TRANSVERSE VIBRATION OF BEAMS
APPLICATION OF A MODIFIED THEORY

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ABSTRACT

In the analysis of transverse vibration of beams the classical Euler-Bernoulli theory is insufficient to describe the behaviour of the beam in the higher modes. Including the effect of rotary inertia and shear deflection of the beam results in the so called Timoshenko theory which extends the applicability of the theory to higher modes.

This study is based on a work by Levinson [9]. The theory developed takes warping of the cross-section into account. In this theory the arbitrariness in the shear coefficient appearing in the equations of motion is removed. The shear coefficient is shown to be equal to 5/6.

Based on this theory, a theoretical analysis of the vibration of beams with simple and homogeneous boundary conditions is presented. The method of separation of variables is used to obtain the solutions to the Euler-Bernoulli beam theory, Timoshenko beam theory and the modified theory. The eigenvalue problem is formulated for each theory and both the eigenvalues (natural frequencies) and the eigenfunctions (normal modes) are determined for the clamped-free, clamped-clamped, hinged-hinged, free-free, and clamped-hinged boundary conditions. The results of these theories are compared.

KİRİŞLERDE ENİNE TİTREŞİMLER GELİŞTİRİLMİŞ TEORİNİN UYGULANMASI

ÖZET

Kirişlerin titreşiminin analizinde klasik Euler-Bernoulli teorisi, kirişlerin yüksek frekanslardaki davranışlarını ifade etmekte yeterli değildir. Kirişin dönme ataletinin ve kayma eğilmesinin etkisi dahil edilince yüksek frekanslardaki uygulanabilirliği sonucunu ortaya çıkarmıştır ki, bu teoriye Timoshenko teorisi denir.

Bu çalışma Levinson'un [9] üzerinde durduğu bir teoriye dayanır. Bu teori kesitte meydana gelen çarpıklığın etkisini işleme alarak geliştirilmiştir. Bu teoride hareket denkleminde ortaya çıkan kayma katsayısı bırakılmıştır ve bu değer 5/6'ya eşit olarak ortaya çıkar.

Bu teoriye dayanarak, basit ve homojen sınır şartlı kirişlerin titreşimlerinin teorik analizi sunulmaktadır. Değişkenlere ayırma metodu kullanılarak Euler-Bernoulli, Timoshenko ve geliştirilmiş teoriler için çözümler elde edilmiştir. Her teoriye göre tabii frekans problemi formüle edilip, ankastre-serbest, ankastre-ankastre, menteşe-menteşe, serbest-serbest, ankastre-menteşe sınır şartlı kirişlerin tabii frekansları ve normal şekilleri belirtilip, bu teorilerden elde edilen sonuçlar karşılaştırıldı.

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LIST OF SYMBOLS

$w(x,t)$	Transverse deflection
$\psi(x,t)$	Angle of rotation due to bending
$\gamma(x,t)$	Angle of distortion due to shear
A	Cross-section area of beam
E	Modulus of elasticity
G	Modulus of rigidity
I	Moment of inertia of beam
ρ	Mass density
t	Time
x	Index of position along beam axis
L	Length of beam
u	Displacement in the longitudinal direction
z	Vertical axis of cartesian coordinates
M	Bending moment
Q	Shearing force
κ	Shear correction factor
$\phi(x,t)$	Warping function
ϵ_{xz}	Shear strain
ϵ_x	Normal strain
h	Depth of beam
σ	Normal stress
τ	Shear stress
ν	Poisson's ratio
ξ	Non-dimensional length of beam
ω	Angular frequency
$W(\xi)$	Non-dimensional transverse deflection
$\Psi(\xi)$	Non-dimensional angle of rotation due to bending
b	Non-dimensional frequency
r =	$\sqrt{I/AL^2}$
s =	$\sqrt{EI/\kappa AGL^2}$
C_1, C_2, C_3, C_4	Constants
α, β	Constants defined by equations III-28 and III-68
λ =	$\dfrac{\alpha}{\beta}$

ASSUMPTIONS

The following assumptions are considered in deriving the equations of motion for the modified theory:

1. Beam is prismatical and straight.

2. The material is homogenous, isotropic and Hookean.

3. Initially, there is no deflection of the beam.

4. The cross-section will warp into a non-planar surface.

5. On the lateral surfaces of the beam the cross-section remains normal to the shear-free surfaces.

6. Poisson ratio effects and stress components other than the longitudinal normal stress and transverse shear stress are neglected.

7. The thickness of the beam is constant all throughout.

I. INTRODUCTION

In the vibration analysis of a large group of engineering structures their components are idealized as beams. Accurate prediction of the natural frequencies and mode shapes of these components is considerably important. Several theoretical approaches have been developed in accordance with the accuracy needed.

It is a well known fact that the classical Euler-Bernoulli [1] theory of flexural vibrations leads to erroneous results at high frequencies. Consideration of the effects of shear and rotary inertia is necessary for the accurate prediction of the flexural frequencies.

First, improvement on the theory is due to Rayleigh [2] where he introduced the effect of rotary inertia. A further improvement was made by Timoshenko [3,4] who presented two papers in 1921 and 1922 concerning the effects of shear deformation on the bending vibration of beams. Thus, the range of applicability of the theory of beams was extended by taking into account the effects of transverse shear deformation and rotary inertia. The equations including these effects are referred to as Timoshenko beam equations. In these equations the effective transverse shear strain is

$$\epsilon = \frac{\tau_{av}}{\kappa G}$$

where τ_{av} is the average shear stress, G is the shear modulus and κ is the shear correction coefficient. The shear coefficient depends on the shape of the cross-section and it is generally defined as the ratio of the average shear strain on a section to the shear strain at the centroid. The numerical value of the shear coefficients based on this definition are given in reference [5].

The definition of shear coefficient given above is not entirely satisfactory. Goodman [6] and Sutherland [7] have shown that the customary values of shear coefficient lead to unsatisfactory results when Timoshenko beam equations are used to calculate high-frequency spectrum of vibrating beams. They have proposed that shear coefficient should be adjusted arbitrarily so that better results can be obtained. According to them the unsatisfactory results arise from using static strain distributions as a basis for calculating shear coefficient instead of the strain distributions which occur in high-frequency motion.

Levinson and Stephen [8] have presented a formulation of a complete second order beam theory which takes into account shear deformation, transverse direct stress and rotary inertia. The governing differential equation is similar in form to the Timoshenko beam equation, but consists of two coefficients. One of which depends on cross-sectional warping and the second includes terms depending on the transverse direct stresses.

Levinson [9] has presented a modified theory which includes the effect of warping of the cross-section. The theory allows the cross-section both to rotate relative to the neutral surface and to warp into a non-planar surface. The usual assumptions of "cross-section of the beam normal to the neutral surface remain normal during motion" is abandoned. Besides that the cross-section is allowed to warp in such a fashion that it remains normal to the shear-free surfaces.

In this thesis Levinson's theory has been used to obtain the naturual frequencies of a beam with simple boundary conditions. Frequency spectrums have been calculated for different boundary conditions and compared with the frequencies obtained from Euler-Bernoulli theory and Timoshenko beam theory. The frequency spectrums are presented graphically. In addition, normal modes have been obtained for the boundary conditions such as clamped-free, clamped-clamped, hinged-hinged, free-free, clamped-hinged. These are also presented in section IV.

II. EQUATIONS OF MOTION

In this section we will present the equations of motion governing the transverse vibration of beams for the three theories namely Euler-Bernoulli, Timoshenko and Levinson theories.

A. EULER-BERNOULLI THEORY

The analytical determination of the transient flexural vibrations of beams generally employs the elementary or Euler-Bernoulli equation. This equation considers only the lateral inertia and the elastic forces caused by bending deflections. The equation of motion for free vibration based on Euler-Bernoulli theory can easily be found in the literature [10,11] that is

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{II-1}$$

where, w(x,t) is the transverse deflection of the neutral axis, EI the flexural rigidity, ρ the mass density, and A the cross-sectional area of the beam.

1. Boundary Conditions

The necessary and sufficient boundary conditions for the beams are :

Clamped end: At the clamped end of a beam the deflection and the slope are zero, i.e.,

$$\begin{aligned} w(x,t)\,|_{x=0} &= w(x,t)\,|_{x=l} = 0 \\ w'(x,t)\,|_{x=0} &= w'(x,t)\,|_{x=l} = 0 \end{aligned} \qquad \text{II-2}$$

Hinged end: At the hinged end the deflection and the moment vanish, i.e.,

$$\begin{aligned} w(x,t)\,|_{x=0} &= w(x,t)\,|_{x=l} = 0 \\ w''(x,t)\,|_{x=0} &= w''(x,t)\,|_{x=l} = 0 \end{aligned} \qquad \text{II-3}$$

Free end: In the case of free ended beam, the moment and the shear forces vanish at the boundary, i.e.,

$$w''(x,t)\,|_{x=0} = w''(x,t)\,|_{x=l} = 0$$

II.4

$$w'''(x,t)\,|_{x=0} = w'''(x,t)\,|_{x=0} = 0$$

B. TIMOSHENKO BEAM THEORY

It is well known that the Euler-Bernoulli theory of flexural vibrations of beams is adequate for relatively long, slender beams at lower modes of vibration. For beams in which higher modes are required, the Timoshenko theory [3] which considers the effect of shear and rotary inertia gives a better approximation to the true behavior of a beam.

The free body diagram of an element dx of a beam with internal reactions is shown in figure 2.1. The total deflection w(x,t) consists of two parts, one caused by bending and one by shear, so the slope of the deflection curve can be written

$$\frac{\partial w(x,t)}{\partial x} = \gamma(x,t) - \psi(x,t) \qquad \text{II-5}$$

where $\psi(x,t)$ is the angle of rotation due to bending $\gamma(x,t)$ is the angle of distortion due to shear.

The displacement of a material point in the longitudinal direction is of the form

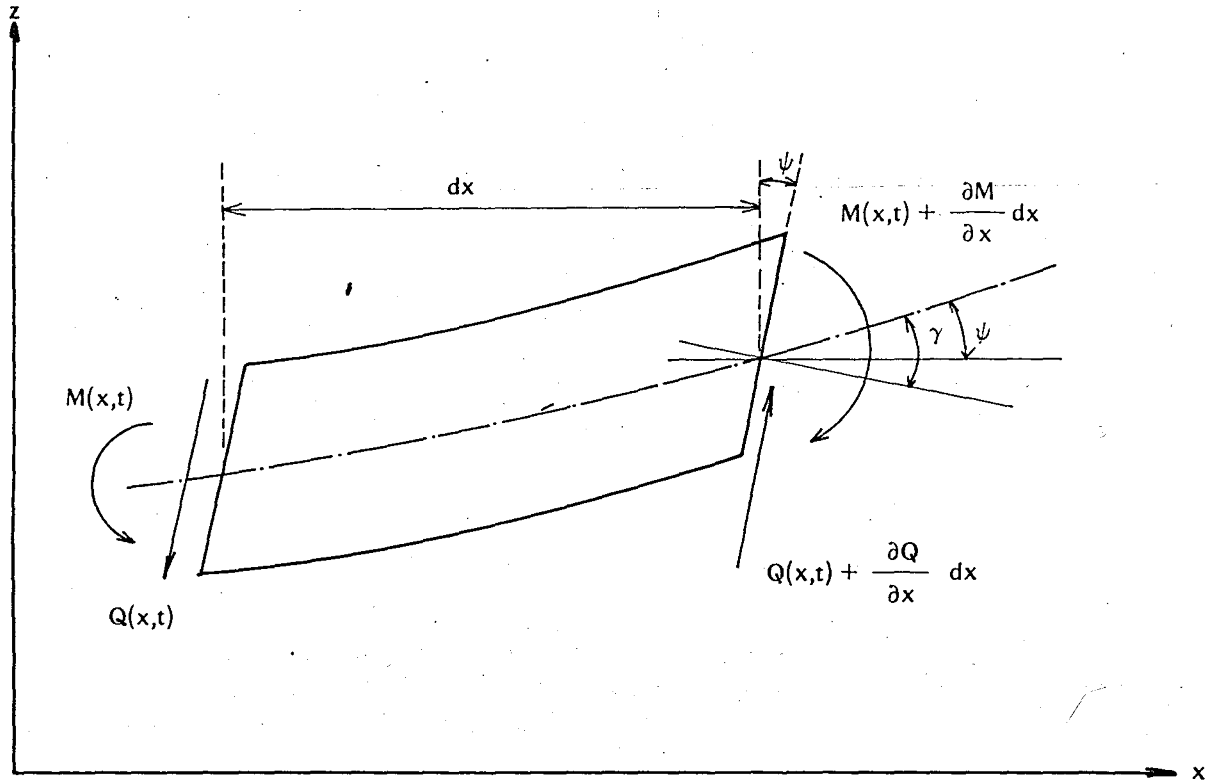
$$u(x,z,t) = z\,\Psi(x,t) \qquad \text{II-6}$$

Integrating the normal stress and shear stress over the area of the cross-section, the bending moment and shear force can be found respectively

$$M = \int_A E\,\frac{\partial u(z,x,t)}{\partial x}\,z\,dA = EI\,\frac{\partial \Psi(x,t)}{\partial x} \qquad \text{II-7}$$

$$Q = \int_A G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) dA = \kappa\left(\frac{\partial w(x,t)}{\partial x} + \psi(x,t)\right) AG \qquad \text{II-8}$$

where A is the cross-sectional area, G the shear modulus, κ the shear correction factor depending on the shape of the cross-section, and EI the bending stiffness. To determine the equation of motion, dynamic equilibrium condition for forces and moments on a beam element (Figure 2.1) must be considered. The dynamic equilibrium condition for the forces in z direction is



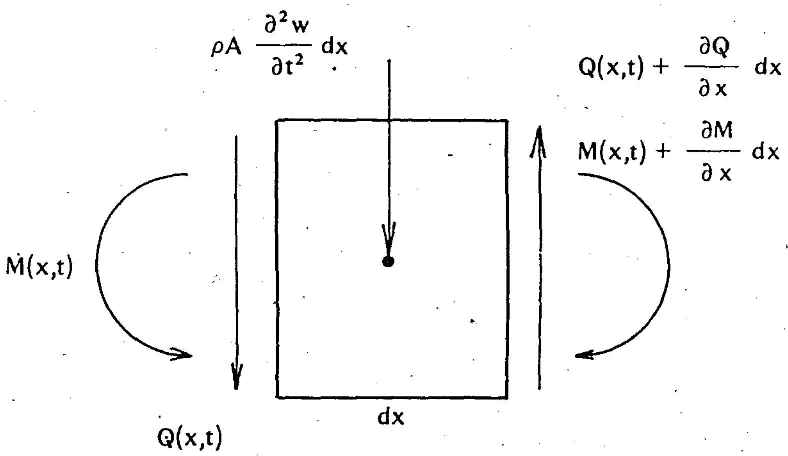


Figure 2.1 Free body diagram of an element of a beam according to Timoshenko theory.

$$Q - Q - \frac{\partial Q}{\partial x} dx + \rho A \frac{\partial^2 w}{\partial t^2} . dx = 0 \qquad \text{II-9}$$

where ρ is the mass density. Substituting equation (II-8) in equation (II-9) we have

$$\kappa AG \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi}{\partial x} \right) - \rho A \frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{II-10}$$

The differential equation for rotation of an element can be written

$$Qdx + M - M - \frac{\partial M}{\partial x} dx + \rho I \frac{\partial^2 \psi}{\partial t^2} = 0 \qquad \text{II-11}$$

Substituting equation (II.7) and equation (II-8) into equation (II-11) we obtain

$$- EI \frac{\partial^2 \psi}{\partial x^2} + \kappa \left(\frac{\partial w}{\partial x} + \psi \right) AG + \rho I \frac{\partial^2 \psi}{\partial t^2} = 0 \qquad \text{II-12}$$

Equation (II-10) and equation (II-12) are the coupled equations of motion for the Timoshenko beam theory. If $\psi(x,t)$ or $w(x,t)$ is eliminated from the coupled equations the equation governing the transverse displacement $w(x,t)$ and $\psi(x,t)$ can be obtained as

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \left(\rho I + \rho I \frac{E}{\kappa G} \right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa G} \frac{\partial^4 w}{\partial t^4} = 0 \qquad \text{II-13}$$

$$EI \frac{\partial^4 \psi}{\partial x^4} + \rho A \frac{\partial^2 \psi}{\partial t^2} - \left(\rho I + \rho I \right) \frac{E}{\kappa G} \frac{\partial^4 \psi}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa G} \frac{\partial^4 \psi}{\partial t^4} = 0 \qquad \text{II-14}$$

1. Boundary Conditions

The appropriate boundary conditions of Timoshenko beam theory are [12]

Clamped end: The deflection and rotation of a point on the clamped end of a beam must vanish.

$$w(x,t)\,|_{x=0} = w(x,t)\,|_{x=l} = 0$$

$$\text{II-15}$$

$$\psi(x,t)\,|_{x=0} = \psi(x,t)\,|_{x=l} = 0$$

Hinged end: When the end of a beam is hinged, the deflection $w(x,t)$ and moment $M(x,t)$ are zero, that is

$$w(x,t)\,|_{x=0} = w(x,t)\,|_{x=l} = 0$$

II-16

$$M(x,t)\,|_{x=0} = M(x,t)\,|_{x=l} = 0$$

Free end: If the boundary of a beam is free then the moment and shear forces should be zero, thus,

$$M(x,t)\,|_{x=0} = M(x,t)\,|_{x=l} = 0$$

II-17

$$Q(x,t)\,|_{x=0} = Q(x,t)\,|_{x=l} = 0$$

C. MODIFIED THEORY

In the modified theory [9] for a beam of narrow rectangular cross-section Euler-Bernoulli hypothesis completely renounced. This theory includes the effect of shear and rotary inertia as Timoshenko theory does. However, difference from the Timoshenko beam theory is that the cross-section of the beam is allowed to warp with the restriction that it remains normal to the shear-free surfaces of the beam.

An element for a beam which is allowed to warp into a non-planar surface is shown in Figures 2.2 and 2.3 with the notations and sign conventions to be used. Here M and O denote the bending moment and shear force respectively while ψ denotes the rotation of the cross-section of the beam at the neutral surface. Thus, the displacement of a material point in the longitudinal direction is considered to be of the form

$$u(z,x,t) = \psi(x,t)\, z + \phi(x,t)\, z^3 \qquad \text{II-18}$$

where ϕ (x,t)is the warping function because it is a measure of the deviation of the cross-section from a plane surface.

The displacement "u" may be chosen as an expansion of series, i.e.,

$$u(z,x,t) = a_1(x,t)\, z + a_2(x,t)\, z^2 + a_3(x,t)\, z^3 + \dots \qquad \text{II-19}$$

where z is the vertical direction of the beam and a_1, a_2, a_3, ... are the coefficients. The first

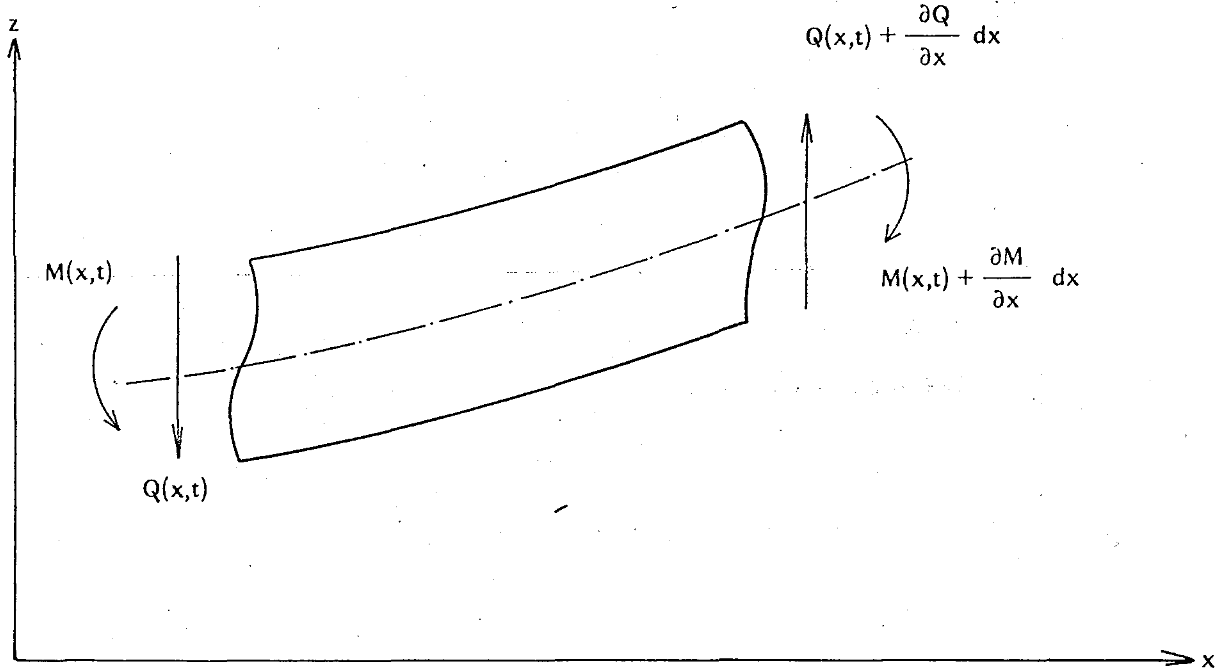


Figure 2.2. Free body diagram of an element of a beam with moments and shears according to the modified theory.

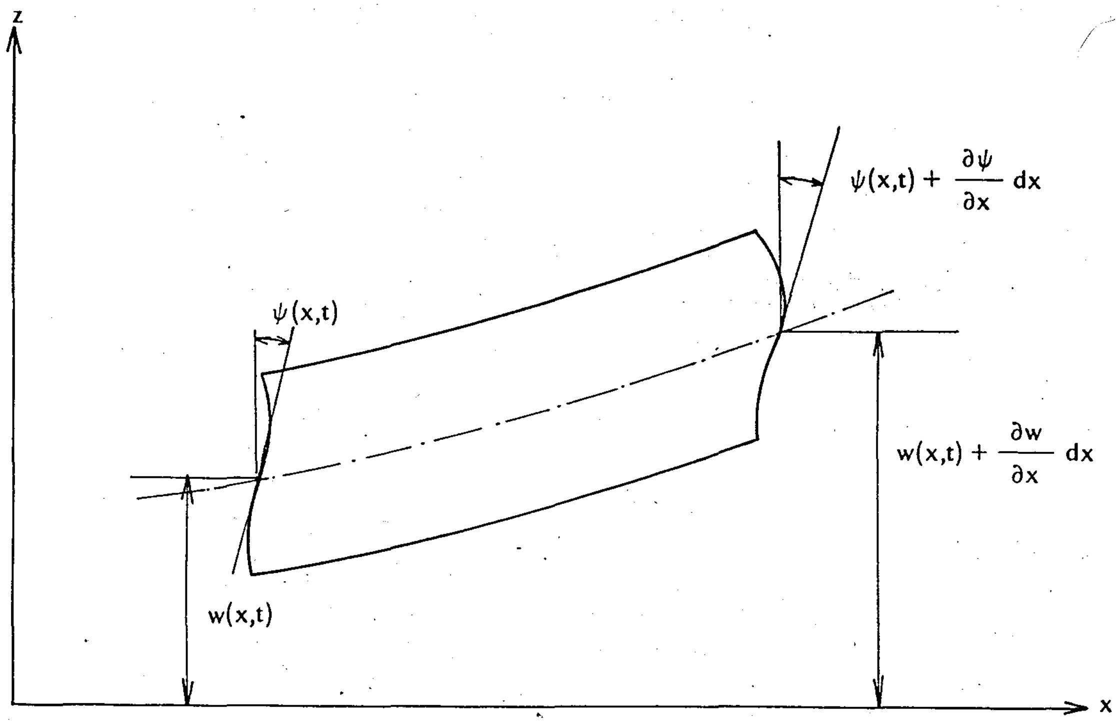


Figure 2.3. Free body diagram of an element of a beam with displacements and rotations according to the modified theory.

term of the above expression corresponds to the case where plane sections normal to the neutral surface remain normal, (Figure 2.4). Actually the cross-section does not remain plane after deformation. Thus, the first term is inadequate. Figure 2.5 shows the contribution of the second term of the series expansion to the deformation. However, this kind of deformation is not possible because there is no extension in the neutral plane, then the second term drops. The third term deforms the cross-section into skew-symmetry as shown in Figure 2.6. Contributions due to the rest of the terms are very small as compared to the first two odd terms. Thus neglecting these high order terms equation II-19 becomes,

$$u(z,x,t) = a_1(x,t)\, z + a_2(x,t)\, z^3 \qquad \text{II-20}$$

Comparing equations (II-18) and (II-20) we see that a_1 and a_2 correspond to ψ and ϕ respectively. In the classical theory and Timoshenko theory, only the first term in the power series expansion equation (II-18) is considered.

Since the shear stress vanishes at the shear-free surface of the beam, so does the shear strain ϵ_{xz}, that is,

$$\epsilon_{xz}\left(x, \pm\frac{h}{2}\right) = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\Bigg|_{z=\pm\frac{h}{2}} = 0 \qquad \text{II-21}$$

Substituting equation (II-18) into the above equation we get,

$$\phi(x,t) = -\frac{4}{3h^2}\left(\psi + \frac{\partial w}{\partial x}\right) \qquad \text{II-22}$$

One can write the one-dimensional Hooke's Law for normal stress and shear stress, as

$$\sigma_x = E\,\epsilon_x \qquad \text{II.23}$$

and

$$\tau = G\,\epsilon_{xz} \qquad \text{II-24}$$

where E is the modulus of elasticity, G the modulus of rigidity, ϵ_x is the normal strain and equals to $\frac{\partial u}{\partial x}$ then,

$$\epsilon_x = \frac{\partial \psi}{\partial x} z - \frac{4}{3h^2}\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) z^3 \qquad \text{II.25}$$

Using equation (II-23) we can write the expression for the bending moment as,

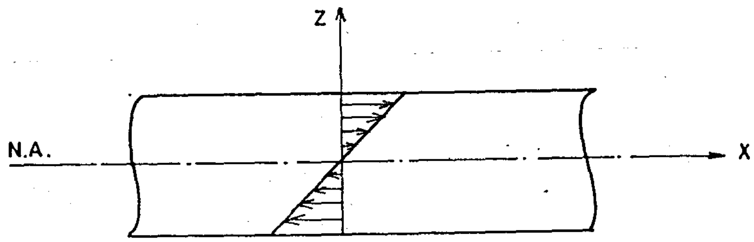


Figure 2.4

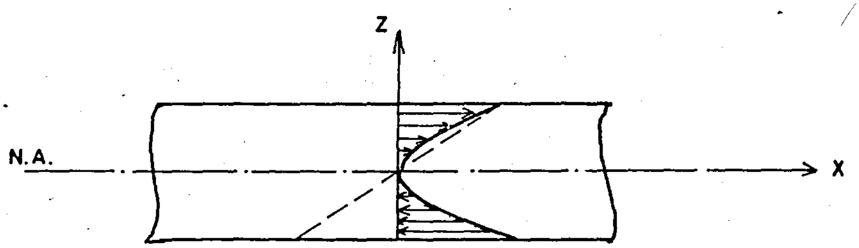


Figure 2.5

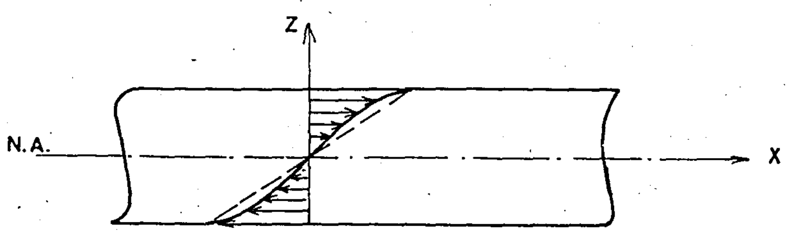


Figure 2.6

$$M = \int_A E \epsilon_x \, z \, dA \qquad \text{II-26}$$

where A is the area for a rectangular beam of depth h and with l. Carrying out the integration, bending moment is obtained as a function of ψ and w,

$$M = \frac{EI}{5} \left(4 \frac{\partial \psi}{\partial x} - \frac{\partial^2 w}{\partial x^2} \right) \qquad \text{II.27}$$

where I is the moment of inertia of the cross-section. In a similar procedure, integrating the shear stress over the area of the cross-section, the transverse shear force can be obtained as

$$Q = \int_A G \epsilon_{xz} dA \qquad \text{II-28}$$

which yields

$$Q = \frac{2}{3} GA \left(\psi + \frac{\partial w}{\partial x} \right) \qquad \text{II-29}$$

Rotary inertia is the moment of the inertial force of an element about an axis through its center of mass and perpendicular to the $x - z$ plane. Thus the rotary inertia effect can be expressed as

$$\int_A \rho \frac{\partial^2 u}{\partial t^2} z \, dA \qquad \text{II.30}$$

where ρ is the mass density of the beam. Integration of the above expression is upon introducing equation (II-18) results in

$$\int_{-h/2}^{h/2} \rho \left[\frac{\partial^2 \psi}{\partial t^2} z - \frac{4}{3h^2} \left(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 w}{\partial x \partial t^2} \right) z^3 \right] z \, dz = \frac{\rho I}{5} \left(4 \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^3 w}{\partial x \partial t^2} \right) \qquad \text{II-31}$$

when the beam is vibrating transversely, the dynamic equilibrium condition for forces can be written in z direction as

$$Q + \frac{\partial Q}{\partial x} = \rho A \frac{\partial^2 w}{\partial t^2} + \dot{Q} \qquad \text{II-32}$$

The moment equilibrium of an element can be expressed as of the form

$$Q dx - \frac{\partial M}{\partial x} dx + \frac{\rho I}{5} \left(4 \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^3 w}{\partial x \partial t^2} \right) dx = 0 \qquad \text{II-33}$$

which yields upon substitution of equation (II-29)

$$\frac{2}{3} AG\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) = \rho A \frac{\partial^2 w}{\partial t^2} \qquad \text{II-34}$$

If we now substitute equation (II-27) and equation (II.29) into equation (II-33) we obtain

$$\frac{2}{3} AG\left(\psi + \frac{\partial w}{\partial x}\right) - \frac{EI}{5}\left(4\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^3 w}{\partial x^3}\right) = -\frac{\rho I}{5}\left(4\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^3 w}{\partial x \partial t^2}\right) \qquad \text{II-35}$$

Eliminating ψ or w from the coupled equations (II-34) and II-35) respectively, yields

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \left(\rho I + \frac{6}{5}\frac{\rho EI}{G}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{6}{5}\rho^2\frac{I}{G}\frac{\partial^4 w}{\partial t^4} = 0 \qquad \text{II-36}$$

$$EI\frac{\partial^4 \psi}{\partial x^4} + \rho A\frac{\partial^2 \psi}{\partial t^2} - \left(\rho I + \frac{6}{5}\frac{\rho EI}{G}\right)\frac{\partial^4 \psi}{\partial x^2 \partial t^2} + \frac{6}{5}\rho^2\frac{I}{G}\frac{\partial^4 \psi}{\partial t^4} = 0 \qquad \text{II-37}$$

This equation is precisely the same equation obtained from Timoshenko beam theory with shear coefficient 5/6.

1. Boundary Conditions:

The appropriate boundary conditions of the modified theory are given for the following cases

Clamped end: At the clamped end of a beam the deflection and the slope are zero. i.e.

$$w(x,t)\,|_{x=0} = w(x,t)\,|_{x=l} = 0$$

$$\psi(x,t)\,|_{x=0} = \psi(x,t)\,|_{x=l} = 0 \qquad \text{II-38}$$

Hinged end: When a beam is hinged ended the deflection and moment vanish i.e.,

$$w(x,t)\,|_{x=0} = w(x,t)\,|_{x=l} = 0$$

$$M(x,t)\,|_{x=0} = M(x,t)\,|_{x=l} = 0 \qquad \text{II-39}$$

Free end: when the end of a beam is free the moment and the shear force are zero. That is,

$$M(x,y)\,|_{x=0} = M(x,y)\,|_{x=l} = 0 \qquad \text{II-40}$$

$$Q(x,y)\,|_{x=0} = Q(x,y)\,|_{x=l} = 0$$

III. SOLUTIONS OF THE EQUATIONS

In this section the problem will be formulated for Euler-Bernoulli theory, Timoshenko theory and the modified theory under simple and homogeneous boundary conditions. The eigenfunctions and characteristic equations for different boundary conditions will be presented.

A. EULER-BERNOULLI THEORY

1. Basic Equation:

Under the appropriate boundary conditions the equation of motion for Euler-Bernoulli theory was given in the preceeding section.

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{III-1}$$

The method of separation of variables may be applied to the above equation by considering a solution of the form

$$w(x,t) = W(x)\, e^{i\omega t} \qquad \text{III-2}$$

and

$$\xi = \frac{x}{L} \qquad \text{III-3}$$

where W (x) = Normal function of w,
ξ = non-dimensional length of beam,
ω = angular frequency
i = $\sqrt{-1}$
L = Length of beam

When the common factor $e^{i\omega t}$ is ommitted equation (III-1) reduces to the following form

$$W^{IV} - b^2 W = 0 \qquad \text{III-4}$$

where

$$b^2 = \frac{\rho A}{EI} L^4 \omega^2 \qquad \text{III-5}$$

and the primes on W represents differentiation with respect to ξ.

2. EIGENVALUE PROBLEM

Equation (III-4) is the associated eigenvalue problem for Euler-Bernoulli theory. Here b^2 is the eigen-value and W the corresponding eigenfunction. The general solution of this linear and homogeneous equation can be written as

$$W = C_1 \ \mathrm{Sinh} \sqrt{b}\,\xi + C_2 \ \mathrm{Cosh} \sqrt{b}\,\xi + C_3 \ \mathrm{Sin} \sqrt{b}\,\xi + C_4 \ \mathrm{Cos} \sqrt{b}\,\xi \qquad \text{III-6}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. Applying the appropriate boundary conditions to the equation (III-6) will yield the eigenfunctions and characteristic equations.

a. Clamped-free beam:

1. Frequency equation (Characteristic equation):

The boundary conditions for clamped-free beam can be imposed on the equation (III-6) yielding

$$\begin{bmatrix} \mathrm{Sinh}\sqrt{b} & \mathrm{Cosh}\sqrt{b} & -\mathrm{Sin}\sqrt{b} & -\mathrm{Cos}\sqrt{b} \\ \mathrm{Cosh}\sqrt{b} & \mathrm{Sinh}\sqrt{b} & -\mathrm{Cos}\sqrt{b} & \mathrm{Sin}\sqrt{b} \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{III-7}$$

For a non-trivial solution the determinant must be zero. Expansion of the determinant yield the characteristic equation which is of the form,

$$1 + \mathrm{Cosh}\sqrt{b}\,\mathrm{Cos}\sqrt{b} = 0 \qquad \text{III-8}$$

2. Normal modes (Eigenfunctions):

Three of the coefficients in equation (III-6) can be written in terms of fourth one, then the eigenfunction will become,

$$W = C_4 \ [\delta \mathrm{Sinh}\sqrt{b}\xi - \mathrm{Cosh}\sqrt{b}\xi - \delta \mathrm{Sin}\sqrt{b}\xi + \mathrm{Cos}\sqrt{b}\xi] \qquad \text{III.9}$$

where

$$\delta = \frac{\mathrm{Sinh}\sqrt{b} + \mathrm{Sin}\sqrt{b}}{\mathrm{Cosh}\sqrt{b} + \mathrm{Cos}\sqrt{b}}$$

and the bending slope may be expressed in the following form

$$W' = H \ [\mathrm{Cosh}\sqrt{b}\xi - \eta \mathrm{Sinh}\sqrt{b}\xi - \mathrm{Cos}\sqrt{b}\xi - \eta \mathrm{Sin}\sqrt{b}\xi]. \qquad \text{III-10}$$

where

$$H = C_1\sqrt{b}$$

and

$$\eta = \frac{\text{Sinh}\sqrt{b} + \text{Sin}\sqrt{b}}{\text{Cosh}\sqrt{b} + \text{Cos}\sqrt{b}}$$

The characteristic equations and eigenfunctions for clamped-clamped, hinged-hinged, free-free and clamped-hinged boundary conditions can beobtained by using a similar procedure as outlined above. We simply present here the resulting equations.

b. Clamped-clamped beam:

1. Frequency equation:

$$\text{Cosh}\sqrt{b}\,\text{Cos}\sqrt{b} - 1 = 0 \qquad \text{III-11}$$

1. Normal modes:

$$W = C_4[\delta\,\text{Sinh}\sqrt{b}\xi - \text{Cosh}\sqrt{b}\xi - \text{Sin}\sqrt{b}\xi + \text{Cos}\sqrt{b}\xi] \qquad \text{III-12}$$

where

$$\delta = \frac{\text{Cosh}\sqrt{b} - \text{Cos}\sqrt{b}}{\text{Sinh}\sqrt{b} - \text{Sin}\sqrt{b}}$$

$$W' = C_1\sqrt{b}\,[\text{Cosh}\sqrt{b}\xi - \eta\text{Sinh}\sqrt{b}\xi - \text{Cos}\sqrt{b}\xi - \eta\text{Sin}\sqrt{b}\xi] \qquad \text{III-13}$$

where

$$\eta = \frac{\text{Cosh}\sqrt{b} - \text{Cos}\sqrt{b}}{\text{Sinh}\sqrt{b} - \text{Sin}\sqrt{b}}$$

c. Hinged-hinged beam:

1. Frequency equation:

$$\text{Sin}\sqrt{b} = 0 \qquad \text{III-14}$$

2. Normal modes:

$$W = C_3 \mathrm{Sin}\sqrt{b}\xi \qquad \text{III-15}$$

$$W' = C_3\sqrt{b}\ \mathrm{Cos}\sqrt{b}\xi \qquad \text{III-16}$$

d. Free-free beam:

1. Frequency equation

$$\mathrm{Cosh}\sqrt{b}\ \mathrm{Cos}\sqrt{b} - 1 = 0 \qquad \text{III-17}$$

2. Normal modes:

$$W = C_4\ [-\delta \mathrm{Sinh}\sqrt{b}\xi + \mathrm{Cosh}\sqrt{b}\xi - \delta \mathrm{Sin}\sqrt{b}\xi + \mathrm{Cos}\sqrt{b}\xi] \qquad \text{III-18}$$

where

$$\delta = \frac{\mathrm{Sinh}\sqrt{b} + \mathrm{Sin}\sqrt{b}}{\mathrm{Cosh}\sqrt{b} - \mathrm{Cos}\sqrt{b}}$$

$$W' = C_1\sqrt{b}\ [\mathrm{Cosh}\sqrt{b}\xi - \eta \mathrm{Sinh}\sqrt{b}\xi + \mathrm{Cos}\sqrt{b}\xi + \eta \mathrm{Sin}\sqrt{b}\xi] \qquad \text{III-19}$$

where

$$\eta = \frac{\mathrm{Sinh}\sqrt{b} + \mathrm{Sin}\sqrt{b}}{\mathrm{Cosh}\sqrt{b} - \mathrm{Cos}\sqrt{b}}$$

e. Clamped-hinged beam:

1. Frequency equation:

$$\tanh\sqrt{b} - \tan\sqrt{b} = 0 \qquad \text{III-20}$$

2. Normal modes:

$$W = C_4\ [\delta \mathrm{Sinh}\sqrt{b}\xi - \mathrm{Cosh}\sqrt{b}\xi - \delta \mathrm{Sin}\sqrt{b}\xi + \mathrm{Cos}\sqrt{b}\xi] \qquad \text{III.21}$$

where

$$\delta = \frac{\mathrm{Cosh}\sqrt{b} + \mathrm{Cos}\sqrt{b}}{\mathrm{Sinh}\sqrt{b} + \mathrm{Sin}\sqrt{b}}$$

$$W' = C_1\sqrt{b}\,[\mathrm{Cosh}\sqrt{b}\xi - \eta\mathrm{Sinh}\sqrt{b}\xi - \mathrm{Cos}\sqrt{b}\xi - \eta\mathrm{Sin}\sqrt{b}\xi] \qquad \text{III-22}$$

where $\eta = \dfrac{\mathrm{Sinh}\sqrt{b} - \mathrm{Sin}\sqrt{b}}{\mathrm{Cosh}\sqrt{b} - \mathrm{Cosh}\sqrt{b}}$

B. TIMOSHENKO BEAM THEORY

1. BASIC EQUTAIONS

Differential equations (III-13) and (III-14) governing the transverse displacement w and angular rotation ψ can be formulated in non-dimensional form

$$W^{IV} + b^2 (r^2 + s^2) W'' - b^2 (1 - b^2 r^2 s^2) W = 0 \qquad \text{III-23}$$

and

$$\Psi^{IV} + b^2 (r^2 + s^2) \Psi'' - b^2 (1 - b^2 r^2 s^2) \Psi = 0 \qquad \text{III-24}$$

where $b^2 = \dfrac{\rho A}{EI} L^4 \omega^2$

$$r^2 = \frac{I}{AL^2}$$

$$s^2 = \frac{EI}{\kappa AGL^2}$$

Boundary conditions in non-dimensional forms are:

Clamped end:

$$w(0) = w(1) = 0$$

$$\Psi(0) = \Psi(1) = 0 \qquad \text{III-25}$$

Hinged end:

$$w(0) = w(1) = 0$$

III-26

$$\Psi'(0) = \Psi(1) = 0$$

Free end:

$$\Psi'(0) = \Psi'(1) = 0$$

III-27

$$\left(\Psi + \frac{1}{L} W'\right)\Big|_{\xi=0} = \left(\Psi + \frac{1}{L} W'\right)\Big|_{\xi=1} = 0$$

2. EIGENVALUE PROBLEM

The corresponding eigenvalue problem for Timoshenko theory is specified by equations (III-23) and (III-24). The general solution of linear and homogeneous equation (III-23) and (III-24) can be found respectively as,

$$W = C_1 \operatorname{Sinh}\alpha\xi + C_2 \operatorname{Cosh}\alpha\xi + C_3 \operatorname{Sin}\beta\xi + C_4 \operatorname{Cos}\beta\xi \qquad \text{III-28}$$

and

$$\Psi = C_1' \operatorname{Cosh}\alpha\xi + C_2' \operatorname{Sinh}\alpha\xi + C_3' \operatorname{Cos}\beta\xi + C_4' \operatorname{Sin}\beta\xi \qquad \text{III-29}$$

where C,s and C′,s are arbitrary constants,

$$\alpha = \frac{b}{\sqrt{2}}\left(\left[(r^2 - s^2)^2 + \frac{4}{b^2}\right]^{1/2} - (r^2 + s^2)\right)^{1/2}$$

and

$$\beta = \frac{b}{\sqrt{2}}\left(\left[(r^2 - s^2)^2 + \frac{4}{b^2}\right]^{1/2} + (r^2 + s^2)\right)^{1/2}$$

For the range of $b < \frac{1}{rs}$, α can be taken as of the form

$$\alpha = i\alpha' \qquad \text{III-30}$$

where $i=\sqrt{-1}$

The constants of equations (III-28) and (III-29) are related as follows:

$$C_1' = -\frac{\alpha^2+b^2s^2}{L\alpha}\;C_1$$

$$C_2' = -\frac{\alpha^2+b^2s^2}{L\alpha}\;C_2 \qquad \text{III.31}$$

$$C_3' = -\frac{\beta^2-b^2s^2}{L\beta}\;C_3$$

$$C_4' = \frac{\beta^2-b^2s^2}{L\beta}\;C_4$$

a. Clamped-free beam:

1. Characteristic equation:

The associated boundary conditions for clamped-free beam can be applied to equations (III-28) and (III-29) yielding

$$\begin{bmatrix} \text{Sinh}\alpha & \text{Cosh}\alpha & -\zeta\text{Sin}\beta & -\zeta\text{Cos}\beta \\ \text{Cosh}\alpha & \text{Sinh}\alpha & -\lambda\text{Cos}\beta & \lambda\text{Sin}\beta \\ 1 & 0 & \lambda\zeta & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{III-32}$$

The corresponding characteristic equation can be obtained by expansion of the determinant, i.e.,

$$2+\left(\frac{1}{\zeta}+\zeta\right)\text{Cosh}\alpha\;\text{Cos}\beta - \left(\frac{1}{\lambda}-\lambda\right)\text{Sinh}\alpha\;\text{Sin}\beta = 0 \qquad \text{III.33}$$

for the range of $b < \frac{1}{rs}$

$$2+\left(\frac{1}{\zeta}+\zeta\right)\text{Cos}\alpha'\;\text{Cos}\beta - \left(-\frac{1}{\lambda'}+\lambda'\right)\text{Sin}\alpha'\;\text{Sin}\beta = 0 \qquad \text{III-34}$$

where

$$\lambda = \frac{\alpha}{\beta}$$

$$\lambda' = \frac{\alpha'}{\beta}$$

$$\zeta = \frac{\beta^2 - b^2 s^2}{\alpha^2 + b^2 s^2}$$

2. Normal modes (Eigenfunctions):

The constants C_1, C_2, C_3 in equation III-28 must be determined in terms of fourth one, then the eigenfunction is of the form

$$W = C_4 \, [\lambda\zeta\delta \, \mathrm{Sinh}\alpha\xi - \mathrm{Cosh}\alpha\xi - \delta \, \mathrm{Sin}\beta\xi + \mathrm{Cos}\beta\xi] \qquad \text{III-35}$$

where

$$\delta = \frac{\frac{1}{\lambda} \, \mathrm{Sinh}\alpha - \mathrm{Sin}\beta}{\zeta \mathrm{Cosh}\alpha + \mathrm{Cos}\beta}$$

when $b < \frac{1}{rs}$ the equation III-35 becomes,

$$W = C_4 \, [-\lambda'\zeta\eta \mathrm{Sin}\alpha'\xi - \mathrm{Cos}\alpha'\xi - \eta \mathrm{Sin}\beta\xi + \mathrm{Cos}\beta\xi] \qquad \text{III-36}$$

where

$$\lambda' = \frac{\alpha'}{\beta}$$

$$\eta = \frac{\frac{1}{\lambda} \, \mathrm{Sin}\alpha' - \mathrm{Sin}\beta}{\zeta \mathrm{Cos}\alpha' + \mathrm{Cos}\beta}$$

The bending slope equation is expressed by using equation III-29 as in the following form

$$\Psi = H \, [-\mathrm{Cosh}\alpha\xi + \frac{\delta}{\lambda\zeta} \, \mathrm{Sinh}\alpha\xi + \mathrm{Cos}\beta\xi + \delta \mathrm{Sin}\beta\xi] \qquad \text{III-37}$$

where

$$\delta = \frac{\lambda \text{Sinh}\alpha + \text{Sin}\beta}{\frac{1}{\zeta}\text{Cosh}\alpha + \text{Cos}\beta} \qquad , \qquad H = \frac{1}{L} C_1$$

in the case of $b < \frac{1}{rs}$ equation III-37 is,

$$\Psi = H\,[-\text{Cos}\alpha'\xi + \frac{\eta}{\lambda'\zeta}\text{Sin}\alpha'\xi + \text{Cos}\beta\xi + \eta\text{Sin}\beta\xi] \qquad \text{III-38}$$

where

$$\eta = \frac{-\lambda'\text{Sin}\alpha' + \text{Sin}\beta}{\frac{1}{\zeta}\text{Cos}\alpha' + \text{Cos}\beta}$$

The same way the frequency equations and the normal mode equations for each case of boundary conditions may be obtained as given above, and the resulting equations are presented.

b. Clamped-clamped beam:

1. Frequency equation.

$b > \frac{1}{rs}$,

$$2 - 2\text{Cosh}\alpha\text{Cos}\beta - \left(\lambda\zeta - \frac{1}{\lambda\zeta}\right)\text{Sinh}\alpha\text{Sin}\beta = 0 \qquad \text{III-39}$$

$b < \frac{1}{rs}$,

$$2 - 2\text{Cos}\alpha'\text{Cos}\beta + \left(\lambda'\zeta + \frac{1}{\lambda'\zeta}\right)\text{Sin}\alpha'\text{Sin}\beta = 0 \qquad \text{III-40}$$

where

$$\lambda' = \frac{\alpha'}{\beta}$$

2. Normal modes:

$b > \frac{1}{rs}$,

$$W = C_4\ [\lambda\zeta\delta\, \text{Sinh}\alpha\xi - \text{Cosh}\alpha\xi - \delta\, \text{Sin}\beta\xi + \text{Cos}\beta\xi] \qquad \text{III-41}$$

$$b < \frac{1}{rs} \quad ,$$

$$W = C_4\ [-\lambda'\zeta\delta'\text{Sin}\alpha'\xi - \text{Cos}\alpha'\xi - \delta'\text{Sin}\beta\xi + \text{Cos}\beta\xi] \qquad \text{III-42}$$

where

$$\delta = \frac{\text{Cosh}\alpha - \text{Cos}\beta}{\lambda\zeta\text{Sinh}\alpha - \text{Sin}\beta}$$

$$\delta' = -\frac{\text{Cos}\alpha' - \text{Cos}\beta}{\lambda'\zeta\text{Sin}\alpha' + \text{Sin}\beta}$$

$$b > \frac{1}{rs} \quad ,$$

$$\Psi = \frac{C_1}{L}\ [-\text{Cosh}\alpha\xi + \frac{\eta}{\lambda\zeta}\ \text{Sinh}\alpha\xi + \text{Cos}\beta\xi + \eta\text{Sin}\beta\xi] \qquad \text{III-43}$$

$$b < \frac{1}{rs}$$

$$\Psi = \frac{C_1}{L}\ [-\text{Cos}\alpha'\xi + \frac{\eta'}{\lambda'\zeta}\ \text{Sin}\alpha'\xi + \text{Cos}\beta\xi + \eta'\text{Sin}\beta\xi] \qquad \text{III-44}$$

where

$$\eta = \frac{\text{Cosh}\alpha - \text{Cos}\beta}{\frac{1}{\lambda\zeta}\ \text{Sinh}\alpha + \text{Sin}\beta}$$

$$\eta = \frac{\text{Cos}\alpha' - \text{Cos}\beta}{\frac{1}{\lambda'\zeta}\ \text{Sin}\alpha' + \text{Sin}\beta}$$

c. **Hinged-hinged beam:**

1. **Frequency equation:**

$$b > \frac{1}{rs} \quad , \qquad \text{Sinh}\alpha\text{Sin}\beta = 0 \qquad \text{III-45}$$

$$b < \frac{1}{rs} \quad , \qquad i\text{Sin}\alpha'\text{Sin}\beta = 0 \qquad \text{III-46}$$

2. Normal modes:

$$W = C_3 \mathrm{Sin}\beta\xi \quad \text{III-47}$$

$$\Psi = \beta C_3 \mathrm{Cos}\beta\xi \quad \text{III-48}$$

d. Free-free beam:

1. Frequency equation:

$$b > \frac{1}{rs},$$

$$2 - 2\mathrm{Cosh}\alpha\mathrm{Cos}\beta + \left(\frac{\zeta}{\lambda} - \frac{\lambda}{\zeta}\right)\mathrm{Sinh}\alpha\mathrm{Sin}\beta = 0 \quad \text{III-49}$$

$$b < \frac{1}{rs}$$

$$2 - 2\mathrm{Cos}\alpha'\mathrm{Cos}\beta + \left(\frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta}\right)\mathrm{Sin}\alpha'\mathrm{Sin}\beta = 0 \quad \text{III-50}$$

2. Normal modes:

$$b > \frac{1}{rs}$$

$$W = C_4\ [-\lambda\zeta\delta\mathrm{Sinh}\alpha\xi + \zeta\mathrm{Cosh}\alpha\xi - \zeta\delta\mathrm{Sin}\beta\xi + \mathrm{Cos}\beta\xi] \quad \text{III-51}$$

$$b < \frac{1}{rs}$$

$$W = C_4\ [\ \lambda'\zeta\delta'\mathrm{Sin}\alpha'\xi + \zeta\mathrm{Cos}\alpha'\xi - \zeta\delta'\mathrm{Sin}\beta\xi + \mathrm{Cos}\beta\xi] \quad \text{III-52}$$

where

$$\delta = \frac{\mathrm{Cosh}\alpha - \mathrm{Cos}\beta}{\lambda\mathrm{Sinh}\alpha - \zeta\mathrm{Sin}\beta}$$

$$\delta' = \frac{-\mathrm{Cos}\alpha' + \mathrm{Cos}\beta}{\lambda'\mathrm{Sin}\alpha' + \zeta\mathrm{Sin}\beta}$$

$b > \frac{1}{rs}$

$$\Psi = \frac{C_1}{L}\left[-\text{Cosh}\alpha\xi + \frac{\zeta\eta}{\lambda}\text{Sinh}\alpha\xi - \zeta\text{Cos}\beta\xi - \zeta\eta\text{Sin}\beta\xi\right] \qquad \text{III-53}$$

$b < \frac{1}{rs}$

$$\Psi = \frac{C_1}{L}\left[-\text{Cos}\alpha'\xi + \frac{\zeta\eta'}{\lambda'}\text{Sin}\alpha'\xi - \zeta\text{Cos}\beta\xi - \zeta\eta'\text{Sin}\beta\xi\right] \qquad \text{III-54}$$

where

$$\eta = \frac{\text{Cosh}\alpha - \text{Cos}\beta}{\frac{\zeta}{\lambda}\text{Sinh}\alpha + \text{Sin}\beta}$$

$$\eta' = \frac{\text{Cos}\alpha' - \text{Cos}\beta}{\frac{\zeta}{\lambda'}\text{Sin}\alpha' + \text{Sin}\beta}$$

e. Clamped-hinged beam:

1. Frequency equation:

$b > \frac{1}{rs}$,

$$\lambda\zeta\tanh\alpha - \tan\beta = 0 \qquad \text{III-55}$$

$b < \frac{1}{rs}$,

$$\lambda'\zeta\tan\alpha' + \tan\beta = 0 \qquad \text{III-56}$$

2. Normal modes:

$b > \frac{1}{rs}$,

$$W = C_4\left[\lambda\delta\text{Sinh}\alpha\xi - \text{Cosh}\alpha\xi - \frac{1}{\zeta}\delta\text{Sin}\beta\xi + \text{Cos}\beta\xi\right] \qquad \text{III-57}$$

$$b > \frac{1}{rs} \quad ,$$

$$W = C_4 \; [-\lambda'\delta'\mathrm{Sin}\alpha'\xi - \mathrm{Cos}\alpha'\xi - \frac{\delta'}{\zeta} \; \mathrm{Sin}\beta\xi + \mathrm{Cos}\beta\xi] \qquad \text{III-58}$$

where

$$\delta = \frac{\mathrm{Cosh}\alpha + \zeta\mathrm{Cos}\beta}{\lambda\mathrm{Sinh}\alpha + \mathrm{Sin}\beta}$$

$$\delta' = \frac{\mathrm{Cos}\alpha' + \zeta\mathrm{Cos}\beta}{-\lambda'\mathrm{Sin}\alpha' + \mathrm{Sin}\beta}$$

$$b > \frac{1}{rs} \quad ,$$

$$\Psi = \frac{C_1}{L} \; [-\mathrm{Cosh}\alpha\xi + \frac{\eta}{\lambda\zeta} \; \mathrm{Sinh}\alpha\xi + \mathrm{Cos}\beta\xi + \eta\mathrm{Sin}\beta\xi] \qquad \text{III-59}$$

$$b < \frac{1}{rs} \quad ,$$

$$\Psi = \frac{C_1}{L} [-\mathrm{Cos}\alpha'\xi + \frac{\eta'}{\lambda'\zeta} \; \mathrm{Sin}\alpha'\xi + \mathrm{Cos}\beta\xi + \eta'\mathrm{Sin}\beta\xi] \qquad \text{III-60}$$

where

$$\eta = \frac{\lambda\zeta\mathrm{Sinh}\alpha - \mathrm{Sin}\beta}{\mathrm{Cosh}\alpha - \mathrm{Cos}\beta}$$

$$\eta' = \frac{-\lambda'\zeta\mathrm{Sin}\alpha' - \mathrm{Sin}\beta}{\mathrm{Cos}\alpha' - \mathrm{Cos}\beta}$$

C. MODIFIED THEORY

1. Basic Equations:

The coupled equations (II-36) and (II-37) for the modified theory can be expressed in non-dimensional form.

$$W'' + s_1^2 b^2 W + L\Psi' = 0 \qquad \text{III-61}$$

$$s^2 \Psi'' - (1 - b^2 r^2 s^2)\, \Psi - \frac{1}{5L} s_1^2 W''' - \left(1 + \frac{1}{5} b^2 r^2 s_1^2\right) \frac{1}{L} W' = 0 \qquad \text{III-62}$$

where

$$b^2 = \frac{\rho A}{EI} L^4 \omega^2$$

$$r^2 = \frac{I}{AL^2}$$

$$s_1^2 = \frac{3}{2} \frac{EI}{AGL^2}$$

$$s^2 = \frac{6}{5} \frac{EI}{AGL^2}$$

The appropriate boundary conditions of the modified theory are given in non-dimensional form for the following cases:

Clamped end:

$$W(0) = W(1) = 0$$

$$\text{III-63}$$

$$\Psi(0) = \Psi(1) = 0$$

Hinged end:

$$W(0) = W(1) = 0$$

$$\left(\frac{4}{L} \Psi' - \frac{1}{L^2} W''\right)\Big|_{\xi=0} = \left(\frac{4}{L} \Psi' - \frac{1}{L^2} W''\right)\Big|_{\xi=1} = 0 \qquad \text{III-64}$$

Free end:

$$\left(\frac{4}{L} \Psi' - \frac{1}{L^2} W''\right)\Big|_{\xi=0} = \left(\frac{4}{L} \Psi' - \frac{1}{L^2} W''\right)\Big|_{\xi=1} = 0$$

$$\text{III-65}$$

$$\left(\Psi + \frac{1}{L} W'\right)\Big|_{\xi=0} = \left(\Psi + \frac{1}{L} W'\right)\Big|_{\xi=1} = 0$$

2. Eigenvalue Problem

The associated time independent eigenvalue problem for the modified theory may be obtained by eliminating W or Ψ from the coupled equations (III-61) and (III-68) that is,

$$W^{IV} + b^2(r^2 + s^2)W'' - b^2(1 - b^2r^2s^2)W = 0 \qquad \text{III-66}$$

$$\Psi^{IV} + b^2(r^2 + s^2)\Psi'' - b^2(1 - b^2r^2s^2)\Psi = 0 \qquad \text{III-67}$$

For each type of beam the roots of the frequency equations b_n (n = 1,2,3, ...,) give the eigenvalue of the problem. For the corresponding eigenvalue problem the general solution can be obtained in the following matrix form.

$$\begin{bmatrix} W \\ \\ \Psi \end{bmatrix} = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ \\ C'_1 & C'_2 & C'_3 & C'_4 \end{bmatrix} \begin{bmatrix} \text{Sinh}\alpha\xi \\ \text{Cosh}\alpha\xi \\ \text{Sin}\beta\xi \\ \text{Cos}\beta\xi \end{bmatrix} \qquad \text{III-68}$$

where C,s and C',s are constants

$$\alpha = \frac{b}{\sqrt{2}}\left(\left[(r^2 - s^2)^2 + \frac{4}{b^2}\right]^{\frac{1}{2}} - (r^2 + s^2)\right)^{\frac{1}{2}} \qquad \text{III-69}$$

$$\beta = \frac{b}{\sqrt{2}}\left(\left[(r^2 - s^2)^2 + \frac{4}{b^2}\right]^{\frac{1}{2}} + (r^2 + s^2)\right)^{\frac{1}{2}} \qquad \text{III-70}$$

The constants C,s and C',s are related each other as of the form.

$$C'_1 = -\frac{\alpha^2 + b^2 s_1^2}{L\alpha} C_1$$

$$C'_2 = -\frac{\alpha^2 + b^2 s_1^2}{L\alpha} C_2 \qquad \text{III-71}$$

$$C'_3 = -\frac{\beta^2 - b^2 s_1^2}{L\beta} C_3$$

$$C'_4 = \frac{\beta^2 - b^2 s_1^2}{L\beta} C_4$$

Applying the boundary conditions given by equations (III-63), (III-64) and (III-65) to equation (III-68) will give the eigenfunctions and characteristic equations.

a. Clamped-free beam:

1. Frequency equation:

The appropriate boundary conditions for clamped-free beam are given by equation (III-63) and applying equation (III-68) we get

$$\begin{bmatrix} \mathrm{Sinh}\alpha & \mathrm{Cosh}\alpha & -\zeta_1 \mathrm{Sin}\beta & -\zeta_1 \mathrm{Cos}\beta \\ \mathrm{Cosh}\alpha & \mathrm{Sinh}\alpha & -\lambda \mathrm{Cos}\beta & \lambda \mathrm{Sin}\beta \\ -1 & 0 & -\lambda\zeta & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{III-72}$$

The corresponding characteristic equation can be obtained by expansion of the determinant.

$$(\zeta + \zeta_1) + (1 + \zeta\zeta_1)\,\mathrm{Cosh}\alpha\,\mathrm{Cos}\beta - \left(\frac{\zeta_1}{\lambda} - \lambda\zeta\right)\mathrm{Sinh}\alpha\,\mathrm{Sin}\beta = 0 \qquad \text{III-73}$$

where

$$\lambda = \frac{\alpha}{\beta}$$

$$\zeta = \frac{\beta^2 - b^2 s_1^2}{\alpha^2 + b^2 s_1^2}$$

$$\zeta_1 = \frac{\beta^2 + 4\,(\beta^2 - b^2 s_1^2}{\alpha^2 + 4\,(\alpha^2 + b^2 s_1^2}$$

For the range of $b < \dfrac{1}{rs}$ the characteristic equation becomes

$$(\zeta + \zeta_1) + (1 + \zeta\zeta_1)\,\mathrm{Cos}\alpha'\,\mathrm{Cos}\beta - \left(\frac{\zeta_1}{\lambda'} + \lambda'\zeta\right)\mathrm{Sin}\alpha'\,\mathrm{Sin}\beta = 0 \qquad \text{III-74}$$

where

$$\alpha = i\alpha'$$

$$\lambda' = \frac{\alpha'}{\beta}$$

2. Normal modes:

The eigenfunctions can be obtained by expressing the constants C_1, C_2, C_3 in terms of C_4 in equation (III-68). Here the subscript "n" is omitted for W, Ψ, b, α, β and the constants.

$$W = C_4\ [\lambda\zeta\delta \operatorname{Sinh}\alpha\xi - \operatorname{Cosh}\alpha\xi - \delta \operatorname{Sin}\beta\xi + \operatorname{Cos}\beta\xi] \qquad \text{III-75}$$

where

$$\delta = \frac{\frac{1}{\lambda}\operatorname{Sinh}\alpha - \operatorname{Sin}\beta}{\zeta \operatorname{Cosh}\alpha + \operatorname{Cos}\beta}$$

when $b<\frac{1}{rs}$ the equation III-75 will be the following form.

$$W = C_4\ [-\lambda'\zeta\delta' \operatorname{Sin}\alpha'\xi - \operatorname{Cos}\alpha'\xi - \delta' \operatorname{Sin}\beta\xi + \operatorname{Cos}\beta\xi] \qquad \text{III-76}$$

where

$$\delta' = \frac{\frac{1}{\lambda'}\operatorname{Sin}\alpha' - \operatorname{Sin}\beta}{\zeta \operatorname{Cos}\alpha' + \operatorname{Cos}\beta}$$

The bending slope Ψ can be expressed as:

$$\Psi = H\ [-\operatorname{Cosh}\alpha\xi + \frac{\eta}{\lambda\zeta} \operatorname{Sinh}\alpha\xi + \operatorname{Cos}\beta\xi + \eta \operatorname{Sin}\beta\xi] \qquad \text{III-77}$$

where

$$\eta = \frac{\lambda\zeta \operatorname{Sinh}\alpha + \zeta_1 \operatorname{Sin}\beta}{\operatorname{Cosh}\alpha + \zeta_1 \operatorname{Cos}\beta}$$

$$H = \frac{I}{L}\ C_1$$

in case $b<\frac{1}{rs}$ the equation III-77 may become of the form.

$$\Psi = H\ [-\operatorname{Cos}\alpha'\xi + \frac{\eta'}{\lambda'\zeta} \operatorname{Sin}\alpha'\xi + \operatorname{Cos}\beta\xi + \eta' \operatorname{Sin}\beta\xi] \qquad \text{III-78}$$

where

$$\eta' = \frac{-\lambda'\zeta \operatorname{Sin}\alpha' + \zeta_1 \operatorname{Sin}\beta}{\operatorname{Cos}\alpha' + \zeta_1 \operatorname{Cos}\beta}$$

The frequency equations and the normal mode equations for clamped-clamped, hinged-hinged, free-free and clamped-hinged boundaries are listed by using a similar procedure in the following form.

b. Clamped-clamped beam:

1. Frequency equation:

$$b > \frac{1}{rs} \quad ,$$

$$2 - 2\operatorname{Cosh}\alpha \operatorname{Cos}\beta + \left(\frac{1}{\lambda\zeta} - \lambda\zeta \right) \operatorname{Sinh}\alpha \operatorname{Sin}\beta = 0 \qquad \text{III-79}$$

$$b < \frac{1}{rs} \quad ,$$

$$2 - 2\operatorname{Cos}\alpha' \operatorname{Cos}\beta + \left(\frac{1}{\lambda'\zeta} + \lambda'\zeta \right) \operatorname{Sin}\alpha' \operatorname{Sin}\beta = 0 \qquad \text{III-80}$$

1. Normal modes:

$$b > \frac{1}{rs} \quad ,$$

$$W = C_4 \left[\lambda\zeta\delta \operatorname{Sinh}\alpha\xi - \operatorname{Cosh}\alpha\xi - \delta \operatorname{Sin}\beta\xi + \operatorname{Cos}\beta\xi \right] \qquad \text{III-81}$$

$$b < \frac{1}{rs} \quad ,$$

$$W = C_4 \left[-\lambda'\zeta\delta' \operatorname{Sin}\alpha'\xi - \operatorname{Cos}\alpha'\xi - \delta' \operatorname{Sin}\beta\xi + \operatorname{Cos}\beta\xi \right] \qquad \text{III-82}$$

where

$$\delta = \frac{\operatorname{Cosh}\alpha - \operatorname{Cos}\beta}{\lambda\zeta \operatorname{Sinh}\alpha - \operatorname{Sin}\beta}$$

$$\delta' = \frac{Cos\alpha' - Cos\beta}{-\lambda'\zeta Sin\alpha' - Sin\beta}$$

$$b > \frac{1}{rs} \quad ,$$

$$\Psi = \frac{C_1}{L}\left[-Cosh\alpha\xi + \frac{\eta}{\lambda\zeta} Sinh\alpha\xi + Cos\beta + \eta Sin\beta\xi\right] \qquad \text{III-83}$$

$$b < \frac{1}{rs} \quad ,$$

$$\Psi = \frac{C_1}{L}\left[-Cos\alpha'\xi + \frac{\eta'}{\lambda'\zeta} Sin\alpha'\xi + Cos\beta\xi + \eta' Sin\beta\xi\right] \qquad \text{III-84}$$

where

$$\eta = \frac{Cosh\alpha - Cos\beta}{\frac{1}{\lambda\zeta} Sinh\alpha + Sin\beta}$$

$$\eta' = \frac{Cos\alpha' - Cos\beta}{\frac{1}{\lambda'\zeta} Sin\alpha' + Sin\beta}$$

c. **Hinged-hinged beam:**

1. **Frequency equation:**

$$b > \frac{1}{rs} \quad , \qquad Sinh\alpha\, Sin\beta = 0 \qquad \text{III-85}$$

$$b < \frac{1}{rs} \quad , \qquad iSin\alpha' Sin\beta = 0 \qquad \text{III-86}$$

1. **Normal modes:**

$$W = C_3 Sin\beta\xi \qquad \text{III-87}$$

$$\Psi = C_3 \beta\, Cos\beta\xi \qquad \text{III-88}$$

d. Free-free beam:

1. Frequency equation:

$$b > \frac{1}{rs} \quad ,$$

$$2 - 2\text{Cosh}\alpha\text{Cos}\beta + \left(\frac{\zeta_1}{\lambda} - \frac{\lambda}{\zeta_1}\right)\text{Sinh}\alpha\text{Sin}\beta = 0$$ III-89

$$b < \frac{1}{rs} \quad ,$$

$$2 - 2\text{Cos}\alpha'\text{Cos}\beta + \left(\frac{\zeta_1}{\lambda'} + \frac{\lambda'}{\zeta_1}\right)\text{Sin}\alpha'\text{Sin}\beta = 0$$ III-90

2. Normal modes:

$$b > \frac{1}{rs} \quad ,$$

$$W = C_4\ [-\lambda\zeta_1\delta\text{Sinh}\alpha\xi + \zeta_1\text{Cosh}\alpha\xi - \delta\xi_1\text{Sin}\beta\xi + \text{Cos}\beta\xi]$$ III-91

$$b < \frac{1}{rs} \quad ,$$

III-92 $$W = C_4\ [\lambda'\zeta_1\delta'\text{Sin}\alpha'\xi + \zeta_1\text{Cos}\alpha'\xi - \delta'\zeta_1\text{Sin}\beta\xi + \text{Cos}\beta\xi]$$ III-92

where

$$\delta = \frac{\text{Cosh}\alpha - \text{Cos}\beta}{\lambda\text{Sinh}\alpha - \zeta_1\text{Sin}\beta}$$

$$\delta' = \frac{\text{Cos}\alpha' - \text{Cos}\beta}{\lambda'\text{Sin}\alpha' + \zeta_1\text{Sin}\beta}$$

$$b > \frac{1}{rs} \quad ,$$

$$\Psi = \frac{C_1}{L}\left[-\text{Cosh}\alpha\xi + \frac{\zeta_1\eta}{\lambda}\ \text{Sinh}\alpha\xi - \zeta_1\text{Cos}\beta\xi - \zeta_1\eta\text{Sin}\beta\xi\right]$$ III-93

$$b < \frac{1}{rs} \quad ,$$

$$\Psi = \frac{C_1}{L}\left[-\text{Cos}\alpha'\xi + \frac{\zeta_1\eta'}{\lambda'}\text{Sin}\alpha'\xi - \zeta_1\text{Cos}\beta\xi - \zeta_1\eta'\text{Sin}\beta\xi\right] \quad \text{III-94}$$

where

$$\eta = \frac{\text{Cosh}\alpha - \text{Cos}\beta}{\frac{1}{\lambda}\text{Sinh}\alpha + \text{Sin}\beta}$$

$$\eta' = \frac{\text{Cos}\alpha' - \text{Sin}\beta}{\frac{1}{\lambda'}\text{Sin}\alpha' + \text{Sin}\beta}$$

e. Clamped-hinged beam:

1. Frequency equation:

$$b > \frac{1}{rs},$$

$$\zeta\lambda\tanh\alpha - \tan\beta = 0 \quad \text{III-95}$$

$$b < \frac{1}{rs},$$

$$\zeta\lambda'\tan\alpha' + \tan\beta = 0 \quad \text{III-96}$$

2. Normal modes:

$$b > \frac{1}{rs},$$

$$W = C_4\left[\lambda\zeta\delta\,\text{Sinh}\alpha\xi - \text{Cosh}\alpha\xi - \delta\,\text{Sin}\beta\xi + \text{Cos}\beta\xi\right] \quad \text{III-97}$$

$$b < \frac{1}{rs},$$

$$W = C_4\left[-\lambda'\zeta\delta'\text{Sin}\alpha'\xi - \text{Cos}\alpha'\xi - \delta'\text{Sin}\beta\xi + \text{Cos}\beta\xi\right] \quad \text{III-98}$$

where

$$\delta = \frac{\text{Cosh}\alpha + \zeta_1\text{Cos}\beta}{\lambda\zeta\text{Sin}\alpha + \zeta_1\text{Sin}\beta}$$

$$\delta' = \frac{\text{Cos}\alpha' + \zeta_1 \text{Cos}\beta}{-\lambda'\zeta\text{Sin}\alpha' + \zeta_1 \text{Sin}\beta}$$

$$b > \frac{1}{rs} \quad ,$$

$$\Psi = \frac{C_1}{L} \left[-\text{Cosh}\alpha\xi + \frac{\eta}{\lambda\zeta} \text{Sinh}\alpha\xi + \text{Cos}\beta\xi + \eta\text{Sin}\beta\xi\right] \qquad \text{III-99}$$

$$b < \frac{1}{rs} \quad ,$$

$$\Psi = \frac{C_1}{L} \left[-\text{Cos}\alpha'\xi + \frac{\eta'}{\lambda'\zeta} \text{Sin}\alpha'\xi + \text{Cos}\beta\xi + \eta'\text{Sin}\beta\xi\right] \qquad \text{III-100}$$

where

$$\eta = \frac{\lambda\zeta\text{Sinh}\alpha - \text{Sin}\beta}{\text{Cosh}\alpha - \text{Cos}\beta}$$

$$\eta' = -\frac{\lambda'\zeta\text{Sin}\alpha' + \text{Sin}\beta}{\text{Cos}\alpha' - \text{Cos}\beta}$$

IV. NUMERICAL RESULTS AND CONCLUSIONS

In this section a group of numerical examples of free transverse vibration of beams with simple end conditions will be presented. The results obtained using Euler-Bernoulli, Timoshenko and the modified theories will be compared. Frequency spectrums and normal modes for each theory will be given.

A. Frequency Spectrum

For a given beam with known E, G and h/l ratio, the characteristic values b_n (n = 1, 2, 3, ...) can be found from the appropriate frequency equations given in the previous section. In the following numerical calculations a beam of rectangular cross-section with E = 210 GN/m², G = 80.77 GN/m², and h/l values equal to 0.05, 0.10, 0.15, 0.20, 0.25, 0.30 was considered. Shear coefficient in the case of Timoshenko beam theory was taken to be 0.822 while the corresponding values in the modified theory are 2/3 and 5/6. For every h/l ratio fifteen different b,s were obtained up to six digit accuracy. The numerical calculations were carried on a Monroe EC 8800 machine and the results obtained are presented in figures (4.1.1. to 4.1.10.).

Non-dimensional frequencies of the beam for homogenous boundary conditions with different homogenous boundary conditions with different height to span ratios are given in the tables (4.1.1. to 4.1.30). together with the corresponding frequencies for Euler-Bernoulli and Timoshenko beam theories. Examination of tables (4.1.1. to 4.1.12) and (4.1.25 to 4.1.30) show that for clamped-free, clamped-clamped and clamped-hinged end conditions the frequencies from the modified theory are slightly lower than those of Timoshenko beam theory with shear coefficient of 0.822. However as seen in tables (4.1.13 to 4.1.24) the frequencies for hinged-hinged and free-free boundary conditions predicted by the modified theory are slightly higher than Timoshenko beam theory. Using the shear coefficient value of 5/6 the frequency equation for the case of a hinged-hinged beam is identical both in the modified and Timoshenko theories.

For a hinged-hinged beam there are two distinct frequency spectra, $b > \frac{1}{rs}$, the frequency equation is $\sinh\alpha \sin\beta = 0$ while for $b < \frac{1}{rs}$ the frequency equation becomes $i \sin\alpha' \sin\beta = 0$. It means that we have two independent characteristic equations, $\sin\alpha' = 0$ and $\sin\beta = 0$ which yield different natural frequency values. Traill-Nash and Collar [13] claimed the presence of a new spectrum of natural frequencies in a Timoshenko beam. Dolph [14] has also showed the presence of the second family of frequencies. There is still questions about this second frequency spectrum and several researches [15, 16, 17] do reject them.

B. Normal Modes

The normal mode equations of beams in free vibration with different boundary conditions were obtained in section III. The first six modes of vibration for an h/l ratio of 0.10 are presented in figures (4.2.1 through 4.2.30).

The amplitude for clamped-free, clamped-clamped, clamped-hinged boundary conditions predicted by the modified theory are greater than those predicted by Timoshenko and the classical theories. However, just the opposite is observed in the case of beams with hinged-hinged and free-free end conditions.

C. CONCLUSION

In this thesis the frequency spectrums for beams with rectangular cross-section as obtained from three different theories, namely the classical, Timoshenko and the modified theories. The method of separation of variables was used to construct the solutions to flexural vibrations of beams with appropriate boundary conditions.

The curves in figures (4.1.1. to 4.1.10) are characteristic frequency spectrums for Timoshenko and the modified theories. In the ragion $b < \frac{1}{rs}$ the curves are smooth while in the ragion $b > \frac{1}{rs}$ are not. This is due to the fact that the effect of h/l ratio is small is in the first ragion.

For a given beam the modified and Timoshenko theories reveal that there are more resonances in a frequency range than are predicted by Euler-Bernoulli theory. This increase in the number of reconances is due to the effect of shear and rotary inertia. Besides that the number of resonance of Timoshenko and the modified theories are equal. The forms of frequency equations of the modified theory are similar to those of Timoshenko beam theory except the case of clamped-free boundary condition. In the modified theory there are two different shear coefficients which are 2/3 and 5/6 as opposed to that of Timoshenko theory where the values for shear coefficient range from 2/3 to 0.870 [18].

It was observed that in the case of clamped-free, clamped-clamped and clamped-hinged boundary conditions the frequencies as obtained from the modified theory are slightly lower (0.5 – 1.5%) in the lower frequency range than those predicted by Timoshenko theory. This trend is reversed in the high frequency range that is the modified theory gives frequencies that are 0.2 – 1% higher. For hinged-hinged and free-free beams it has been shown that the frequencies using the modified theory are slightly higher by 0.05 – 0.85% than the ones based on Timoshenko theory. In the case of hinged-hinged beam there is no difference in the frequency spectra between the Timoshenko theory and the modified theory if 5/6 is used as the shear coefficient in Timoshenko theory.

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	3.516412	3.509356	3.507464
2.	22.035062	21.731694	21.701703
3.	61.699344	59.741206	59.617360
4.	120.904164	114.130618	113.831578
5.	199.860282	182.958725	182.411291
6.	298.555892	263.958712	263.115589
7.	416.991061	355.012299	353.863541
8.	555.166336	454.244222	452.820739
9.	713.081238	560.061864	557.348946
10.	890.736977	671.148083	669.418843
11.	1088.133526	786.429764	784.724832
12.	1305.256806	905.039061	903.497480
13.	1542.134549	1026.276052	1025.039855
14.	1798.737198	1149.575013	1148.782658
15.	2075.097408	1274.476163	1174.257680

Table 4.1.1. – Nondimensional Frequency for Clamped-Free Boundary
h/l = 0.05

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	3.516412	3.488219	3.482216
2.	22.035062	20.896082	20.789753
3.	61.699344	54.929842	54.547731
4.	120.904164	99.582341	98.816772
5.	199.860282	151.509964	150.391328
6.	298.555892	208.128264	206.809763
7.	416.991061	267.753350	266.448134
8.	555.166336	329.259084	328.184283
9.	713.081238	391.872875	391.214655
10.	890.736977	455.016204	454.923919
11.	1088.133526	518.185766	518.781196
12.	1305.256806	580.756023	582.164416
13.	1542.134549	641.405983	643.813612
14.	1798.737198	683.006243	687.898617
15.	2075.097401	701.309306	704.032840

Table 4.1.2. – Nondimensional Frequency for Clamped–Free Boundary
h/l = 0.1

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	3.516412	3.454429	3.441428
2.	22.035062	19.701944	19.501332
3.	61.699344	49.144963	48.550614
4.	120.904164	84.770413	83.837149
5.	199.860282	123.874517	122.852938
6.	298.555892	164.679462	163.856658
7.	416.991061	206.188857	205.782070
8.	555.166336	247.434345	247.629542
9.	713.081238	286.970289	287.889397
10.	890.736977	308.595191	311.034076
11.	1088.133526	326.930943	327.554427
12.	1305.256806	340.242653	343.565150
13.	1542.134549	368.049679	368.002379
14.	1798.937198	383.433576	387.656617
15.	2075.097401	413.481712	412.964797

Table 4.1.3. – Nondimensional Frequency for Clamped-Free Boundary
h/l = 0.15

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	3.516412	3.408812	3.386869
2.	22.035062	18.334488	18.048979
3.	61.699344	43.597432	42.912906
4.	120.904164	72.326807	71.512021
5.	199.860282	102.796075	102.171656
6.	298.555892	133.331636	133.188120
7.	416.991061	162.582152	162.958884
8.	555.166336	177.127573	178.728270
9.	713.081238	193.599449	193.518444
10.	890.736977	204.532444	206.959068
11.	1088.133526	227.056368	226.485660
12.	1305.256806	238.224732	241.124565
13.	1542.134549	264.060579	263.389853
14.	1798.737198	272.900985	275.971417
15.	2075.097401	301.897995	301.495167

Table 4.1.4 – Nondimensional Frequency for Clamped-Free Boundary
h/l = 0.20

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	3.516412	3.353443	3.320696
2.	22.035062	16.934734	16.589243
3.	61.699344	38.736881	38.066429
4.	120.904164	62.318650	61.750637
5.	199.860282	86.683046	86.443021
6.	298.555892	107.464777	108.201430
7.	416.991061	120.524719	120.425199
8.	555.166336	129.233367	130.911080
9.	713.081238	147.114100	146.507697
10.	890.736977	156.741841	159.191882
11.	1088.133526	174.683470	173.605392
12.	1305.256806	187.727294	191.515421
13.	1542.134549	202.907567	200.763476
14.	1798.737198	218.380643	222.740376
15.	2075.097401	234.193249	231.770363

Table 4.1.5. – Nondimensional Frequency for Clamped-Free Boundary
h/l = 0.25

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	3.516412	3.289643	3.245293
2.	22.035062	15.588896	15.212473
3.	61.699344	34.618984	34.020924
4.	120.904164	54.165974	53.881180
5.	199.860282	73.506234	73.469553
6.	298.555892	82.482696	83.483305
7.	416.991061	95.917923	95.436534
8.	555.166336	103.005903	104.533413
9.	713.081238	120.742962	120.071896
10.	890.736977	126.027391	127.813737
11.	1088.133526	145.412665	144.878423
12.	1305.256806	151.970192	154.254489
13.	1542.134549	168.076294	167.175961
14.	1798.737198	180.824457	211.316456
15.	2075.097401	191.351101	216.252342

Table 4.6 – Nondimensional Frequency for Clamped-Free Boundary
h/l = 0.30

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	22.373297	21.990580	21.931120
2.	61.672828	59.317540	59.063491
3.	120.903395	113.130561	112.504505
4.	199.859462	181.016466	179.842690
5.	298.555547	260.764891	258.905937
6.	416.990857	350.305920	347.685817
7.	555.165299	447.831179	444.446295
8.	713.078941	551.819471	547.737798
9.	890.731904	661.021406	656.373296
10.	1088.123960	774.424560	769.387649
11.	1305.255216	891.213165	885.995265
12.	1542.125831	1010.730257	1005.552798
13.	1798.735501	1132.445220	1127.529526
14.	2075.084372	1255.927352	1251.484561
15.	3271.172452	1380.825025	1364.704787

Table 4.1.7. – Nondimensional Frequency of Clamped-Clamped Boundary
h/l = 0.05

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	22.373297	20.956121	20.749920
2.	61.672828	53.667981	52.915779
3.	120.903395	96.939592	95.399636
4.	199.859462	147.120916	144.779853
5.	298.555547	201.937805	198.996871
6.	416.990857	259.882983	256.669980
7.	555.165299	319.974093	316.846720
8.	713.078941	381.538893	378.821747
9.	890.731904	444.099402	442.060685
10.	1088.123960	507.297176	506.158351
11.	1305.255216	570.844308	570.814095
12.	1542.125831	634.393311	635.796436
13.	1798.735501	686.218844	690.731560
14.	2075.084372	696.550933	700.797616
15.	3271.172452	720.594725	723.376018

Table 4.1.8. – Frequency of Clamped-Clamped Boundary
h/l = 0.1

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	22.373297	19.526420	19.150601
2.	61.672828	47.115444	45.988106
3.	120.903395	81.122378	79.250867
4.	199.859462	118.562682	116.309215
5.	298.555547	158.137149	155.961047
6.	416.990857	198.990960	197.300578
7.	555.165299	240.601450	239.703004
8.	713.078941	282.347804	282.681134
9.	890.731904	311.337422	313.141785
10.	1088.123960	323.210797	323.869543
11.	1305.255216	343.278945	343.571078
12.	1542.125831	364.915074	369.119029
13.	1798.735501	387.299665	386.586339
14.	2075.084372	408.269101	412.634343
15.	3271.172452	437.266132	436.978067

Table 4.1.9. – Frequency for Clamped-Clamped Boundary
h/l = 0.15

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	22.373297	17.953760	17.433120
2.	61.672828	41.049197	39.774891
3.	120.903395	68.371369	66.646560
4.	199.859462	97.652903	96.048365
5.	298.555547	128.193092	127.132889
6.	416.990857	158.988603	159.054978
7.	555.165299	179.989409	180.816268
8.	713.078941	189.648143	191.426130
9.	890.731904	208.896504	208.760772
10.	1088.123960	221.642749	223.987379
11.	1305.255216	245.883597	245.783713
12.	1542.125831	254.400993	256.571596
13.	1798.735501	286.765743	287.653601
14.	2075.084372	320.431134	321.404526
15.	3271.172457	331.868921	332.319923

Table 4.1.10. – Frequency for Clamped-Clamped Boundary
h/l = 0.20

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	22.373297	16.407880	15.785280
2.	61.672828	35.894258	34.648472
3.	120.903395	58.553581	57.167857
4.	199.859462	82.374917	81.513956
5.	298.555547	107.100666	107.048115
6.	416.990857	118.014358	119.191096
7.	555.165299	133.370476	133.315527
8.	713.078941	142.742361	144.431629
9.	890.731904	160.058564	159.540759
10.	1088.123960	173.108870	176.209726
11.	1305.255216	187.710479	185.976284
12.	1542.125831	203.860504	208.444427
13.	1798.735501	218.024769	215.230571
14.	2075.084372	232.252816	235.588350
15.	2371.172457	252.310732	251.420492

Table 4.1.11. – Clamped-Clamped Boundary
h/l = 0.25

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	22.373297	14.976320	14.292820
2.	61.672828	31.641403	30.524868
3.	120.903395	50.985557	49.966236
4.	199.859462	70.750732	70.717599
5.	298.555547	85.716420	85.866655
6.	416.990857	91.375364	92.326581
7.	555.165299	109.269977	109.224787
8.	713.078941	113.070077	114.038275
9.	890.731904	135.048358	135.656898
10.	1088.123960	136.914802	137.652847
11.	1305.255216	157.239830	157.191935
12.	1542.125831	165.812048	168.280055
13.	1798.735501	180.229248	178.865275
14.	2075.084372	193.086081	197.396208
15.	2371.172457	206.018181	203.368924

Table 4.1.12. – Clamped-Clamped Boundary
h/l = 0.30

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	9.869604	9.827690	9.828139
2.	39.478417	38.823337	38.829921
3.	88.826439	85.630770	85.661940
4.	157.913670	148.294538	148.384700
5.	246.740110	224.581183	224.779527
6.	355.305758	312.252815	312.619202
7.	483.610615	409.246589	409.846894
8.	631.654681	513.762689	514.664643
9.	799.437956	624.286616	625.556629
10.	986.960440	739.573122	741.274335
11.	1194.222132	858.613133	860.804299
12.	1421.223033	980.595924	983.330922
13.	1667.963144	1104.873264	1108.200891
14.	1934.442462	1230.927916	1234.892100
15.	2220.660990	1358.347575	1362.987621

Table 4.1.13. – Nondimensional Frequency of Hinged-Hinged Boundary
h/l = 0.05

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	9.869604	9.705839	9.707479
2.	39.478417	37.073638	37.096174
3.	88.826439	78.063214	78.154816
4.	157.913670	128.440686	128.666164
5.	246.740110	184.813293	185.318586
6.	355.305758	245.148986	245.832732
7.	483.610615	307.731976	308.723039
8.	631.654681	371.700896	373.038609
9.	799.437956	436.457365	438.172796
10.	986.960440	501.620898	503.738244
11.	1194.222132	566.950047	569.487846
12.	1421.223033	632.292668	635.264964
13.	1667.963144	674.745939	679.370117
14.	1934.442462	686.119255	690.715714
15.	2220.660990	697.554547	700.971771

Table 4.1.14 – Nondimensional Frequency for Hinged-Hinged Boundary
h/l = 0.1

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	9.869604	9.514539	9.517990
2,	39.478417	34.694765	34.735480
3.	88.826439	69.365193	69.506294
4.	157.913670	108.955104	109.258997
5.	246.740110	150.927516	151.443067
6.	355.305758	193.981051	194.743478
7.	483.610615	237.485981	238.489553
8.	631.654681	281.018981	282.339988
9.	799.437956	299.887446	301.944009
10.	986.960440	311.073117	313.096288
11.	1194.222132	324.505103	326.124071
12.	1421.223033	341.228994	343.170632
13.	1667.963144	367.837624	369.761058
14.	1934.442462	384.017290	385.870569
15.	2220.660990	410.886273	413.217894

Table 4.1.15. – Nondimensional Frequency for Hinged-Hinged Boundary
h/l = 0.15

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	9.869604	9.268420	9.274060
2.	39.478417	32.110184	32.166540
3.	88.826439	61.287246	61.458186
4.	157.913670	92.925238	93.259667
5.	246.740110	125.405230	125.934564
6.	355.305758	158.073165	158.816250
7.	483.610615	168.686521	169.842523
8.	631.654681	179.625175	180.749312
9.	799.437956	190.669806	191.637245
10.	986.960440	207.390924	208.449778
11.	1194.222132	223.101294	224.298636
12.	1421.223033	244.480787	245.475742
13.	1667.963144	255.344154	256.773819
14.	1934.442462	286.654440	289.066525
15.	2220.660990	331.891948	332.767481

Table 4.1.16. – Nondimensional Frequency for Hinged-Hinged Boundary
h/l = 0.20

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	9.869604	8.983260	8.991190
2.	39.478417	29.582932	29.650992
3.	88.826439	54.833921	54.519521
4.	157.913670	80.259361	80.598137
5.	246.740110	106.390074	106.901058
6.	355.305758	107.925745	108.699215
7.	483.610615	118.609355	119.318849
8.	631.654681	132.421549	133.113977
9.	799.437956	144.069224	144.726077
10.	986.960440	158.272694	159.150438
11.	1194.222132	176.491543	177.099102
12.	1421.223033	183.938519	185.002593
13.	1667.963144	209.437964	210.687959
14.	1934.442462	212.410836	212.971179
15.	2220.660990	234.795272	236.230110

Table 4.1.17. – Nondimensional Frequency for Hinged-Hinged Boundary
h/l = 0.25

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	9.869604	8.673699	8.633879
2.	39.478417	27.238777	27.314758
3.	88.826439	48.495264	48.685866
4.	157.913670	70.254757	70.585005
5.	246.740110	74.971814	75.485570
6.	355.305758	85.307264	85.792655
7.	483.610615	91.959417	92.440261
8.	631.654681	108.658131	109.100344
9.	799.437956	113.486293	114.121707
10.	986.960440	134.831196	135.621844
11.	1194.222132	137.319748	137.721863
12.	1421.223033	156.019365	156.964541
13.	1667.963144	168.513299	168.877147
14.	1934.442462	177.079321	178.177858
15.	2220.660990	198.036286	199.286717

Table 4.1.18. – Nondimensional Frequency for Hinged-Hinged Boundary
h/l = 0.30

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	0.00	0.00	0.00
2.	22.373297	22.170938	22.172434
3.	61.672828	60.113529	60.125322
4.	120.903395	115.656884	115.202175
5.	199.859462	184.950533	185.067996
6.	298.555547	267.244398	267.485876
7.	416.990857	359.863342	360.289535
8.	555.165299	460.860016	461.530231
9.	713.078941	568.565056	569.556993
10.	890.731904	681.587627	682.959303
11.	1088.123960	798.789163	800.611156
12.	1305.255216	919.246594	921.555568
13.	1542.125831	1042.214740	1045.072124
14.	1798.735501	1167.091768	1170.544645
15.	2075.084372	1293.389465	1297.480291

Table 4.1.19. – Nondimensional Frequency for Free-Free Boundary
h/l = 0.05

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	0.00	0.00	0.00
2.	22.373297	21.604220	21.608600
3.	61.672828	56.171242	56.207886
4.	120.903395	102.404394	102.525746
5.	199.859462	156.171718	156.441222
6.	298.555542	214.677595	215.156834
7.	416.990857	276.005851	276.749003
8.	555.165299	338.888410	339.491293
9.	713,078941	402.446854	403.847106
10.	890.731904	466.005742	467.785777
11.	1088.123960	528.892980	531.084634
12.	1305.255216	590.140730	592.786401
13.	1542.125831	647.043177	650.283746
14.	1798.735501	687.080469	691.549983
15.	2075.084372	693.951293	698.565683

Table 4.1.20. – Nondimensional Frequency for Free-Free Boundary
h/l = 0.1

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	0.00	0.00	0.00
2.	22.373297	20.771923	20.780200
3.	61.672828	51.213839	51.274402
4.	120.903950	88.790117	88.463146
5.	199.859462	129.534838	129.877377
6.	298.555542	171.517147	172.073351
7.	416.990857	213.387238	214.194684
8.	555.165299	254.009611	255.102631
9.	713.078941	289.440447	290.967057
10.	890.731904	313.474731	315.371128
11.	1088.123960	318.271238	320.091639
12.	1305.255216	352.241051	354.058356
13.	1542.125831	355.763481	357.599139
14.	1798.735501	444.888836	447.057131
15.	2075.084372	449.337744	451.527902

Table 4.1.21. – Nondimensional Frequency for Free-Free Boundary
h/l = 0.15

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	0.00	0.00	0.00
2.	22.373297	19.786681	19.798800
3.	61.672828	46.197146	46.275188
4.	120.903395	76.765118	76.963339
5.	199.859462	107.988449	108.355773
6.	298.555542	138.536673	139.102531
7.	416.990857	163.362340	164.258756
8.	555.165299	182.595624	183.611543
9.	713.073190	185.591420	186.649439
10.	890.731904	215.097974	216.136353
11.	1088.123960	217.248974	218.297917
12.	1305.255216	250.238619	255.632412
13.	1542.125831	290.741005	291.246119
14,	1798,735501	323.558417	325.167980
15.	2075.084372	326.794001	328.419.860

Table 4.1.22. – Nondimensional Frequency for Free-Free Boundary
h/l = 0.20

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	0.00	0.00	0.00
2.	22.373297	18.741940	18.757402
3.	61.672828	41.529019	41.618374
4.	120.903395	66.642209	66.849357
5.	199.859462	89.986151	90.372651
6.	298.555547	111.107633	111.660977
7.	416.990857	115.326809	116.041987
8.	555.165299	165.595157	166.449007
9.	713.078941	167.251129	168.133697
10.	890.731904	194.309280	195.364834
11.	1088.123960	196.935303	197.786082
12.	1305.255216	224.259416	225.318713
13.	1542.125831	227.252270	228.272131
14.	1798.735501	255.330513	256.430452
15.	2075.084372	257.883838	258.994495

Table 4.1.23. – Nondimensional Frequency for Free-Free Boundary
h/l = 0.25

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	0.00	0.00	0.00
2.	22.373297	17.701481	17.719581
3.	61.672828	37.296734	37.395594
4.	120.903395	58.117882	58.324329
5.	199.859462	72.809020	73.223973
6.	298.555547	110.998123	111.795012
7.	416.990857	111.454211	112.913163
8.	555.165299	160.112011	160.905894
9.	713.078941	161.941931	162.617353
10.	890.731904	184.908690	185.799727
11.	1088.123960	188.027197	188.827324
12.	1305.255216	210.562569	211.595597
13.	1542.125831	213.524476	214.506753
14.	1798.735501	237.276420	238.220021
15.	2075.084372	239.649205	240.604441

Table 4.1.24. – Nondimensional Frequency for Free-Free Boundary
h/l = 0.30

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	15.418211	15.263940	15.246640
2.	49.964865	48.632879	48.541586
3.	104.247693	99.074888	98.831502
4.	178.269735	164.492597	164.021657
5.	272.030990	242.635723	241.883814
6.	385.531418	331.340940	330.289412
7.	518.771093	428.669989	427.340566
8.	671.749929	532.964609	531.418851
9.	844.468023	642.847635	641.181260
10.	1036.925331	757.195792	755.529793
11.	1249.121782	875.101870	873.572491
12.	1481.057514	995.836488	994.585515
13.	1732.732450	1118.813593	1117.980611
14.	2004.146605	1243.561729	1243.278370
15.	2295.299876	1369.700573	1370.087210

Table 4.1.25. – Nondimensional Frequency for Clamped-Hinged Boundary
h/l = 0.05

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	15.418211	14.829400	14.765880
2.	49.964865	45.254114	44.960679
3.	104.247693	87.576495	86.921465
4.	178.269735	137.954880	136.934460
5.	272.030990	193.606149	192.346925
6.	385.531418	252.682570	251.388214
7.	518.771093	313.981836	312.871156
8.	671.749929	376.712394	375.979348
9.	844.468023	440.341298	440.142201
10.	1036.925331	504.341298	504.960223
11.	1249.121782	568.922716	570.155525
12.	1481.057514	633.365643	635.533461
13.	1732.732450	674.733939	679.369455
14.	2004.146605	681.481299	686.163170
15.	2295.299876	695.919240	700.660698

Table 4.1.26. – Nondimensional Frequency for Clamped-Hinged Boundary
h/l = 0.1

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	15.418211	14.185620	14.060440
2.	49.964865	41.012816	40.532044
3.	104.247693	75.423984	74.572365
4.	178.269735	113.921244	112.916308
5.	272.030990	154.636636	153.764551
6.	385.531418	196.544143	196.047837
7.	518.771093	239.056294	234.102235
8.	671.749929	281.728297	282.518398
9.	844.468023	299.881760	301.941982
10.	1036.925331	302.880598	304.961421
11.	1249.121782	321.815164	325.043296
12.	1481.057514	327.054195	328.293749
13.	1732.732450	359.776277	362.932786
14.	2004.146605	369.582160	370.147194
15.	2295.299876	408.560361	410.370868

Table 4.1.27. – Nondimensional Frequency for Clamped-Hinged Boundary
h/l = 0.15

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	15.418211	13.417600	13.228940
2.	49.964865	36.780230	36.194959
3.	104.247693	64.970752	64.167499
4.	178.269735	95.384740	94.705434
5.	272.030990	126.824307	126.539588
6.	385.531418	158.598330	158.949104
7.	518.771093	168.683493	169.842365
8.	671.749929	171.554648	172.682399
9.	844.468023	188.892695	191.477522
10.	1036.925331	193.885262	193.392308
11.	1249.121782	220.681005	257.091501
12.	1481.057514	227.609225	265.737516
13.	1732.732450	254.248698	289.107771
14.	2004.146605	266.023105	310.216149
15.	2295.299876	260.033406	261.651831

Table 4.1.28. – Nondimensional Frequency for Clamped-Hinged Boundary
h/l = 0.20

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	15.418211	12.597140	12.355210
2.	49.964865	32.954371	32.350092
3.	104.247693	56.516295	55.886563
4.	178.269735	81.386138	81.079648
5.	272.030990	110.967859	111.539287
6.	385.531418	128.757038	130.843270
7.	518.771093	133.896148	133.012953
8.	671.749929	155.952990	156.421402
9.	844.468023	162.069700	162.839250
10.	1036.925331	183.213117	185.278343
11.	1249.121782	194.548908	194.505526
12.	1481.057514	209.703097	210.605651
13.	1732.732450	229.129648	230.384568
14.	2004.146605	237.025224	238.218342
15.	2295.299876	260.033406	261.651831

Table 4.1.29. – Nondimensional Frequency for Clamped-Hinged Boundary
h/l = 0.25

Mode No.	Euler–Bernoulli Theory	Timoshenko Theory (κ = 0.822)	Modified Theory
1.	15.418211	11.778860	11.497030
2.	49.964865	29.640688	29.077310
3.	104.247693	49.747755	49.325143
4.	178.269735	70.586953	70.670284
5.	272.030990	74.970442	75.485497
6.	385.531418	77.834207	78.282592
7.	518.771093	91.230229	92.479328
8.	671.749929	96.823291	96.383150
9.	844.468023	113.051564	114.236073
10.	1036.925331	122.796540	122.896884
11.	1249.121782	135.046985	135.555463
12.	1481.057514	151.131655	154.459907
13.	1732.732450	157.725271	156.004516
14.	2004.146605	176.160524	178.831221
15.	2295.299876	184.694231	185.039934

Table 4.1.30. – Nondimensional Frequency for Clamped-Hinged Boundary
h/l = 0.30

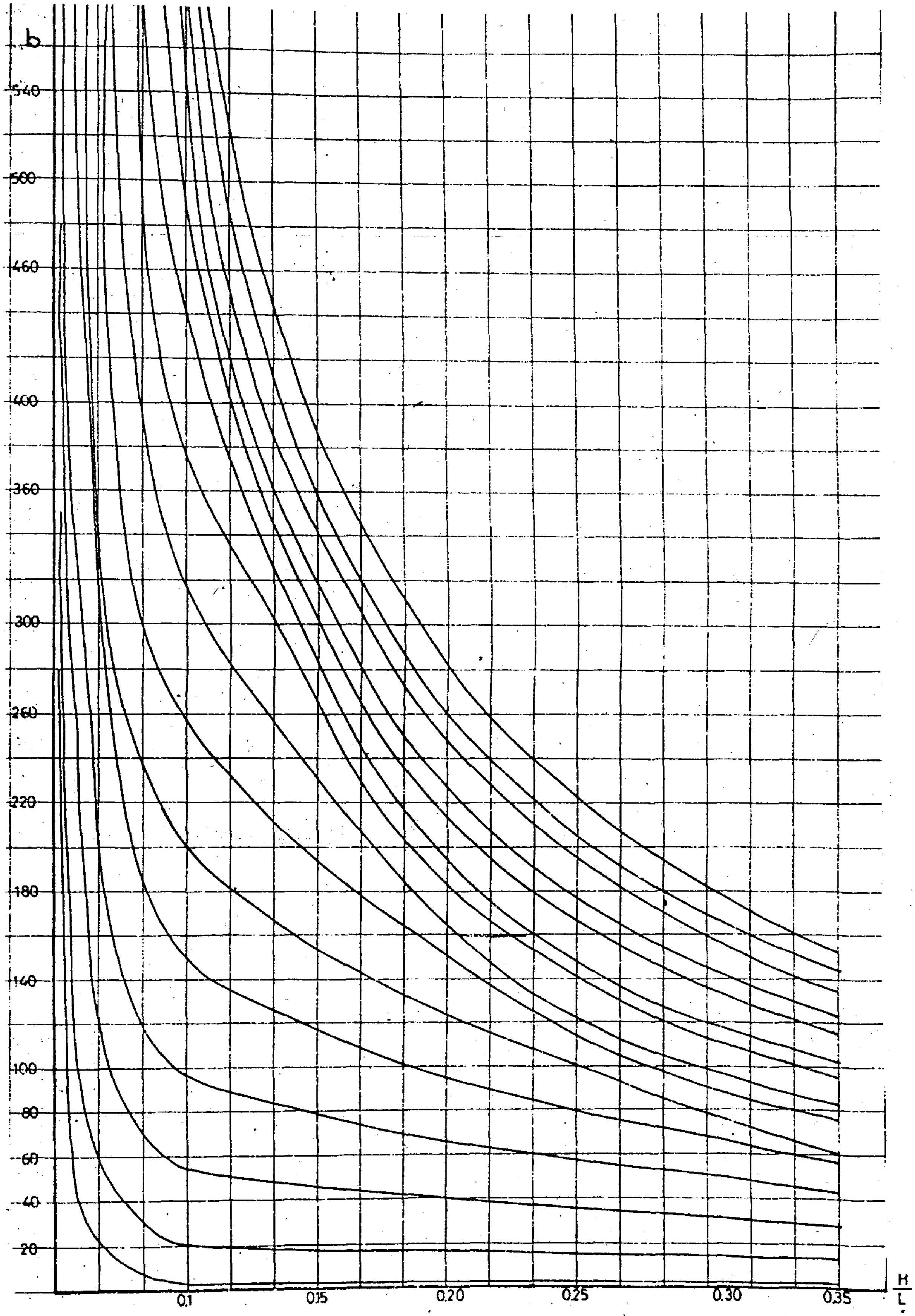


Figure 4.1.1. Frequency spectrum of a clamped-free beam according to the modified theory

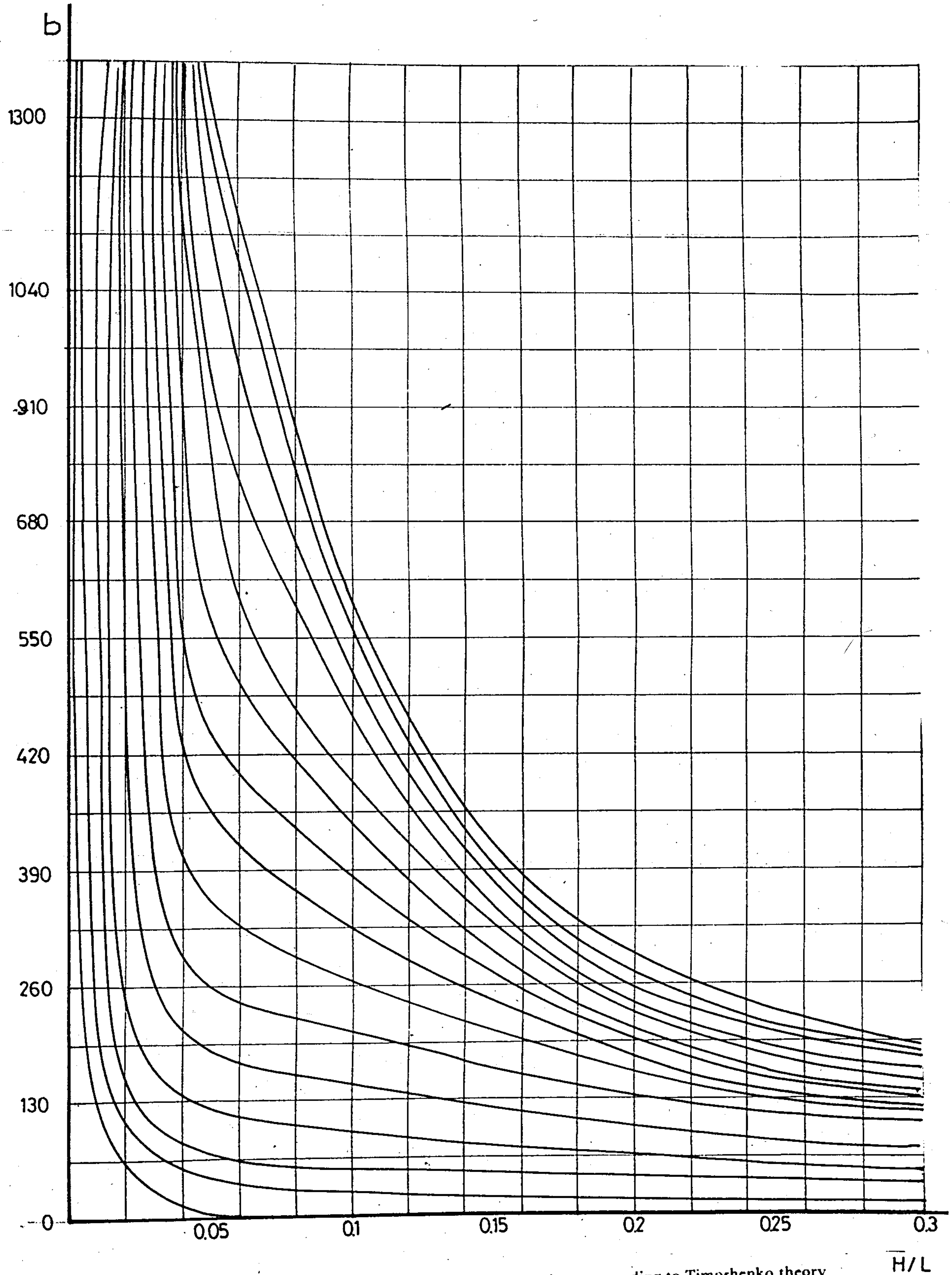


Figure 4.1.2. Frequency spectrum of a clamped-free beam according to Timoshenko theory

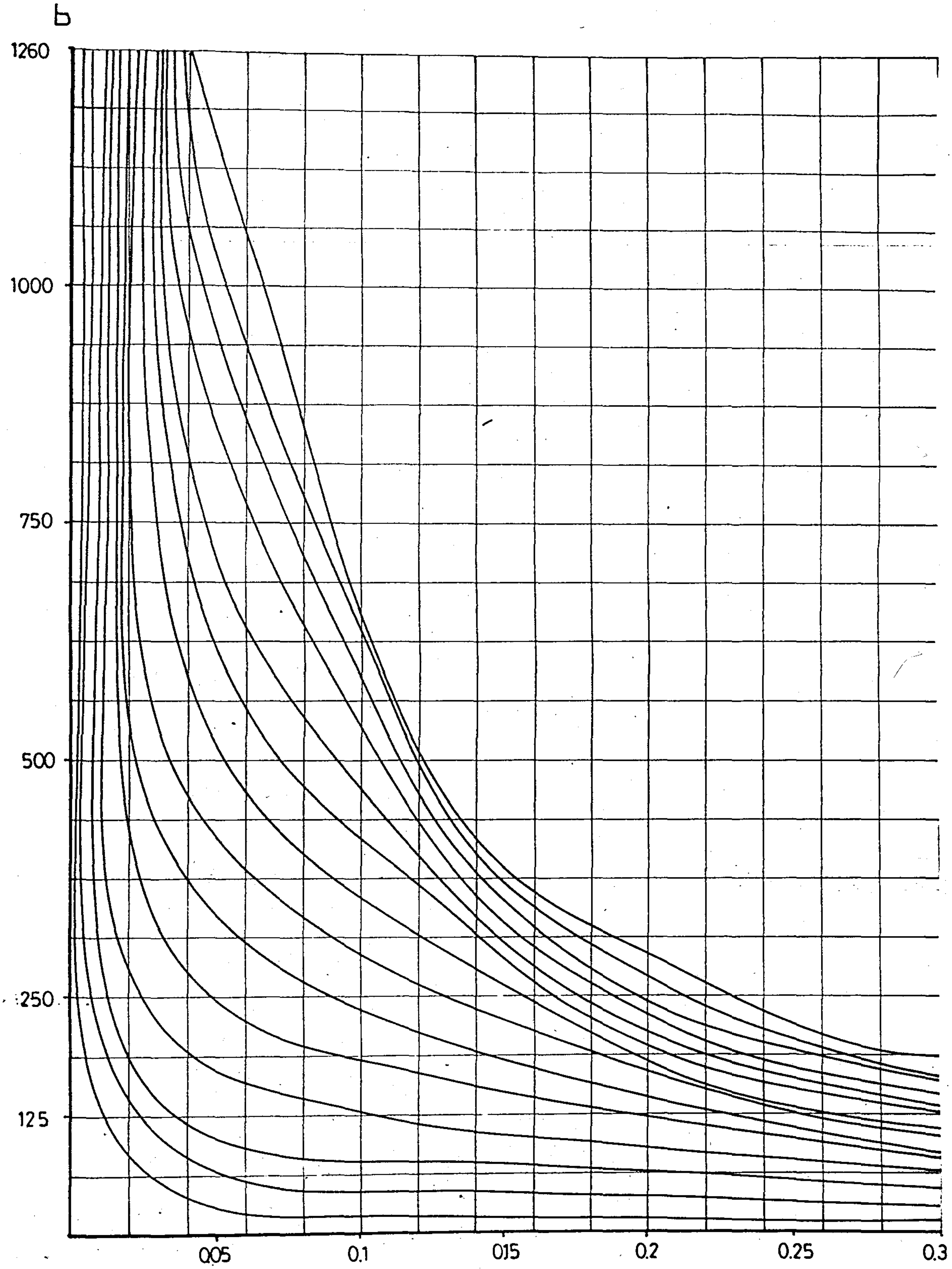


Figure 4.1.3. Frequency spectrum of a clamped-clamped beam according to the modified theory

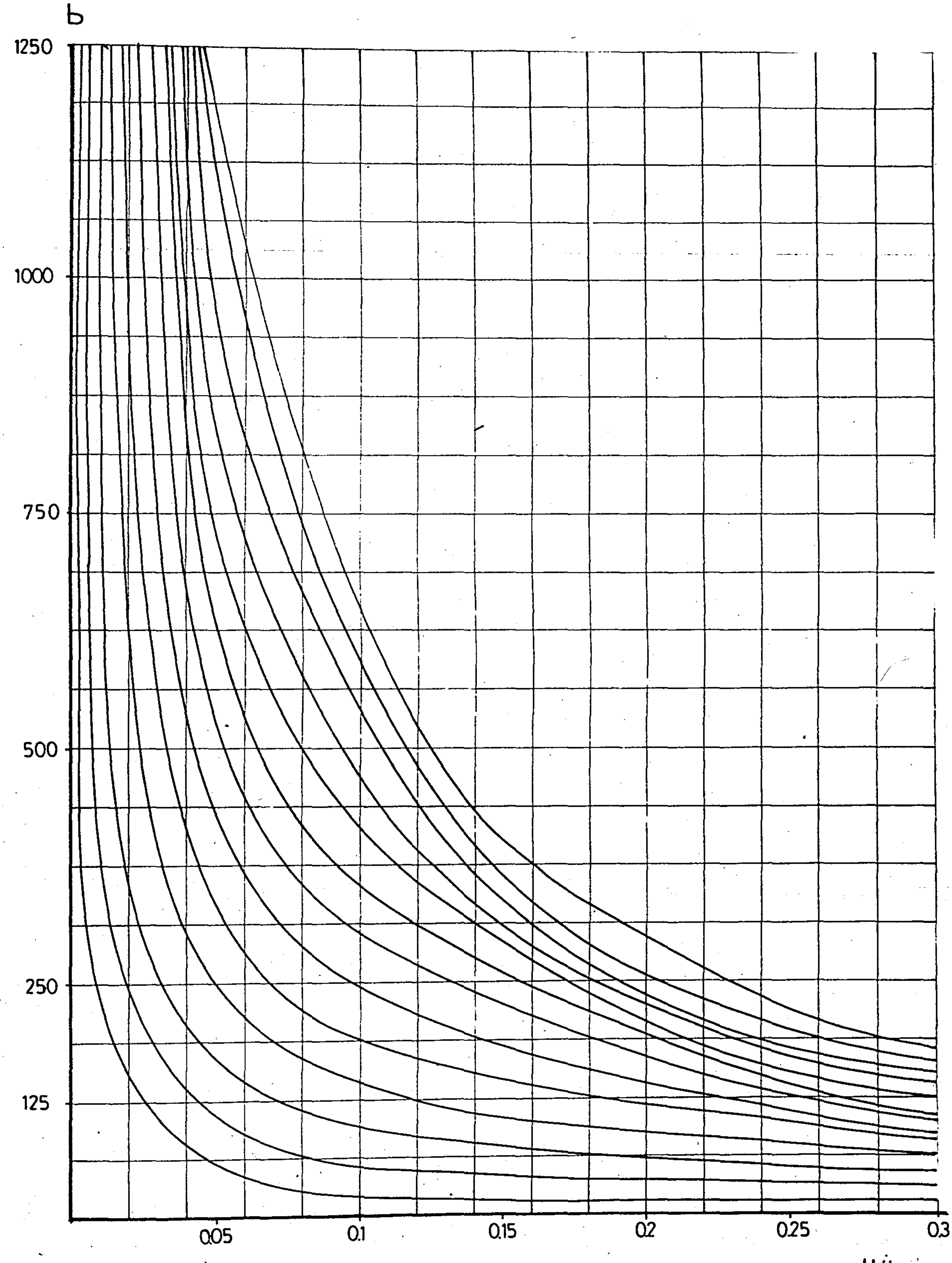


Figure 4.1.4. Frequency spectrum of a clamped-clamped beam according to Timoshenko theory

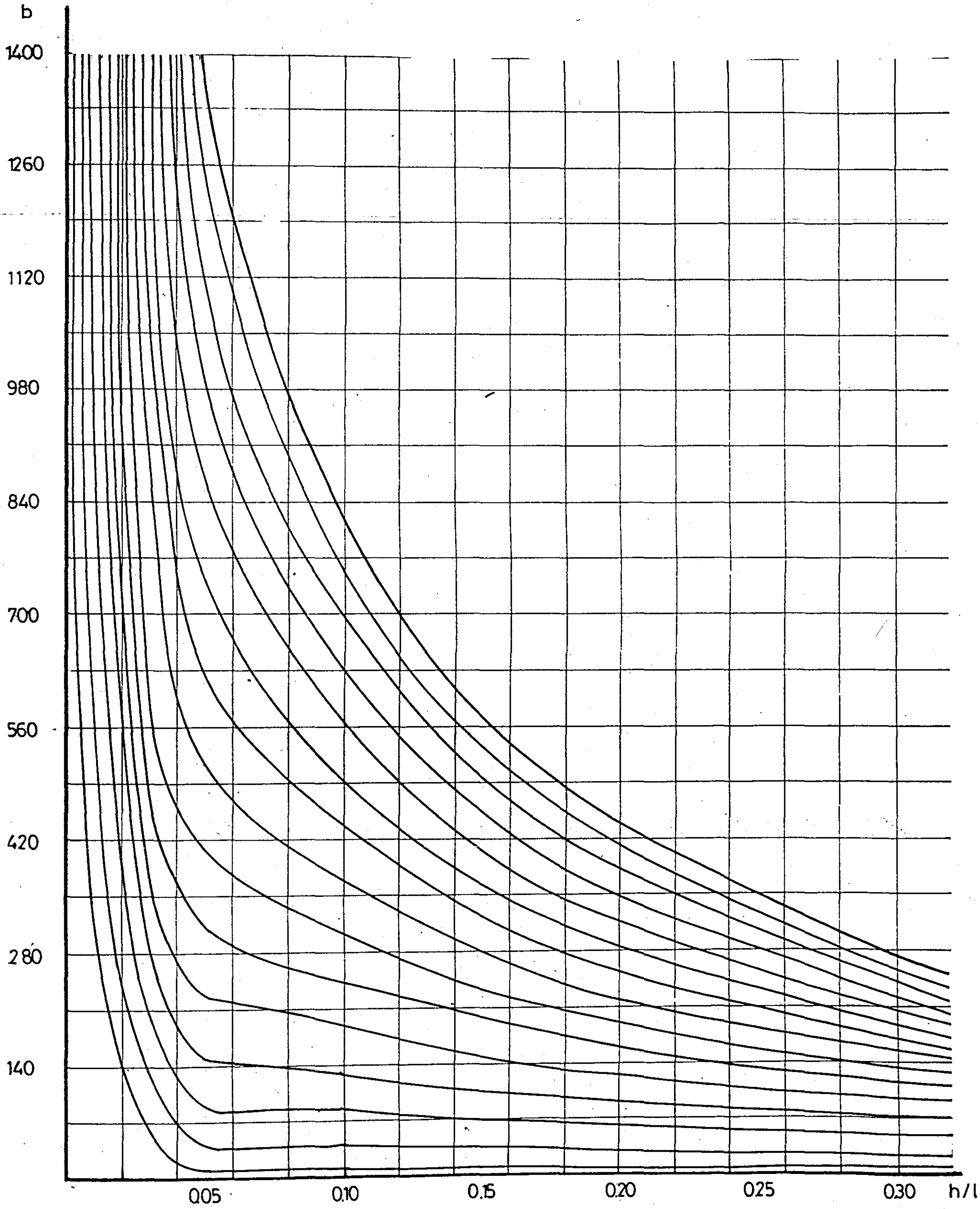


Figure 4.1.5. Frequency spectrum of a hinged-hinged beam according to the modified theory

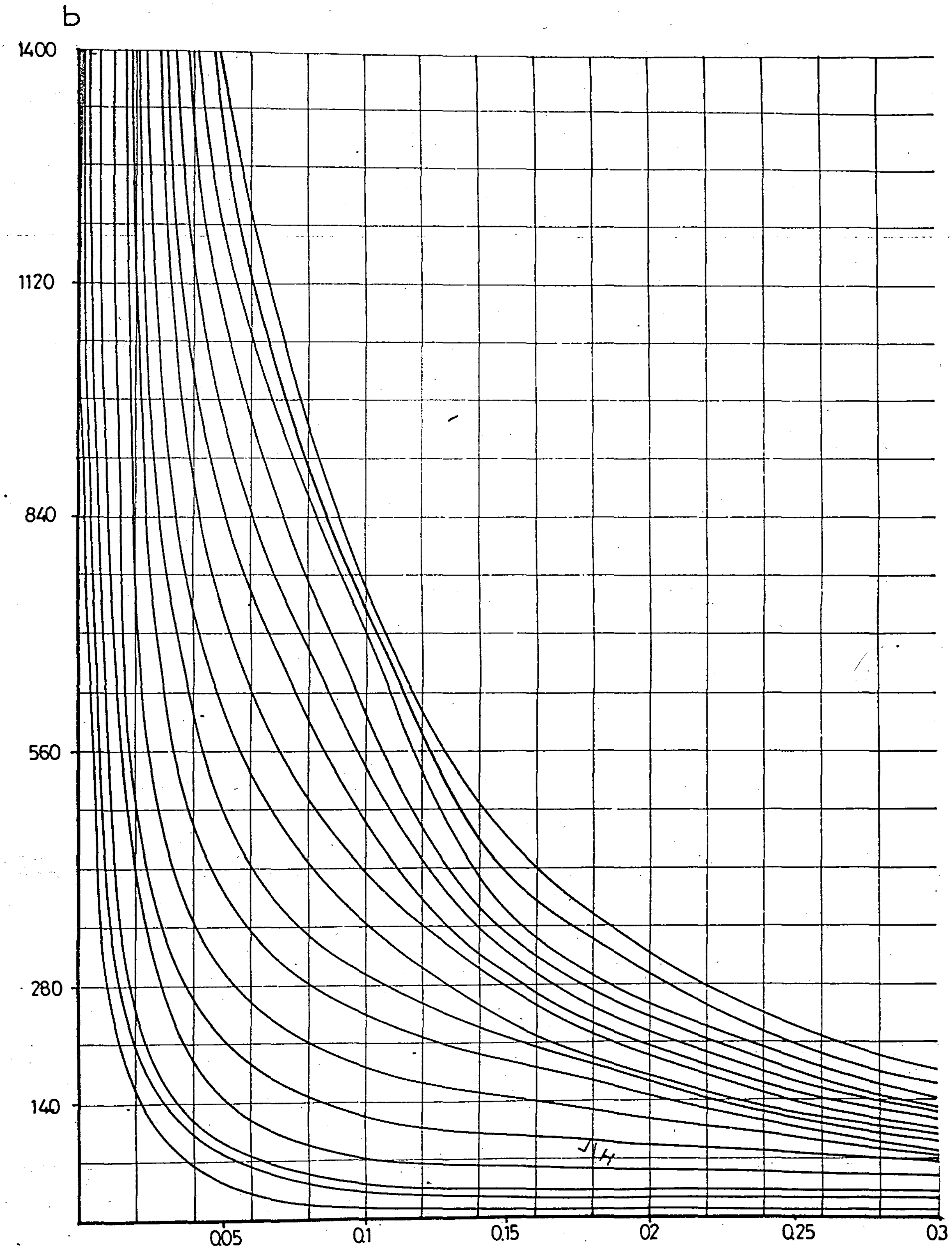


Figure 4.1.6. Frequency spectrum of a hinged-hinged beam according to Timoshenko theory

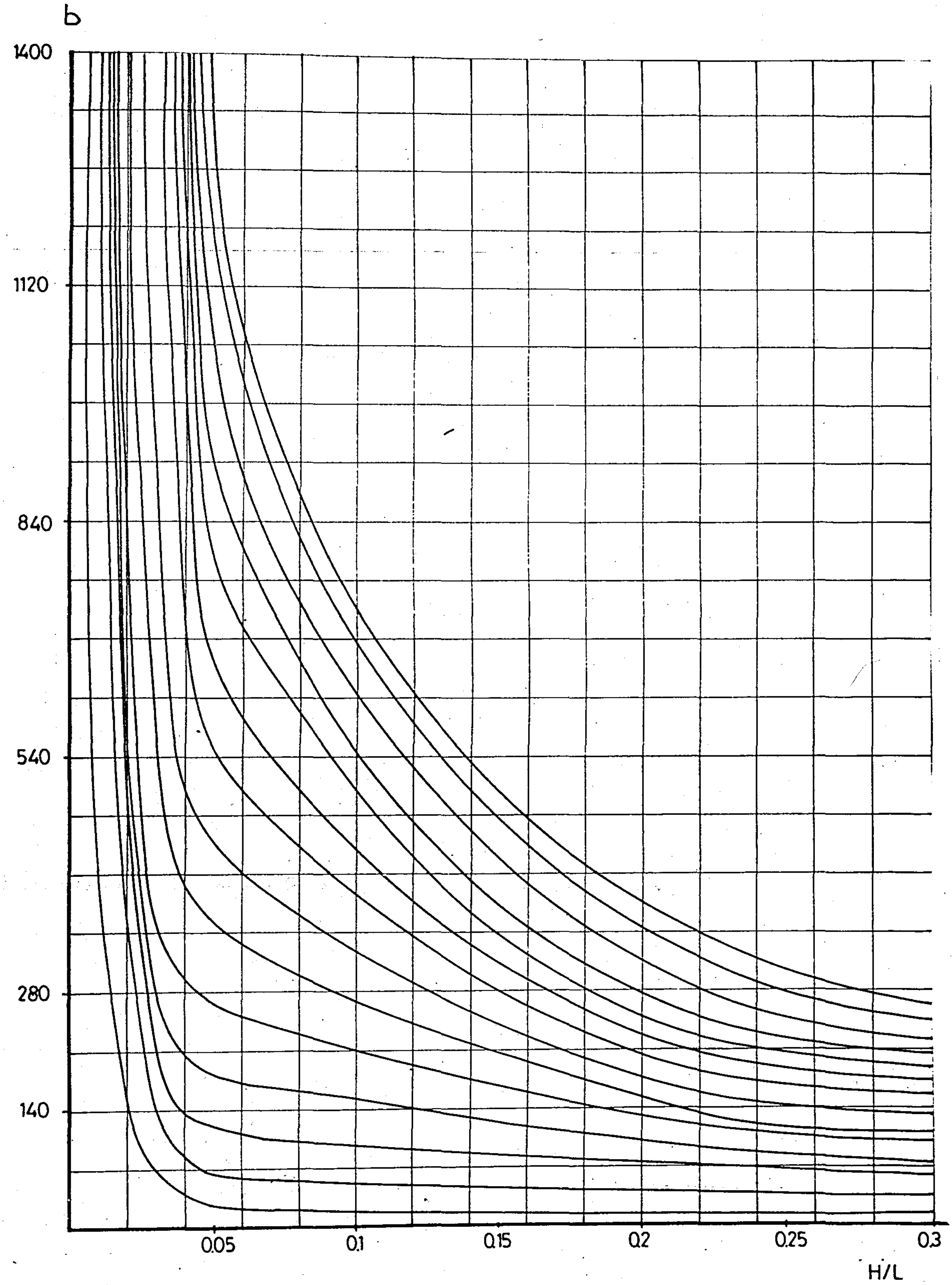


Figure 4.1.7. Frequency spectrum of a free-free beam according to the modified theory

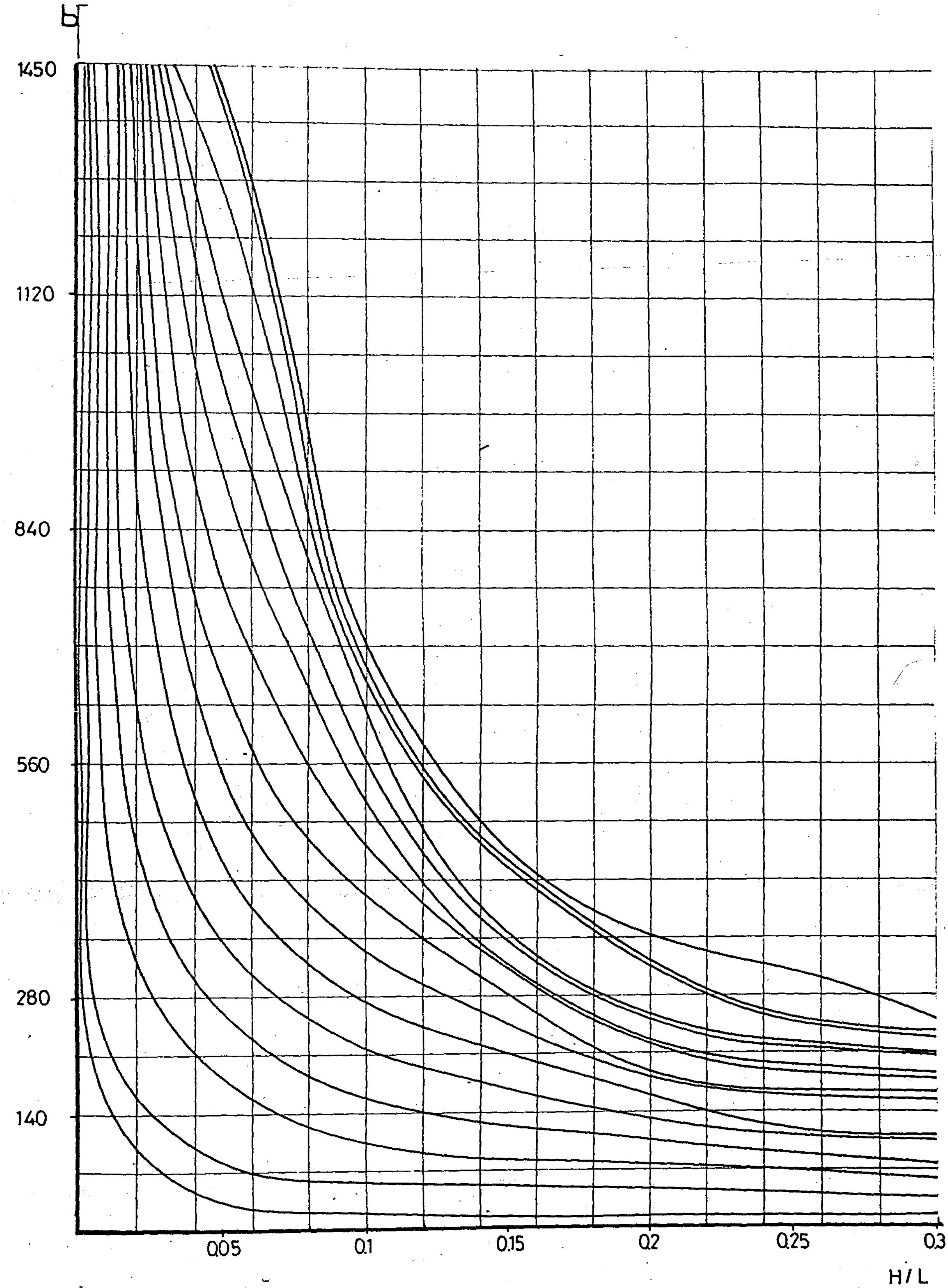


Figure 4.1.8. Frequency spectrum of a free-free beam according to Timoshenko theory

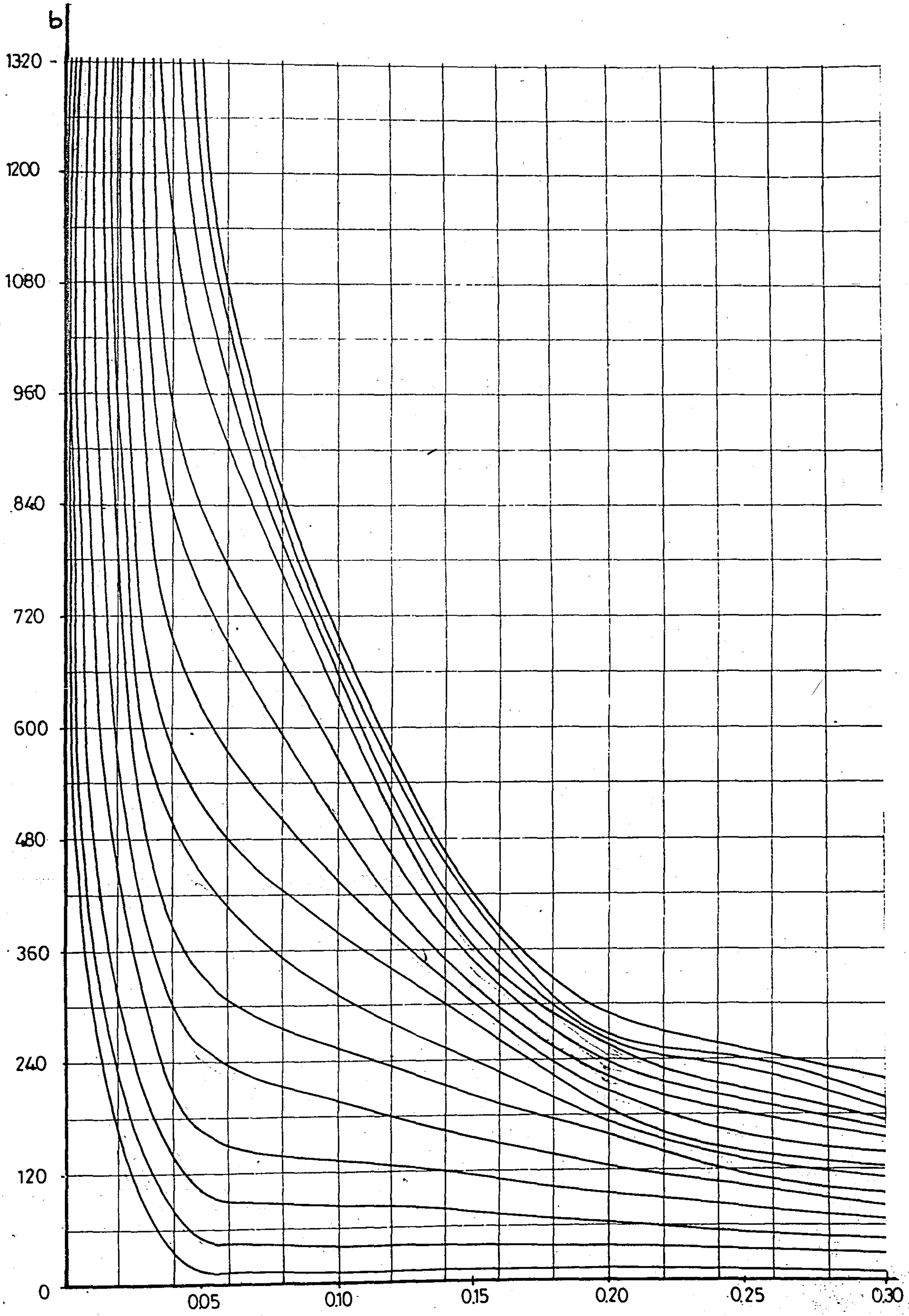


Figure 4.1.9. Frequency spectrum of a clamped-hinged beam according to the modified theory

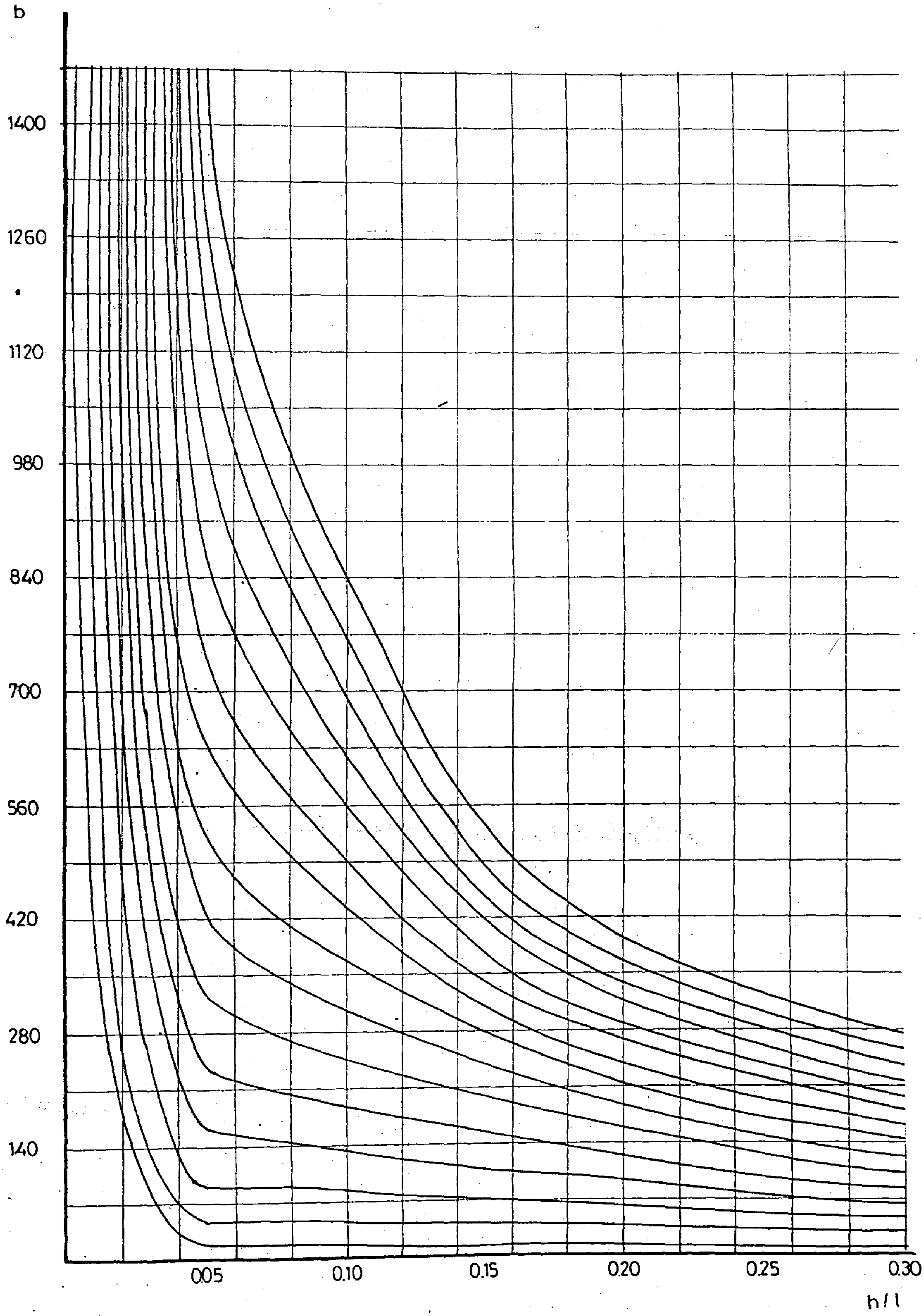


Figure 4.1.10. Frequency spectrum of a clamped-hinged beam according to Timoshenko theory

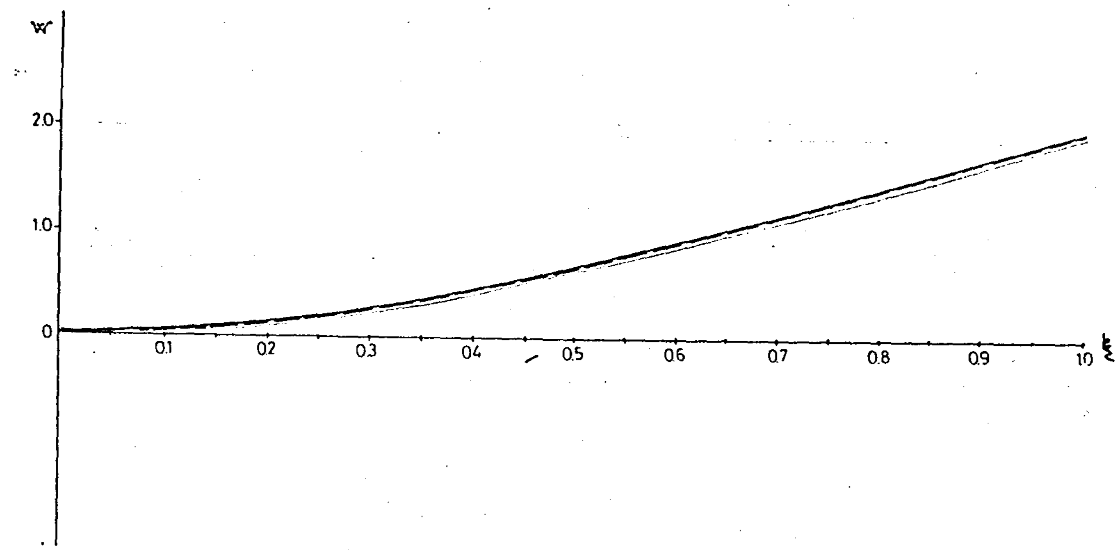


Figure 4.2.1. Deflection curve of the first mode for clamped-free beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

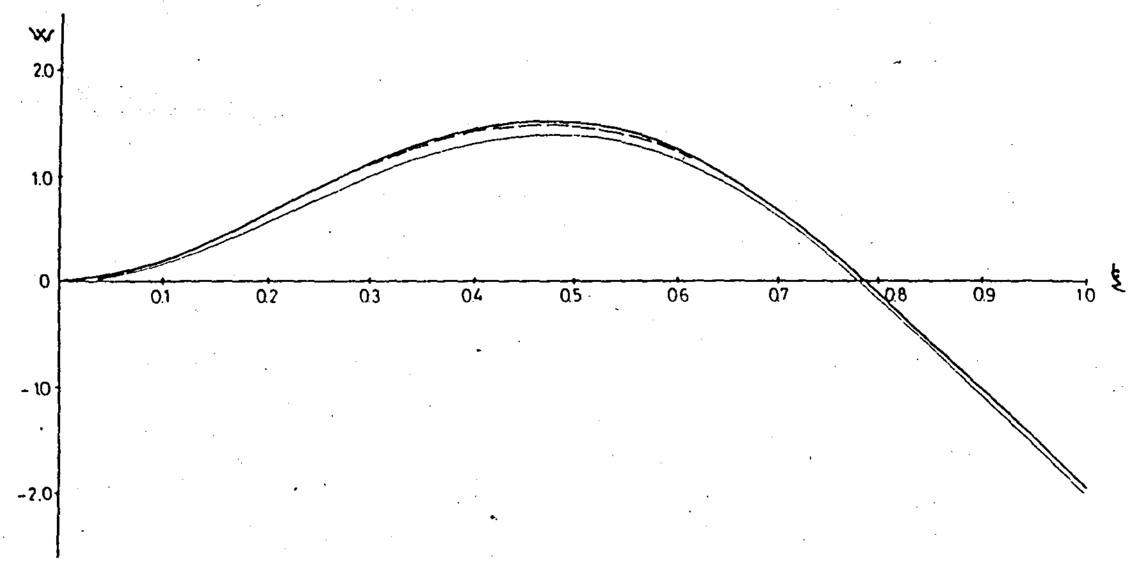


Figure 4.2.2. Deflection curve of the second mode for clamped-free beam (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

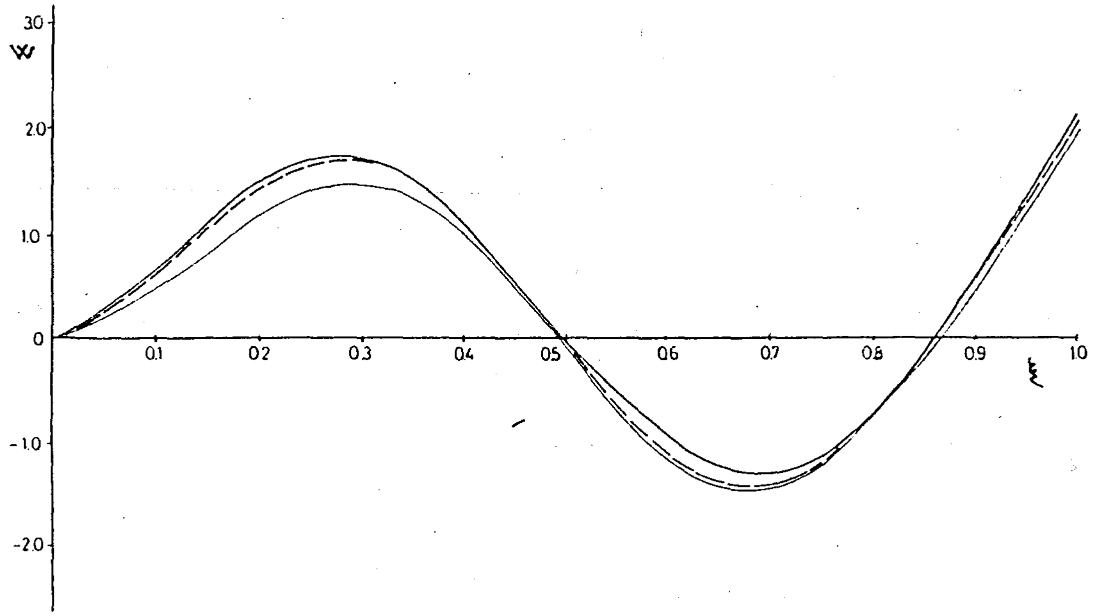


Figure 4.2.3. Deflection curve of the third mode for clamped-free beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

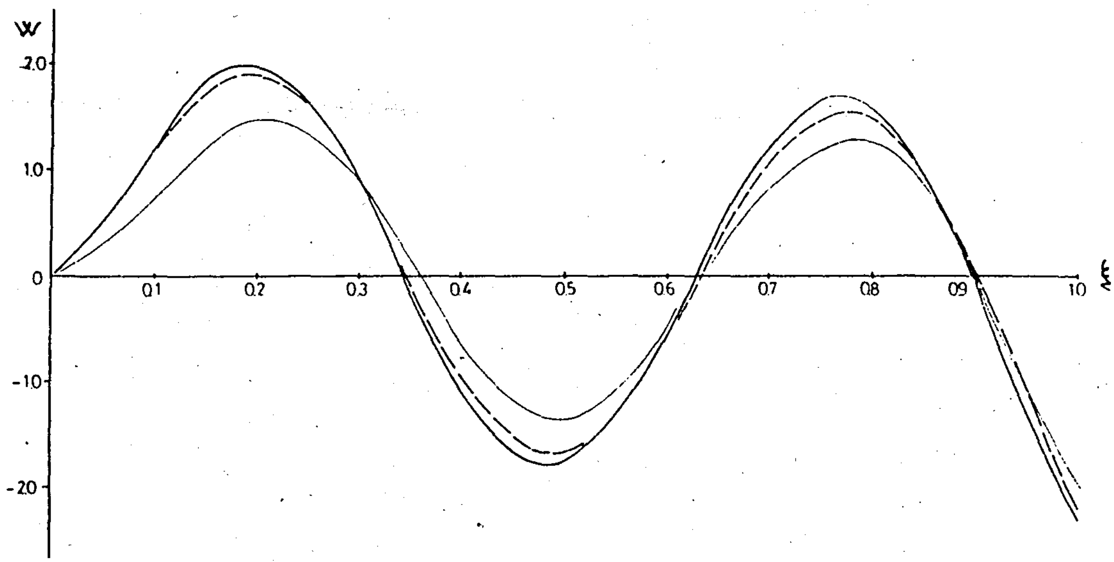


Figure 4.2.4. Deflection curve of the fourth mode for clamped-free beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

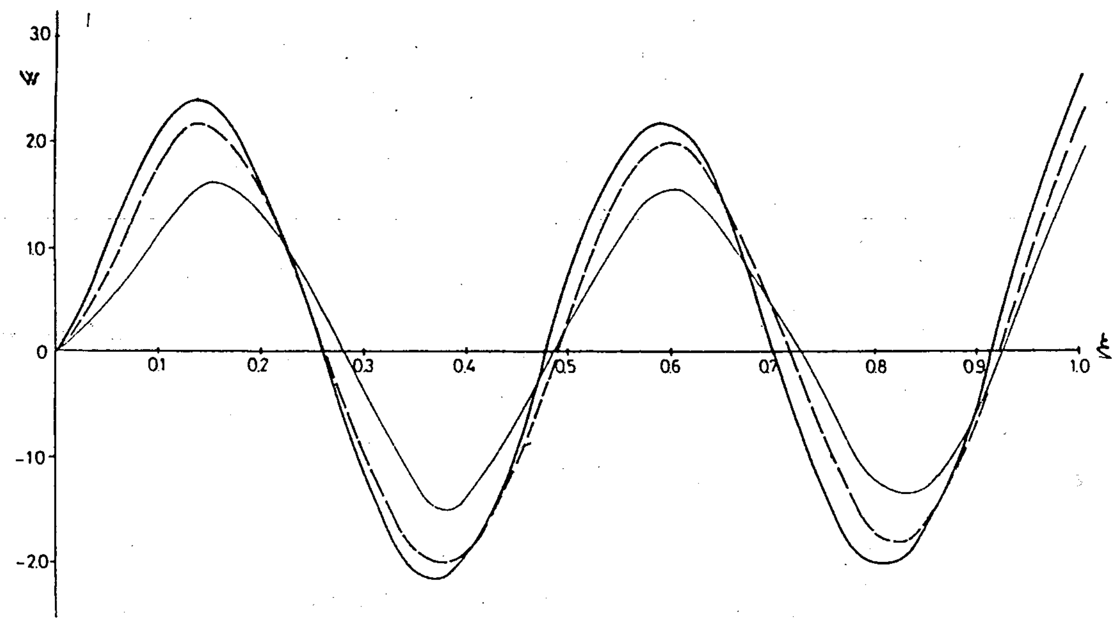


Figure 4.2.5. Deflection curve of the fifth mode for clamped-free beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

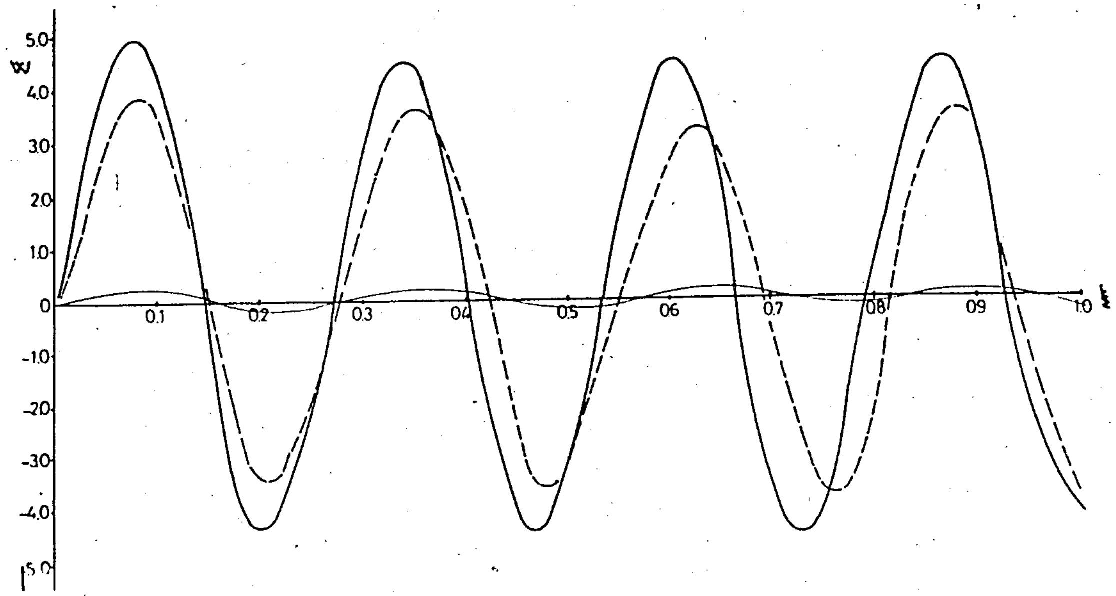


Figure 4.2.6. Deflection curve of the eight mode for clamped-free beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

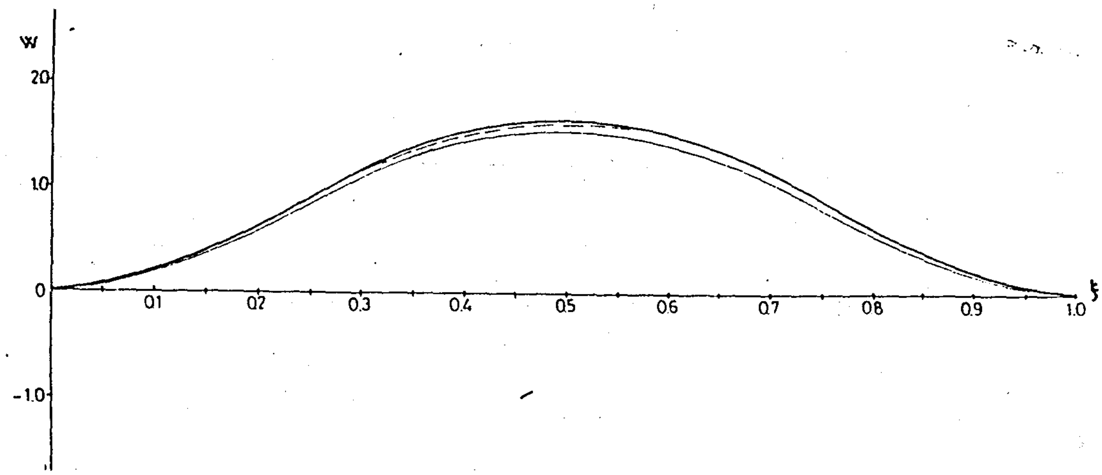


Figure 4.2.7. Deflection curve of the first mode for clamped-clamped beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

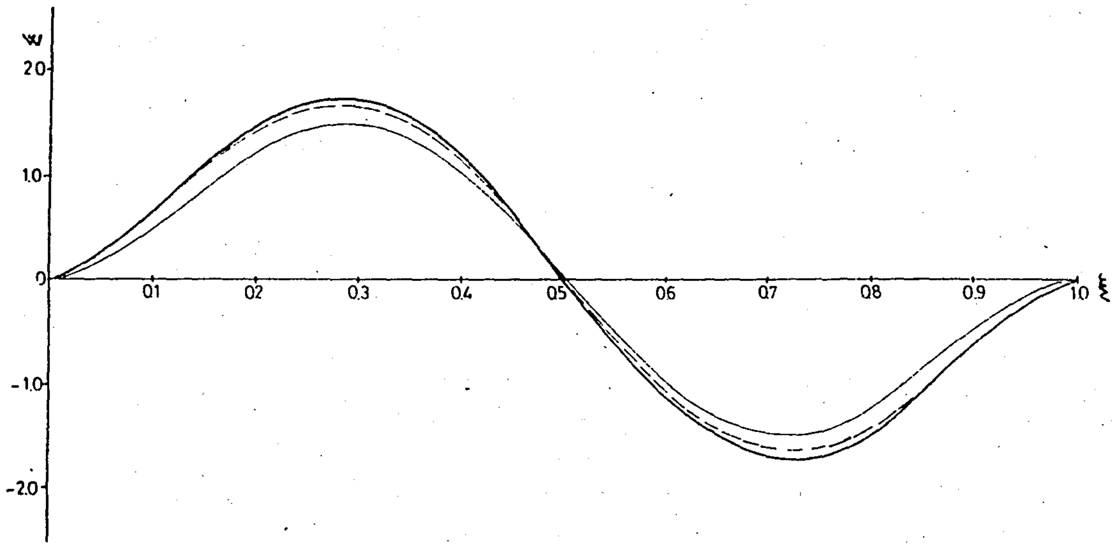


Figure 4.2.8. Deflection curve of the second mode for clamped-clamped beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

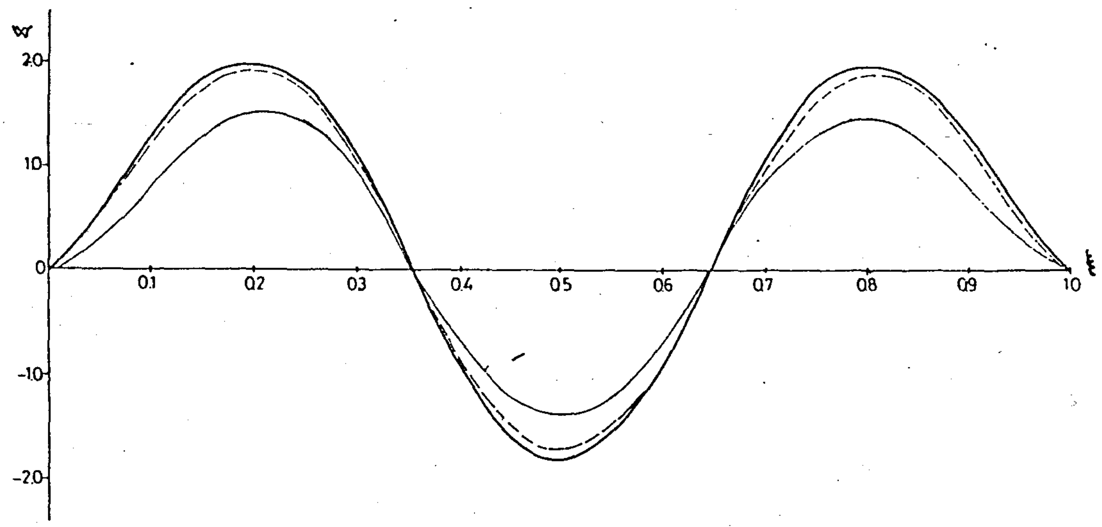


Figure 4.2.9. Deflection curve of the third mode for clamped-clamped beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

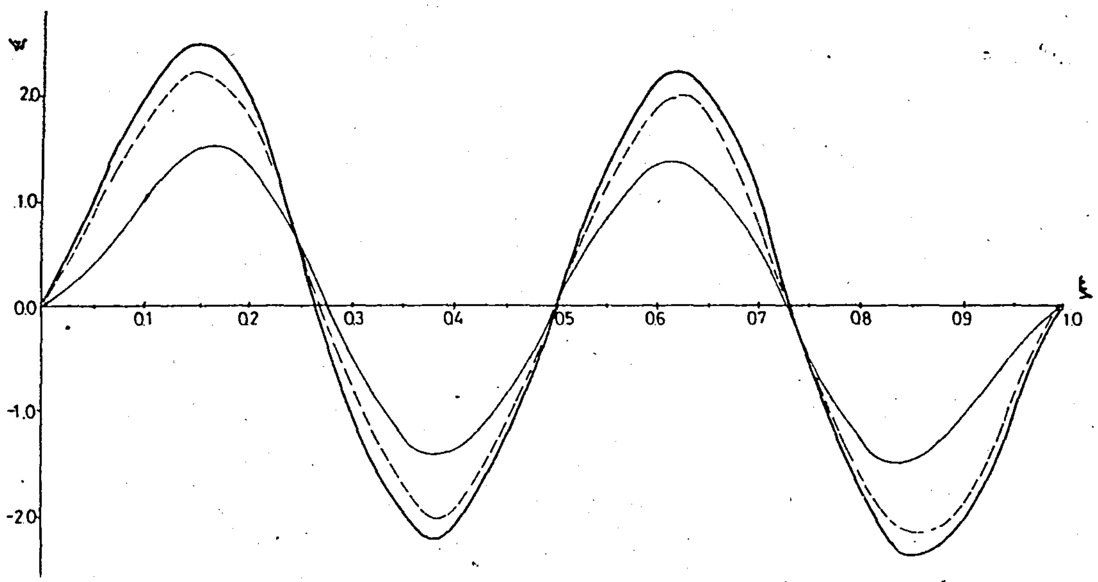


Figure 4.2.10. Deflection curve of the fourth mode for clamped-clamped beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

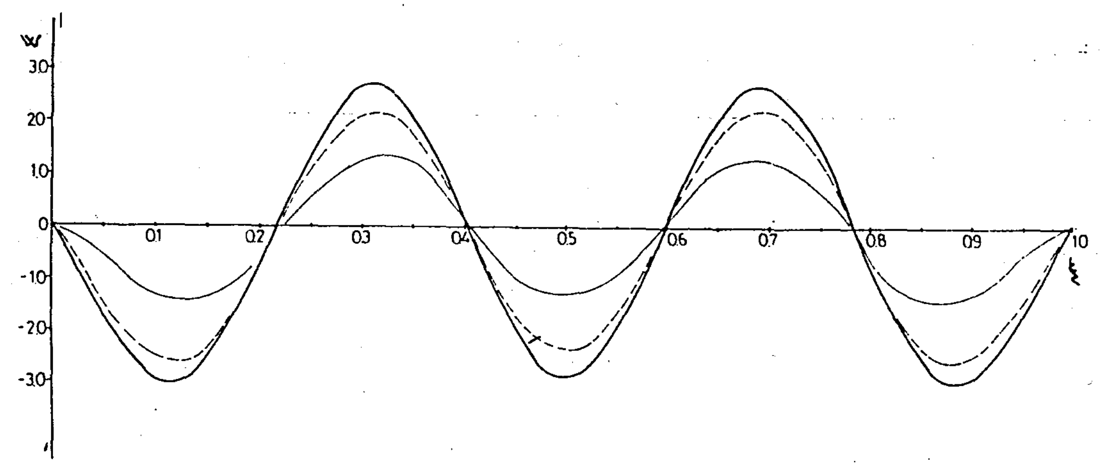


Figure 4.2.11 Deflection curve of the fifth mode for clamped-clamped beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

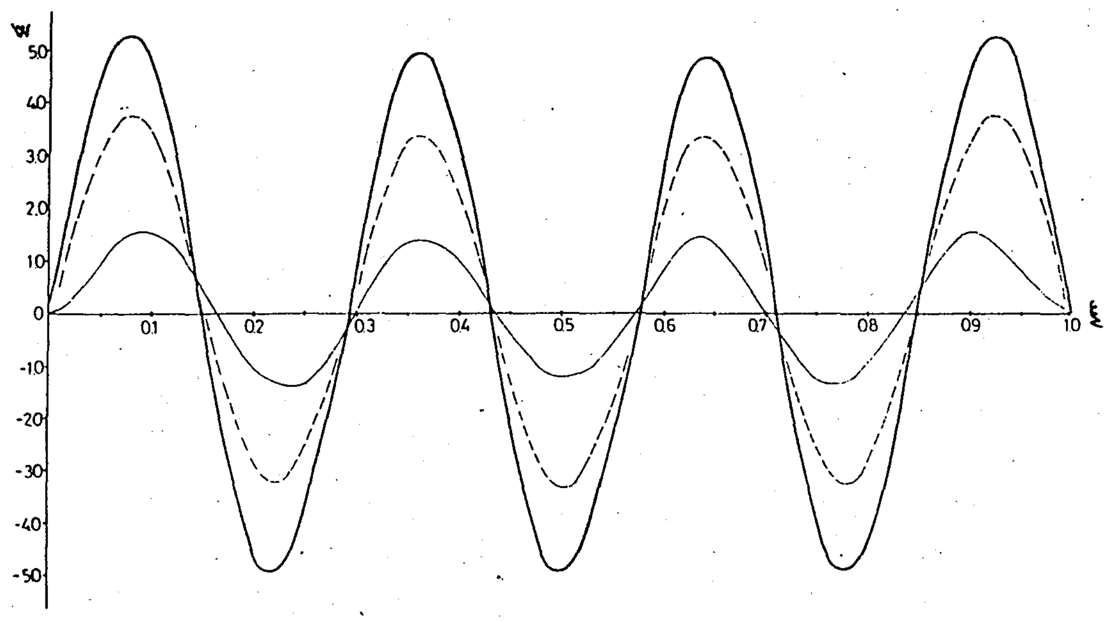


Figure 4.2.12. Deflection curve of the seventh mode for clamped-clamped beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

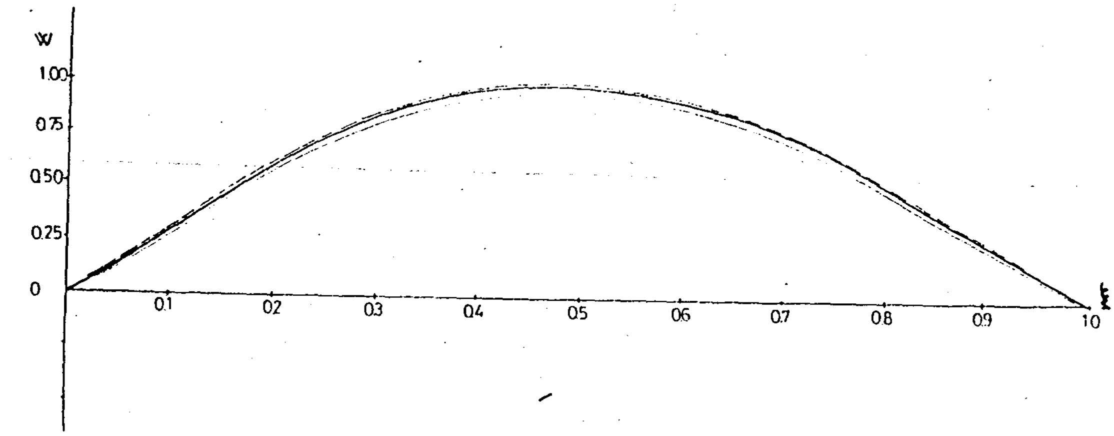


Figure 4.2.13. Deflection curve of the first mode for hinged-hinged beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

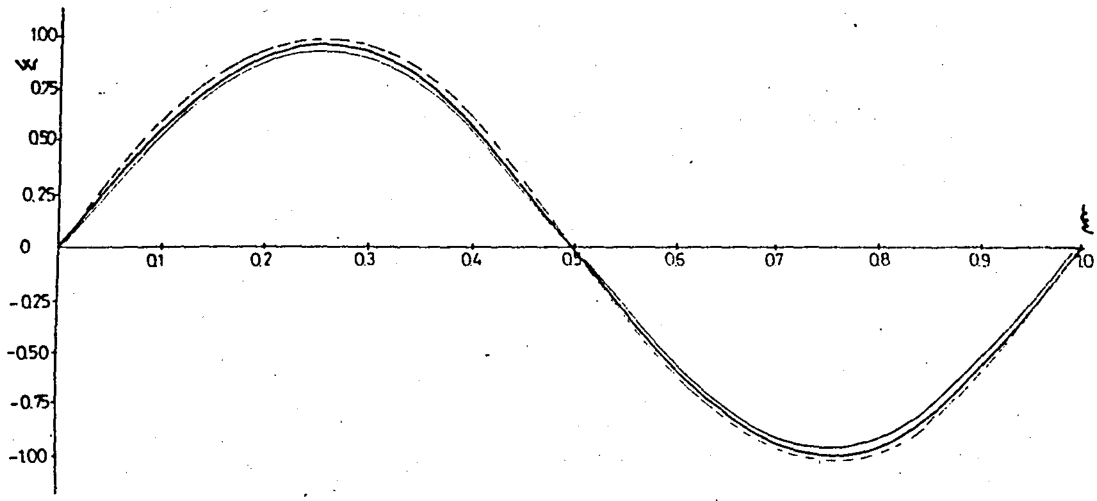


Figure 4.2.14. Deflection curve of the second mode for hinged-hinged beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

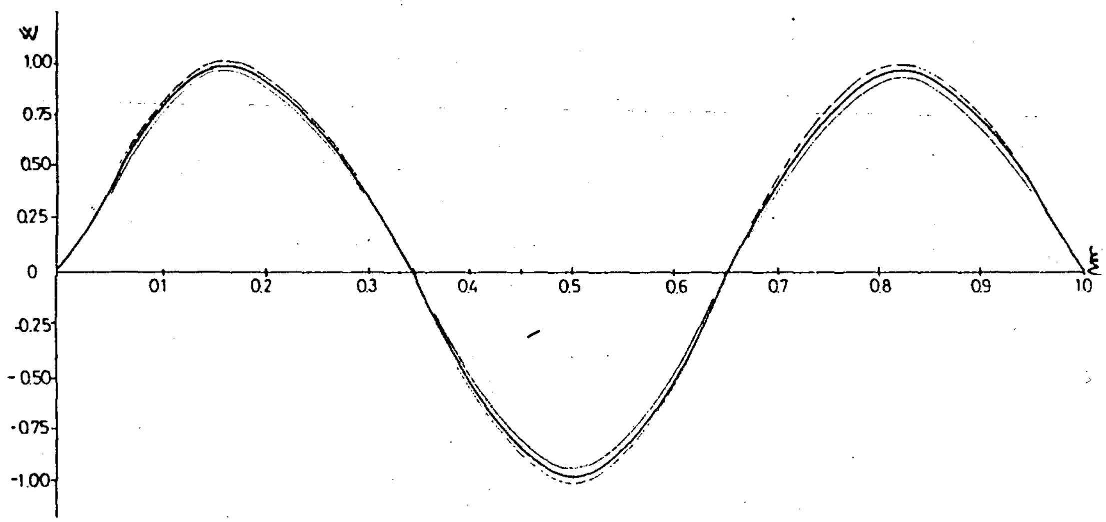


Figure 4.2.15 Deflection curve of the third mode for hinged-hinged beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

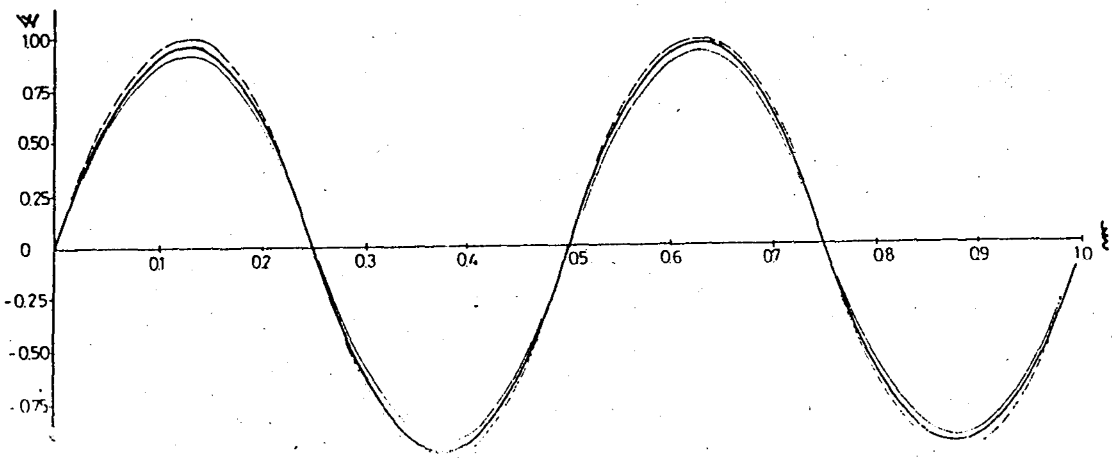


Figure 4.2.16. Deflection curve of the fourth mode for hinged-hinged beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

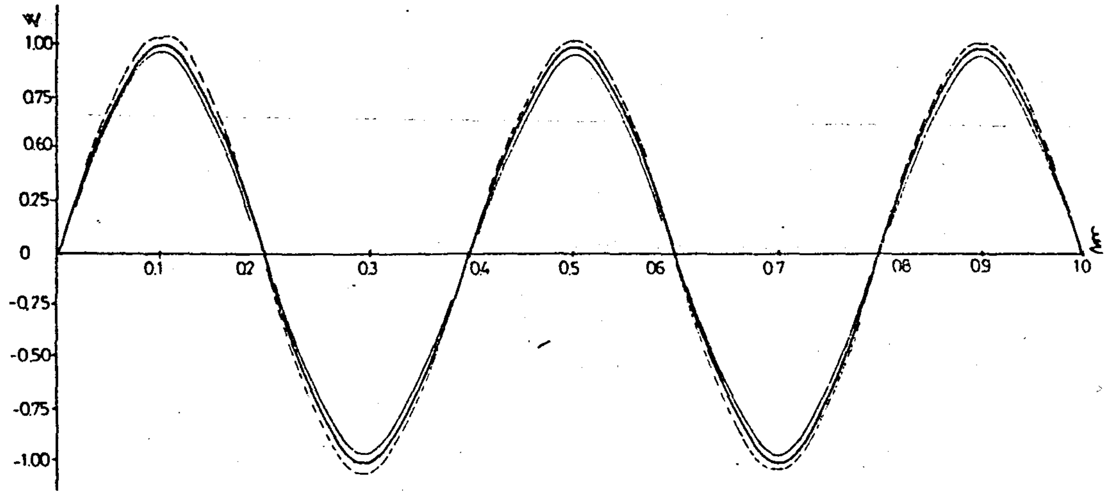


Figure 4.2.17. Deflection curve of the fifth mode for hinged-hinged beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

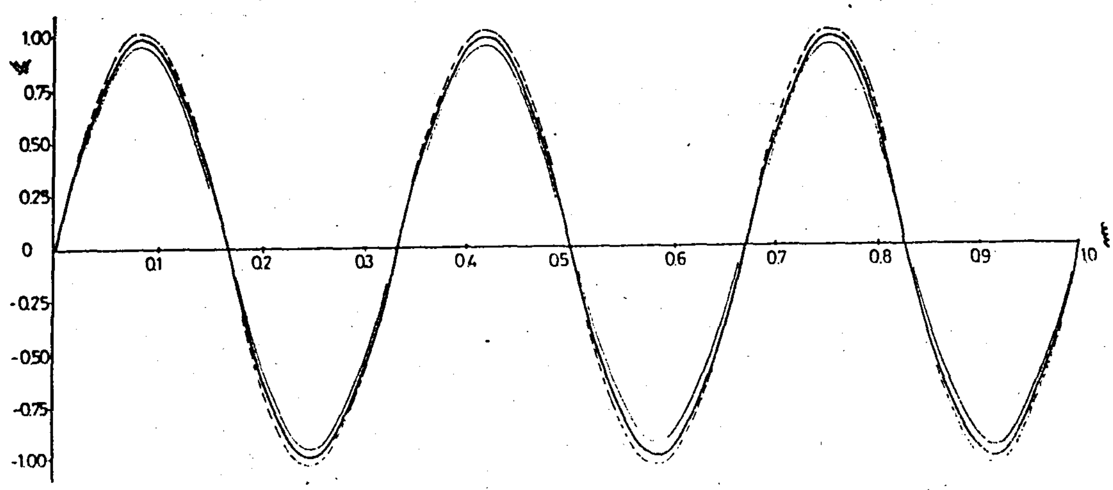


Figure 4.2.18. Deflection curve of sixth mode for hinged-hinged beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

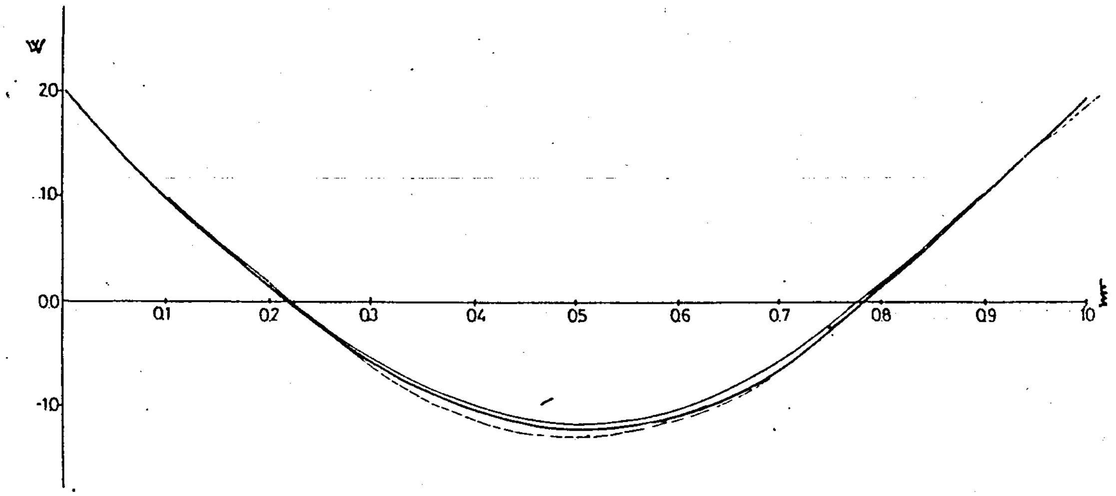


Figure 4.2.19. Deflection curve of the first mode for free-free beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

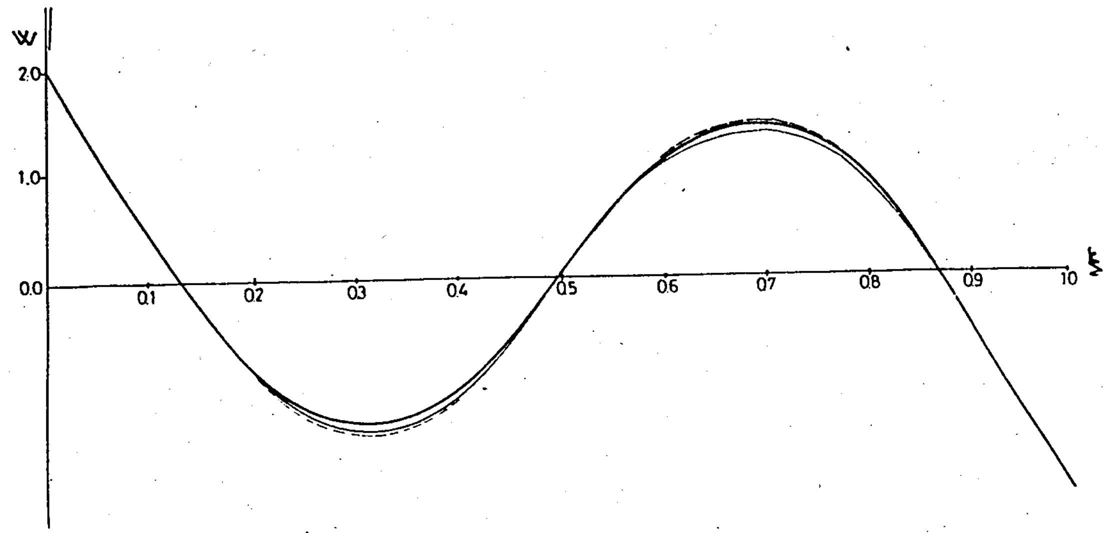


Figure 4.2.20. Deflection curve of the second mode for free-free beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

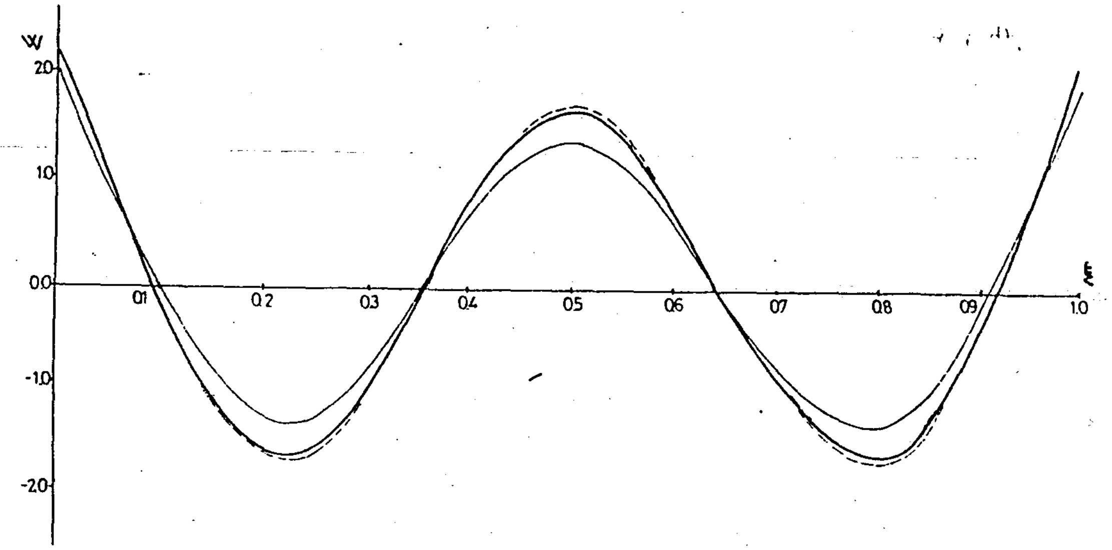


Figure 4.2.21. Deflection curve of the third mode for free-free beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

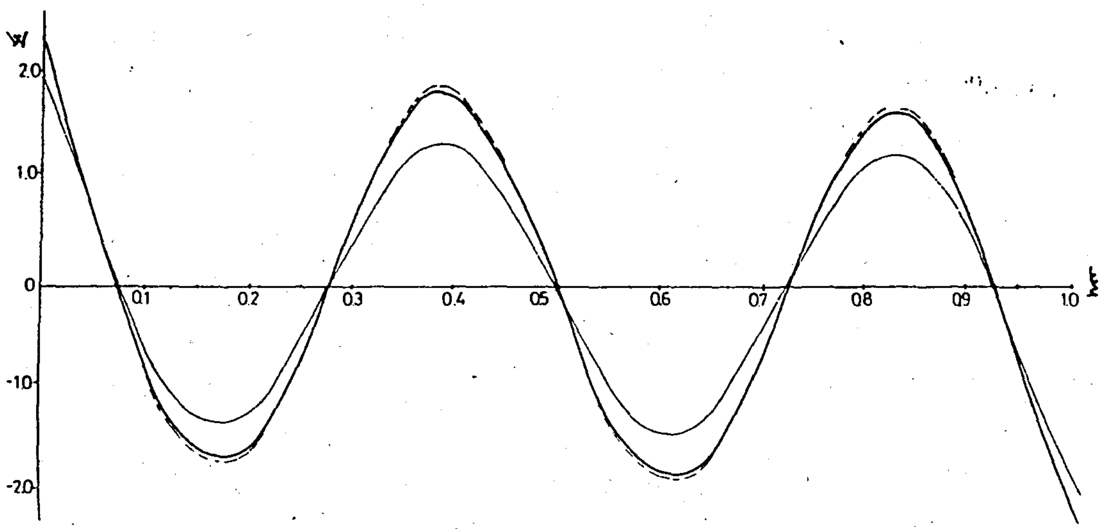


Figure 4.2.22. Deflection curve of the fourth mode for free-free beam, (—— for the modified theory, ... for Timoshenko theory, —— for Euler-Bernoulli theory)

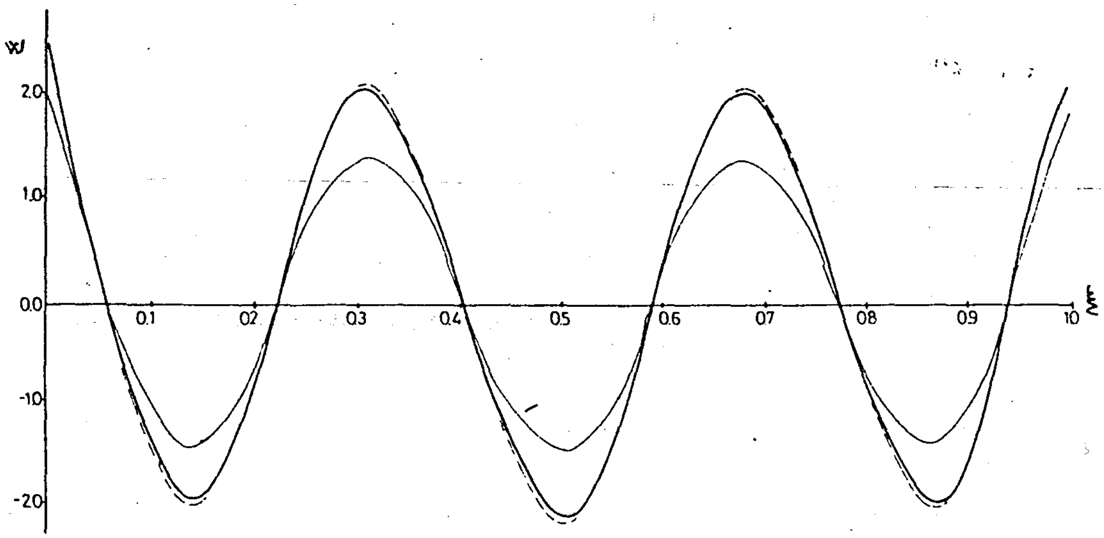


Figure 4.2.23. Deflection curve of the fifth mode for free-free beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

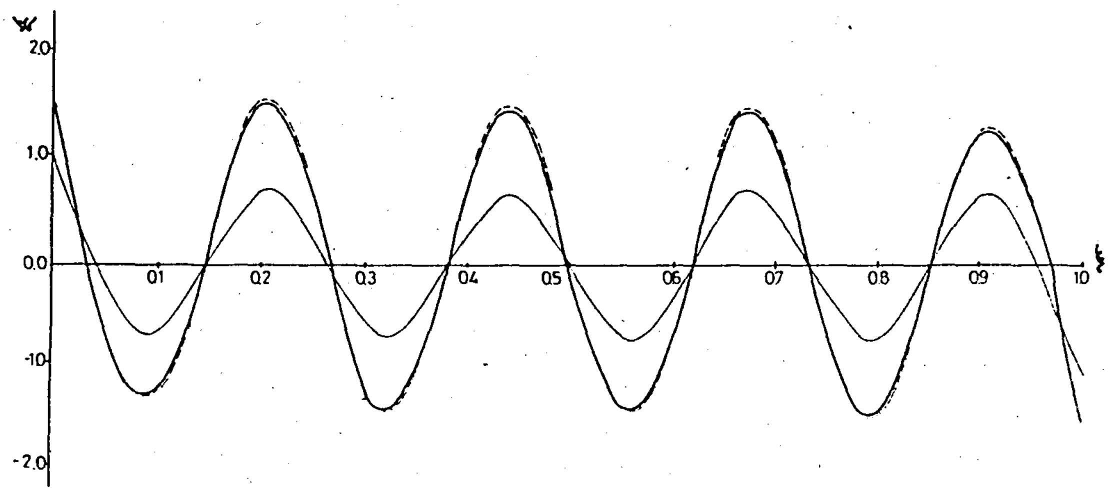


Figure 4.2.24. Deflection curve of the eight mode for free-free beam (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

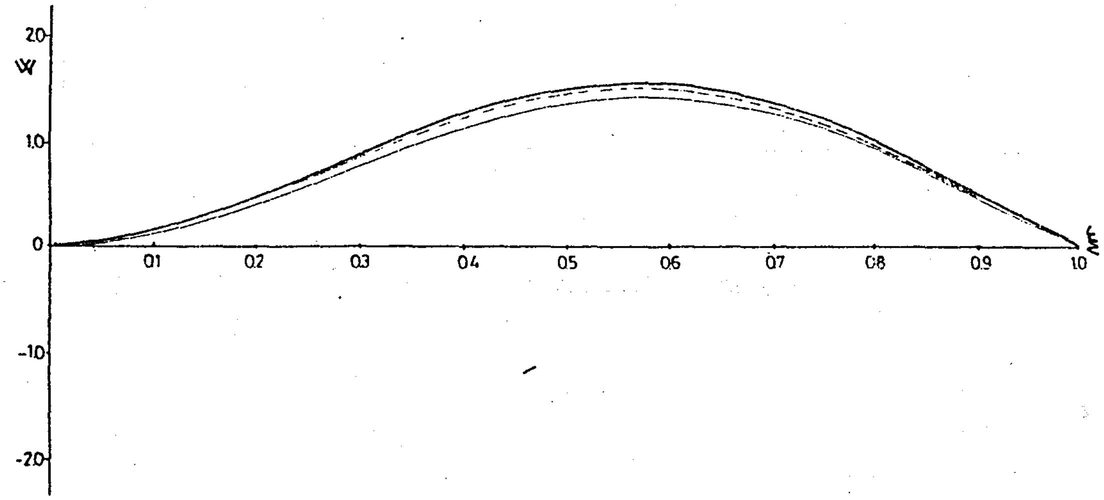


Figure 4.2.25. Deflection curve of the first mode for clamped-hinged beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

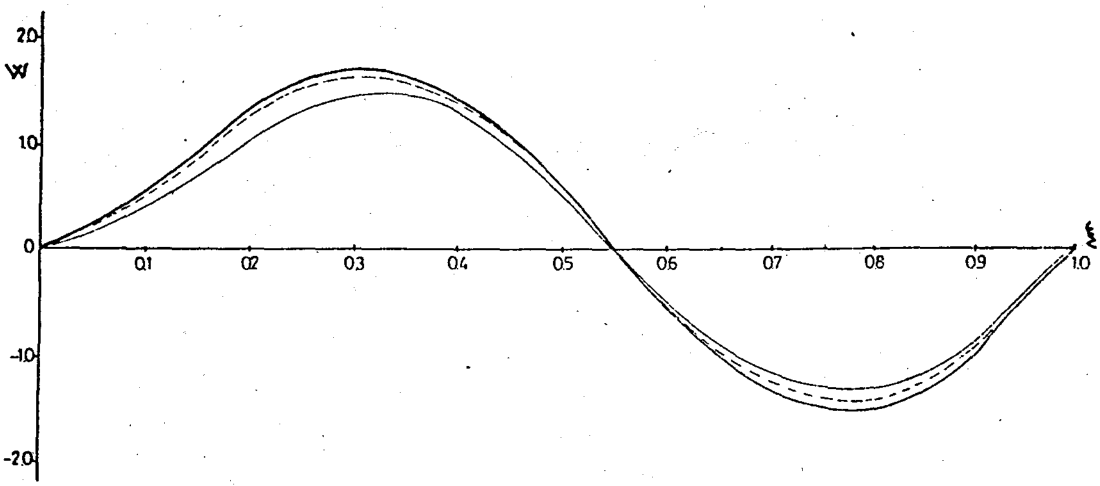


Figure 4.2.26. Deflection curve of the second mode for clamped-hinged beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

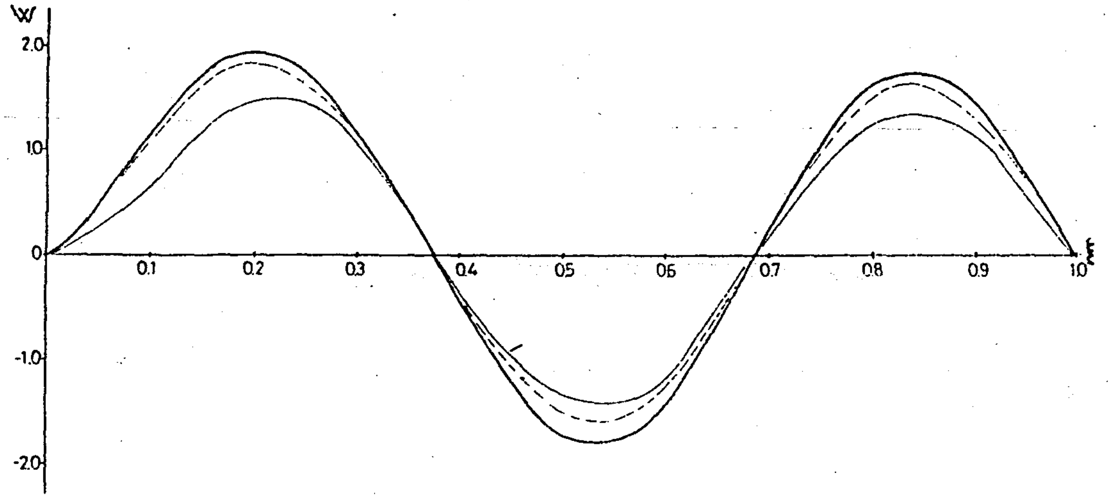


Figure 4.2.27. Deflection curve of the third mode for clamped-hinged beam (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

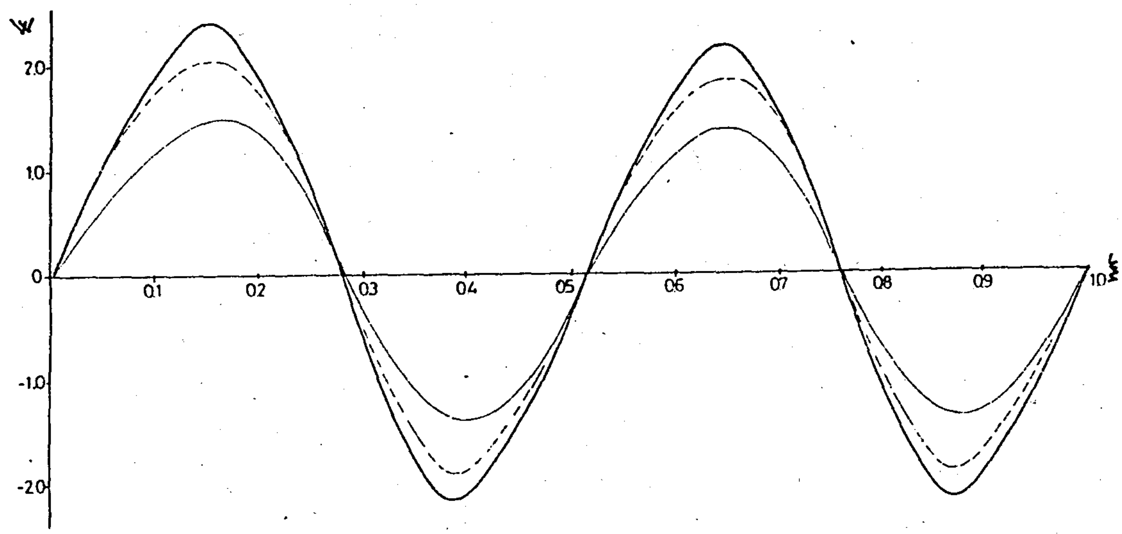


Figure 4.2.28. Deflection curve of the fourth mode for clamped-hinged beam, (— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

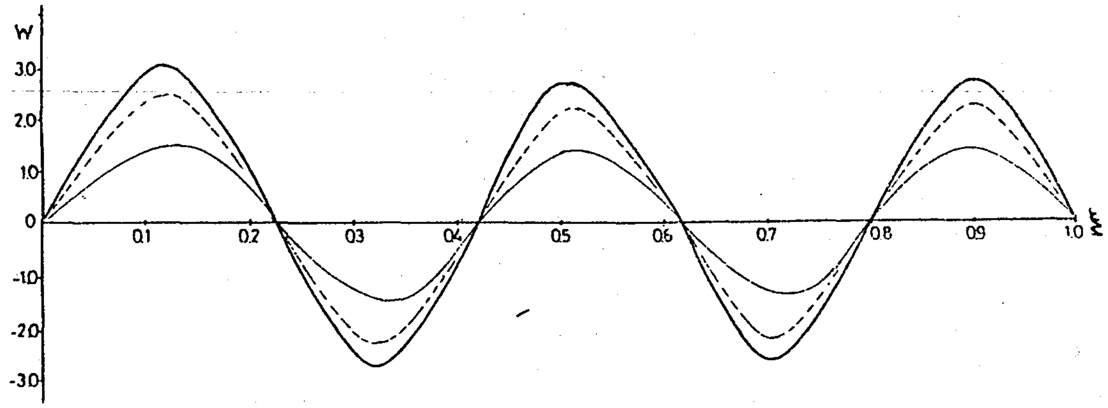


Figure 4.2.29. Deflection curve of the fifth mode for clamped-hinged beam, (—— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

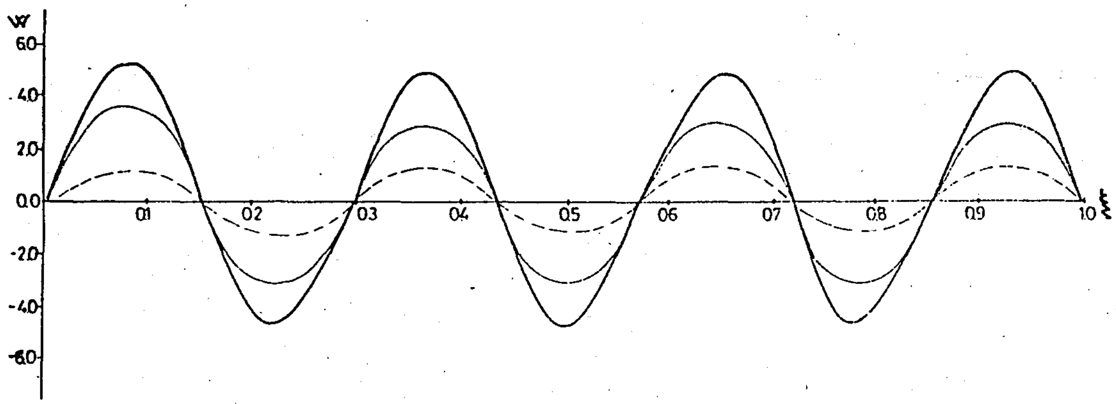


Figure 4.2.30. Deflection curve of the sixth mode for clamped-hinged beam, (—— for the modified theory, ... for Timoshenko theory, — for Euler-Bernoulli theory)

APPENDIX

SHEAR COEFFICIENT

The shear coefficient is a non-dimensional quantity depending on the shape of cross-section.

It has been shown that Timoshenko beam theory is equivalent to the modified theory provided that Timoshenko shear coefficient has the value of 5/6. Unlike Timoshenko beam theory there is no need for additional calculations to get the shear coefficient in the modified theory. Since shear coefficient was directly determined in the derivation of the equations. Usually the values for the shear coefficients are obtained either by matching the high frequency spectrum of the approximate theory by a few known exact results [8] or making use of certain approximations within the theory of elasticity [18]. A sample of the values of shear coefficient for rectangular beam are presented in Table (A.1.1.)

AUTHORS	SHEAR COEFFICIENT
1. Timoshenko	2/8
2. Midlin	0.822
3. Roark	0.833
4. Goodman ($\nu = 1/3$)	0.870
5. Cowper ($\nu = 0.3$)	0.850
6. Stephen ($\nu = 0.3$)	0.866

Table A.1.1. Comparison of the shear coefficients for beams of rectangular cross-section

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