

ON MINIMAL DEFINING SETS OF FULL DESIGNS

by

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B.S., in Mathematics, Boğaziçi University, 2007

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University

2009

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my thesis supervisors, Assoc. Prof. Emine Şule Yazıcı and Prof. Haluk Oral, for their endless support in preparation of this study.

I am thankful to Selda Küçükçifçi for her support and help during my graduate studies. Also, I would like to thank Müge Taşkın Aydın and Tınaz Ekim Aşıcı for participating in my thesis committee.

I thank the audience of my thesis defence and previous talks about the subject for their fruitful discussions.

I am deeply thankful to my friends Halil İbrahim Baştürk and Halim Sekizkardeş for their true friendship and motivation. I am thankful to my officemates for a friendly atmosphere.

Most importantly, I am indebted to my family for their self-sacrifice and patience through my education.

ABSTRACT

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A defining set of a t - (v, k, λ) design is a subset of the block set of the design which is not contained in any other design with the same parameters. A defining set is said to be minimal if none of its proper subsets is a defining set. A defining set is said to be smallest if no other defining set has a smaller cardinality. A t - (v, k, λ) design $D = (V, \mathcal{B})$ is called a full design if \mathcal{B} is the collection of all possible k -subsets of V . Every simple t -design is contained in a full design and the intersection of a defining set of a full design with a simple t -design contained in it, gives a defining set of the corresponding t -design. With this motivation, in this thesis, the full designs are studied when the block size is 3 and several families of non-isomorphic minimal defining sets of full designs are given. Also, it is proven that there exists some sizes in the spectrum of the full design on v elements such that the number of non-isomorphic minimal defining sets on each of that sizes goes to infinity as $v \rightarrow \infty$. Moreover, the lower bound on the size of the defining sets of the full designs is improved with finding the size of the smallest defining sets of the full designs on 8 and 9 points. Also, all smallest defining sets of the full designs on 8 and 9 points are classified.

ÖZET

TAM TASARIMLARIN MİNİMAL BELİRLEYİCİ KÜMELERİ ÜZERİNE

Tek bir $t-(v, k, \lambda)$ tasarımı tarafından içerilen kısmi tasarıma, bu $t-(v, k, \lambda)$ tasarımının belirleyici kümesi denir. Hiçbir öz kısmi tasarımı belirleyici küme olmayan belirleyici kümelere minimal belirleyici küme denir. En az sayıda blok içeren belirleyici kümelere en küçük belirleyici küme denir. Bir $t-(v, k, \lambda)$ tasarımı $D = (V, \mathcal{B})$ 'ye, eğer \mathcal{B} kümeler topluluğu V kümesinin tüm k elemanlı alt kümelerinden oluşuyorsa, tam tasarım denir. Tam tasarımların belirleyici kümeleri, tam tasarımın içinde bulundurduğu bütün tasarımların belirleyici kümelerinin bulunabilmesini sağlamaktadır. Bu motivasyonla, tam tasarımların blok büyüklüğü 3 olanları incelenmiş ve bu tam tasarımlar için birkaç izomorfik olmayan minimal belirleyici küme sınıfları bulunmuştur. Ayrıca, v noktalı tam tasarımların bazı büyüklükteki izomorfik olmayan belirleyici kümelerinin sayısının v sonsuza giderken sonsuza gittiği ispatlanmıştır. Bunlara ek olarak, tam tasarımların belirleyici kümelerinin büyüklüğü üzerindeki alt sınır, 8 ve 9 noktalı tam tasarımların en küçük belirleyici kümelerinin büyüklüğü bulunarak geliştirilmiştir. Aynı zamanda, 8 ve 9 noktalı tam tasarımların en küçük belirleyici kümelerinin hepsi sınıflandırılmıştır.

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LIST OF SYMBOLS/ABBREVIATIONS

$d(v)$	a defining set of $F(V)$ on the v -set
$d_m(v)$	a minimal defining set of $F(V)$ on the v -set
$d_s(v)$	a smallest defining set of $F(V)$ on the v -set
$D_i(V)$	a $d_m(v)$ defined by Construction i on the underlying set V
$D_i^C(V)$	$F(V) \setminus D_i(V)$
$E_i(V)$	a $d_m(v)$ defined by Extension of Construction i on the underlying set V
$\mathbb{E}_i(V)$	a $d_m(v)$ defined by Double Extension of Construction i on the underlying set V
$E_i^C(V)$	$F(V) \setminus E_i(V)$
$\mathbb{E}_i^C(V)$	$F(V) \setminus \mathbb{E}_i(V)$
$F(V)$	the full design on the set V with block size 3
BIBD	balanced incomplete block design

1. INTRODUCTION

The history of design theory goes all the way back to 1782 Euler's officer problem. The first example of designs; Kirkman schoolgirl problem was given and studied by T. P. Kirkman in 1850. Even though at the beginning block designs were the product of mathematical curiosity, after their applications to statistical experimental designs were discovered in mid 20th century, they became a widely studied subject. Block designs have strong connections to other mathematical areas like algebra, coding theory, finite geometries, cryptography, graph theory; and nowadays they are widely used in agriculture to computer science, chemistry to DNA decoding. More information can be found in [1].

In 1984, Curtis [2] examined a Steiner System namely $S(5, 8, 24)$ and discovered that one can choose 8 blocks of this design that defines the rest of the 751 blocks. This was the first result on defining sets of designs. In 1990, Gray [3] considered the idea of defining sets of designs in a more general context, proving several basic results. After that, to find some or all defining sets in families of designs, some designs with small number of blocks have been searched using a combination of theoretical structure analyzes and computer searches. In 2000, it was suggested to use defining sets in combinatorial designs for secret sharing algorithms by Seberry and Street [4][5]. After that, the research on this subject continued with an increasing speed. Further information about defining set concept in a general context can be found in [6].

Suppose V is a nonempty finite set of v objects. A *block design* based on V is a collection of k -subsets of V with each object belonging to r of the k -subsets. The members of V are called *points*, the k -subsets are called *blocks* and the number r is *the repetition number* of the design. If $k = v$, then each block is identical to V and the design is called *complete*. The interesting case, when $k < v$, the design is called *incomplete*. An incomplete design is called *balanced incomplete block design (BIBD)* if each pair of elements of V occurs

exactly a positive integer λ times in the design.

One of the most important types of combinatorial designs are t -designs which are a generalization of BIBDs and defined as follows.

Definition 1.1. *Let v, k, t and λ_t be positive integers such that $v > k \geq t$. A t - (v, k, λ_t) design $D = (V, \mathcal{B})$ is an ordered pair, where V is a set of v elements and \mathcal{B} is a collection of k -subsets of V such that each t -subset of V belongs to exactly λ_t blocks in the design.*

The subscript of the parameter λ_t is sometimes omitted and simply written as λ . Now, an example to a t -design will be given. Let $D = (V, \mathcal{B})$ be a 2 - $(7, 3, 1)$ design, where $V = \{0, 1, 2, 3, 4, 5, 6\}$ and $\mathcal{B} = \{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$. Here, there are 7 points and each block contains 3 points of V with any two elements of V belonging to exactly one block.

Now, the following lemma illustrates the relation between parameters v, b, r, k, t and λ and shows that the parameters v, k, t and λ are enough to define a t -design.

Lemma 1.2. *Let v, k, t and λ be positive integers such that $v > k \geq t$. For a t - (v, k, λ) design D with b blocks and the repetition number r , the followings hold.*

- (i) $vr = bk$.
- (ii) $\lambda \binom{v}{t} = b \binom{k}{t}$.
- (iii) D is an s - (v, k, λ_s) design for all $0 < s \leq t$, where $\lambda_s = \lambda \binom{v-s}{t-s} / \binom{k-s}{t-s}$.
Note that, $\lambda_t = \lambda$, $\lambda_1 = r$ and for $s = 0$, $\lambda_0 = b$.

Proof. Let $D = (V, \mathcal{B})$ be a t - (v, k, λ) design with b blocks and the repetition number r .

- (i) There are v points, each belonging to exactly r blocks in D . Also, there are b blocks, each containing k points in the design. Hence, $vr = bk$.
- (ii) There are $\binom{v}{t}$ t -subsets, each occurs λ times in the design. Also, there are b blocks, each contains $\binom{k}{t}$ t -subsets in D . Hence, $\lambda \binom{v}{t} = b \binom{k}{t}$.

(iii) Let α be a s -subset of V . Then, there are $\binom{v-s}{t-s}$ t -subsets of V containing α . Each such t -subset of V occurs in λ blocks. Also, each block containing α , contains $\binom{k-s}{t-s}$ such t -subsets. Hence, the result follows. \square

A t -(v, k, λ) design D is called *simple* if each block of D occurs only once in D , i.e. D does not contain any repeated blocks. A t -(v, k, λ_t) design $D = (V, \mathcal{B})$ is called a *full* design if \mathcal{B} is the collection of all possible k -subsets of V . Note that, for the full designs, $\lambda_t = \binom{v-t}{k-t}$ and the full design is the unique largest possible simple design on V .

For example, $D = (V, \mathcal{B})$ is the 2-(7, 3, 5) full design if \mathcal{B} is the collection all possible 3-subsets of $V = \{0, 1, 2, \dots, 6\}$. Note that, D is the unique simple design with these parameters. However, there are other designs with these parameters which are not simple. Repeating each block of a 2-(7, 3, 1) design 5 times will give a 2-(7, 3, 5) design which is not simple.

Let $D_1 = (V_1, \mathcal{B}_1)$ and $D_2 = (V_2, \mathcal{B}_2)$ be two t -(v, k, λ) designs with the same parameters. Then, D_1 and D_2 are said to be *isomorphic* if there exists a bijection $\psi : V_1 \rightarrow V_2$ such that $\psi(\mathcal{B}_1) = \mathcal{B}_2$. Otherwise, D_1 and D_2 are said to be *non-isomorphic*. Examples of two non-isomorphic 2-(7, 3, 5) designs are given above.

A *defining set* of a t -design is a subset of the block set of the design which is not contained in any other design with the same parameters. A defining set is said to be *minimal* if none of its proper subsets is a defining set. A defining set is said to be *smallest* if no other defining set has a smaller cardinality. Lastly, a defining set of a t -design is said to be a *largest minimal defining set*, if no minimal defining set of the design contains more blocks. The *spectrum* of the minimal defining sets of a t -design D is defined as $\text{spec}(D) = \{|M| : M \text{ is a minimal defining set of } D\}$.

For example, 2-(7, 3, 1) design has only one minimal defining set up to isomorphism which is $\{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}\}$. However, 2-(7, 3, 5) full design, say D , has 109 non-isomorphic minimal defining sets with 2 of size 15, 4 of size 16, 10 of size 17, 31 of size 18, 50 of size 19, 10 of size 20 and 2 of size 21. Hence, D has two smallest defining sets with size 15 and two largest minimal defining sets with size 21 and $\text{spec}(D) = \{15, 16, 17, 18, 19, 20, 21\}$.

Definition 1.3. *Let V be a set on v elements. Let T_1 and T_2 be collections of k -subsets of V with m elements. If each t -subset of V occurs in the k -subsets belonging to T_1 with precisely the same multiplicity that it occurs in the k -subsets belonging to T_2 , then T_1 and T_2 are said to be mutually t -balanced. If T_1 and T_2 are disjoint, then $T = (T_1, T_2)$ is said to be a t -(v, k) trade of volume m . Sometimes, T_1 is said to be a trade and T_2 is said to be its trade mate.*

The smallest possible trade on 6 distinct points is known as the *Pasch trade* (T_1, T_2) with $T_1 = \{\{u, w, x\}, \{u, y, z\}, \{v, w, y\}, \{v, x, z\}\}$ and $T_2 = \{\{v, w, x\}, \{v, y, z\}, \{u, w, y\}, \{u, x, z\}\}$. An other example is on 7 distinct points which will be referred as *type 2 trade* (T_1^*, T_2^*) in this thesis with $T_1^* = \{\{t, w, x\}, \{t, v, y\}, \{t, v, z\}, \{u, y, z\}, \{u, v, w\}\}, \{u, v, x\}$ and trade mate $T_2^* = \{\{u, w, x\}, \{u, v, y\}, \{u, v, z\}, \{t, y, z\}, \{t, v, w\}\}, \{t, v, x\}$.

The following lemma [3] explains the relation between minimal defining sets and trades.

Lemma 1.4. *Let D be a subset of a design (V, \mathcal{B}) , and let $D^C = \mathcal{B} \setminus D$. Then,*

- (i) *D is a defining set of (V, \mathcal{B}) if, and only if, there is no trade $T \subseteq D^C$; that is, every trade contained in \mathcal{B} intersects D .*
- (ii) *Suppose that D is a defining set of (V, \mathcal{B}) . Then, D is minimal if, and only if, for every block $X \in D$ there exists a trade T satisfying $T \setminus D^C = \{X\}$.*

Proof. (i) Contrapositive of the statement will be proven. First, assume D is a not a defining set of (V, \mathcal{B}) , then there exists at least two designs with the same parameters

which contains D , say (V, \mathcal{B}) and (V, \mathcal{B}^*) . Now, let $T_1 = \mathcal{B} \setminus \mathcal{B}^*$ and $T_2 = \mathcal{B}^* \setminus \mathcal{B}$, then T_1 and T_2 are mutually t -balanced and disjoint. Hence, T_1 is a trade which does not intersect D with trade mate T_2 . For the other side, let T_1 be a trade in D^C with a trade mate T_2 . Then, $T_2 \cup (\mathcal{B} \setminus T_1)$ is an other design on the same parameters which contains D . Hence, D is not a defining set of (V, \mathcal{B}) .

- (ii) Contrapositive of the statement will be proven. Firstly, assume D is not minimal. Then, for some block $X \in D$, $D \setminus \{X\}$ is a defining set. Hence, there is no trade T satisfying $T \setminus D^C = \{X\}$ by part (i). Now, assume for some block $X \in D$ there is no trade T satisfying $T \setminus D^C = \{X\}$. Then, $D^C \cup \{X\}$ does not contain any trade and hence, $D \setminus \{X\}$ is a defining set by part (i). Thus, D is not minimal defining set. \square

In 1993, Greenhill [7] constructed an algorithm for finding the smallest defining sets of t -designs which works only for single-transposition-free designs, i.e. designs whose automorphism group contains no single transposition. In 1995, to find a smallest defining set of t -designs, Khodkar [8] used integer programming to minimize the number of blocks in a t -design that intersect each occurrence of every trade in a list of trades contained in the design. In 1997, Ramsay [9] constructed an algorithm for finding all minimal trades contained in a simple t -design to use in Khodkar's Algorithm. In 2003, the Minimal Defining Set Algorithm by Havas, Lawrance, Ramsay, Street and Yazıcı [10], is used to find all minimal defining sets of t -designs by intersecting the designs with the same parameters and then simply checking whether a partial design is a subset of any other design with the same parameters or not. Lastly, in 2007, The Finding Minimal Defining Set Algorithm by Kolotoğlu and Yazıcı [11], is constructed as a refinement of the previous algorithm with the following main idea; instead of taking the intersection of the design with all possible designs with the same parameters, which causes redundant work, the algorithm takes completions of the partial designs of a given design and then takes the intersection of these completions.

Algorithms on minimal defining sets is shortly described above, however only the last two algorithms can find all minimal defining sets of a given t -design and these algorithms are applicable only for small designs. For example, for a t -design with 35 blocks the last two algorithms needs 2^{35} bits, i.e. 4 Gigabytes RAM memory to allocate.

This thesis mainly focus on full designs. The following two lemmas which are stated in [12] and [13], respectively, are the main motivation of this emphasis.

Lemma 1.5. *Let (V_A, \mathcal{A}) and (V_B, \mathcal{B}) be two t -designs with $\mathcal{A} \subseteq \mathcal{B}$. If D is a defining set of (V_B, \mathcal{B}) , then $D \cap \mathcal{A}$ is a defining set of (V_A, \mathcal{A}) .*

Proof. Any trade in \mathcal{A} is also in \mathcal{B} so intersects with D , hence intersects with $D \cap \mathcal{A}$. Therefore, $D \cap \mathcal{A}$ is a defining set of (V_A, \mathcal{A}) by Lemma 1.4. \square

Lemma 1.6. *Let (V_A, \mathcal{A}) and (V_B, \mathcal{B}) be two t -designs with $\mathcal{A} \subseteq \mathcal{B}$. If D is a minimal defining set of (V_A, \mathcal{A}) , then there exists a minimal defining set D^* of (V_B, \mathcal{B}) such that $D^* \cap \mathcal{A} = D$.*

Proof. Let $\mathcal{B} \setminus \mathcal{A} = \{X_1, X_2, \dots, X_n\}$. Then, $M_0 = D \cup \{X_1, X_2, \dots, X_n\}$ is a defining set of (V_B, \mathcal{B}) since there is no trade T in M_0^C . Now, define M_i recursively for $1 \leq i \leq n$, as follows: $M_i = M_{i-1}$ if there exists a trade $T \in \mathcal{B}$ such that $T \cap M_{i-1} = \{X_i\}$, and $M_i = M_{i-1} \setminus \{X_i\}$ otherwise. Thus, $M_n \cap \mathcal{A} = D$ and M_n is a minimal defining set of (V_B, \mathcal{B}) by Lemma 1.4. \square

Recall that the full design is the largest possible simple design on v elements and every simple design is contained in a full design with same block size. With the above results, the minimal defining sets of full designs give valuable information about the minimal defining sets of simple t -designs.

In this thesis, the defining sets of the full designs with block size 3 will be considered, i.e.; the full 2 - $(v, 3, v - 2)$ designs. *Full design* will refer to a full design with block size 3

and *block* will refer to a 3-subset. The notations will be identical to [13] and [14]. $F(V)$ will denote the full design on the set V with block size 3. So, $F(V) = \{\{x, y, z\} | x, y, z \in V, x, y, z \text{ distinct}\}$. If D is a defining set of $F(V)$, then the complement of D is denoted as $D^C = F(V) \setminus D$. $d(v)$ will denote a defining set of the full design on the v -set. $d_m(v)$ and $d_s(v)$ will denote a minimal and a smallest defining set of the full design on the v -set, respectively.

For $v \leq 5$, the minimal defining set of the full design on v points is only the empty set since there are no trades on 5 points or less. There are 8 non-isomorphic minimal defining sets with sizes 6, 7, 8 and 10 of the full design on 6 points [10]. There are 109 non-isomorphic minimal defining sets with sizes 15, 16, 17, 18, 19, 20 and 21 of the full design on 7 points [11]. The known algorithms can not be applicable for finding all the minimal defining sets of full designs on $v \geq 8$ points, however some classes of these minimal defining sets can be constructed theoretically for all v .

In the next chapter, previous constructions will be stated and in Chapter 3 new constructions will be given and it will be shown that there exists some sizes in the spectrum of the full design on v elements such that the number of non-isomorphic minimal defining sets on each of that sizes goes to infinity as $v \rightarrow \infty$. Bounds on the size of the defining sets, the spectrum analysis and also all smallest defining sets of the full designs on 8 and 9 points will be given in Chapter 4. Lastly, in Chapter 5, the concluding remarks and future directions will be stated.

2. PREVIOUS CONSTRUCTIONS

In this chapter, all constructions previously suggested for finding minimal defining sets of full designs with block size 3 will be stated. Also, we present some lemmas which enabled us to extend some families of known minimal defining sets.

Akbari, Maimani and Maysoori [15] proved that the following construction gives a minimal defining set for the full design on v points for all $v \geq 6$.

Lemma 2.1 (Construction 1). *Let $V = \{0, 1, \dots, v - 1\}$, where $v \geq 6$. Define*

$$D_1(V) = F(V \setminus \{0\}) \setminus \left\{ \{1, 2, \alpha\} \mid 3 \leq \alpha \leq v - 1 \right\}$$

and $|D_1(V)| = \binom{v-1}{3} - (v-3) = (v^3 - 6v^2 + 5v + 12)/6$. Then, the set $D_1(V)$ is a $d_m(v)$ of $F(V)$.

Kolotoğlu and Yazıcı [11, 17] found a similar construction; after removing all blocks that contain the identified point 0, the full design on the set $\{1, 2, 3, 4\}$ is removed instead of removing the blocks that contain the pair $\{1, 2\}$. A more general version of this construction with block sizes greater than 3 is obtained independently by Gray and Street [16].

Lemma 2.2 (Construction 2). *Let $V = \{0, 1, \dots, v - 1\}$, where $v \geq 5$. Define*

$$D_2(V) = F(V \setminus \{0\}) \setminus F(\{1, 2, 3, 4\})$$

and $|D_2(V)| = \binom{v-1}{3} - \binom{4}{3} = (v^3 - 6v^2 + 11v - 30)/6$. Then, the set $D_2(V)$ is a $d_m(v)$ of $F(V)$.

Two families of minimal defining sets for the full design are given above with the common property that one point does not occur in the corresponding minimal defining sets. The next theorem [13] shows that up to isomorphism only these two families of minimal defining sets have this property when $v \geq 6$.

Theorem 2.3. *Let D be a $d_m(v)$ of $F(V)$, with $v = |V| \geq 6$, and let $D_1(V)$ and $D_2(V)$ be the minimal defining sets given by Constructions 1 and 2, respectively. Suppose that $D \subseteq F(V \setminus \{a\})$, for some $a \in V$. Then, D is isomorphic to either $D_1(V)$ or $D_2(V)$.*

Now, the next well-known lemma is a powerful tool for showing whether a subcollection of blocks form a defining set. The main idea is that; let $F(U \cup W) \supseteq D^*$ be an extension of $F(U) \supseteq D$, D be a defining set of $F(U)$, $D^C = F(U) \setminus D$ and $(D^*)^C = F(U \cup W) \setminus D^*$. Then, to prove D^* is $d(u+w)$ of $F(U \cup W)$, it is only necessary to consider the blocks in $(D^*)^C \setminus D^C$. If the blocks in $(D^*)^C \setminus D^C$ are uniquely determined by D^* , then D^C can be determined uniquely by D and hence $(D^*)^C$ is forced. Also, for the minimality, we can take advantage of a similar argument and Lemma 1.4.

Lemma 2.4. *Let U and W be disjoint sets of size u and w , respectively. Let D be a $d_m(u)$ of $F(U)$, and let D^* be a subcollection of blocks of $U \cup W$ such that $D^* \cap F(U) = D$. Then,*

- (i) *D^* is a $d(u+w)$ of $F(U \cup W)$ if, and only if, all blocks in $(D^*)^C \setminus D^C$ are uniquely determined by D^* .*
- (ii) *D^* is a $d_m(u+w)$ of $F(U \cup W)$ if, and only if, it is a $d(u+w)$ and for every block $X \in D^* \setminus D$ there exists a trade T with $T \setminus (D^*)^C = \{X\}$.*

A simple way to construct minimal defining sets is the following; start with a full design $F(U)$ and its minimal defining set D and then, add a set of new points to the vertex set U , say W , and add corresponding new blocks to the minimal defining set D . Now, $D^* = D \cup (F(U \cup W) \setminus F(U))$ is clearly a $d(u+w)$ of $F(U \cup W)$. To get a $d_m(u+w)$ of $F(U \cup W)$, some blocks which contains new points need to be removed from D^* as in Lemma 1.6. However, selection of these blocks is actually quite complicated.

The next lemma [13] is a powerful tool to extend minimal defining sets of full designs.

Lemma 2.5 (Extension Construction). *Let U and W be disjoint sets of size $u \geq 5$ and $w \geq 0$, respectively. Let D be a $d_m(u)$ of $F(U)$ of size n , and let $a \in U$. Let $V = U \cup W$ and $v = |V|$. Define*

$$E = D \cup \left(F(V \setminus \{a\}) \setminus F(U) \right).$$

Then,

- (i) E is a $d(v)$ of $F(V)$ of size $n + \binom{v-1}{3} - \binom{u-1}{3}$.
- (ii) E is a $d_m(v)$ of $F(V)$ if for every $\alpha_1, \alpha_2 \in U \setminus \{a\}$, for an arbitrary $p \notin U$, there exists a trade T with

$$T \setminus \left(D^C \cup \left\{ \{p, a, \alpha\} \mid \alpha \in U \setminus \{a\} \right\} \right) = \left\{ \{p, \alpha_1, \alpha_2\} \right\}.$$

Note that, for the minimality, selection of initial minimal defining set and the pivot point in U makes a difference.

The next construction [13] is an application of Lemma 2.5.

Lemma 2.6 (Extension of Construction 1). *Let U and W be disjoint sets with $U = \{0, 1, \dots, u-1\}$ and $w = |W|$ where $u \geq 6$ and $w \geq 0$, let $V = U \cup W$ and $v = |V| = u + w$. Define*

$$E_1(V) = D_1(U) \cup \left(F(V \setminus \{1\}) \setminus F(U) \right)$$

and $|E_1(V)| = (u^3 - 6u^2 + 5u + 12)/6 + \binom{v-1}{3} - \binom{u-1}{3} = (v^3 - 6v^2 + 5v + 12)/6 + w$. Then, the set $E_1(V)$ is a $d_m(v)$ of $F(V)$.

Similar to Lemma 2.5, the next lemma [13] shows that extension of a known minimal defining set can be done by adding two different sets of points, W and Q .

Lemma 2.7 (Double Extension Construction). *Let U , W and Q be disjoint sets of size $u \geq 5$, $w \geq 2$ and $q \geq 1$, respectively. Let $V = U \cup W \cup Q$ and $v = |V|$. Let $a, b \in U$ and $c \in Q$ with $a \neq b$. Assume D is a $d_m(u)$ of $F(U)$ of size n , and let $D^C = F(U) \setminus D$. Define*

$$\mathbb{E}^C = D^C \cup \left\{ \{a, \beta_1, \alpha\}, \{a, \beta_1, \beta_2\}, \{a, \beta_1, c\} \mid \alpha \in U \setminus \{a\}, \beta_1, \beta_2 \in W \right\} \\ \cup \left\{ \{b, \gamma_1, \alpha\}, \{b, \gamma_1, \gamma_2\}, \{b, \gamma_1, \beta\} \mid \alpha \in U \setminus \{b\}, \beta \in W, \gamma_1, \gamma_2 \in Q \right\}.$$

Then, $\mathbb{E}^C \subseteq F(V)$ and

- (i) $\mathbb{E} = F(V) \setminus \mathbb{E}^C$ is a $d(v)$ of $F(V)$ of size $n + \binom{v-1}{3} - \binom{u-1}{3} - w$.
- (ii) \mathbb{E} is a $d_m(v)$ of $F(V)$; if for every $\alpha_1, \alpha_2 \in U \setminus \{x\}$, where $x \in \{a, b\}$, for an arbitrary $p \notin U$, there exists a trade T with

$$T \setminus \left(D^C \cup \left\{ \{p, x, \alpha\} \mid \alpha \in U \setminus \{x\} \right\} \right) = \left\{ \{p, \alpha_1, \alpha_2\} \right\}.$$

The intersection of a minimal defining set of a larger full design with a smaller full design contained in it, is necessarily a defining set by Lemma 1.5 and is not necessarily a minimal defining set, in general. However, for a $d_m(v)$ of $F(V)$, say E , constructed as described in Lemma 2.5 or Lemma 2.7, $E \cap F(U) = D$ where D is a $d_m(u)$ of $F(U) \subseteq F(V)$. Also, these two extensions can be used to construct minimal defining sets having consecutive sizes. This issue will be analyzed deeply in Chapter 4.

The next construction [13] is an application of Lemma 2.7.

Lemma 2.8 (Double Extension of Construction 2). *Let U , W and Q be disjoint sets with $U = \{0, 1, 2, 3, 4\}$, $w = |W| \geq 2$, $q = |Q| \geq 1$ and $V = U \cup W \cup Q$ with $v = |V| = 5 + w + q$.*

Let c be an arbitrary element of Q . Define

$$\begin{aligned} \mathbb{E}_2^C(V) = & \left\{ \{0, \beta_1, \alpha\}, \{0, \beta_1, \beta_2\}, \{0, \beta_1, c\} \mid \alpha \in U \setminus \{0\}, \beta_1, \beta_2 \in W \right\} \\ & \cup \left\{ \{1, \gamma_1, \alpha\}, \{1, \gamma_1, \gamma_2\}, \{1, \gamma_1, \beta\} \mid \alpha \in U \setminus \{1\}, \beta \in W, \gamma_1, \gamma_2 \in Q \right\} \end{aligned}$$

and $\mathbb{E}_2(V) = F(V) \setminus \mathbb{E}_2^C(V)$. Then, $|\mathbb{E}_2(V)| = \binom{v-1}{3} - \binom{4}{3} - w = (v^3 - 6v^2 + 11v - 30)/6 - w$ and the set $\mathbb{E}_2(V)$ is a $d_m(v)$ of $F(V)$.

The next construction [13] shows that we can remove more blocks and find a minimal defining set with smaller size than the defining set suggested in the Lemma 2.5 by choosing the right initial defining set and the pivot point.

Lemma 2.9 (Construction 3). *Let U and W be disjoint sets with $U = \{0, 1, \dots, 6\}$ and $w = |W| \geq 0$. Let $V = U \cup W$ and $v = |V| = 7 + w$. Let*

$$D = \left\{ \{i, i+1, i+3\}, \{i, i+2, j\} \mid 0 \leq i \leq 4, \text{ addition mod } 5, j \in \{5, 6\} \right\}.$$

Define

$$D_3(V) = D \cup \left(F(V \setminus \{0\}) \setminus F(U) \right) \setminus \left\{ \{\beta, 2, 3\} \mid \beta \in W \right\}$$

and $|D_3(V)| = 15 + \binom{v-1}{3} - \binom{7}{3} - w = (v^3 - 6v^2 + 5v + 6)/6$. Then, the set $D_3(V)$ is a $d_m(v)$ of $F(V)$.

Remember that, the minimal defining set of the full design on 5 points is the empty set. The following construction [14] make use of these information: the construction removes blocks of several copies of full design on 5 points having two common points.

Lemma 2.10 (Construction 4). *Let $\{0, 1\}$ and U_1, U_2, \dots, U_k be disjoint sets with $|U_i| = 3$,*

$1 \leq i \leq k$. Let $V = \{0, 1\} \cup \bigcup_{i=1}^k U_i$, and $v = |V| = 2 + 3k$. Define

$$D_4(V) = F(V) \setminus \bigcup_{i=1}^k F(\{0, 1\} \cup U_i)$$

and $|D_4(V)| = \binom{v}{3} - k \binom{5}{3} = (v^3 - 3v^2 - 18v + 40)/6$. Then, the set $D_4(V)$ is a $d_m(v)$ of $F(V)$.

Construction 4 can be enlarged for all $v \geq 5$ by adding new set of points, say S . As in Lemma 1.6, some blocks that contain points in S will be removed as described in the next construction [14] to get a minimal defining set.

Lemma 2.11 (Construction 5). *Let $\{0, 1\}, U_1, U_2, \dots, U_k$ and S be disjoint sets with $|U_i| = 3$, $U_i = \{x_i, y_i, z_i\}$, $1 \leq i \leq k$, and $s = |S| \geq 0$. Let $V = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, and $v = |V| = 2 + 3k + s$. Define*

$$D_5(V) = F(V) \setminus \left(\bigcup_{i=1}^k F(\{0, 1\} \cup U_i) \cup \bigcup_{\sigma \in S} \left\{ \{0, 1, \sigma\}, \{0, x_1, \sigma\}, \{1, x_1, \sigma\} \right\} \right)$$

and $|D_5(V)| = \binom{v}{3} - k \binom{5}{3} - 3s = (v^3 - 3v^2 - 16v + 36)/6 - k$. Then, the set $D_5(V)$ is a $d_m(v)$ of $F(V)$.

The next construction [14] illustrates that removing a different set of blocks that contain points in S from the one described in Construction 5, enable us to get non-isomorphic minimal defining sets.

Lemma 2.12 (Construction 6). *Let $\{0, 1\}, U_1, U_2, \dots, U_k$ and S be disjoint sets with $|U_i| = 3$, $U_i = \{x_i, y_i, z_i\}$, $1 \leq i \leq k$, and $s = |S| \geq 0$. Let $V = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, and*

$v = |V| = 2 + 3k + s$. Define

$$D_6(V) = F(V) \setminus \left(\bigcup_{i=1}^k \left(F(\{0, 1\} \cup U_i) \cup \left\{ \{x_i, y_i, \sigma\}, \{x_i, z_i, \sigma\}, \{y_i, z_i, \sigma\} \mid \sigma \in S \right\} \right) \cup \bigcup_{i=2}^k \left\{ \{x_1, x_i, \sigma\} \mid \sigma \in S \right\} \right)$$

and $|D_6(V)| = \binom{v}{3} - k \binom{5}{3} - 3ks - s(k-1) = (v^3 - 3v^2 + 8v - 24kv + 72k^2 - 30k - 12)/6$.
Then, the set $D_6(V)$ is a $d_m(v)$ of $F(V)$.

The following two lemmas [14] shows that the family of minimal defining sets constructed by Construction 5 can be extended by using Lemma 2.5 and Lemma 2.7, respectively.

Lemma 2.13 (Extension of Construction 5). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, S$ and W be disjoint sets with $|U_i| = 3$, $U_i = \{x_i, y_i, z_i\}$, $1 \leq i \leq k$, $s = |S| \geq 0$ and $w = |W| \geq 0$. Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 2 + 3k + s$ and $V = U \cup W$, $v = |V| = u + w$. Define*

$$E_5(V) = D_5(U) \cup \left(F(V \setminus \{0\}) \setminus F(U) \right)$$

and $|E_5(V)| = (v^3 - 3v^2 - 16v + 36)/6 - w(2v - 9 - w)/2 - k$. Then, the set $E_5(V)$ is a $d_m(v)$ of $F(V)$.

Lemma 2.14 (Double Extension of Construction 5). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, S, W$ and Q be disjoint sets with $|U_i| = 3$, $U_i = \{x_i, y_i, z_i\}$, $1 \leq i \leq k$, $s = |S| \geq 0$, $w' = |W| \geq 2$ and $q = |Q| \geq 1$. Let c be an arbitrary element of Q , and let $w = w' + q$. Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 2 + 3k + s$ and $V = U \cup W \cup Q$ and $v = |V| = u + w$.*

Define

$$\mathbb{E}_5^C(V) = D_5^C(U) \cup \left\{ \{0, \beta_1, \alpha\}, \{0, \beta_1, \beta_2\}, \{0, \beta_1, c\} \mid \alpha \in U \setminus \{0\}, \beta_1, \beta_2 \in W \right\} \\ \cup \left\{ \{1, \gamma_1, \alpha\}, \{1, \gamma_1, \gamma_2\}, \{1, \gamma_1, \beta\} \mid \alpha \in U \setminus \{1\}, \beta \in W, \gamma_1, \gamma_2 \in Q \right\}$$

and $\mathbb{E}_5(V) = F(V) \setminus \mathbb{E}_5^C(V)$. Then, $|\mathbb{E}_5(V)| = (v^3 - 3v^2 - 16v + 36)/6 - w(2v - 9 - w)/2 - w' - k$ and the set $\mathbb{E}_5(V)$ is a $d_m(v)$ of $F(V)$.

In this chapter, 10 different constructions are stated. For some values of v , some of the defining sets obtained by these constructions are isomorphic but, in general, they are all non-isomorphic.

3. NEW CONSTRUCTIONS

In this chapter, firstly, we enlarge the minimal defining sets defined by Construction 4. Then, a new family of minimal defining sets and some of their enlargements and corresponding extensions and double extensions via Lemma 2.5 and Lemma 2.7 will be given. Also, we will show that there exists some sizes in the spectrum of the full design on v elements such that the number of non-isomorphic minimal defining sets on each of that sizes goes to infinity as $v \rightarrow \infty$.

Lemma 3.1 (Construction 7). *Let $\{0, 1\}, U_1, U_2, \dots, U_k$ and S be disjoint sets with $|U_i| = 3$, $U_i = \{x_i, y_i, z_i\}$, $1 \leq i \leq k$, and $s = |S| \geq 0$. Let $V = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, and $v = |V| = 2 + 3k + s$. Define*

$$D_7(V) = F(V) \setminus \left(\bigcup_{i=1}^k F(\{0, 1\} \cup U_i) \cup \left\{ \{0, x_1, \sigma\}, \{0, y_1, \sigma\}, \{x_1, y_1, \sigma\} \mid \sigma \in S \right\} \right. \\ \left. \cup \bigcup_{i=2}^k \left\{ \{0, x_i, \sigma\} \mid \sigma \in S \right\} \right).$$

Then, the set $D_7(V)$ is a $d_m(v)$ of $F(V)$ of size $(v^3 - 3v^2 - 10v - 6kv + 18k^2 - 12k + 24)/6$.

Proof.

$$\begin{aligned} |D_7(V)| &= \binom{v}{3} - k \binom{5}{3} - 3s - s(k-1) \\ &= (v^3 - 3v^2 + 2v - 60k - 6ks - 12s)/6 \\ &= (v^3 - 3v^2 - 10v - 6kv + 18k^2 - 12k + 24)/6. \end{aligned}$$

Let $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i$. Note that, $D_7(V) \cap F(U) = D_4(U)$. Thus, by Lemma 2.4 and Construction 4, it is enough to consider blocks containing at least one element of S to prove $D_7(V)$ is a $d_m(v)$ of $F(V)$.

First, we will show that all blocks in $D_7^C(V) \setminus D_4^C(U)$ are uniquely determined by $D_7(V)$. Now, let $\sigma \in S$ be arbitrary. σ occurs twice with x_1 , twice with y_1 , once with x_i for $2 \leq i \leq k$ and $2 + (k - 1)$ times with 0 in $D_7^C(V)$.

A block that contains both σ and x_i should also contain 0 since; x_i and x_1 or; x_i and y_1 or; x_i and x_j , $i \neq j$, does not occur in a block in $D_7^C(V)$. Hence, the blocks $\{0, x_i, \sigma\}$ are uniquely determined for $2 \leq i \leq k$.

Now, for the remaining blocks, σ occurs exactly twice with x_1 , twice with y_1 and twice with 0. Also, σ does not occur with any other points of V in $D_7^C(V)$. Hence, the blocks $\{0, x_1, \sigma\}$, $\{0, y_1, \sigma\}$, $\{x_1, y_1, \sigma\}$ are uniquely determined. Thus, $D_7^C(V)$ is uniquely determined by $D_7(V)$.

To show minimality of $D_7(V)$, it is enough to show for every block $X \in D_7(V) \setminus D_4(U)$, there exists a trade T satisfying $T \setminus D_7^C(V) = \{X\}$ by Lemma 2.4.

First, consider the blocks X where X is a subset of $\{0, 1\} \cup U_1 \cup S$. Note that, $D_7(V) \cap F(\{0, 1\} \cup U_1 \cup S)$ is isomorphic to $D_5(\{0, 1\} \cup U_1 \cup S)$, with a permutation which maps 1 to y_1 and fixes other points. By Construction 5, $D_5(\{0, 1\} \cup U_1 \cup S)$ is minimal, so, the required trades exist.

Note that, y_i and z_i , $2 \leq i \leq k$, and; x_1 and y_1 have a symmetric structure. Hence, without loss of generality, for arbitrary $\sigma_1, \sigma_2 \in S$, $X \in \left\{ \{0, y_i, \sigma_1\}, \{1, x_i, \sigma_1\}, \{1, y_i, \sigma_1\}, \{x_1, x_i, \sigma_1\}, \{x_1, y_i, \sigma_1\}, \{z_1, x_i, \sigma_1\}, \{z_1, y_i, \sigma_1\}, \{x_i, y_i, \sigma_1\}, \{x_i, x_j, \sigma_1\}, \{x_i, y_j, \sigma_1\}, \{y_i, z_i, \sigma_1\}, \{y_i, y_j, \sigma_1\}, \{x_i, \sigma_1, \sigma_2\}, \{y_i, \sigma_1, \sigma_2\} \right\}$ for $2 \leq i < j \leq k$.

For $X \in \left\{ \{1, y_i, \sigma_1\}, \{x_1, x_i, \sigma_1\}, \{x_1, y_i, \sigma_1\}, \{z_1, x_i, \sigma_1\}, \{z_1, y_i, \sigma_1\}, \{x_i, y_j, \sigma_1\}, \{y_i, z_i, \sigma_1\}, \{y_i, y_j, \sigma_1\} \right\}$, the required (Pasch) trades are $\left\{ \{1, y_i, \sigma_1\}, \{0, x_i, \sigma_1\}, \{1, x_i, z_i\}, \{0, y_i, z_i\} \right\}$, $\left\{ \{x_1, x_i, \sigma_1\}, \{0, y_1, \sigma_1\}, \{1, x_1, y_1\}, \{0, 1, x_i\} \right\}$, $\left\{ \{x_1, y_i, \sigma_1\}, \{0, y_1, \sigma_1\}, \{1, x_1, y_1\}, \{0, 1, y_i\} \right\}$,

$\left\{ \{z_1, x_i, \sigma_1\}, \{0, x_1, \sigma_1\}, \{1, x_1, z_1\}, \{0, 1, x_i\} \right\}$, $\left\{ \{z_1, y_i, \sigma_1\}, \{0, x_1, \sigma_1\}, \{1, x_1, z_1\}, \{0, 1, y_i\} \right\}$,
 $\left\{ \{x_i, y_j, \sigma_1\}, \{0, x_j, \sigma_1\}, \{1, x_j, y_j\}, \{0, 1, x_i\} \right\}$, $\left\{ \{y_i, z_i, \sigma_1\}, \{0, x_i, \sigma_1\}, \{1, x_i, z_i\}, \{0, 1, y_i\} \right\}$
 and $\left\{ \{y_i, y_j, \sigma_1\}, \{0, x_j, \sigma_1\}, \{1, x_j, y_j\}, \{0, 1, y_i\} \right\}$, respectively.

For $X \in \left\{ \{x_i, y_i, \sigma_1\}, \{x_i, x_j, \sigma_1\}, \{y_i, \sigma_1, \sigma_2\} \right\}$, the required (type 2) trades are
 $\left\{ \{x_i, y_i, \sigma_1\}, \{0, x_1, \sigma_1\}, \{0, y_1, \sigma_1\}, \{1, x_1, y_1\}, \{0, 1, x_i\}, \{0, 1, y_i\} \right\}$, $\left\{ \{x_i, x_j, \sigma_1\}, \{0, x_1, \sigma_1\}, \{0, y_1, \sigma_1\}, \{1, x_1, y_1\}, \{0, 1, x_j\}, \{0, 1, x_i\} \right\}$ and $\left\{ \{y_i, \sigma_1, \sigma_2\}, \{0, x_i, \sigma_1\}, \{0, x_i, \sigma_2\}, \{0, 1, z_i\}, \{1, x_i, y_i\}, \{x_i, y_i, z_i\} \right\}$, respectively.

For $X = \{0, y_i, \sigma_1\}$, the required trade is $\left\{ \{0, y_i, \sigma_1\}, \{x_1, y_1, \sigma_1\}, \{0, x_i, \sigma_1\}, \{1, x_i, y_i\}, \{0, 1, x_1\}, \{0, 1, y_1\} \right\}$ with trade mate $\left\{ \{x_i, y_i, \sigma_1\}, \{0, x_1, \sigma_1\}, \{0, y_1, \sigma_1\}, \{0, 1, y_i\}, \{0, 1, x_i\}, \{1, x_1, y_1\} \right\}$. For $X = \{1, x_i, \sigma_1\}$, the required trade is $\left\{ \{1, x_i, \sigma_1\}, \{0, x_1, \sigma_1\}, \{0, y_1, \sigma_1\}, \{1, x_1, y_1\}, \{0, 1, x_i\}, \{x_i, y_i, z_i\}, \{0, 1, y_i\}, \{0, 1, z_i\} \right\}$ with trade mate $\left\{ \{0, 1, \sigma_1\}, \{0, x_i, \sigma_1\}, \{x_1, y_1, \sigma_1\}, \{0, 1, x_1\}, \{0, 1, y_1\}, \{0, y_i, z_i\}, \{1, x_i, y_i\}, \{1, x_i, z_i\} \right\}$. Lastly, for $X = \{x_i, \sigma_1, \sigma_2\}$, the required trade is $\left\{ \{x_i, \sigma_1, \sigma_2\}, \{0, y_1, \sigma_1\}, \{0, y_1, \sigma_2\}, \{0, 1, x_1\}, \{0, 1, z_1\}, \{x_1, y_1, z_1\}, \{0, x_i, y_i\}, \{0, y_i, z_i\}, \{1, y_i, z_i\} \right\}$ with trade mate $\left\{ \{y_1, \sigma_1, \sigma_2\}, \{0, x_i, \sigma_1\}, \{0, x_i, \sigma_2\}, \{x_i, y_i, z_i\}, \{0, 1, y_i\}, \{0, 1, z_i\}, \{1, x_1, z_1\}, \{0, x_1, y_1\}, \{0, x_1, z_1\} \right\}$. \square

The family of minimal defining sets constructed by the Construction 7 can be extended by Lemma 2.5.

Lemma 3.2 (Extension of Construction 7). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, S$ and W be disjoint sets with $|U_i| = 3$, $U_i = \{x_i, y_i, z_i\}$, $1 \leq i \leq k$, $s = |S| \geq 0$ and $w = |W| \geq 0$. Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 2 + 3k + s$ and $V = U \cup W$, $v = |V| = u + w$. Define*

$$E_7(V) = D_7(U) \cup \left(F(V \setminus \{0\}) \setminus F(U) \right).$$

Then, the set $E_7(V)$ is a $d_m(v)$ of of size $(v^3 - 6v^2 + 11v)/6 + (u^2 - 7u)/2 + (3k^2 - ku - 2k + 4)$.

Proof. By Lemma 2.5,

$$\begin{aligned}
|E_7(V)| &= \binom{u}{3} - k \binom{5}{3} - 3s - s(k-1) + \binom{v-1}{3} - \binom{u-1}{3} \\
&= \binom{v-1}{3} + \binom{u}{3} - \binom{u-1}{3} - 10k - ks - 2s \\
&= (v^3 - 6v^2 + 11v - 6)/6 + (3u^2 - 9u + 6)/2 - 10k - ks - 2s \\
&= (v^3 - 6v^2 + 11v)/6 + (u^2 - 7u)/2 + (3k^2 - ku - 2k + 4).
\end{aligned}$$

Also, by Lemma 2.5, to prove $E_7(V)$ is a $d_m(v)$ of $F(V)$, it is only necessary to find, for an arbitrary $p \notin U$ and for every $\alpha_1, \alpha_2 \in U \setminus \{0\}$, a trade T satisfying

$$T \setminus \left(D_7^C(V) \cup \left\{ \{p, 0, \alpha\} \mid \alpha \in U \setminus \{0\} \right\} \right) = \left\{ \{p, \alpha_1, \alpha_2\} \right\}.$$

First, if $\{\alpha_1, \alpha_2\} \subseteq \{1\} \cup U_1 \cup S$, then corresponding trades exist by Extension of Construction 5.

For $\{\alpha_1, \alpha_2\} \in \left\{ \{1, x_i\}, \{1, y_i\}, \{x_1, x_i\}, \{x_1, y_i\}, \{z_1, x_i\}, \{z_1, y_i\}, \{x_i, y_i\}, \{x_i, x_j\}, \{x_i, y_j\}, \{y_i, z_i\}, \{y_i, y_j\}, \{\sigma, y_i\} \right\}$, where $2 \leq i < j \leq k$ and $\sigma \in S$, the required trades are $\left\{ \{1, x_i, p\}, \{0, y_i, p\}, \{0, 1, z_i\}, \{x_i, y_i, z_i\} \right\}$, $\left\{ \{1, y_i, p\}, \{0, z_i, p\}, \{1, x_i, z_i\}, \{0, x_i, y_i\} \right\}$, $\left\{ \{x_1, x_i, p\}, \{0, 1, x_1\}, \{0, z_i, p\}, \{1, x_i, z_i\} \right\}$, $\left\{ \{x_1, y_i, p\}, \{1, x_1, z_1\}, \{0, z_1, p\}, \{0, 1, y_i\} \right\}$, $\left\{ \{z_1, x_i, p\}, \{1, y_1, z_1\}, \{0, y_1, p\}, \{0, 1, x_i\} \right\}$, $\left\{ \{z_1, y_i, p\}, \{1, y_1, z_1\}, \{0, y_1, p\}, \{0, 1, y_i\} \right\}$, $\left\{ \{x_i, y_i, p\}, \{1, x_i, z_i\}, \{0, y_i, z_i\}, \{0, 1, p\} \right\}$, $\left\{ \{x_i, x_j, p\}, \{0, z_i, p\}, \{1, x_i, z_i\}, \{0, 1, x_j\} \right\}$, $\left\{ \{x_i, y_j, p\}, \{1, x_i, y_i\}, \{0, y_i, p\}, \{0, 1, y_j\} \right\}$, $\left\{ \{y_i, z_i, p\}, \{0, x_i, z_i\}, \{1, x_i, y_i\}, \{0, 1, p\} \right\}$, $\left\{ \{y_i, y_j, p\}, \{0, z_i, p\}, \{1, y_i, z_i\}, \{0, 1, y_j\} \right\}$ and $\left\{ \{\sigma, y_i, p\}, \{0, x_i, \sigma\}, \{0, 1, p\}, \{1, x_i, y_i\} \right\}$, respectively.

Lastly, for $\{\alpha_1, \alpha_2\} = \{\sigma, x_i\}$, where $\sigma \in S$ and $2 \leq i \leq k$, the required trade is $\left\{ \{\sigma, x_i, p\}, \{0, x_1, \sigma\}, \{0, y_1, \sigma\}, \{0, 1, x_i\}, \{1, x_1, y_1\}, \{0, 1, p\} \right\}$ with trade mate $\left\{ \{0, \sigma, p\}, \{1, x_i, p\}, \{0, x_i, \sigma\}, \{x_1, y_1, \sigma\}, \{0, 1, x_1\}, \{0, 1, y_1\} \right\}$. \square

In Construction 4, a family of full designs on 5 points are removed from the full design. Now, a family of full designs on 4 points will be removed from the full design in a similar way: Let $|V| = v = 2k + 2$, then fix two points of V , say $\{0, 1\}$ and partition the rest $2k$ points as $U_i = \{x_i, y_i\}$ for $1 \leq i \leq k$. Then, remove the full design on $\{0, 1\} \cup U_i$, i.e. $F(\{0, 1\} \cup U_i)$. The remaining blocks form a defining set of $F(V)$, however it is not minimal. Then, for the minimality, remove some extra blocks as described below.

Lemma 3.3 (Construction 8). *Let $\{0, 1\}$ and U_1, U_2, \dots, U_k be disjoint sets with $|U_i| = 2$ and $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k$. Let $V = \{0, 1\} \cup \bigcup_{i=1}^k U_i$, and $v = |V| = 2 + 2k$. Define*

$$D_8(V) = F(V) \setminus \left(\bigcup_{i=1}^k F(\{0, 1\} \cup U_i) \cup \bigcup_{\substack{i=2 \\ j < i}}^k \left\{ \{0, x_j, x_i\}, \{0, y_j, x_i\}, \{1, x_j, y_i\}, \{1, y_j, y_i\} \right\} \right).$$

Then, the set $D_8(V)$ is a $d_m(v)$ of $F(V)$ of size $(v^3 - 6v^2 + 8v)/6$.

Proof.

$$\begin{aligned} |D_8(V)| &= \binom{v}{3} - 4k - 4 \binom{k}{2} \\ &= \binom{v}{3} - 2k^2 - 2k \\ &= (v^3 - 6v^2 + 8v)/6. \end{aligned}$$

Now, we will prove that $D_8(V)$ is a $d_m(v)$ of $F(V)$ by induction on k . For $k = 1$, $D_8(\{0, 1\} \cup U_1) = \emptyset$ which is a $d_m(4)$. Also, for $k = 2$, $D_8(\{0, 1\} \cup U_1 \cup U_2) = \left\{ \{0, x_1, y_2\}, \{0, y_1, y_2\}, \{1, x_1, x_2\}, \{1, y_1, x_2\}, \{x_1, y_1, x_2\}, \{x_1, y_1, y_2\}, \{x_1, x_2, y_2\}, \{y_1, x_2, y_2\} \right\}$ which is a $d_m(6)$ with size 8.

Assume for $k = n$, $D_8(\{0, 1\} \cup \bigcup_{i=1}^n U_i)$ is a $d_m(2 + 2n)$ of $F(\{0, 1\} \cup \bigcup_{i=1}^n U_i)$. Then,

$D_8(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i)$ is a $d_m(4 + 2n)$ of $F(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i)$ will be shown. By Lemma 2.4, it is enough to consider the blocks containing at least one element of U_{n+1} to prove $D_8(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i)$ is a minimal defining set.

Firstly, we will show that $D_8^C(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i) \setminus D_8^C(\{0, 1\} \cup \bigcup_{i=1}^n U_i)$ is uniquely determined by $D_8(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i)$.

The blocks that contain x_{n+1} and y_{n+1} should contain 0 or 1. Otherwise, one block that contains x_{n+1} and y_{n+1} should contain both 0 and 1 since x_{n+1} occurs once without 0 and y_{n+1} occurs once without 1 in $D_8^C(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i) \setminus D_8^C(\{0, 1\} \cup \bigcup_{i=1}^n U_i)$. That is a contradiction. So, the blocks $\{0, x_{n+1}, y_{n+1}\}$ and $\{1, x_{n+1}, y_{n+1}\}$ are uniquely determined.

Then, for the remaining blocks, the blocks that contain x_{n+1} should also contain 0 and the blocks that contain y_{n+1} should also contain 1. Hence, the blocks $\{0, x_{n+1}, p\}$ are uniquely determined for $p \in V \setminus \{0, x_{n+1}, y_{n+1}\}$ and the blocks $\{1, y_{n+1}, p\}$ are uniquely determined for $p \in V \setminus \{1, x_{n+1}, y_{n+1}\}$.

Thus, the blocks in $D_8^C(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i) \setminus D_8^C(\{0, 1\} \cup \bigcup_{i=1}^n U_i)$ are uniquely determined by $D_8(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i)$.

To show minimality of $D_8(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i)$, by Lemma 2.4, it is enough to show for every block $X \in D_8(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i) \setminus D_8(\{0, 1\} \cup \bigcup_{i=1}^n U_i)$, there exists a trade T satisfying $T \setminus D_8^C(\{0, 1\} \cup \bigcup_{i=1}^{n+1} U_i) = \{X\}$.

First, consider the case where X contains only one of the points x_{n+1} and y_{n+1} . Then, without loss of generality, $X \in \left\{ \{1, x_i, x_{n+1}\}, \{1, y_i, x_{n+1}\}, \{p_i, p_j, x_{n+1}\} \mid p_i \in U_i, p_j \in U_j, p_i \neq p_j, 1 \leq i, j \leq n \right\}$ and the required (Pasch) trades are $\left\{ \{1, x_i, x_{n+1}\}, \{1, y_i, x_{n+1}\}, \{0, x_i, y_i\}, \{0, x_{n+1}, y_{n+1}\} \right\}$, $\left\{ \{1, y_i, x_{n+1}\}, \{1, x_i, x_{n+1}\}, \{0, x_i, y_i\}, \{0, x_{n+1}, y_{n+1}\} \right\}$ and $\left\{ \{p_i, p_j, x_{n+1}\}, \{1, p_i, y_{n+1}\}, \{0, 1, p_j\}, \{0, x_{n+1}, y_{n+1}\} \right\}$, respectively.

Now, consider the case where X contains both x_{n+1} and y_{n+1} . Then, up to generality $X = \{x_i, x_{n+1}, y_{n+1}\}$, $1 \leq i \leq n$, and the required (Pasch) trade is $\left\{ \{x_i, x_{n+1}, y_{n+1}\}, \{1, y_i, y_{n+1}\}, \{0, x_i, y_i\}, \{0, 1, x_{n+1}\} \right\}$. \square

The family of minimal defining sets constructed by the Construction 8 can be extended both by Lemma 2.5 and by Lemma 2.7.

Lemma 3.4 (Extension of Construction 8). *Let $\{0, 1\}, U_1, U_2, \dots, U_k$ and W be disjoint sets with $|U_i| = 2$, $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k$, $k \geq 2$ and $w = |W| \geq 0$. Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i$, $u = |U| = 2 + 2k$ and $V = U \cup W$, $v = |V| = u + w$. Define for $1 \leq i \leq k$,*

$$E_{8,i}(V) = D_8(U) \cup \left(F(V \setminus \{x_i\}) \setminus F(U) \right).$$

Then, the sets $E_{8,i}(V)$, $1 \leq i \leq k$, are $d_m(v)$ s of size $(v^3 - 6v^2 + 11v - 6)/6 - k$.

Proof. The set $D_8(U)$ is a $d_m(u)$ of size $(u^3 - 6u^2 + 8u)/6$. By Lemma 2.5,

$$\begin{aligned} |E_{8,i}(V)| &= \frac{u^3 - 6u^2 + 8u}{6} + \binom{v-1}{3} - \binom{u-1}{3} \\ &= \frac{u^3 - 6u^2 + 8u}{6} + \frac{v^3 - 6v^2 + 11v - 6}{6} - \frac{u^3 - 6u^2 + 11u - 6}{6} \\ &= \frac{v^3 - 6v^2 + 11v}{6} - \frac{u}{2} \\ &= \frac{v^3 - 6v^2 + 11v - 6}{6} - k. \end{aligned}$$

By Lemma 2.5 with D, E and a corresponding to $D_8(U), E_{8,i}(V)$ and x_i , respectively, to prove that $E_{8,i}(V)$ is a $d_m(v)$, it is only necessary to find, for an arbitrary $p \notin U$ and for every $\alpha_1, \alpha_2 \in U \setminus \{x_i\}$, a trade T satisfying,

$$T \setminus \left(D_8^C(U) \cup \left\{ \{p, x_i, \alpha\} \mid \alpha \in U \setminus \{x_i\} \right\} \right) = \left\{ \{p, \alpha_1, \alpha_2\} \right\}.$$

For $\{\alpha_1, \alpha_2\} \in \left\{ \{0, 1\}, \{0, x_m\}, \{0, y_m\}, \{0, y_i\}, \{1, x_m\}, \{1, y_m\}, \{1, y_i\} \right\}$, with $1 \leq m \leq k$ and $m \neq i$, the required trades are $\left\{ \{0, 1, p\}, \{y_m, x_i, p\}, \{1, x_m, y_m\}, \{0, x_m, x_i\} \right\}$, $\left\{ \{0, x_m, p\}, \{y_m, x_i, p\}, \{1, x_m, y_m\}, \{0, 1, x_i\} \right\}$, $\left\{ \{0, y_m, p\}, \{x_i, y_i, p\}, \{1, y_m, y_i\}, \{0, 1, x_i\} \right\}$, $\left\{ \{0, y_i, p\}, \{y_m, x_i, p\}, \{1, x_i, y_i\}, \{0, 1, y_m\} \right\}$, $\left\{ \{1, x_m, p\}, \{x_i, y_i, p\}, \{0, x_m, x_i\}, \{0, 1, y_i\} \right\}$, $\left\{ \{1, y_m, p\}, \{0, x_i, p\}, \{0, x_i, y_i\}, \{1, y_m, y_i\} \right\}$ and $\left\{ \{1, y_i, p\}, \{x_m, x_i, p\}, \{0, x_i, y_i\}, \{0, 1, x_m\} \right\}$, respectively.

For $\{\alpha_1, \alpha_2\} \in \left\{ \{x_m, x_n\}, \{x_m, y_n\}, \{x_m, y_i\}, \{y_m, y_i\}, \{y_m, y_n\} \right\}$, with $1 \leq m, n \leq k$, $m \neq i$ and $n \neq i$, the required trades are $\left\{ \{x_m, x_n, p\}, \{1, x_i, p\}, \{0, x_m, x_i\}, \{0, 1, x_n\} \right\}$, $\left\{ \{x_m, y_n, p\}, \{1, x_i, p\}, \{0, x_m, x_i\}, \{0, 1, y_n\} \right\}$, $\left\{ \{x_m, y_i, p\}, \{1, x_i, p\}, \{0, x_m, x_i\}, \{0, 1, y_i\} \right\}$, $\left\{ \{y_m, y_i, p\}, \{1, x_i, p\}, \{0, x_i, y_i\}, \{0, 1, y_m\} \right\}$ and $\left\{ \{y_m, y_n, p\}, \{1, y_i, p\}, \{0, y_m, y_i\}, \{0, 1, y_n\} \right\}$, respectively. \square

Lemma 3.5 (Double Extension of Construction 8). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, W$ and Q be disjoint sets with $|U_i| = 2$, $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k, k \geq 2$, $w = |W| \geq 2$ and $q = |Q| \geq 1$. Let c be a distinguished element of Q . Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i$, $u = |U| = 2 + 2k$ and $V = U \cup W \cup Q$, $v = |V| = u + w + q$. Define for any $1 \leq i \leq j \leq k$,*

$$\begin{aligned} \mathbb{E}_8^C(V) = & D_8^C(U) \cup \left\{ \{x_i, \beta_1, \alpha\}, \{x_i, \beta_1, \beta_2\}, \{x_i, \beta_1, c\} \mid \alpha \in U \setminus \{x_i\}, \beta_1, \beta_2 \in W \right\} \\ & \cup \left\{ \{y_j, \gamma_1, \alpha\}, \{y_j, \gamma_1, \gamma_2\}, \{y_j, \gamma_1, \beta\} \mid \alpha \in U \setminus \{y_j\}, \beta \in W, \gamma_1, \gamma_2 \in Q \right\}. \end{aligned}$$

Then, the set $\mathbb{E}_8(V) = F(V) \setminus \mathbb{E}_8^C(V)$ is a $d_m(v)$ of size $(v^3 - 6v^2 + 11v - 6)/6 - k - w$.

Proof. By Lemma 2.7,

$$\begin{aligned} |\mathbb{E}_8(V)| &= \frac{u^3 - 6u^2 + 8u}{6} + \binom{v-1}{3} - \binom{u-1}{3} - w \\ &= \frac{v^3 - 6v^2 + 11v - 6}{6} - k - w. \end{aligned}$$

Now, the required trades for the pivot point x_i are given in Extension of Construction 8. Also, the required trades for the pivot point y_j can be obtained by interchanging 0 and 1; and x_i and y_i for $1 \leq i \leq k$, for the given trades in Extension of Construction 8. Thus, by Lemma 2.7 with D, \mathbb{E}, a and b corresponding to $D_8(U), \mathbb{E}_8(V), x_i$ and y_j , respectively, it follows that $\mathbb{E}_8(V)$ is also a $d_m(v)$. \square

Lemma 3.6. *The minimal defining sets $E_{8,i}(V)$ and $E_{8,j}(V)$ are non-isomorphic if $1 \leq i \neq j \leq k$.*

Proof. Assume for a contradiction, $E_{8,m}(V)$ and $E_{8,n}(V)$ are isomorphic for some m and n with $1 \leq m < n \leq k$. Let $w = |W| \geq 1$. Then, only the pairs $\{x_m, \alpha\}$ do not occur in $E_{8,m}(V)$ and only the pairs $\{x_n, \beta\}$ do not occur in $E_{8,n}(V)$ where $\alpha, \beta \in W$. Hence, for a fixed $\alpha \in W$, $\{x_m, \alpha\}$ should be mapped to $\{x_n, \beta\}$ for some $\beta \in W$. Also, all $\beta \in W$ occurs exactly once with all points of $V \setminus \{x_n, \beta\}$ in $E_{8,n}^C(V)$. Thus, x_m should be mapped to x_n .

Now, the pair $\{x_m, 0\}$ occurs $(v-2) - 2 - 2(m-1) - (k-m) = v - k - m - 2$ times, the pair $\{x_m, 1\}$ occurs $(v-2) - 2 - (k-m) = v - k + m - 4$ times, the pair $\{x_m, y_m\}$ occurs $(v-2) - 2 = v - 4$ times and the pairs $\{x_m, p\}$ for $p \in U \setminus \{0, 1, x_m, y_m\}$ occur $(v-2) - 1 = v - 3$ times in $E_{8,m}(V)$. Also, the pair $\{x_n, 0\}$ occurs $v - k - n - 2$ times, the pair $\{x_n, 1\}$ occurs $v - k + n - 4$ times, the pair $\{x_n, y_n\}$ occurs $v - 4$ times and the pairs $\{x_n, p\}$ for $p \in U \setminus \{0, 1, x_n, y_n\}$ occur $v - 3$ times in $E_{8,n}(V)$. Thus, the pair $\{x_m, 0\}$ can not be mapped to one of the pairs $\{x_n, 0\}, \{x_n, 1\}, \{x_n, y_n\}$ and $\{x_n, p\}$

for $p \in U \setminus \{0, 1, x_n, y_n\}$ since if so, we need to have $m = n$, $m + n = 2$, $k + m = 2$ or $k + m = 1$, respectively. Hence, none of these cases are possible and the result follows. \square

Theorem 3.7. *There exist values in the spectrum of the full design on v elements such that the number of non-isomorphic minimal defining sets with each of these sizes goes to infinity as $v \rightarrow \infty$.*

Proof. Extension of Construction 8 and Lemma 3.6 requires $k \geq 2$ and $w \geq 1$, i.e. $k \leq (v - 3)/2$, respectively. Hence, for a fixed $v \geq 7$, there are at least k non-isomorphic minimal defining sets with size $(v^3 - 6v^2 + 11v - 6)/6 - k$ for all $2 \leq k \leq (v - 3)/2$ by Extension of Construction 8 and Lemma 3.6. Thus, as $v \rightarrow \infty$, the result follows. \square

Note that, Construction 8 is defined for $v = 2 + 2k$ where $k \geq 2$. Now, by adding a new set of points to V , say S , Construction 8 can be enlarged for all $v \geq 6$. Some blocks that contain points in S will be removed to get a minimal defining set. In Construction 9, 10 and 11, this idea will be used with removing different sets of blocks containing points in S . Also, extensions of these constructions via Lemma 2.5 and Lemma 2.7 will be given, respectively.

Lemma 3.8 (Construction 9). *Let $\{0, 1\}, U_1, U_2, \dots, U_k$ and S be disjoint sets with $|U_i| = 2$ and $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k$, $k \geq 2$ and $s = |S| \geq 0$. Let $V = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$ and $v = |V| = 2 + 2k + s$. Define*

$$D_9(V) = F(V) \setminus \left(\bigcup_{i=1}^k F(\{0, 1\} \cup U_i) \cup \bigcup_{\substack{i=2 \\ j < i}}^k \left\{ \{0, x_j, x_i\}, \{0, y_j, x_i\}, \{1, x_j, y_i\}, \{1, y_j, y_i\} \right\} \right. \\ \left. \cup \left\{ \{0, x_k, \sigma_1\}, \{1, y_{k-1}, \sigma_1\}, \{1, x_k, \sigma_1\}, \{y_{k-1}, x_k, \sigma_1\}, \{x_k, \sigma_1, \sigma_2\} \mid \sigma_1, \sigma_2 \in S \right\} \right).$$

Then, the set $D_9(V)$ is a $d_m(v)$ of $F(V)$ of size $(v^3 - 6v^2 - 7v + 30)/6 + k(2v - 4k + 1)$.

Proof.

$$\begin{aligned}
|D_9(V)| &= \binom{v}{3} - 4k - 4 \frac{k(k-1)}{2} - 4s - \binom{s}{2} \\
&= \frac{v^3 - 6v^2 + 2v}{6} - 2k^2 - 2k - 4(v-2-2k) - \frac{(v-2k-2)(v-2k-3)}{2} \\
&= \frac{v^3 - 6v^2 - 7v + 30}{6} + k(2v - 4k + 1).
\end{aligned}$$

Let $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i$. Then, $D_9(V) \cap F(U) = D_8(U)$. Thus, by Lemma 2.4 and Construction 8, it is enough to consider blocks that contains at least one element of S .

Let $\sigma \in S$ be arbitrary. σ occurs once without x_k , twice with 1 and twice with y_{k-1} in $D_9^C(V)$. Hence, the blocks $\{1, x_k, \sigma\}$ and $\{y_{k-1}, x_k, \sigma\}$ are uniquely determined.

Now, y_k occurs once without 1, twice with 0 and twice with x_k in $D_9^C(V)$. Hence, the blocks $\{0, 1, y_k\}$ and $\{1, x_k, y_k\}$ are also uniquely determined. Note that, for the remaining blocks in $D_9^C(V)$, the pairs $\{1, x_k\}$ and $\{y_{k-1}, x_k\}$ occur only once.

Next, the blocks that contain at least two elements of S will be considered. $\{0, \sigma_i, \sigma_j\}$, $\{1, \sigma_i, \sigma_j\}$, $\{\sigma_m, \sigma_i, \sigma_j\}$, $\{y_{k-1}, \sigma_i, \sigma_j\}$ and $\{x_k, \sigma_i, \sigma_j\}$ are the only blocks where $\sigma_i, \sigma_j, \sigma_m \in S$, all distinct, that can be occur in a possible completion. For the first three cases, if they occur then, all blocks that contain σ_i or σ_j should also contain x_k . So, both the blocks $\{\sigma_i, y_{k-1}, x_k\}$ and $\{\sigma_j, y_{k-1}, x_k\}$ will be forced. But, this is not possible. If a block of the form $\{y_{k-1}, \sigma_i, \sigma_j\}$ occurs in a possible completion then, the blocks that contain the pairs $\{1, \sigma_i\}$ or $\{1, \sigma_j\}$ should also contain the point x_k which is not possible. Hence, the blocks $\{x_k, \sigma_i, \sigma_j\}$ are uniquely determined.

For the remaining blocks, the block that contains x_k also contains 0. Hence, the blocks $\{0, x_k, \sigma\}$ are uniquely determined for all $\sigma \in S$. Lastly, the blocks $\{1, y_{k-1}, \sigma\}$ are

determined uniquely for all $\sigma \in S$. Thus, $D_9(V)$ is a $d(v)$ of $F(V)$.

To show minimality of $D_9(V)$, by Lemma 2.4, it is enough to show for every block $X \in D_9(V) \setminus D_8(U)$, there exists a trade T satisfying $T \setminus D_9^C(V) = \{X\}$.

Let $\sigma \in S$ be arbitrary. For the block $\{0, 1, \sigma\}$, the required trade is $\left\{ \{0, 1, \sigma\}, \{y_{k-1}, x_k, \sigma\}, \{0, x_k, y_k\}, \{1, y_{k-1}, y_k\} \right\}$. For the block $\{0, x_i, \sigma\}$, $1 \leq i \leq k-1$, the required trade is $\left\{ \{0, x_i, \sigma\}, \{1, x_k, \sigma\}, \{0, x_k, y_k\}, \{1, x_i, y_k\} \right\}$. For the block $\{0, y_i, \sigma\}$, $1 \leq i \leq k$, the required trade is $\left\{ \{0, y_i, \sigma\}, \{1, x_k, \sigma\}, \{0, x_1, x_k\}, \{1, x_1, y_i\} \right\}$. For the block $\{1, p, \sigma\}$ where $p \in \{x_i, y_j | 1 \leq i, j \leq k, i \neq k, j \neq k-1\}$, the required trade is $\left\{ \{1, p, \sigma\}, \{y_{k-1}, x_k, \sigma\}, \{0, p, x_k\}, \{0, 1, y_{k-1}\} \right\}$. For the block $\{p_1, p_2, \sigma\}$ where $\{p_1, p_2\} \subset \{x_i, y_j | 1 \leq i, j \leq k, i \neq k\}$, the required trade is $\left\{ \{p_1, p_2, \sigma\}, \{1, x_k, \sigma\}, \{0, p_1, x_k\}, \{0, 1, p_2\} \right\}$. For the block $\{x_k, p, \sigma\}$ where $p \in \{x_i, y_j | 1 \leq i, j \leq k, i \neq k, j \neq k-1\}$, the required trade is $\left\{ \{x_k, p, \sigma\}, \{1, y_{k-1}, \sigma\}, \{0, 1, p\}, \{0, y_{k-1}, x_k\} \right\}$.

Now, let $\sigma_1, \sigma_2 \in S$ be arbitrary. For the blocks $\{0, \sigma_1, \sigma_2\}$ and $\{1, \sigma_1, \sigma_2\}$, the required trades are $\left\{ \{0, \sigma_1, \sigma_2\}, \{1, x_k, \sigma_1\}, \{y_{k-1}, x_k, \sigma_2\}, \{0, 1, y_{k-1}\} \right\}$ and $\left\{ \{1, \sigma_1, \sigma_2\}, \{0, x_k, \sigma_1\}, \{y_{k-1}, x_k, \sigma_2\}, \{0, 1, y_{k-1}\} \right\}$, respectively. For the block $\{p, \sigma_1, \sigma_2\}$ where $p \in \{x_i, y_j | 1 \leq i, j \leq k, i \neq k\}$, the required trade is $\left\{ \{p, \sigma_1, \sigma_2\}, \{0, x_k, \sigma_1\}, \{1, x_k, \sigma_2\}, \{0, 1, p\} \right\}$.

Lastly, let $\sigma_1, \sigma_2, \sigma_3 \in S$ be arbitrary. Then, the required trade is $\left\{ \{\sigma_1, \sigma_2, \sigma_3\}, \{1, x_k, \sigma_1\}, \{y_{k-1}, x_k, \sigma_2\}, \{1, y_{k-1}, \sigma_3\} \right\}$. \square

Lemma 3.9 (Extension of Construction 9). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, S$ and W be disjoint sets with $|U_i| = 2$, $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k, k \geq 2$ and $s = |S| \geq 0$ and $w = |W| \geq 0$. Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 2 + 2k + s$ and $V = U \cup W$, $v = |V| = u + w$.*

Define,

$$E_9(V) = D_9(U) \cup \left(F(V \setminus \{x_k\}) \setminus F(U) \right).$$

Then, the set $E_9(V)$, is a $d_m(v)$ of size $(v^3 - 6v^2 + 11v + 30)/6 + k(2u - 4k + 1) - 3u$.

Proof. By Construction 9, the set $D_9(U)$ is a $d_m(u)$ of size $(u^3 - 6u^2 - 7u + 30)/6 + k(2u - 4k + 1)$. By Lemma 2.5,

$$\begin{aligned} |E_9(V)| &= \frac{u^3 - 6u^2 - 7u + 30}{6} + k(2u - 4k + 1) + \binom{v-1}{3} - \binom{u-1}{3} \\ &= \frac{v^3 - 6v^2 + 11v + 30}{6} + k(2u - 4k + 1) - 3u. \end{aligned}$$

By Lemma 2.5 with D, E and a corresponding to $D_9(U), E_9(V)$ and x_k , respectively, to prove that $E_9(V)$ is a $d_m(v)$, it is only necessary to find, for an arbitrary $p \notin U$ and for every $\alpha_1, \alpha_2 \in U \setminus \{x_k\}$, a trade T satisfying,

$$T \setminus \left(D_9^C(U) \cup \left\{ \{p, x_k, \alpha\} \mid \alpha \in U \setminus \{x_k\} \right\} \right) = \left\{ \{p, \alpha_1, \alpha_2\} \right\}.$$

Firstly, if $\{\alpha_1, \alpha_2\} \subseteq \{0, 1\} \cup \bigcup_{i=1}^k U_i$, then the corresponding trades exist by Extension of Construction 8.

Without loss of generality, for $\{\alpha_1, \alpha_2\} \in \left\{ \{0, \sigma_1\}, \{1, \sigma_1\}, \{x_i, \sigma_1\}, \{y_i, \sigma_1\}, \{x_{k-1}, \sigma_1\}, \{y_{k-1}, \sigma_1\}, \{y_k, \sigma_1\}, \{\sigma_1, \sigma_2\} \right\}$ where $\sigma_1, \sigma_2 \in S$ and $1 \leq i \leq k-2$, the required trades are $\left\{ \{0, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{0, 1, x_k\}, \{1, y_{k-1}, \sigma_1\} \right\}$, $\left\{ \{1, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{0, 1, y_{k-1}\}, \{0, x_k, \sigma_1\} \right\}$, $\left\{ \{x_i, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{1, x_i, y_{k-1}\}, \{1, x_k, \sigma_1\} \right\}$, $\left\{ \{y_i, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{1, y_i, y_{k-1}\}, \{1, x_k, \sigma_1\} \right\}$, $\left\{ \{x_{k-1}, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{1, x_{k-1}, y_{k-1}\}, \{1, x_k, \sigma_1\} \right\}$, $\left\{ \{y_{k-1}, \sigma_1, p\}, \{0, x_k, p\}, \{0, 1, y_{k-1}\}, \{1, x_k, \sigma_1\} \right\}$, $\left\{ \{y_k, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{1, x_k, y_k\}, \{1, y_{k-1}, \sigma_1\} \right\}$ and $\left\{ \{\sigma_1, \sigma_2, p\}, \{y_{k-1}, x_k, p\}, \{1, x_k, \sigma_1\}, \{1, y_{k-1}, \sigma_2\} \right\}$, respectively. \square

Lemma 3.10 (Double Extension of Construction 9). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, S, W$ and Q be disjoint sets with $|U_i| = 2$, $U_i = \{x_i, y_i\}$, $1 \leq i \leq k$, $k \geq 2$, $s = |S| \geq 0$ and $w = |W| \geq 2$ and $q = |Q| \geq 1$. Let c be a distinguished element of Q . Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 2 + 2k + s$ and $V = U \cup W \cup Q$ and $v = |V| = u + w + q$. Define*

$$\begin{aligned} \mathbb{E}_9^C(V) = & D_9^C(U) \cup \left\{ \{x_k, \beta_1, \alpha\}, \{x_k, \beta_1, \beta_2\}, \{x_k, \beta_1, c\} \mid \alpha \in U \setminus \{x_k\}, \beta_1, \beta_2 \in W \right\} \\ & \cup \left\{ \{y_{k-1}, \gamma_1, \alpha\}, \{y_{k-1}, \gamma_1, \gamma_2\}, \{y_{k-1}, \gamma_1, \beta\} \mid \alpha \in U \setminus \{y_{k-1}\}, \beta \in W, \gamma_1, \gamma_2 \in Q \right\}. \end{aligned}$$

Then, the set $\mathbb{E}_9(V) = F(V) \setminus \mathbb{E}_9^C(V)$ is a $d_m(v)$ of size $(v^3 - 6v^2 + 11v + 30)/6 + k(2u - 4k + 1) - 3u - w$.

Proof. By Lemma 2.7,

$$\begin{aligned} |\mathbb{E}_9(V)| &= \frac{u^3 - 6u^2 - 7u + 30}{6} + k(2u - 4k + 1) + \binom{v-1}{3} - \binom{u-1}{3} - w \\ &= \frac{v^3 - 6v^2 + 11v + 30}{6} + k(2u - 4k + 1) - 3u - w. \end{aligned}$$

The trades for the pivot point x_k are given in Extension of Construction 9 and the trades for the pivot point y_{k-1} , are the trades which are given in Extension of Construction 9 and the following one: $\left\{ \{x_k, \sigma, p\}, \{0, y_{k-1}, p\}, \{0, 1, x_k\}, \{1, y_{k-1}, \sigma\} \right\}$. Therefore, $\mathbb{E}_9(V)$ is also a $d_m(v)$ by Lemma 2.7 with D, \mathbb{E}, a and b corresponding to $D_9(U)$, $\mathbb{E}_9(V)$, x_k and y_{k-1} , respectively, . \square

Lemma 3.11 (Construction 10). *Let $\{0, 1\}, U_1, U_2, \dots, U_k$ and S be disjoint sets with $|U_i| = 2$ and $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k$, $k \geq 2$ and $s = |S| \geq 0$. Let $V = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$ and $v = |V| = 2 + 2k + s$. Define*

$$\begin{aligned} D_{10}(V) = & F(V) \setminus \left(\bigcup_{i=1}^k F(\{0, 1\} \cup U_i) \cup \bigcup_{\substack{i=2 \\ j < i}}^k \left\{ \{0, x_j, x_i\}, \{0, y_j, x_i\}, \{1, x_j, y_i\}, \{1, y_j, y_i\} \right\} \right. \\ & \left. \cup \left\{ \{0, x_k, \sigma_1\}, \{0, y_k, \sigma_1\}, \{1, y_k, \sigma_1\}, \{x_k, y_k, \sigma_1\}, \{y_k, \sigma_1, \sigma_2\} \mid \sigma_1, \sigma_2 \in S \right\} \right). \end{aligned}$$

Then, the set $D_{10}(V)$ is a $d_m(v)$ of $F(V)$ of size $(v^3 - 6v^2 - 7v + 30)/6 + k(2v - 4k + 1)$.

Proof.

$$\begin{aligned} |D_{10}(V)| &= \binom{v}{3} - 4k - 4\frac{k(k-1)}{2} - 4s - \binom{s}{2} \\ &= \frac{v^3 - 6v^2 - 7v + 30}{6} + k(2v - 4k + 1). \end{aligned}$$

Let $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i$. Then $D_{10}(V) \cap F(U) = D_8(U)$. Thus, by Lemma 2.4 and Construction 8, it is enough to consider blocks that contains at least one element of S .

Let $\sigma \in S$ be arbitrary. σ occurs once without y_k , twice with 0 and twice with x_k in $D_{10}^C(V)$. Hence, the blocks $\{0, y_k, \sigma\}$ and $\{x_k, y_k, \sigma\}$ are uniquely determined.

Now, x_k occurs once without 0, twice with 1 and twice with y_k in $D_{10}^C(V)$. Hence, the blocks $\{0, 1, x_k\}$ and $\{0, x_k, y_k\}$ are also uniquely determined. Note that, for the remaining blocks in $D_{10}^C(V)$, the pairs $\{0, y_k\}$ and $\{x_k, y_k\}$ occur only once.

Now, the blocks that contain at least two elements of S will be considered. $\{0, \sigma_i, \sigma_j\}$, $\{1, \sigma_i, \sigma_j\}$, $\{\sigma_m, \sigma_i, \sigma_j\}$, $\{x_k, \sigma_i, \sigma_j\}$ and $\{y_k, \sigma_i, \sigma_j\}$ are the only possible blocks for a possible completion where $\sigma_i, \sigma_j, \sigma_m \in S$, all distinct. For the first three cases, if they occur then, all blocks that contain σ_i or σ_j should also contain y_k . So, $\{\sigma_i, x_k, y_k\}$ and $\{\sigma_j, x_k, y_k\}$ will be forced. But, this is not possible. For the fourth case, if it occurs in a possible completion then, the blocks that contain the pairs $\{0, \sigma_i\}$ or $\{0, \sigma_j\}$ should also contain the point y_k which is not possible. Hence, the blocks $\{y_k, \sigma_i, \sigma_j\}$ are determined uniquely.

For the remaining blocks, the block that contains y_k also contains 1. Hence, the blocks $\{1, y_k, \sigma\}$ are uniquely determined for all $\sigma \in S$. Lastly, the blocks $\{0, x_k, \sigma\}$ are forced for all $\sigma \in S$. Thus, $D_{10}(V)$ is a $d(v)$ of $F(V)$.

To show minimality of $D_{10}(V)$, by Lemma 2.4, it is enough to show for every block $X \in D_{10}(V) \setminus D_8(U)$, there exists a trade T satisfying $T \setminus D_{10}^C(V) = \{X\}$.

Let $\sigma \in S$ be arbitrary. For the block $\{0, 1, \sigma\}$, the required trade is $\left\{ \{0, 1, \sigma\}, \{x_k, y_k, \sigma\}, \{0, x_1, x_k\}, \{1, x_1, y_k\} \right\}$. For the block $\{0, p, \sigma\}$ where $p \in \{x_i, y_j | 1 \leq i, j \leq k-1\}$, the required trade is $\left\{ \{0, p, \sigma\}, \{x_k, y_k, \sigma\}, \{0, 1, x_k\}, \{0, p, y_k\} \right\}$. For the block $\{1, p, \sigma\}$ where $p \in \{x_i, y_j | 1 \leq i, j \leq k, j \neq k\}$, the required trade is $\left\{ \{1, p, \sigma\}, \{0, y_k, \sigma\}, \{0, p, x_k\}, \{1, x_k, y_k\} \right\}$. For the block $\{p_1, p_2, \sigma\}$ where $\{p_1, p_2\} \subset \{x_i, y_j | 1 \leq i, j \leq k, j \neq k\}$, the required trade is $\left\{ \{p_1, p_2, \sigma\}, \{0, y_k, \sigma\}, \{1, p_1, y_k\}, \{0, 1, p_2\} \right\}$. For the block $\{y_k, p, \sigma\}$ where $p \in \{x_i, y_j | 1 \leq i, j \leq k-1\}$, the required trade is $\left\{ \{y_k, p, \sigma\}, \{1, x_k, \sigma\}, \{0, 1, y_k\}, \{0, p, x_k\} \right\}$.

Now, let $\sigma_1, \sigma_2 \in S$ be arbitrary. For the blocks $\{0, \sigma_1, \sigma_2\}$ and $\{1, \sigma_1, \sigma_2\}$, the required trades are $\left\{ \{0, \sigma_1, \sigma_2\}, \{1, y_k, \sigma_1\}, \{x_k, y_k, \sigma_2\}, \{0, 1, x_k\} \right\}$ and $\left\{ \{1, \sigma_1, \sigma_2\}, \{0, x_k, \sigma_1\}, \{x_k, y_k, \sigma_2\}, \{0, 1, y_k\} \right\}$, respectively. For the block $\{p, \sigma_1, \sigma_2\}$ where $p \in \{x_i, y_j | 1 \leq i, j \leq k, j \neq k\}$, the required trade is $\left\{ \{p, \sigma_1, \sigma_2\}, \{0, y_k, \sigma_1\}, \{1, y_k, \sigma_2\}, \{0, 1, p\} \right\}$.

Lastly, let $\sigma_1, \sigma_2, \sigma_3 \in S$ be arbitrary. Then, the required trade is $\left\{ \{\sigma_1, \sigma_2, \sigma_3\}, \{0, x_k, \sigma_1\}, \{0, y_k, \sigma_2\}, \{x_k, y_k, \sigma_3\} \right\}$. \square

Lemma 3.12 (Extension of Construction 10). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, S$ and W be disjoint sets with $|U_i| = 2$, $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k, k \geq 2$ and $s = |S| \geq 0$ and $w = |W| \geq 0$. Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 2 + 2k + s$ and $V = U \cup W$, $v = |V| = u + w$. Define,*

$$E_{10}(V) = D_{10}(U) \cup \left(F(V \setminus \{x_k\}) \setminus F(U) \right).$$

Then, the set $E_{10}(V)$ is a $d_m(v)$ of size $(v^3 - 6v^2 + 11v + 30)/6 + k(2u - 4k + 1) - 3u$.

Proof. By Lemma 2.5,

$$\begin{aligned} |E_{10}(V)| &= \frac{u^3 - 6u^2 - 7u + 30}{6} + k(2u - 4k + 1) + \binom{v-1}{3} - \binom{u-1}{3} \\ &= \frac{v^3 - 6v^2 + 11v + 30}{6} + k(2u - 4k + 1) - 3u. \end{aligned}$$

Also, by Lemma 2.5 with D, E and a corresponding to $D_{10}(U)$, $E_{10}(V)$ and x_k , respectively, to prove that $E_{10}(V)$ is a $d_m(v)$, it is only necessary to find, for an arbitrary $p \notin U$ and for every $\alpha_1, \alpha_2 \in U \setminus \{x_k\}$, a trade T satisfying,

$$T \setminus \left(D_{10}^C(U) \cup \left\{ \{p, x_k, \alpha\} \mid \alpha \in U \setminus \{x_k\} \right\} \right) = \left\{ \{p, \alpha_1, \alpha_2\} \right\}.$$

If $\{\alpha_1, \alpha_2\} \subseteq \{0, 1\} \cup \bigcup_{i=1}^k U_i$, then the corresponding trades exist by Extension of Construction 8.

Without loss of generality, for $\{\alpha_1, \alpha_2\} \in \left\{ \{0, \sigma_1\}, \{1, \sigma_1\}, \{x_i, \sigma_1\}, \{y_i, \sigma_1\}, \{y_k, \sigma_1\}, \{\sigma_1, \sigma_2\} \right\}$ where $\sigma_1, \sigma_2 \in S$ and $1 \leq i \leq k-1$, the required trades are

$$\begin{aligned} &\left\{ \{0, \sigma_1, p\}, \{x_k, y_k, p\}, \{0, 1, x_k\}, \{1, y_k, \sigma_1\} \right\}, \left\{ \{1, \sigma_1, p\}, \{x_k, y_k, p\}, \{0, 1, x_k\}, \{0, y_k, \sigma_1\} \right\}, \\ &\left\{ \{x_i, \sigma_1, p\}, \{x_k, y_k, p\}, \{0, x_i, x_k\}, \{0, y_k, \sigma_1\} \right\}, \left\{ \{y_i, \sigma_1, p\}, \{x_k, y_k, p\}, \{0, y_i, x_k\}, \{0, y_k, \sigma_1\} \right\}, \\ &\left\{ \{y_k, \sigma_1, p\}, \{1, x_k, p\}, \{0, 1, y_k\}, \{0, x_k, \sigma_1\} \right\}, \left\{ \{\sigma_1, \sigma_2, p\}, \{x_k, y_k, p\}, \{0, x_k, \sigma_1\}, \{0, y_k, \sigma_2\} \right\}. \end{aligned}$$

□

Lemma 3.13 (Double Extension of Construction 10). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, S, W$ and Q be disjoint sets with $|U_i| = 2$, $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k, k \geq 2, s = |S| \geq 0$ and $w = |W| \geq 2$ and $q = |Q| \geq 1$. Let c be a distinguished element of Q . Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 2 + 2k + s$ and $V = U \cup W \cup Q$ and $v = |V| = u + w + q$.*

Define

$$\begin{aligned} \mathbb{E}_{10}^C(V) = & D_{10}^C(U) \cup \left\{ \{x_k, \beta_1, \alpha\}, \{x_k, \beta_1, \beta_2\}, \{x_k, \beta_1, c\} \mid \alpha \in U \setminus \{x_k\}, \beta_1, \beta_2 \in W \right\} \\ & \cup \left\{ \{y_k, \gamma_1, \alpha\}, \{y_k, \gamma_1, \gamma_2\}, \{y_k, \gamma_1, \beta\} \mid \alpha \in U \setminus \{y_k\}, \beta \in W, \gamma_1, \gamma_2 \in Q \right\}. \end{aligned}$$

Then, the set $\mathbb{E}_{10}(V) = F(V) \setminus \mathbb{E}_{10}^C(V)$ is a $d_m(v)$ of size $(v^3 - 6v^2 + 11v + 30)/6 + k(2u - 4k + 1) - 3u - w$.

Proof. By Lemma 2.7,

$$\begin{aligned} |\mathbb{E}_{10}(V)| &= \frac{u^3 - 6u^2 - 7u + 30}{6} + k(2u - 4k + 1) + \binom{v-1}{3} - \binom{u-1}{3} - w \\ &= \frac{v^3 - 6v^2 + 11v + 30}{6} + k(2u - 4k + 1) - 3u - w. \end{aligned}$$

The trades for the pivot point x_k are given in Extension of Construction 10 and the trades for the pivot point y_k , are the trades which are given in Extension of Construction 10 and the following one: $\left\{ \{x_k, \sigma, p\}, \{1, y_k, p\}, \{0, 1, x_k\}, \{0, y_k, \sigma\} \right\}$. Therefore, $\mathbb{E}_{10}(V)$ is also a $d_m(v)$ by Lemma 2.7 with D, \mathbb{E}, a and b corresponding to $D_{10}(U), \mathbb{E}_{10}(V), x_k$ and y_k , respectively, . \square

Lemma 3.14 (Construction 11). *Let $\{0, 1\}, U_1, U_2, \dots, U_k$ and S be disjoint sets with $|U_i| = 2$ and $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k$, $k \geq 2$ and $s = |S| \geq 0$. Let $V = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, and $v = |V| = 2 + 2k + s$. Define*

$$\begin{aligned} D_{11}(V) = & F(V) \setminus \left(\bigcup_{i=1}^k F(\{0, 1\} \cup U_i) \cup \bigcup_{\substack{i=2 \\ j < i}}^k \left\{ \{0, x_j, x_i\}, \{0, y_j, x_i\}, \{1, x_j, y_i\}, \{1, y_j, y_i\} \right\} \right. \\ & \cup \left\{ \{1, y_k, \sigma_1\}, \{x_{k-1}, y_{k-1}, \sigma_1\}, \{x_{k-1}, y_k, \sigma_1\}, \{y_{k-1}, y_k, \sigma_1\}, \{x_k, y_k, \sigma_1\}, \right. \\ & \left. \left. \{y_k, \sigma_1, \sigma_2\} \mid \sigma_1, \sigma_2 \in S \right\} \right). \end{aligned}$$

Then, the set $D_{11}(V)$ is a $d_m(v)$ of $F(V)$ of size $(v^3 - 6v^2 - 13v + 42)/6 + k(2v - 4k + 3)$.

Proof.

$$\begin{aligned}
|D_{11}(V)| &= \binom{v}{3} - 4k - 4\frac{k(k-1)}{2} - 5s - \binom{s}{2} \\
&= \frac{v^3 - 6v^2 - 13v + 42}{6} + k(2v - 4k + 3).
\end{aligned}$$

Let $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i$. Then, $D_{11}(V) \cap F(U) = D_8(U)$. Thus, by Lemma 2.4 and Construction 8, it is enough to consider blocks that contains at least one element of S .

Let $\sigma \in S$ be arbitrary. σ occurs once without y_k , twice with x_{k-1} and twice with y_{k-1} in $D_{11}^C(V)$. Hence, the blocks $\{x_{k-1}, y_k, \sigma\}$ and $\{y_{k-1}, y_k, \sigma\}$ are uniquely determined. Note that, for the remaining blocks in $D_{11}^C(V)$, the pairs $\{x_{k-1}, y_k\}$ and $\{y_{k-1}, y_k\}$ occur only once.

Now, we will consider the blocks that contain at least two elements of S . $\{1, \sigma_i, \sigma_j\}$, $\{x_{k-1}, \sigma_i, \sigma_j\}$, $\{y_{k-1}, \sigma_i, \sigma_j\}$, $\{x_k, \sigma_i, \sigma_j\}$, $\{\sigma_m, \sigma_i, \sigma_j\}$ and $\{y_k, \sigma_i, \sigma_j\}$ are only possible blocks where $\sigma_i, \sigma_j, \sigma_m \in S$, all distinct, for a possible completion. For the first two cases, if they occur then, all blocks that contain σ_i or σ_j should also contain y_k . So, both the blocks $\{\sigma_i, y_{k-1}, y_k\}$ and $\{\sigma_j, y_{k-1}, y_k\}$ will be forced. But, this is not possible. For the third, fourth and fifth cases, if they occur then, the blocks that contain the pairs $\{x_{k-1}, \sigma_i\}$ or $\{x_{k-1}, \sigma_j\}$ should also contain the point y_k which is not possible. Hence, the blocks $\{y_k, \sigma_i, \sigma_j\}$ are uniquely determined.

For the remaining blocks, y_k occurs $3 + 2(k-1) + 2s$ times, the pair $\{x_k, y_k\}$ occurs $2 + s$ times and the pair $\{1, y_k\}$ occurs $2 + 2(k-1) + s$ times. Since $\{1, x_k\}$ occurs twice in $D_{11}^C(V)$, every block containing y_k should also contain 1 or x_k , possible one exception. Hence, the blocks $\{1, y_k, \sigma\}$ and $\{x_k, y_k, \sigma\}$ are uniquely determined for all $\sigma \in S$, with possible exception of $\{1, y_k, \sigma_0\}$ or $\{x_k, y_k, \sigma_0\}$ for some $\sigma_0 \in S$. Now, it will be shown that

this exception is not possible. If the block $\{1, y_k, \sigma_0\}$ does not occur, i.e. $\{x_{k-1}, y_k, \sigma_0\}$ or $\{y_{k-1}, y_k, \sigma_0\}$ occurs, then for the remaining blocks, if one block contains y_k then it should also contain 1 and x_k occurs only once without 0. Thus, one of the two blocks that contain the pair $\{x_k, y_k\}$ should contain both 0 and 1, which is not possible. Now, if the block $\{x_k, y_k, \sigma_0\}$ does not occur, then for the remaining blocks, y_k occurs once without 1 and x_k occurs only once without 0. Thus, one of the three blocks that contain the pair $\{x_k, y_k\}$ should contain both 0 and 1, which is the desired contradiction.

Since an exception is not possible above, the blocks $\{x_{k-1}, y_{k-1}, \sigma\}$ are also uniquely determined for all $\sigma \in S$. Thus, $D_{11}(V)$ is a $d(v)$ of $F(V)$.

To show minimality of $D_{11}(V)$, by Lemma 2.4, it is enough to show for every block $X \in D_{11}(V) \setminus D_8(U)$, there exists a trade T satisfying $T \setminus D_{11}^C(V) = \{X\}$.

Let $\sigma \in S$ be arbitrary. For the block $\{0, 1, \sigma\}$, the required trade is $\left\{ \{0, 1, \sigma\}, \{x_{k-1}, y_k, \sigma\}, \{0, x_{k-1}, x_k\}, \{1, x_k, y_k\} \right\}$. For the block $\{0, p, \sigma\}$, where $p \in \{x_i, y_j | 1 \leq i, j \leq k-1\}$, the required (Pasch) trade is $\left\{ \{0, p, \sigma\}, \{x_k, y_k, \sigma\}, \{0, 1, x_k\}, \{1, p, y_k\} \right\}$. For the blocks $\{0, x_k, \sigma\}$ and $\{0, y_k, \sigma\}$, the required (Pasch) trades are $\left\{ \{0, x_k, \sigma\}, \{x_{k-1}, y_k, \sigma\}, \{0, 1, x_{k-1}\}, \{1, x_k, y_k\} \right\}$ and $\left\{ \{0, y_k, \sigma\}, \{x_{k-1}, y_{k-1}, \sigma\}, \{0, 1, x_{k-1}\}, \{1, y_{k-1}, y_k\} \right\}$, respectively. For the block $\{1, p, \sigma\}$ where $p \in \{x_i, y_j | 1 \leq i, j \leq k-1\}$, the required (Pasch) trade is $\left\{ \{1, p, \sigma\}, \{x_k, y_k, \sigma\}, \{0, p, x_k\}, \{0, 1, y_k\} \right\}$. For the block $\{1, x_k, \sigma\}$, the required (Pasch) trade is $\left\{ \{1, x_k, \sigma\}, \{x_{k-1}, y_k, \sigma\}, \{0, x_{k-1}, x_k\}, \{0, 1, y_k\} \right\}$. For the block $\{p_1, p_2, \sigma\}$ where $\{p_1, p_2\} \subset \{x_i, y_j | 1 \leq i \leq k, 1 \leq j \neq k-2\}$, the required (Pasch) trade is $\left\{ \{p_1, p_2, \sigma\}, \{y_{k-1}, y_k, \sigma\}, \{1, p_1, y_{k-1}\}, \{1, p_2, y_k\} \right\}$. For the block $\{y_{k-1}, p, \sigma\}$ where $p \in \{x_i, y_j | 1 \leq i \leq k, i \neq k-1, 1 \leq i \leq k-2\}$, the required (Pasch) trade is $\left\{ \{y_{k-1}, p, \sigma\}, \{x_{k-1}, y_k, \sigma\}, \{1, p, y_k\}, \{1, x_{k-1}, y_{k-1}\} \right\}$. For the block $\{y_k, p, \sigma\}$ where $p \in \{x_i, y_j | 1 \leq i, j \leq k-2\}$, the required trade is $\left\{ \{y_k, p, \sigma\}, \{x_{k-1}, y_{k-1}, \sigma\}, \{1, p, y_{k-1}\}, \{1, x_{k-1}, y_k\} \right\}$.

Now, let $\sigma_1, \sigma_2 \in S$ be arbitrary. For the block $\{p, \sigma_1, \sigma_2\}$ where $p \in \{0, 1\}$, the

required trades is $\left\{ \{p, \sigma_1, \sigma_2\}, \{x_{k-1}, y_k, \sigma_1\}, \{y_{k-1}, y_k, \sigma_2\}, \{1, x_{k-1}, y_{k-1}\} \right\}$. For the block $\{p, \sigma_1, \sigma_2\}$ where $p \in \{x_i, y_j \mid 1 \leq i, j \leq k-1, j \neq k-1\}$, the required trade is $\left\{ \{p, \sigma_1, \sigma_2\}, \{y_{k-1}, y_k, \sigma_1\}, \{1, y_k, \sigma_2\}, \{1, p, y_{k-1}\} \right\}$. For the blocks $\{y_{k-1}, \sigma_1, \sigma_2\}$ and $\{x_k, \sigma_1, \sigma_2\}$, the required trades are $\left\{ \{y_{k-1}, \sigma_1, \sigma_2\}, \{x_{k-1}, y_k, \sigma_1\}, \{1, y_k, \sigma_2\}, \{1, x_{k-1}, y_{k-1}\} \right\}$ and (type 2) $\left\{ \{x_k, \sigma_1, \sigma_2\}, \{x_{k-1}, y_k, \sigma_2\}, \{y_{k-1}, y_k, \sigma_2\}, \{1, x_{k-1}, y_{k-1}\}, \{1, y_k, \sigma_1\}, \{1, x_k, y_k\} \right\}$, respectively.

Lastly, let $\sigma_1, \sigma_2, \sigma_3 \in S$ be arbitrary. Then, the required trade is $\left\{ \{\sigma_1, \sigma_2, \sigma_3\}, \{y_{k-1}, y_k, \sigma_2\}, \{x_{k-1}, y_k, \sigma_3\}, \{x_{k-1}, y_{k-1}, \sigma_1\} \right\}$. \square

Lemma 3.15 (Extension of Construction 11). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, S$ and W be disjoint sets with $|U_i| = 2$, $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k, k \geq 2$ and $s = |S| \geq 0$ and $w = |W| \geq 0$. Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 2 + 2k + s$ and $V = U \cup W$, $v = |V| = u + w$. Define,*

$$E_{11}(V) = D_{11}(U) \cup \left(F(V \setminus \{x_k\}) \setminus F(U) \right).$$

Then, the set $E_{11}(V)$ is a $d_m(v)$ of size $(v^3 - 6v^2 + 11v + 30)/6 + k(2u - 4k + 3) - 4u$.

Proof. By Lemma 2.5,

$$\begin{aligned} |E_{11}(V)| &= \frac{u^3 - 6u^2 - 13u + 42}{6} + k(2u - 4k + 3) + \binom{v-1}{3} - \binom{u-1}{3} \\ &= \frac{v^3 - 6v^2 + 11v + 30}{6} + k(2u - 4k + 3) - 4u. \end{aligned}$$

To prove that $E_{11}(V)$ is a $d_m(v)$, by Lemma 2.5 with D, E and a corresponding to $D_{11}(U)$, $E_{11}(V)$ and x_k , respectively, it is only necessary to find, for an arbitrary $p \notin U$ and for every $\alpha_1, \alpha_2 \in U \setminus \{x_k\}$, a trade T satisfying,

$$T \setminus \left(D_{11}^C(U) \cup \left\{ \{p, x_k, \alpha\} \mid \alpha \in U \setminus \{x_k\} \right\} \right) = \left\{ \{p, \alpha_1, \alpha_2\} \right\}.$$

If $\{\alpha_1, \alpha_2\} \subseteq \{0, 1\} \cup \bigcup_{i=1}^k U_i$, then the corresponding trades exist by Extension of Construction 8.

For $\{\alpha_1, \alpha_2\} \in \left\{ \{0, \sigma_1\}, \{1, \sigma_1\}, \{x_i, \sigma_1\}, \{y_i, \sigma_1\}, \{x_{k-1}, \sigma_1\}, \{y_{k-1}, \sigma_1\}, \{y_k, \sigma_1\}, \{\sigma_1, \sigma_2\} \right\}$ where $\sigma_1, \sigma_2 \in S$ and $1 \leq i \leq k-2$, the required trades are $\left\{ \{0, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{0, 1, x_k\}, \{1, y_{k-1}, \sigma_1\} \right\}$, $\left\{ \{1, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{0, 1, y_{k-1}\}, \{0, x_k, \sigma_1\} \right\}$, $\left\{ \{x_i, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{1, x_i, y_{k-1}\}, \{1, x_k, \sigma_1\} \right\}$, $\left\{ \{y_i, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{1, y_i, y_{k-1}\}, \{1, x_k, \sigma_1\} \right\}$, $\left\{ \{x_{k-1}, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{1, x_{k-1}, y_{k-1}\}, \{1, x_k, \sigma_1\} \right\}$, $\left\{ \{y_{k-1}, \sigma_1, p\}, \{0, x_k, p\}, \{0, 1, y_{k-1}\}, \{1, x_k, \sigma_1\} \right\}$, $\left\{ \{y_k, \sigma_1, p\}, \{y_{k-1}, x_k, p\}, \{1, x_k, y_k\}, \{1, y_{k-1}, \sigma_1\} \right\}$ and $\left\{ \{\sigma_1, \sigma_2, p\}, \{y_{k-1}, x_k, p\}, \{1, x_k, \sigma_1\}, \{1, y_{k-1}, \sigma_2\} \right\}$, respectively. \square

Lemma 3.16 (Double Extension of Construction 11). *Let $\{0, 1\}, U_1, U_2, \dots, U_k, S, W$ and Q be disjoint sets with $|U_i| = 2$, $U_i = \{x_i, y_i\}$ where $1 \leq i \leq k, k \geq 2$, $s = |S| \geq 0$ and $w = |W| \geq 2$ and $q = |Q| \geq 1$. Let c be a distinguished element of Q . Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 2 + 2k + s$ and $V = U \cup W \cup Q$ and $v = |V| = u + w + q$. Define*

$$\begin{aligned} \mathbb{E}_{11}^C(V) &= D_{11}^C(U) \cup \left\{ \{x_k, \beta_1, \alpha\}, \{x_k, \beta_1, \beta_2\}, \{x_k, \beta_1, c\} \mid \alpha \in U \setminus \{x_k\}, \beta_1, \beta_2 \in W \right\} \\ &\cup \left\{ \{y_{k-1}, \gamma_1, \alpha\}, \{y_{k-1}, \gamma_1, \gamma_2\}, \{y_{k-1}, \gamma_1, \beta\} \mid \alpha \in U \setminus \{y_{k-1}\}, \beta \in W, \gamma_1, \gamma_2 \in Q \right\}. \end{aligned}$$

Then, the set $\mathbb{E}_{11}(V) = F(V) \setminus \mathbb{E}_{11}^C(V)$ is a $d_m(v)$ of size $(v^3 - 6v^2 + 11v + 30)/6 + k(2u - 4k + 3) - 4u - w$.

Proof. By Lemma 2.7,

$$\begin{aligned} |\mathbb{E}_{11}(V)| &= \frac{u^3 - 6u^2 - 13u + 42}{6} + k(2u - 4k + 3) + \binom{v-1}{3} - \binom{u-1}{3} - w \\ &= \frac{v^3 - 6v^2 + 11v + 30}{6} + k(2u - 4k + 3) - 4u - w. \end{aligned}$$

The trades for the pivot point x_k are given in Extension of Construction 11 and the trades

for the pivot point y_{k-1} , are the trades which are given in Extension of Construction 11 and the following one: $\left\{ \{x_k, \sigma, p\}, \{0, y_{k-1}, p\}, \{0, x_k, y_k\}, \{y_{k-1}, y_k, \sigma\} \right\}$. Therefore, $\mathbb{E}_{11}(V)$ is also a $d_m(v)$ by Lemma 2.7 with D, \mathbb{E}, a and b corresponding to $D_{11}(U), \mathbb{E}_{11}(V), x_k$ and y_{k-1} , respectively. \square

In this chapter several new constructions are presented. Now, we will briefly describe that they are non-isomorphic with the previous constructions. The minimal defining sets constructed by Construction 7 give new sizes in the spectrum, for some appropriate values of k . Moreover, Construction 8 and its enlargements Construction 9, 10 and 11 define minimal defining sets which do not contain the full design on 5 points in their complements and hence they are non-isomorphic with Construction 4 and its enlargements Construction 5, 6 and 7.

4. BOUNDS AND SPECTRUM ANALYSIS

4.1. BOUNDS ON THE SIZE OF $d_m(v)$

In this section, lower and upper bounds on the size of $d_m(v)$ of full designs on v points will be given. The lower bound stated in [13] will be improved for $v \geq 8$ by finding the size of $d_s(v)$ for $v = 8$ and $v = 9$. Also, all $d_s(v)$ s for $v = 8$ will be classified by integer programming.

Firstly, we will give an upper bound on the size of $d_m(v)$ by using the following lemma [13].

Lemma 4.1. *Let $v = |V| \geq 4$ and D be a $d_m(v)$ of $F(V)$. Then, every element in V occurs in at least three blocks of D^C .*

Proof. To get a contradiction, assume for some $\alpha \in V$ there are at most two blocks in D^C that contain α , without loss of generality, say $\{\alpha, \beta_1, \gamma\}$ and $\{\alpha, \beta_2, \sigma\}$ where $\alpha, \beta_1, \beta_2, \gamma$ and σ are all distinct, only possibly $\beta_1 = \beta_2$.

Now, the block $\{\alpha, \gamma, \sigma\}$ is in D and there exists a trade T_1 with the trade mate T_2 such that $T_1 \setminus D^C = \{\alpha, \gamma, \sigma\}$. Since T_1 and T_2 are disjoint, there is a block containing the pair $\{\alpha, \gamma\}$ and a block containing the pair $\{\alpha, \sigma\}$ in T_2 . So, α occurs at least twice in T_2 , hence in T_1 . If α occurs exactly two times in T_1 , say $\{\alpha, \gamma, \sigma\}$ and $\{\alpha, \beta_1, \gamma\}$, then there are at least 3 blocks that contains α in T_2 , which is not possible. If α occurs exactly three times in T_1 , say $\{\alpha, \gamma, \sigma\}$, $\{\alpha, \beta_1, \gamma\}$ and $\{\alpha, \beta_2, \sigma\}$, then there should be 4 blocks that contains α in T_2 which gives the desired contradiction. \square

Corollary 4.2. *If D is a $d_m(v)$, then $|D| \leq \binom{v}{3} - v$, for $v \geq 4$.*

Proof. By Lemma 4.1, for every points in V , there are at least three blocks in D^C . Since each block contains three points, there are at least v blocks in D^C . Hence, the result

follows. □

Now, a lower bound on the size of $d(v)$ will be given. The following lemma is proven in a general context in [13], but here, it will be stated only for full designs.

Lemma 4.3. *Let $u = |U|$ and $v = |V|$ with $u \leq v$. Let D_u and D_v be smallest defining sets of $F(U)$ and $F(V)$, respectively. Then, $|D_v| \geq \frac{|D_u|}{\binom{u}{3}} \binom{v}{3}$.*

Proof. There are $\frac{v!}{u!}$ many different isomorphic copies of $F(U)$ contained in $F(V)$ and every block of $F(V)$ is contained in exactly $n = \frac{v! \binom{u}{3}}{u! \binom{v}{3}}$ isomorphic copies of $F(U)$. The intersections of D_v with all isomorphic copies of $F(U)$ is a defining set of the corresponding copy of $F(U)$, and hence should contain at least $|D_u|$ blocks. In this counting argument, each block of D_v is counted n times. Therefore, $|D_v| \geq \frac{v!}{u!} |D_u| \frac{1}{n}$. □

Note that, $|d_s(7)| = 15$ and hence, $|d_s(v)| \geq \frac{15}{35} \binom{v}{3}$ for $v \geq 7$ by the previous lemma. With finding the size of $d_s(v)$ for some $v > 7$, this lower bound could be improved.

Remember that, Khodkar [8] introduced an algorithm that uses integer programming to minimize the number of blocks in a t -design that intersects each occurrence of every trade in a list of trades contained in the design, to find a smallest defining set of t -designs. Now, this algorithm will be discussed in detail for simple t -designs and will be extended for finding all minimal defining sets.

Consider a simple t -design with b blocks and all trades contained in it are given. First, enumerate the blocks and the trades, then construct a matrix A with entries $a_{ij} = 1$ if the i th trade contains the j th block and $a_{ij} = 0$ otherwise. Then, the corresponding integer programming problem can be constructed as follows: Minimize $\sum_{i=1}^b x_i$ subject to $Ax \geq y$ where x is a binary vector and y is a vector with all entries equal to 1. Hence, a solution vector x of this integer program represents a set of blocks which intersects all

trades that are represented as rows of the matrix A . Thus, this vector represents a defining set of the corresponding t -design, indeed, a smallest defining set since the magnitude of x is minimized.

The main problem of this algorithm is actually finding all trades contained in the design. In addition, due to the large number of such trades, the corresponding integer program may not give a solution in a reasonable time, even if all trades contained in the design can be classified. For example, there are 3 non-isomorphic 2-(6, 3) trades, 8 non-isomorphic 2-(7, 3) trades and 15011 non-isomorphic 2-(8, 3) trades which are classified in [18] and [19]. To find the size of the smallest defining set of the full design on 8 points, the corresponding integer program should contain constraints for these trades and all of their isomorphic copies. The size of this integer program is too large to be solved in a reasonable time. However, the integer programming problem could be constructed by considering only some families of trades to find a lower bound on the size of defining sets.

Note that, there are $\binom{8}{3} = 56$ blocks and $8!/48 = 840$ distinct Pasch trades contained in the full design on 8 points. The corresponding integer program which contains constraints for only Pasch trades gives a solution vector with a magnitude 29. Also, it is known that there are defining sets of size 29 of the full design on 8 points by Double Extension of Construction 2 with $w = v - 6$ and Construction 3. Therefore, $|d_s(8)| = 29$ which is stated in the following lemma.

Lemma 4.4. *If D is a smallest defining set of the full design on 8 points, then $|D| = 29$.*

With a similar method of finding the value of $|d_s(8)|$, also $|d_s(9)|$ can be found. Note that, there are $\binom{9}{3} = 84$ blocks and $9!/144 = 2520$ distinct Pasch trades contained in the full design on 9 points. The corresponding integer program which contains constraints for only these Pasch trades gives an answer that at least 45 blocks are needed to intersect with all these Pasch trades. However, there is no known defining set of size 45. Of course, this

does not mean that $|d_s(9)| > 45$. An other integer program is constructed by considering not only Pasch trades but also some additional families of trades and we can check that at least 48 blocks are needed to intersect with all these trades. Also, a defining set of size 48 is found which will be given in the proof of the next lemma and hence $|d_s(9)| = 48$.

Lemma 4.5. *If D is a smallest defining set of the full design on 9 points, then $|D| = 48$.*

Proof. By the integer programming that is described above, $|D| \geq 48$ holds. Also, by exhaustive computer search $D^* = \{\{5, 7, 8\}, \{5, 6, 8\}, \{5, 6, 7\}, \{4, 7, 8\}, \{4, 6, 8\}, \{4, 6, 7\}, \{4, 5, 8\}, \{4, 5, 7\}, \{4, 5, 6\}, \{3, 6, 7\}, \{3, 5, 7\}, \{3, 5, 6\}, \{3, 4, 7\}, \{3, 4, 6\}, \{2, 6, 7\}, \{2, 5, 7\}, \{2, 5, 6\}, \{2, 4, 7\}, \{2, 4, 6\}, \{2, 3, 7\}, \{2, 3, 6\}, \{1, 4, 8\}, \{1, 4, 7\}, \{1, 4, 6\}, \{1, 3, 8\}, \{1, 3, 7\}, \{1, 3, 6\}, \{1, 3, 4\}, \{1, 2, 8\}, \{1, 2, 7\}, \{1, 2, 6\}, \{1, 2, 4\}, \{1, 2, 3\}, \{0, 5, 8\}, \{0, 5, 7\}, \{0, 5, 6\}, \{0, 3, 8\}, \{0, 3, 7\}, \{0, 3, 6\}, \{0, 3, 5\}, \{0, 2, 8\}, \{0, 2, 7\}, \{0, 2, 6\}, \{0, 2, 5\}, \{0, 2, 3\}, \{0, 1, 8\}, \{0, 1, 3\}, \{0, 1, 2\}\}$ is a $d(9)$ of $F(V)$ where $V = \{0, 1, \dots, 8\}$ and $|D^*| = 48$. Hence, $|d_s(9)| = 48$. \square

Corollary 4.6. *If D is a $d(v)$, then $|D| \geq \frac{48}{84} \binom{v}{3}$, for $v \geq 9$.*

Proof. The result follows directly by Lemma 4.3 and Lemma 4.5. \square

We used ILOG CPLEX 11 [20] for our integer programming problems. Now, with *populating the solution pool* in ILOG CPLEX 11, all optimal solutions of an integer program can be generated. And, this option is used to find all solutions of the above two integer programs which found $|d_s(8)|$ and $|d_s(9)|$. Note that, each solution does not necessarily represent a defining set. However, applying an isomorphism test to these solutions and then checking whether the corresponding partial designs are defining sets or not by a completion routine, all smallest defining sets of the full designs on 8 and 9 points are found. There are 4 non-isomorphic smallest defining sets of the full design on 8 points and 1 non-isomorphic smallest defining set of the full design on 9 points which are given in Appendix A.

Now, a similar argument can be used to extend Khodkar's algorithm for finding all minimal defining sets of a given simple t -design (actually for all t -designs but the details

for the repeated blocks will not be given here) as follows. First we find all solution vectors with magnitude s via an integer program that contains constraints for all trades contained in the design as described above, i.e. minimize $\sum_{i=1}^b x_i$ subject to $Ax \geq y$ where x is a binary vector and y is a vector with all entries are equal to 1. Then, we construct a new matrix, say B , where solution vectors of the previous integer program are the rows of B . Then, we construct the following integer programming problem: minimize $\sum_{i=1}^b x_i$ subject to $Ax \geq y$ and $Bx < z$ where x is a binary vector, y is a vector with all entries equal to 1 and z is a vector with all entries equal to s . The solutions of this new integer program, if it exists, represent a defining set since they intersect with all trades. Moreover, they represent minimal defining sets since any proper subset of them can not be a defining set which was guaranteed by the constraint $Bx < z$. By repeating this argument, until the corresponding integer program has no solution, i.e. all solution vectors that represent a minimal defining set are found, all minimal defining sets can be found. Note that, this algorithm may not give all minimal defining sets in a reasonable time, if there is a large number of trades contained in the design or there is a large number of minimal defining sets. But if this is not the case, all minimal defining sets can be found even if the number of blocks is considerably large. Remember that, the algorithms in [10] and [11] need 2^b bits RAM memory, where b is the number of blocks of the design and for $b = 56$, this means that 8000000 Gigabytes RAM memory is needed to allocate.

4.2. SPECTRUM ANALYSIS OF $d_m(v)$

In the previous section lower and upper bounds on $d_m(v)$ are given. Now, the known largest and smallest values of the spectrum of the full designs on v points will be stated explicitly and these values will be compared with the bounds.

Construction 5 defines a minimal defining set that has size $(v^3 - 3v^2 - 16v + 30)/6$ for $k = 1$. Also, Construction 6 and 7 construct defining sets with the same size for $k = 1$, however these are all isomorphic to each other. This defining set has the largest known

size and the difference between its size and the upper bound is equal to $2v - 5$.

The smallest size of a minimal defining set that is known is equal to $(v^3 - 6v^2 + 5v + 6)/6$ which can be obtained by Double Extension of Construction 2 with $w = v - 6$, by Construction 3 and by Double Extension of Construction 5 with $k = 1$, $w' = v - 5$ and $w = v - 4$. Actually, the defining sets that are constructed by Double Extension of Construction 2 and 5 for this size are identical and the other defining sets are all non-isomorphic. The difference between this size and the lower bound is $(v^3 - 10v^2 + 9v + 14)/14$ for $v \geq 9$ and the difference between the known largest and smallest value in the spectrum is $(3v^2 - 21v + 24)/6$.

In general, for a fixed v , considering different values of the sizes of the sets U and W in Lemma 2.5, and different values of the sizes of the sets U , W and Q in Lemma 2.7, minimal defining sets with different sizes can be obtained. Also, these enlargements of some families of defining sets give consecutive sizes in the spectrum. Extension of Construction 5 and Double Extension of Construction 5 give approximately $9v^2/50$ consecutive sizes for $v \geq 12$ which is stated in the next theorem .

Theorem 4.7. [14] *Let $v \geq 12$ and $r \equiv -2v \pmod{5}$ with $0 \leq r < 5$. Then, there exists a $d_m(v)$ of size n provided that*

$$\frac{v^3 - 6v^2 + 5v + 6}{6} \leq n \leq \frac{25v^3 - 123v^2 + v(50 - 18r) + (3r^2 + 75r - 300)}{150}$$

For a fixed v , Construction 5, 6 and 7 which are enlargements of Construction 4 construct different minimal defining sets for all $1 \leq k \leq \lfloor \frac{v-2}{3} \rfloor$. Also, Construction 9, 10 and 11 construct distinct minimal defining sets for all $2 \leq k \leq \lfloor \frac{v-2}{2} \rfloor$. Hence, the number of minimal defining sets constructed by these new constructions is larger than the number of the minimal defining sets obtained by the constructions stated in Chapter 2. However, the sizes of these defining sets given in Chapter 3 are mostly overlapped with each other and lie between the bounds given in Theorem 4.2.

5. CONCLUSION

In this thesis, several constructions that define minimal defining sets of the full designs are given. The intersections of these minimal defining sets with simple t -designs are defining sets of the corresponding t -designs but not necessarily minimal by Lemma 1.5. Moreover, this method can be applied to all isomorphic copies of the constructed minimal defining sets of the full designs. Now, we will compare these intersections with the minimal defining set of some simple t -designs. The minimal defining sets of these t -designs are classified previously and can be found in [21].

The constructions stated in Chapter 2 and 3, construct 4 non-isomorphic minimal defining sets of the full design on 6 points. The 2 -(6, 3, 2) has only one minimal defining set up to isomorphism. This defining set can be found by intersecting the 2 -(6, 3, 2) design with an isomorphic copy of any 4 non-isomorphic minimal defining sets of the full design on 6 points.

The constructions given in this thesis, define 11 non-isomorphic minimal defining sets of the full design on 7 points. We intersected all isomorphic copies of these 11 non-isomorphic minimal defining sets with the 2 -(7, 3, 1) design, the simple 2 -(7, 3, 2) design, the simple 2 -(7, 3, 3) design and the simple 2 -(7, 3, 4) design. The 2 -(7, 3, 1) design has only one minimal defining set up to isomorphism and this defining set can be found by these intersections. The simple 2 -(7, 3, 2) design has 17 non-isomorphic minimal defining sets and 16 of them can be found by the intersection method described above and the other minimal defining set is not identical but contained in some of these intersections. The simple 2 -(7, 3, 3) design has 20 non-isomorphic minimal defining sets and all of them are contained in an intersection of the design with some isomorphic copy of a constructed minimal defining sets of the full design and 7 of them can be obtained with the intersection method. The simple 2 -(7, 3, 4) design has 163 non-isomorphic minimal defining sets and 69

of them are contained in some of these intersections and 24 of them are identical to some of these intersections that are constructed by this method.

The constructions given in this thesis, define 37 non-isomorphic minimal defining sets of the full design on 9 points. We used intersection method with all isomorphic copies of these 37 non-isomorphic minimal defining sets for the 2-(9, 3, 1) design, #24 simple 2-(9, 3, 2) design and #36 simple 2-(9, 3, 2) design. The numbering we use is identical to CRC Handbook of Combinatorial Designs [1]. The 2-(9, 3, 1) design has only two non-isomorphic minimal defining sets. The one with size 5 is identical to some of these intersections, however the one with size 4 is only contained by some of these intersections. #24 simple 2-(9, 3, 2) design has 216 non-isomorphic minimal defining sets and 24 of them can be found by the intersection method and the other minimal defining sets are not identical but are contained in some of these intersections. Lastly, #36 simple 2-(9, 3, 2) design has 1148 non-isomorphic minimal defining sets and 19 of them are identical to some of these intersections and all of them are contained in some of these intersections.

Above results show that the minimal defining sets of full designs give valuable information about the minimal defining sets of simple t -designs. However, more information could be obtained by constructing new minimal defining sets especially which have blocks containing the points more regularly and which fill the gaps in the spectrum. Also, this idea can be generalized for block size greater than 3. Actually, Street and Gray stated a construction for full designs with arbitrary block size in [16]. In addition, minimal defining sets of multiple copies of full designs can be constructed. Hence, this intersection method can be used for every t -designs not necessarily for simple t -designs.

In Chapter 4, an upper and a lower bound on the size of the minimal defining sets of full designs are given. The upper bound is considerably good however, the lower bound should be improved. For $v = 6, v = 7$ and $v = 8$ intersecting minimally with only all Pasch trades is enough for being a defining set, however Lemma 4.5 shows that for $v \geq 9$ finding

the size of smallest defining sets is getting more complicated. This also shows that trade structures are getting more complicated for $v \geq 9$. Also, with the intersection method, an upper bound for the size of smallest defining sets and a lower bound for the size of largest defining sets of simple t -designs can be obtained.

Lastly, an algorithm is suggested for finding all minimal defining sets of t -designs by using integer programming in Chapter 4. This algorithm may give the solution in a reasonable time for t -designs which do not contain a large number of trades and do not have a large number of minimal defining sets, even if the number of blocks in the design is considerably large. Also, the constraints in the integer program can be further reduced by theoretical results about substructures.

APPENDIX A: ALL NON-ISOMORPHIC $d_s(8)$ s and $d_s(9)$

There are 4 non-isomorphic $d_s(8)$ s of $F(V)$ where $V = \{0, 1, \dots, 7\}$ and they are given below as $S_{8_1}, S_{8_2}, S_{8_3}$ and S_{8_4} .

$$S_{8_1} = \{\{5, 6, 7\}, \{3, 6, 7\}, \{3, 5, 7\}, \{3, 4, 7\}, \{3, 4, 5\}, \{2, 6, 7\}, \{2, 5, 7\}, \{2, 4, 7\}, \{2, 4, 5\}, \\ \{2, 3, 7\}, \{2, 3, 4\}, \{1, 6, 7\}, \{1, 5, 6\}, \{1, 4, 6\}, \{1, 4, 5\}, \{1, 3, 4\}, \{1, 2, 4\}, \{0, 6, 7\}, \\ \{0, 3, 7\}, \{0, 3, 6\}, \{0, 3, 4\}, \{0, 2, 7\}, \{0, 2, 6\}, \{0, 2, 4\}, \{0, 2, 3\}, \{0, 1, 6\}, \{0, 1, 4\}, \\ \{0, 1, 3\}, \{0, 1, 2\}\}$$

$$S_{8_2} = \{\{4, 5, 7\}, \{3, 6, 7\}, \{3, 5, 7\}, \{3, 4, 6\}, \{3, 4, 5\}, \{2, 6, 7\}, \{2, 5, 7\}, \{2, 4, 6\}, \{2, 4, 5\}, \\ \{2, 3, 6\}, \{1, 5, 7\}, \{1, 4, 5\}, \{1, 3, 6\}, \{1, 3, 5\}, \{1, 2, 6\}, \{1, 2, 5\}, \{0, 6, 7\}, \{0, 4, 6\}, \\ \{0, 4, 5\}, \{0, 3, 6\}, \{0, 3, 4\}, \{0, 2, 6\}, \{0, 2, 4\}, \{0, 2, 3\}, \{0, 1, 6\}, \{0, 1, 5\}, \{0, 1, 4\}, \\ \{0, 1, 3\}, \{0, 1, 2\}\}$$

$$S_{8_3} = \{\{4, 5, 7\}, \{3, 5, 7\}, \{3, 5, 6\}, \{3, 4, 6\}, \{3, 4, 5\}, \{2, 5, 7\}, \{2, 3, 6\}, \{2, 3, 5\}, \{1, 6, 7\}, \\ \{1, 5, 7\}, \{1, 4, 7\}, \{1, 4, 6\}, \{1, 3, 6\}, \{1, 2, 7\}, \{1, 2, 6\}, \{0, 6, 7\}, \{0, 4, 7\}, \{0, 4, 6\}, \\ \{0, 3, 6\}, \{0, 3, 4\}, \{0, 2, 7\}, \{0, 2, 6\}, \{0, 2, 4\}, \{0, 2, 3\}, \{0, 1, 7\}, \{0, 1, 6\}, \{0, 1, 4\}, \\ \{0, 1, 3\}, \{0, 1, 2\}\}$$

$$S_{8_4} = \{\{4, 5, 6\}, \{3, 5, 6\}, \{3, 4, 6\}, \{3, 4, 5\}, \{2, 4, 6\}, \{2, 4, 5\}, \{2, 3, 6\}, \{2, 3, 5\}, \{2, 3, 4\}, \\ \{1, 4, 6\}, \{1, 4, 5\}, \{1, 3, 6\}, \{1, 3, 5\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}, \{0, 6, 7\}, \{0, 4, 6\}, \\ \{0, 3, 6\}, \{0, 3, 4\}, \{0, 2, 7\}, \{0, 2, 6\}, \{0, 2, 4\}, \{0, 2, 3\}, \{0, 1, 7\}, \{0, 1, 6\}, \{0, 1, 4\}, \\ \{0, 1, 3\}, \{0, 1, 2\}\}$$

There is one $d_s(9)$ of $F(V)$ where $V = \{0, 1, \dots, 8\}$ up to isomorphism and it is given below as S_9 .

$$\begin{aligned}
 S_9 = & \{\{5, 7, 8\}, \{5, 6, 8\}, \{5, 6, 7\}, \{4, 7, 8\}, \{4, 6, 8\}, \{4, 6, 7\}, \{4, 5, 8\}, \{4, 5, 7\}, \{4, 5, 6\}, \\
 & \{3, 6, 7\}, \{3, 5, 7\}, \{3, 5, 6\}, \{3, 4, 7\}, \{3, 4, 6\}, \{2, 6, 7\}, \{2, 5, 7\}, \{2, 5, 6\}, \{2, 4, 7\}, \\
 & \{2, 4, 6\}, \{2, 3, 7\}, \{2, 3, 6\}, \{1, 4, 8\}, \{1, 4, 7\}, \{1, 4, 6\}, \{1, 3, 8\}, \{1, 3, 7\}, \{1, 3, 6\}, \\
 & \{1, 3, 4\}, \{1, 2, 8\}, \{1, 2, 7\}, \{1, 2, 6\}, \{1, 2, 4\}, \{1, 2, 3\}, \{0, 5, 8\}, \{0, 5, 7\}, \{0, 5, 6\}, \\
 & \{0, 3, 8\}, \{0, 3, 7\}, \{0, 3, 6\}, \{0, 3, 5\}, \{0, 2, 8\}, \{0, 2, 7\}, \{0, 2, 6\}, \{0, 2, 5\}, \{0, 2, 3\}, \\
 & \{0, 1, 8\}, \{0, 1, 3\}, \{0, 1, 2\}\}
 \end{aligned}$$

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