main fields Littlewood Richardson coefficients appear in the literature. In Section 2.1, after some results in the representation theory of finite groups are stated, we will move on to the representation of symmetric groups. Our treatment of the representation of symmetric groups builds upon Bruce Sagan's treatment of the subject in his book [5] "The symmetric group: representations, combinatorial algorithms, and symmetric functions". Then in Section 2.2, we will follow with the introduction of symmetric functions, Schur basis, and demonstration of their connection to the representation of symmetric groups. Lastly, in Section 2.3, we will investigate the geometric interpretation of Littlewood Richardson coefficients. A comprehensive treatment of the subjects introduced in this section can be found in Fulton's works [6].

2.1. Representation of Symmetric Groups

Let G be a finite group of n elements, V be a vector space over complex numbers, and $GL(V)$ be the group of all invertible linear transformations from V to itself. A group homomorphism $\rho : G \to GL(V)$ is called a *represantation* of G on V. In this case, V is called a G-module, and any $g \in G$ can be identified with an automorphism on V.

We define a G-homomorphism $\theta : V \to W$ between two G-modules V and W as a linear transformation such that $\theta(gv) = g\theta(v)$, for $g \in G$ and $v \in V$. If θ is also a bijection, we call it a G-isomorphism and say V and W are equivalent.

We define a submodule of G-module V as a subspace W that is a G-module itself. As every module V has two trivial submodules: V and $\{0\}$, we say V is irreducible if it has no nontrivial submodule.

We want to classify the irreducible submodules because when G is a finite group, Maschke states every G-module can be written as a direct sum of irreducibles such that $V = \bigoplus W_i$, where W_i irreducible submodules. For this aim, characters of representations can be effectively used.

We remark that after we fixed a basis for V, we can identify $\rho(g)$ with a matrix $X_\rho(g) \in GL(V)$. Then the character of $\rho(g)$ is defined by the trace of the corresponding matrix, that is, $\chi_\rho(g) := Tr(X_\rho(g))$. Now the function

$$\chi_\rho : G \to \mathbb{C}$$

is called the character of the representation ρ (and equivalently V) and it has many interesting properties:

- For e is the identity element of G, we have $\chi_\rho(e) = Tr(I_{dim(V)}) = dim(V)$.
- If σ and ρ are two equivalent representations of G then their characters χ_σ and χ_ρ are clearly equal functions.
- Note that the group elements lying in the same conjugacy classes are identified with similar matrices, and hence their characters are equal. Therefore $\chi_\rho : G \to \mathbb{C}$ is, in fact, a class function.
- Denoting by $R(G)$ the vector space of all class functions on G, characters of irreducible representations becomes an orthonormal basis for $R(G)$ with respect to the following inner product:
$$< \chi, \psi >= \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$
- If V is decomposed as $V = \bigoplus_{i=0}^{k} m_i W_i$, where W_i is an irreducible G-module with character χ_i, then:
  - $\chi_\rho = \sum_i m_i \chi_i$ where χ_i is the character of W_i
  - $< \chi_\rho, \chi_i >= m_i$
  - $< \chi_\rho, \chi_\rho >= \sum_i m_i^2$
  - χ_ρ is irreducible iff $< \chi_\rho, \chi_\rho >= 1$.

Hence characters allow us to study group representations efficiently since equivalent representations give rise to the same character; a representation is irreducible if and only if its character is irreducible. This equivalence, together with the inner product on $R(G)$, allows us to determine whether the given representation is irreducible.

Below we explain how to find the irreducible representation of finite groups. Let G has a group action on an arbitrary finite set S. The vector space generated by elements of S over complex numbers: $\mathbb{C}S$ is a G-module with this action, and the corresponding representation is *permutation representation* of G.

In fact, any group G acts on itself by left multiplication. In this case, on the other hand, G-module $\mathbb{C}G$ has a ring structure where the ring multiplication comes from the linear extension of group multiplication. So we have not only a module but also the group algebra $\mathbb{C}[G] = \{c_1g_1 + c_2g_2 + \ldots + c_ng_n : \ c_i \in \mathbb{C}, \ g_i \in G\}$. We call it *regular representation.*

We calculate the character of regular representation with module $V = \mathbb{C}[G]$. Let's denote it with χ_{reg}. Then for any $g \in G$, action of g permutes the bases $\{g_1, g_2, \ldots g_n\}$ of V. Therefore its matrix representation $X_{reg}(g)$ is just the permutation matrix. Then for any $g \in G$, character $\chi_{reg}(g)$ is the number of g_i such that $gg_j = g_j$. Therefore:

$$\chi_{reg}(g) = \begin{cases} |G| \text{ if g is identity,} \\ \\ 0 \text{ otherwise.} \end{cases}$$

Now we consider the decomposition of regular representation; $C[G] = \bigoplus_i m_iW_i$. Then $m_i =< \chi_{reg}, \chi_i >= dim(W_i)$. So every irreducible representation is in the direct sum with multiplicity equal to its dimension. Moreover, the number of these irreducible representations equals the dimension of $R(G)$, which is the number of conjugacy classes of G. Therefore to find the irreducible representations of a group, one only needs to decompose its regular representation.

Now we will see some methods to construct new representations from the given ones. Firstly, We can also construct representations of the group $G \times H$ from representations of individual groups. Let V and W be G-module and H-module, respectively. Then the tensor product $V \bigotimes W$ is $G \times H$ module with the action

$$(g,h)(v \bigotimes w) = (gv) \bigotimes (hw).$$

Moreover, if χ and ψ are characters of these representations, then $\chi \bigotimes \psi(g,h) = \chi(g)\psi(h)$ is the character of the *tensor product representation.*

Any representation $\rho : G \to GL(V)$ can naturally be restricted to a subgroup of H of G. The resulting representation of H is denoted by $\rho \downarrow_H^G$ where $\rho \downarrow_H^G (h) := \rho(h)$ for all $h \in H$. On the other hand, for any subgroup, H of G and any representation σ of H on V induces a representation of G. Then induced representation $\sigma \uparrow_H^G = \mathbb{C}[G] \bigotimes_{\mathbb{C}[h]} V$. We can make this definition more concrete in our case with matrix representations. Let $g_1, g_2, \ldots, g_k$ be transversal for the left cosets of H. And $Y_\sigma(h) \in GL(V)$ be matrix representation as before. Then the matrix of the induced representation is defined as

$$X_\sigma \uparrow_H^G (g) = \begin{pmatrix} Y(g_1^{-1}gg_1) & Y(g_1^{-1}gg_2) & \ldots & Y(g_1^{-1}gg_k) \\ Y(g_2^{-1}gg_1) & Y(g_2^{-1}gg_2) & \ldots & Y(g_2^{-1}gg_k) \\ & & \vdots & \\ Y(g_k^{-1}gg_1) & Y(g_k^{-1}gg_2) & \ldots & Y(g_k^{-1}gg_k) \end{pmatrix},$$

where $Y(g_i^{-1}gg_j)$ is the zero matrices if $g_i^{-1}gg_j \notin H$. Here, for example, if one considers the trivial representation of H, that is $1 : H \mapsto \mathbb{C}$ where $1(h) = 1\mathbb{C}$, then $1 \uparrow_H^G = \mathbb{C}[G] \bigotimes_{\mathbb{C}[H]} V = \mathbb{C}[G/H]$ and hence give *the left coset representation.*

Now we can look into irreducible representations of the symmetric group S_n. For a finite group, the number of irreducible representations is equal to the number of conjugacy classes. Thus for S_n, it is the number of partitions of n. We will map each partition of n to an irreducible representation of S_n.

To relate each partition to an irreducible representation of S_n formally, we follow the Specht module construction. We start by associating a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_i)$ with a subgroup of S_n. The corresponding subgroup is named *Young Subgroup* and defined as $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_i}$.

We can represent any Young subgroup with a Young tabloid which is an equivalence class of Young tableaus of shape λ and content $\{1, 2, \ldots, n\}$ whose rows have the same set of content. For partition $\lambda = (4, 2, 1)$, a Young tabloid is shown below.S_n has a natural action on Young tableaus: Given a tableau $t = (t_{ij})$ and $\pi \in S_n$, we have $\pi t = (\pi(t_{ij}))$. Clearly, this action induces an action of S_n on tabloids naturally.

$$\begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & & \\ \cline{1-2} 7 & & & \\ \cline{1-1} \end{array} = \left\{ \begin{array}{cccc} 1 & 4 & 2 & 3 \\ 6 & 5 & & \\ 7 & & & \end{array} , \begin{array}{cccc} 1 & 3 & 2 & 4 \\ 6 & 5 & & \\ 7 & & & \end{array} , \quad \cdots \quad \right\}$$

Figure 2.1. Young tabloid.

Now, for the Young subgroup S_λ, we consider left coset with transversals $\pi_1, \ldots, \pi_k$. Then the usual group action on these cosets gives that induced representation $1_{S_\lambda}^{S_n}$ is the module $\mathbb{C}\{\pi_1 S_\lambda, \ldots, \pi_k S_\lambda\}$ where the coset $\pi_i S_\lambda$ can be replaced by a tabloid denoted by $\{t_i\}$. Hence we obtain an isomorphic module $M^\lambda = \mathbb{C}\{\{t_1\}, \ldots, \{t_k\}\}$ which is called *permutation module of* λ. These induced representations are not guaranteed to be irreducible. On the other hand, a special linear combination of tabloids in M^λ gives an irreducible submodule S^λ, which is called the *Specht module* corresponding to the partition λ. Moreover, any permutation module M^λ has the decomposition into Specht modules

$$M^\lambda = \bigoplus_{\mu \unlhd \lambda} K_{\lambda\mu} S^\mu.$$

The coefficients of Specht modules S^μ in M^λ are given by the *Kostka number*, $K_{\lambda\mu}$ which is the number of semi-standard tableaux of shape λ and content μ.

Let μ and ν be partitions such that $|\mu| = k$ and $|\nu| = m$ where $k + m = n$. We are interested in the tensor product $S^\mu \bigotimes S^\nu$ which is an irreducible module of $S_k \times S_m$, but its induced representation $(S^\mu \bigotimes S^\nu) \uparrow^{S_n}$ does not need to be irreducible anymore. On the other hand, the resulting module can be decomposed into Specht modules associated with partitions of n as follows:

$$(S^\mu \bigotimes S^\nu) \uparrow^{S_n} = \bigoplus_{\lambda} c_{\mu\nu}^\lambda S^\lambda,$$

where the coeffcients $c^{\lambda}_{\mu\nu}$ is nothing but LR coefficients. Moreover the above equality can be transfered over the respective characters as follows:

$$\chi^{\mu}\chi^{\nu} = \sum_{\lambda} c^{\lambda}_{\mu\nu}\chi^{\lambda}.$$

2.2. Symmetric Functions

Here we explain how LR coefficients exist in the theory of symmetric functions. In the ring of formal power series $\mathbb{C}[[x_1, x_2, \ldots]]$ an element which is fixed by every permutation $\pi \in \bigcup_{n\geq 1} S_n$ is called a *symmetric function*. The subset of all symmetric functions is, in fact, a subring of $\mathbb{C}[[x_1, x_2, \ldots]]$ and denoted by Λ.

We say $m_{\lambda} = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_k}^{\lambda_k}$ is the monomial symmetric function corresponds to partition λ. As any symmetric function can be written as a linear combination of monomial symmetric functions, we can define the ring of symmetric functions as $\Lambda = \mathbb{C}\{m_{\lambda}\}_{\lambda}$. This is a graded ring and we have $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$, where Λ_n is generated by m_{λ} such that $\lambda \vdash n$.

We define n-th power sum symmetric function by $p_n = \sum_{i\geq 1} x_i^n$. Then for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, we associate power sum symmetric function indexed by λ by the rule: $p_{\lambda} = p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_l}$. Moreover the set $\{p_{\lambda} : \lambda \vdash n\}$ provides another basis for Λ_n. We remark that $\{p_{\lambda}\}_{\lambda}$ provides another basis for Λ and the rule

$$< p_{\lambda}, p_{\mu} >' = \delta_{\lambda\mu} k_{\lambda},$$

where k_{μ} is the size of the conjugacy class of S_n corresponding to $\mu \vdash n$, defines an inner product on Λ.

A special basis allows us to connect symmetric functions to the representation of the symmetric groups: *Schur functions*. There are many definitions of them. Here we provide a combinatorial definition for Shur functions: Given a partition $\lambda \vdash n$, the filling of its boxes with positive integers gives us a semi-standard tableau T of shape λ if the labels are nondecreasing along the rows and increasing along the columns.

Moreover any such T gives a monomial X^T where

$$T \longrightarrow X^T = x_{i_1}^{l_1}..x_{i_k}^{l_k}$$

and l_j is the number of occurrence of i_j in T. Hence the rule $s_\lambda := \sum_T X^T$ where the sum is over all semi-standard tableau of shape λ defines Schur function of $\lambda \vdash n$. As an example, consider $\lambda = (2,1)$. Then possible choices for T are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \ldots$$

and so $s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + 2x_1x_2x_3 + \ldots$ respectively.

The following theorem of Frobenius connects Schur's symmetric functions with power sum symmetric functions.

Theorem 2.1. Given $\lambda, \mu \vdash n$, we have

$$s_\lambda = \frac{1}{n!} \sum_{\mu} k_\mu \, \chi_\mu^\lambda \, p_\mu$$

where k_μ be the size of the conjugacy class of S_n corresponding to μ and χ_μ^λ be the character of Specht module S^λ evaluated on this conjugacy class.

As another outcome, this teorem shows that $\{s_\lambda\}_\lambda$ is infact an orthonormal basis for Λ since we have $< s_\lambda, s_\mu >' = \delta_{\lambda,\mu}$. As we will see now, the Frobenius theorem is the main tool that connects symmetric functions and representations of symmetric groups. Define the characteristic map $ch^n : R(S_n) \to \Lambda_n$ with the rule that

$$ch^n(\chi) = \frac{1}{n!} \sum_{|\mu|=n} k_\mu \, \chi_\mu \, p_\mu,$$

where k_μ be the size of the conjugacy class of S_n corresponding to $\mu \vdash n$ and χ_μ is the value of the class fucntion χ on the conjugacy class corresponding to μ. $R(S_n)$ and Λ_n have the same dimension which is the number of partition n, and moreover ch^n maps orthonormal basis $\{\chi^\lambda\}_\lambda$ of $R(S_n)$ to the orthonormal basis $\{s^\lambda\}_\lambda$ of Λ_n. Thus, it is an isomorphism of vector spaces; in fact, it is an isometry.

We want to demonstrate that the characteristic map is an algebra isomorphism. We have already seen Λ is a graded algebra with the usual product of functions. We can see $R = \bigoplus_n R(S_n)$ also a graded algebra with the product;

$$\chi \cdot \phi = (\chi \otimes \phi) \uparrow_{S_n \times S_m}^{S_{n+m}} .$$

In order to show the algebra isomorphism between them, recall that for two class functions $\chi, \psi : S_n \to \mathbb{C} \subset \Lambda^n$, we have the inner product

$$< \chi, \psi >= \frac{1}{|S_n|} \sum_{\pi \in S_n} \chi(\pi)\psi(\pi^{-1}).$$

In fact, this inner product can be extended over the vector space of all functions $f : S_n \mapsto \Lambda^n$ by the rule

$$< f, h >= \frac{1}{|S_n|} \sum_{\pi \in S_n} f(\pi)h(\pi^{-1}).$$

Moreover, the characteristic map can be represented in terms of this extended inner product: For any $\pi \in S_n$, let $\lambda(\pi)$ be the partition representing the conjugacy class of S_n that contains π, and let $p : S_n \to \Lambda_n$, such that $p(\pi) = p_{\lambda(\pi)}$, that is $p(\pi)$ is the power sum symmetric function given by $\lambda(\pi)$. Then

$$ch^n(\chi) = \frac{1}{n!} \sum_{\mu \vdash n} k_\mu \, \chi_\mu \, p_\mu = \frac{1}{n!} \sum_{\mu \vdash n} \sum_{\pi : \lambda(\pi) = \mu} \chi_\mu \, p_\mu = \frac{1}{n!} \sum_{\pi \in S_n} \chi(\pi)p(\pi) =< \chi, p >,$$

since $\lambda(\pi) = \lambda(\pi^{-1})$ in general.

Hence using the Frobenius reciprocity formula, we see that

$$\begin{aligned}
ch(\chi \cdot \psi) &=< \chi \cdot \psi, p > \\
&=< (\chi \otimes \psi) \uparrow_{S_n \times S_m}^{S_{n+m}}, p > \\
&=< (\chi \otimes \psi), p \downarrow_{S_n \times S_m}^{S_{n+m}} > \\
&= \frac{1}{(n+m)!} \sum_{\pi \in S_{n+m}} (\chi \otimes \psi)(\pi) p \downarrow_{S_n \times S_m}^{S_{n+m}} (\pi) \\
&= \frac{1}{n!m!} \sum_{\pi\sigma \in S_n \times S_m} (\chi \otimes \psi)(\pi\sigma) p(\pi\sigma) \\
&= \frac{1}{n!m!} \sum_{\pi \in S_n, \sigma \in S_m} \chi(\pi)\psi(\sigma)p(\pi)p(\sigma) \\
&= \frac{1}{n!} \sum_{\pi \in S_n} \chi(\pi)p(\pi) \frac{1}{m!} \sum_{\sigma \in S_m} \psi(\sigma)p(\sigma) \\
&= ch(\chi)ch(\psi).
\end{aligned}$$

Above, we demonstrated that $ch : R \to \Lambda$ is an isomorphism of algebras that also respect products. Thus it engenders another characterization of LR coefficients: As $\chi^\mu \chi^\nu = \sum_\lambda c^\lambda_{\mu\nu}\chi^\lambda$ true for the characters of two irreducible representation of the symmetric group, we have

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda,$$

and hence the coefficients, appearing in the decomposition of the product of two Schur functions into a sum of symmetric functions, are nothing but LR coefficients.

Schur functions are related to the representation of other groups than symmetric groups, such as general linear groups. The relation between representations of these groups is described by Schur-Weyl duality.

The content of this chapter can be summarized as how representation of the S_{m+n} can be decomposed into irreducible representations in the subgroup $S_m \times S_n$. A natural question is how this decomposition occurs for other groups and subgroups. When G is a compact group, and H is a closed subgroup, we can write any irreducible representation of G as a direct sum of irreducible representations of H. How this decomposition occurred is described by *branching rules.* When G is a semisimple Lie Group, these decompositions separate into four types (type A, type B, type C, and type D Coxeter groups) and some well-defined exceptions based on their root system. Under some restrictions, the branching rules of all of them can be described in terms of Littlewood-Richardson coefficients, which can be found in studies of Berenstein and Zelevinky. [7] However, the scope of this thesis would not permit us to discuss all the types. Thus we will focus on only the first type, type A, that contains representations of $GL_n(\mathbb{C})$ and symmetric groups, as presented in this section.

2.3. Intersection Theory of Schubert Varieties

In this section, we will show that LR coefficients play an important role in the theory of Schubert varieties of Grasmannians and thus has many applications in enumerative geometry. A comprehensive study on the subject can be found in Manival's work. [15]

2.3.1. Grassmannian Varieties

We will start by defining Grasmanianian varieties, investigate the matrix representation of these varieties with the help of Plücker coordinates and move on to the Schubert varieties.

Definition 2.2. The Grassmannian $G_{k,n}$ is the set of all k-dimensional linear subspaces of vector space $\mathbb{C}^n$.

Let $e_1, .., e_n$ be standard basis of $\mathbb{C}^n$. And let $V \in G_{k,n}$ be a k dimensional subspace such that $V =< v_1, ..., v_k >$. These basis vectors $v_1, ..., v_k$ are mapped onto $k \times n$ matrix whose rows are coordinates of $v_1, ..., v_k$ in standard basis. Then we can define a function from the space of full rank $k \times n$ matrices $M^*_{k\times n}$ to $G_{k,n}$.

This map is surjective since in the full-rank matrices, the rows are linearly independent, hence forming a basis for a k-dimensional vector space. However, it is not injective since the rows of such different matrices form a basis for the same k dimensional subspace. To overcome this inefficiency, we construct a polynomial function on the linear spaces V of $G_{k,n}$ that is invariant under a change of basis.

Given a basis $\alpha = \{v_1, ..., v_k\}$ of V, let $M_\alpha = (v_{ij})$ be the matrix which admits $\vec{v_i}$ as its i-th row. Then for any k-tuppes $J = (j_1, j_2, \ldots, j_n)$ choosen as a subsequence of $(1, 2, \ldots, n)$, we define P_J as the determinant of $k \times k$ submatrix of M_α associated to $J = (j_1, j_2, \ldots, j_n)$.

Then we define $P(M_\alpha)$ as a vector $\mathbb{C}^{\binom{n}{k}}$ by the formula

$$P(M_\alpha) = (P_{J_1}, \ldots, P_{J_{\binom{n}{k}}}).$$

Here observe that for any basis α and β of V we have $M_\beta = g \cdot M_\alpha$ for a suitable choice of an invertible matrix $g \in GL(k)$. Then, it is easy to see that for any k-tupple J, we have $P_J(M_\beta) = \det(g).P_J(M_\alpha)$. Hence $P(M_\beta) = \det(g)P(M_\alpha)$, and therefore $P(M_\beta)$ and $P(M_\alpha)$ represent the same point in the projective space $\mathbb{P}^{\binom{n}{k}-1}$. Thus we have the next definition.

Definition 2.3. Given $V \in G_{k,n}$ with basis α, we define Plücker coordinates of V as a point in $\mathbb{P}^{\binom{n}{k}-1}$ by the correspondence

$$P(V) := P(M_\alpha) := [P_{J_1} : \ldots : P_{J_{\binom{n}{k}}}] \in \mathbb{P}^{\binom{n}{k}-1}.$$

Plücker coordinates are free from the choice of basis, and the well-defined injection $P : G_{k,n} \mapsto \mathbb{P}^{\binom{n}{k}-1}$ is called the *Plücker embedding of* $G_{k,n}$.

Example 2.4. We will compute the Plücker coordinates of $V \in G_{2,4}$ where $V =< e_2 + 2e_3 + 3e_4, e_1 + e_3 >$. Then we have the basis matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

thus Plücker coordinates

$$\begin{aligned} P(V) &= [P_{12} : P_{13} : P_{14} : P_{23} : P_{24} : P_{34}] \\ &= [-1 : -2 : -3 : 1 : 0 : -3]. \end{aligned}$$

The next theorem states the embbedding $P : G_{k,n} \mapsto \mathbb{P}^{\binom{n}{k}-1}$ gives Grassmannian $G_{k,n}$ the structure of projective variety.

Theorem 2.5. Points $[... : P(V)_J : ...] \in \mathbb{P}^{\binom{n}{k}-1}$ that are the image of subspaces $V \in G_{k,n}$ under Plücker map are precisely the ones in the zero loci of following equations, which are called Plücker relations:

$$\sum_{\lambda=1}^{k+1} (-1)^{\lambda-1} p(j_1 ... j_{k-1} l_\lambda) p(l_1 ... \hat{l}_\lambda ... l_{k+1}) = 0$$

where $j_1 ... j_{k-1}$ and $l_1 ... l_{k+1}$ any sequences such that $1 \leq j_\alpha, l_\beta \leq n$ and $\hat{l}_\lambda$ means omitting l_λ from sequence.

Example 2.6. Consider $G_{2,4}$. So for $k = 2$, sets of sequences $j_1 ... j_{k-1}$ and $l_1 ... l_{k+1}$ in the theorem are of length 1 and 3, namely $\{(1), (2, 3, 4)\}$, $\{(2), (1, 3, 4)\}$, $\{(3), (1, 2, 4)\}$, $\{(4), (1, 2, 3)\}$. For the first set of sequences $\{(1), (2, 3, 4)\}$, it can be observed that we have the homogenous polynomial:

$$p(12)p(34) - p(13)p(24) + p(14)p(23) = 0.$$

The remaining sequences with the given formula also result in the same equation. Therefore Plücker relations, in this case, is a single polynomial equation. Hence, we can consider $G_{2,4}$ as a projective variety in $\mathbb{P}^{\binom{4}{2}-1}$ since it is expressed as the zeros of a polynomial equation.

2.3.2. Schubert Varieties

We will investigate a special set of subvarieties of Grassmannian varieties called Schubert varieties. For that, we decompose Grasmannianian into Schubert cells indexed by Young diagrams.

A *complete flag* $\mathcal{A}$ in $\mathbb{C}^n$ is a sequence of subspaces $A_1 \subset A_2 \subset \ldots \subset A_n$ with dimension of A_i is i. In this case, we can also use the notation $\mathcal{A} = (A_1, A_2, \ldots, A_n)$. Moreover if we choose A_i to be the space generated by first i standard bases such that $< e_1, e_2 > ... \subset < e_1, e_2, ..., e_n >= \mathbb{C}^n$ it is called a *standart flag.*

Let $V \in G_{k,n}$ and $\mathcal{A}$ be the complete flag as described above. We classify V based on dimensions of $V \cap A_i$. Dimension of these intersections is a sequence of non-decreasing integers that raise at most 1 in each step up to $dim(V) = k$. So every V has a unique sequence for a given flag, and we can decompose $G_{k,n}$ into disjoint subsets, which we call Schubert cells. Moreover, we can identify this sequence with a partition λ whose Young diagram fits inside the rectangle of size $k \times (n-k)$. So given a partition and complete flag we define Schubert cell as

$$\Omega_\lambda = \{V \in G_{k,n} : \dim(V \cap A_j) = i,\ n-k+i-\lambda_i \leq j \leq n-k+i-\lambda_{i+1},\ 0 \leq j \leq k\}.$$

Remark that $(n - k + i - \lambda_i)$ corresponds to dimension jump in the intersection with the flag. After fixing a basis, we can also represent Schubert cells as matrices in a certain form. Remember that we can represent a point $V \in G_{k,n}$ with a matrix whose rows span V in the given basis. Then we say $V \in \Omega_\lambda$ if it can be spanned by the row of a matrix whose entries in the $(n-k+i-\lambda_i)$-th column of the ith row is 1, and all other entries in these columns and at the right of these entries in the same row are 0. Schubert cells may also be defined directly by the sequence $J = (j_1, j_2, \ldots)$ where $j_i = n-k+i-\lambda_i$ and denoted as Ω_J. The sequence J is called the Schubert symbol or jump sequence.

Example 2.7. Let $V \in \Omega_{(5,2,1)} \subset G_{3,10}$ with the standard flag. Then V is the row span of the following matrix:

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{pmatrix},$$

where $* \in \mathbb{C}$.

The dimension of a Schubert's cell is equal to number $*$ in its matrix form as in the example, which can be seen as equal to $k(n-k)-|\lambda|$ in general. So we have the isomorphism $\Omega_\lambda = \mathbb{C}^{k(n-k)-|\lambda|}$.

We have the decomposition $G_{k,n} = \bigcup_\lambda \Omega_\lambda$ as mentioned before. In the following example, we give all the Schubert cells for a given Grassmannian.

Example 2.8. For $G_{2,4}$, we have the Schubert cells;

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix},$$
$$\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}, \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}.$$

For another equivalent characterization of Schubert cells, after fixing a basis $\{f_1, \ldots, f_n\}$ of $\mathbb{C}^n$, consider the point $V_\lambda =< f_{n-k+1-\lambda_1}, \ldots, f_{n-\lambda_k} >$ in $G_{k,n}$. Let B be the subspace of $GL(\mathbb{C}^n)$ that stabilize the given flag $\mathcal{A}$. Then Schubert cell Ω_λ is the orbit of V_λ under the action of B on the $G_{k,n}$ by left multiplication of matrices.

Remark that Schubert cells are not algebraic varieties. But we can define Schubert varieties as their closure in the Zariski topology such that $X_\lambda = \bar{\Omega_\lambda} = \bigcup_{\lambda \subset \mu} \Omega_\mu$. For a given complete flag $\mathcal{A}$, this definition is equivalent to

$$X_\lambda = \{V \in \mathbb{G}_{k,n} : \dim(V \cap A_{n-k+i-\lambda_i}) \geq i\}.$$

To observe this is indeed an algebraic variety, we turn back to Plücker embedding. Consider the linear spaces whose Plücker embedding is in the closure of the image of Schubert cell Ω_λ under plücker map in $\mathbb{P}^{\binom{n}{k}-1}$. We consider which minors are equal to zero on the set of matrices corresponding Ω_λ. For example, remember the $\Omega_{(5,2,1)}$, which is the space generated by the rows of matrix

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{pmatrix}.$$

We want to choose three columns, indexed by j_1, j_2, j_3, of a matrix in this form whose determinant is zero regardless of the values of $*$. Then for such a set $\{j_1, j_2, j_3\}$ it should be the case $4 \leq j_1 \leq 8,\ j_2 = 9,\ j_3 = 10$ or $j_1 = 3,\ 8 \leq j_2 \leq 9,\ j_3 = 10$ or $j_1 = 3,\ j_2 = 7,\ j_3 = 10$. Matrices that satisfy these conditions are of the forms

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 \end{pmatrix},$$

where the first $*$ next to a 0 in each row is non-zero. Then the spaces generated by the rows of these matrices constitute Schubert variety $X_{(5,2,1)}$.

2.3.3. Intersection Theory

Schubert varieties were first discovered to answer problems of enumerative geometry, which include questions such as how many lines would pass through given two points. However, even this elementary case can yield infinitely many answers if the two points coincide. Special cases, such as the one we mentioned, engender the definition of general positions. It can be said that if we randomly choose some subspaces with given conditions in an ambient space, they would be in general position with the probability of 1. For instance, two points are in general position if they do not coincide. If we go back to our question above, the answer to how many lines intersect them is 1 regardless of what these points are as long as they are in general position. Schubert asserts this idea rigorously as the principle of conservation of numbers. This approach leads to the solution of many enumerative geometric problems.

For an example, consider the question of how many lines intersect given four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in $\mathbb{R}^3$. We can consider this question as if the lines are in the projective space $\mathbb{P}^3$ since we assume them to be in general position. The set of all lines in $\mathbb{P}^3$ is given by projective Grassmannian $\mathbb{G}_{1,3}$. Now we go back to complex space and consider lines $\mathbb{P}^3$ as 2-dimensional subspaces of $\mathbb{C}^4$, i.e. we can assume $\mathbb{G}_{1,3} = G_{2,4}$. Recall the relation of Schubert cells and dimension jump with the intersection of the given flag in the previous chapter. Let $\mathcal{F} = (F_1, F_2, F_3, F_4)$ be a complete flag in $\mathbb{C}^4$. Then defining conditions for Schubert varieties in $G_{2,4}$ with respect to this flag obtained from the definition are

$$\begin{aligned}
X_{(1)}(\mathcal{F}) &= \{V \in G_{2,4} : dim(V \cap F_2) \geq 1\}, \\
X_{(1,1)}(\mathcal{F}) &= \{V \in G_{2,4} : V \subset F_3\}, \\
X_{(2)}(\mathcal{F}) &= \{V \in G_{2,4} : F_1 \subset V\}, \\
X_{(2,1)}(\mathcal{F}) &= \{V \in G_{2,4} : F_1 \subset V \subset F_3\}, \\
X_{(2,2)}(\mathcal{F}) &= \{V \in G_{2,4} : dim(F_1 \cap V) \geq 1; dim(F_2 \cap V) \geq 2\} = \{F_2\}.
\end{aligned}$$

We consider the line $\ell \in \mathbb{P}^3$, which actually represents the plane F_2 in the complete flag $\mathcal{F}$. Then $X_{(1)}(\mathcal{F})$ of $G_{2,4}$ actually represents the set of lines in $\mathbb{P}^3$ (equivalently planes in $\mathbb{C}^4$) that intersect the line $\ell \in \mathbb{P}^3$. So for given four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in $\mathbb{P}^3$ we can choose complete flags $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ in $\mathbb{C}^4$ so that $X_{(1)}(\mathcal{F}_i)$ represents the set of lines that intersect the line $\ell_i \in \mathbb{P}^3$. Then the intersection $X_{(1)}(\mathcal{F}_1) \cap X_{(1)}(\mathcal{F}_2) \cap X_{(1)}(\mathcal{F}_3) \cap X_{(1)}(\mathcal{F}_4)$ clearly gives us number of lines in $\mathbb{P}^3$ that intersect the given lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4.

To calculate such an intersection, the cohomology theory of Schubert varieties can be used, which takes us one step forward by letting us express these intersections as the multiplication in the cohomology ring and removes dependence on the flags. Schubert varieties of $G_{k,n}$ are indexed by the partitions which fit inside $k \times (n-k)$ rectangle, the cohomology ring $H^*(G_{k,n}; Z)$ admits the set of corresponding cohomology classes $[X_\lambda]$ as a basis. Moreover, the cup product $\smile$ within $H^*(G_{k,n}; Z)$ plays a vital role in the intersection theory of these Schubert varieties, namely;

$$[X_\lambda] \smile [X_\mu] = [X_\lambda(\mathcal{F}) \bigcap X_\mu(\mathcal{F}')] = \sum_{|\lambda|+|\mu|=|\nu|} c^\nu_{\lambda\mu}[X_\nu],$$

where $\mathcal{F}'$ is chosen as the opposite flag of $\mathcal{F}$ and $c^\nu_{\lambda\mu} = 0$ when $\{X_\nu \not\subset X_\lambda(\mathcal{F}) \bigcap X_\mu(\mathcal{F}')\}$. The convention is to denote cohomology class $[X_\lambda]$ with σ_λ which is called the Schubert cycle and $[X_\lambda] \smile [X_\mu]$ with $\sigma_\lambda\sigma_\mu$. Using this setup, the equality

$$\sigma_\lambda\sigma_\mu = \sum c^\nu_{\lambda\mu}\sigma_\nu$$

is obtained and it is known that the coefficients $c^\nu_{\lambda\mu}$ are nothing but LR coefficients. Hence in the terminology of Schur symmetric functions the number $c^\nu_{\lambda\mu}$ in $s_\lambda s_\mu = \sum c^\nu_{\lambda\mu}s_\nu$ gives us number of appearances of Shubert variety X_μ in the intersection $X_\lambda(\mathcal{F}) \bigcap X_\mu(\mathcal{F}')$.

To apply this setup in our question $X_{(1)}(\mathcal{F}_1) \cap X_{(1)}(\mathcal{F}_2) \cap X_{(1)}(\mathcal{F}_3) \cap X_{(1)}(\mathcal{F}_4)$, first recall that Schubert varieties of $G_{2,4}$ are indexed by the partitions which fit inside 2×2 rectangle. Hence the Shubert varieties in $X_{(1)}(\mathcal{F}_1) \cap X_{(1)}(\mathcal{F}_2) \cap X_{(1)}(\mathcal{F}_3) \cap X_{(1)}(\mathcal{F}_4)$ are indexed by such partitions. On the other hand, to find these partitions, all we need to write $s_{(1)}s_{(1)}s_{(1)}s_{(1)}$ as a sum of Schur functions and among them, we need to take the ones which are indexed by partitions that fit inside 2×2 rectangle.

For this aim, we can use Monk's rule on Schur symmetric functions tableau, which says that given s_λ,

$$s_\lambda s_{(1)} = \sum s_{\lambda^*},$$

where the partition λ^* is obtained from λ by adding only a single box. For example for $\lambda = (2,1)$ we have

$$s_{(2,1)}s_{(1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}.$$

Iterating the Monk's rule on $s_{(1)}s_{(1)}s_{(1)}s_{(1)}$ we obtained that

$$s_{(1)}s_{(1)}s_{(1)}s_{(1)} = s_{(4)} + 3s_{(3,1)} + 2s_{(2,2)} + 3s_{(2,1,1)} + s_{(1,1,1,1)}$$

but only the partition $s_{(2,2)}$ fits 2×2 rectangle. Hence the intersection $X_{(1)}(\mathcal{F}_1) \cap X_{(1)}(\mathcal{F}_2) \cap X_{(1)}(\mathcal{F}_3) \cap X_{(1)}(\mathcal{F}_4)$ consist of only $X_{(2,2)}$ with multiplicit 2. As $X_{(2,2)}$ consists of only a singleton point of $G_{2,4}$, we can conclude that there are only two lines intersecting all of the given four lines in general position.

3. COMPUTATION OF LITTLEWOOD-RICHARDSON COEFFICIENT

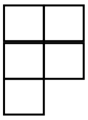
In this chapter, different combinatorial interpretations of LR coefficients are introduced, and how they relate to each other is explained. Each of these combinatorial tools has a distinct visual representation that highlights different characteristics of the LR coefficient. Although they are introduced and proved in different contexts, some of which were mentioned in the previous chapter, our aim here is to show their equivalence with linear bijections, following Pak and Vallejo's study [16]. The classical combinatorial method for computing LR coefficients is specific labeling of skewed shape Young tableaux that are known as Littlewood-Richardson tableaux.

Berenstein and Zelevinsky change how LR coefficients are computed in a fundamental way with the tableau approach. With the tableau approach, distinct objects labeled with integers that satisfy certain conditions are counted. Berenstein and Zelevinsky introduce a set of inequalities that define a cone whose integer points give LR coefficients. Their construction prompts generalized LR coefficients that describe tensor product multiplicity of any complex semisimple Lie algebra. Moreover, this leap enables the integration of analytical tools to study LR coefficients. The other constructions, Hive and Honeycombs, also follow this approach. Therefore, we can describe all of them as cones in real vector space, and it is meaningful to talk about linear maps between them. LR tableaux constitute the only exception to this approach. However, in section 3.2, we will introduce an intermediary object, LR triangles, that has an intuitive connection with tableau and is defined by inequalities like other constructions.

3.1. Littlewood-Richardson Tableaux

A *partition* of a positive integer n is a finite non-increasing sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ of positive integers such that $\sum \lambda_i = n$. We can picture a partition λ with a diagram by placing λ_i many boxes for $1 \leq i \leq k$ from top to bottom as below:

$$\lambda = (2,2,1) =$$



.

Let $\mu = (\mu_1, \dots, \mu_k)$ be another partition where $l \in \mathbb{Z}^+$. If $\mu_i \leq \lambda_i$ for all $1 \leq i \leq k$, we can define a skew-diagram λ/μ by removing boxes of μ from the diagram of λ.

Labeling the diagram λ/μ with positive integers is called *Young tableau* of shape λ/μ. Moreover, if a tableau T has labels weakly increasing in its rows from left to right and strictly increasing in its columns from top to bottom, we say T is a semi-standard tableau of the shape λ/μ.

A word $w = w_1 w_2 \dots w_m$ consisting of positive integers is called a *lattice word* if in each subword $w_1 w_2 \dots w_i$, the number of occurrences of w_i is not longer than w_{i-1}.

The *row word* of a semi-standard tableau T is obtained by reading the labels of T from right to left in each row, starting from the top row and continuing to the bottom, successively. If the row word of a semi-standard tableau T is a lattice word, we call T a Littlewood-Richardson(LR) tableau.

The *content* of a Young tableau T is a sequence $(\nu_1, \nu_2, \dots, \nu_k)$ where ν_i is the number of occurrence of label i in T. If T is an LR tableau, then content ν is a partition since the row word of T is a lattice word.

Theorem 3.1. Number of LR tableaux with shape λ/μ and content ν gives LR coefficient $c^{\lambda}_{\mu,\nu}$.

Various proofs for this theorem are proposed in the literature but discussing them would exceed the scope of this study. For further examinations regarding some of the said proofs [5, 6] can be consulted.

3.2. Littlewood-Richardson Triangles

For a given $n \in \mathbb{N}$ we define a triangle in $\mathbb{R}^3$:

$$\{(x, y, z) \in \mathbb{R}^3 : x + y + z = n, \quad x, y, z \geq 0\},$$

and we divide the triangle into sub-triangles whose corners are either $[(x+1, y, z),\ (x, y+1, z),\ (x, y, z+1)]$ or $[(x-1, y, z),\ (x, y-1, z),\ (x, y, z-1)]$, whenever the corners are in the triangle. The resulting triangular grid consists of $\binom{k+2}{2}$ many vertices with labels $t_{i,j}$, where $1 \leq i \leq j \leq n$ as shown in figure 3.1. We denote this grid with tri_n.

$$\begin{array}{ccccccc}
 & & & t_{0,0} & & & \\
 & & t_{0,1} & & t_{1,1} & & \\
 & t_{0,2} & & t_{1,2} & & t_{2,2} & \\
t_{0,3} & & t_{1,3} & & t_{2,3} & & t_{3,3}
\end{array}$$

Figure 3.1. tri_n.

Labels of tri_n; $t_{i,j}$ will be real numbers where $t_{0,0} = 0$ is fixed. We denote the vector space of these labels with T_n, which has dimension $\binom{n+2}{2} - 1$.

We define *Littlewood-Richardson triangles* for given n as a tri_n whose labels satisfy conditions:

$$t_{i,j} \geq 0 \text{ for all } 1 \leq i < j \leq n, \tag{3.1}$$

$$\sum_{p=0}^{i-1} t_{p,j} \geq \sum_{p=0}^{i} t_{p,j+1} \text{ for all } 1 \leq i \leq j < n, \tag{3.2}$$

$$\sum_{q=i}^{j} t_{i,q} \geq \sum_{q=i+1}^{j+1} t_{i+1,q} \text{ for all } 1 \leq i \leq j < n. \tag{3.3}$$

The set of real numbers that satisfies these conditions is Littlewood Richardson Cone and will be denoted as LR_n.

Let λ, μ and ν be three partitions with length n. We say an LR triangle is of type (λ, μ, ν) and denote as $LR_n(\lambda, \mu, \nu)$ if it satisfies conditions;

$$\mu_j = t_{0,j}, \qquad \lambda_j = \sum_{p=0}^{j} t_{p,j}, \qquad \nu_i = \sum_{q=i}^{n} t_{i,q}$$

which are also shown in the figure below.

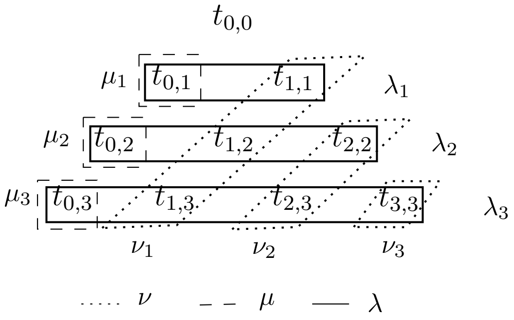


Figure 3.2. LR triangle of type (λ, μ, ν).

Then LR coefficient $c^{\lambda}_{\mu,\nu}$ can be computed in two equivalent ways with the help of LR triangles, as we mentioned previously:

- Number of different $LR_n(\lambda, \mu, \nu)$ triangles with integer coefficients,
- Number of integer points contained in convex cone created by the subspace of vector space T consist of real-valued labels satisfying the above conditions.

This useful equivalence allows us to relate LR tableaux to the concept of integer points in a certain polyhedron which are also relied on by the other methods, such as hives, honeycombs, and Berenstein-Zelevinsky cones. We refer to these concepts interchangeably throughout this thesis.

Next, we define the map between LR tableaux and LR triangles and prove that it is a bijection.

Theorem 3.2. Let τ be a function from LR tableaux with shape λ/μ and content ν to $LR_n(\lambda, \mu, \nu)$ with integer labels as defined below:

- $t_{0,j} = \mu_j$ for $1 \leq j \leq n$,
- $t_{i,j}$ is number of label i in j^{th} row of λ/μ.

Then τ is a bijection.

We verify the fact that τ is a bijection in a short example below.

Example 3.3. Let $n = 4$, and we denote the number i in j^{th} column as $t_{i,j}$. In terms of this notation, inequalities 3.2, 3.3 can be written explicitly, as demonstrated below.

Table 3.1. Second set of inequalities for tri_3.

	$i = 1$	$i = 2$	$i = 3$
$j = 1$	$\mu_1 \geq \mu_2 + t_{1,2}$	-	-
$j = 2$	$\mu_2 \geq \mu_3 + t_{1,3}$	$\mu_2 + t_{1,2} \geq \mu_3 + t_{1,3} + t_{2,3}$	-
$j = 3$	$\mu_3 \geq \mu_4 + t_{1,4}$	$\mu_3 + t_{1,3} \geq \mu_4 + t_{1,4} + t_{2,4}$	$\mu_3 + t_{1,3} + t_{2,3} \geq \mu_4 + t_{1,4} + t_{2,4} + t_{3,4}$

For instance, inequality for $i = 1$ and $j = 3$ states that the number of cells labeled with 1 in the fourth row of λ/μ is less than the number of cells in the same row, which are the first cells in their columns. And the inequality for $i = 2$ and $j = 3$ considered together with this inequality states the number of cells labeled with 2 in the fourth row of λ/μ is less than cells labeled with 1 in the third row. It can be seen these inequalities together are equivalent to T having strictly increasing labels in its columns. The next set of inequalities is shown below.

Table 3.2. Third set of inequalities for tri_3.

	$i = 1$	$i = 2$	$i = 3$
$j = 1$	$t_{1,1} \geq t_{2,2}$	-	-
$j = 2$	$t_{1,1} + t_{1,2} \geq t_{2,2} + t_{2,3}$	$t_{2,2} \geq t_{3,3}$	-
$j = 3$	$t_{1,1} + t_{1,2} + t_{1,3} \geq t_{2,2} + t_{2,3} + t_{2,4}$	$t_{2,2} + t_{2,3} \geq t_{3,3} + t_{3,4}$	$t_{3,3} \geq t_{4,4}$

Since lattice word counts labels from the left in each row of T, and rows are non-decreasing, this set of inequalities corresponds to the ballot condition on tableau T.

Proof. In light of the previous example, it is easy to see for any n, inequalities 3.2 and 3.3 on LR triangles correspond to increasing columns and ballot conditions on LR tableau. The only remaining condition we should check is that when a given triangle maps to a tableau, its rows are non-decreasing. When we choose $i = j$ in the inequality 3.2, we obtain

$$\sum_{p=0}^{j-1} t_{p,j} \geq \sum_{p=0}^{j} t_{p,j+1} \text{ for all } 1 \leq j < n.$$

That completes the proof.

3.3. Hives

We define a *hive* H_n on tri_n similar to LR triangles. This time we label the vertices with $(h_{i,j})_{0 \leq i \leq j \leq n}$. Then a hive is tri_n such that on each rhombus composed of two unit triangles, the sum of the corners on the short diagonal is greater than or equal to the sum of the corners on the long diagonal.

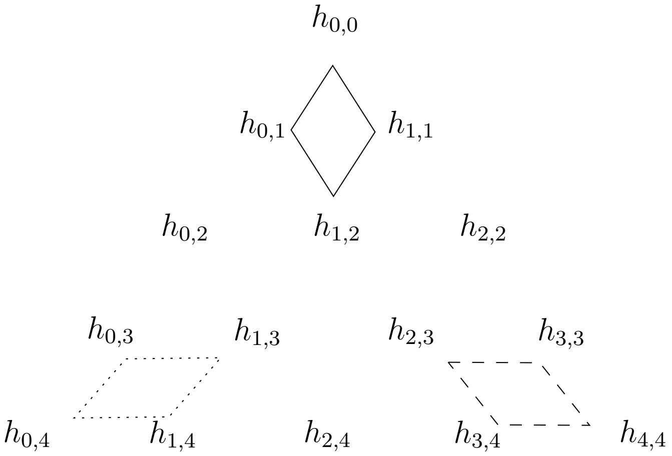


Figure 3.3. Hive H_4.

Since there is three unit rhombus based on orientation, as shown in the below figure, we can explicitly write those inequalities as

$$h_{i,j} - h_{i,j-1} \geq h_{i-1,j} - h_{i-1,j-1} \text{ for } 1 \leq i < j \leq n, \tag{3.4}$$

$$h_{i-1,j} - h_{i-1,j-1} \geq h_{i,j+1} - h_{i,j} \text{ for } 1 \leq i \leq j \leq n, \tag{3.5}$$

$$h_{i,j} - h_{i-1,j} \geq h_{i+1,j+1} - h_{i,j+1} \text{ for } 1 \leq i \leq j \leq n. \tag{3.6}$$

Let λ, μ, ν be partitions with length n such that $|\lambda| = |\mu| + |\nu|$. The type of a given hive can be determined by the entries on the boundaries alone. The difference between consecutive entries on each side gives us a partition in the type. We say a hive H_n is of type (λ, μ, ν) if the conditions

$$\mu_j = h_{0,j} - h_{0,j-1}, \text{ for } 1 \leq j \leq n, \tag{3.7}$$

$$\lambda_j = h_{j,j} - h_{j-1,j-1}, \text{ for } 1 \leq j \leq n, \tag{3.8}$$

$$\nu_i = h_{i,k} - h_{i-1,k}, \text{ for } 1 \leq i \leq n \tag{3.9}$$

are satisfied. Next we define the map $\Phi_n \colon LR_n \to H_n$ between LR triangles and hive triangles with the same size n as

$$h_{i,j} = \Phi_n(t_{i,j}) = \sum_{p=0}^{i} \sum_{q=p}^{j} t_{p,q}.$$

The map Φ_n can be visualized on the LR triangles nicely. For instance, in figure 3.4, $h_{1,2} = \Phi_3(t_{1,2})$ is equal to some of the labels on the dashed arrow in LR_4. The value of other entries of H_n can be similarly obtained from the LR triangles.

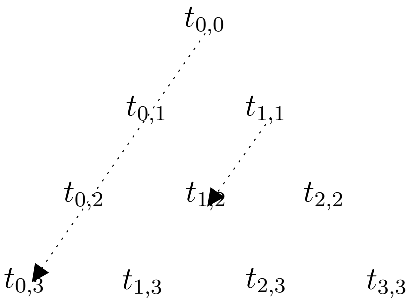


Figure 3.4. $\Phi_3(t_{12})$.

Theorem 3.4. Φ_n is a bijection. Then $c^{\nu}_{\lambda,\mu}$ is equal to number of hives $H_n(\lambda,\mu,\nu)$ with integer coefficients or integer points in the convex polytope $H_n(\lambda,\mu,\nu)$.

Before we prove the theorem, we look at the relationship between conditions that define LR tableaux, LR triangles, and hives with an example.

Example 3.5. Let $\lambda = (5,5,4,3)$, $\mu = (3,2,2,0)$ and $\nu = (4,3,2,1)$. Littlewood-Richardson tableaux with shape λ/μ with the content ν are given in the figure below alongside hives, and LR triangles of type (λ,μ,ν) correspond to these tableaux.

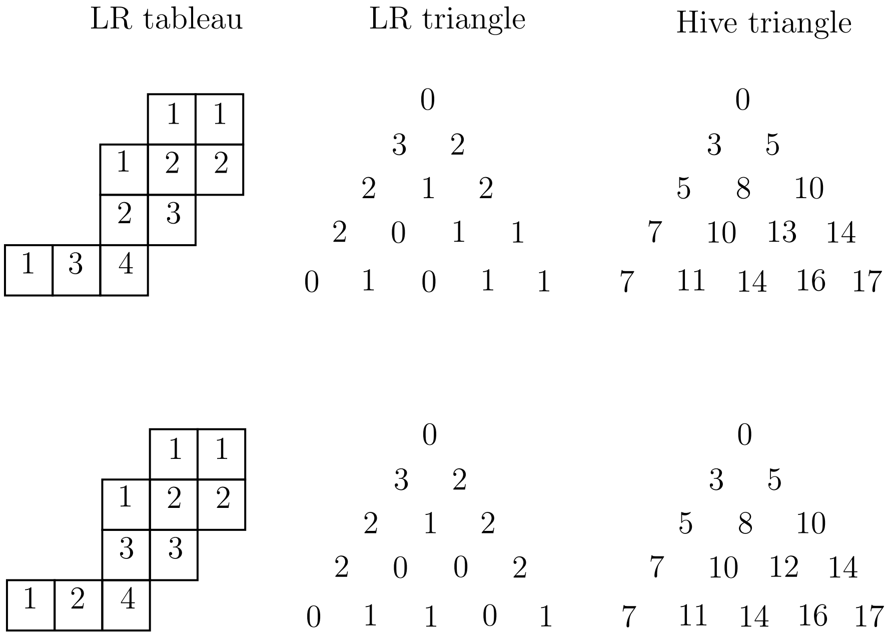


Figure 3.5. LR triangles and the hives correspond to the same LR tableaux.

We remark that inequalities 3.2, 3.3 for LR triangles, and 3.5, 3.6 for hive triangles correspond to strictly increasing columns and ballot conditions in LR tableaux, respectively. In the figures below, we show a configuration that fails to satisfy increasing columns or ballot conditions in the LR tableau and also does not satisfy given inequalities for LR and hive triangle.

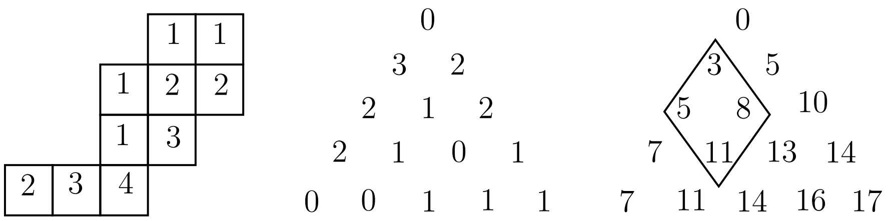


Figure 3.6. LR triangle and the hive correspond to the tableau with a non-increasing column.

In the first figure, $t_{1,1} \ngeq t_{2,2}$ in the LR triangle, and the sum of the labels corresponds to obtuse angles less than the sum of the labels at acute angles for specified rhombus in the hive triangle.

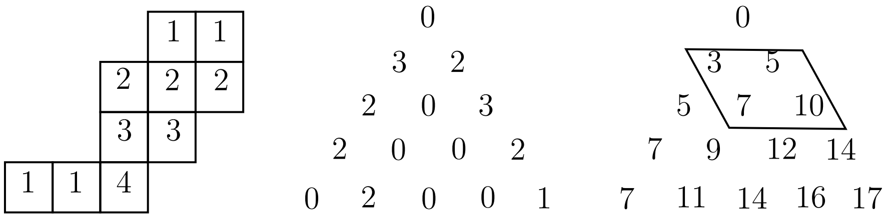


Figure 3.7. LR triangle and the hive correspond to the tableau that does not satisfy ballot condition.

In the second figure, $t_{0,2} \ngeq t_{0,3}+t_{1,3}$ in the LR triangle, and inequality corresponds to rhombus in the figure does not hold in the hive triangle.

Proof. We will consider Φ as a linear map from the vector space of LR triangles to the vector space of hives with the same dimension. Then we will show it is onto and preserves volumes. Consequently, two cones contain the same number of integer points as desired.

First, we will consider both spaces as subspaces of the triangular array T with real labels $t_{i,j}$, where $0 \leq i \leq j \leq n$ and $t_{0,0} = 0$, without further restrictions. Then we can find a base for T whose elements are square matrices of size $(n+1) \times (n+1)$. For every label $t_{i,j}$, we define matrix E_{ij} whose only non-zero entry is 1 in the $(i+1)-$th row of $(j+1)-$th column. For example, a base for a triangular array T_3 of size $n=3$ is $\mathbb{B}=\{E_{01},E_{02},E_{03},E_{11},E_{12},E_{13},E_{22},E_{23},E_{33}\}$. Then we can express Φ_3 in this base with the matrix

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&0\\
1&1&0&1&1&0&0&0&0\\
1&1&1&1&1&1&0&0&0\\
1&1&0&1&1&0&1&0&0\\
1&1&1&1&1&1&1&1&0\\
1&1&1&1&1&1&1&1&1
\end{pmatrix}.$$

As seen in the example above, it is a triangular matrix with 1's on the diagonal. Since the transformation matrix of Φ_n has a non-zero determinant, it is a bijection. Moreover, since the determinant is equal to 1, it is a volume-preserving map. If we substitute $\Phi_n(t_{i,j})$ instead of $h_{i,j}$ in inequalities 3.4, 3.5 and 3.6, we obtain the exact inequalities 3.1, 3.2 and 3.3 respectively. The same can be done with inequalities 3.7,3.8, and 3.9 to validate that Φ_n preserves the types.

Since Φ_n is a bijection, its inverse also exists and is defined as

$$t_{i,j}=\Phi_n^{-1}(h_{i,j})=\begin{cases} h_{0,j}-h_{0,j-1}, & \text{if } i=0 \text{ and } 1\le j\le n,\\ h_{j,j}-h_{j-1,j}, & \text{if } 1\le i=j\le n,\\ h_{i,j}-h_{i,j-1}-h_{i-1,j}+h_{i-1,j-1}, & \text{if } 1\le i<j\le n. \end{cases} \tag{3.10}$$

It can be seen that the inequalities that define hives and the LR triangles are equivalent with Φ_n and its inverse. Moreover, equalities define the type of hives, and the LR triangles are also mapped to each other.

3.4. Berenstein-Zelevinsky Triangle

We constructed a new graph from the triangle grid that we used to define LR triangles and hives. We take a triangular grid size $n+1$. This time, we take the middle points of all the edges not on the boundary as vertices and add edges as shown in the Figure 3.8. Let's call this graph Δ_n.

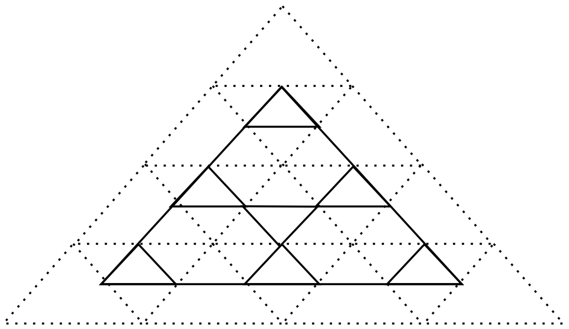


Figure 3.8. Brenstein-Zelevinsky triangle.

We label the vertices of Δ_n with $x_{i,j},\ y_{i,j},\ z_{i,j}$ where $1 \leq i \leq j \leq n$ such that i-th unit triangle from the left whose base is at the j-th row has labels $x_{i,j},\ y_{i,j},\ z_{i,j}$ on its corners as shown in the Figure 3.9.

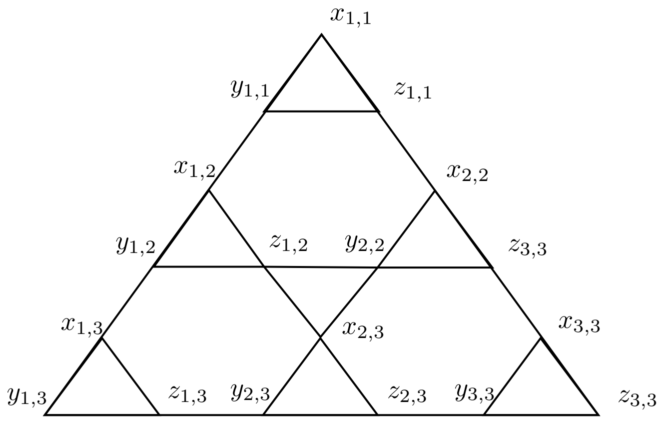


Figure 3.9. BZ_3.

Let V_n be the vector space consisting of real-valued labels of Δ_n similar to T_n is the real-valued labeling of tri_n. Consider the labelings of Δ_n such that on each hexagon, the sum of labels on the parallel edges is equal, i.e., for all $1 \leq i \leq j < n$;

$$y_{i,j} + z_{i,j} = y_{i+1,j+1} + z_{i,j+1}, \tag{3.11}$$

$$x_{i,j+1} + y_{i,j} = x_{i+1,j+1} + y_{i+1,j+1}, \tag{3.12}$$

$$x_{i+1,j+1} + z_{i,j} = x_{i,j+1} + z_{i,j+1}. \tag{3.13}$$

We denote the subspace containing all the labels satisfying the above equalities as $W_n \subset V_n$. Finally, we define a *Berenstein-Zelevinsky triangle* as the labeling of Δ_n with positive real numbers in W_n. We denote the cone of all BZ triangles with size n with BZ_n.

The type of a Berenstein-Zelevinsky triangle is determined by the entries on the boundary edges of Δ_n, similar to hive triangles and tri_n. We say a Berenstein-Zelevinsky triangle B is of type (λ, μ, ν) if equations

$$x_{1,j} + y_{1,j} = \mu_j - \mu_{j+1} \text{ for } 1 \leq j \leq n, \tag{3.14}$$

$$x_{j,j} + z_{j,j} = \lambda_j - \lambda_{j+1} \text{ for } 1 \leq j \leq n, \tag{3.15}$$

$$y_{i,n} + z_{i,n} = \nu_i - \nu_{i+1} \text{ for } 1 \leq i \leq n \tag{3.16}$$

are satisfied. Note that given a Berenstein-Zelevinsky triangle, its type cannot be decided uniquely, unlike LR and hive triangles, without further restriction on the partition. Because for given partitions, we only know the difference between the components of the partition. But given partitions λ, μ, ν, we can calculate the LR coefficient $c^{\lambda}_{\mu,\nu}$ by the number of Berenstein-Zelevinsky triangles in $BZ_n(\lambda, \mu, \nu)$ with integer coefficients.

Remember that T_n is the space of labels on the grid tri_n such that $t_{0,0} = 0$. We define a linear map $\Psi_n : T_n \to W_{n-1}$ on T_n such that $\Psi_n(h_{i,j}) = (x_{i,j}, y_{i,j}, z_{i,j})$ where

$$x_{i,j} = h_{i,j} + h_{i-1,j} - h_{i-1,j-1} - h_{i,j+1}, \tag{3.17}$$

$$y_{i,j} = h_{i-1,j} + h_{i,j+1} - h_{i,j} - h_{i-1,j+1}, \tag{3.18}$$

$$z_{i,j} = h_{i,j} + h_{i,j+1} - h_{i-1,j} - h_{i+1,j+1}. \tag{3.19}$$

Remark that Ψ_n maps the difference between the two sides of the inequalities that define hive triangles. In the previous section, we showed that the linear map Φ_n maps inequalities define hives and LR triangles to each other. So the composition $\Psi_n \circ \Phi_n : T_n \to W_{n-1}$ is obtained by mapping the differences in the inequalities 3.1, 3.2, 3.3 to W_{n-1} such that

$$x_{i,j} = \sum_{p=0}^{i-1} t_{p,j} - \sum_{p=0}^{i} t_{p,j+1}, , \tag{3.20}$$

$$y_{i,j} = t_{i,j+1}, \tag{3.21}$$

$$z_{i,j} = \sum_{q=i}^{j} t_{i,q} - \sum_{q=i+1}^{j+1} t_{i+1,q}. \tag{3.22}$$

Next, we will establish the equivalence of the BZ triangles with LR triangles by demonstrating that $\Psi_n \circ \Phi_n$ maps $LR_n(\lambda, \mu, \nu)$ bijectively onto $BZ_{n-1}(\lambda, \mu, \nu)$. Before we state the main theorem, we first need to determine the relationship between the dimensions of vector spaces with the next lemma.

Lemma 3.6. Dimension of W_n equals to $dim(T_{n+1}) - 2$.

Proof. Equalities 3.12 and 3.13 form a system of linear equations that consist of $2\binom{n}{2}$ many equations in $3\binom{n+1}{2}$ variable. Because there are $\binom{n}{2}$ many choices of i, j smaller than n for two sets of equations, and $\binom{n+1}{2}$ many unit triangles with 3 variable attached to them. When we arrange those variables as $x_{11},\ y_{11},\ z_{11},\ x_{12},\ y_{12},\ z_{12},\ x_{22},\ y_{22},\ z_{22},\ x_{21},\ y_{21},\ z_{21},\ \ldots$ and arrange equalities in proper order, we obtain a coefficient matrix in row echelon form whose rank is $2\binom{n}{2}$. So BZ_n has dimension $3\binom{n+1}{2} - 2\binom{n}{2}$.

For example, the system of linear equation for $n = 3$ with labels as in Figure 3.9 is given by the equation

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{1,1} \\ y_{1,1} \\ z_{1,1} \\ \vdots \\ x_{3,3} \\ y_{3,3} \\ z_{3,3} \end{pmatrix} = 0.$$

Lemma 3.7. The linear operator $\Phi_n \circ \Psi_n : T_n \to W_{n-1}$ is surjective.

Proof. We can rearrange the equalities 3.20, 3.21, 3.22 to obtain $(t_{i,j})$ such that

$$\begin{aligned} t_{0,j} &= p + \sum_{k=j}^{n-1} x_{1,k} + y_{1,k} \text{ for } 1 \leq j < n, \text{ and } t_{0,n} = p, \\ t_{i,j} &= y_{i,j-1}, \text{ for } 1 \leq i < j \leq n, \qquad (3.23) \\ t_{j,j} &= r + \sum_{k=j}^{n-1} z_{k,k} \text{ for } 1 \leq j < n, \text{ and } t_{n,n} = r. \end{aligned}$$

Hence for any $B = (x_{i,j}, y_{i,j}, z_{i,j}) \in W_{n_1}$ we have a labeling $T = (t_{i,j}) \in T_n$ as required for any choice of real numbers s and t. Moreover, equation 3.23 describes the set of all labelings in the preimage $(\Phi_n \circ \Psi_n)^{-1}(B)$ by Lemma 3.7.

Now we can prove the main result of this section.

Theorem 3.8. $\Phi_n \circ \Psi_n$ maps LR_n to BZ_{n-1} surjectively. Moreover it maps $LR_n(\lambda, \mu, \nu)$ bijectively to $BZ_{n-1}(\lambda, \mu, \nu)$.

Remember that we have defined the cone BZ_{n-1} as the subspace of $B \in W_n$ with positive labelings. And by equalities 3.20, 3.21, 3.22; $B = (\Phi_n \circ \Psi_n)(T)$ is positive only if the inequalities 3.1, 3.2, 3.3 are true for $T \in T_n$ i.e. $T \in LR_n$. This concludes the first part of the proof. For the second part, notice that $(\Phi_n \circ \Psi_n)$ map an LRT $L \in LR_n$ whose type (λ, μ, ν) is given by equalities 3.2 to $B \in BZ_{n-1}$ with the same type that satisfies the equalities 3.15, 3.14, 3.16 for the same partitions λ, μ, ν. But since $(\Phi_n \circ \Psi_n)$ is not injective, we need to check that LR triangles with different types are not mapped to the BZ triangles with the same type. But elements in the preimage of an BZT $B \in BZ_{n-1}(\lambda, \mu, \nu)$ under $(\Phi_n \circ \Psi_n)$ have different types by equalities 3.23.

Since we have established an isomorphism between cones LR_n and $BZ_{n-1} \times \mathbb{R}^2$, and define an bijective linear map from $LR_n(\lambda, \mu, \nu)$ to $BZ_{n-1}(\lambda, \mu, \nu)$, we can define an embedding of BZ_{n-1} into LR_n and the inverse to the map $(\Phi_n \circ \Psi_n) : LR_n(\lambda, \mu, \nu) \rightarrow BZ_{n-1}(\lambda, \mu, \nu$. We have almost defined it in the equation 3.23. If we choose $t_{0,n} = t_{n,n} = 0$ to define the map $\Omega_n(x_{i,j}, y_{i,j}, z_{i,j}) = (t_{i,j})$ such that

$$\begin{aligned} t_{0,j} &= \sum_{k=j}^{n-1} x_{1,k} + y_{1,k}, \\ t_{i,j} &= y_{i,j-1}, \\ t_{j,j} &= \sum_{k=j}^{n-1} z_{k,k}, \end{aligned} \tag{3.24}$$

we have the desired inverse for any set partitions $|\lambda| = |\mu| + |\nu|$ such that $\mu_k = \nu_k = 0$.

3.5. Honeycombs

Honeycombs were first introduced by Allen Knutson and Terence Tao in 1999 in order to prove saturation conjecture, in their article "The Honeycomb Model of $GL_n(\mathbb{C})$ Tensor Product I: Proof of the Saturation Conjecture". In this article, in which an introduction to hive methods is also provided, they show a correspondence between Berenstein Zelevinsky cones and honeycombs.

The innovation brought by the honeycomb approach is its ability to reflect not just the relationship between variables of inequalities but also inequalities themselves. In the honeycomb diagram, edges not only show the relations between labels but also edges lengths demonstrate the inequalities intrinsically.

In this chapter, we define the honeycomb model in a way that reflects its usage in this study. After that, we will move on to a demonstration of the bijection between honeycombs and hives with dual graphs.

We define honeycombs on the plane $\mathbb{R}^3_{\sum=0} := \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. We will be interested in the triangular lattice $\mathbb{Z}^3_{\sum=0}$ contained in this plane, where each integral point is related to others on a triangle. We construct a directed graph on this triangular lattice. Let Γ be a infinite directed graph such that its vertices are $V_\Gamma = \{(i, j, k) \in \mathbb{Z}^3_{\sum=0} : 3 \nmid 2i + j\}$. Edges of Γ are determined by vertices as follows; for every vertex $v = (i, j, k)$ in V_Γ such that $2i + j \equiv 2 \mod 3$ we place three edges with directions $(0, -1, 1), (1, 0, -1), (-1, 1, 0)$ whose tails are v and whose heads are respectively $(i, j - 1, k + 1), (i + 1, j, k - 1), (i - 1, j + 1, k)$. Then a vertex $w = (k, l, m)$ such that $2k + l \equiv 1 \mod 3$ has to be the head of three edges. This graph is called *infinite honeycomb tinkertoy.*

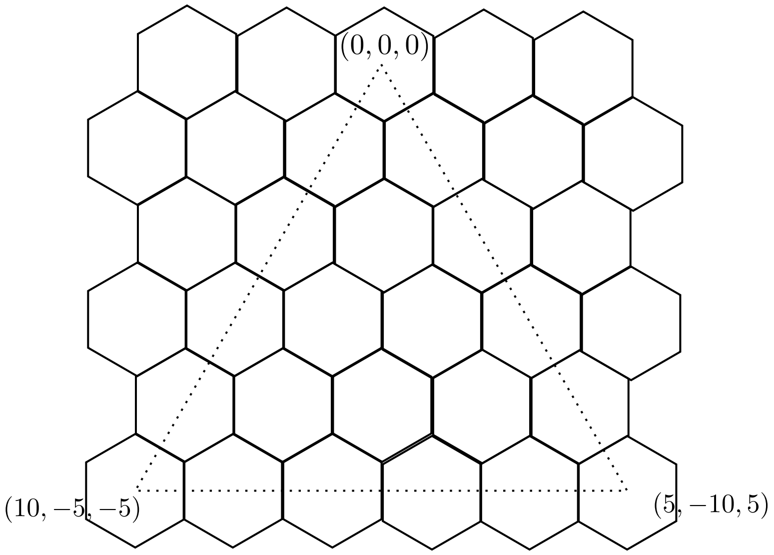


Figure 3.10. Infinite honeycomb tinkertoy.

Now consider the subsection of infinite honeycomb tinkertoy consisting of vertices $V_\Delta = \{(i, j, k) \in V_\Gamma : j + 3n \geq i \geq k \geq j\}$ and all the vertices in E_Γ whose tail or head is in V_Δ. We will call this subtinkertoy GL_n honeycomb tinkertoy and denote it as τ_n.

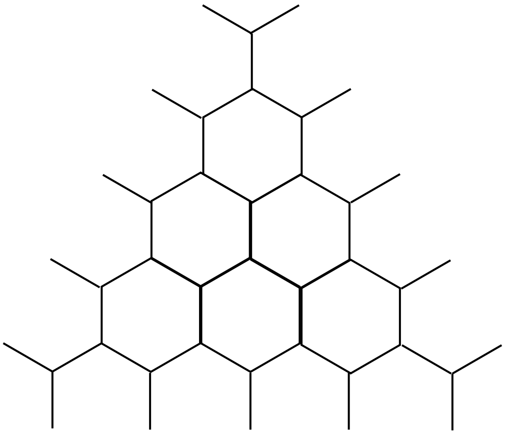


Figure 3.11. GL_5 honeycomb tinkertoy.

Let's define the direction of the edges with a map such that for $e \in E_\Gamma$; $d(e) \in \{(0,-1,1),(1,0,-1),(-1,1,0),\}$. Next we define map $h : V_\Gamma \to \mathbb{Z}^3_{\sum=0}$ such that each finite edge $e \in \Gamma$ with two vertex *head(e)* and *tail(e)*, satisfy; $h(tail(e)) - h(head(e)) \in d(e) \cup \{0\}$. In other words, map h should preserve the direction of edges attached to them in the tinkertoy.

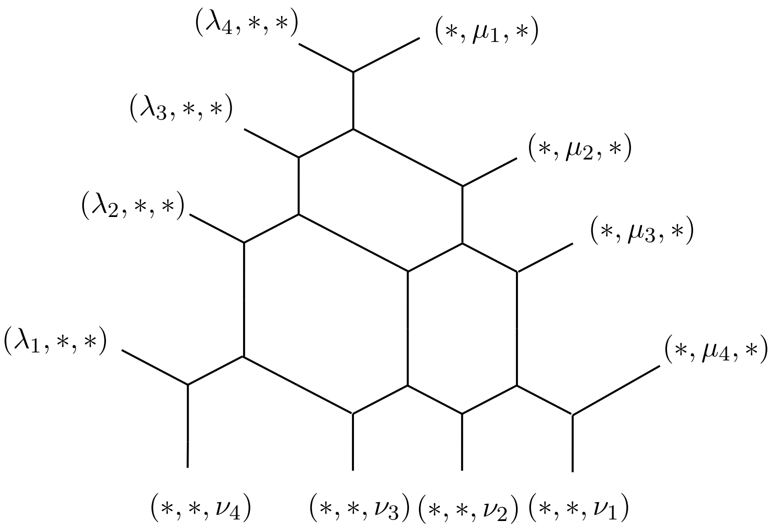


Figure 3.12. Honeycomb with boundary conditions.

Before we show how to calculate the LR coefficients with honeycombs, one last definition is required. Remark that in τ_n, there are $3n$ many semi-infinite edges whose tails are not in V_Δ. We will call them boundary edges. There is a coordinate that remains constant along its coordinate direction for each boundary edge, which we will call *boundary condition.*

Theorem 3.9. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be three partitions. Then LR coefficient $c^{\gamma}_{\lambda,\mu}$ is equal to number of $\tau_n -$*honeycombs* with integer vertices whose boundary conditions $\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_n, -\gamma_1, -\gamma_2, \ldots, -\gamma_n$ are assigned by starting from the bottom-left of the triangular graph and continue clockwise with order stated above.

We will prove this theorem by defining a bijection between hive triangles and honeycombs. To achieve this goal, we define *dual graph* of a hive triangle H_n on the same plane GL_n honeycomb tinkertoy is defined such that every hexagon in τ_n contains a label of hive diagram H_n, as shown in the figure.

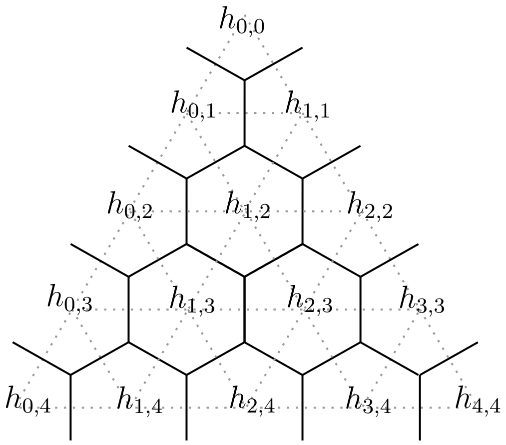


Figure 3.13. Dual graph of H_4.

We define *weight* of an edge in the dual graph as the positive difference of hive labels on each side of the edge, as shown in the figure below. To connect dual graphs to honeycombs, we will assign a coordinate in $\mathbb{R}^3$ to each vertex by setting the weight of the edges connected to the vertex.

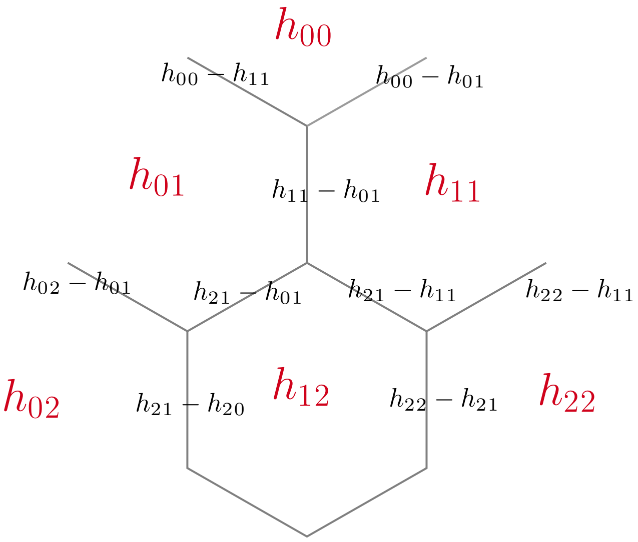


Figure 3.14. Weighted dual graph.

In a dual graph, each vertex has three edges with their weights. To find coordinates $(x, y, z) \in \mathbb{R}^3$ of a vertex; first, we assign the largest weight on edges connected to that vertex to z, since honeycombs that we are looking for lies on the plane $x + y = z$. Then x coordinate will be the weight of the next edge counterclockwise to the edge with the highest weight. Lastly, the weight of the remaining edge will be the y coordinate. We take the first hive in the example from the previous chapter and apply the above algorithm to find the corresponding honeycomb.

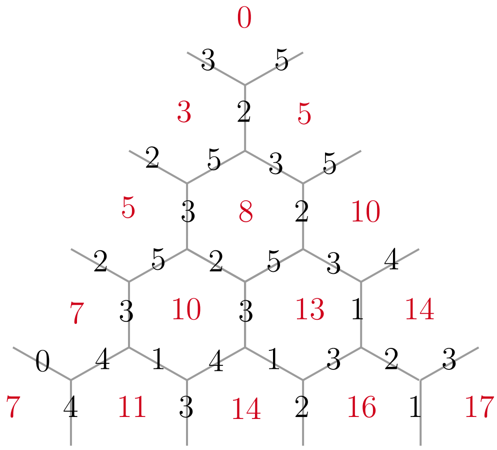


Figure 3.15. A hive with its dual graph.

In the figure above, hive entries, displayed in red, and the dual graph with weights written on the corresponding edges can be seen. And in the figures below weighted dual graph with coordinates and the honeycomb with constant coordinates on the boundaries are shown.

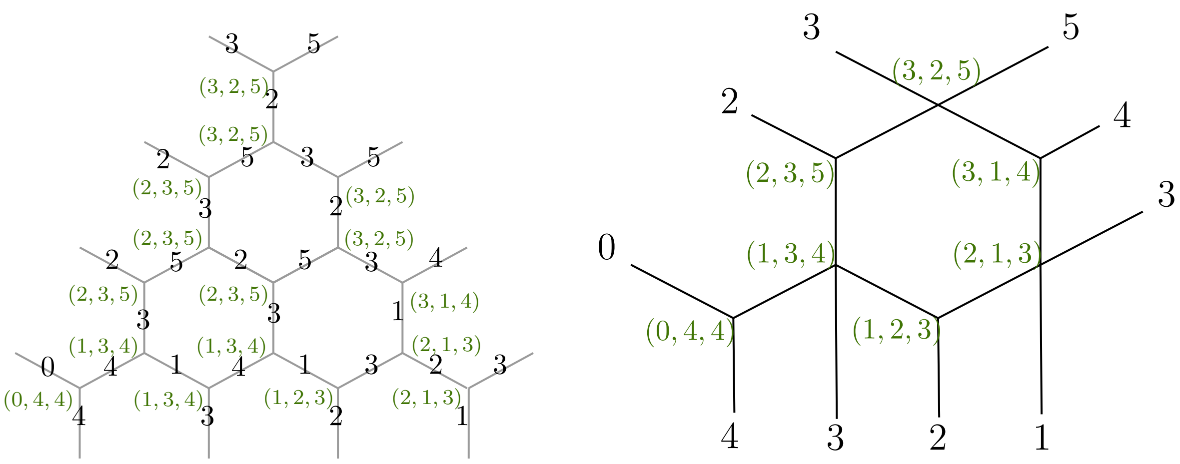


Figure 3.16. A dual graph and corresponding honeycomb.

Assigning weights of the edges in a dual graph to coordinates corresponds to a configuration map on the dual graph, which is a GL_n honeycomb tinkertoy. Remark that some vertices may have the same coordinates. Edges between those vertices are degenerate edges in the configuration.

Moreover, we can find an algorithm from honeycombs with integer coefficients to hives. Once we have a dual graph with coordinates assigned to vertices, we can find the corresponding weighted dual graph. Then since we know that the first entry in a hive is always zero, we can find the remaining entries from top to bottom by given weights.

Hence, to obtain an inverse of the map from hives to honeycombs, the only remaining challenge is retrieving degenerate edges and obtaining the dual graph from a honeycomb. We achieve this by adding edges whenever a transverse crossing is present on a vertex of a honeycomb. In the figure below, three types of transverse crossing based on the directions of edges and how to 'fix' them are shown.

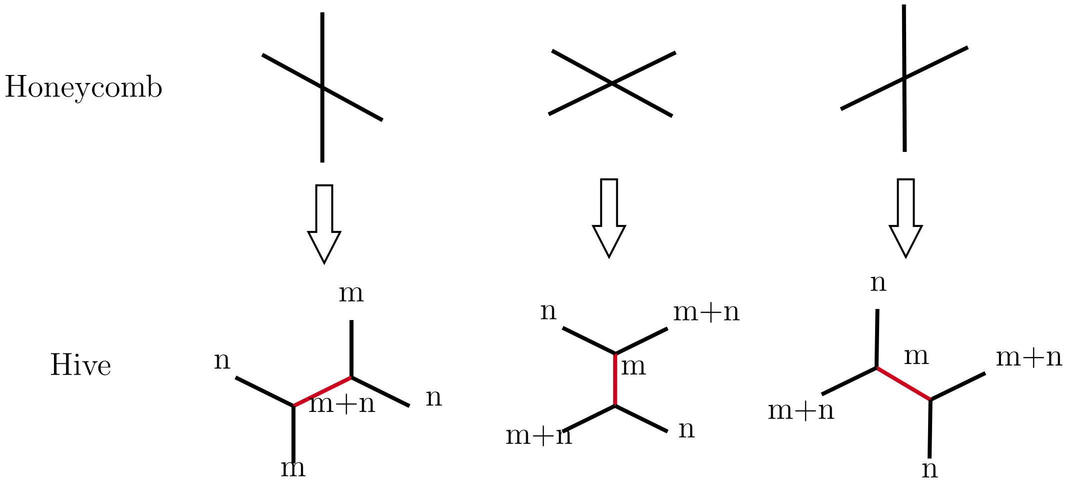


Figure 3.17. Tranverse crossings.

We have shown a method to map honeycombs and hives with integer coefficients to each other with the algorithms above. The rigorous proof of the bijection between them can be found in [8].

4. A NEW APPROACH ON CALCULATIONS OF LITTLEWOOD-RICHARSON COEFFECIENT

In this part, we will explain our contribution to the calculations of Littlewood Richardson coefficients. We note that our result, in fact, coincides with the result of Peter Mcnamara [11], but uses another technical tool that we call *ballot sequences.* Since it is a totally new technique, it may be useful for the solutions to other questions existing in this area.

4.1. Lower Bounds for Littlewood Richardson Coefficients

To explain our method, we start by introducing a generalization of dominance order. Dominance order given on the partitions, hence on Young diagrams, has an important role in the areas of mathematics in which these objects are vital. The dominance order relates two partitions of the same positive integer in the following manner: Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ of n, we set $\lambda \leq_{Dom} \mu$ if

$$\lambda_1 + \lambda_2 + \ldots + \lambda_i \leq \mu_1 + \mu_2 + \ldots + \mu_i \text{ for all } 1 \leq i \leq \min\{k, l\}.$$

The method developed in this paper needs the relation given by the dominance order but is defined on partitions of different sizes. This is why we call this order the *generalized dominance order.*

Definition 4.1. Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n and $\mu = (\mu_1, \ldots, \mu_l)$ of m and $n \leq m$, we set $\lambda \leq_{GDom} \mu$ in *the generalized dominance order* if

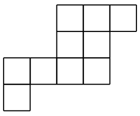
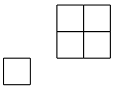
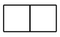
$$\lambda_1 + \lambda_2 + \ldots + \lambda_i \leq \mu_1 + \mu_2 + \ldots + \mu_i \text{ for all } 1 \leq i \leq \min\{k, l\}.$$

4.1.1. Lower Derivations on Skew Partitions and the Main Theorem

The lower derivations that we introduce in this section play a crucial role in determining the lower and upper boundaries for non-zero LRC coefficients.

Definition 4.2. Let γ/λ be given. Then the skew partition obtained by deleting top i (possibly empty) cells in each column of γ/λ is called *i-th lower derivation* of γ/λ and denoted by $\gamma/\lambda_{(i)}$.

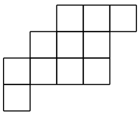
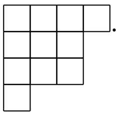
Example 4.3. Consider γ/λ where $\gamma = (5,4,4,1)$ and $\lambda = (2,2)$. Then

$\gamma/\lambda =$

$\Longrightarrow \gamma/\lambda_{(1)} =$

$\Longrightarrow \gamma/\lambda_{(2)} =$

.

For every skew partition γ/λ, one can obtain a canonical partition in the following manner: Let $r_1, r_2, \ldots, r_n$ be an enumeration of the rows of γ/λ according to their sizes in non-increasing fashion and let $l(r_i)$ be the size of the i-th row with respect to this enumeration, that is $l(r_i) \geq l(r_{i+1})$ for each i. Now the partition $(l(r_1), l(r_2), \ldots, l(r_n))$ is called *left-top standardization* of γ/λ and denoted by

$$\uparrow_{\leftarrow} (\gamma/\lambda) = (l(r_1), l(r_2), \ldots, l(r_n))$$

Example 4.4. Consider γ/λ where $\gamma = (5,4,4,1)$ and $\lambda = (2,1)$. Then

$\gamma/\lambda =$

$\Longrightarrow \uparrow_{\leftarrow} (\gamma/\lambda) = (4,3,3,1) =$

.

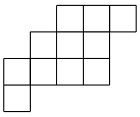
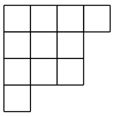
Now using lower derivations and left-top standardization on skew partition γ/λ we define a sequence of partitions $\partial_{(1)}(\gamma/\lambda), \partial_{(2)}(\gamma/\lambda), \ldots$ which, in fact, defines the lower boundary for non-zero LR coefficients.

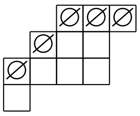
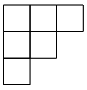
Definition 4.5. Let γ/λ be given. Then *i-th lower standardization* of γ/λ is denoted by $\partial_{(i)}(\gamma/\lambda)$ and obtained by the following rules:

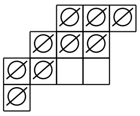
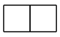
$$\partial_{(1)}(\gamma/\lambda) = \uparrow_{\leftarrow} (\gamma/\lambda),$$
$$\partial_{(i)}(\gamma/\lambda) = \uparrow_{\leftarrow} (\gamma/\lambda_{(i-1)}) \text{ for } i \geq 2.$$

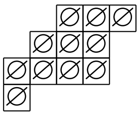
Observe that when $\lambda = \varnothing$ then $\partial_{(1)}(\gamma/\varnothing) = \gamma = (\gamma_1, \gamma_2, \ldots)$ and $\partial_{(i)}(\gamma/\varnothing) = (\gamma_i, \gamma_{i+1}, \ldots)$ in general.

Example 4.6. Consider γ/λ given below.

$\gamma/\lambda =$  $\longrightarrow \partial_{(1)}(\gamma/\lambda) =$ 

$\gamma/\lambda_{(1)} =$  $\longrightarrow \partial_{(2)}(\gamma/\lambda) =$ 

$\gamma/\lambda_{(2)} =$  $\longrightarrow \partial_{(3)}(\gamma/\lambda) =$ 

$\gamma/\lambda_{(3)} =$  $\longrightarrow \partial_{(4)}(\gamma/\lambda) = \varnothing$

Now we are ready to state our main theorem in a rigorous manner:

Theorem 4.7 (Main Theorem). Given a skew partition γ/λ, if $\mu = (\mu_1, \mu_2, \ldots, \mu_i, \ldots)$ satisfy $c^{\gamma}_{\lambda,\mu} \neq 0$ then $\mid \mu \mid = \mid \gamma \mid - \mid \lambda \mid$ and for all $i \geq 1$,

$$\partial_{(i)}(\gamma/\lambda) \leq_{GDom} (\mu_i, \mu_{i+1}, \ldots).$$

Moreover, for $\mu = \partial_{(1)}(\gamma/\lambda)$ we have $c^{\gamma}_{\lambda,\mu} = 1$, that is $\partial_{(1)}(\gamma/\lambda)$ is the minimum partition satisfying $c^{\gamma}_{\lambda,\mu} \neq 0$ with respect to generalized dominance order.

4.1.2. Proof of the Main Theorem

The proof of the main theorem depends on several other characterizations of LR tableaux that we provide below. Recall that definition of LR tableaux depends on whether its row word is the reverse lattice word or not. Here we provide two characterizations that are free from the row word.

To start with, observe that $\lambda \subset \gamma = (\gamma_1, ..., \gamma_n)$ the skew partition γ/λ can be realized in the grid, say $G_{\gamma/\lambda}$, of size $n \times \gamma_1$. Hence the cells of γ/λ can be identified with their coordinates in $G_{\gamma/\lambda}$, that is, (i, j) refers to the cell which lies in the i-th row and j-th column of $G_{\gamma/\lambda}$. For each $c = (i, j) \in G_{\gamma/\lambda}$, whether it belongs to γ/λ or not, we denote by $\mathrm{NE}_{\gamma/\lambda}(c)$ the set consisting of all the cells in the northeast of c which also lies in γ/λ. That is,

$$\mathrm{NE}_{\gamma/\lambda}(c) = \{(s, t) \in \gamma/\lambda : 1 \le s \le i, \;\; \max(j, \lambda_s) \le t \le \gamma_s\}.$$

Now, for a given semi-standard tableau T of shape γ/λ, we define the *north east counting operation*

$$\mathrm{NE}\#_T \colon G_{\gamma/\lambda} \times [n] \to \mathbb{N},$$

where $\mathrm{NE}\#_T(c, k)$ counts the number of cells in $\mathrm{NE}_{\gamma/\lambda}(c)$ that T labels by k.

Definition 4.8. We say that a semi-standard tableau T of shape γ/λ satisfies *cardinality condition*, if T satisfies the following additional condition: For any cell $c \in \gamma/\lambda$, if the label of c is $k > 1$ then

$$\mathrm{NE}\#_T(c, k) \le \mathrm{NE}\#_T(c, k-1). \tag{4.1}$$

The following lemma gives an equivalent definition of the cardinality condition, which will be useful in many places throughout this section.

Lemma 4.9. Let T be a semi-standard tableau of shape γ/λ. Then T satisfies the cardinality condition if and only if

$$\mathrm{NE}\#_T(c, l) \le \mathrm{NE}\#_T(c, l-1) \text{ for all } c \in G_{\gamma/\lambda} \text{ and for all } \; l > 1. \tag{4.2}$$

Proof. It is clear that the assertion (4.2) implies cardinality condition (4.1). So we suppose that T satisfies the cardinality condition, that is

$$\mathrm{NE}\#_T(c, k) \le \mathrm{NE}\#_T(c, k-1) \text{ for any cell } c \in \gamma/\lambda \text{ with the label } k > 1. \tag{4.3}$$

First, observe that the first row of T consists of only the label 1 and so the assertion (4.2) holds for any cell c lying in the first row of $G_{\gamma/\lambda}$ and for all $l > 1$. To proceed by induction, we order the cells of $G_{\gamma/\lambda}$ from right to left in each row, starting from the top row. Now we can assume the assertion for the first n cells of $G_{\gamma/\lambda}$ under this ordering.

Suppose that $(n+1)$-th cell is located at $c = (i, j)$ of the grid $G_{\gamma/\lambda}$. Then by induction, the assertion (4.2) is also true for $c_e = (i, j-1)$ and $c_n = (i-1, j)$. Now if $\mathrm{NE}\#_T(c, l) > \mathrm{NE}\#_T(c, l-1)$ for some $l > 1$ then the following conditions must be true at the same time, since c_e and c_n come earlier in this order.

i. $\mathrm{NE}\#_T(c_e, l) = \mathrm{NE}\#_T(c_e, l-1)$ and c or a cell directly above c is labelled by l but none of them is labelled by $l-1$.
ii. $\mathrm{NE}\#_T(c_n, l) = \mathrm{NE}\#_T(c_n, l-1)$ and c or a cell directly to the right of c is labelled by l but none of them is labelled by $l-1$.

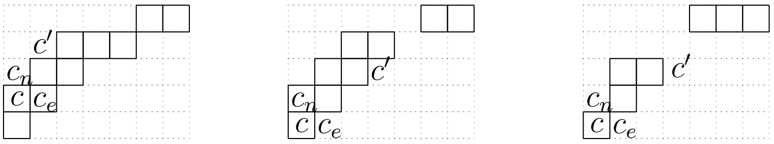


Figure 4.1. Possible configurations for the cell c'.

First, observe that if this cell lies outside of γ/λ then we have three configurations as illustrated in the above figures by c'. Observe that in each case, there can not be a cell in γ/λ directly above c' and directly to the right of c' at the same time. So (i) and (ii) can not be true at the same time. Hence $\mathrm{NE}\#_T(c', l) \leq \mathrm{NE}\#_T(c', l-1)$ for this $l > 1$.

Now suppose that c belongs to γ/λ. If c has one of the three configurations above, then because of the conditions (i) and (ii), c must be labeled by l. But T satisfies the cardinality condition, hence $\mathrm{NE}\#_T(c, l) \leq \mathrm{NE}\#_T(c, l-1)$ for this $l > 1$. So we assume the following configuration for c, that is, both c_e and c_n lie in γ/λ.

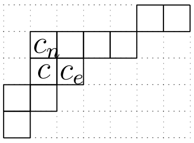


Figure 4.2. Configuration of c.

If c is labeled with l then cardinality condition directly gives $\mathrm{NE}\#_T(c,l) \leq \mathrm{NE}\#_T(c,l-1)$. Otherwise, assertion (i) implies that a cell directly above c must be labeled by l; hence, c is labeled by a number strictly greater than l. But in this case, any cell directly right to c is also labeled by a number strictly greater than l; hence, the assertion (ii) can not be true. So both (i) and (ii) can not be true at the same time. Therefore in this case we still have $\mathrm{NE}\#_T(c,l) \leq \mathrm{NE}\#_T(c,l-1)$ for this $l > 1$.

Lemma 4.10. Let T be a semistandard tableau of shape γ/λ. Then the reverse lattice word of T satisfies the ballot condition; that is, T is an LR tableau if and only if T satisfies the cardinality condition.

Proof. Suppose T satisfies the cardinality condition. Recall that the reverse lattice word of T is obtained by reading the labels of each cell, from right to left along rows, starting from the top row. In order to arrive at a cell $c = (i,j)$ with label k in this reading, we need to read the labels of all cells in γ/λ located in NE of $c = (i,j)$ and in NW of $d = (i-1, j-1)$ along the way. Observe that since T is semistandard and c is labeled by k, the NW of $d = (i-1, j-1)$ can not contain the label k. On the other hand $\mathrm{NE}\#_T(c,k) \leq \mathrm{NE}\#_T(c,k-1)$ by cardinality condition. Therefore, in the subword of the reverse lattice word up to this label k, the number of $k-1$ is always greater than or equal to the number of k. Hence reverse lattice word directly satisfies the ballot condition. For the other direction, assume that the cardinality condition does not hold in T. Now among the cells of γ/λ on which the cardinality condition fails, let $c = (i,j)$ be the one lying in the rightmost position of the highest possible row. That is $\mathrm{NE}\#_T(c,k) > \mathrm{NE}\#_T(c,k-1)$.

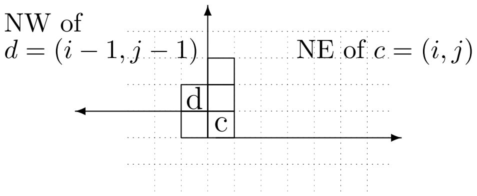


Figure 4.3. Location of labels in the row word until the cell c.

First, we assume that c does not line in the last column or first row of γ/λ. Then $c_n = (i+1, j)$ and $c_e = (i, j+1)$ lie in the grid $G_{\gamma/\lambda}$, and since they satisfy the cardinality condition assertions

i. $\mathrm{NE}\#_T(c_e, k) = \mathrm{NE}\#_T(c_e, k-1)$ and none of the cells directly above c is labelled by $k-1$,

ii. $\mathrm{NE}\#_T(c_n, k) = \mathrm{NE}\#_T(c_n, k-1)$ and none of cells directly to the right of c is labelled by $k-1$

must hold. But the assertion i. implies that the label of the cells in the NW of $d = (i-1, j-1)$, if they exist, must be strictly smaller than $k-1$. Hence in the sub tableau of T lying in NE of c or NW of d, the number of appearances of label k must be strictly smaller than that of label $k-1$. Therefore the reverse lattice word of T does not satisfy the ballot condition. The cases that c lies in the first row or the last column can be dealt with in a similar manner.

To proceed to the next Lemma, we need the following definition.

Definition 4.11. Let T be skew tableau. A sequence $\{(c_1, 1), (c_2, 2), \ldots, (c_k, k)\}$ is called a ballot sequence if for all $1 \leq i \leq k$, the cells c_i can be chosen according to the rules

i. i is the label of c_i in T.

ii. for $i > 1$, c_i always lies below c_{i-1} in T.

Lemma 4.12. Any LR tableau T can be partitioned in to ballot sequences.

Proof. We first explain the algorithm to construct ballot sequences. Suppose that T has shape γ/λ and k is the largest label in T. Among the cells of T labeled by k, let c_k be the one located in the rightmost position of the highest possible row of T. This means that c_k lies in the right boundary of γ/λ and that k does not label any cell at the NE of c_k except c_k itself.

Now, the fact that T satisfies the cardinality condition (by Lemma 4.2) implies that there is at least one cell with label $k-1$ in NE of c_k. Among the cells labeled with $k-1$ in NE of c_k, we again choose the one located in the rightmost position of the highest possible row, say c_{k-1}. We pursue this algorithm until we reach a cell with label 1, hence in that way, we construct a ballot sequence $\{(c_1, 1), (c_2, 2), \ldots, (c_k, k)\}$ in T.

Observe that each cell c_i in this sequence lies in the right boundary of γ/λ. Moreover, its label i is the largest label appearing in NE of c_i, and i does not label any cell lying above c_i or to the right of c_i. It is clear that the removal of these cells from T still results in a skew semistandard tableau, say T'. Once we show that T' is an LR tableau, the construction of ballot sequences and hence the proof of the theorem is completed by induction.

Now by Lemma 4.2, it is enough to show T' satisfies the cardinality condition. Assume by contradiction that T' does not satisfy the cardinality condition. Among the cells of T' on which the cardinality condition fails, let c be the one located in the rightmost position of the highest possible row. Since c in the original tableau T satisfies the cardinality condition, there must be at least one label removed from NE of c in producing T'. Among these labels, let m be the largest one. Observe that $m \neq k$ since removing the largest label from a tableau does not terminate the cardinality condition on any cell. Hence $m < k$ and not only m but also whole sequence of labels $m-1, m-2, ..., 1$ must be removed from NE of c according to the algorithm explained above.

Hence cardinality condition fails at c in T' if and only if $\mathrm{NE}\#_{T'}(c, m+1) > \mathrm{NE}\#_{T'}(c, m)$. On the other hand, since $m < k$ and since the algorithm starts with the removal of the largest label k, it must also remove a cell, say $d \neq c$, with label $m+1$. Below, by diving the grid $G_{\gamma/\lambda}$ into four regions $G_1,\ G_2,\ G_3,\ G_4$, we show that such a cell can not exist and hence arrive at a contradiction to the assumption that T' is not an LR tableau.

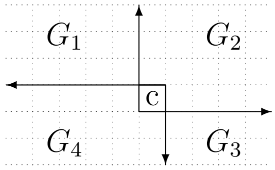


Figure 4.4. Regions G_1, G_2, G_3, G_4 on the grid.

First, d can not lie in G_2 since the largest label removed from the NE of c is m. Now recall that d lies in the right boundary of λ/μ, $m+1$ is the largest label appearing in NE of d, and $m+1$ does not label any cell lying above d or to the right of d. This shows that d can not lie in G_4 since otherwise NE of d contains NE of c and this gives $\mathrm{NE}\#_{T'}(c, m+1) \leq \mathrm{NE}\#_{T'}(c, m)$, since $\mathrm{NE}\#_{T}(c, m+1) \leq \mathrm{NE}\#_{T}(c, m)$ originally. Similarly, d can not lie in G_3 otherwise, no row above d contains the label $m+1$ and this yields again $\mathrm{NE}\#_{T'}(c, m+1) \leq \mathrm{NE}\#_{T'}(c, m)$. Now, if d lies in G_1, then the cells to the right of c must be empty; hence any cell below these empty cells must be empty. But this contradicts the fact that c lies in the skew diagram λ/μ.

The following result is crucial in the proof of the main theorem. To state the result, we need some definitions.

For a skew diagram γ/λ, let $r_1,\ r_2,\ ...r_n$ be an enumeration of its rows according to their sizes in a non-increasing fashion and let $l(r_i)$ be the size of the i-th row with respect to this enumeration. Hence the assertion $l(r_i) \geq l(r_{i+1})$ for $i = 1, .., n-1$ always true. Also, for a skew semistandard tableau T of shape γ/λ, we define $\#_T\colon [n] \to \mathbb{N}$ so that $\#_T(k)$ is the number of times that the label k is used in T.

Proposition 4.13. Let T be a LR tableau of shape γ/λ with n rows. Then for all $m \leq n$,

$$\#_T(1) + \#_T(2) + ... + \#_T(m) \geq l(r_1) + l(r_2) + ... + l(r_m).$$

In particular, if $c^{\gamma}_{\lambda,\mu} \neq 0$ then we have $\mu_1 + \mu_2 + ... + \mu_m \geq l(r_1) + l(r_2) + ... + l(r_m)$ for all $m \leq n$.

Proof. By Lemma 4.10, T can be partitioned into disjoint ballot sequences. Let P be the set consisting of these disjoint ballot sequences. Now for a fixed $m \leq n$ and any $I \subset [m]$, let P^m_I be the set consisting of ballot sequences that contain a label from each row r_i if $i \in I$, but not from r_j if $j \in [m] - I$. Observe that such a ballot sequence may or may not contain labels from the rows $r_{m+1}, \ldots, r_n$. It is easy to see that, for this fixed m,

$$P = \bigcup_{I \subset [m]} P^m_I = \bigcup_{i \leq m} \bigcup_{\substack{I \subset [m] \\ |I| = i}} P^m_I$$

and this union is disjoint. Now any ballot sequence in a fixed set P^m_I contains at least $|I|$ many distinct labels chosen from different rows; hence it always contains the labels $1, 2, \ldots, |I|$. Denoting the size of P^m_I by p^m_I we see that, p^m_I many times the labels $1, 2, \ldots, |I|$ appear in the ballot sequences lying in P^m_I. Now define

$$p^m(i) = \sum_{\substack{I \subset [m] \\ |I| = i}} p^m_I$$

to deduce that each label $1, 2, \ldots, i$ exists $p^m(i)$ many times in $\bigcup_{\substack{I \subset [m] \\ |I| = i}} P^m_I$ and total size of these labels in this union is clearly $ip^m(i)$.

Since P is the partition of T into ballot sequences and since $P = \bigcup_{i \leq m} \bigcup_{\substack{I \subset [m] \\ |I| = i}} P^m_I$, as well as the ballot sequences, are disjoint we conclude that

$$\#_T(1) + \#_T(2) + \ldots + \#_T(m) \geq \sum_{i=1}^{m} ip^m(I).$$

We split the sum on the right-hand side into two parts; sequences that contain only one label among rows $r_1, r_2, \ldots, r_m$, and all the other sequences;

$$\begin{aligned}\sum_{i=1}^{m} ip^m(i) &= p^m(1) + \sum_{i=2}^{m} ip^m(i)\\ &= \sum_{j\in[m]} p^m_{\{j\}} + \sum_{i=2}^{m} ip^m(I).\end{aligned}$$

We know sequences in $P = \bigcup_{I\subset[m]} P^m_I$ exhaust labels in T. For $j \in [m]$, take a ballot sequence that contains a label in the row r_j. Then this sequence either lies in $P^m_{\{j\}}$ or in P^m_I for some $I \subset [m]$ with $j \in I$ and $|I| > 1$. Therefore the number of labels in the row r_j satisfies that

$$l(r_j) = p^m_{\{j\}} + \sum_{\substack{I\subset[m]\\ |I|>1\\ j\in I}} p^m_I.$$

Putting this identity in the equation above, we get

$$\begin{aligned}\sum_{i=1}^{m} ip^m(i) &= \sum_{j\in[m]} \left(l(r_j) - \sum_{\substack{I\subset[m]\\ |I|>1\\ j\in I}} p^m_I \right) + \sum_{i=2}^{m} ip^m(i)\\ &= \sum_{j\in[m]} l(r_j) - \sum_{j\in[m]} \sum_{\substack{I\subset[m]\\ |I|>1\\ j\in I}} p^m_I + \sum_{i=2}^{m} ip^m(i).\end{aligned}$$

Observe that in the summation $\sum_{j\in[m]} \sum_{\substack{I\subset[m]\\ |I|>1\\ j\in I}} p^m_I$ the index I with $|I| = i > 1$ appears i many times. That is

$$\sum_{j\in[m]} \sum_{\substack{I\subset[m]\\ |I|>1\\ j\in I}} p^m_I = \sum_{\substack{I\subset[m]\\ |I|>1}} |I| p^m_I = \sum_{i=2}^{m} \sum_{\substack{I\subset[m]\\ |I|=i}} ip^m_I = \sum_{i=2}^{m} ip^m(I).$$

Hence $\sum_{i=1}^{m} ip^m(i) = \sum_{j\in[m]} l(r_j)$ and this proves

$$\#(1) + \#(2) + ... + \#(m) \geq \sum_{i=1}^{m} ip^m(i) = l(r_1) + l(r_2) + ... + l(r_m).$$

4.1.3. The Proof of Theorem 4.7

Now we are ready the prove the main theorem given at the beginning of this chapter. Given a skew partition γ/λ and a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_i, \ldots)$, we assume that $c^{\gamma}_{\lambda,\mu} \neq 0$. Then we want to show:

(i) For all $i \geq 1$, $\partial_{(i)}(\gamma/\lambda) \leq_{GDom} (\mu_i, \mu_{i+1}, \ldots)$,
(ii) For $\mu = \partial_{(1)}(\gamma/\lambda)$ we have $c^{\gamma}_{\lambda,\mu} = 1$, that is $\partial_{(1)}(\gamma/\lambda)$ is the minimum partition satisfying $c^{\gamma}_{\lambda,\mu} \neq 0$ with respect to generalized dominance order.

We start with the first statement. So let T be an LRC tableau of shape γ/λ and with the content μ. Then by Lemma 4.12, for all $m \leq n$,

$$l(r_1) + l(r_2) + ... + l(r_m) \leq \#_T(1) + \#_T(2) + ... + \#_T(m) = \mu_1 + \ldots + \mu_m,$$

where the sequence $r_1,\ r_2,\ ...r_n$ is an ordering of the rows of γ/λ according to their sizes in non-increasing fashion and $l(r_i)$ is the size of the i-th row in this ordering. Now it is easy to see that the partition $(l(r_1), l(r_2), \ldots, l(r_n))$ is nothing but $\partial_{(1)}(\gamma/\lambda)$. Hence the case $i = 1$, that is

$$\partial_{(1)}(\gamma/\lambda) \leq_{GDom} (\mu_1, \mu_2, \ldots, \mu_n)$$

follows directly by Lemma 4.12.

For the case $i > 1$, first observe that removing all the cells with label 1,2,..$i-1$ from T still gives a semi-standard tableau, say T' of shape γ'/λ' with labels greater than or equal to i. Moreover, T' satisfies the cardinality condition for any cell labeled with $j \geq i+1$. In other words, T' can be considered as an LR tableau of shape γ'/λ' with the content $(\mu_i, \ldots, \mu_n)$ where the lowest label is allowed to be i. Hence Lemma 4.12 gives

$$\partial_{(1)}(\gamma'/\lambda') \leq_{GDom} (\mu_i, \mu_{i+1}, \ldots, \mu_n).$$

Now the theorem follows when we show that $\partial_{(i)}(\gamma/\lambda) \leq_{GDom} \partial_{(1)}(\gamma'/\lambda')$. Recall that

$$\partial_{(i)}(\gamma/\lambda) = \uparrow_{\leftarrow} (\gamma/\lambda_{(i-1)}) = (l(t_1), l(t_2), \ldots, l(t_m)),$$

where $\gamma/\lambda_{(i-1)}$ is obtained by deleting top i cells in each column of γ/λ whenever possible. Here the sequence $t_1, t_2, \ldots, t_m$ is the ordering of the rows of $\gamma/\lambda_{(i-1)}$ according to their sizes in non-increasing fashion and $l(t_i)$ is the size of the i-th row in this ordering. On the other hand

$$\partial_{(1)}(\gamma'/\lambda') = \uparrow_{\leftarrow} (\gamma'/\lambda') = (l(s_1), l(s_2), \ldots, l(s_k)),$$

where γ'/λ' is the shape of the tableau T' obtained from T by removing the cells labeled by 1,2,..$i-1$ and again the sequence $s_1, s_2, \ldots, s_k$ represents the ordering of the rows of γ'/λ' according to their sizes in non-increasing fashion. Observe that these cells removed can be located only in the first $i-1$ rows of each column of γ/λ. That is

$$\gamma/\lambda_{(i-1)} \subset \gamma'/\lambda' \subset \gamma/\lambda$$

and each row t_i of $\gamma/\lambda_{(i-1)}$ lies in a unique row, say s_{j_i} of γ'/λ'. Hence for each i,

$$l(t_1) + l(t_2) + \ldots + l(t_i) \leq l(s_{j_1}) + l(s_{j_2}) + \ldots + l(s_{j_i}) \leq l(s_1) + l(s_2) + \ldots + l(s_i).$$

This proves $\partial_{(i)}(\gamma/\lambda) \leq_{GDom} \partial_{(1)}(\gamma'/\lambda') \leq_{GDom} (\mu_i, \mu_{i+1}, \ldots, \mu_n)$ as required.

Now we will show that for $\mu = \partial_{(1)}(\gamma/\lambda) = (l(r_1), l(r_2), \ldots, l(r_n))$, $c^{\gamma}_{\lambda,\mu} = 1$. Observe that each index $1 \leq i \leq n$ appears in the content $\mu = (l(r_1), l(r_2), \ldots, l(r_n))$ at least $l(r_n)$ many times and γ/λ has n rows, each of which has at least $l(r_n)$ cells. We can extract each index i from μ, $l(r_n)$ many times and place them into the right most $l(r_n)$ cells of the i-th row of γ/λ from the top; hence we construct $l(r_n)$ many ballot sequences in γ/λ consisting of the labels $1, 2, \ldots, n$.

Assuming that k is the largest index such that $l(r_k) - l(r_n) > 0$, we then have the reduced content $\mu' = (l(r_1) - l(r_n), \ldots, l(r_k) - l(r_n))$ and a sub skew shape γ'/λ' consisting of the unlabelled cells of γ/λ. Here γ'/λ' also has k rows, each of which has at least $l(r_k) - l(r_n)$ cells, and by repeating the same process above, we can construct $l(r_k) - l(r_n)$ many ballot sequences consisting of the labels $1, 2, \ldots, k$. A continuation of this process terminates by producing a tableau, say T, of shape γ/λ with content $\mu = (l(r_1), l(r_2), \ldots, l(r_n))$, which can be partitioned into ballot sequences. By Lemma 4.10, T is an LR tableau and this proves $c^{\gamma}_{\lambda,\mu} \geq 1$.

Now we will show that $c^{\gamma}_{\lambda,\mu} \not> 1$. First observe that if S is an LR tableau of γ/λ and content $\mu = (\mu_1, \ldots, \mu_l)$ then S must be partitioned into ballot sequences among which μ_l of them consisting of $1, 2, \ldots, l$, $(\mu_{l-1} - \mu_l)$ of them consisting of $1, 2, \ldots, l-1$ so on so fort. Now if $\mu = (l(r_1), l(r_2), \ldots, l(r_n))$ then first of all $l(r_n)$ many ballot sequences having $1, 2, \ldots, n$ must be located in S. Now since γ/λ has n rows and since the largest label that can appear in the i-th row from the top is i, we see that in S, the right most $l(r_n)$ cells of the i-th row must be labeled by i. Hence both S and the tableau T that we construct above have the same labeling the right of $l(r_n)$ cells in each row. Now by iterating this argument, we can conclude that $S = T$ and hence $c^{\gamma}_{\lambda,\mu} = 1$.

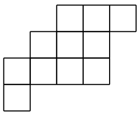
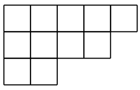
4.2. Upper Bounds for LRC

As Theorem 4.7 declares the lower bounds for a partition μ to appear in $s_{\gamma/\lambda} = \sum c^{\gamma}_{\lambda,\mu} s_{\mu}$ with non zero coefficients, it is natural to expect some set of upper bounds for μ, together with lower bounds, defines sufficient conditions for $c^{\gamma}_{\lambda,\mu} > 0$. In fact we know that if $\mu = (\mu_1, \mu_2, \ldots, \mu_i, \ldots)$ satisfy $c^{\gamma}_{\lambda,\mu} \neq 0$ then $\mid \mu \mid = \mid \gamma \mid - \mid \lambda \mid$ and

$$(\mu_1, \mu_2, \ldots) \leq_{GDom} {}^{\leftarrow}\uparrow (\gamma/\lambda),$$

where ${}^{\leftarrow}\uparrow (\gamma/\lambda)$ is obtained from γ/λ by sliding its boxes first in the north and then in the west direction. We also know that when $\mu = {}^{\leftarrow}\uparrow (\gamma/\lambda)$ we have $c^{\gamma}_{\lambda,\mu} = 1$. On the other hand, to arrive at sufficient conditions for nonzero LR coefficients, one needs to define additional upper bounds by defining some derivations similar to the upper derivations used in Theorem 4.7.

Example 4.14. Consider γ/λ where $\gamma = (5, 4, 4, 1)$ and $\lambda = (2, 1)$. Then

$\gamma/\lambda =$  $\Longrightarrow \leftarrow_{\uparrow} (\gamma/\lambda) = (5, 4, 2) =$ .

Theorem 4.15. Given a skew partition γ/λ, let $\nu = (\nu_1, \nu_2, \ldots, \nu_i)$ be the top-left standardization $\leftarrow_{\uparrow} (\gamma/\lambda)$. Then $c^{\gamma}_{\lambda,\nu} = 1$. Moreover if $\mu = (\mu_1, \mu_2, \ldots, \mu_j)$ satisfies $c^{\gamma}_{\lambda,\mu} \neq 0$ then $\nu \geq_{GDom} \mu$.

Proof. Observe that top-left standardization preserves the number of columns in γ/λ and the number of cells in each column. Since any LRC tableau is semi-standard and has increasing labels on its columns, only the first cell of each column of γ/λ can be labeled by 1. These cells correspond to cells in the first row of ν. So for any content μ with $c^{\gamma}_{\lambda,\mu} \geq 0$, $\mu_1 \leq \nu_1$. Similarly, the only cells that can be labeled with 1 or 2 are the first two cells of each column of γ/λ. And a number of those cells are $\nu_1 + \nu_2$. So μ should satisfy $\mu_1 + \mu_1 \leq \nu_1 + \nu_2$. We can generalize this argument for all the rows in ν. Since $\nu_1 + \nu_2 + \ldots + \nu_i = \mu_1 + \mu_2 + \ldots + \mu_j$, the result follows.

Now we will show $c^{\gamma}_{\lambda,\nu} = 1$ where $\nu = \leftarrow_{\uparrow} (\gamma/\lambda)$ as above. Firstly, $c^{\gamma}_{\lambda,\nu} \neq 0$. γ/λ labeled with the content ν as described is semi-standard. Moreover, its reverse lattice word satisfies the ballot condition since in every column, when we reach a cell labeled, let's say by k, the same column will contain all the labels $k-1, k-2, \ldots, 1$, and those labels will have been in the reverse lattice word.

Now, we will show that the labeling described above is unique with γ/λ and ν. Assume there is another tableau with shape γ/λ and content ν. Remark that in the above labeling, every label k is in the k^{th} cell in all the columns. So to obtain different labeling with the same content ν, some cells should have labels bigger than their position in their respective columns, like m^{th} cell in a column with label $m+1$, which can't be possible if we want columns with strictly increasing labels.

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