IMPLEMENTATION OF THE HEATH-JARROW-MORTON MODEL ON THE TURKISH GOVERNMENT ZERO-COUPON BONDS

by

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ABSTRACT

IMPLEMENTATION OF THE HEATH-JARROW-MORTON MODEL ON THE TURKISH GOVERNMENT ZERO-COUPON BONDS

The term structure of interest rates explains the relationship among interest rates on default-free securities that vary in their time to maturity. The evolution of the term structure of interest rates through time reflects the future expectations of the market. There is a wide range of approaches including the equilibrium models and arbitrage-free models, all of which try to describe the term structure of interest rates and price interest rate sensitive financial instruments. This thesis focuses on a specific arbitrage-free model called Heath-Jarrow-Morton (HJM), and tries to structure the term structure based on the forward rates. The forward rates are obtained from the prices of zero-coupon bonds issued by the Turkish Government. The evolution of forward rates is described by a multi-factor HJM model with pre-specified drift and volatility functions. The parameters of the models are estimated by an historical procedure using a statistical Principal Component Analysis (PCA). The estimated parameters are used in a simulation algorithm to forecast the future zero-coupon bond prices. The computational results show that the model slightly underprices the real zero-coupon bonds. This deviation between the model and real prices are tried to be explained by a multiple linear regression analysis. Some global and domestic economic indicators are used to understand the relations. The performance evaluations indicate the success of the HJM model together with an error forecast based on specific economic parameters.

ÖZET

HEATH-JARROW-MORTON MODELİNİN KUPONSUZ TÜRK DEVLET TAHVİLLERİNE UYGULANMASI

Faiz oranlarının vade yapısı, vade tarihine kalan zamanları farklı olan ve temerrüt riski taşımayan menkul kıymetlerin faiz ilişkilerini açıklar. Faiz oranlarının vade yapısının zaman içerisindeki gelişimi piyasanın gelecekle ilgili beklentilerini yansıtır. Vade yapısını açıklamaya ve faiz oranına hassas olan finansal araçları fiyatlamaya çalışan, denge modellerinin ve arbitraj fırsatı bulundurmayan modellerin dahil olduğu birçok yaklaşım mevcuttur. Bu çalışma, Heath-Jarrow-Morton (HJM) olarak adlandırılan, arbitraj fırsatı taşımayan bir modele odaklanmış ve faizlerin vade yapısını vadeli kurlar üzerine kurmaya çalışmıştır. Vadeli kurlar, Türk Hükümeti tarafından ihraç edilen kuponsuz tahviller kullanılarak elde edilmiştir. Vadeli kurların gelişimi, önceden belirlenmiş drift ve volatilite fonksiyonlarına sahip çok faktörlü bir HJM modeli ile tanımlanmıştır. Model parametrelerinin değerlemesi, geçmiş tarihli verilerin dahil edildiği bir değerleme yöntemi ve istatistiksel bir analiz olan PCA kullanılarak yapılmıştır. İleri tarihteki tahvil fiyatları, bulunan parametre değerlerinin dahil edildiği bir benzetim algoritması ile tahmin edilmiştir. Sayısal sonuçlar, model fiyatlarının gerçek tahvil fiyatlarını bir miktar düşük seviyede değerlediğini göstermiştir. Model fiyatları ve gerçek fiyatlar arasındaki bu sapma çoklu lineer regresyon analizi ile açıklanmaya çalışılmıştır. İlişkileri anlamak için bazı küresel ve ulusal ekonomik göstergeler kullanılmıştır. Performans değerlendirmeleri, HJM modelinin bazı ekonomik göstergelerle yapılan hata tahminiyle beraber başarılı olduğunu göstermektedir.

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LIST OF SYMBOLS

$\boldsymbol{a_k}$	Eigenvector for factor k
$\boldsymbol{A}$	Eigenvector matrix having columns $\boldsymbol{a_k}$
$BIST^i$	BIST 100 index for instance i
C	a large positive constant
CB^i	currency basket rate for instance i
DR^i	deposit rate for instance i
$Error^i$	error in zero-coupon bond prices for instance i
$f(t,T)$	instantaneous forward rate at time T as seen from date t
$\boldsymbol{L}$	Eigenvalue matrix having diagonal elements λ_k
M	number of economic factors in regression model
n	number of simulation runs
N	number of factors in forward rate model
p_k	proportion of variance explained by factor k
$P(t,T)$	time t price of zero-coupon bond having maturity date T
$P_m(\delta,\tau_3)$	market price of bond δ-ahead time for relative maturity τ_3
$P_{res}(\delta,\tau_3)$	model price of bond δ-ahead time for relative maturity τ_3
$Pr(>\|t\|)$	two-tail p-value of regression variables
R^2	coefficient of determination
$Repo^i$	overnight repo rate for instance i
t	trading date
T	maturity date
T^*	ending date of finite time horizon
VIX^i	volatility index for instance i
$W(t)$	Brownian motion
$x_{j,k}$	forward rate changes at observation date t_j for maturity τ_k
$\boldsymbol{X}$	observation matrix for forward rate changes
y^i	response observation for instance i in regression model

z_m^i	independent variable for instance i and economic factor m
α	intercept parameter of regression model
β_m	regression coefficient of economic factor m
δ	small time shift
ε^i	random error term for instance i in regression model
γ	parameter of exponentially decaying volatility function
λ_k	Eigenvalue for factor k
$\mu\big(t,T,f(t,T)\big)$	drift function in forward rate dynamics
$\sigma_i\big(t,T,f(t,T)\big)$	volatility function of factor i in forward rate dynamics
$\boldsymbol{\Sigma}$	covariance matrix of $\boldsymbol{X}$
τ	relative maturity

LIST OF ACRONYMS/ABBREVIATIONS

AEM	Average Error Model
BDT	Black-Derman-Toy
BIST	Borsa İstanbul
CBOE	Chicago Board Options Exchange
CBRT	Central Bank of the Republic of Turkey
CIR	Cox-Ingersoll-Ross
cov	covariance of
exp	exponential of
HJM	Heath-Jarrow-Morton
ISIN	International Securities Identification Number
LIBOR	London Interbank Offered Rate
min	minimum of
PCA	Principal Component Analysis
REM	Relative Error Model
TED	T-Bill and Eurodollar futures contract
TL	Turkish Lira
US	United States
VIX	Volatility Index

1. INTRODUCTION

The term structure of interest rates is the relationship among interest rates or the yields to maturity on default-free securities that vary only in their time to maturity. By describing the entire evolution of interest rates through time, the term structure, which is also known as the *yield curve*, reflects the expectations of the market about future changes in interest rates. The uncertainty of future market conditions generates a term structure having a stochastic behavior in nature. An explanatory analysis of the term structure makes it possible to gain information about how alternations in underlying variables may affect the term structure within the stochastic nature of the problem.

One of the most prominent research areas in financial economics has been the term structure models of interest rates and its relations with the pricing of interest rate sensitive instruments. Many academic researches have been focused on the problem of modeling the term structure since the pricing of interest rate sensitive financial products is dependent on the term structure and its movements [24]. Therefore, the ability of a financial model to capture the stochastic movements of the term structure is of fundamental importance.

In the last forty years, a wide spectrum of term structure models has been developed in the academic literature and used by practitioners. The models possess various aspects from the determination of interest rate processes to the arbitrage-free valuation schemes of interest rate sensitive instruments. Depending on the structure, the term structure models may be classified into two major groups as *equilibrium term structure models* and *no-arbitrage term structure models* [34].

In equilibrium term structure models, the current term structure is an output of the model. These models do not possess no-arbitrage property, meaning that the model prices may deviate from the market prices due to the involvement of unobservable parameters, especially the market price of risk. These parameters have to be specified exogenously or derived explicitly. Alongside these problems, the advantage of these models is that they can be utilized to price all interest rate contingent claims in a consistent way [26]. No-arbitrage term structure models, on the other hand, take the current term structure of interest rates as an input and utilize the full information of the market yield curve so that the market price of risk is eliminated. The model yield curve exactly matches the market term structure.

The existing term structure models may also be classified according to the number of factors used in the model [37]. The factors represent the sources of uncertainty to describe the term structure movements over time. Single-factor models postulate a perfect correlation among different points in the term structure, whereas multi-factor models facilitate to describe the absence of perfect correlation and to explain a greater variety of the term structure movements over time. On the other hand, the relationship among factors has to be explicitly considered in order to have a consistency with the arbitrage-free requirement.

In a wide variety of the equilibrium term structure models, bond price modeling has been investigated assuming that the bond prices are contingent on only *the short rate*. For these models, another specification is the distributional assumption that is incorporated in the model. Short rate process may follow normal, log-normal or non-central chi-square distribution so that they have different structural properties [12]. These short rate models attempt to propose a process for short-term interest rate and to describe the movements of the yield curve by using a one-factor model. They do not consider other rates on the yield curve, therefore all bond prices move in the same direction at the same time. Since the long-term movements may behave independent of short-term rates at a certain level, a more realistic approach [10] is to model the whole term structure by not only the short rate but also by additional factors, for instance the long-term interest, a spread between the short-term and long-term interest rates, the volatility of the short rate, the mean of the short rate, or the inflation rate.

There may be several approaches to build no-arbitrage term structure models. In these approaches, the exogenous stochastic structure is mainly built upon the short rate or the discount bond prices or the instantaneous forward rates [25]. Both single-factor models and multi-factor models may be incorporated into these models.

More recent studies represent diversifications in the classical term structure models in several aspects. Due to the fact that the instantaneous short rates or the forward rates are unobservable in the real market and difficult to estimate [19], many researchers directly incorporate discrete market rates, such as the London Interbank Offered Rates (shortly *LIBOR*) or forward swap rates. This type of models is called *LIBOR and swap market models* [3]. The complex calibration of classical models to value more advanced instruments like caps, floors and swaptions may be dissolved by the pricing formulas of these market models.

This thesis presents a real-life implementation of a no-arbitrage term structure interest rate model, specifically the model proposed by Heath *et al.* [22]. For the application, a discrete-time approximation to the continuous time economy is adopted. The model is structured upon the forward rates by using the Turkish Government zero-coupon bond price data. The evolution of the term structure of forward rates is described with a three-factor stochastic model and pre-specified volatility and drift functions. In order to estimate the volatility parameters of the model, *Principal Component Analysis*, in short PCA, is conducted for the forward rate changes that are calculated based on the historical zero-coupon bond price data. The estimated parameters are used in a simulation algorithm to compare the model prices with a set of real zero-coupon bond price data that are not used in the estimation procedure. Then, the errors between the real prices and model prices are tried to be explained by a multiple linear regression analysis. Some global and domestic economic factors are used in order to investigate the relations and draw possible conclusions about the deviations in the bond prices. The performances of the regression analyses are discussed on a different set of test data consisting of real zero-coupon bond prices.

2. LITERATURE SURVEY

2.1. Equilibrium Term Structure Models

The traditional approach in equilibrium term structure models has been to propose a model for the short rate and deduct the current term structure and its evolutionary movements from this model. The parameters of the model are calibrated so that the model prices fit the market data as much as possible.

Merton proposes a one-factor model of discount bond prices assuming that the only source of uncertainty to describe the term structure movements is the short rate [35]. The stochastic process for short rate is basically a Brownian motion with a drift parameter. Another assumption of the model is that the market price of risk is constant. Explicit formulas for the valuation of put and call options, warrants and down-and-out type of options are derived. A major deficiency of this model is that it implies negative interest rates with a positive probability.

The model built by Vasicek [42] is a partial equilibrium model of the term structure of interest rates. As in Merton model, the only state variable is the short rate for the entire term structure. Although the general model cannot provide the bond prices in closed form, the specific case in which the market price of risk is assumed to be constant presents an explicit solution. The specific case assumes an Ornstein-Uhlenbeck process for the short rate implying a mean-reverting Gaussian interest rate process. Even though the special case has been used in the subsequent researches to price bond options, futures, futures options and other types of interest rate sensitive instruments [14], the possibility of negative interest rates lingers on as a problem.

In both Merton and Vasicek models, the short rate may take negative values due to Gaussian specifications of interest rate process. Dothan postulates a log-normally distributed process so that the short rate can never be negative [17]. However, the model does not have an explicit formula for the bond price. Brennan and Schwartz also provide a log-normal process with a mean-reverting interest rate process. As Dothan model, the valuation of zero-coupon bonds and bond options requires numerical procedures since explicit formulas do not exist [37].

Cox *et al.* [16] deals with the negative interest rate problem of Merton and Vasicek by proposing an alternative one-factor short rate model in which the interest rate process follows a non-central chi-square distribution. The model is known as *square-root process* since the volatility of the short rates is proportional to the square root of this rate so that it never takes negative values. Closed form solutions for zero-coupon bonds and European call options are already given with the model. Later on, CIR model is frequently used to develop pricing methods, such as mortgage-backed security valuation model, futures and futures option pricing models, swap pricing model, and yield option valuation model [14].

Longstaff [32] extends the framework of CIR and derives an alternative equilibrium term structure model by assuming a *double square-root process* for the short rate. The closed-form solutions for the zero-coupon bond price are obtained in which yields are nonlinear functions of the short rate.

Brennan and Schwartz postulate a two-factor model for the term structure of interest rates [11]. In the model, alongside the short rate, long-term rate of interest is included. The short rate has a mean-reverting structure to the long rate. Both the short rate and the long rate follow a log-normal distribution. Longstaff and Schwartz provide a similar two-factor method assuming that the underlying factors are the short rate and its volatility so that the yields depend on the volatility of the short rates [33]. The idea behind the model is that the volatility may be an important factor for the valuation of interest rate sensitive products. They derive closed-form solutions for the prices of discount bonds and discount bond options.

Subrahmanyam provides a brief summary of the two-factor term structure models [40]. Beside the models of Brennan and Schwartz, and Longstaff and Schwartz, there are various models varying in the choice of factors. These models may include the short rate and a spread between the short-term and the long-term rate, the short-term rate and the inflation rate, and the short-term rate and its mean as the model parameters.

2.2. No-Arbitrage Term Structure Models

The main logic behind no-arbitrage term structure models is to embed the current yield curve and the bond prices as input. These models do not make any statements about the current term structure, meaning that they just hang on to this yield curve. Since the market price of risk is already reflected in the current term structure, the information about the yield curve is fully contained in these models and the estimation of the market risk is eliminated.

The exogenous stochastic dynamics of the term structure may be dependent on the short rate, zero-coupon bond prices or forward rates. One-factor or multi-factor models may be embodied within these models. Upon the structural construction, arbitrage-free movements of the term structure or bond prices through time are modeled and interest rate sensitive instruments are priced in a consistent way with the initial curve structure. Due to the fact that these models are consistent with the current yield curves unlike the equilibrium models, they are also called *term-structure-consistent models* [2].

The model proposed by Ho and Lee is the first approach to the implementation of no-arbitrage term structure models [24]. The model is originally formulated as a discrete-time model with a single factor. The approach possessed is to fit an exact binomial tree structure onto the discount bond prices while maintaining the restriction that the bond price converges to unity at maturity. They clearly show the dynamics that do not permit arbitrage opportunities along the tree.

The short rate process in the continuous-time version of Ho-Lee model [44] may be seen as a special case of Vasicek model without mean-reversion. The mean-reverting drift is replaced by a time-dependent deterministic function in Ho-Lee model. The short rate process follows a normal distribution as in the case of Vasicek model; therefore the possibility of negative future interest rates stays as a problem in this model.

Black *et al.* [4] postulate a similar one-factor discrete-time model, which is known as BDT model, with the assumption that the short rate process follows a log-normal distribution. A log-normal binomial tree is implemented, and the prices of bonds and

European style bond options are found. Since the models have a log-normal property, negative interest rates are discarded. On the other hand, the continuous-time version of BDT model has the drawback that the prices of bonds and European type bond options cannot be found analytically [26].

Black and Karasinski develop a continuous-time single-factor model which has the same representation in the discrete-time version with BDT model [5]. The model, as in the continuous-time version of BDT model, includes a mean-reversion process for the short rate that follows a log-normal distribution. However, the model depends on time in three deterministic functions, namely the speed of the mean reversion, the drift and the volatility of the short rate whereas BDT model has the drift and the volatility terms that are dependent on time.

Hull and White [26] propose extensions of Vasicek model and CIR model. Both of these single-factor extensions have coherence with the current term structure of interest rates and the volatility of all interest rates. In the extended Vasicek model, the current volatility of spot interest rates is specified, whereas the current volatility of forward interest rates is incorporated in the extended CIR model.

In both of the extended models, Hull and White add time dependence to the parameters of the original Vasicek model assuming that the anticipations about future interest rates may represent a time-dependent nature. The extension includes time-dependency to the interest rate process through allowing the drift, volatility and reversion rate to be functions of time. For the extended Vasicek model, European bond options can be priced analytically, and numerical procedures can be used to value American style securities. On the other hand, for a narrow span of interest rate sensitive instruments such as coupon bonds and European options on bonds have analytical derivations in the extended CIR model if the time-dependent parameters are assumed to be constant [34].

A trinomial tree procedure to construct a wide spectrum of single-factor models of short rate is postulated by Hull and White [25]. The procedure provides a way of implementing other models, such as extended Vasicek, extended CIR and Black-Karasinski models. It is assumed that the short-term interest rate process is specified in terms of time-dependent functions that are unknown. Then, the model uses the trinomial tree approach to determine these functions to value bond options and other non-path dependent interest rate sensitive instruments.

Heath *et al.* [22] propose a new methodology, shortly HJM model, in which a complete model of the term structure is described in an arbitrage-free way that does not explicitly depend on the market price of risk. They assume that a multi-factor interest rate model may be built upon the evolution of the forward rate curve while preserving the consistency with the existing term structure and any specified volatility structure. The exogenously specified initial forward rate curve is taken as an input and a continuous-time stochastic process is developed to describe its random fluctuations such that the process represents a no-arbitrage term structure model to price interest rate sensitive instruments. The conditions on forward rate process are characterized by the existence of a unique and equivalent martingale measure.

The model is also represented as the binomial approximation to the original HJM model in [21], providing analytical derivations for some specific volatility functions. This discrete binomial model, in fact, generalizes the original model of Ho-Lee by utilizing the forward rates and allowing multiple factors. As a practical tool for the empirical implementations of the continuous-time model, the discrete-time model may be used to value interest rate sensitive products, such as callable treasury bonds, treasury futures, or options on treasury futures. A real-life application of discrete-time model to treasury bond futures and treasury bond futures options is presented in [23]. A historic estimation procedure is adopted and PCA is applied in order to determine the number of factors to be used and estimate the volatility functions.

In more recent term structure studies, many variations are proposed based on former models. Jara considers an alternative interest rate model for the evolution of prices of future contracts rather than forward rates, and investigates the no-arbitrage conditions [28]. With the introduction of LIBOR and swap market models, a noteworthy switch from instantaneous, continuously compounding forward rates to simple discrete market rates is observed.

The first study in this direction is undertaken by Sandmann and Sondermann, who structure a discrete binomial model based on the effective annual interest rate to price Eurodollar future contracts [38]. Miltersen *et al*. provide an HJM-type LIBOR market model with the inclusion of log-normal LIBOR rates. This model of simple forward LIBOR rates gives closed-form solutions for bond options, and caps and floors [36]. Another HJM-type LIBOR market model with log-normal volatility is postulated by Brace *et al.* [9]. This model begins with the continuous-time HJM framework and proposes a full forward LIBOR term structure model; therefore it may be seen as a discrete-time version of the HJM model. They derive pricing formulas for swaptions, caps and floors at different compounding frequencies. A different framework than HJM, in which a forward LIBOR model is used for swap market is studied by Jamshidian [27]. The prices of LIBOR and swap derivatives are examined and appropriate practical models are proposed for these derivatives. One may also refer to the in-depth survey of studies about complex exotic derivatives given in [37].

3. HJM MODEL

The problem studied in this thesis is a real-life implementation of a no-arbitrage term structure interest rate model, specifically the HJM model. For the application, a discrete-time approximation to the continuous-time model is adopted. The model is built upon the instantaneous forward rates that are found by using the zero-coupon bond prices.

The market prices of risk are not explicitly modeled in order to price interest-rate sensitive instruments. Instead of estimating market prices of risk directly, an exogenously specified initial forward rate curve and the volatilities of the forward rates are incorporated into the model. A risk-neutral probability measure is used in the process to provide the arbitrage-free property.

The evolution of the term structure of forward rates is described with a multi-factor stochastic model. The multiple random factors to describe the fluctuations of the forward rates are in the form of independent standard Brownian motions. These factors provide positive but not perfectly correlated returns for bonds having different maturities.

3.1. Continuous-Time Model

In order to provide the continuous-time version of the HJM model, some preliminary definitions and structures are needed. The definitions are followed from [3, 29].

Definition 3.1. *A zero-coupon bond with maturity date T is a contract which bears no coupon and guarantees the holder 1 unit of money to be paid at the maturity date T. The trading dates t and the maturities T go from date 0 to date $T^* \geq T$.*

It is assumed that the zero-coupon bonds are default-free and the market is sufficiently liquid, i.e. the default and liquidity risks are discarded. Therefore, the only risk driver for the zero-coupon bonds is the interest rate risk.

Definition 3.2. *The price of a zero-coupon bond at time t that pays a certain unit of money at maturity date $T \geq t$ is denoted by $P(t,T)$.*

Assumptions 3.1. *The following assumptions about bond prices are made:*

- *All bond prices are strictly positive, meaning that*

$$P(t,T) > 0 \text{ for } \forall t \in [0,T] \text{ and } \forall T \in [0,T^*].$$

- *All bond prices are equal to 1 at maturity T, i.e.*

$$P(T,T) = 1 \text{ for } \forall T \in [0,T^*].$$

Definition 3.3. *The instantaneous forward rate at time T as seen from date t is denoted by $f(t,T)$. It is the rate one can contract for at time t on a riskless loan over the period $[T, T + dT]$.*

The relationship between bond prices $P(t,T)$ and forward rates $f(t,T)$ is

$$P(t,T) = exp\left\{-\int_t^T f(t,u)du\right\} \tag{3.1}$$

which can be equivalently re-written with the ending condition $P(t,T) = 1$ as

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}.$$

In order to structure the random evolution of the interest rates, one may use either bond prices or forward rates interchangeably.

Definition 3.4. *A stochastic process $W = \{W(t), t \geq 0\}$ is called a Brownian motion if the following conditions hold.*

- $W(0) = 0$.
- *The process* W *has independent increments, i.e. if* $r < s \leq t < u$ *then* $W(u) - W(t)$ *and* $W(s) - W(r)$ *are independent stochastic variables.*
- *For* $s < t$ *the stochastic variable* $W(t) - W(s)$ *has the normal distribution with mean* 0 *and variance* $t - s$, $N(0, t - s)$.
- W *has continuous trajectories.*

The multi-factor HJM model of the forward rate process assuming that N is the number of stochastic factors included into the model has the following differential structure:

$$d\boldsymbol{f}(t,T) = \boldsymbol{\mu}\big(t,T,\boldsymbol{f}(t,T)\big)dt + \sum_{i=1}^{N} \boldsymbol{\sigma}_i\big(t,T,\boldsymbol{f}(t,T)\big)d\boldsymbol{W}_i(t). \tag{3.2}$$

Assume that the zero-coupon bond prices have the maturity dates T_k for $k = 1, \ldots, K$ such that $0 < T_1 < T_2 < \cdots < T_k < \cdots < T_K \leq T^*$. In Equation 3.2, the drift term of the instantaneous forward rates $\boldsymbol{\mu}\big(t,T,\boldsymbol{f}(t,T)\big)$ is a $Kx1$ vector and the representative form of this vector is

$$\boldsymbol{\mu}\big(t,T,\boldsymbol{f}(t,T)\big) = \begin{bmatrix} \mu\big(t,T_1,f(t,T_1)\big) \\ \mu\big(t,T_2,f(t,T_2)\big) \\ \vdots \\ \mu\big(t,T_K,f(t,T_K)\big) \end{bmatrix}.$$

Each volatility function of the instantaneous forward rates $\boldsymbol{\sigma}_i\big(t,T,\boldsymbol{f}(t,T)\big)$ for $i = 1, \ldots, N$ represents a $Kx1$ vector of the form

$$\boldsymbol{\sigma}_i\big(t,T,\boldsymbol{f}(t,T)\big) = \begin{bmatrix} \sigma_i\big(t,T_1,f(t,T_1)\big) \\ \sigma_i\big(t,T_2,f(t,T_2)\big) \\ \vdots \\ \sigma_i\big(t,T_K,f(t,T_K)\big) \end{bmatrix}.$$

The independent Brownian motion term seen in Equation 3.2, shortly $\boldsymbol{W}_i(t)$ for $i = 1, \dots, N$ consists of independent normal random variables admitting a $1xN$ dimensional vector

$$\boldsymbol{W}_i(t) = \begin{bmatrix} W_i^1(t) \\ W_i^2(t) \\ \vdots \\ W_i^N(t) \end{bmatrix}.$$

In the generic working mechanism of HJM model, the initial forward rate curve is taken as given. The random fluctuations of forward rates across time are modeled by multiple random factors represented as the volatility functions. The fluctuations in the forward rates are structured upon independent Brownian motions; therefore the changes in the forward rates are normally distributed.

Suppose that the initial forward rate curve $\boldsymbol{f}(0,T)$ for $T \in [0,T^*]$ is known at time 0. Equation 3.2 in differential form may be written in integral form as

$$\boldsymbol{f}(t,T) = \boldsymbol{f}(0,T) + \int_0^t \boldsymbol{\mu}\big(s,T,\boldsymbol{f}(s,T)\big)ds + \sum_{i=1}^{N} \int_0^t \boldsymbol{\sigma}_i\big(s,T,\boldsymbol{f}(s,T)\big)d\boldsymbol{W}_i(s). \quad (3.3)$$

In order to ensure that this multi-factor model is consistent with arbitrage-free prices, Heath *et al.* provide the conditions on the forward rate process that can be seen in Theorem 3.1 without proof. The detailed mathematical background can be seen in the original paper [22]. Behind the complex mathematical derivations, they show that there exists a unique risk-neutral probability measure such that the model admits no-arbitrage. The consistency is satisfied by a specific condition on the relation between the drift and volatility terms.

Theorem 3.1. *HJM drift condition imposes the following relation between the drift and volatility terms, for every t and $T \geq t$:*

$$\boldsymbol{\mu}\big(t,T,\boldsymbol{f}(t,T)\big) = \sum_{i=1}^{N} \boldsymbol{\sigma}_i\big(t,T,\boldsymbol{f}(t,T)\big) \int_t^T \boldsymbol{\sigma}_i\big(t,u,\boldsymbol{f}(t,u)\big)du. \quad (3.4)$$

Embedding the drift condition, Equation 3.4 transforms into

$$d\boldsymbol{f}(t,T) = \sum_{i=1}^{N} \boldsymbol{\sigma}_i\big(t,T,\boldsymbol{f}(t,T)\big) \int_t^T \boldsymbol{\sigma}_i\big(t,u,\boldsymbol{f}(t,u)\big)du\,dt + \sum_{i=1}^{N} \boldsymbol{\sigma}_i\big(t,T,\boldsymbol{f}(t,T)\big)d\boldsymbol{W}_i(t) \tag{3.5}$$

whose integration from 0 to t yields

$$\boldsymbol{f}(t,T) - \boldsymbol{f}(0,T) = \sum_{i=1}^{N} \int_0^t \boldsymbol{\sigma}_i\big(s,T,\boldsymbol{f}(s,T)\big) \int_s^T \boldsymbol{\sigma}_i\big(s,u,\boldsymbol{f}(s,u)\big)du\,ds + \sum_{i=1}^{N} \int_0^t \boldsymbol{\sigma}_i\big(s,T,\boldsymbol{f}(s,T)\big)d\boldsymbol{W}_i(s).$$

Some studies that incorporate the continuous-time model may be found in the literature. Heath *et al.* provide closed form solutions for the bonds and European type bond options for specific choices of volatility functions in [22]. For instance, they assume a single-factor model with the volatility function $\sigma_1\big(t,T,f(t,T)\big) = \sigma$, where σ is a positive constant. The model with this specification implies the continuous-time version of Ho-Lee model. They show for this specific case that the stochastic forward rate process is reduced to the following form:

$$f(t,T) - f(0,T) = \sigma^2 t\big(T - {t}/{2}\big) + \sigma W(t)$$

and the bond price equation they obtain has this expression:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} exp\left\{-\left({\sigma^2}/{2}\right)Tt(T-t) - \sigma(T-t)W(t)\right\}.$$

In the second example, HJM *et al.* assume a two-factor model with the volatility structures $\sigma_1\big(t,T,f(t,T)\big) = \sigma_1$ and $\sigma_2\big(t,T,f(t,T)\big) = \sigma_2 exp\{-(\gamma/2)(T-t)\}$ where σ_1, σ_2 and γ are strictly positive constants. They deduce the forward rate process as

$$f(t,T) - f(0,T) = \sigma_1^2 t(T - t/2)$$
$$-2(\sigma_2/\gamma)^2\left[e^{-\gamma T}(e^{\gamma t} - 1) - 2e^{-(\gamma/2)T}\left(e^{(\gamma/2)t} - 1\right)\right]$$
$$+\sigma_1 W_1(t) + \sigma_2 \int_0^t e^{-(\gamma/2)(T-u)}\, dW_2(u).$$

In these two examples, the forward rate evolutions and bond prices can be found by simple simulation algorithms.

3.2. Discrete-Time Approximation

In the continuous-time version of the multi-factor HJM model, the closed form expressions are only available for some specific cases for volatility structures and for limited number of factors. It is argued that [20] the exact simulation of Equation 3.5 is infeasible with the exception of very specific choices of $\sigma_i\big(t,T,f(t,T)\big)$. Therefore, a different approach must be utilized in order to apply multi-factor HJM models with various volatility functions. In this thesis, this problem is resolved by incorporating a discrete approximation in the implementation processes. This approximate discrete-time version of the original model facilitates the description of the forward rate dynamics. A very similar procedure is proposed and utilized in [29].

Assuming that $t + \delta < T$ as $\delta \to 0$, the discrete-time approximation may be written in the following form

$$\Delta\boldsymbol{f}(t,T) \approx \boldsymbol{\mu}\big(t,T,\boldsymbol{f}(t,T)\big)\delta + \sum_{i=1}^{N} \boldsymbol{\sigma}_i\big(t,T,\boldsymbol{f}(t,T)\big)\Delta\boldsymbol{W}_i(t). \tag{3.6}$$

In Equation 3.6, $\Delta\boldsymbol{f}(t,T) = \boldsymbol{f}(t+\delta,T) - \boldsymbol{f}(t,T)$ is the change in the forward rate over the interval t to $t+\delta$, and $\Delta\boldsymbol{W}_i(t) = \boldsymbol{W}_i(t+\delta) - \boldsymbol{W}_i(t)$ for $i = 1,\dots,N$ are normally distributed with mean 0 and variance δ.

Assume that the initial forward rate curve $\boldsymbol{f}(0,T)$ is obtained at some reference date $t = 0$. Then, the following relation implies the evolution of the forward rates given the initial forward rate:

$$\boldsymbol{f}(\delta,T) \approx \boldsymbol{f}(0,T) + \boldsymbol{\mu}\big(0,T,\boldsymbol{f}(0,T)\big)\delta + \sum_{i=1}^{N} \boldsymbol{\sigma}_i\big(0,T,\boldsymbol{f}(0,T)\big)\big(\boldsymbol{W}_i(\delta) - \boldsymbol{W}_i(0)\big). \tag{3.7}$$

4. SOLUTION METHODOLOGY

The construction of the discrete-time approximation version of the original model necessitates the specification of the initial forward rate curve and the parameters of the model. These parameters, namely the drift and the volatility terms that depend upon the volatility structure chosen, are estimated from the observed prices of the zero-coupon bonds. Taking these parameters as inputs, the valuation of interest rate sensitive instruments is modeled.

Many different methods may be used for the initial forward rate construction from the zero-coupon bond prices. The essence of this construction is to determine the continuously compounded forward rates of all maturities. Due to the fact that the observed prices may be insufficient to describe the continuous forward rate curve, an approximate piece-wise constant step function may be utilized [29]. An alternative way may be the modification of forward rates in such a way that a particular set of computed interest rate sensitive instrument prices are consistent with the real market prices as much as possible. This matching technique is incorporated for call provisions on treasury bonds in [21] and for treasury futures in [1].

There are mainly two different approaches to estimate the volatility functions, namely *historic volatility estimation* and *implicit volatility estimation*. In the former approach, a time-series of past observations of forward rates are employed to estimate the volatility functions. On the other hand, implicit volatility estimation uses the current market prices of traded interest rate sensitive instruments and the computed price formulas are inverted to acquire the volatility functions. The major aim of this approach is to find the best volatility function in which the model prices are consistent with the market prices; hence this technique is sometimes called *curve-fitting*. This approach estimates the values of the volatility functions that best fit to market prices, frequently by an error minimizing technique.

The implicit volatility estimation may be realized by two different methods [29]. In the first method, there are no restrictions on the volatility matrix:

$$\begin{bmatrix} \sigma_1\big(t, T_1, f(t, T_1)\big) & \cdots & \sigma_N\big(t, T_1, f(t, T_1)\big) \\ \vdots & \ddots & \vdots \\ \sigma_1\big(t, T_K, f(t, T_K)\big) & \cdots & \sigma_N\big(t, T_K, f(t, T_K)\big) \end{bmatrix}.$$

The second method of implicit volatility estimation restricts the number of parameters by assuming functional forms for the volatility functions. In the method, only the parameters of the functional forms are estimated from real market prices. For instance, a restriction may assume $\sigma_i\big(t, T, f(t, T)\big) = \sigma_i$ for $i = 1, \dots, N$ where σ_i are positive constants. This restriction limits the volatility estimation to only N constant parameters.

A large spectrum of volatility structures is available in the literature for either historic or implicit volatility estimations. These volatility structures may be differentiated with respect to several aspects. Some volatility structures are deterministic functions, meaning that the volatility does not depend on the term $f(t, T)$. On the other hand, there are other forms of volatility structures that include this dependence through different functional forms. A second aspect of the volatility functions is the *time-invariance* property implying that $\sigma_i\big(t, T, f(t, T)\big)$ for $i = 1, \dots, N$ depends on t and T only through $T - t$. For the volatility functions having time-invariance property, given a term structure at time t, the imminent random evolution depends only on the term structure, not on the specific date t. The term $T - t$ is sometimes called *relative maturity* and symbolized as τ such that $\tau = T - t$. Another differentiation for volatility functions arises when volatilities are bounded above by positive large numbers. If the forward rates become too large, then these bounds insure that forward rates do not explode to infinity.

A historic estimation procedure including a principal component analysis (PCA) is proposed and implemented for two specific cases of volatility functions in [29]. An implicit volatility estimation technique is used for six forms of volatility functions in [1]. A brief summary of the forward rate volatility structures in the existing literature is given below:

- *Absolute:* $\sigma_i\big(t, T, f(t, T)\big) = \sigma_i$ where $\sigma_i > 0$ for $i = 1, \dots, N$ are positive constants.
- *Exponential decaying:* $\sigma_i\big(t, T, f(t, T)\big) = \sigma_i exp\{-\gamma_i (T - t)\}$ where $\sigma_i > 0$ and $\gamma_i > 0$ for $i = 1, \dots, N$ are positive constants.

- *Proportional:* $\sigma_i\big(t,T,f(t,T)\big) = \sigma_i f(t,T)$ where for $i = 1,\dots,N$ are positive constants.
- *Square root:* $\sigma_i\big(t,T,f(t,T)\big) = \sigma_i f(t,T)^{1/2}$ where $\sigma_i > 0$ for $i = 1,\dots,N$ are positive constants.
- *Deterministic:* $\sigma_i\big(t,T,f(t,T)\big) = \sigma_i(T-t)$ where $\sigma_i(T-t)$ for $i = 1,\dots,N$ are deterministic functions of $T-t$.
- *Linear absolute:* $\sigma_i\big(t,T,f(t,T)\big) = \sigma_{0i} + \sigma_{1i}(T-t)$ where $\sigma_{0i} > 0$ for $i = 1,\dots,N$ are positive constants and $\sigma_{1i}(T-t)$ for $i = 1,\dots,N$ are deterministic functions of $T-t$.
- *Linear proportional:* $\sigma_i\big(t,T,f(t,T)\big) = [\sigma_{0i} + \sigma_{1i}(T-t)]f(t,T)$ where $\sigma_{0i} > 0$ for $i = 1,\dots,N$ are positive constants and $\sigma_{1i}(T-t)$ for $i = 1,\dots,N$ are deterministic functions of $T-t$.
- *Nearly proportional:* $\sigma_i\big(t,T,f(t,T)\big) = \sigma_i(T-t)\min[f(t,T),C]$ where $\sigma_i(T-t)$ for $i = 1,\dots,N$ are deterministic functions of $T-t$ and C is a large positive constant.

It is proven that the absolute volatility function with a single factor HJM model corresponds to the continuous-time version of Ho-Lee model [23]. For this type of model, closed form solutions for European bond options are available.

The exponential decaying model has the property that the volatility depends on time to maturity in a proportionality of exponential decaying form. This type of volatility function is used in a single-factor HJM model and numerically tested on Eurodollar futures and option prices by Kuo *et al.* [31]. The parameters are found by implicit volatility estimation technique and the results are compared with those of absolute and square root volatility functions. They reach the conclusion that the exponential decaying function outperforms the other two functions.

Amin and Morton study the implicit volatility estimation by six different forms of volatility functions in a single-factor HJM model using again Eurodollar futures and option prices [1]. Among the set of volatility functions they investigate, they assert that the two-parameter models like exponential decaying, linear absolute, and linear proportional functions have a tendency to fit the market prices better.

Heath *et al.* analytically show for a single-factor HJM model that the forward rates may explode in a finite time if proportional volatility function is used [22]. They also provide an historic estimation procedure with nearly proportional volatility functions for a multi-factor HJM model in [23]. A numerical application to treasury futures and treasury futures options is illustrated and they found that the model prices do not fit the market prices exactly.

The deterministic volatility function is not frequently used in the applications including a historic volatility estimation procedure. Therefore, this type of volatility function along with an historic volatility estimation method is analytically investigated for a multi-factor HJM model in this thesis. Further, the numerical efficiency of the model with selected volatility structure is also tested on real zero-coupon bond prices.

4.1. Initial Forward Rate Curve

The first step to build the model is the specification of the initial forward rate curve. In this thesis, an approximate piece-wise constant step function over chosen maturity intervals is utilized in order to eliminate the possibility of the insufficiency of observed zero-coupon bond price data to construct the continuously compounded forward rates of all maturities.

The zero-coupon bond prices having maturity dates T_k for $k = 1, \ldots, K$ such that $0 < T_1 < T_2 < \cdots < T_k < \cdots < T_K \leq T^*$ are used to form the initial curve at the reference date $t = 0$.

Definition 4.1. *Continuously compounded forward rates imply the following relation for* $k = 1, \ldots, K-1$ [29]

$$\frac{P(t,T_k)}{P(t,T_{k+1})} = exp\left\{\int_{T_k}^{T_{k+1}} f(t,u)du\right\}.$$

If $f(t,u)$ is constant between times T_k and T_{k+1} for $k = 1, \ldots, K-1$, then the following equality holds:

$$\frac{P(t,T_k)}{P(t,T_{k+1})} = exp\{f(t,T_k)(T_{k+1} - T_k)\} \tag{4.1}$$

where $f(t,u) = f(t,T_k)$ for $T_k \leq u < T_{k+1}$.

Equation 4.1 can be transformed into

$$f(t,T_k) = ln\left(\frac{P(t,T_k)}{P(t,T_{k+1})}\right)\frac{1}{(T_{k+1} - T_k)}.$$

The observed zero-coupon bond prices for a set of pre-specified maturities at time $t = 0$ represent the following $Kx1$ vector:

$$\begin{bmatrix} P(0,T_1) \\ P(0,T_2) \\ . \\ . \\ P(0,T_K) \end{bmatrix}. \tag{4.2}$$

The approximate piece-wise constant step function for the initial forward rate curve obtained by the observed prices has this form for $k = 1, \ldots, K-1$:

$$f(0,T_k) = ln\left(\frac{P(0,T_k)}{P(0,T_{k+1})}\right)\frac{1}{(T_{k+1} - T_k)}.$$

If the concept of relative maturities $\tau_k = T_k - t$ is possessed at the reference date $t = 0$ for $k = 1, \ldots, K$, then the price Equation 4.2 turns into

$$\begin{bmatrix} P(0,\tau_1) \\ P(0,\tau_2) \\ . \\ . \\ P(0,\tau_K) \end{bmatrix} \tag{4.3}$$

and initial forward rate curve for $k = 1, \ldots, K-1$ becomes

$$f(0,\tau_k) = ln\left(\frac{P(0,\tau_k)}{P(0,\tau_{k+1})}\right)\frac{1}{(\tau_{k+1}-\tau_k)}. \tag{4.4}$$

4.2. Parameter Estimation

In the construction of the model, the next step is to estimate the model parameters, namely the drift and the volatility terms. In this thesis, a historic estimation procedure is utilized using a deterministic type of volatility function which has a time-invariance property. Time-invariance property brings the relative maturity concept into the estimation procedure, i.e. given a term structure at time t the next random evolution depends only on the term structure, not on the specific date t. The volatility structures chosen for $i = 1, \dots, N$ are

$$\boldsymbol{\sigma}_i\big(t, T, \boldsymbol{f}(t,T)\big) = \boldsymbol{\sigma}_i(T-t) \text{ where } \boldsymbol{\sigma}_i(T-t) \text{ are deterministic functions of } T-t.$$

and the drift structure corresponding to the volatility functions above is

$$\boldsymbol{\mu}(t, T, \boldsymbol{f}(t,T)) = \boldsymbol{\mu}(T-t) \text{ where } \boldsymbol{\mu}(T-t) \text{ is a deterministic function of } T-t.$$

Given the volatility and drift terms chosen, the approximate discrete model proposed in the Section 3.2. turns into:

$$\Delta\boldsymbol{f}(t,T) \approx \boldsymbol{\mu}(T-t)\delta + \sum_{i=1}^{N}\boldsymbol{\sigma}_i(T-t)\Delta\boldsymbol{W}_i(t) \tag{4.5}$$

which can be equivalently written as

$$\boldsymbol{f}(t+\delta,T) - \boldsymbol{f}(t,T) \approx \boldsymbol{\mu}(\tau)\delta + \sum_{i=1}^{N}\boldsymbol{\sigma}_i(\tau)\Delta\boldsymbol{W}_i(t). \tag{4.6}$$

Assume that the observations of the forward rates are realized at times t_j for $j = 1, \dots, J$ in the past such that $t_1 < t_2 < \cdots < t_j < \cdots < t_J < 0$. And also suppose that the forward rate realizations are observed for the relative maturities $\tau_k = T_k - t_j$ for $k = 1, \dots, K$ and $j = 1, \dots, J$. It means that the forward rate observations $f(t_j, t_j + \tau_k)$ are realized at times t_j for the relative maturities τ_k for $j = 1, \dots, J$ and $k = 1, \dots, K$. Further, it is supposed that the observations of forward rates $f(t_j + \delta, t_j + \tau_k)$ are also available for some sufficiently small positive δ satisfying $t_j + \delta < t_{j+1}$ for $j = 1, \dots, J - 1$ and $t_J + \delta \leq 0$. Given the assumptions, the model in Equation 4.6 transforms into

$$f\left(t_j + \delta, t_j + \tau_k\right) - f(t_j, t_j + \tau_k) \approx \mu(\tau_k)\delta + \sum_{i=1}^{N} \sigma_i(\tau_k) \left(W_i\left(t_j + \delta\right) - W_i\left(t_j\right)\right). \tag{4.7}$$

In order to have N factors in the model, there must be at least N different continuously compounded forward rates. Due to the fact that forward rates are observed according to K relative maturities, the number of factors N is assumed to be equal to the number of relative maturities K henceforth.

Assume that the observed forward rate changes are defined by $x_{j,k}$ in the following form:

$$x_{j,k} = f\left(t_j + \delta, t_j + \tau_k\right) - f\left(t_j, t_j + \tau_k\right).$$

A total of J past observations of K different relative maturities give the observation matrix $\boldsymbol{X}$ having the elements $x_{j,k}$

$$\boldsymbol{X} = \begin{bmatrix} x_{1,1} & \cdots & x_{J,1} \\ \vdots & \ddots & \vdots \\ x_{1,K} & \cdots & x_{J,K} \end{bmatrix} \in R^{KxJ}.$$

It turns out that $\boldsymbol{X}$ is a normally distributed process with a $Kx1$ mean vector

$$\boldsymbol{\mu}(\tau) = \delta \begin{bmatrix} \mu(\tau_1) \\ \mu(\tau_2) \\ \vdots \\ \mu(\tau_K) \end{bmatrix} \in R^K.$$

Knowing the difficulty to construct the relation given in Equation 3.4, the drift term is directly estimated from the forward rates changes. The estimator used for this term for $k = 1, \dots, K$ is similar to the one proposed in [41]:

$$\hat{\mu}(\tau_k) = \frac{1}{\delta J} \sum_{j=1}^{J} x_{j,k}. \tag{4.8}$$

Using the observation data matrix $\boldsymbol{X}$, the sample covariance matrix $\widehat{\boldsymbol{\Sigma}}$ whose dimension is KxK can be obtained. Since this matrix is positive semi-definite, it has a principal component decomposition

$$cov(\boldsymbol{X}) = \widehat{\boldsymbol{\Sigma}} = \boldsymbol{ALA}^T \tag{4.9}$$

where the KxKmatrix $\boldsymbol{A} = (\boldsymbol{a_1}, \dots, \boldsymbol{a_K})$ gives the K eigenvectors $\boldsymbol{a}_k$ for $k = 1, \dots, K$ of $\widehat{\boldsymbol{\Sigma}}$ and the KxK diagonal matrix $\boldsymbol{L} = diag(\lambda_1, \dots, \lambda_K)$ gives the K eigenvalues such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K$ for $k = 1, \dots, K$. These matrices can be described as follows:

$$\boldsymbol{A} = [\boldsymbol{a_1}, \dots, \boldsymbol{a_K}] = \begin{bmatrix} a_{1,1} & \cdots & a_{1,K} \\ \vdots & \ddots & \vdots \\ a_{K,1} & \cdots & a_{K,K} \end{bmatrix} \tag{4.10}$$

$$\boldsymbol{L} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_K \end{bmatrix}. \tag{4.11}$$

For the estimates of the volatility functions, the decomposition obtained by PCA can be used as a best approximation [39] in the form

$$\widehat{\boldsymbol{\sigma}}(\tau) = \frac{1}{\sqrt{\delta}} \boldsymbol{A}\sqrt{\boldsymbol{L}} \tag{4.12}$$

which can be re-written in explicit form as

$$\begin{bmatrix} \hat{\sigma}_1(\tau_1) & \hat{\sigma}_2(\tau_1) & \cdots & \hat{\sigma}_K(\tau_1) \\ \hat{\sigma}_1(\tau_2) & \hat{\sigma}_2(\tau_2) & \cdots & \hat{\sigma}_K(\tau_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_1(\tau_K) & \hat{\sigma}_2(\tau_K) & \cdots & \hat{\sigma}_K(\tau_K) \end{bmatrix} = \frac{1}{\sqrt{\delta}} \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,K} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ a_{K,1} & a_{K,2} & \cdots & a_{K,K} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_K} \end{bmatrix}. \tag{4.13}$$

PCA also provides the contribution of the principal components to the total variation of the data. The proportion of variation explained by the k^{th} factor, which is designated as p_k for $k = 1, \dots, K$ is given by the equation [18]:

$$p_k = \frac{\lambda_k}{\sum_{i=1}^{K} \lambda_i}. \tag{4.14}$$

The approximate discrete model becomes fully described after the initial forward rate curve is constructed and the volatility parameters are estimated by PCA for chosen drift and volatility functions. Given the initial forward rate, the forward rates evolve through time according to the estimated parameters. As a result, the underlying stochastic model given below turns out to be a modified version of the model proposed in Section 3.2 according to the chosen functional structures for the drift and volatility terms:

$$\boldsymbol{f}(\delta,\tau) \approx \boldsymbol{f}(0,\tau) + \widehat{\boldsymbol{\mu}}(\tau)\delta + \sum_{i=1}^{N} \widehat{\boldsymbol{\sigma}}_i(\tau)\big(\boldsymbol{W}_i(\delta) - \boldsymbol{W}_i(0)\big). \tag{4.15}$$

5. MODEL IMPLEMENTATION

5.1. Data

The discrete-time approximation model discussed in Section 3.2 with the specifications for the initial forward rate curve, the drift and volatility functions given in Section 4 is implemented to a set of real-life zero coupon bonds issued by the Turkish Government. The prices of zero-coupon bonds are announced in *Debt Securities Market Daily Bulletins* [6], which is prepared by İstanbul Stock Exchange (*Borsa İstanbul*) on a daily basis. The prices are given in a separate report in the bulletin as the *Outright Purchases and Sales Market Daily Bulletin – Government Securities*. In the report, the opening, closing, lowest, highest and weighted average daily prices of the zero-coupon government bonds are given. The bulletin includes further information such as yields, price and yield changes, nominal values, traded values, and etc. A representative and condensed form of the bulletin having sufficient information from the day 4th of January, 2010 is given in Table 5.1.

Table 5.1. A Sample Bulletin for Zero-Coupon Bonds.

Government Bonds	Currency	Days to Maturity	Opening Price	Lowest Price	Highest Price	Closing Price	Weighted Average Price
TRT130110T10	TLR	9	99.834	99.834	99.839	99.839	99.834
TRT030310T10	TLR	58	98.884	98.877	98.884	98.883	98.880
TRT140410T16	TLR	100	98.039	98.013	98.039	98.013	98.032
TRT050510T16	TLR	121	97.621	97.574	97.621	97.587	97.601
TRT230610T13	TLR	170	96.625	96.538	96.625	96.568	96.570
TRT031110T10	TLR	303	94.065	93.743	94.065	93.751	93.940
TRT081210T14	TLR	338	93.183	92.942	93.223	92.942	93.122
TRT020211T11	TLR	394	91.677	91.442	91.759	91.460	91.611
TRT110511T17	TLR	492	89.288	88.967	89.364	88.967	89.168
TRT030811T14	TLR	576	87.382	87.130	87.503	87.142	87.345

The government debt securities are represented by *International Securities Identification Numbers*, shortly ISIN codes. These codes consist of 12 alpha-numeric digits to identify securities. The first and second digits stand for the country codes. *TR* is used as an abbreviation for the securities of Turkey. The third digit specifies the type of the security; *T* is used for treasury bonds, *B* denotes treasury bills, etc. The next six digits explain the maturity date of the security in terms of the day, month and year. For instance the first bond on the Table 5.1 reaches its maturity at the date 13th of January, 2010. The tenth digit carries the information that whether the security is issued in local or foreign currencies. *T* is used for the securities issued in Turkish Lira and *F* stands for those issued in foreign currency. This digit also explains whether the price belongs to the full security, principal or coupon. For the securities in Turkish Lira, *T* means the full bond price, *A* represents the principal value and *K* stands for the coupon value. On the other hand, *F* means the full bond price, *P* symbolizes the principal value and *C* is an abbreviation of the coupon value for the securities issued in foreign currencies. The eleventh digit sets the securities that have the same maturity date and properties in order. The last digit is used as a control digit and a standard procedure is used to calculate this number.

Days to maturity gives the remaining time until the maturity of bonds. The opening, closing, lowest and highest prices carries the information of market prices on the specific trading day. The weighted average price of bond is calculated based on the cumulative trade of bonds on that specific date. In this thesis, the model is implemented based on the weighted average daily prices of the government bonds.

The initial starting date for the analysis is set as 4th of January, 2010. The time horizon to put the model into practice extends to 30th of December, 2011. There are 24 zero-coupon bonds having different maturities during this period of time. The maturities vary through the time horizon chosen; they have an average value of 283 days (~0.78 year) and a maximum value of 679 days (~1.86 years). Since the maximum maturity is about 2 years, the model is implemented based on a time interval having a length of 2 years in order to observe the changes in forward rates in a full cycle.

The zero-coupon bond price observations in the first and the second trading days of each month are used for the calculations in order to investigate the changes in forward rates at different points in the time horizon chosen. For the quality of the approximation, the model is implemented based on the daily changes as suggested in [29], due to the fact that weekly or monthly observation interval may be too long. Therefore, the daily changes in the forward rates at the observation days of 24 months are calculated assuming that $\delta = 1$ day. The zero-coupon bond prices are used to construct the term structure explaining the time interval of 1 year since the zero-coupon bonds have an average time-to-maturity of about 0.8 year. The term structure is built upon 3 different forward rates having the relative maturities 3, 6 and 12 months in order to cover the term structure of a year.

If there are more than one price observations in the first or second trading day of each month, then the one which is closest to the mid-point of the relative maturity is used in the calculations (e.g. for 3 months, the one whose time-to-maturity is closest to 1.5 months is chosen). If any bond price data is not available on the selected dates, these values are estimated by linear interpolation or extrapolation. For instance, on 1st of April, 2010, there is no price observation for the maturity interval of 6 months but the prices for the relative maturities of 3 and 12 months are available. The price of a bond within 6-month interval is interpolated according to the direct proportion of time-to-maturities of bonds within 3 and 12-month intervals assuming that a synthetic bond having a time-to-maturity of 4.5 months exists. In such a way, 8 of 144 price observations, which are approximately 5.6% of the data, are synthetically produced.

5.2. Initial Forward Rate Curve

The initial forward rate curve with respect to the relative maturities chosen is constructed following Equation 4.4. At each observation date, the reference date is rolled over to the observation date t_j and $t_j + \delta$ for $j = 1, \dots, 24$ and the forward rates $f\left(t_j + \delta, t_j + \tau_k\right)$ and $f(t_j, t_j + \tau_k)$ for $k = 1, 2, 3$ are calculated accordingly. All calculations are done assuming that the time is based on a yearly scale.

The model assumes that the changes in the forward rates $x_{j,k}$ for $j = 1, \dots, 24$ and $k = 1, 2, 3$ are distributed normally. The histograms of daily changes in the forward rates versus frequencies over the time horizon chosen for the maturity intervals 3, 6 and 12 months are displayed in Figures 5.1 through 5.3.

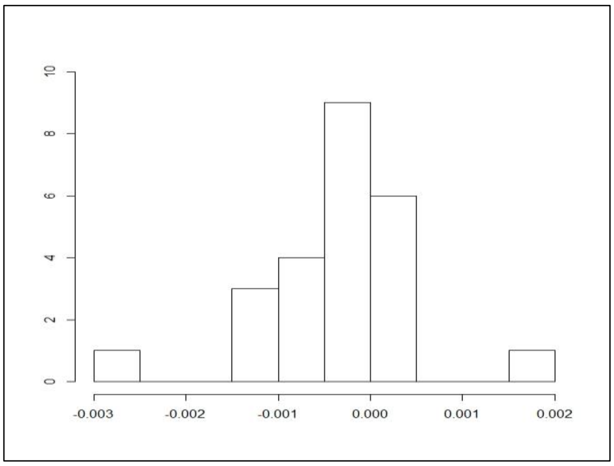

Figure 5.1. 3-Month Forward Rate Changes.

From Figure 5.1, it can be assumed that the changes in the forward rates are distributed around zero since the mean changes for 3-month is -0.0003. The distribution of 3-month changes has longer left tail; therefore it is left-skewed. Despite the changes admit an asymmetric distribution, the skewness statistics stay within the range ± 1. Similarly, the kurtosis statistic of 3-month changes implies a peaked distribution with a long tail.

The histogram seen in Figure 5.2 shows that the changes in 6-month forward rates are distributed around zero since the mean changes is -0.0003. The distribution of 6-month is left-skewed similar to 3-month. 6-month changes admit an asymmetric distribution but the skewness statistics stay within the range ± 1. Also, the kurtosis of 6-month changes expresses a relatively flat distribution with shorter tails.

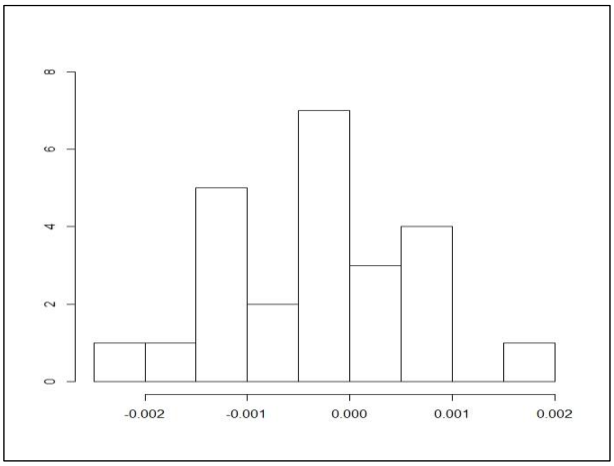

Figure 5.2. 6-Month Forward Rate Changes.

The Figure 5.3 implies that the changes in 12-month forward rates are distributed around zero. The distribution of 12-month changes has a longer right tail and the distribution is right-skewed. Even though the changes admit an asymmetric distribution, the skewness statistic does not go beyond the range ± 1. Similarly, the kurtosis statistic expresses a relatively flat distribution with shorter tail.

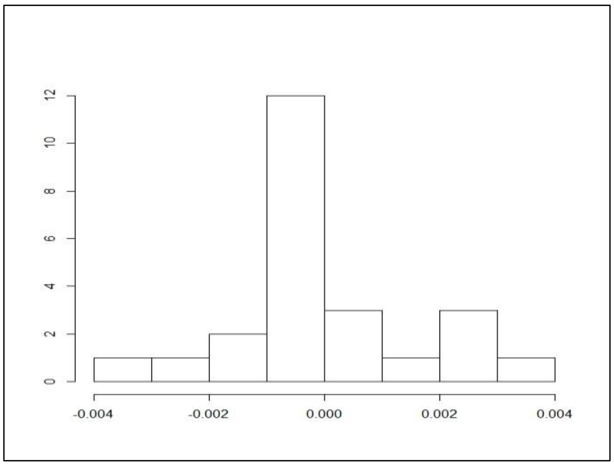

Figure 5.3. 12-Month Forward Rate Changes.

For all the three cases, it can be assumed that the changes in the forward rates are distributed around zero. The skewness statistics stay within the range ± 1 supporting the normality assumption. Despite the fact that 3-months changes have a peaked distribution with a long left tail, 6 and 12-month changes having kurtosis statistics within the range ± 1 support the assumption. While knowing this pitfall, the normal distribution is still assumed for the forward rate changes. The summary statistics of changes calculated by *Excel* are given in Table 5.2.

Table 5.2. Summary Statistics of Forward Rate Changes.

Statistics	3-Month	6-Month	12-Month
Minimum	-0.0028	-0.0022	-0.0034
Average	-0.0003	-0.0003	0.0000
Maximum	0.0016	0.0016	0.0033
Standard Deviation	0.0008	0.0009	0.0017
Skewness	-0.8922	-0.0577	0.2535
Excess Kurtosis	4.3119	-0.1060	0.1885

Examining the correlation matrix for the forward rate changes given in Table 5.3, it can be observed that the correlations among different maturity intervals are small. Changes in 3-month forward rates are negatively correlated with 6 and 12-month forward rate changes. Changes in 6-month forward rates are also negatively correlated with 12-month changes. Since all these correlations have insignificant values, it may be assumed that the forward rate changes for different maturity intervals move independent of each other.

Table 5.3. Correlations among Forward Rate Changes.

Correlation Matrix	**3-Month**	**6-Month**	**12-Month**
3-Month	1.0000	-0.0231	-0.0376
6-Month	-0.0231	1.0000	-0.1704
12-Month	0.0376	-0.1704	1.0000

5.3. Parameter Estimation

The parameter estimation for the model construction is made using the drift and volatility functions explained in Section 4.1. Since the number of forward rates observed is chosen for 3 different relative maturities τ_k for $k = 1, \ldots, K$, the number of factors in the model N is assumed to be equal to K.

From 24 monthly observations and the daily changes in the forward rates $x_{j,k}$ for $j = 1, \ldots, 24$ and $k = 1, 2, 3$ are used to form the matrix $\boldsymbol{X} \in R^{3x24}$. Using this matrix $\boldsymbol{X}$, the drift estimator $\hat{\mu}(\tau_k)$ for $k = 1, 2, 3$ having dimension $3x1$ is obtained following Equation 4.8 and shown below:

$$\begin{bmatrix} \hat{\mu}(\tau_1) \\ \hat{\mu}(\tau_2) \\ \hat{\mu}(\tau_3) \end{bmatrix} = \begin{bmatrix} \text{-12.02\%} \\ \text{-11.39\%} \\ \text{-1.06\%} \end{bmatrix}.$$

The matrix $\boldsymbol{X}$ is decomposed by PCA using statistical software named R. The function *prcomp* returns the decomposition given in Equation 4.9. Within the function, the sample covariance matrix $\hat{\boldsymbol{\Sigma}}$ having dimensions $3x3$ is formed from the matrix $\boldsymbol{X}$:

$$\widehat{\boldsymbol{\Sigma}} = \begin{bmatrix} 6.41\text{E-}07 & -1.72\text{E-}08 & 5.06\text{E-}08 \\ -1.72\text{E-}08 & 8.60\text{E-}07 & -2.66\text{E-}07 \\ 5.06\text{E-}08 & -2.66\text{E-}07 & 2.82\text{E-}06 \end{bmatrix}.$$

The decomposition of $\widehat{\boldsymbol{\Sigma}}$ for the eigenvectors and eigenvalues is also returned by the function and the following matrices are obtained:

$$\boldsymbol{A} = \begin{bmatrix} 0.0235 & 0.0556 & -0.9982 \\ -0.1319 & -0.9896 & -0.0572 \\ 0.9910 & -0.1330 & 0.0160 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 2.86\text{E-}06 & 0 & 0 \\ 0 & 8.26\text{E-}07 & 0 \\ 0 & 0 & 6.39\text{E-}07 \end{bmatrix}.$$

The estimates of the volatility functions are obtained using the decomposition obtained by PCA and following Equation 4.13:

$$\begin{bmatrix} \hat{\sigma}_1(\tau_1) & \hat{\sigma}_2(\tau_1) & \hat{\sigma}_3(\tau_1) \\ \hat{\sigma}_1(\tau_2) & \hat{\sigma}_2(\tau_2) & \hat{\sigma}_3(\tau_2) \\ \hat{\sigma}_1(\tau_3) & \hat{\sigma}_2(\tau_3) & \hat{\sigma}_3(\tau_3) \end{bmatrix} = \begin{bmatrix} 0.0760\% & 0.0964\% & -1.5424\% \\ -0.4259\% & -1.7179\% & -0.0889\% \\ 3.2005\% & -0.2309\% & 0.0244\% \end{bmatrix}.$$

The individual and cumulative proportions of variances explained by the factors are summarized in Table 5.4. The first factor explains approximately 66% of the total variation in the original data. The contribution of the second factor is about 19%. Although the first two factors explain 85% of the total variance, the full model with three factors is used in the further analyses.

Table 5.4. Variance Explained by Principal Components.

Variance	1st Factor	2nd Factor	3rd Factor
Individual Proportion	66.12%	19.10%	14.78%
Cumulative Proportion	66.12%	85.22%	100.00%

The approximate discrete model proposed in Section 3.2 becomes fully structured and numerically described after the initial forward rate curve is constructed and the volatility parameters are estimated by PCA for chosen drift and volatility functions. The evolution of the forward rates through time given the estimated drift and volatility parameters has the following relation:

$$\boldsymbol{f}(\delta,\tau) \approx \boldsymbol{f}(0,\tau) + \widehat{\boldsymbol{\mu}}(\tau)\delta + \sum_{i=1}^{N} \widehat{\boldsymbol{\sigma}}_i(\tau)\big(\boldsymbol{W}_i(\delta) - \boldsymbol{W}_i(0)\big). \tag{5.1}$$

6. BOND PRICE SIMULATION

The drift and volatility estimations explained in Section 5 are found based on the time horizon starting from 4th of January, 2010 and ending at 30th of December, 2011. These estimations are tested by the zero-coupon bond price simulations for the period 2nd of January, 2012 – 30th of November, 2012. The simulations are used as a forecasting tool for the bond prices and the model prices are compared with the real market prices of zero-coupon bonds. The real market prices are obtained by the same electronic source [6].

At the first trading day of each week within the interval, piece-wise constant initial forward rates $f(0, \tau_k)$ for $k = 1, 2, 3$ are constructed following the procedure explained in Section 4.1. If there does not exist any real market price for zero-coupon bonds, then this time point is not included into the analysis. In total, 35 time points are selected as the observation dates for the comparisons of simulated prices and market prices.

The evolution of forward rates is investigated for 1-day ahead, 1 week ahead and 1-month ahead prices according to Equation 5.1. In other words, δ is chosen as 1 day, 1 week and 1 month for the evolutions of 3-month, 6-month and 12-month forward rates, $f(\delta, \tau_k)$ for $k = 1, 2, 3$.

The bond price equation given in Equation 3.1 may be discretized in a similar fashion proposed in [20]. Assuming that $\tau_0 = \delta$, the following equation represents a discrete version of zero coupon bond prices in which the forward rates generated by the simulation algorithm can be used:

$$P(\delta, \tau_K) = exp\left(-\sum_{k=1}^{K} f(\delta, \tau_k)\,(\tau_k - \tau_{k-1})\right). \tag{6.1}$$

In the simulation, the forward rates $f(\delta, \tau_k)$ for $k = 1, 2, 3$ are generated and they are used to forecast the prices of bonds that have a year to maturity, shortly $P(\delta, \tau_3)$. Since the whole term structure is explained by three different relative maturities having a maximum time horizon of 12 months, the prices of bonds having maturity 1 year are estimated by the simulation. The average of the generated bond prices is used as the forecast result for the model bond price $P_{res}(\delta, \tau_3)$. A generic representation of the simulation algorithm run for $n = 10000$ is given Figure 6.1 and detailed R codes are attached to the appendix A.3.

Inputs: $n = 10000$

$\hat{\mu}(\tau_k)$ for $k = 1, 2, 3$

$\hat{\sigma}_i(\tau_k)$ for $i = 1, 2, 3$ and $k = 1, 2, 3$

δ in terms of days

$f(0, \tau_k)$ for $k = 1, 2, 3$

Initializations: $f(\delta, \tau_k) \leftarrow 0$ for $k = 1, 2, 3$

$P(\delta, \tau_3) \leftarrow 0$

Compute n simulated $f(\delta, \tau_k)$ for $k = 1, 2, 3$ using Equation 5.1;

Find n simulated $P(\delta, \tau_3)$ using Equation 6.1;

Compute $P_{res}(\delta, \tau_3) \leftarrow average\big(P(\delta, \tau_3)\big)$;

$Standard\ Error \leftarrow standard\ deviation\big(P(\delta, \tau_3)\big)/\sqrt{n}$;

Return: $P_{res}(\delta, \tau_3)$, $Standard\ Error$

Figure 6.1. Bond Price Simulation Algorithm.

The real market prices of zero-coupon bonds, which are abbreviated as $P_m(\delta, \tau_3)$ and the simulated prices $P_{res}(\delta, \tau_3)$ are summarized in Tables A.1 through A.3 for 1-day ahead, 1-week ahead and 1-month ahead prices. The summary statistics of three different time steps each having 35 instances for the deviations of simulated bond prices from real market prices are shown in Table 6.1. The deviations are calculated based on the following equation:

$$\%\ Deviation = \frac{P_m(\delta, \tau_3) - P_{res}(\delta, \tau_3)}{P_m(\delta, \tau_3)}. \tag{6.2}$$

Table 6.1. Summary Statistics of Simulated Bond Prices.

% $\boldsymbol{Deviation}$	$\boldsymbol{\delta = 1}$ **Day**	$\boldsymbol{\delta = 1}$ **Week**	$\boldsymbol{\delta = 1}$ **Month**
Minimum	0.06%	-0.05%	-0.11%
Average	0.89%	0.99%	1.41%
Maximum	2.44%	2.90%	4.01%

The standard errors of the simulations are also reported in the appendix in Tables A.1 through A.3. The standard errors found are very small relative to the price differences $P_m(\delta, \tau_3) - P_{res}(\delta, \tau_3)$. In all the instances of three cases (1-day, 1-week and 1-month), the simulated prices are far away from the range of ± 2 or ± 3 standard errors.

In almost all of the instances, it is observed that the simulated bond prices underprice the real market prices of zero-coupon bonds. Another finding is that as δ increases, the average and maximum deviations increase since the precision of the approximation relatively lose its power as the time step becomes larger.

It is found that all the instances of the simulations for 1-day ahead prices underprice the real market prices. The simulated prices stay below the real prices by approximately 0.9% on the average. With the exceptions of 2 instances, 1-week ahead model prices also underprice the real values by an average of 1.0%. For the simulations of 1-month ahead prices, it is again observed that the model prices are below the real prices with the exception of 1 instance. The average deviation is found approximately 1.4% for this case.

The errors resulting from the underpricing of the model may be related to the discretization or approximation methods applied to the original continuous HJM model since the errors get larger while the time shift δ increases. Another alternative source for the errors may be the changes in economic factors affecting the market conditions through different time intervals. The model parameters are estimated from the time horizon 2010-2011 and these parameters are used to forecast bond prices of year 2012. Varying economic conditions due to this time shift may affect the volatility and the prices of zero-coupon bonds.

7. ERROR ANALYSIS

The average price values obtained from the simulation algorithm given the estimated parameters for the drift and volatility terms underprice the real market prices of the zero-coupon bonds as it can be seen in Tables A.1 through A.3. The differences may be due the discretization or approximation methods used in the implementation of the HJM model. Another possible cause for the errors may be related to the changes in the market conditions that may affect the bond prices. In this section of the thesis, the errors between the real prices and model prices are tried to be explained by the changing market conditions. A *multiple linear regression analysis* is utilized in which some domestic and global economic indicators are used in order to explain the relations and draw possible end results.

Some researches that carry out similar analyses for the HJM model can be found in the literature. Amin and Morton study the HJM model with six different volatility structures using Eurodollar futures and options data from 1987-1992 [1]. They compute the forward rates each day by using the previous day's estimation in order to forecast the option price. Then, they analyze the forecast error by a multiple linear regression model that includes the amount by which the option is in-the money, the maturity of the option, the forward yield in the underlying Eurodollar futures contract, and the spread between 3-month treasury bill and 3-month Eurodollar yield (shortly *TED* spread). A similar study is conducted by Kuo *et al.* for the empirical performance of the HJM model for Eurodollar futures and options data from 2000-2002 [31]. They use the model to predict 1-day ahead option prices. Then, they try to identify the prediction errors by a multiple linear regression model in which moneyness, maturity of the option contract, TED spread, Eurodollar futures rate, and the level of volatility are used as economic variables.

The variables affecting the zero-coupon bond prices may differ from those related to Eurodollar futures and options. Despite the fact that there are vastly many economic factors in financial markets that may have implicit impacts on the bond prices, the study has to be narrowed down to a limited number of variables in order to clearly understand the relations in the regression analyses. For the case of zero-coupon bonds issued by the

Turkish Government, Euro/TL and US Dollar/TL exchange rates, overnight repo rates of the *Central Bank of The Republic of Turkey (CBRT)*, interest rates for TL deposits and Borsa İstanbul *BIST 100* index are used as domestic economic indicators in this thesis. In order to reflect the possible effects of the volatility in global financial markets, the volatility index of *Chicago Board Options Exchange (CBOE)*, shortly *VIX*, is included into the analyses.

In the domestic financial market, the fundamental foreign currencies that are traded in huge volumes are Euro and US Dollar. Financial players in the market frequently use these currencies as investment instruments. Since these variables imply the value of TL relative to other currencies, there may be possible effects of these foreign exchange rates on the interest rate perceptions of the market and accordingly on the bond prices. The foreign currency exchange rates for Euro/TL and US Dollar/TL are announced daily in the *Electronic Data Delivery System* of the CBRT [13]. Within the system, the exchange rates for selling and buying of these currencies for any desired period of time are given in *Exchange Rates* section. In the regression analyses, the simple average of the buying and selling rates is taken into consideration. Then, these individual currency exchange rates are used to calculate the *currency basket* rate which is equivalent to the average of Euro/TL and US Dollar/TL exchange rates. This variable is included into the regression models in order to investigate the possible effects of these most frequently used foreign currencies.

The overnight repo of the CBRT is a debt instrument of the government; therefore it may be seen as an alternative investment option to the government bonds. Hence, there may exist a possible relation between overnight repo rates and the zero-coupon bond prices. The overnight repo rates are released daily in the *Debt Securities Market Data – Traded Value Data*, which is prepared by Borsa İstanbul [7]. The weighted average of simple repo rates given in the *Overnight Repo Interest Rates and Traded Value* reports are used in the analyses.

The interest rates for TL deposits may reflect the interest rate expectation of the market over the terms of deposits. This interest rate anticipation may have a connection with the bond prices. The interest rate for TL deposits data can be found in *Electronic Data Delivery System* of the CBRT [13]. These rates are announced weekly under the section

Interest Rates – Weekly Average of Interest rates for Turkish Lira Deposits in the system. The system allows gathering the weighted average of interest rates for TL deposits for different terms, such as 1-month, 3-month, 6-month and 1-year. In the regression analyses, the weighted average rates for deposits having a term up to 1-year are used since the term structure is built upon the forward rates with a maximum term of a year in Section 5.

The stock exchange index, BIST 100 may be seen as a major indicator of the financial activities in Turkish financial market. Possible scenarios such as the well-being in the market and the confidence of financial players may be inferred by looking at this index. Thus, this indicator may have a relation with the prevailing interest rate perceptions in the market. The BIST 100 index is obtained through *Equity Market Data – Index Data*, which is prepared by Borsa İstanbul [8]. The historical daily opening, lowest, highest and closing prices for the index are given in the *Price Indices* reports. The closing prices are used in the regression analyses to reflect the daily progress of this index.

Lastly, the VIX index is incorporated in the error analysis since it may represent the market expectations globally, and therefore it may imply concerns about the volatility in the domestic market. The VIX index announced by the CBOE is attained by [15]. The menu *Products – Index Microsites* lists the cash-settled index options. Under the VIX index, a spreadsheet named *Historical Daily Prices – Spreadsheets with Closing Prices for Several Indices* includes the daily closing prices of this index. These prices are used in the regression analyses to investigate the possible relations between the domestic volatility in the bond prices and the global financial volatility.

The errors found in Section 6 for three cases each having 35 instances are regressed against the variables mentioned above. The cases 1-day, 1-week and 1-month errors are analyzed separately for all combinations of the variables in order to draw possible conclusions about the relations. Different error structures and regression models are investigated and the dominant variables to explain the errors are identified by analyzing the statistics of regression tests. Then, the estimates of these dominant variables are used to study the regression model performances against the errors of simulated prices. For the period starting at 3rd of December, 2012 and ending at 12th of March, 2013, models with the dominant variables are tested for the real zero-coupon bond prices in order to understand whether these variables can explain the errors between the model and real bond prices.

7.1. Multiple Linear Regression Model

In multiple linear regression models, the main objective is to explain the relations between a dependent variable and independent variables through a linear structure. The generic form of the multiple linear regression model given the observations $(y^i, z_1^i, z_2^i, \dots, z_M^i)$ of factors $m = 1, \dots, M$ for instances $i = 1, \dots, I$ is as follows:

$$y^i = \alpha + \beta_1 z_1^i + \beta_2 z_2^i + \cdots + \beta_M z_M^i + \varepsilon^i$$

where y^i are the observed *response*s to the values $z_1^i, z_2^i, \dots, z_M^i$ of M observed independent variables. α is the unknown *intercept* parameter and $\beta_1, \beta_2, \dots, \beta_M$ are the unknown *regression coefficients*. ε^i represent the *random error* terms which are assumed to be independent and identically distributed with mean 0 and variance σ^2. The identification of the relations is done through the estimation of the *regression coefficients* α and $\beta_1, \beta_2, \dots, \beta_M$ by a method called *the least square estimation* [43].

In our case, the responses y^i are the errors between the model prices and real prices for instances $i = 1, \dots, 35$. The economic indicators constitute the independent variables $z_1^i, z_2^i, \dots, z_M^i$ which are used to explain the errors. Despite the fact that the economic variables mentioned in Section 7 may have implicit relations among them, it is assumed that these variables move independent of each other.

HJM model is an arbitrage-free term structure model that includes the volatility of interest rates into its dynamics so that the model term structure exactly matches the real market term structure. The deviation between the model and market prices may arise due to unexpected extreme changes in the economic indicators. Assuming a constant intercept parameter in the regression analyses to explain the errors may be meaningless since this constant error should have already been captured in the historic volatility estimation.

Therefore, the intercept parameter α is discarded from the regression models, i.e. it is forced to zero.

For the separate cases of δ for 1-day, 1-week and 1-month, the full multiple linear regression model for instances $i = 1, \ldots, 35$ turns out to be

$$Error^i = \beta_1.Repo^i + \beta_2.CB^i + \beta_3.DR^i + \beta_4.BIST^i + \beta_5.VIX^i + \varepsilon^i \qquad (7.1)$$

where $Error^i = P_m^i(\delta, \tau_3) - P_{res}^i(\delta, \tau_3)$ and variable $Repo$ stands for the overnight repo rate, CB symbolizes the currency basket rate, DR represents the interest rate for TL deposits, $BIST$ designates the index BIST 100 and VIX indicates the volatility index as mentioned earlier.

The regression model seen in Equation 7.1 is named as the *absolute error model (AEM)* henceforth, since the errors are explained as the difference between real prices and model prices. In AEM's, the values of economic factors are taken as the nominal values of the independent variables on the future date for which the errors are forecasted. For instance, if the errors are forecasted for the day 9th of January, 2012 on the calculation date 2nd of January, 2012, then the values of independent variables of the day 9th of January, 2012 are used in the regression analyses.

Beside the AEM, the *relative error model (REM)* in which the definitions of error and economic factors are slightly changed, is also tried to explain the errors. The errors are defined as the % deviation that is mentioned in Section 6 and the values of the independent variables are taken as the *relative changes* between the time at which the calculations are done and the future time for which the bond prices are forecasted. For example, if the bond prices and errors are forecasted for the day 9th of January, 2012 on the calculation date 2nd of January, 2012, then the relative change in independent variables is found as difference between the value on 9th of January, 2012 and the value on 2nd of January, 2012, divided by the value on 2nd of January, 2012. The full REM turns out to be

$$Error_{rel}^{i} = \beta_1.Repo_{rel}^{i} + \beta_2.CB_{rel}^{i} + \beta_3.DR_{rel}^{i} + \beta_4.BIST_{rel}^{i} + \beta_5.VIX_{rel}^{i} + \varepsilon^{i} \tag{7.2}$$

where $Error_{rel}^{i} = \left(P_m^i(\delta,\tau_3) - P_{res}^i(\delta,\tau_3)\right)/P_m^i(\delta,\tau_3)$ for the cases δ 1-day, 1-week and 1-month, and subscript *rel* indicates the relative changes in the independent variable values and the errors.

The regression tests are conducted by built-in command *lm* of statistical software *R*. The summary statistics include estimates of the regression coefficients, standard errors and corresponding *t-values*. The significance of the independent variables are identified by analyzing the two-tail p-values of the parameters, shortly $Pr(>|t|)$. The coefficient of determination R^2 that shows how much variance of the response is explained by the independent variables, is computed separately since the regression models without an intercept parameter necessitate a correction in R^2 calculations.

All the possible combinations of the independent variables that can be seen in the full models are analyzed in separate regression tests. Then, the dominant combinations of variables are identified if all the independent variables in a combination represent statistical significance considering a confidence level of 95%. If corrected R^2 value of a combination is less than zero, then this combination is discarded from further analyses.

7.1.1. 1-Day Errors

The AEM and REM are implemented on the 35 instances of 1-day errors with the only exception that the independent variable DR is not included in the regression analyses. Since DR data are announced weekly, it is not sensible to analyze its effect on the daily bond price changes or errors. The possible combinations represent 2^4-1 distinct regression tests.

7.1.1.1. AEM. The AEM is applied to 1-day absolute errors with the exceptions of DR parameter. The summary statistics for all combinations are given in Table A.4. The dominant models whose independent variables are statistically significant at a 95% confidence interval are given in Table 7.1 along with the regression estimates. Although model 14 admits a statistical significance, it is not included into further analyses due to its negative corrected R^2 value.

Table 7.1. Dominant Models for 1-Day AEM Regressions.

		Regression Coefficients				$Pr(>\|t\|)$			
Model	**Combinations**	β_1	β_2	β_4	β_5	$Repo$	CB	$BIST$	VIX
8	$CB, BIST$		2.136	-5.79E-05			0.0003	0.0021	
12	$Repo$	10.350				3.57E-10			
13	CB		0.402				5.58E-09		
14	$BIST$			1.29E-05				4.47E-08	
15	VIX				0.046				2.17E-09

<u>7.1.1.2. REM.</u> The REM is applied to 1-day relative errors. All the combinations of the independent variables are tried in different regression tests. The summary statistics for all combinations are given in Table A.5. The models 12 and 15 represent significance at a 95% confidence interval and they are given in Table 7.2 along with the regression estimates. Despite these models represent statistical significance, they are excluded from further analyses due to their very negative corrected R^2 values. It should also be noticed that $Repo$ seems to have a significant impact on the errors since it has a two-tail p-value less than 5% for all the regression tests including this parameter.

Table 7.2. Dominant Models for 1-Day REM Regressions.

		Regression Coefficients				$Pr(>\|t\|)$			
Model	**Combinations**	β_1	β_2	β_4	β_5	$Repo$	CB	$BIST$	VIX
12	$Repo$	-0.1203				0.0109			
15	VIX				-0.0817				0.0352

7.1.2. 1-Week Errors

The two models explained in Section 7.1 are implemented on the 35 instances of 1-week errors. All the combinations 2^5-1 are tried in multiple linear regression analyses similar to 1-day errors.

7.1.2.1. AEM. The model given by Equation 7.1 is applied to 1-week absolute errors. The summary statistics for all combinations are given in Table A.6. The dominant models, whose independent variables are statistically significant at a 95% confidence interval are given in Table 7.3 along with the regression estimates. Due to the fact that models 29 and 30 give negative corrected coefficient of determination values, they are discarded from performance evaluations although they have statistically significant variables.

Table 7.3. Dominant Models for 1-Week AEM Regressions.

		Regression Coefficients					$Pr(>\|t\|)$				
Model	**Combinations**	β_1	β_2	β_3	β_4	β_5	$Repo$	CB	DR	$BIST$	VIX
22	$DR, BIST$			26.06	-2.6E-05				0.0025	0.0441	
24	$CB, BIST$		1.99		-5.1E-05			0.0047		0.0247	
27	$Repo$	11.50					3.9E-09				
28	DR			9.57					7.0E-09		
29	CB		0.45					2.2E-08			
30	$BIST$				1.4E-05					1.0E-07	
31	VIX					0.05					3.4E-09

7.1.2.2. REM. The REM is applied to 1-week relative errors. All combinations of the independent variables are used in different regression tests and the summary statistics for all combinations are given in Table A.7. It should be noted that $BIST$ seems to have a significant impact on the errors since it has a two-tail p-value less than 5% for all the regression tests including this parameter. There are only one dominant combination, whose independent variable is statistically significant at a 95% confidence interval. The model 30 is given in Table 7.4 along with the regression estimates. Despite this model represents statistical significance, it is excluded from further analyses due to its very negative corrected R^2 value.

Table 7.4. Dominant Models for 1-Week REM Regressions.

		Regression Coefficients					$Pr(>\|t\|)$				
Model	**Combinations**	β_1	β_2	β_3	β_4	β_5	$Repo$	CB	DR	$BIST$	VIX
30	$BIST$				0.2308					0.004	

7.1.3. 1-Month Errors

Both of the AEM and REM are applied to the 35 instances of 1-month errors. All the combinations 2^5-1 are analyzed in multiple linear regression analyses similar to 1-week errors.

7.1.3.1. AEM. The AEM is applied to 1-month absolute errors. The summary statistics for all combinations are given in Table A.8. The dominant models, whose independent variables are statistically significant at a 95% confidence interval, are given in Table 7.5 along with the regression estimates. Due to the fact that models 27, 29 and 30 give negative corrected coefficient of determination values, they are discarded from performance evaluations although they have statistically significant variables. It is also important to note that DR seems to have a significant impact on the errors since it has a two-tail p-value less than 5% for all the regression tests including this parameter.

Table 7.5. Dominant Models for 1-Month AEM Regressions.

		Regression Coefficients					$Pr(>\|t\|)$				
Model	**Combinations**	β_1	β_2	β_3	β_4	β_5	$Repo$	CB	DR	$BIST$	VIX
15	$Repo, DR, BIST$	-33.27		83.14	-6.60E-05		0.0090		0.0004	0.0022	
16	$Repo, CB, DR$	-22.60	-2.86	91.82			0.0312	0.0027	0.0005		
21	CB, DR		-1.76	51.00				0.0294	0.0037		
27	$Repo$	16.71					2.9E-09				
28	DR			13.94					2.6E-10		
29	CB		0.64					1.8E-09			
30	$BIST$				2.03E-05					4.9E-09	
31	VIX					0.08					1.3E-09

7.1.3.2. REM. The REM is applied to 1-month relative errors. All the combinations of the independent variables are tried in distinct regression tests and the summary statistics are given in Table A.9. The dominant models are given in Table 7.6 along with the regression estimates. Despite these models represent statistical significance, they are excluded from further analyses due to their very negative corrected R^2 values. It should be noticed that $BIST$ seems to have a significant impact on the errors due to its very low two-tail p-values that are less than 5% for all the regression tests including this parameter.

Table 7.6. Dominant Models for 1-Month REM Regressions.

		Regression Coefficients					$Pr(>\|t\|)$				
Model	**Combinations**	β_1	β_2	β_3	β_4	β_5	$Repo$	CB	DR	$BIST$	VIX
12	$Repo, CB, VIX$	-0.032	-0.302			-0.033	0.025	0.010			0.024
18	$Repo, CB$	-0.035	-0.325				0.018	0.009			
19	$Repo, BIST$	-0.026			0.148		0.044			2.65E-05	
20	$Repo, VIX$	-0.046				-0.036	0.002				0.022
25	CB, VIX		-0.406			-0.037		0.001			0.017
27	$Repo$	-0.052					0.001				
29	CB		-0.004					0.001			
30	$BIST$				0.176					5.60E-07	
31	VIX					-0.004					0.013

7.2. Performance Evaluation

The multiple regression analyses for 3 different cases (1-day, 1-week and 1-month) provide statistical evidence that the AEM is more significant than the REM. Therefore, the model to conduct the performance evaluations is selected as the AEM regressions.

The dominant combinations of the economic factors (excluding the ones that have discarded due to their negative R^2 values) explained in Section 7.1 are used to make the performance evaluations. The estimates of these dominant combinations are studied for the time period starting at 3rd of December, 2012 and ending at 12th of March, 2013. Within this horizon, the simulation errors and the errors given by forecasts of the dominant models are compared.

The zero-coupon bond prices are simulated and the simulation errors are found by the same way with the procedure given in Section 6. Then, the errors are forecasted by the estimates of the dominant combinations in order to understand whether the variables can explain the differences between the real and model bond prices. For each case of 1-day, 1-week and 1-month errors, 10 instances of zero-coupon bond prices are found by the simulation within the selected time horizon. The numerical results for all the instances of 3 cases are given in the appendix in Tables A.10 through A.13. The performances are assessed based on the percentage of simulation error that can be explained by the errors forecasted by the delected dominant combinations. The summary of the performances of these regression tests is given in Table 7.7.

Table 7.7. Performance Summary of the Selected Regression Models.

Case	Model	Combination	% Error Explained by Regression
1-Day	8	$CB, BIST$	-12.8%
	12	$Repo$	30.6%
	13	CB	48.4%
	15	VIX	39.2%
1-Week	22	$DR, BIST$	-2.0%
	24	$CB, BIST$	1.0%
	27	$Repo$	34.3%
	28	DR	44.6%
	31	VIX	45.3%
1-Month	15	$Repo, DR, BIST$	-83.8%
	16	$Repo, CB, DR$	4.5%
	21	CB, DR	16.4%
	28	DR	61.5%
	31	VIX	60.0%

It can be seen that the models with the parameter $BIST$ provide larger errors than the simulation errors with the exception that the model 24 of 1-week case forecasts approximately the same amount of error with the simulation. Therefore, this factor does not serve efficiently to explain the errors.

It should be noticed that the individual parameter CB performs well to express the errors for 1-day case (model 13). On the other hand, the relations of this parameter for 1-week and 1-month-errors is not supportive. Although the economic factor CB seems reasonable for 1-day errors, the performance does not imply consistency for all the cases.

The parameters $Repo$ and DR represent medium efficiencies relative to the parameter VIX in general. Despite these factors have positive coefficients in their individual dominant regression models (in Tables 7.1, 7.3 and 7.5), their performances in explaining the errors are less sufficient relative to the individual VIX variable.

For all the three cases, the VIX variable gives reasonable amounts of explanation for the errors since approximately half of the errors can be explained by the single parameter VIX. For 1-day case, about 40% of the errors can be disclosed on the average. The prices obtained from the simulation algorithm, real prices and price forecasts (simulated prices plus the error forecasts of the selected multiple linear regression model) for ten instances of 1-day case are given in Figure 7.1.

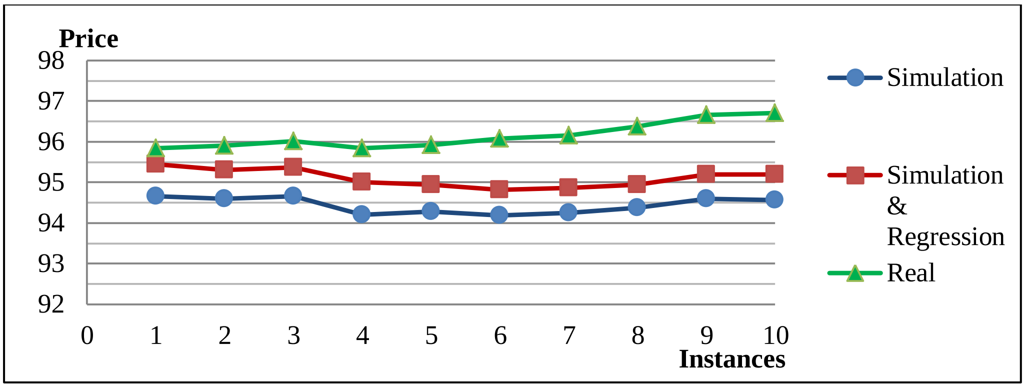

Figure 7.1. 1-Day Ahead Forecasted Prices for VIX.

In the 1-week case, the single parameter VIX explains 45% of the errors approximately. 3 out of ten instances give nearly the same forecast for the real zero-coupon bond prices. The prices are depicted in Figure 7.2.

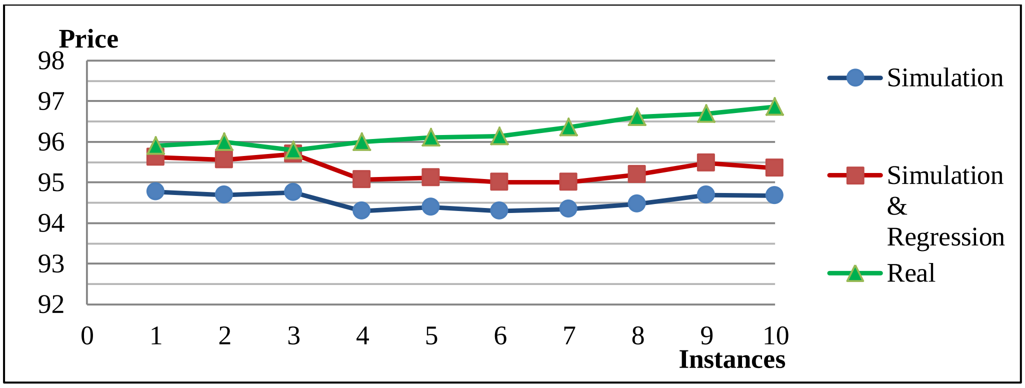

Figure 7.2. 1-Week Ahead Forecasted Prices for VIX.

The performance of VIX parameter for the case 1-month is better than those of 1-day and 1-week cases. 60% of the deviation between simulated prices and real market prices can be explained by this individual parameter. The prices of ten instances are shown in Figure 7.3. Although error forecast of one instance along with the simulated price overestimates the real market value, the general performance over 10 instances is reasonable.

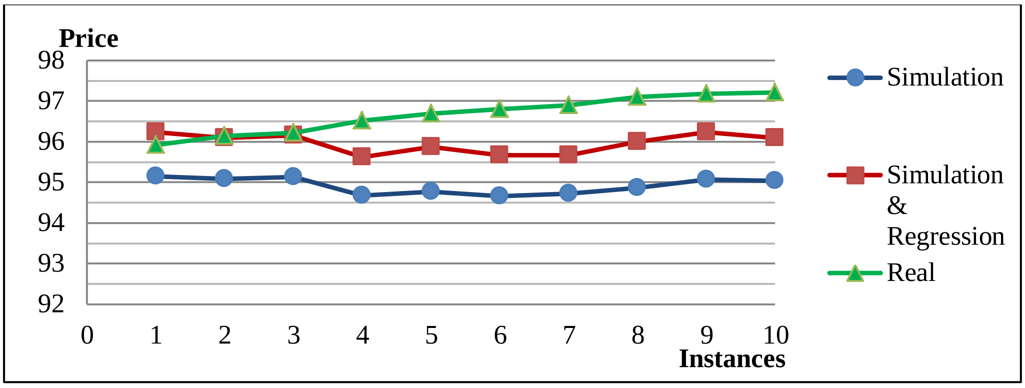

Figure 7.3. 1-Month Ahead Forecasted Prices for VIX.

The common outcome of the three cases may be natural due to the fact that as the global financial volatility increases, the volatility in the domestic market also increases. Therefore, the volatility in the domestic zero-coupon bond prices may rise in a similar fashion and the errors become larger. This relation between errors and the VIX parameter may also be observed from its positive regression coefficients of the VIX parameter in the corresponding dominant AEM regressions.

To conclude, the only sensible variable whose performance in explaining the errors may be appreciated is the economic factor VIX. It is consistent for all the three cases and provides sufficient explanatory power for the errors between the real zero-coupon bond prices and simulated model prices. The remaining unexplained error may be due to the discretization or approximation methods applied to the HJM model or because of the economic factors which are not considered within the limits of this thesis. Thefore, the simulation model in which a multi-factor HJM model is used, and a reliable forecast for the VIX parameter provide sufficient power to determine the future zero-coupon bond prices.

8. CONCLUSION

In this thesis, we implement a three-factor HJM model on the prices of zero-coupon bonds issued by the Turkish Government. The model adopts a discrete-time approximation to the original continuous-time version of the HJM model. The drift and volatility functions are specified in advance and deterministic type of structures are utilized. We estimate the parameters of the model from 2-year-data of zero-coupon bond prices using an historical estimation procedure.

The estimated parameters are used to find the 1-day, 1-week and 1-month ahead prices of bonds in a simulation algorithm. The computational results indicate that the model slightly undervalue the real zero-coupon bond prices. We try to explain this deficiency by a multiple linear regression analysis whose indepenendent variables are selected as domestic and global economic indicators. The regression analyses of all three cases show that the AEM provide statistical significance to explain the errors. Among all the possible combinations of economic indicators, the global volatility parameter VIX gives reasonable amounts of explanation for all the three cases.

We conclude from the analyses that the discrete-time approximation of the multi-factor HJM model accompanied by an error forecast obtained from a multiple linear regression model, in which a single economic indicator is used, almost gives the real market prices of zero-coupon bonds. It should be kept in mind that obtaining a reliable forecast for the VIX parameter may also necessitate a separate statistical and/or stochastic model to understand its own dynamics through time.

With the problem formulation we have in hand, the future work may also focus on different volatility structures. Along with an historic or implied volatility estimation procedure, performances of other volatility functions may be compared. Lets not forget that the time horizon to estimate the model parameters should be extended whether our model parameters can be improved.

Although we deal with a limited number of zero-coupon bonds having short time-to-maturities (up to 2 years), coupon bearing bonds may also be included in order to study longer time intervals for the term structure movements. It should be noted in such applications that a coupon bond stripping procedure for inferring the zero-coupon bond prices from coupon bearing bonds is necessary.

Finally, our model enables us to price several types of interest rate sensitive instruments. There may be several directions that can be possessed in order to implement the model on various financial securities and experiment with the pricing performances of the model.

APPENDIX A: Experiment Results

A.1. Simulation Results

Table A.1. Simulated Prices and Real Market Prices for 1-Day.

$\boldsymbol{\delta = 1\ Day}$			
$\boldsymbol{P_{res}(\delta, \tau_3)}$	**Standard Error**	$\boldsymbol{P_m(\delta, \tau_3)}$	% $\boldsymbol{Deviation}$
89.904	0.09%	91.091	1.30%
89.262	0.09%	91.159	2.08%
90.056	0.07%	91.950	2.06%
90.301	0.08%	92.491	2.37%
91.586	0.10%	93.643	2.20%
91.877	0.08%	94.177	2.44%
91.664	0.09%	91.749	0.09%
91.682	0.10%	92.006	0.35%
91.356	0.09%	91.979	0.68%
91.087	0.06%	91.707	0.68%
91.309	0.07%	91.598	0.32%
91.062	0.09%	91.529	0.51%
90.838	0.08%	91.644	0.88%
90.838	0.09%	91.943	1.20%
90.869	0.08%	92.229	1.47%
91.028	0.06%	91.195	0.18%
90.892	0.09%	91.515	0.68%
91.198	0.07%	91.806	0.66%
91.281	0.07%	92.079	0.87%
91.481	0.07%	92.486	1.09%
91.801	0.07%	93.087	1.38%
92.419	0.10%	93.440	1.09%
92.299	0.09%	93.561	1.35%
92.500	0.09%	92.561	0.07%
93.016	0.10%	93.221	0.22%
93.176	0.10%	93.525	0.37%
93.014	0.10%	93.850	0.89%
93.246	0.09%	93.985	0.79%
93.315	0.06%	94.302	1.05%
93.506	0.09%	93.563	0.06%
93.490	0.07%	93.557	0.07%
93.348	0.07%	93.630	0.30%
93.412	0.11%	93.731	0.34%
93.608	0.06%	94.047	0.47%
93.742	0.07%	94.319	0.61%

Table A.2. Simulated Prices and Real Market Prices for 1-Week.

$\delta = 1\,Week$			
$P_{res}(\delta, \tau_3)$	**Standard Error**	$P_m(\delta, \tau_3)$	$\%\,Deviation$
90.004	0.19%	91.096	1.20%
89.353	0.22%	91.854	2.72%
90.152	0.19%	92.250	2.27%
90.401	0.19%	93.101	2.90%
91.681	0.21%	94.163	2.64%
91.971	0.22%	94.142	2.31%
91.763	0.22%	91.964	0.22%
91.778	0.21%	91.802	0.03%
91.450	0.21%	91.762	0.34%
91.184	0.20%	92.028	0.92%
91.403	0.21%	91.537	0.15%
91.154	0.23%	91.535	0.42%
90.936	0.21%	91.784	0.92%
90.934	0.22%	92.203	1.38%
90.962	0.22%	92.430	1.59%
91.120	0.19%	91.217	0.11%
90.986	0.24%	91.649	0.72%
91.295	0.23%	92.046	0.82%
91.378	0.22%	92.402	1.11%
91.576	0.22%	92.850	1.37%
91.895	0.21%	93.462	1.68%
92.515	0.24%	93.514	1.07%
92.394	0.22%	93.777	1.47%
92.596	0.20%	93.303	0.76%
93.114	0.22%	93.513	0.43%
93.272	0.23%	93.386	0.12%
93.112	0.24%	94.038	0.98%
93.349	0.23%	94.241	0.95%
93.416	0.22%	94.480	1.13%
93.602	0.24%	93.557	-0.05%
93.589	0.23%	93.541	-0.05%
93.446	0.20%	93.677	0.25%
93.509	0.22%	93.968	0.49%
93.704	0.22%	94.255	0.58%
93.840	0.22%	94.603	0.81%

Table A.3. Simulated Prices and Real Market Prices for 1-Month.

$\delta = 1\,Week$			
$P_{res}(\delta, \tau_3)$	**Standard Error**	$P_m(\delta, \tau_3)$	% $Deviation$
90.357	0.43%	93.156	3.00%
89.730	0.46%	93.479	4.01%
90.515	0.46%	93.643	3.34%
90.749	0.44%	94.177	3.64%
92.057	0.43%	94.275	2.35%
92.343	0.44%	94.305	2.08%
92.129	0.46%	92.203	0.08%
92.150	0.43%	92.051	-0.11%
91.820	0.44%	92.226	0.44%
91.548	0.47%	92.313	0.83%
91.767	0.44%	92.203	0.47%
91.525	0.47%	92.051	0.57%
91.298	0.45%	92.236	1.02%
91.304	0.44%	92.399	1.18%
91.330	0.46%	93.074	1.87%
91.485	0.46%	92.241	0.82%
91.354	0.42%	92.850	1.61%
91.668	0.47%	93.499	1.96%
91.754	0.44%	93.669	2.04%
91.943	0.46%	93.879	2.06%
92.266	0.45%	94.468	2.33%
92.890	0.44%	94.552	1.76%
92.765	0.44%	94.593	1.93%
92.969	0.44%	93.711	0.79%
93.488	0.44%	94.038	0.59%
93.655	0.45%	94.135	0.51%
93.489	0.45%	94.572	1.15%
93.716	0.46%	94.621	0.96%
93.791	0.46%	94.817	1.08%
93.985	0.46%	94.108	0.13%
93.966	0.48%	94.319	0.37%
93.825	0.46%	94.603	0.82%
93.886	0.47%	94.908	1.08%
94.088	0.46%	95.202	1.17%
94.218	0.47%	95.442	1.28%

A.2. Regression Results

Table A.4. 1-Day AEM Regressions.

		$Pr(>\|t\|)$				
Model	**Combinations**	$\boldsymbol{Repo}$	$\boldsymbol{CB}$	$\boldsymbol{BIST}$	$\boldsymbol{VIX}$	$\boldsymbol{R^2}$
1	$Repo, CB, BIST, VIX$	0.4363	0.0328	0.0250	0.3692	28.74%
2	$CB, BIST, VIX$		0.0085	0.0047	0.4401	27.31%
3	$Repo, BIST, VIX$	0.0921		0.4145	0.5503	17.27%
4	$Repo, CB, VIX$	0.0638	0.6713		0.7491	15.98%
5	$Repo, CB, BIST$	0.5329	0.0398	0.0347		26.83%
6	$BIST, VIX$			0.2794	0.0088	9.47%
7	CB, VIX		0.8180		0.1760	6.31%
8	$CB, BIST$		0.0003	0.0021		25.92%
9	$Repo, VIX$	0.0648			0.9651	15.50%
10	$Repo, BIST$	0.0021		0.5710		16.33%
11	$Repo, CB$	0.0217	0.7744			15.71%
12	$Repo$	3.57E-10				15.50%
13	CB		5.58E-09			0.88%
14	$BIST$			4.47E-08		-11.8%
15	VIX				2.17E-09	6.15%

Table A.5. 1-Day REM Regressions.

		$Pr(>\|t\|)$				
Model	**Combinations**	$Repo$	CB	$BIST$	VIX	R^2
1	$Repo, CB, BIST, VIX$	0.027	0.693	0.109	0.127	-86%
2	$CB, BIST, VIX$		0.978	0.104	0.033	-118%
3	$Repo, BIST, VIX$	0.027		0.07	0.083	-87%
4	$Repo, CB, VIX$	0.024	0.354		0.19	-102%
5	$Repo, CB, BIST$	0.007	0.379	0.163		-100%
6	$BIST, VIX$			0.085	0.026	-118%
7	CB, VIX		0.619		0.06	-137%
8	$CB, BIST$		0.590	0.190		-152%
9	$Repo, VIX$	0.032			0.11	-108%
10	$Repo, BIST$	0.008		0.09		-106%
11	$Repo, CB$	0.007	0.188			-114%
12	$Repo$	0.011				-125%
13	CB		0.342			-166%
14	$BIST$			0.123		-155%
15	VIX				0.035	-139%

Table A.6. 1-Week AEM Regressions.

		$Pr(>\|t\|)$					
Model	**Combinations**	$Repo$	DR	CB	$BIST$	VIX	R^2
1	$Repo, DR, CB, BIST, VIX$	0.4056	0.1039	0.7463	0.5199	0.0952	26.86%
2	$DR, CB, BIST, VIX$		0.1495	0.3599	0.9739	0.0972	25.12%
3	$Repo, CB, BIST, VIX$	0.8090		0.7590	0.4900	0.1680	20.00%
4	$Repo, DR, BIST, VIX$	0.2293	0.0991		0.0212	0.0620	26.60%
5	$Repo, DR, CB, VIX$	0.5915	0.0953	0.0255		0.0354	25.82%
6	$Repo, DR, CB, BIST$	0.4440	0.1850	0.3990	0.1600		19.62%
7	$CB, BIST, VIX$			0.8480	0.3870	0.1580	19.84%
8	$DR, BIST, VIX$		0.2510		0.0300	0.1410	23.03%
9	DR, CB, VIX		0.0889	0.0182		0.0284	25.12%
10	$DR, CB, BIST$		0.2540	0.6590	0.1610		18.08%
11	$Repo, BIST, VIX$	0.9958			0.0901	0.0547	19.75%
12	$Repo, CB, VIX$	0.5728		0.1154		0.0699	18.74%
13	$Repo, CB, BIST$	0.7680		0.1770	0.1780		14.85%
14	$Repo, DR, VIX$	0.3970	0.6550			0.3790	12.64%
15	$Repo, DR, BIST$	0.7989	0.0884		0.1028		17.72%
16	$Repo, DR, CB$	0.4670	0.2080	0.2440			14.24%
17	$Repo, DR$	0.2410	0.6150				10.47%
18	$Repo, CR$	0.0705		0.9062			9.81%
19	$Repo, BIST$	0.0128			0.9558		9.78%
20	$Repo, VIX$	0.4440				0.3580	12.09%
21	DR, CB		0.0368	0.1358			12.78%
22	$DR, BIST$		0.0025		0.0441		17.56%
23	DR, VIX		0.8220			0.2320	10.63%
24	$CB, BIST$			0.0047	0.0247		14.61%
25	CB, VIX			0.0934		0.0119	17.92%
26	$BIST, VIX$				0.0596	0.0016	19.75%
27	$Repo$	3.9E-09					9.77%
28	DR		7.0E-09				6.61%
29	CB			2.2E-08			-2.27%
30	$BIST$				1.0E-07		-9.17%
31	VIX					3.4E-09	10.49%

Table A.7. 1-Week REM Regressions.

		$Pr(>\|t\|)$					
Model	**Combinations**	$Repo$	DR	CB	$BIST$	VIX	R^2
1	$Repo, DR, CB, BIST, VIX$	0.2761	0.8590	0.1280	0.0193	0.5079	-74%
2	$DR, CB, BIST, VIX$		0.8550	0.1344	0.0166	0.5712	-81%
3	$Repo, CB, BIST, VIX$	0.2680		0.1205	0.0168	0.5047	-74%
4	$Repo, DR, BIST, VIX$	0.3006	0.8255		0.0109	0.5238	-88%
5	$Repo, DR, CB, VIX$	0.2543	0.7704	0.0731		0.2115	-109%
6	$Repo, DR, CB, BIST$	0.2956	0.8755	0.1254	0.0093		-77%
7	$CB, BIST, VIX$			0.1050	0.0152	0.5644	-81%
8	$DR, BIST, VIX$		0.5554		0.0094	0.5839	-95%
9	DR, CB, VIX		0.9384	0.0762		0.2448	-118%
10	$DR, CB, BIST$		0.8507	0.1308	0.0084		-83%
11	$Repo, BIST, VIX$	0.2360			0.0098	0.5125	-88%
12	$Repo, CB, VIX$	0.2616		0.0711		0.2067	-110%
13	$Repo, CB, BIST$	0.2857		0.1168	0.0080		-77%
14	$Repo, DR, VIX$	0.2800	0.8680			0.2030	-133%
15	$Repo, DR, BIST$	0.3205	0.8071		0.0050		-91%
16	$Repo, DR, CB$	0.2981	0.7914	0.0681			-120%
17	$Repo, DR$	0.3280	0.8380				-145%
18	$Repo, CR$	0.3050		0.0650			-121%
19	$Repo, BIST$	0.2514			0.0044		-91%
20	$Repo, VIX$	0.2240				0.1940	-133%
21	DR, CB		0.9430	0.0704			-128%
22	$DR, BIST$		0.5492		0.0045		-97%
23	DR, VIX		0.5840			0.2320	-141%
24	$CB, BIST$			0.1018	0.0076		-83%
25	CB, VIX			0.0600		0.2380	-119%
26	$BIST, VIX$				0.0086	0.5781	-97%
27	$Repo$	0.2640					-145%
28	DR		0.5810				-152%
29	CB			0.0553			-128%
30	$BIST$				0.0040		-99%
31	VIX					0.2260	-144%

Table A.8. 1-Month AEM Regressions.

		$Pr(>\|t\|)$					
Model	**Combinations**	$\boldsymbol{Repo}$	$\boldsymbol{DR}$	$\boldsymbol{CB}$	$\boldsymbol{BIST}$	$\boldsymbol{VIX}$	$\boldsymbol{R^2}$
1	$Repo, DR, CB, BIST, VIX$	0.0591	0.0017	0.7347	0.5375	0.8700	33.51%
2	$DR, CB, BIST, VIX$		0.0071	0.1047	0.2694	0.9184	24.98%
3	$Repo, CB, BIST, VIX$	0.3820		0.1790	0.1810	0.8820	7.20%
4	$Repo, DR, BIST, VIX$	0.0128	0.0006		0.0048	0.9570	33.25%
5	$Repo, DR, CB, VIX$	0.0337	0.0007	0.0056		0.7671	32.65%
6	$Repo, DR, CB, BIST$	0.0555	0.0014	0.7592	0.5008		33.45%
7	$CB, BIST, VIX$			0.2480	0.2360	0.9030	4.84%
8	$DR, BIST, VIX$		0.0135		0.1511	0.5192	18.22%
9	DR, CB, VIX		0.0057	0.0606		0.7861	21.92%
10	$DR, CB, BIST$		0.0062	0.0772	0.2501		24.95%
11	$Repo, BIST, VIX$	0.6100			0.9100	0.4700	1.54%
12	$Repo, CB, VIX$	0.5550		0.8640		0.6640	1.60%
13	$Repo, CB, BIST$	0.3770		0.1240	0.1560		7.13%
14	$Repo, DR, VIX$	0.6168	0.0442			0.2765	13.38%
15	$Repo, DR, BIST$	0.0090	0.0004		0.0022		33.25%
16	$Repo, DR, CB$	0.0312	0.0005	0.0027			32.46%
17	$Repo, DR$	0.6363	0.0287				10.06%
18	$Repo, CR$	0.3820		0.1970			1.00%
19	$Repo, BIST$	0.1270			0.2540		-0.10%
20	$Repo, VIX$	0.5500				0.1760	1.50%
21	DR, CB		0.0037	0.0294			21.73%
22	$DR, BIST$		0.0036		0.0894		17.13%
23	DR, VIX		0.0387			0.2759	12.69%
24	$CB, BIST$			0.0466	0.1536		4.80%
25	CB, VIX			0.8690		0.4390	0.50%
26	$BIST, VIX$				0.7500	0.1070	0.73%
27	$Repo$	2.9E-09					-4.20%
28	DR		2.6E-10				9.44%
29	CB			1.8E-09			-1.35%
30	$BIST$				4.9E-09		-7.53%
31	VIX					1.3E-09	12.42%

Table A.9. 1-Month REM Regressions.

		$Pr(>\|t\|)$					
Model	**Combinations**	$\boldsymbol{Repo}$	$\boldsymbol{DR}$	$\boldsymbol{CB}$	$\boldsymbol{BIST}$	$\boldsymbol{VIX}$	$\boldsymbol{R^2}$
1	$Repo, DR, CB, BIST, VIX$	0.0988	0.9166	0.1929	8.06E-03	0.3810	-15.53%
2	$DR, CB, BIST, VIX$		0.6976	0.0816	3.23E-03	0.3908	-26.70%
3	$Repo, CB, BIST, VIX$	0.0855		0.1744	5.99E-03	0.3763	-15.57%
4	$Repo, DR, BIST, VIX$	0.0429	0.7480		4.82E-04	0.5230	-22.36%
5	$Repo, DR, CB, VIX$	0.0397	0.5795	0.0098		0.0235	-46.56%
6	$Repo, DR, CB, BIST$	0.0966	0.9798	0.2423	6.61E-04		-18.58%
7	$CB, BIST, VIX$			0.0838	1.77E-03	0.4114	-27.34%
8	$DR, BIST, VIX$		0.9090		2.30E-05	0.6070	-39.95%
9	DR, CB, VIX		0.3242	0.0005		0.0158	-68.35%
10	$DR, CB, BIST$		0.7912	0.1045	1.80E-04		-29.80%
11	$Repo, BIST, VIX$	0.0419			4.07E-04	0.4952	-22.78%
12	$Repo, CB, VIX$	0.0250		0.0102		0.0237	-48.03%
13	$Repo, CB, BIST$	0.0875		0.2114	5.02E-04		-18.58%
14	$Repo, DR, VIX$	0.0020	0.8756			0.0257	-82.40%
15	$Repo, DR, BIST$	0.0435	0.6982		3.46E-05		-24.01%
16	$Repo, DR, CB$	0.0269	0.7135	0.0103			-73.38%
17	$Repo, DR$	0.0010	0.7308				-113.6%
18	$Repo, CB$	0.0181		0.0093			-74.12%
19	$Repo, BIST$	0.0435			2.65E-05		-24.60%
20	$Repo, VIX$	0.0017				0.0222	-82.54%
21	DR, CB		0.4126	0.0004			-102.5%
22	$DR, BIST$		0.8660		8.13E-07		-41.13%
23	DR, VIX		0.8281			0.0135	-147.1%
24	$CB, BIST$			0.1019	1.04E-04		-30.09%
25	CB, VIX			0.0007		0.0172	-73.62%
26	$BIST, VIX$				1.68E-05	0.5920	-40.01%
27	$Repo$	0.0009					-114.4%
28	DR		0.9730				-198.1%
29	CB			0.0005			-106.8%
30	$BIST$				5.60E-07		-41.26%
31	VIX					0.0125	-147.5%

Table A.10. 1-Day Error Forecasts for Models 8, 12, 13 and 15.

Model	Instance	$P_{res}(\delta, \tau_3)$	$P_m(\delta, \tau_3)$	Simulation Error	Error Forecast	% Error Explained
8	1	94.666	95.843	1.178	0.017	1.4%
	2	94.594	95.904	1.310	-0.073	-5.6%
	3	94.660	96.014	1.354	-0.074	-5.5%
	4	94.199	95.836	1.637	-0.044	-2.7%
	5	94.286	95.913	1.627	-0.223	-13.7%
	6	94.188	96.084	1.895	-0.265	-14.0%
	7	94.247	96.159	1.912	-0.365	-19.1%
	8	94.371	96.369	1.998	-0.579	-29.0%
	9	94.596	96.661	2.065	-0.340	-16.5%
	10	94.565	96.704	2.139	-0.241	-11.3%
	Average			**1.712**	**-0.219**	**-12.8%**
12	1	94.666	95.843	1.178	0.532	45.2%
	2	94.594	95.904	1.310	0.527	40.2%
	3	94.660	96.014	1.354	0.597	44.1%
	4	94.199	95.836	1.637	0.526	32.1%
	5	94.286	95.913	1.627	0.518	31.8%
	6	94.188	96.084	1.895	0.518	27.3%
	7	94.247	96.159	1.912	0.519	27.1%
	8	94.371	96.369	1.998	0.518	25.9%
	9	94.596	96.661	2.065	0.493	23.9%
	10	94.565	96.704	2.139	0.493	23.0%
	Average			**1.712**	**0.524**	**30.6%**
13	1	94.666	95.843	1.178	0.826	70.2%
	2	94.594	95.904	1.310	0.823	62.9%
	3	94.660	96.014	1.354	0.828	61.2%
	4	94.199	95.836	1.637	0.837	51.2%
	5	94.286	95.913	1.627	0.830	51.0%
	6	94.188	96.084	1.895	0.824	43.5%
	7	94.247	96.159	1.912	0.829	43.4%
	8	94.371	96.369	1.998	0.824	41.2%
	9	94.596	96.661	2.065	0.832	40.3%
	10	94.565	96.704	2.139	0.830	38.8%
	Average			**1.712**	**0.828**	**48.4%**
15	1	94.666	95.843	1.178	0.779	66.2%
	2	94.594	95.904	1.310	0.709	54.1%
	3	94.660	96.014	1.354	0.709	52.3%
	4	94.199	95.836	1.637	0.812	49.6%
	5	94.286	95.913	1.627	0.663	40.7%
	6	94.188	96.084	1.895	0.620	32.7%
	7	94.247	96.159	1.912	0.617	32.3%
	8	94.371	96.369	1.998	0.566	28.3%
	9	94.596	96.661	2.065	0.606	29.3%
	10	94.565	96.704	2.139	0.625	29.2%
	Average			**1.712**	**0.670**	**39.2%**

Table A.11. 1-Week Error Forecasts for Models 22, 24, 27, 28 and 31.

Model	Instance	$P_{res}(\delta, \tau_3)$	$P_m(\delta, \tau_3)$	Simulation Error	Error Forecast	% Error Explained
22	1	94.764	95.905	1.140	0.093	8.2%
	2	94.693	96.002	1.309	0.199	15.2%
	3	94.758	95.794	1.036	-0.029	-2.8%
	4	94.295	95.997	1.702	0.003	0.2%
	5	94.386	96.113	1.727	-0.068	-3.9%
	6	94.290	96.145	1.855	0.001	0.1%
	7	94.347	96.358	2.012	-0.227	-11.3%
	8	94.469	96.606	2.137	-0.079	-3.7%
	9	94.697	96.684	1.986	-0.148	-7.4%
	10	94.667	96.870	2.203	-0.087	-4.0%
	Average			**1.711**	**-0.034**	**-2.0%**
24	1	94.764	95.905	1.140	0.155	13.6%
	2	94.693	96.002	1.309	0.125	9.6%
	3	94.758	95.794	1.036	0.185	17.9%
	4	94.295	95.997	1.702	0.042	2.5%
	5	94.386	96.113	1.727	-0.066	-3.8%
	6	94.290	96.145	1.855	-0.099	-5.3%
	7	94.347	96.358	2.012	-0.284	-14.1%
	8	94.469	96.606	2.137	-0.040	-1.9%
	9	94.697	96.684	1.986	-0.007	-0.4%
	10	94.667	96.870	2.203	0.168	7.6%
	Average			**1.711**	**0.018**	**1.0%**
27	1	94.764	95.905	1.140	0.611	53.6%
	2	94.693	96.002	1.309	0.649	49.6%
	3	94.758	95.794	1.036	0.657	63.4%
	4	94.295	95.997	1.702	0.576	33.9%
	5	94.386	96.113	1.727	0.578	33.4%
	6	94.290	96.145	1.855	0.578	31.1%
	7	94.347	96.358	2.012	0.578	28.7%
	8	94.469	96.606	2.137	0.548	25.6%
	9	94.697	96.684	1.986	0.548	27.6%
	10	94.667	96.870	2.203	0.551	25.0%
	Average			**1.711**	**0.587**	**34.3%**
28	1	94.764	95.905	1.140	0.779	68.3%
	2	94.693	96.002	1.309	0.825	63.0%
	3	94.758	95.794	1.036	0.740	71.4%
	4	94.295	95.997	1.702	0.774	45.5%
	5	94.386	96.113	1.727	0.763	44.2%
	6	94.290	96.145	1.855	0.796	42.9%
	7	94.347	96.358	2.012	0.744	37.0%
	8	94.469	96.606	2.137	0.759	35.5%
	9	94.697	96.684	1.986	0.729	36.7%
	10	94.667	96.870	2.203	0.718	32.6%
	Average			**1.711**	**0.763**	**44.6%**
31	1	94.764	95.905	1.140	0.852	74.7%
	2	94.693	96.002	1.309	0.868	66.3%
	3	94.758	95.794	1.036	0.947	91.4%
	4	94.295	95.997	1.702	0.780	45.8%
	5	94.386	96.113	1.727	0.733	42.5%
	6	94.290	96.145	1.855	0.718	38.7%
	7	94.347	96.358	2.012	0.660	32.8%
	8	94.469	96.606	2.137	0.721	33.7%
	9	94.697	96.684	1.986	0.779	39.2%
	10	94.667	96.870	2.203	0.687	31.2%
	Average			**1.711**	**0.774**	**45.3%**

Table A.12. 1-Month Error Forecasts for Models 15, 16 and 21.

Model	Instance	$P_{res}(\delta, \tau_3)$	$P_m(\delta, \tau_3)$	Simulation Error	Error Forecast	% Error Explained
15	1	95.151	95.913	0.763	-0.216	-28.4%
	2	95.077	96.135	1.058	-1.664	-157.2%
	3	95.135	96.213	1.077	-1.667	-154.7%
	4	94.676	96.514	1.838	-1.580	-86.0%
	5	94.769	96.684	1.915	-1.584	-82.7%
	6	94.663	96.802	2.139	-1.584	-74.0%
	7	94.724	96.898	2.175	-1.587	-73.0%
	8	94.860	97.097	2.237	-1.567	-70.1%
	9	95.074	97.180	2.106	-1.500	-71.3%
	10	95.041	97.208	2.168	-1.700	-78.4%
	Average			**1.747**	**-1.465**	**-83.8%**
16	1	95.151	95.913	0.763	0.399	52.3%
	2	95.077	96.135	1.058	0.343	32.4%
	3	95.135	96.213	1.077	0.629	58.4%
	4	94.676	96.514	1.838	0.182	9.9%
	5	94.769	96.684	1.915	0.007	0.4%
	6	94.663	96.802	2.139	0.001	0.0%
	7	94.724	96.898	2.175	-0.107	-4.9%
	8	94.860	97.097	2.237	-0.166	-7.4%
	9	95.074	97.180	2.106	-0.158	-7.5%
	10	95.041	97.208	2.168	-0.348	-16.1%
	Average			**1.747**	**0.078**	**4.5%**
21	1	95.151	95.913	0.763	0.485	63.6%
	2	95.077	96.135	1.058	0.457	43.2%
	3	95.135	96.213	1.077	0.615	57.1%
	4	94.676	96.514	1.838	0.334	18.2%
	5	94.769	96.684	1.915	0.236	12.3%
	6	94.663	96.802	2.139	0.232	10.9%
	7	94.724	96.898	2.175	0.174	8.0%
	8	94.860	97.097	2.237	0.132	5.9%
	9	95.074	97.180	2.106	0.111	5.3%
	10	95.041	97.208	2.168	0.082	3.8%
	Average			**1.747**	**0.286**	**16.4%**

Table A.13. 1-Month Error Forecasts for Models 28, 29 and 31.

Model	Instance	$P_{res}(\delta, \tau_3)$	$P_m(\delta, \tau_3)$	Simulation Error	Error Forecast	% Error Explained
28	1	95.151	95.913	0.763	1.128	147.9%
	2	95.077	96.135	1.058	1.111	105.0%
	3	95.135	96.213	1.077	1.160	107.7%
	4	94.676	96.514	1.838	1.085	59.0%
	5	94.769	96.684	1.915	1.062	55.5%
	6	94.663	96.802	2.139	1.062	49.7%
	7	94.724	96.898	2.175	1.046	48.1%
	8	94.860	97.097	2.237	1.039	46.4%
	9	95.074	97.180	2.106	1.036	49.2%
	10	95.041	97.208	2.168	1.023	47.2%
	Average			**1.747**	**1.075**	**61.5%**
29	1	95.151	95.913	0.763	1.328	174.2%
	2	95.077	96.135	1.058	1.316	124.4%
	3	95.135	96.213	1.077	1.323	122.8%
	4	94.676	96.514	1.838	1.325	72.1%
	5	94.769	96.684	1.915	1.331	69.5%
	6	94.663	96.802	2.139	1.333	62.3%
	7	94.724	96.898	2.175	1.332	61.2%
	8	94.860	97.097	2.237	1.338	59.8%
	9	95.074	97.180	2.106	1.342	63.7%
	10	95.041	97.208	2.168	1.336	61.6%
	Average			**1.747**	**1.330**	**76.1%**
31	1	95.151	95.913	0.763	1.091	143.1%
	2	95.077	96.135	1.058	1.011	95.6%
	3	95.135	96.213	1.077	1.017	94.4%
	4	94.676	96.514	1.838	0.951	51.7%
	5	94.769	96.684	1.915	1.099	57.4%
	6	94.663	96.802	2.139	1.012	47.3%
	7	94.724	96.898	2.175	0.949	43.6%
	8	94.860	97.097	2.237	1.141	51.0%
	9	95.074	97.180	2.106	1.162	55.2%
	10	95.041	97.208	2.168	1.050	48.4%
	Average			**1.747**	**1.048**	**60.0%**

A.3. Simulation Algorithm R Codes

```
SIM_ALGO←function(n,mu,sigma,delta,f_initial){
#Simulation algorithm for forward rates of 3-factor HJM model
to calculate delta time ahead zero-coupon bond prices

#Inputs
#n...............number of simulation runs
#mu..............estimated drift (vector of dimensions 3x1)
#sigma...........estimated volatilities (matrix of dimensions
3x3)
#delta...........time progress in terms of days
#f_initial.......initial forward rate curve (vector of
dimensions 3x1)

#Initializations
f_3←0
f_6←0
f_12←0
p←0
res←0

#Computations of n simulated forward rates
f_3←f_initial[1]+delta/365*mu[1]+(sigma[1,1]*rnorm(n,0,sqrt(
delta/365))+sigma[1,2]*rnorm(n,0,sqrt(delta/365))+sigma[1,3]*
rnorm(n,0,sqrt(delta/365)))
f_6←f_initial[2]+delta/365*mu[2]+(sigma[2,1]*rnorm(n,0,sqrt(
delta/365))+sigma[2,2]*rnorm(n,0,sqrt(delta/365))+sigma[2,3]*
rnorm(n,0,sqrt(delta/365)))
f_12←f_initial[3]+delta/365*mu[3]+(sigma[3,1]*rnorm(n,0,sqrt
(delta/365))+sigma[3,2]*rnorm(n,0,sqrt(delta/365))+sigma[3,3]
*rnorm(n,0,sqrt(delta/365)))
```

```
#Finding n simulated zero-coupon bond prices
tau_1←3/12
tau_2←6/12
tau_3←12/12

p←exp(-(f_3*tau_1+f_6*(tau_2-tau_1)+f_12*(tau_3-tau_2)))*100

#Result calculations
p_res←mean(p)
st_error←sd(p)/sqrt(n)
results←c(p_res,st_error)
names(results)←c("p_res","st_error")

#Return of outputs
results

}
```

REFERENCES

1. Amin, K. I., and A. Morton, “Implied Volatility Functions in Arbitrage-Free Term Structure Models”, *Journal of Financial Economics*, Vol. 35, No. 2, pp. 141-180, 1994.

2. Baz, J., and G. Chacko, *Financial Derivatives: Pricing, Applications, and Mathematics*, Cambridge University Press, New York, 2009.

3. Björk, T., *Arbitrage Theory in Continuous Time*, Oxford University Press, New York, 3rd edition, 2009.

4. Black, F., E. Derman, and W. Toy, "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options", *Financial Analysts Journal*, Vol. 46, No. 1, pp. 33-39, 1990.

5. Black, F., and P. Karasinski, "Bond and Option Pricing when Short Rates are Lognormal", *Financial Analysts Journal*, Vol. 47, No. 4, pp. 52-59, 1991.

6. Borsa İstanbul, *Data / Debt Securities Market Data / Bulletin Data*, 2013, http://borsaistanbul.com/en/data/data/debt-securities-market-data/bulletin-data, 13 March 2013.

7. Borsa İstanbul, *Data / Debt Securities Market Data / Traded Value Data*, 2013, http://borsaistanbul.com/en/data/data/debt-securities-market-data/traded-value-data, 13 March 2013.

8. Borsa İstanbul, *Data / Equity Market Data / Index Data*, 2013, http://borsaistanbul.com/en/data/data/equity-market-data/index-data, 13 March 2013.

9. Brace, A., D. Gątarek, and M. Musiela, “The Market Model of Interest Rate Dynamics”, *Mathematical Finance*, Vol. 7, No. 2, pp. 127-155, 1997.

10. Brennan, M. J., and E. S. Schwartz, “A Continuous Time Approach to the Pricing of Bonds”, *Journal of Banking & Finance*, Vol. 3, No. 2, pp. 133-155, 1979.

11. Brennan, M. J., and E. S. Schwartz, “An Equilibrium Model of Bond Pricing and a test of Market Efficiency”, *Journal of Financial & Quantitative Analysis*, Vol. 17, No. 3, pp. 301-329, 1982.

12. Brigo, D., and F. Mercurio, "A Deterministic–Shift Extension of Analytically–Tractable and Time–Homogeneous Short–Rate Models", *Finance and Stochastics*, Vol. 5, No. 3, pp. 369-387, 2001.

13. Central Bank of Repuclic of Turkey, *Electronic Data Delivery System*, 2013 http://evds.tcmb.gov.tr/yeni/cbt-uk.html, 13 March 2013.

14. Chan, K. C., A. Karolyi, F. A. Longstaff, and A. B. Sanders, "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate", *The Journal of Finance*, Vol. 47, No. 3, pp. 1209-1227, 1992.

15. Chicago Board Options Exchange, *Products / Index Microsites*, 2013, http://www.cboe.com/micro/VIX/vixintro.aspx, 13 March 2013.

16. Cox, J. C., J. E. Ingersoll, and S. A. Ross, "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol. 53, No. 2, pp. 385-408, 1985.

17. Dothan, L. U., "On the Term Structure of Interest Rates", *Journal of Financial Economics*, Vol. 6, No. 1, pp. 59-69, 1978.

18. Everitt, B., and T. Hothorn, *An Introduction to Applied Multivariate Analysis with R*, Springer, New York, Dordrecht, Heidelberg, London, 2011.

19. Filipović, D., *Term-Structure Models: A Graduate Course*, Springer, Dordrecht, Heidelberg, London, New York, 2009.

20. Glasserman, P., *Monte Carlo Methods in Financial Engineering*, Springer, New York, 2004.

21. Heath, D., R. Jarrow, and A. Morton, “Bond pricing and the Term Structure of Interest Rates: A Discrete Time Approximation”, *The Review of Futures Markets*, Vol. 9, No. 1, pp. 54-76, 1990.

22. Heath, D., R. Jarrow, and A. Morton, “Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation”, *Econometrica*, Vol. 60, No. 1, pp. 77-105, 1992.

23. Heath, D., R. A. Jarrow, and A. Morton, “Contingent Claim Valuation with a Random Evolution of Interest Rates”, *The Review of Futures Markets*, Vol. 9, No. 1, pp. 54-76, 1990.

24. Ho, T. S. Y., and S. B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims", *The Journal of Finance*, Vol. 41, No. 5, pp. 1011-1029, 1986.

25. Hull, J., and A. White, "One-Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities", *Journal of Financial & Quantitative Analysis*, Vol. 28, No. 2, pp. 235-254, 1993.

26. Hull, J., and A. White, "Pricing Interest-Rate-Derivative Securities", The *Review of Financial Studies*, Vol. 3, No. 4, pp. 573-592, 1990.

27. Jamshidian, F., “LIBOR and Swap Market Models and Measures”, *Finance and Stochastics*, Vol. 1, No. 4, pp. 293-330, 1997.

28. Jara, D., *An Extension of Lévy’s Theorem and Applications to Financial Models Based on Futures Prices*, Ph.D. Thesis, Carnegie Mellon University, 2000.

29. Jarrow, R. A., *Modelling Fixed-Income Securities and Interest Rate Options*, Stanford University Press, Stanford, 2nd edition, 2002.

30. Jarrow, R., and S. Turnbull, *Derivative Securities*, South-Western College Publishing, Cincinnati, 2nd edition, 2000.

31. Kuo, I. D., Y. N. Lin, Y. C. Chang, "Which Interest Rate Option Model?", *International Research Journal of Finance and Economics*, Vol. 10, pp. 65-75, 2007.

32. Longstaff, F. A., "A Nonlinear General Equilibrium Model of the Term Structure of Interest Rates", *Journal of Financial Economics*, Vol. 23, No. 2, pp. 195-224, 1989.

33. Longstaff, F. A., and E. S. Schwartz, "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model", *The Journal of Finance, Vol.* 47, No. 4, pp. 1259-1282, 1992.

34. Lyuu, Y. D., *Financial Engineering and Computation: Principles, Mathematics, Algorithms*, Cambridge University Press, New York, 2002.

35. Merton, R. C., "Theory of Rational Option Pricing", *The Bell Journal of Economics and Management Science*, Vol. 4, No. 1, pp. 141-183, 1973.

36. Miltersen, K., K. Sandmann, and D. Sondermann, "Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates", *The Journal of Finance*, Vol. 52, No. 1, pp. 409-430, 1997.

37. Musiela, M., and M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer, Berlin, Heidelberg, 2nd edition, 2005.

38. Sandmann, K., and D.S ondermann, "A Note on the Stability of Log-Normal Interest Rate Models and the Pricing of Eurodollar Futures", *Mathematical Finance*, Vol. 7, No. 2, pp. 119-125, 1997.

39. Shreve, S. E., *Stochastic Calculus for Finance II: Continuous Time Models*, Springer, New York, 2005.

40. Subrahmanyam, M. G., "The Term Structure of Interest Rates: Alternative Approaches and Their Implications for the Valuation of Contingent Claims", *The Geneva Papers on Risk and Insurance Theory*, Vol. 21, No. 1, pp. 7-28, 1996.

41. Suvaj, E. B., and W. Aleksander, “Calibration of the Multi-Factor HJM Model for Energy Markets”, *Acta Physica Polonica B*, Vol. 37, No. 5, pp. 1455-1466, 2006.

42. Vasicek, O., “An Equilibrium Characterization of the Term Structure”, *Journal of Financial Economics*, Vol. 5, No. 2, pp. 177-188, 1977.

43. Walpole, R. E., R. H. Myers, S. L. Myers, and K. Ye, *Probability & Statistics for Engineers & Scientists*, Pearson Prentice Hall, New Jersey, 8th edition, 2007.

44. Wilmott, P., *Paul Wilmott on Quantitative Finance*, John Wiley & Sons, West Sussex, 2nd edition, 2006.