

**A SYSTEM OF CRACKS  
IN  
AN ELASTIC HALF-PLANE**

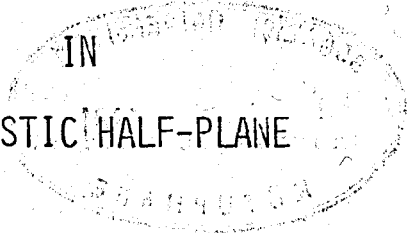
by  
**M. Oktay ALNIAK**

**Submitted to the Faculty of Engineering  
in Partial Fulfillment of the Requirements  
for the Degree of  
MASTER OF SCIENCE  
in  
MECHANICAL ENGINEERING**

**Boğaziçi University  
1981**

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A SYSTEM OF CRACKS



AN ELASTIC HALF-PLANE

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B.S. in M.E., Ege Üniversitesi, 1971

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M. Oktay ALNIAK

## ABSTRACT

The plane elastostatic problem of an isotropic elastic half-plane containing an arbitrary system of cracks is considered. Fourier transform technique is employed to obtain the edge dislocation solutions. Using the edge dislocation solutions as Green's functions, the problem is formulated in terms of a system of singular integral equations. The system of singular integral equations is solved numerically for various crack combinations and external loads. The stress intensity factors and the probably cleavage angles are studied in detail.

## ÖZET

Bu çalışmada, içinde gelişmiş güzel oluşmuş çatlaklar bulunan isotrop, elastik bir yarım düzlemin elastostatik problemi incelenmiştir. Fourier integral transform tekniği kullanılarak kenar dislokasyon çözümleri bulunmuştur. Kenar dislokasyon çözümleri çekirdek fonksiyonları olarak kullanılıp, problem bir dizi tekil integral denkleme indirgenmiş ve bu denklemler değişik çatlak geometrileri ve dış yükler için nümerik olarak çözülmüştür. Gerilme şiddeti faktörleri ve muhtemel ayrılma yüzeyleri detaylı olarak incelenmiştir.

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## LIST OF SYMBOLS

<u>Symbol</u>	<u>Definition</u>
$a_0$	Horizontal distance of the edge dislocation
$(-a, a)$	End points of a single arbitrarily oriented crack
$(-a, b)$	End points of a single radial crack
$(-a_i, a_i)$	End points of arbitrarily oriented cracks
$(a_i, b_i)$	End points of radial cracks
$\vec{b}_x$	Burgers vector of an edge dislocation parallel to the X axis
$\vec{b}_y$	Burgers vector of an edge dislocation parallel to the Y axis
$\vec{b}_s, \vec{b}_q$	Burgers vectors in $\{s, q\}$ coordinate system
$k(s, t)$	Kernels of the singular integral equations
$h(s, t)$	Kernels of the singular integral equations
$m(s, t)$	Kernels of the singular integral equations
$n(s, t)$	Kernels of the singular integral equations
$k^*(s, t)$	Bounded kernels
$n^*(s, t)$	Bounded kernels
$k_1, k_2$	Stress intensity factors
$k'_1, k'_2$	Stress intensity factor ratios

$u$	Displacement in $s$ direction
$v$	Displacement in $q$ direction
$\theta$	Angle between the $X$ axis and the axis of the crack
$f(s)$	Dislocation density functions
$g(s)$	
$U(X,Y)$	Airy stress function
$\phi(X,\alpha)$	Fourier transform of $U(X,Y)$
$\phi$	Cleavage angle
$\mu$	Shear modulus
$\nu$	Poisson's ratio
$\kappa$	$3-4\nu$ for plane strain $\frac{3-\nu}{1+\nu}$ for generalized plane stress
$\{X,Y\}$	
$\{x,y\}$	Cartesian coordinate systems
$\{s,q\}$	
$\sigma_{XX}^*, \sigma_{YY}^*, \sigma_{XY}^*$	Stress components due to an edge dislocation in an infinite plane
$\sigma_{XX}^1, \sigma_{YY}^1, \sigma_{XY}^1$	Stress components due to an edge dislocation with $\vec{b}_X$ in a half-plane
$\sigma_{XX}^2, \sigma_{YY}^2, \sigma_{XY}^2$	Stress components due to an edge dislocation with $\vec{b}_Y$ in a half-plane
$\sigma_{qq}, \sigma_{qs}$	Normal and shear stress components in $\{s,q\}$ coordinate system

## CHAPTER 1

### INTRODUCTION

One of the fundamental requirements of any engineering structure is that it should not fail in service. This necessitates a sufficient knowledge about materials containing cracks and the behavior of cracked structures.

Crack problems do not only have practical importance in fracture mechanics but are also of mathematical interest. Many investigators have treated the problems involving cracks from different points of view. In this chapter, the practical importance of the problem is emphasized along with the mathematical approach.

Recent publications [5,14,15] indicate that fracture phenomenon which is one of the possible modes of mechanical failure is an important problem. The fracture behavior of materials has been considered in two groups as ductile and brittle fracture. Ductile fracture is characterized by appreciable plastic deformation and the amount of gross deformation which is usually presented

at the fracture surfaces of the body. The limiting case of flow until the cross section vanishes may be called rupture. Griffith [11] proposed that fracture in brittle solids arises from many fine flaws in the surface of the material. Brittle fracture in metals is characterized by a rapid rate of crack propagation, with no gross deformation and very little or no microdeformation. Brittle fracture occurs without warning and usually causes disastrous consequences.

Fracture occurs in characteristic ways, depending on the state of stress, the rate of application of stress, and the temperature. In addition, more complex effects such as torsion, fatigue, creep, low temperature, temper embrittlement, hydrogen embrittlement and others may cause fracture.

The most important cause of failure for all real structures is known to be due to the material imperfections which are already present inside the body or at the surface. This kind of imperfections may exist as flaws or cracks, the largest of which most likely causes the failure.

The structural strength of materials generally depends on the material properties, the shape and the size of defects as well as together with the orientation of

flaws in the medium. In the application of fracture mechanics, other kinds of imperfections due to manufacturing are particularly important, such as flat cavities which develop during casting, small cracks resulting from residual stresses in welded materials, and fatigue cracks under external loads.

Thus, in a design dealing with materials, it is necessary to have a good estimate of the disturbed stress state caused by these flaws. This general summary deals with only a minute part of the available investigations in the theory and practice of fracture mechanics from the macroscopic and microscopic points of view. In addition, to these investigations there has been a large number of theoretical and mathematical studies on the same subject.

In the last few decades a considerable progress has been made in the application of linear elastic fracture mechanics by employing plane elasticity theory. Many infinite plane problems containing cracks with arbitrary orientations and locations were solved. Various researchers have also investigated the crack problems in a semi-infinite medium. In general, the half-plane problems have been presented with a boundary which is either free or bounded to a second half-plane.

Among the recent publications dealing with a system of cracks the paper of Savruk and Datsyshin [17] deals with the problem on elastic, isotropic infinite medium containing arbitrarily arranged cuts. They obtained the solution of crack problems by means of singular integral equations.

The problem of bonded half-planes containing an arbitrarily oriented crack was first investigated by Erdogan and Aksogan [8]. They reduce the problem to a system of singular integral equation by using the Mellin transform technique. The stress intensity factors and the probable crack propagation angles for various crack orientations were given. Later on Ashbaugh [1] solved the same problem by employing a different technique.

A similar study was presented by Atkinson [2] who used stress field for an edge dislocation in bimetallic medium given by Head [12] to generate integral equations for the dislocation densities associated with cracks and dislocation pile ups. The integral equations were solved numerically.

The problem of interaction between a circular inclusion and an arbitrarily oriented crack has been solved by Erdogan, Gupta and Ratwani [7]. They used the existing solutions for the edge dislocations as Green's functions.

The integral equations for the line crack were obtained as a system of singular integral equations with simple Cauchy kernels. The problem was solved for various typical crack orientations and the corresponding stress intensity factors were given. Although there is no direct similarity between the subject of this thesis and the paper [7], the singular integral equation formulation, edge dislocations, Green's functions and the stress intensity factors have the same important theoretical basis.

Among the papers concerning crack problems in an elastic half-plane the paper by Ioakimidis and Theocaris [13] presents a general formulation for a system of curvilinear cracks by using complex variable methods. Even though the formulation is quite general the numerical examples are given only for single straight or arc-shaped cracks.

In this study, dislocation solutions are used as Green's functions to formulate the system of cracks problem. For this purpose first the stress field of an elastic half-plane due to a pair of edge dislocations with Burgers vectors parallel and perpendicular to the free surface are determined from the infinite plane solution by using the Fourier transform technique. The stress field due to an edge dislocation with a Burgers vector forming an angle with the free surface can be obtained by

resolving the Burgers vector into two components one parallel and the other being perpendicular to the free surface and summing the stresses set up by these components. To simulate a line crack in a given direction stress solutions for a pair of edge dislocations with Burgers vectors in the specified direction and in the direction perpendicular to it can be used as Green's functions. Replacing Burgers vectors by dislocation densities and integrating these density functions along the crack yields the stresses created by introducing the crack. Stress distributions due to several existing cracks can be superimposed. Expressing that the total stresses on the crack surfaces are equal to zero one obtains a system of singular integral equations with dislocation densities as unknown functions. If  $n$  is the number of cracks then there are  $2n$  unknown functions and hence  $2n$  singular integral equations can be expressed. A set of singular integral equations for a system of radial cracks can be readily obtained. However, for cracks whose axes do not intersect on the free surface, a translation of axes is necessary to be able to express the singular integral equations. The system of singular integral equations are solved numerically by using the Gauss-Chebyshev integration formulas. Numerical results are first obtained for a single crack problem. It is observed that the results match all the existing single crack solutions. Then to characterize the crack interactions two crack problems

are solved by considering radial cracks and cracks perpendicular or parallel to the free surface. The effect of variations of distances and angles between the cracks on the stress intensity factors and cleavage angles are investigated. The system of singular integral equations are solved for the cases of symmetric and anti-symmetric uniform tractions or uniform tension applied at infinity, parallel to the free surface. In all the above-mentioned problems, cracks are located in the vicinity of the free surface since the solutions for cracks away from the free surface can be approximated by infinite plane solutions.

## CHAPTER 2

### FORMULATION OF THE PROBLEM

First the stress field in an isotropic half-plane (which extends to the right of the free surface  $X=0$ ) due to a pair of edge dislocations at the point  $(X=a_0, Y=0)$  with Burgers vectors  $\vec{b}_X$  and  $\vec{b}_Y$  will be determined. The desired solution can be constructed by modifying the elastic stress fields of edge dislocations in an infinite plate.

An edge dislocation, of Burgers vector  $\vec{b}_X$ , parallel to X axis, at the point  $(a_0, 0)$  of an infinite plane (see Figure 1) has the following stress field [12].

$$\begin{aligned}\sigma_{XX}^* &= -\frac{D_X}{\pi} \frac{Y[Y^2 + 3(X - a_0)^2]}{[(X - a_0)^2 + Y^2]^2} \\ \sigma_{YY}^* &= \frac{D_X}{\pi} \frac{Y[(X - a_0)^2 - Y^2]}{[(X - a_0)^2 + Y^2]^2} \\ \sigma_{XY}^* &= \frac{D_X}{\pi} \frac{(X - a_0)[(X - a_0)^2 - Y^2]}{[(X - a_0)^2 + Y^2]^2}\end{aligned}\quad (1a, b, c)$$

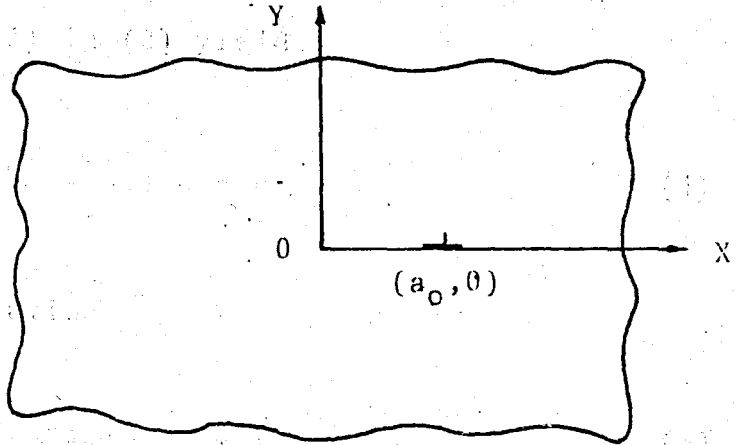


FIGURE 1: An Edge Dislocation at the Position  $(a_0, 0)$  with Burgers vector  $\vec{b}_X$  in an infinite plane.

where,  $D_X = \frac{2\mu b_X}{(1+\kappa)}$

$$\kappa = \begin{cases} 3-4\nu & \text{for plane strain} \\ \frac{3-\nu}{1+\nu} & \text{for generalized plane stress} \end{cases}$$

$\mu$  being shear modulus,  $\nu$  being Poisson's ratio.

The residual loading problem for the half-plane is formulated in terms of the biharmonic function  $U$  which satisfies

$$\nabla^4 U = 0 \tag{2}$$

It is convenient to express

$$U(X, Y) = \frac{2}{\pi} \int_0^\infty \phi(X, \alpha) \sin \alpha Y d\alpha \tag{3}$$

Substitution of (3) in (2) yields

$$\left(\frac{d^2}{dX^2} - \alpha^2\right)^2 \phi = 0 \quad (4)$$

which has the solution

$$\phi(X, \alpha) = (A_1 + XA_2)e^{-\alpha X} + (A_3 + XA_4)e^{\alpha X} \quad (5)$$

Since the stresses and displacements must be bounded at infinity the Airy stress function yields as

$$U(X, Y) = \frac{2}{\pi} \int_0^{\infty} (A_1 + XA_2)e^{-\alpha X} \sin \alpha Y d\alpha \quad (6)$$

where  $A_1(\alpha)$ ,  $A_2(\alpha)$  are unknown functions to be determined from the boundary conditions. The stress components could be expressed as

$$\sigma_{XX} = \frac{\partial^2 U}{\partial Y^2}, \quad \sigma_{YY} = \frac{\partial^2 U}{\partial X^2}, \quad \sigma_{XY} = -\frac{\partial^2 U}{\partial X \partial Y} \quad (7a, b, c)$$

Upon differentiation one obtains

$$\begin{aligned} \sigma_{XX} &= -\frac{2}{\pi} \int_0^{\infty} \alpha^2 (A_1 + XA_2) e^{-\alpha X} \sin \alpha Y d\alpha \\ \sigma_{YY} &= \frac{2}{\pi} \int_0^{\infty} \alpha [\alpha A_1 + (\alpha X - 2)A_2] e^{-\alpha X} \sin \alpha Y d\alpha \\ \sigma_{XY} &= \frac{2}{\pi} \int_0^{\infty} \alpha [\alpha A_1 + (\alpha X - 1)A_2] e^{-\alpha X} \cos \alpha Y d\alpha. \end{aligned} \quad (8a, b, c)$$

Boundary conditions of the residual loading problem are

$$\sigma_{XX}(0, Y) = - \overset{*}{\sigma}_{XX}(0, Y) ,$$

$$\sigma_{XY}(0, Y) = - \overset{*}{\sigma}_{XY}(0, Y) .$$

From these conditions one can express

$$-\frac{2}{\Pi} \int_0^{\infty} \alpha^2 A_1 \sin \alpha Y d\alpha = \frac{D_X}{\Pi} \frac{Y(Y^2 + 3a_0^2)}{(a_0^2 + Y^2)^2} ,$$

$$\frac{2}{\Pi} \int_0^{\infty} (\alpha^2 A_1 - \alpha A_2) \cos \alpha Y d\alpha = \frac{D_X}{\Pi} \frac{a_0(a_0^2 - Y^2)}{(a_0^2 + Y^2)^2} . \quad (9a, b)$$

Multiplying Equation (9a) by  $\sin \alpha Y dY$ , Equation (9b) by  $\cos \alpha Y dY$  and integrating from zero to infinity gives

$$-\alpha^2 A_1 = \frac{D_X}{\Pi} \int_0^{\infty} \frac{Y(Y^2 + 3a_0^2)}{(a_0^2 + Y^2)^2} \sin \alpha Y dY ,$$

$$\alpha^2 A_1 - \alpha A_2 = \frac{D_X}{\Pi} \int_0^{\infty} \frac{a_0(a_0^2 - Y^2)}{(a_0^2 + Y^2)^2} \cos \alpha Y dY ,$$

evaluating the integrals appearing on the right yields

$$-\alpha^2 A_1 = \frac{D_X}{2} e^{-\alpha a_0} (1 + \alpha a_0) ,$$

$$\alpha^2 A_1 - \alpha A_2 = \frac{D_X}{2} \alpha a_0 e^{-\alpha a_0} ,$$

from which  $A_1$  and  $A_2$  can be determined as

$$A_1 = - \frac{D_X}{2\alpha^2} e^{-\alpha a_0} (1 + \alpha a_0) ,$$

$$A_2 = - \frac{D_X}{2\alpha} e^{-\alpha a_0} (1 + 2\alpha a_0) .$$

Hence substituting  $A_1, A_2$  in Equations (8a,b,c) results in

$$\sigma_{XX} = \frac{D_X}{\Pi} \int_0^\infty [1 + \alpha a_0 + \alpha X(1 + 2\alpha a_0)] e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha ,$$

$$\sigma_{YY} = - \frac{D_X}{\Pi} \int_0^\infty [1 + \alpha a_0 + (\alpha X - 2)(1 + 2\alpha a_0)] e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha ,$$

$$\sigma_{XY} = - \frac{D_X}{\Pi} \int_0^\infty [1 + \alpha a_0 + (\alpha X - 1)(1 + 2\alpha a_0)] e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha ,$$

or

$$\begin{aligned} \sigma_{XX} = & \frac{D_X}{\Pi} \left[ \int_0^\infty e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha + (X+a_0) \int_0^\infty \alpha e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha \right. \\ & \left. + 2a_0 X \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha \right] , \end{aligned}$$

$$\begin{aligned} \sigma_{YY} = & \frac{D_X}{\Pi} \left[ \int_0^\infty e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha + (3a_0 - X) \int_0^\infty \alpha e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha \right. \\ & \left. - 2a_0 X \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha \right] , \end{aligned}$$

$$\begin{aligned} \sigma_{XY} = & \frac{D_X}{\Pi} \left[ (a_0 - X) \int_0^\infty \alpha e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha \right. \\ & \left. - 2a_0 X \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha \right] , \end{aligned}$$

and evaluation of infinite integrals (see Appendix) gives

$$\sigma_{XX} = \frac{D_X}{\Pi} \left\{ \frac{Y}{(X+a_0)^2+Y^2} + \frac{2Y(X+a_0)^2}{[(X+a_0)^2+Y^2]^2} + \frac{4a_0XY[3(X+a_0)^2-Y^2]}{[(X+a_0)^2+Y^2]^3} \right\},$$

$$\sigma_{YY} = \frac{D_X}{\Pi} \left\{ \frac{Y}{(X+a_0)^2+Y^2} - \frac{2Y(X-3a_0)(X+a_0)}{[(X+a_0)^2+Y^2]^2} - \frac{4a_0XY[3(X+a_0)^2-Y^2]}{[(X+a_0)^2+Y^2]^3} \right\},$$

$$\sigma_{XY} = \frac{D_X}{\Pi} \left\{ (a_0-X) \frac{[(X+a_0)^2-Y^2]}{[(X+a_0)^2+Y^2]^2} - 4a_0X \frac{(X+a_0)[(X+a_0)^2-3Y^2]}{[(X+a_0)^2+Y^2]^3} \right\}.$$

(10a,b,c)

Thus the stress field of the half-plane can be obtained by superposition of expressions (1a,b,c) and (10a,b,c) as

$$\begin{aligned} \sigma_{XX}^1 &= \frac{D_X}{\Pi} \left\{ - \frac{Y[Y^2+3(X-a_0)^2]}{[(X-a_0)^2+Y^2]^2} + \frac{Y}{(X+a_0)^2+Y^2} \right. \\ &\quad \left. + \frac{2Y(X+a_0)^2}{[(X+a_0)^2+Y^2]^2} + \frac{4a_0XY[3(X+a_0)^2-Y^2]}{[(X+a_0)^2+Y^2]^3} \right\}, \end{aligned}$$

$$\begin{aligned} \sigma_{YY}^1 &= \frac{D_X}{\Pi} \left\{ \frac{Y[(X-a_0)^2-Y^2]}{[(X-a_0)^2+Y^2]^2} + \frac{Y}{(X+a_0)^2+Y^2} \right. \\ &\quad \left. - \frac{2Y(X-3a_0)(X+a_0)}{[(X+a_0)^2+Y^2]^2} - \frac{4a_0XY[3(X+a_0)^2-Y^2]}{[(X+a_0)^2+Y^2]^3} \right\}, \end{aligned}$$

$$\sigma_{XY} = \frac{D_X}{\pi} \left\{ \frac{(x-a_0) [(x-a_0)^2 - Y^2]}{[(x-a_0)^2 + Y^2]^2} + \frac{(a_0 - x) [(x+a_0)^2 - Y^2]}{[(x+a_0)^2 + Y^2]^2} - 4a_0 x \frac{(x+a_0) [(x+a_0)^2 - 3Y^2]}{[(x+a_0)^2 + Y^2]^3} \right\} \quad (11a,b,c)$$

An edge dislocation, of Burgers vector  $\vec{b}_Y$ , parallel to Y axis, at the point  $(a_0, 0)$  of an infinite plane (see Figure 2) has the following stress field [12]

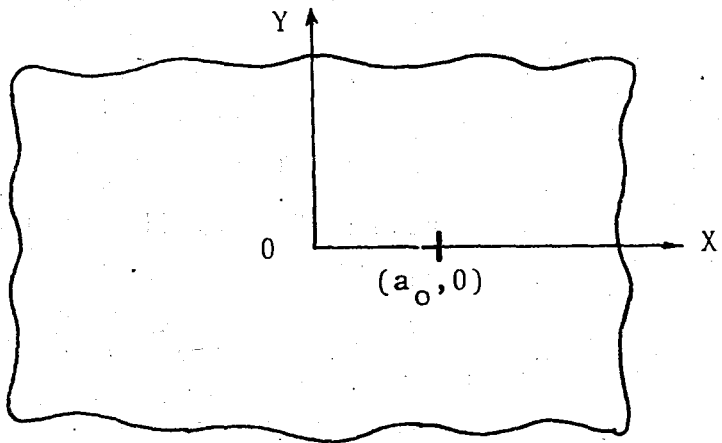


FIGURE 2. An Edge Dislocation at the Position  $(a_0, 0)$  with Burgers Vector  $\vec{b}_Y$  in an Infinite Plane.

$$\sigma_{XX} = \frac{D_Y}{\pi} \frac{(X-a_0) [(X-a_0)^2 - Y^2]}{[(X-a_0)^2 + Y^2]^2},$$

$$\sigma_{YY} = \frac{D_Y}{\pi} \frac{(X-a_0) [(X-a_0)^2 + 3Y^2]}{[(X-a_0)^2 + Y^2]^2}.$$

$$\sigma_{XY} = \frac{D_Y}{\Pi} \frac{Y[(X-a_0)^2 - Y^2]}{[(X-a_0)^2 + Y^2]^2} \quad (12a,b,c)$$

where,

$$D_Y = \frac{2\mu b_Y}{(1+\kappa)}$$

Again the residual loading problem will be formulated in terms of the biharmonic function  $U(X,Y)$  which may be expressed as

$$U(X,Y) = \frac{2}{\Pi} \int_0^{\infty} \phi(X,\alpha) \cos \alpha Y d\alpha \quad (13)$$

for this case.

Substitution of (13) in (2) gives

$$\left(\frac{d^2}{dX^2} - \alpha^2\right)^2 \phi = 0$$

A solution which is bounded at infinity can be substituted in (13). Hence one obtains

$$U(X,Y) = \frac{2}{\Pi} \int_0^{\infty} (A_1 + X A_2) e^{-\alpha X} \cos \alpha Y d\alpha$$

Again  $A_1(\alpha)$ ,  $A_2(\alpha)$  will be determined from the boundary conditions. The stress components are derivable from (7a,b,c) as

$$\sigma_{XX} = - \frac{2}{\Pi} \int_0^{\infty} \alpha^2 (A_1 + X A_2) e^{-\alpha X} \cos \alpha Y d\alpha ,$$

$$\sigma_{YY} = \frac{2}{\Pi} \int_0^{\infty} \alpha [\alpha A_1 + (\alpha X - 2) A_2] e^{-\alpha X} \cos \alpha Y d\alpha ,$$

$$\sigma_{XY} = - \frac{2}{\Pi} \int_0^{\infty} \alpha [\alpha A_1 + (\alpha X - 1) A_2] e^{-\alpha X} \sin \alpha Y d\alpha . \quad (14a,b,c)$$

Now expressing the boundary conditions of the residual loading problem

$$\sigma_{XX}(0, Y) = - \sigma_{XX}^*(0, Y) ,$$

$$\sigma_{XY}(0, Y) = - \sigma_{XY}^*(0, Y) ,$$

one obtains

$$- \frac{2}{\Pi} \int_0^{\infty} \alpha^2 A_1 \cos \alpha Y d\alpha = \frac{D_Y}{\Pi} \frac{a_0 (a_0^2 - Y^2)}{(a_0^2 + Y^2)^2} ,$$

$$- \frac{2}{\Pi} \int_0^{\infty} (\alpha^2 A_1 - \alpha A_2) \sin \alpha Y d\alpha = - \frac{D_Y}{\Pi} \frac{Y (a_0^2 - Y^2)}{(a_0^2 + Y^2)^2} . \quad (15a,b)$$

Multiplying Equation (15a) by  $\cos \alpha Y dY$ , Equation (15b) by  $\sin \alpha Y dY$  and integrating both sides from zero to infinity gives

$$- \alpha^2 A_1 = \frac{D_Y}{\Pi} \int_0^{\infty} \frac{a_0 (a_0^2 - Y^2)}{(a_0^2 + Y^2)^2} \cos \alpha Y dY ,$$

$$- \alpha^2 A_1 + \alpha A_2 = \frac{D_Y}{\Pi} \int_0^{\infty} \frac{Y (Y^2 - a_0^2)}{(a_0^2 + Y^2)^2} \sin \alpha Y dY$$

evaluating the integrals appearing on the right yields

$$-\alpha^2 A_1 = \frac{D_Y}{2} \cdot \alpha a_0 e^{-\alpha a_0} ,$$

$$-\alpha^2 A_1 + \alpha A_2 = \frac{D_Y}{2} e^{-\alpha a_0} (1 - \alpha a_0) ,$$

from which  $A_1$  and  $A_2$  can be determined as

$$A_1 = - \frac{D_Y}{2\alpha} a_0 e^{-\alpha a_0} ,$$

$$A_2 = \frac{D_Y}{2\alpha} e^{-\alpha a_0} (1 - 2\alpha a_0) .$$

Substitution of  $A_1$ ,  $A_2$  in Equations (14a,b,c) results in

$$\sigma_{XX} = \frac{D_Y}{\Pi} \int_0^\infty [\alpha a_0 + \alpha X(2\alpha a_0 - 1)] e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha ,$$

$$\sigma_{YY} = \frac{D_Y}{\Pi} \int_0^\infty [3\alpha a_0 - 2 + \alpha X(1 - 2\alpha a_0)] e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha ,$$

$$\sigma_{XY} = \frac{D_Y}{\Pi} \int_0^\infty [\alpha a_0 + (\alpha X - 1)(2\alpha a_0 - 1)] e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha ,$$

or

$$\begin{aligned} \sigma_{XX} = \frac{D_Y}{\Pi} & \left[ (a_0 - X) \int_0^\infty \alpha e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha \right. \\ & \left. + 2a_0 X \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha \right] , \end{aligned}$$

$$\begin{aligned} \sigma_{YY} = & \frac{D_Y}{\Pi} \left[ -2 \int_0^{\infty} e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha \right. \\ & + (X+3a_0) \int_0^{\infty} \alpha e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha \\ & \left. - 2a_0 X \int_0^{\infty} \alpha^2 e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha \right], \end{aligned}$$

$$\begin{aligned} \sigma_{XY} = & \frac{D_Y}{\Pi} \left[ \int_0^{\infty} e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha \right. \\ & - (X+a_0) \int_0^{\infty} \alpha e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha \\ & \left. + 2a_0 X \int_0^{\infty} \alpha^2 e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha \right], \end{aligned}$$

and evaluation of infinite integrals (see Appendix) gives

$$\sigma_{XX} = \frac{D_Y}{\Pi} \left\{ (a_0 - X) \frac{[(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^2} + 4a_0 X \frac{(X+a_0)[(X+a_0)^2 - 3Y^2]}{[(X+a_0)^2 + Y^2]^3} \right\},$$

$$\begin{aligned} \sigma_{YY} = & \frac{D_Y}{\Pi} \left\{ -2 \frac{(X+a_0)}{(X+a_0)^2 + Y^2} + (X+3a_0) \frac{[(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^2} \right. \\ & \left. - 4a_0 X \frac{(X+a_0)[(X+a_0)^2 - 3Y^2]}{[(X+a_0)^2 + Y^2]^3} \right\}, \end{aligned}$$

$$\sigma_{XY} = \frac{D_Y}{\Pi} \left\{ \frac{Y}{(X+a_0)^2 + Y^2} - \frac{2Y(X+a_0)^2}{[(X+a_0)^2 + Y^2]^2} \right.$$

$$\left. + 4a_0 XY \frac{[3(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^3} \right\}. \quad (16a, b, c)$$

Consequently the stress field of the half-plane can be

obtained by superposition of expressions (12a,b,c) and (16a,b,c) as

$$\begin{aligned} \sigma_{XX}^2 &= \frac{D_Y}{\pi} \left\{ \frac{(X-a_0) [(X-a_0)^2 - Y^2]}{[(X-a_0)^2 + Y^2]^2} + (a_0 - X) \frac{[(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^2} \right. \\ &\quad \left. + 4a_0 X \frac{(X+a_0) [(X+a_0)^2 - 3Y^2]}{[(X+a_0)^2 + Y^2]^3} \right\}, \\ \sigma_{YY}^2 &= \frac{D_Y}{\pi} \left\{ (X-a_0) \frac{[(X-a_0)^2 + 3Y^2]}{[(X-a_0)^2 + Y^2]^2} - 2 \frac{(X+a_0)}{(X+a_0)^2 + Y^2} \right. \\ &\quad \left. + (X+3a_0) \frac{[(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^2} - 4a_0 X \frac{(X+a_0) [(X+a_0)^2 - 3Y^2]}{[(X+a_0)^2 + Y^2]^3} \right\}, \\ \sigma_{XY}^2 &= \frac{D_Y}{\pi} \left\{ \frac{Y [(X-a_0)^2 - Y^2]}{[(X-a_0)^2 + Y^2]^2} + \frac{Y}{(X+a_0)^2 + Y^2} \right. \\ &\quad \left. - \frac{2Y(X+a_0)^2}{[(X+a_0)^2 + Y^2]^2} + 4a_0 XY \frac{[3(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^3} \right\}. \quad (17a, b, c) \end{aligned}$$

A powerful method of analysis is based on the re-

presentation of a crack by a continuous distribution of dislocations. To simulate a line crack in s direction

(see Figure 3) stress solutions for a pair of edge dislocations with Burgers vectors  $\vec{b}_s$  and  $\vec{b}_q$  can be used as

Green's functions. Stress distributions due to  $\vec{b}_s$  and  $\vec{b}_q$  can be obtained by resolving these vectors into components in X and Y directions and summing the stresses set up by

these components.

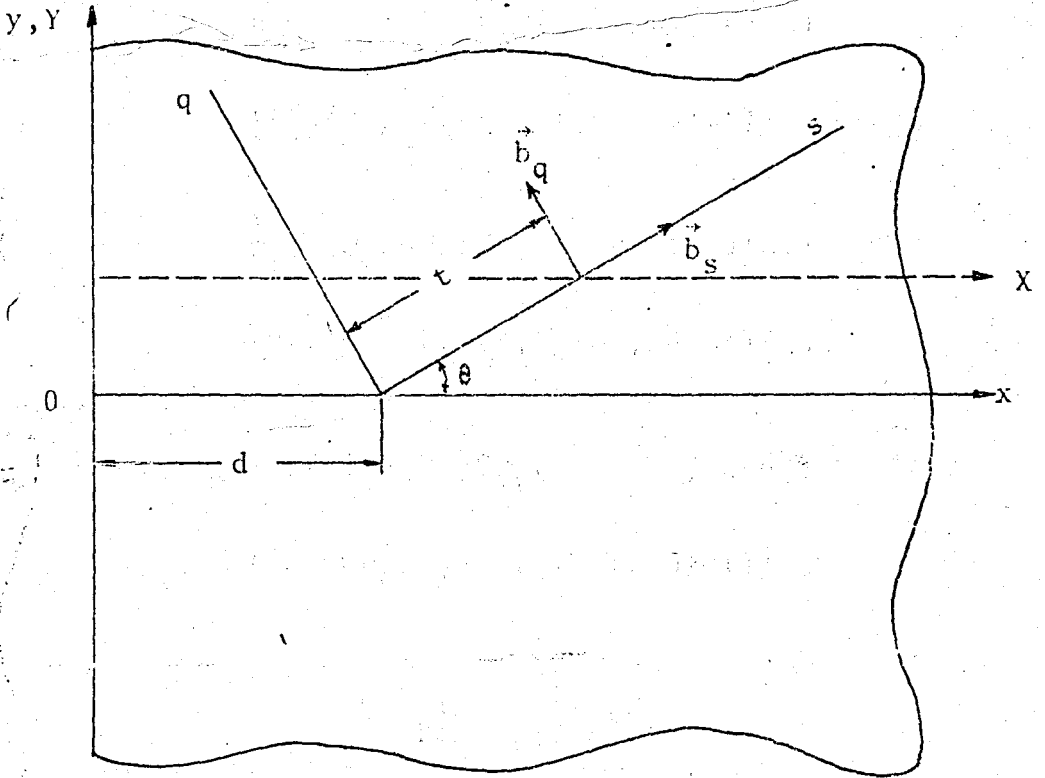


FIGURE 3. A Pair of Edge Dislocations with Burgers Vectors  $\vec{b}_s$  and  $\vec{b}_q$  in a Half-Plane.

Hence introducing the notation

$$\alpha = \cos\theta, \quad \beta = \sin\theta,$$

$$D_s = \frac{2\mu}{1+\kappa} b_s, \quad D_q = \frac{2\mu}{1+\kappa} b_q$$

we obtain the stresses due to edge dislocations with

Burgers vectors  $\vec{b}_s$  and  $\vec{b}_q$  as

$$\sigma_{XX}(X, y) = \frac{1}{\Pi} [(\beta D_q - \alpha D_s) G_{XX}(x, y - \beta t, d + \alpha t) - (\beta D_s + \alpha D_q) F_{XX}(x, y - \beta t, d + \alpha t)] ,$$

$$\sigma_{YY}(X, y) = \frac{1}{\Pi} [(\beta D_q - \alpha D_s) G_{YY}(x, y - \beta t, d + \alpha t) - (\beta D_s + \alpha D_q) F_{YY}(x, y - \beta t, d + \alpha t)] ,$$

$$\sigma_{XY}(X, y) = \frac{1}{\Pi} [(\beta D_q - \alpha D_s) G_{XY}(x, y - \beta t, d + \alpha t) - (\beta D_s + \alpha D_q) F_{XY}(x, y - \beta t, d + \alpha t)] ,$$

where

$$G_{XX}(X, Y, a_0) = - \frac{\Pi \sigma_{XX}^1(X, Y, a_0)}{D_X} ,$$

$$G_{YY}(X, Y, a_0) = - \frac{\Pi \sigma_{YY}^1(X, Y, a_0)}{D_X} ,$$

$$G_{XY}(X, Y, a_0) = - \frac{\Pi \sigma_{XY}^1(X, Y, a_0)}{D_X} ,$$

$$F_{XX}(X, Y, a_0) = - \frac{\Pi \sigma_{XX}^2(X, Y, a_0)}{D_Y} ,$$

$$F_{YY}(X, Y, a_0) = - \frac{\Pi \sigma_{YY}^2(X, Y, a_0)}{D_Y} ,$$

$$F_{XY}(X, Y, a_0) = - \frac{\Pi \sigma_{XY}^2(X, Y, a_0)}{D_Y} .$$

(18a-f)

At a point along the  $s$ -axis since  $x=d+\alpha s$ ,  $y=\beta s$  the normal and shear stress components  $\sigma_{qq}$  and  $\sigma_{qs}$  in the  $\{s,q\}$  coordinate system can be expressed as

$$\sigma_{qq}(0,s) = \beta^2 \sigma_{xx}(d+\alpha s, \beta s) + \alpha^2 \sigma_{yy}(d+\alpha s, \beta s) - 2\alpha\beta \sigma_{xy}(d+\alpha s, \beta s) \quad ,$$

$$\sigma_{qs}(0,s) = (\alpha-\beta) \sigma_{xy}(d+\alpha s, \beta s) + \alpha\beta [\sigma_{yy}(d+\alpha s, \beta s) - \sigma_{xx}(d+\alpha s, \beta s)] \quad .$$

A crack along the  $s$  axis can be represented by continuous arrays of dislocations with density functions

$$f(s) = \frac{2\mu}{1+\kappa} \frac{\partial}{\partial s} [v(0^+,s) - v(0^-,s)] \quad , \quad (19)$$

$$g(s) = \frac{2\mu}{1+\kappa} \frac{\partial}{\partial s} [u(0^+,s) - u(0^-,s)] \quad , \quad (20)$$

where  $u$  and  $v$  are the displacement components in the  $\{s,q\}$  coordinate system. As the crack extends from  $s=-a$  to  $s=a$  we obviously have

$$f(s) = g(s) = 0 \quad \text{for} \quad |s| > \bar{a} \quad .$$

To obtain the stresses due to distributions of

edge dislocations we replace  $D_s$  and  $D_q$  by  $-f(t)dt$  and  $-g(t)dt$  respectively and integrate from  $-a$  to  $a$ . The unknown density functions  $f(t)$  and  $g(t)$  will be determined from the conditions that stresses are prescribed on the crack surface which yield

$$\sigma_{qq}(0,s) = \frac{1}{\pi} \int_{-a}^a k(s,t)f(t)dt + \frac{1}{\pi} \int_{-a}^a h(s,t)g(t)dt = p(s),$$

$-a < s < a,$

$$\sigma_{qs}(0,s) = \frac{1}{\pi} \int_{-a}^a m(s,t)f(t)dt + \frac{1}{\pi} \int_{-a}^a n(s,t)g(t)dt = q(s),$$

$-a < s < a$

(21a,b)

where

$$k(s,t) = \beta^2 [\alpha F_{XX}(d+as, \beta s - \beta t, d+at) - \beta G_{XX}(d+as, \beta s - \beta t, d+at)] + \alpha^2 [\alpha F_{YY}(d+as, \beta s - \beta t, d+at) - \beta G_{YY}(d+as, \beta s - \beta t, d+at)] - 2\alpha\beta [\beta F_{XY}(d+as, \beta s - \beta t, d+at) + \alpha G_{XY}(d+as, \beta s - \beta t, d+at)],$$

$$h(s,t) = \beta^2 [\beta F_{XX}(d+as, \beta s - \beta t, d+at) + \alpha G_{XX}(d+as, \beta s - \beta t, d+at)] + \alpha^2 [\beta F_{YY}(d+as, \beta s - \beta t, d+at) + \alpha G_{YY}(d+as, \beta s - \beta t, d+at)] - 2\alpha\beta [\beta F_{XY}(d+as, \beta s - \beta t, d+at) + \alpha G_{XY}(d+as, \beta s - \beta t, d+at)],$$

$$\begin{aligned}
 m(s,t) = & (\alpha - \beta)^2 \left[ \alpha F_{XY}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right. \\
 & \left. - \beta G_{XY}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right] \\
 & + \alpha \beta \left\{ \alpha \left[ F_{YY}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right. \right. \\
 & \left. \left. - F_{XX}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right] \right. \\
 & \left. - \beta \left[ G_{YY}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right. \right. \\
 & \left. \left. - G_{XX}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right] \right\} ,
 \end{aligned}$$

$$\begin{aligned}
 n(s,t) = & (\alpha - \beta)^2 \left[ \beta F_{XY}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right. \\
 & \left. + \alpha G_{XY}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right] \\
 & + \alpha \beta \left\{ \beta \left[ F_{YY}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right. \right. \\
 & \left. \left. - F_{XX}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right] \right. \\
 & \left. + \alpha \left[ G_{YY}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right. \right. \\
 & \left. \left. - G_{XX}(d+\alpha s, \beta s - \beta t, d+\alpha t) \right] \right\} .
 \end{aligned}$$

To provide the continuity of the material outside the crack we must impose the two conditions

$$\begin{aligned}
 \int_{-a}^a f(t) dt &= 0 , \\
 \int_{-a}^a g(t) dt &= 0 . \qquad (22a, b)
 \end{aligned}$$

It can easily be shown that the kernels  $k(s,t)$ ,  $n(s,t)$  have Cauchy singularities, namely

$$k(s,t) = \frac{1}{t-s} + \tilde{k}(s,t) ,$$

$$n(s,t) = \frac{1}{t-s} + \dot{n}^*(s,t) ,$$

where  $\dot{k}^*(s,t)$ ,  $\dot{n}^*(s,t)$  are bounded functions.

Thus the Equations (21a,b) form two singular integral equations of the first kind. These singular integral equations take simple forms when the crack is perpendicular or parallel to the free surface. If the crack is perpendicular to the free surface we take  $\alpha=1$ ,  $\beta=0$  in the expressions (21a,b) and we obtain

$$\frac{1}{\pi} \int_{-a}^a \left[ \frac{1}{t-s} + \frac{s-t}{(t+s+2d)^2} + \frac{4(t+d)(s+d)}{(t+s+2d)^3} \right] f(t) dt = p(s), \quad -a < s < a$$

$$\frac{1}{\pi} \int_{-a}^a \left[ \frac{1}{t-s} + \frac{s-t}{(t+s+2d)^2} + \frac{4(t+d)(s+d)}{(t+s+2d)^3} \right] g(t) dt = q(s), \quad -a < s < a$$

(23a,b)

Similarly for the case of the crack being parallel to the interface we take  $\alpha=0$ ,  $\beta=1$  and we could express

$$\frac{1}{\pi} \int_{-a}^a \left\{ \frac{1}{t-s} - \frac{t-s}{(t-s)^2 + 4d^2} - \frac{8d^2(t-s)}{[(t-s)^2 + 4d^2]^2} - \frac{4d^2(t-s)[12d^2 - (t-s)^2]}{[(t-s)^2 + 4d^2]^3} \right\} f(t) dt$$

$$- \frac{1}{\pi} \int_{-a}^a \left\{ 8d^3 \frac{[4d^2 - 3(t-s)^2]}{[(t-s)^2 + 4d^2]^3} \right\} g(t) dt = p(s), \quad -a < s < a$$

$$\frac{1}{\pi} \int_{-a}^a \{ 8d^3 \frac{[4d^2 - 3(t-s)^2]}{[(t-s)^2 + 4d^2]^3} \} f(t) dt + \frac{1}{\pi} \int_{-a}^a \left\{ \frac{1}{t-s} - \frac{t-s}{(t-s)^2 + 4d^2} \right. \\ \left. + \frac{8d^2(t-s)}{[(t-s)^2 + 4d^2]^2} - \frac{4d^2(t-s)[12d^2 - (t-s)^2]}{[(t-s)^2 + 4d^2]^3} \right\} g(t) dt = q(s), \\ -a < s < a$$

(24a, b)

These expressions are in agreement with those obtained by Asbaugh [1] and also with the expressions given by Erdogan [9] except the sign discrepancy which was remarked by Asbaugh [1].

Now we can easily extend the discussion to a system of randomly oriented cracks (see Figure 4).

For this case using the same formulation described for a single crack and using  $\{s_i, q_i\}, (i=1, \dots, n)$  rectangular coordinate systems, expressions of crack surface tractions yield the following system of singular integral equations

$$\frac{1}{\pi} \sum_{j=1}^n \int_{-a_j}^{a_j} k_{ij}(s_i, t_j) f_j(t_j) dt_j + \int_{-a_j}^{a_j} h_{ij}(s_i, t_j) g_j(t_j) dt_j \\ = p(s_i), \\ -a_i < s_i < a_i, \\ (i=1, \dots, n),$$

$$\frac{1}{\Pi} \sum_{j=1}^n \left[ \int_{-a_j}^{a_j} m_{ij}(s_i, t_j) f_j(t_j) dt_j + \int_{-a_j}^{a_j} n_{ij}(s_i, t_j) g_j(t_j) dt_j \right]$$

$$= q(s_i) ,$$

$$- a_i < s_i < a_i ,$$

$$(i=1, \dots, n) ,$$

(25a,b)

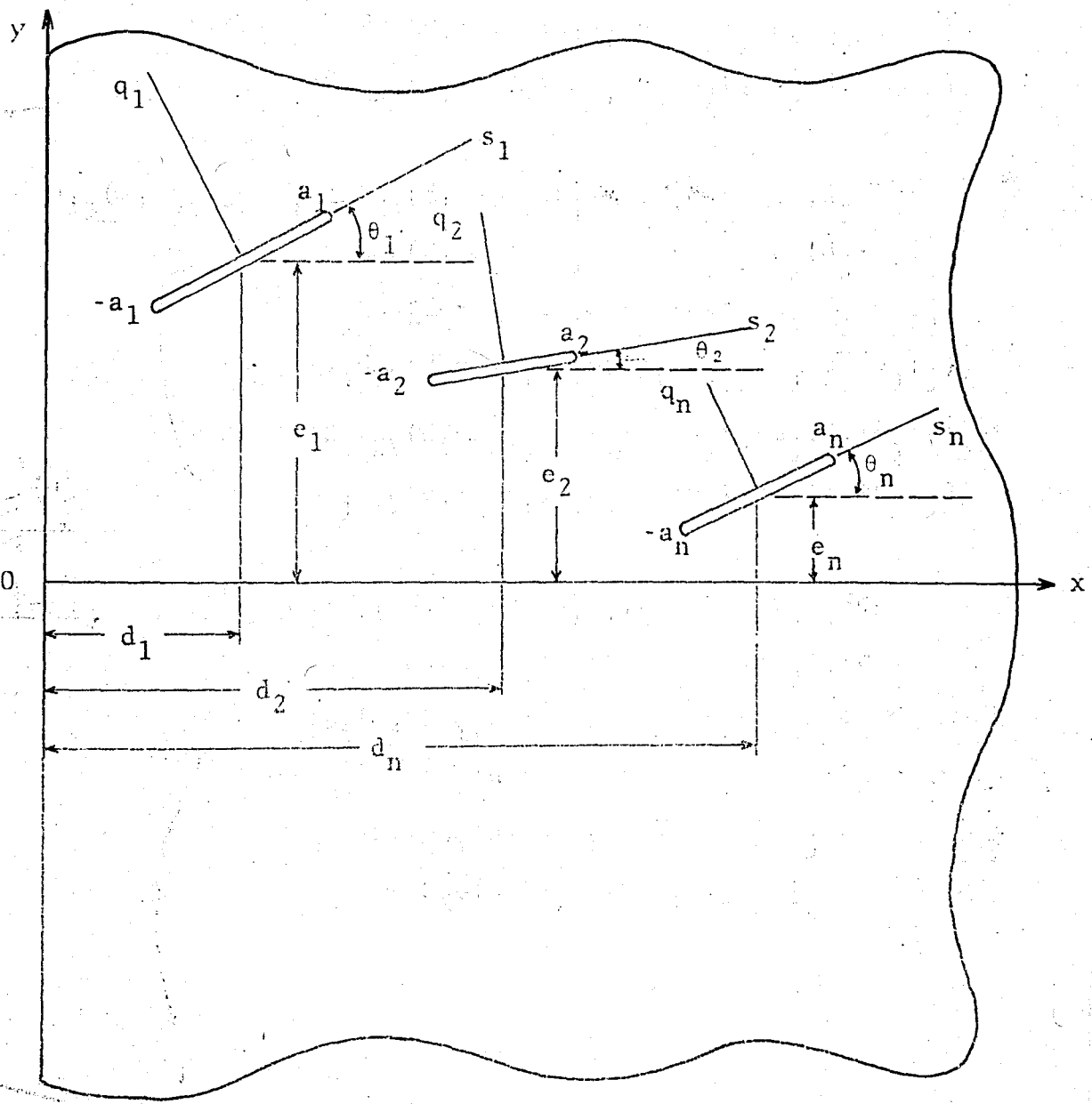


FIGURE 4. A System of Randomly Oriented Cracks in a Half-Plane.

where

$$\begin{aligned}
 k_{ij}(s_i, t_j) = & \beta_i^2 [\alpha_j F_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & - \beta_j G_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] \\
 & + \alpha_i^2 [\alpha_j F_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & - \beta_j G_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] \\
 & - 2\alpha_i \beta_i [\alpha_j F_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & - \beta_j G_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] ,
 \end{aligned}$$

$$\begin{aligned}
 h_{ij}(s_i, t_j) = & \beta_i^2 [\beta_j F_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & + \alpha_j G_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] \\
 & + \alpha_i^2 [\beta_j F_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & + \alpha_j G_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] \\
 & - 2\alpha_i \beta_i [\beta_j F_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & + \alpha_j G_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] ,
 \end{aligned}$$

$$\begin{aligned}
 m_{ij}(s_i, t_j) = & (\alpha_i - \beta_i)^2 [\alpha_j F_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & - \beta_j G_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] , \\
 & + \alpha_i \beta_i \{ \alpha_j [F_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & - F_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] , \\
 & - \beta_j [G_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & - G_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] \} ,
 \end{aligned}$$

$$\begin{aligned}
 n_{ij}(s_i, t_j) = & (\alpha_i^2 - \beta_i^2) [\beta_j F_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & + \alpha_j G_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] \\
 & + \alpha_i \beta_i \{ \beta_j [F_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & - F_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] \\
 & + \alpha_j [G_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j) \\
 & - G_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)] \}
 \end{aligned}$$

Again to provide the continuity of the material outside the cracks we must impose the single-valuedness conditions

$$\int_{-a_j}^{a_j} f_j(t_j) dt_j = 0, \quad (j=1, \dots, n),$$

$$\int_{-a_j}^{a_j} g_j(t_j) dt_j = 0, \quad (j=1, \dots, n). \quad (26a, b)$$

For a system of radial cracks the singular integral equations take simpler forms and they can be readily expressed (see Figure 5) as

$$\begin{aligned}
 \frac{1}{\pi} \sum_{j=1}^n \left[ \int_{a_j}^{b_j} k_{ij}(s_i, t_j) f_j(t_j) dt_j + \right. \\
 \left. \int_{a_j}^{b_j} h_{ij}(s_i, t_j) g_j(t_j) dt_j \right] = p(s_i), \\
 a_i < s_i < b_i, \quad (i=1, \dots, n),
 \end{aligned}$$

$$\frac{1}{\Pi} \sum_{j=1}^n \int_{a_j}^{b_j} m_{ij}(s_i, t_j) f_j(t_j) dt_j + \int_{a_j}^{b_j} n_{ij}(s_i, t_j) g_j(t_j) dt_j = q(s_i)$$

$$a_i < s_i < b_i, \quad (i=1, \dots, n),$$

(27a,b)

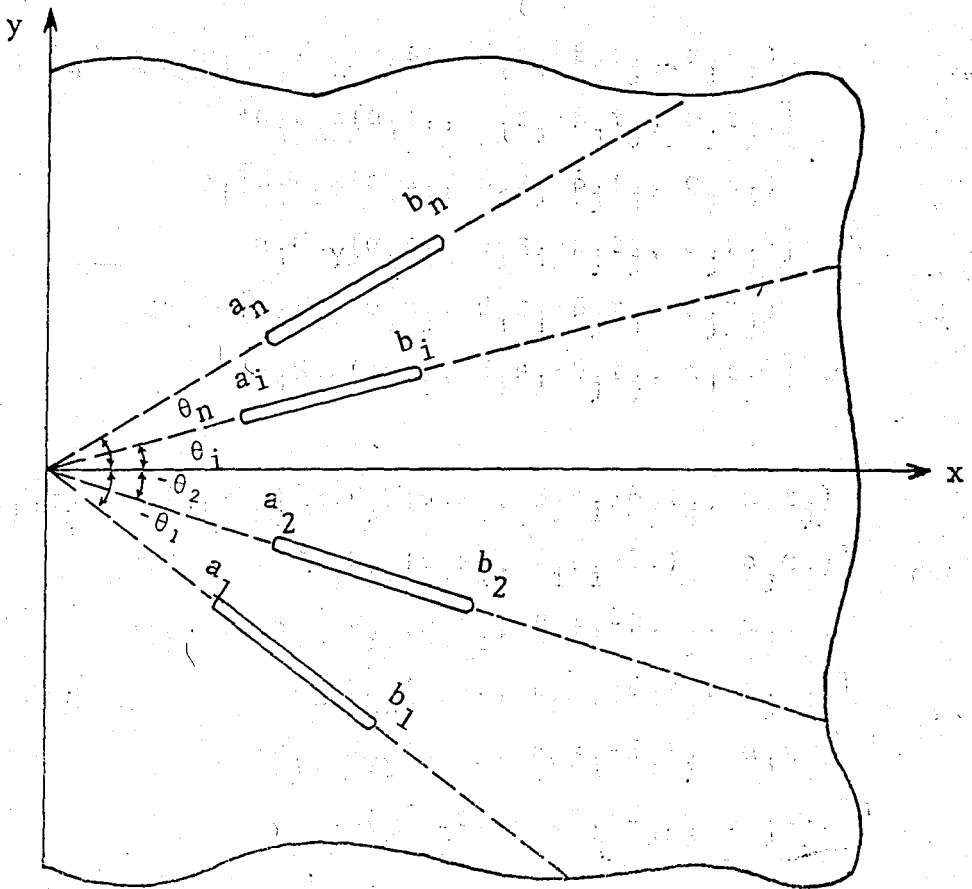


FIGURE 5. A System of Radial Cracks in a Half-Plane.

where

$$k_{ij}(s_i, t_j) = \beta_i^2 [\alpha_j F_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) - \beta_j G_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] + \alpha_i^2 [\alpha_j F_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) - \beta_j G_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] - 2\alpha_i \beta_i [\alpha_j F_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) - \beta_j G_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] ,$$

$$h_{ij}(s_i, t_j) = \beta_i^2 [\beta_j F_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) + \alpha_j G_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] + \alpha_i^2 [\beta_j F_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) + \alpha_j G_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] - 2\alpha_i \beta_i [\beta_j F_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) + \alpha_j G_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] ,$$

$$m_{ij}(s_i, t_j) = (\alpha_i^2 - \beta_i^2) [\alpha_j F_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) - \beta_j G_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] + \alpha_i \beta_i \{ \alpha_j [ F_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) - F_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] - \beta_j [ G_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) - G_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] \} ,$$

$$\begin{aligned}
 n_{ij}(s_i, t_j) = & (\alpha_i^2 - \beta_i^2) [\beta_j F_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) \\
 & + \alpha_j G_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] \\
 & + \alpha_i \beta_i \{ \beta_j [F_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) \\
 & - F_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] \\
 & + \alpha_j [G_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j) \\
 & - G_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] \} .
 \end{aligned}$$

Equations (27a,b) are subjected to the following single-valuedness conditions

$$\int_{a_j}^{b_j} f_j(t_j) dt_j = 0, \quad (j=1, \dots, n),$$

$$\int_{a_j}^{b_j} g_j(t_j) dt_j = 0, \quad (j=1, \dots, n). \quad (28a,b)$$

The system of singular integral equations (21a,b), (25a,b) and (27a,b) will be solved numerically. The numerical procedure will be described in the next chapter.

## CHAPTER 3

### NUMERICAL SOLUTION OF SINGULAR INTEGRAL EQUATIONS

The quadrature method developed [10] to solve integral equations with Cauchy type kernels will be used. Now the method will be described for equations (21a,b).

First the intervals of integration are normalized by introducing the new variables

$$\tau = \frac{t}{a}, \quad \xi = \frac{s}{a}$$

and the unknown functions can be expressed as

$$\begin{aligned} f(t) &= F(\tau)(1-\tau^2)^{-1/2}, \\ g(t) &= G(\tau)(1-\tau^2)^{-1/2}. \end{aligned}$$

Defining the new kernels by

$$\begin{aligned} K(\xi, \tau) &= ak(s, t), & H(\xi, \tau) &= ah(s, t), \\ M(\xi, \tau) &= am(s, t), & N(\xi, \tau) &= an(s, t) \end{aligned}$$

equations (21a,b) and (22a,b) become

$$\begin{aligned} & \frac{1}{\pi} \int_{-1}^1 K(\xi, \tau) F(\tau) (1-\tau^2)^{-1/2} d\tau + \\ & \frac{1}{\pi} \int_{-1}^1 H(\xi, \tau) G(\tau) (1-\tau^2)^{-1/2} d\tau = p(\xi) \quad -1 < \xi < 1, \\ & \frac{1}{\pi} \int_{-1}^1 M(\xi, \tau) F(\tau) (1-\tau^2)^{-1/2} d\tau + \\ & \frac{1}{\pi} \int_{-1}^1 N(\xi, \tau) G(\tau) (1-\tau^2)^{-1/2} d\tau = q(\xi) \quad -1 < \xi < 1, \end{aligned} \quad (29a, b)$$

$$\int_{-1}^1 F(\tau) (1-\tau^2)^{-1/2} d\tau = 0,$$

$$\int_{-1}^1 G(\tau) (1-\tau^2)^{-1/2} d\tau = 0. \quad (30a, b)$$

Applying the Gauss-Chebyshev quadrature [10] to these equations yields the following system of linear equations for the determination of  $F(\tau_k)$  and  $G(\tau_k)$

$$\begin{aligned} & \sum_{k=1}^N W_k [K(\xi_j, \tau_k) F(\tau_k) + H(\xi_j, \tau_k) G(\tau_k)] = p(\xi_j), \\ & \sum_{k=1}^N W_k [M(\xi_j, \tau_k) F(\tau_k) + N(\xi_j, \tau_k) G(\tau_k)] = q(\xi_j), \\ & \sum_{k=1}^N W_k F(\tau_k) = 0, \quad \sum_{k=1}^N W_k G(\tau_k) = 0, \end{aligned} \quad (31a-d)$$

where

$$W_1 = W_n = \frac{1}{2(N-1)}, \quad W_k = \frac{1}{N-1}, \quad k=2, \dots, N-1,$$

$$\tau_k = \cos\left(\pi \frac{k-1}{N-1}\right), \quad k=1, \dots, N,$$

$$\xi_\ell = \cos\left(\pi \frac{2\ell-1}{2N-2}\right), \quad \ell=1, \dots, N-1. \quad (32a, b, c)$$

The quantities of main importance are stress intensity factors which are defined as

$$k_1(a) = \lim_{s \rightarrow a} \sqrt{2(s-a)} \sigma_{qq}(0, s),$$

$$k_1(-a) = \lim_{s \rightarrow -a} \sqrt{2(s+a)} \sigma_{qq}(0, s),$$

$$k_2(a) = \lim_{s \rightarrow a} \sqrt{2(s-a)} \sigma_{qs}(0, s),$$

$$k_2(-a) = \lim_{s \rightarrow -a} \sqrt{2(s+a)} \sigma_{qs}(0, s). \quad (33a-d)$$

The stress intensity factors are proportional to the values of  $F(\tau)$  and  $G(\tau)$  at the corresponding end points through the relations

$$k_1(-a) = F(-1) \sqrt{a}, \quad k_2(-a) = G(-1) \sqrt{a},$$

$$k_1(a) = -F(1) \sqrt{a}, \quad k_2(a) = -G(1) \sqrt{a}.$$

For a system of randomly oriented cracks the new variables are defined as

$$\tau_j = \frac{t_j}{a_j}, \quad \xi_i = \frac{s_i}{a_i} \quad (i, j=1, \dots, n)$$

and the unknown functions can be expressed as

$$f_j(t_j) = F_j(\tau_j)(1-\tau_j^2)^{-1/2},$$

$$g_j(t_j) = G_j(\tau_j)(1-\tau_j^2)^{-1/2} \quad (j=1, \dots, n)$$

introducing the new kernels

following system of linear algebraic equations for unknowns  $F_j(\tau_j)$  and  $G_j(\tau_j)$  ( $j=1, \dots, n$ ), ( $k=1, \dots, N$ ).

$$K_{ij}(\xi_i, \tau_j) = a_j k_{ij}(s_i, t_j),$$

$$H_{ij}(\xi_i, \tau_j) = a_j h_{ij}(s_i, t_j),$$

$$M_{ij}(\xi_i, \tau_j) = a_j m_{ij}(s_i, t_j),$$

$$N_{ij}(\xi_i, \tau_j) = a_j n_{ij}(s_i, t_j)$$

(34a-d)

equations (25a,b) and (26a,b) become

$$\frac{1}{\pi} \sum_{j=1}^n \left[ \int_{-1}^1 K_{ij}(\xi_i, \tau_j) F_j(\tau_j) (1-\tau_j^2)^{-1/2} d\tau_j \right. \\ \left. + \int_{-1}^1 H_{ij}(\xi_i, \tau_j) G_j(\tau_j) (1-\tau_j^2)^{-1/2} d\tau_j \right] = P_i(\xi_i)$$

$-1 < \xi_i < 1$  (36a-d)

$(i=1, \dots, n)$

where  $\prod_{j=1}^n \left[ \int_{-1}^1 M_{ij}(\xi_i, \tau_j) F_j(\tau_j) (1-\tau_j^2)^{-1/2} d\tau_j \right]$  were previously defined.

The stress intensity factors can be defined with expressions similar to (33a-d) and they can be obtained

from the values of  $F_j(\tau_j)$  and  $G_j(\tau_j)$  at  $-1 < \xi_i < 1$ , corresponding end points.  
( $i=1, \dots, n$ )

$$\int_{-1}^1 F_j(\tau_j) (1-\tau_j^2)^{-1/2} d\tau_j = 0,$$

For a system of radial cracks the new variables are defined as  $\int_{-1}^1 G_j(\tau_j) (1-\tau_j^2)^{-1/2} d\tau_j = 0, (j=1, \dots, n)$  (35a-d)

$$\tau_j = \frac{2s_j - (b_j + a_j)}{b_j - a_j}$$

Now applying the Gauss-Chebyshev quadrature gives the following system of linear algebraic equations for unknowns  $F_j(\tau_k)$  and  $G_j(\tau_k)$  ( $j=1, \dots, n$ ), ( $k=1, \dots, N$ ).

and the unknown functions can be expressed as

$$\sum_{j=1}^n \sum_{k=1}^N W_k F_j(\tau_k) K_{ij}(\xi_\ell, \tau_k) + G_j(\tau_k) H_{ij}(\xi_\ell, \tau_k) = P_i(\xi_\ell),$$

$$s_j(\tau_j) = G_j(\tau_j) (1-\tau_j^2)^{-1/2}, \quad (j=1, \dots, n)$$

$$\sum_{j=1}^n \sum_{k=1}^N W_k F_j(\tau_k) M_{ij}(\xi_\ell, \tau_k) + G_j(\tau_k) N_{ij}(\xi_\ell, \tau_k) = q_i(\xi_\ell),$$

Introducing the new kernels

$$(i=1, \dots, n), (\ell=1, \dots, N)$$

$$K_{ij}(\xi_i, \tau_j) = \frac{b_j - a_j}{2} k_{ij}(s_i, \tau_j),$$

$$\sum_{k=1}^N W_k F_j(\tau_k) = 0,$$

$$H_{ij}(\xi_i, \tau_j) = \frac{b_j - a_j}{2} h_{ij}(s_i, \tau_j),$$

$$\sum_{k=1}^N W_k G_j(\tau_k) = 0, \quad (j=1, \dots, n), \quad (36a-d)$$

$$M_{ij}(\xi_i, \tau_j) = \frac{b_j - a_j}{2} m_{ij}(s_i, \tau_j),$$

where  $\tau_k$ ,  $\xi_\ell$ ,  $W_k$  ( $k, \ell=1, \dots, N$ ) were previously defined.

The stress intensity factors can be defined with expressions similar to (33a-d) and they can be obtained from the values of  $F_j(\tau_j)$  and  $G_j(\tau_j)$  at the corresponding end points.

For a system of radial cracks the new variables are defined as

$$\tau_j = \frac{2t_j}{b_j - a_j} - \frac{b_j + a_j}{b_j - a_j},$$

$$\xi_i = \frac{2s_i}{b_i - a_i} - \frac{b_i + a_i}{b_i - a_i}, \quad (i, j=1, \dots, n),$$

and the unknown functions can be expressed as

$$\begin{aligned} f_j(t_j) &= F_j(\tau_j) (1 - \tau_j^2)^{-\frac{1}{2}}, \\ g_j(t_j) &= G_j(\tau_j) (1 - \tau_j^2)^{-\frac{1}{2}}. \end{aligned} \quad (j=1, \dots, n)$$

Introducing the new kernels

$$K_{ij}(\xi_i, \tau_j) = \frac{b_j - a_j}{2} k_{ij}(s_i, t_j),$$

$$H_{ij}(\xi_i, \tau_j) = \frac{b_j - a_j}{2} h_{ij}(s_i, t_j),$$

$$M_{ij}(\xi_i, \tau_j) = \frac{b_j - a_j}{2} m_{ij}(s_i, t_j),$$

$$N_{ij}(\xi_i, \tau_j) = \frac{b_j - a_j}{2} n_{ij}(s_i, t_j) \quad (37a-d)$$

then equations (27a,b) and (28a,b) will take the forms identical with (35a-d). Again by applying the Gauss-Chebyshev quadrature formula a system of linear algebraic equations will be obtained to determine the unknowns  $F_j(\tau_j)$  and  $G_j(\tau_k)$  ( $j=1, \dots, n; k=1, \dots, N$ ).

For a single radial crack  $n=1$  and the subscripts of all the quantities will be removed. Then for a crack with end points  $(a,b)$  the stress intensity factors may be defined as

$$k_1(a) = \lim_{s \rightarrow a} \sqrt{2(a-s)} \sigma_{qq}(0, s) ,$$

$$k_1(b) = \lim_{s \rightarrow b} \sqrt{2(s-b)} \sigma_{qq}(0, s) ,$$

$$k_2(a) = \lim_{s \rightarrow a} \sqrt{2(a-s)} \sigma_{qs}(0, s) ,$$

$$k_2(b) = \lim_{s \rightarrow b} \sqrt{2(s-b)} \sigma_{qs}(0, s) . \quad (38a-d)$$

It is then observed that the stress intensity factors are proportional to the values of  $F(\tau)$  and  $G(\tau)$  at the end points through the relations

$$k_1(a) = F(-1) \sqrt{\frac{b-a}{2}} ,$$

$$k_1(b) = -F(1) \sqrt{\frac{b-a}{2}},$$

$$k_2(a) = G(-1) \sqrt{\frac{b-a}{2}},$$

$$k_2(b) = -G(1) \sqrt{\frac{b-a}{2}}. \quad (39a-d)$$

The stress intensity factors of a system of radial cracks can be obtained from the values  $F_j(+1)$  and  $G_j(+1)$  with expressions similar to (39a-d).

## CHAPTER 4

### NUMERICAL RESULTS AND DISCUSSION

The numerical results are obtained first for a single crack. If the crack is perpendicular to the free surface for the pressurized crack we take  $p(s) = -\sigma_0$  in (23a) and after normalizing the singular integral equation we solve it numerically. The calculated stress intensity factors are given in Table 1 (see Figure 7).

These results are identical with those given by Cook and Erdogan [4]. Since the singular integral equation (23b) governing the mode II solution is the same with the equation for mode I case, the results for mode I apply to an analogous mode II problem.

Table 2 shows some numerical results for both symmetric and anti-symmetric loadings of a crack parallel to the free surface obtained by solving (24a,b) with  $p(s) = -\sigma_0$ ,  $q(s) = 0$  and  $p(s) = 0$ ,  $q(s) = -\tau_0$  (Figure 8). Thus the values given in the table

TABLE 1

Stress Intensity Factors for a Crack  
Perpendicular to the Free Surface

$\frac{d}{a}$	$\frac{k_1(-a)}{\sigma_0 \sqrt{a}}$	$\frac{k_1(a)}{\sigma_0 \sqrt{a}}$
1.1	1.7585	1.2108
1.2	1.4637	1.1626
1.3	1.3316	1.1331
1.4	1.2544	1.1123
1.5	1.2035	1.0967
2.0	1.0913	1.0539
3.0	1.0345	1.0246
4.0	1.0182	1.0141
5.0	1.0112	1.0092
10.0	1.0027	1.0024
$\infty$	1.0000	1.0000

$$k_1'(-a) = k_1(-a)/\sigma_0 \sqrt{a}, \quad k_1'(a) = k_1(a)/\sigma_0 \sqrt{a},$$

$$k_2'(-a) = k_2(-a)/\tau_0 \sqrt{a}, \quad k_2'(a) = k_2(a)/\tau_0 \sqrt{a}$$

are essentially the normalized stress intensity factors. Additionally the angles corresponding to the planes of maximum cleavage stress at the crack tips are given (Figure 6). These angles are shown to be the angles of crack propagation and may be calculated from [9]

$$k_1 \sin \phi + k_2 (3 \sin \phi - 1) = 0,$$

$$3k_2 \sin \phi - k_1 \cos \phi < 0.$$

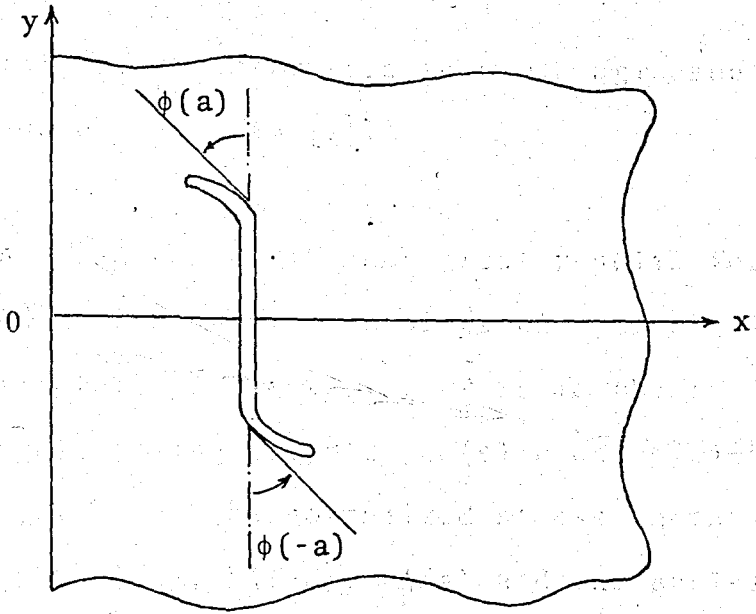


FIGURE 6. Probable Crack Propagation Angles.

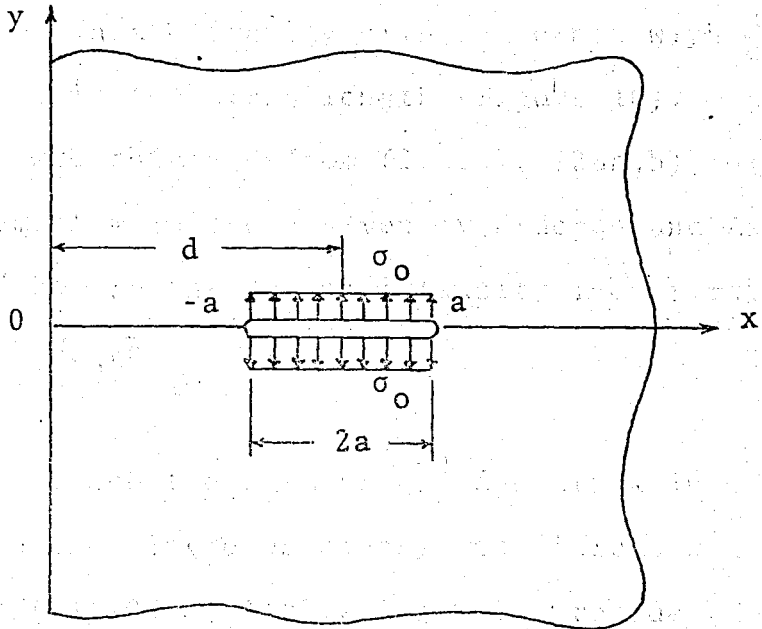


FIGURE 7. Pressurized Crack Perpendicular to the Free Surface of a Half-Plane.

The stress intensity factors are in close agreement with those obtained by Ashbaugh [1].

Tables 3 and 4 show the numerical results for both symmetric and anti-symmetric loadings of a crack with different orientations and a fixed  $\frac{d}{a}$  ratio obtained by solving (21a,b), taking  $p(s)=-\sigma_0$ ,  $q(s)=0$  and  $p(s)=0$ ,  $q(s)=-\tau_0$  (Figure 9). The normalized stress intensity factors  $k_1'(-a)$ ,  $k_1'(a)$ ,  $k_2'(-a)$ ,  $k_2'(a)$  and the angles for the maximum cleavage stress  $\phi(-a)$ ,  $\phi(a)$  are displayed for  $\frac{d}{a} = 1.5$  or  $\frac{d}{a} = 2$ . ratios.

Table 5 shows the calculated stress intensity factor ratios for an arbitrarily oriented crack with  $\frac{a}{c} = 1$ ,  $\frac{b}{c} = 3$  where  $2c$  is the crack length (Figure 10). These values which are obtained from (27a,b), (28a,b) for  $n = 1$  are in agreement with those given by Erdogan and Aksogan [8].  $k_1'$  and  $k_2'$  are the stress intensity factor ratios  $k_1'/\sigma_0\sqrt{c}$  and  $k_2'/\sigma_0\sqrt{c}$ .

If we let  $a=0$  the case of an edge crack is obtained. For this case the single-valuedness conditions will be replaced by  $F(-1)=0$ ,  $G(-1)=0$  which will provide the boundedness of the unknown functions (see [3]). The variations of the stress intensity factors with  $\theta$  are displayed in Figure 12 for uniform loading of the half-plane at infinity

TABLE 2

Stress Intensity Factor Ratios and Cleavage Angles for a Crack Parallel to the Free Surface

	$\frac{d}{a}$	0.1	0.2	0.3	0.4	0.5	1.	2.	3.	4.	5.	10.
$p = -\sigma_0$	$k_1^1(-a)$	14.259	5.9499	3.8326	2.9055	2.3974	1.5110	1.1633	1.0779	1.0451	1.0292	1.0075
	$k_1^1(a)$	14.259	5.9499	3.8326	2.9055	2.3974	1.5110	1.1633	1.0779	1.0451	1.0292	1.0075
	$k_2^1(-a)$	8.9640	3.0301	1.5929	0.9940	0.6805	0.1849	0.0368	0.0124	0.0055	0.0029	0.0004
$q = 0$	$k_2^1(a)$	-8.9640	-3.0301	-1.5929	-0.9940	-0.6805	-0.1849	-0.0368	-0.0124	-0.0055	-0.0029	-0.0004
	$\phi(-a)$	-44.94	-40.60	-36.20	-31.93	-27.94	-13.55	-3.612	-1.313	0.600	0.320	0.042
	$\phi(a)$	44.94	40.60	36.20	31.93	27.94	13.56	3.612	1.313	-0.600	-0.320	-0.042
$p = 0$	$k_1^1(-a)$	-0.8697	-0.6151	-0.4801	-0.3876	-0.3182	-0.1331	-0.0330	-0.0117	-0.0053	-0.0028	-0.0004
	$k_1^1(a)$	0.8697	0.6151	0.4801	0.3876	0.3182	0.1331	0.0330	0.0117	0.0053	0.0028	0.0004
	$k_2^1(-a)$	3.0301	1.2077	1.1551	1.1329	1.1209	1.0874	1.0432	1.0233	1.0141	1.0094	1.0025
$q = -\tau_0$	$k_2^1(a)$	-3.0301	-1.2077	-1.1551	-1.1329	-1.1209	-1.0874	-1.0432	-1.0233	-1.0141	-1.0094	-1.0025
	$\phi(-a)$	-82.79	-80.44	-78.61	-77.16	-76.02	-72.88	-71.13	-70.75	-70.63	-70.58	-70.54
	$\phi(a)$	-59.12	-61.18	-62.83	-64.15	-65.21	-68.21	-69.93	-70.31	-70.43	-70.48	-70.52

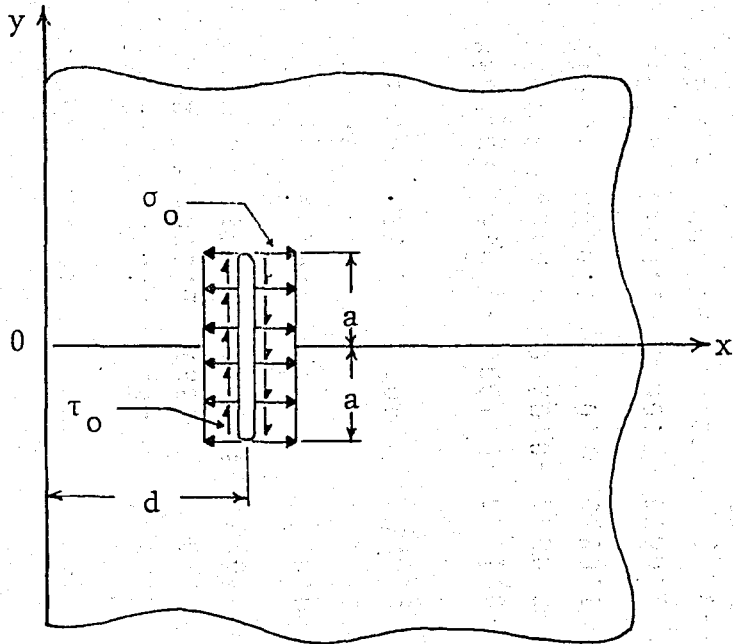


FIGURE 8. Crack Parallel to the Free Surface of a Half-Plane with Uniform Traction Applied on the Crack Surfaces.

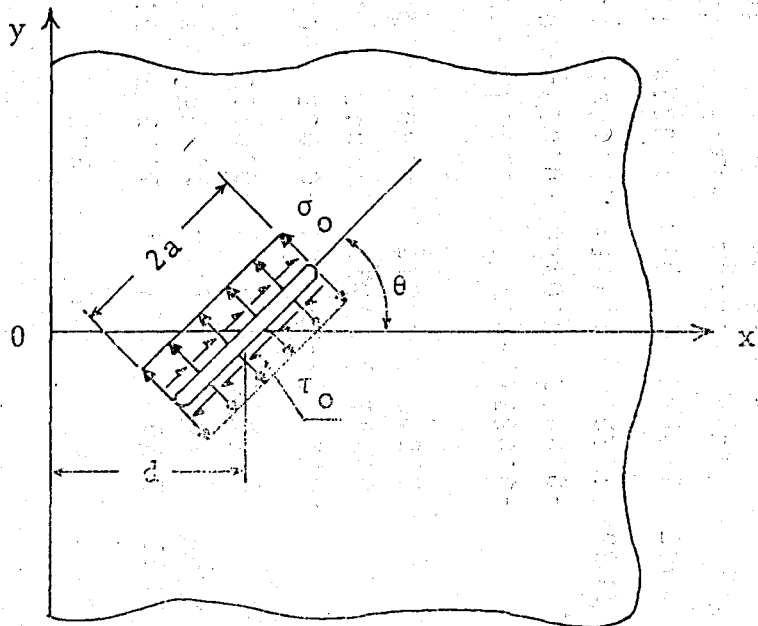


FIGURE 9. Arbitrarily Oriented Crack in a Half-Plane with Uniform Traction Applied on the Crack Surfaces.

TABLE 3

Stress Intensity Factor Ratios and Cleavage Angles for an Arbitrarily Oriented Crack ( $\frac{d}{a}=1.5$ )

$\theta$	$0^\circ$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$	$70^\circ$	$80^\circ$	$90^\circ$	
$p=-\sigma_0$	$k'_1(-a)$	1.2035	1.2080	1.2204	1.2377	1.2558	1.2710	1.2807	1.2835	1.2790	1.2677
	$k'_1(a)$	1.0967	1.1006	1.1119	1.1297	1.1523	1.1777	1.2038	1.2287	1.2505	1.2677
	$k'_2(-a)$	0.	-0.0083	-0.0129	-0.0113	-0.0031	0.0104	0.0271	0.0447	0.0611	0.0750
$q=0$	$k'_2(a)$	0.	-0.0245	-0.0469	-0.0657	-0.0797	-0.0887	-0.0925	-0.0913	-0.0853	-0.0750
	$\phi(-a)$	0.	0.791	1.212	1.045	0.281	-0.941	-2.424	-3.978	-5.449	-6.728
	$\phi(a)$	0.	2.549	4.817	6.609	7.842	8.519	8.688	8.407	7.735	6.728
$p=0$	$k'_1(-a)$	0.	-0.0571	-0.1031	-0.1322	-0.1438	-0.1411	-0.1283	-0.1090	-0.0864	-0.0628
	$k'_1(a)$	0.	-0.0083	-0.0146	-0.0174	-0.0156	-0.0086	-0.0034	-0.0200	-0.0402	0.0628
	$k'_2(-a)$	1.2035	1.1958	1.1754	1.1491	1.1229	1.1006	1.0836	1.0719	1.0648	1.0615
$q=-\tau_0$	$k'_2(a)$	1.0967	1.0953	1.0914	1.0858	1.0795	1.0733	1.0678	1.0637	1.0614	1.0615
	$\phi(-a)$	-70.53	-71.44	-72.21	-72.74	-72.99	-72.99	-72.80	-72.48	-72.09	-71.66
	$\phi(a)$	-70.53	-70.67	-70.78	-70.83	-70.80	-70.68	-70.47	-70.17	-69.81	-69.40

TABLE 4

Stress Intensity Factor Ratios and Cleavage Angles for an Arbitrarily Oriented Crack ( $\frac{d}{a}=2$ )

	$\theta$	$0^\circ$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$	$70^\circ$	$80^\circ$	$90^\circ$
$p=-\sigma_0$	$k_1^1(-a)$	1.0913	1.0949	1.1049	1.1192	1.1349	1.1494	1.1604	1.1667	1.1676	1.1633
	$k_1^1(a)$	1.0539	1.0566	1.0644	1.0766	1.0918	1.1087	1.1257	1.1414	1.1543	1.1633
	$k_2^1(-a)$	0.0	-0.0068	-0.0114	-0.0124	-0.0093	-0.0026	0.0067	0.0172	0.0276	0.0368
$q=0$	$k_2^1(a)$	0.0	-0.0130	-0.0251	-0.0354	-0.0431	-0.0480	-0.0497	-0.0483	-0.0439	-0.0363
	$\phi(-a)$	0.0	0.709	1.183	1.272	0.941	0.258	-0.662	-1.689	-2.704	-3.612
	$\phi(a)$	0.0	1.414	2.700	3.755	4.512	4.941	5.041	4.830	4.340	3.612
$p=0$	$k_1^1(-a)$	0.0	-0.0236	-0.0440	-0.0588	-0.0669	-0.0685	-0.0646	-0.0563	-0.0453	-0.0330
	$k_1^1(a)$	0.0	-0.0053	-0.0095	-0.0116	-0.0109	-0.0073	-0.0003	0.0091	0.0205	0.0330
	$k_2^1(-a)$	1.0913	1.0894	1.0842	1.0768	1.0686	1.0607	1.0540	1.0488	1.0452	1.0432
$q=-\tau_0$	$k_2^1(a)$	1.0539	1.0535	1.0523	1.0505	1.0484	1.0463	1.0444	1.0431	1.0426	1.0432
	$\phi(-a)$	-70.53	-70.94	-71.31	-71.57	-71.73	-71.77	-71.70	-71.56	-71.36	-71.13
	$\phi(a)$	-70.53	-70.63	-70.70	-70.74	-70.73	-70.66	-70.54	-70.36	-70.15	-69.93

TABLE 5

Stress Intensity Factor Ratios for an Arbitrarily Oriented Crack ( $\frac{a}{c}=1, \frac{b}{c}=3$ ).

$\theta^\circ$	$k_1'(a)$	$k_1'(b)$	$k_2'(a)$	$k_2'(b)$
0	1.0913	1.0539	0	0
5	1.0838	1.0468	0.0918	0.0849
10	1.0613	1.0255	0.1802	0.1674
15	1.0245	0.9908	0.2646	0.2448
20	0.9741	0.9437	0.3422	0.3150
25	0.9112	0.8857	0.4110	0.3757
30	0.8374	0.8186	0.4693	0.4253
35	0.7540	0.7444	0.5154	0.4623
40	0.6630	0.6655	0.5480	0.4854
45	0.5662	0.5842	0.5657	0.4940
50	0.4658	0.5028	0.5677	0.4879
55	0.3642	0.4238	0.5530	0.4670
60	0.2642	0.3492	0.5209	0.4320
65	0.1691	0.2806	0.4720	0.3838
70	0.0834	0.2191	0.4041	0.3241
75	0.0127	0.1647	0.3210	0.2548
80	-0.0356	0.1159	0.2254	0.1789
85	-0.0507	0.0689	0.1237	0.0996

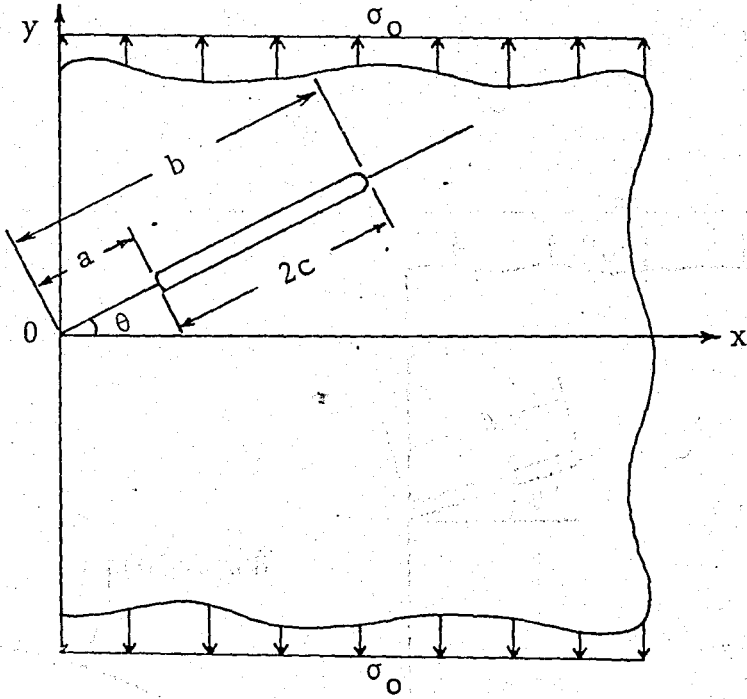


FIGURE 10. Arbitrarily Oriented Crack in a Half-Plane with Uniform Tension Applied at Infinity.

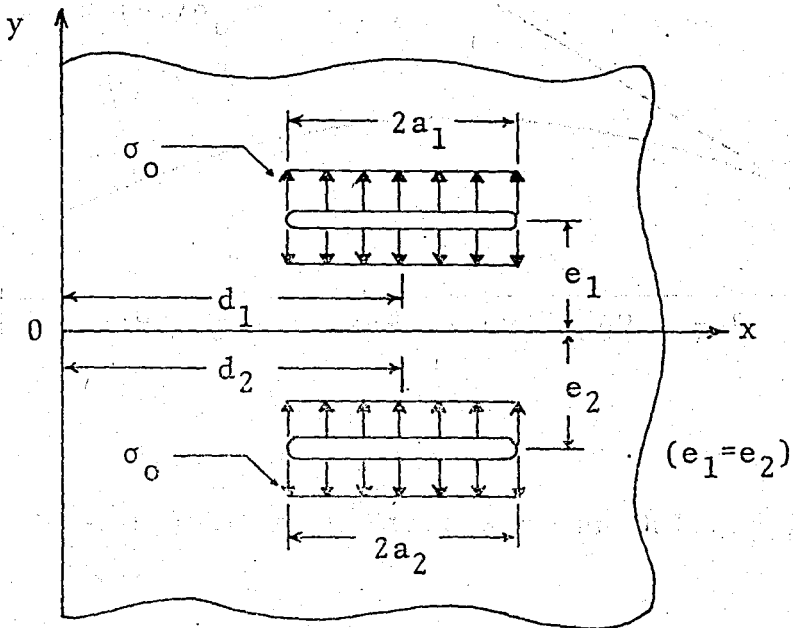


FIGURE 11. Two Cracks Perpendicular to the Free Surface of a Half-Plane with Uniform Pressure Acting on the Crack Surfaces.

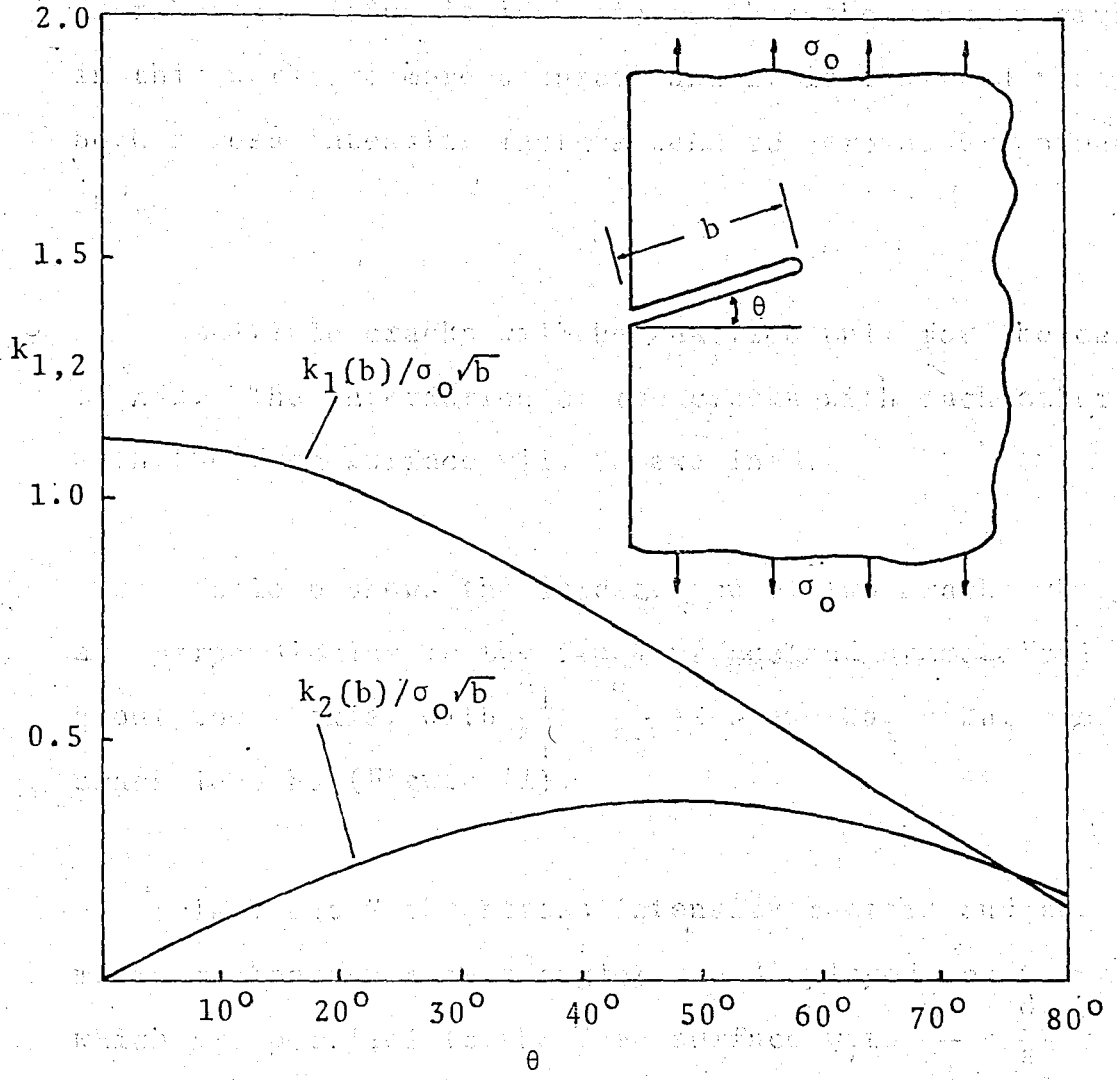


FIGURE 12. Stress Intensity Factors of an Oblique Edge Crack.

These curves are in close agreement with those given by Ioakimidis and Theocaris [13] except the results for large values of  $\theta$ . Even though the convergence is very poor beyond  $\theta=80^\circ$  it is believed that the results given in this work are more accurate and it is observed that both stress intensity factors tend to zero as  $\theta$  approaches  $90^\circ$ .

Multiple cracks will be analyzed only for the case of  $n=2$ . The interaction of two cracks with each other and with the free surface will be examined.

Table 6 shows the interaction of two cracks which are perpendicular to the free surface and symmetrical about the x axis, with  $\frac{d_1}{a_1} = \frac{d_2}{a_2} = 2$  where  $2a_1 = 2a_2$  are the crack lengths (Figure 11).

In Table 7 the stress intensity factors and the maximum cleavage stress angles are displayed for two cracks which are parallel to the free surface with  $\frac{d_1}{a_1} = \frac{d_2}{a_2} = 2$  where  $2a_1 = 2a_2$  are the crack lengths (Figure 13). From both tables it is observed that when the cracks are sufficiently away from each other single crack solutions are obtained.

In Tables 8, 9 and 10 some radial crack interactions are shown. Table 8 shows the interaction of two radial

TABLE 6

Stress Intensity Factor Ratios and Cleavage Angles for Two Cracks Perpendicular to the Free Surface ( $\frac{d_1}{a_1} = \frac{d_2}{a_2} = 2$ ,  $p(s) = -\sigma_0$ ,  $q(s) = 0$ ,  $k' = k/\sigma_0\sqrt{a_1}$ )

$\frac{e_1}{a_1}$	$k_1'(-a_1)$	$k_1'(a_1)$	$k_2'(-a_1)$	$k_2'(a_1)$	$\phi(-a_1)$	$\phi(a_1)$
0.05	0.7562	0.7254	0.2052	-0.2094	-26.999	28.288
0.10	0.7732	0.7391	0.1899	-0.2009	-24.98	27.04
0.20	0.8012	0.7633	0.1613	-0.1818	-21.21	24.38
0.30	0.8246	0.7846	0.1382	-0.1652	-18.08	22.02
0.40	0.8449	0.8040	0.1197	-0.1498	-15.53	19.85
0.50	0.8634	0.8219	0.1048	-0.1351	-13.46	17.77
1.00	0.9466	0.9012	0.0586	-0.0692	-7.027	8.684
2.00	1.0310	0.9956	0.0092	-0.0164	-1.028	1.888
3.00	1.0486	1.0213	-0.0064	-0.0118	0.679	1.324
4.00	1.0570	1.0286	-0.0078	-0.0104	0.845	1.163
5.00	1.0638	1.0327	-0.0063	-0.0083	0.673	0.926
10.00	1.0813	1.0449	-0.0014	-0.0021	0.153	0.233
20.00	1.0885	1.0513	-0.0002	-0.0003	0.022	0.035
40.00	1.0906	1.0532	-0.0003	-0.0004	0.003	0.005

TABLE 7

Stress Intensity Factor Ratios and Cleavage Angles for Two Cracks Parallel  
to the Free Surface ( $\frac{d_1}{a_1} = \frac{d_2}{a_2} = 2$ ,  $p(s) = -\sigma_0$ ,  $q(s) = 0$ ,  $k' = k/\sigma_0\sqrt{a_1}$ )

$\frac{e_1}{a_1}$	$k_1'(-a_1)$	$k_1'(a_1)$	$k_2'(-a_1)$	$k_2'(a_1)$	$\phi(-a_1)$	$\phi(a_1)$
1.1	1.9153	1.3360	-0.0253	-0.0982	1.510	8.320
1.2	1.5856	1.2679	-0.0258	-0.0819	1.867	7.327
1.3	1.4378	1.2293	-0.0224	-0.0707	1.785	6.536
1.4	1.3510	1.2042	-0.0169	-0.0619	1.436	5.857
1.5	1.2995	1.1871	-0.0104	-0.0549	0.919	5.269
2.0	1.1753	1.1551	0.0218	-0.0350	-2.123	3.464
3.0	1.1528	1.1561	0.0435	-0.0298	-4.307	2.953
4.0	1.1587	1.1606	0.0418	-0.0329	-4.125	3.238
5.0	1.1619	1.1623	0.0394	-0.0348	-3.873	3.425
10.0	1.1633	1.1633	0.0369	-0.0366	-3.625	3.601
20.0	1.1634	1.1634	0.0368	-0.0368	-3.612	3.612

TABLE 8

Stress Intensity Factor Ratios and Cleavage Angles for Two Radial Cracks in a Half-Plane,  
 Symmetrical About x Axis Loaded at Infinity with  $\sigma_0$  Parallel to the  
 Free Surface ( $\frac{a_1}{c_1} = 1, \frac{b_1}{c_1} = 3, \frac{a_2}{c_2} = 1, \frac{b_2}{c_2} = 3, c_1 = c_2, k' = k/\sigma_0\sqrt{c_1}$ )

$\theta^\circ$	$k'_1(a_1)$	$k'_1(b_1)$	$k'_2(a_1)$	$k'_2(b_1)$	$\phi(a_1)$	$\phi(b_1)$
5	0.7407	0.7831	0.2882	-0.0859	-34.75	12.24
10	0.7451	0.8158	0.3185	0.0146	-36.81	- 2.05
20	0.7285	0.8160	0.3797	0.1948	-41.09	-24.41
30	0.6785	0.7485	0.4429	0.3394	-45.68	-38.10
40	0.5799	0.6316	0.5018	0.4302	-50.81	-46.50
50	0.4316	0.4884	0.5291	0.4566	-56.02	-52.08
60	0.2545	0.3437	0.4986	0.4174	-61.16	-55.90
70	0.0825	0.2173	0.3957	0.3191	-66.60	-58.24
80	-0.0353	0.1155	0.2241	0.1781	-73.56	-58.79
85	-0.0506	0.0689	0.1234	0.0994	-78.50	-58.05

TABLE 9

Stress Intensity Factor Ratios and Cleavage Angles for Two Symmetrically Located Cracks in a Half-Plane Forming Angles  $\pm 45^\circ$  with x Axis, Loaded at Infinity with  $\sigma_0$  Parallel to the Free Surface ( $k' = k/\sigma_0\sqrt{c_1}$ )

$\frac{a_1+b_1}{2c_1}$	$k'_1(a_1)$	$k'_1(b_1)$	$k'_2(a_1)$	$k'_2(b_1)$	$\phi(a_1)$	$\phi(b_1)$
1.1	0.5984	0.6818	0.8315	0.3758	-57.60	-42.26
1.2	0.5433	0.6505	0.6880	0.3886	-56.45	-43.91
1.3	0.5245	0.6297	0.6255	0.4001	-55.66	-45.15
1.4	0.5166	0.6139	0.5901	0.4103	-55.07	-46.14
1.5	0.5131	0.6013	0.5674	0.4195	-54.61	-46.96
2.0	0.5111	0.5617	0.5212	0.4518	-53.43	-49.55
3.0	0.5101	0.5290	0.5032	0.4794	-52.92	-51.58
4.0	0.5077	0.5163	0.5000	0.4893	-52.90	-52.29
5.0	0.5057	0.5104	0.4993	0.4936	-52.93	-52.62
10.0	0.5018	0.5025	0.4995	0.4987	-53.06	-53.02

cracks which are symmetrical about x axis with  $\frac{a_1}{c_1} = 1$ ,  $\frac{b_1}{c_1} = 3$  and  $\frac{a_2}{c_2} = 1$ ,  $\frac{b_2}{c_2} = 3$  where  $2c_1 = 2c_2$  are the crack lengths (Figure 14). Table 9 displays the interaction of two cracks of equal length and of equal radial distance forming angles  $\theta = \pm 45^\circ$  with x axis (figure 15). In Table 10 the results are given for two radial cracks, one on x axis and the other one forming different angles with x axis where  $\frac{a_1}{c_1} = 1$ ,  $\frac{b_1}{c_1} = 3$  and  $\frac{a_2}{c_2} = 1$ ,  $\frac{b_2}{c_2} = 3$  (Figure 16).

Figure 17 shows the stress intensity factor ratios for two collinear cracks perpendicular to the free surface when the uniform tension is applied at infinity parallel to the free surface. It is assumed that one of the cracks is an edge crack. It is observed from the figure that when the neighboring crack tips get close the stress intensity factors become very large which implies that the cracks will run into each other with a slight loading. But if the crack tips are significantly away from each other then the stress intensity factor ratios are very close and one can determine at which load level the cracks will run into each other from the highest stress intensity factor ratio, namely,  $k_1(b_1)/\sigma_1\sqrt{b_1}$ .

The results presented here show the effects of crack orientations on the stress intensity factors. It is seen that as the cracks move away from the free surface the results approach to the values corresponding to the infinite plane.

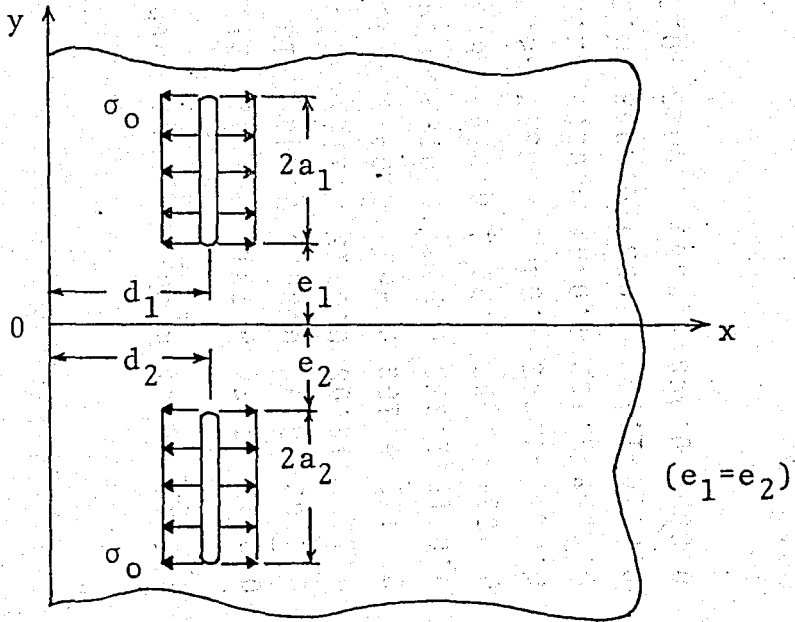


FIGURE 13. Two Cracks Parallel to the Free Surface of a Half-Plane with Uniform Pressure Acting on Crack Surfaces.

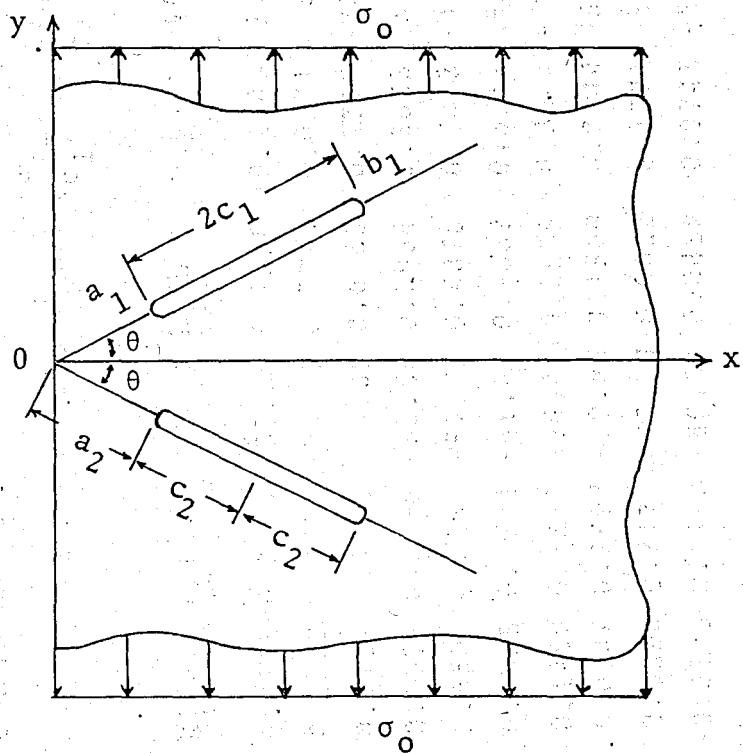


FIGURE 14. Two Symmetrical Cracks in a Half-Plane with Uniform Tension Applied at Infinity.

TABLE 10

Stress Intensity Factor Ratios and Cleavage Angles for Two Radial Cracks on a Half-Plane, One on x Axis and the Other One Arbitrarily Oriented, Loaded at Infinity with  $\sigma_0$  Parallel to the Free Surface ( $\frac{a_1}{c_1}=1.$ ,  $\frac{b_1}{c_1}=3.$ ,  $\frac{a_2}{c_2}=1.$ ,  $\frac{b_2}{c_2}=3.$ ,  $k'=k/\sigma_0\sqrt{c_1}$ )

$\theta_1$	$\theta_2$	$k_1'(a_1)$	$k_2'(a_1)$	$\phi(a_1)$	$k_1'(b_1)$	$k_2'(b_1)$	$\phi(b_1)$	$k_1'(a_2)$	$k_2'(a_2)$	$\phi(a_2)$	$k_1'(b_2)$	$k_2'(b_2)$	$\phi(b_2)$
5.00	0.00	0.7565	0.3004	-35.20	0.7268	-0.1184	17.62	0.7125	-0.2416	31.74	0.7817	0.1642	-21.98
10.00	0.00	0.7652	0.3605	-38.93	0.7210	-0.0321	5.07	0.7058	-0.2102	28.95	0.8343	0.1404	-18.15
20.00	0.00	0.7192	0.4870	-46.39	0.6765	0.1315	-20.58	0.7301	-0.1223	18.08	0.9115	0.1151	-13.97
30.00	0.00	0.6077	0.5836	-52.50	0.5954	0.2635	-37.57	0.7878	-0.0427	6.168	0.9637	0.0975	-11.33
40.00	0.00	0.4605	0.6267	-57.35	0.4876	0.3462	-47.29	0.8587	0.0115	-1.533	0.9997	0.0789	-8.913
50.00	0.00	0.3005	0.6104	-61.48	0.3682	0.3712	-53.26	0.9297	0.0381	-4.682	1.0242	0.0587	-6.515
60.00	0.00	0.1450	0.5343	-65.44	0.2535	0.3381	-57.10	0.9922	0.0413	-4.746	1.0402	0.0387	-4.246
70.00	0.00	0.0142	0.3997	-69.85	0.1569	0.2544	-59.33	1.0417	0.0288	-3.159	1.0491	0.0207	-2.261
80.00	0.00	-0.0590	0.2172	-75.79	0.0822	0.1371	-59.62	1.0760	0.0110	-1.169	1.0530	0.0068	-0.744
85.00	0.00	-0.0576	0.1182	-80.01	0.0488	0.0743	-58.64	1.0861	0.0039	-0.408	1.0538	0.0023	-0.251

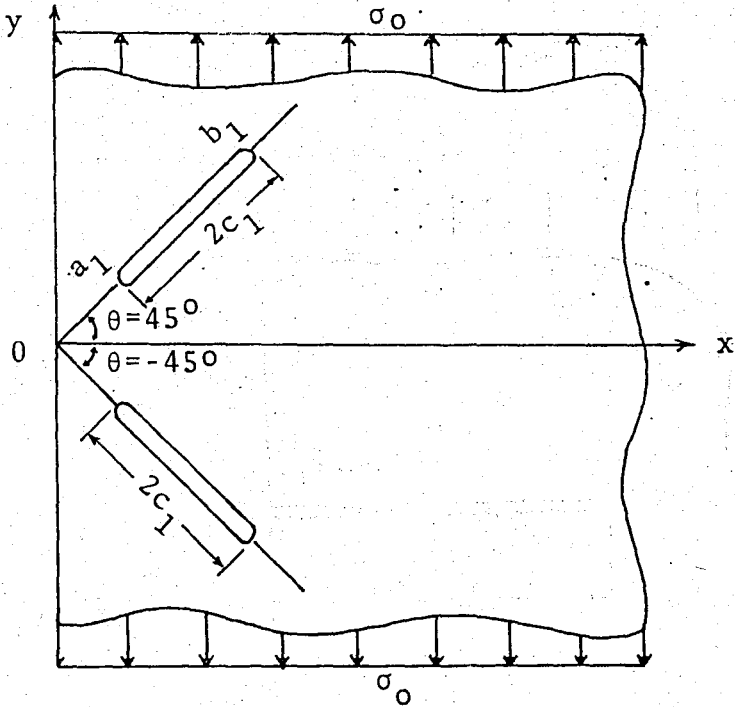


FIGURE 15. Two Symmetrical Cracks in a Half-Plane Forming Angles  $\pm 45^\circ$  with  $x$  Axis.

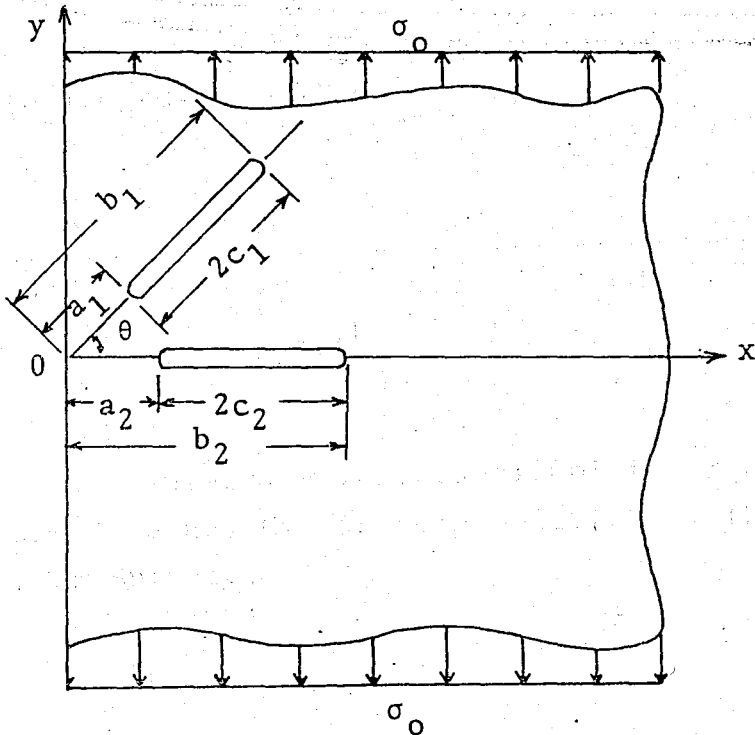


FIGURE 16. Two Radial Cracks in a Half-Plane, One Located on  $x$  Axis, the Other One Arbitrarily Oriented.

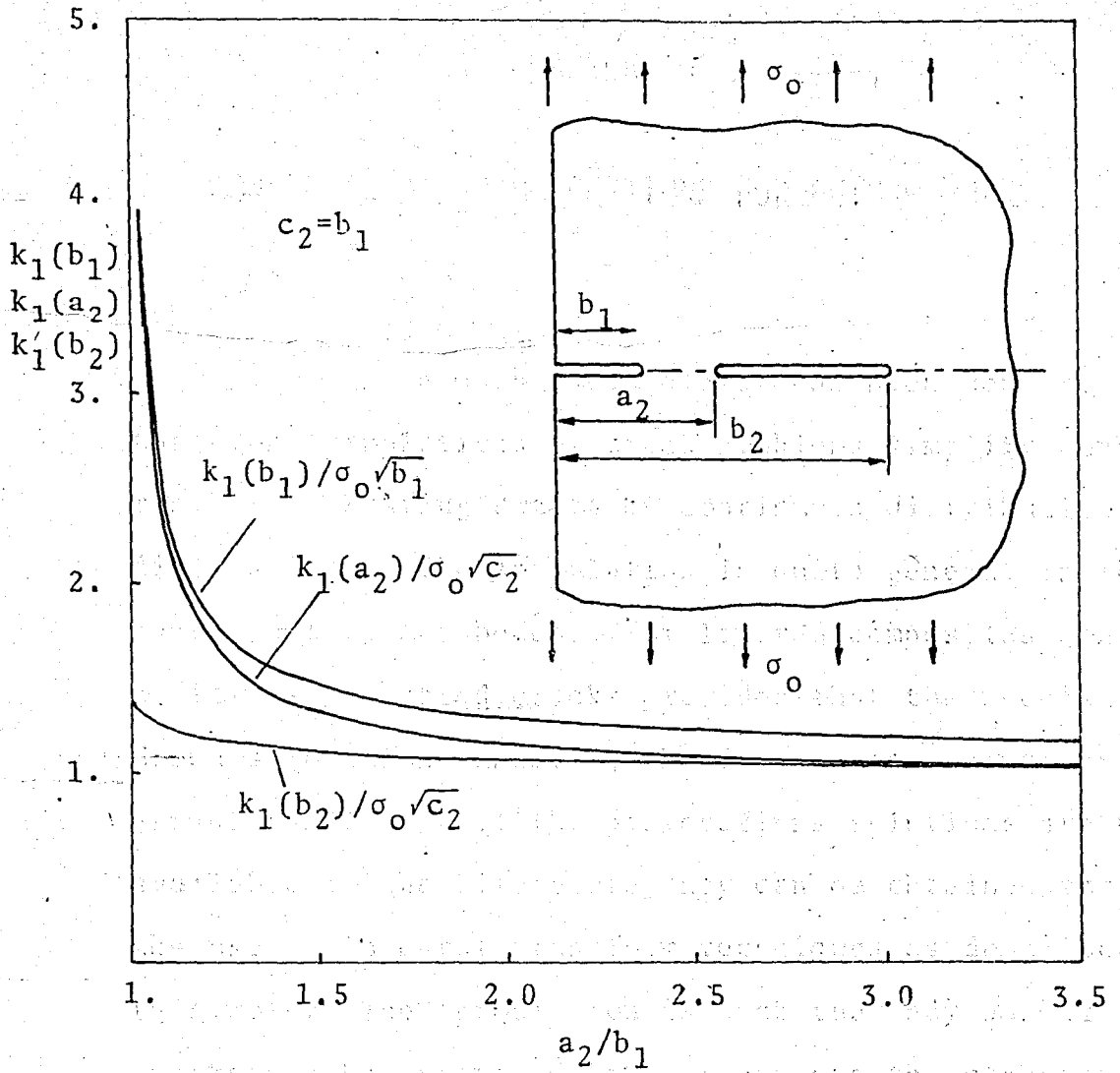


FIGURE 17. Stress Intensity Factors of Collinear Edge and Internal Cracks Perpendicular to the Free Surface.

## CHAPTER 5

### CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

In the preceding sections it has been demonstrated that the formulations of crack problems simplify considerably representing cracks by continuous distributions of dislocations. The formulation is quite general in the sense that it may be used for layered composites containing arbitrarily located cracks provided that the necessary Green's functions, namely, the dislocation solutions are established. If the dislocation solutions are not available in the literature they can be obtained through the use of integral transform techniques as described in this work. The formulation is such that any number of cracks can be treated. Limitations are the computer's memory and the execution time.

It is suggested that this formulation should be applied to bonded elastic half-planes and bonded elastic strips next for a better understanding of the applications of linear elastic fracture mechanics in layered composites.

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## APPENDIX

### EVALUATION OF CERTAIN INFINITE INTEGRALS

$$I_1 = \int_0^{\infty} \frac{Y(Y^2 + 3a_0)^2}{(a_0^2 + Y^2)^2} \sin \alpha Y dY = \int_0^{\infty} \frac{Y}{a_0^2 + Y^2} \sin \alpha Y dY$$

$$- a_0 \frac{d}{da_0} \int_0^{\infty} \frac{Y}{a_0^2 + Y^2} \sin \alpha Y dY$$

From Tables [6]

$$\int_0^{\infty} \frac{Y}{a_0^2 + Y^2} \sin \alpha Y dY = \frac{\pi}{2} e^{-\alpha a_0}$$

$$I_1 = \frac{\pi}{2} e^{-\alpha a_0} - a_0 \frac{d}{da_0} \left( \frac{\pi}{2} e^{-\alpha a_0} \right)$$

$$= \frac{\pi}{2} e^{-\alpha a_0} + \frac{\pi}{2} a_0 e^{-\alpha a_0}$$

$$= \frac{\pi}{2} e^{-\alpha a_0} (1 + \alpha a_0)$$

(A.1)

$$I_2 = \int_0^{\infty} \frac{Y(Y^2 - a_0^2)}{(a_0^2 + Y^2)} \sin \alpha Y dY = \int_0^{\infty} \frac{Y}{a_0^2 + Y^2} \sin \alpha Y dY$$

$$+ a_0 \frac{d}{da_0} \int_0^{\infty} \frac{Y}{a_0^2 + Y^2} \sin \alpha Y dY$$

From Tables [6]

$$\int_0^{\infty} \frac{Y}{a_0^2 + Y^2} \sin \alpha Y dY = \frac{\Pi}{2} e^{-\alpha a_0}$$

$$I_2 = \frac{\Pi}{2} e^{-\alpha a_0} + a_0 \frac{d}{da_0} \left( \frac{\Pi}{2} e^{-\alpha a_0} \right)$$

$$= \frac{\Pi}{2} e^{-\alpha a_0} - \frac{\Pi}{2} a_0 e^{-\alpha a_0}$$

$$= \frac{\Pi}{2} e^{-\alpha a_0} (1 - \alpha a_0)$$

(A.2)

$$I_3 = \int_0^{\infty} \frac{a_0 (a_0^2 - Y^2)}{(a_0^2 + Y^2)} \cos \alpha Y dY$$

$$= \int_0^{\infty} a_0 \left[ \frac{1}{a_0^2 + Y^2} - \frac{2Y^2}{(a_0^2 + Y^2)} \right] \cos \alpha Y dY$$

$$= a_0 \left[ \int_0^{\infty} \frac{1}{a_0^2 + Y^2} \cos \alpha Y dY - 2 \frac{d}{d\alpha} \int_0^{\infty} \frac{Y}{(a_0^2 + Y^2)} \sin \alpha Y dY \right]$$

$$\int_0^{\infty} \frac{1}{a_0^2 + Y^2} \cos \alpha Y dY = \frac{\Pi}{2a_0} e^{-\alpha a_0}$$

$$\int_0^{\infty} \frac{Y}{(a_0^2 + Y^2)^2} \sin \alpha Y dY = \frac{\Pi}{4a_0} \alpha e^{-\alpha a_0}$$

$$I_3 = a_0 \left[ \frac{\Pi}{2a_0} e^{-\alpha a_0} - 2 \frac{d}{d\alpha} \left( \frac{\Pi}{4a_0} \alpha e^{-\alpha a_0} \right) \right]$$

$$= a_0 \left[ \frac{\Pi}{2a_0} e^{-\alpha a_0} - 2 \left( \frac{\Pi}{4a_0} e^{-\alpha a_0} - \frac{\Pi}{4a_0} \alpha a_0 e^{-\alpha a_0} \right) \right]$$

$$= \frac{\Pi}{2} \alpha a_0 e^{-\alpha a_0}$$

(A.3)

$$I_4 = \int_0^{\infty} e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha = \frac{Y}{(X+a_0)^2 + Y^2} \quad (A.4)$$

$$I_5 = \int_0^{\infty} \alpha e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha = \frac{2Y(X+a_0)}{[(X+a_0)^2 + Y^2]^2} \quad (A.5)$$

$$I_6 = \int_0^{\infty} \alpha^2 e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha = \frac{2Y[3(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^3} \quad (A.6)$$

$$I_7 = \int_0^{\infty} e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha = \frac{X+a_0}{(X+a_0)^2 + Y^2} \quad (A.7)$$

$$I_8 = \int_0^{\infty} \alpha e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha = \frac{(X+a_0)^2 - Y^2}{[(X+a_0)^2 + Y^2]^2} \quad (A.8)$$

$$I_9 = \int_0^{\infty} \alpha^2 e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha = \frac{2(X+a_0)[(X+a_0)^2 - 3Y^2]}{[(X+a_0)^2 + Y^2]^3} \quad (A.9)$$

Integrals  $I_4$  and  $I_7$  can be written from Tables [6] and others are obtained by differentiation.

$$I_4 = \int_0^{\infty} e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha = \frac{Y}{(X+a_0)^2 + Y^2}$$

$$I_5 = \int_0^{\infty} \alpha e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha = -\frac{d}{dX} \int_0^{\infty} e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha$$

$$= \frac{2Y(X+a_0)}{[(X+a_0)^2+Y^2]^2}$$

$$I_6 = \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha = \frac{d^2}{dX^2} \int_0^\infty e^{-\alpha(X+a_0)} \sin \alpha Y d\alpha$$
$$= \frac{2Y[3(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^3}$$

$$I_7 = \int_0^\infty e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha = \frac{X+a_0}{(X+a_0)^2 + Y^2}$$

$$I_8 = \int_0^\infty \alpha e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha = -\frac{d}{dX} \int_0^\infty e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha$$
$$= \frac{(X+a_0)^2 - Y^2}{[(X+a_0)^2 + Y^2]^2}$$

$$I_9 = \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha = \frac{d^2}{dX^2} \int_0^\infty e^{-\alpha(X+a_0)} \cos \alpha Y d\alpha$$
$$= \frac{2(X+a_0)[(X+a_0)^2 - 3Y^2]}{[(X+a_0)^2 + Y^2]^3}$$