

# QUOTIENTS OF HOM-FUNCTORS

by

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**ABSTRACT****QUOTIENTS OF HOM-FUNCTORS**

Quotients of Hom-functors are functors of the form  $\text{Hom}_R(P, -)/\text{Hom}_R(P, -)J$  where  $P$  is a projective  $R$ -module and  $J$  is a certain ideal of the endomorphism ring of  $P$ . These functors were used by R. Dipper in the articles *On Quotients of Hom-Functors and Representations of Finite General Linear Groups I-II*, to obtain a classification of the irreducible  $l$ -modular representations of  $GL_n(q)$  for primes  $l$  not dividing  $q$ . In this thesis, the general properties of these functors are examined following Dipper's articles [6] and [7]. Besides, the relation between the quotients of Hom-functors and the Harish-Chandra theory is investigated.

## ÖZET

### HOM-İZLEÇLERİN BÖLÜMLERİ

Hom-izleçlerin bölümleri, projektif bir  $R$ -modülü  $P$  ve  $P$ 'nin endomorfizma halkasının bir ideali  $J$  için  $\text{Hom}_R(P, -)/\text{Hom}_R(P, -)J$  şeklinde tanımlanan izleçlerdir. Bu izleçler R. Dipper'in *On Quotients of Hom-Functors and Representations of Finite General Linear Groups I-II* adlı makalelerinde,  $q$ 'yu bölmeyen  $l$  asal sayıları için  $GL_n(q)$ 'nin indirgenemez  $l$ -modüler temsillerinin sınıflandırılmasında kullanılmıştır. Bu tezde, Dipper'in makaleleri ([6] ve [7]) kullanılarak, bu izleçlerin genel özellikleri incelenmiştir. Ayrıca, Hom-izleçlerin bölümleri ile Harish-Chandra kuramı arasındaki ilişki çalışılmıştır.

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## LIST OF SYMBOLS

$c_{K,G}^f$	The $(K, G)$ -biset $K$ for groups $G$ and $K$ and a group isomorphism $f : G \rightarrow K$ with the left action of $K$ by multiplication, and right action of $G$ by taking image by $f$ , and then multiplying on the right in $K$
$C_{K,G}^f$	Transport of structure functor from the category of $FG$ -modules to the category of $FK$ -modules
$\text{def}_{G/N}^G$	The $(G/N, G)$ -biset $G/N$ for a group $G$ and a normal subgroup of $G$ with the left action of $G/N$ by multiplication, and the right action of $G$ by projection to $G/N$ , and then right multiplication in $G/N$
$\text{Def}_{G/N}^G$	Deflation functor from the category of $FG$ -modules to the category of $F[G/N]$ -modules
$\text{End}_T(M)$	The set of $T$ -module endomorphisms of $M$
$(\text{End}_T(P))_\beta$	The set of $T$ -module endomorphisms of $P$ under which $\ker\beta$ is invariant
$FX$	Permutation $FG$ -module with permutation basis $X$ where $X$ is a $G$ -set
${}^xG$	Conjugate group $x^{-1}Gx$ for a group $G$
$hd(V)$	Head of $V$
$(H, K)_x$	Stabilizer of $x$ in $H \times K$
$\text{Hom}_R(A, B)$	The set of $R$ -linear maps from $A$ to $B$
$\text{im}f$	Image of $f$
$\text{ind}_H^G$	The $(G, H)$ -biset $G$ for a group $G$ and a subgroup $H$ of $G$ with actions left and right multiplications in $G$
$\text{Ind}_H^G$	Induction functor from the category of $FH$ -modules to the category of $FG$ -modules
$\text{inf}_{G/N}^G$	The $(G, G/N)$ -biset $G/N$ for a group $G$ and a normal subgroup $N$ of $G$ with the left action of $G$ by projection to $G/N$ , and then left multiplication in $G/N$ , and the right action of $G/N$ by multiplication

$\text{Inf}_{G/N}^G$	Inflation functor from the category of $F[G/N]$ -modules to the category of $FG$ -modules
$\text{Irr}T$	The complete set of non-isomorphic irreducible $T$ -modules
$J_\beta$	The set of $T$ -module homomorphisms of $P$ whose images contained in $\ker\beta$
$\text{Jac}(V)$	Jacobson radical of $V$
$\ker f$	Kernel of $f$
$\ker P$	Kernel of $P$
<b>M</b>	Mackey system
$M^x$	Conjugate module for the conjugate group $xGx^{-1}$ for an $FG$ -module $M$
$\text{mod}_R$	Category of finitely generated right $R$ -modules
${}_R\text{mod}$	Category of finitely generated left $R$ -modules
$P\backslash G/Q$	$P$ - $Q$ -double coset of representatives in $G$
$\text{res}_H^G$	The $(H, G)$ -biset $G$ for a group $G$ and a subgroup $H$ of $G$ with actions given by left and right multiplications in $G$
$\text{Res}_H^G$	Restriction functor from the category of $FG$ -modules to the category of $FH$ -modules
$R_{P/U}^G$	Harish-Chandra induction from $F[P/U]$ -modules to $FG$ -modules for a field $F$
$\text{soc}(V)$	Socle of $V$
$T_{P/U}^G$	Harish-Chandra truncation from $FG$ -modules to $F[P/U]$ -modules for a field $F$
$\text{tor}_P(V)$	$P$ -torsion submodule of $V$
$\text{tr}_{V_1}(V_2)$	Trace of $V_1$ in $V_2$
$YM$	The submodule of $M$ generated by the images of homomorphisms in $Y$ for a subset $Y$ of $\text{End}_T(M)$
$V \times_H U$	the composition of $V$ and $U$ for an $(H, G)$ -biset $U$ and $(K, H)$ -biset $V$
$1_M$	Identity map on $M$

## 1. INTRODUCTION

Quotients of Hom-functors are functors of the form  $\text{Hom}(P, -)/\text{Hom}(P, -)J$  where  $P$  is projective and  $J$  is a certain ideal of the endomorphism ring of  $P$ . Their terminology and properties were developed by R. Dipper in the articles [6] and [7], and they were used to obtain a classification of the irreducible  $l$ -modular representations of  $GL_n(q)$  for primes  $l$  not dividing  $q$ , and to obtain information on decomposition numbers in terms of Hecke algebras and  $q$ -Schur algebras, in [7].

For a Noetherian commutative ring  $R$ , semiperfect  $R$ -algebra  $T$  with a multiplicative identity, and a projective presentation  $\beta : P \rightarrow M$  where  $P$  and  $M$  are  $T$ -modules, the map

$$H = \text{Hom}_T(P, -)/\text{Hom}_T(P, -)J_\beta$$

where  $J_\beta$  is the ideal of  $\text{End}_T(P)$  consists of endomorphisms of  $P$  under which  $\ker \beta$  is invariant, is a functor from the category of  $T$ -modules to the category of  $\text{End}_T(M)$ -modules. After studying the properties of that functor in [6] and [7], Dipper considered a more specialized situation; taking a discrete complete valuation ring  $O$  with quotient field  $K$  and residue field  $F$ , he replaced the algebra  $T$  with the  $R$ -algebra  $T_R$  where  $R = K, O, F$ , and constructed  $H$  using this  $T_R$  and obtained results similar to the general case.

It was stated in Dipper [7] that, for a finite reductive group  $G$  and  $R = F, K$ , the irreducible  $RG$ -modules are determined using the following method: For any Levi subgroup  $L$  of  $G$ , the irreducible  $RL$ -modules are found. Then, for any Levi subgroup  $L$  and a cuspidal irreducible  $RL$ -module  $C$ , the irreducible  $\text{End}_{RG}(R_L^G(C))$ -modules are found where  $R_L^G$  is the Harish-Chandra induction. Then using the bijection between the isomorphism classes of the irreducible  $RG$ -modules occurring in the head of  $R_L^G(C)$  and a set of representatives of the isomorphism classes of the irreducible  $\text{End}_{RG}(R_L^G(C))$ -modules, the classification of the irreducible  $RG$ -modules is achieved.

As an application to the general theory, it was proved in Dipper [7] that, in the case  $G = GL_n(q)$ , the endomorphism ring  $\text{End}_{RG}(R_L^G(C))$  is isomorphic to a product of some Hecke algebras associated with symmetric groups. Therefore, the representation theory of  $GL_n(q)$  is related to Hecke algebras associated with symmetric groups through the functor

$$H = \text{Hom}_{RG}(P_R, -) / \text{Hom}_{RG}(P_R, -) J_{\beta_R}.$$

Using this method, the classification of non-isomorphic irreducible  $RGL_n(q)$ -modules was achieved in [7], and also, a complete set of non-isomorphic cuspidal irreducible  $FGL_n(q)$ -modules was given.

The aim of this thesis is to examine the properties of quotients of Hom-functors and their connection with the Harish-Chandra theory, and to understand the application of the theory of Hom-functors to the classification of representations of general linear groups, using Dipper [6] and [7]. The thesis is organized as follows:

In Chapter 2, some preliminary definitions and results which are required to construct quotients of Hom-functors are stated.

In Chapter 3, the theory of quotients of Hom-functors is introduced and the properties of those functors are examined in a detailed way.

In Chapter 4, the connection between quotients of Hom-functors and the Harish-Chandra theory is studied. Besides, the notion of bisets is introduced and Mackey Decomposition Theorem (Dipper [7, 2.2.1]) is proved using biset functors.

## 2. PRELIMINARIES

We start with defining what a semiperfect ring is. Firstly, we need some preliminary definitions. A module  $P$  over a ring  $R$  is said to be *projective* if given any diagram of  $R$ -module homomorphisms  $f$  and  $g$

$$\begin{array}{ccccc} & & P & & \\ & \swarrow h & \downarrow f & & \\ A & \xrightarrow{g} & B & \longrightarrow & 0 \end{array}$$

with bottom row exact (that is,  $g$  is an epimorphism), there exists an  $R$ -module homomorphism  $h : P \rightarrow A$  such that the diagram commutes, that is  $gh = f$ . A submodule  $S$  of a module  $M$  is *superfluous* if, whenever  $L$  is a submodule of  $M$  with  $L + S = M$ , then  $L = M$ . A *projective cover* of a module  $M$  is an ordered pair  $(P, \varphi)$ , where  $P$  is a projective module and  $\varphi : P \rightarrow M$  is a surjective map with  $\ker \varphi$  a superfluous submodule of  $P$ .

A ring  $R$  is *semiperfect* if every finitely generated right  $R$ -module has a projective cover.

For a ring  $R$ , the category of finitely generated right  $R$ -modules is denoted by  $\text{mod}_R$  and the category of finitely generated left  $R$ -modules is denoted by  ${}_R\text{mod}$ . Let  $M \in \text{mod}_R$ ,  $P \in \text{mod}_R$  and  $P$  be projective. Let  $\beta : P \rightarrow M$  be an epimorphism of right  $R$ -modules. Then  $\beta$  is called a *projective presentation* of  $M$ .

An  $R$ -module  $M$  is said to satisfy the *ascending chain condition on submodules* (or is *Noetherian*) if for every chain

$$M_1 \subseteq M_2 \subseteq M_2 \subseteq M_3 \subseteq \dots$$

of submodules of  $M$ , there is an integer  $n$  such that  $M_i = M_n$  for all  $i \geq n$ .

An  $R$ -module  $N$  is said to satisfy the *descending chain condition on submodules* (or is *Artinian*) if for every chain

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \dots$$

of submodules of  $N$ , there is an integer  $m$  such that  $N_i = N_m$  for all  $i \geq m$ .

A ring  $R$  is said to be *left* [resp. *right*] *Noetherian* if  $R$  satisfies the ascending chain condition on left [resp. right] ideals.  $R$  is said to be *Noetherian* if  $R$  is both left and right Noetherian.

A ring  $R$  is said to be *left* [resp. *right*] *Artinian* if  $R$  satisfies the descending chain condition on left [resp. right] ideals.  $R$  is said to be *Artinian* if  $R$  is both left and right Artinian.

**Definiton 2.1.** *Let  $V$  be an  $R$ -module. The Jacobson radical of  $V$  is defined as the intersection of all maximal submodules of  $V$ , denoted by  $\text{Jac}(V)$ .*

*The head of  $V$  is the factor module  $V/\text{Jac}(V)$ , denoted by  $\text{hd}(V)$ . Therefore  $\text{hd}(V)$  is the largest semisimple factor module of  $V$ .*

*The socle of  $V$  is the largest semisimple submodule of  $V$ , denoted by  $\text{soc}(V)$ .*

**Definiton 2.2.** *Let  $R$  be a ring.*

- (i) *A nonzero element  $e$  of  $R$  is called an idempotent if  $e^2 = e$ .*
- (ii) *Two idempotents  $e_1$  and  $e_2$  of  $R$  are said to be orthogonal if  $e_1e_2 = e_2e_1 = 0$ .*
- (iii) *An idempotent is called primitive if it is not the sum of two orthogonal idempotents.*
- (iv) *An idempotent decomposition of 1 in  $R$  is a set of pairwise orthogonal idempotents  $e_1, \dots, e_r$  such that  $1 = \sum_{i=1}^r e_i$ . An idempotent decomposition is called primitive if all the involved idempotents are primitive.*

**Lemma 2.3.** *(Fitting's Lemma) Let  $R$  be a ring and  $M$  be an  $R$ -module. Then there*

is a one to one correspondence between idempotent decompositions of  $1 = \sum_{i \in I} e_i$  in  $\text{End}_R(M)$ , where  $I$  is finite, and decompositions  $M = \sum_{i \in I} M_i$ , characterized by the fact that  $e_j$  is the projection of  $M$  onto  $M_j$  with kernel  $\sum_{i \neq j} M_i$ .

*Proof.* See [9, I.1.4]. □

**Proposition 2.4.** *Let  $R$  be a ring.*

- (i) *Let  $P$  be a projective  $R$ -module and  $\phi$  be in  $\text{End}_R(P)$ . Then  $\phi$  is in  $\text{Jac}(\text{End}_R(P))$  if and only if  $\text{im}\phi$  is superfluous in  $P$ .*
- (ii) *If  $R$  is left Artinian, then  $\text{Jac}(R)$  is nilpotent.*

*Proof.* (i) See [1, 17.11].

(ii) See [9, I.3.6(i)] □

**Proposition 2.5.** *Let  $R$  be a right Artinian ring and let  $\{e_i\}$  be a set of primitive idempotents of  $R$ . Set  $P_i = e_i R$ . Then,  $P_i$  contains a unique maximal submodule, namely  $e_i \text{Jac}(R)$ .*

*Proof.* See [9, I.3.14]. □

- Definiton 2.6.**
- (i) *A ring  $R$  is called self-injective if the regular  $R$ -module  $R$  is injective.*
  - (ii) *A ring  $R$  is called quasi-Frobenius if it is Noetherian and injective as an  $R$ -module.*
  - (iii) *If a ring  $R$  is a direct sum of indecomposable modules, say  $R = \bigoplus_i L_i$ , then any module  $M$  isomorphic to some  $L_i$  is called a principal indecomposable module.*

**Proposition 2.7.** *If  $R$  is quasi-Frobenius, then there is a bijection between its minimal left ideals and its principal indecomposable modules.*

*Proof.* See [11, 4.48]. □

**Proposition 2.8.** *Let  $R$  be a ring, and let  $M$  and  $N$  be  $R$ -modules.*

(i) *We have*

$$\text{Jac}(M) = \sum \{L \leq M \mid L \text{ is superfluous in } M\}.$$

(ii) *If  $f : M \rightarrow N$  is an epimorphism and  $\ker f$  is a submodule of  $\text{Jac}(M)$ , then  $\text{Jac}(N) = f(\text{Jac}(M))$ .*

*Proof.* (i) See [1, 9.13].

(ii) See [1, 9.15]. □

**Proposition 2.9.** *Let  $S$  and  $T$  be rings,  $U$  be an  $S$ – $T$ -bimodule,  $N$  be a left  $T$ -module and  $P$  be a projective left  $T$ -module. Then, there is a natural homomorphism*

$$\eta : \text{Hom}_S(P, U) \otimes_T N \rightarrow \text{Hom}_S(P, (U \otimes_T N))$$

*defined by*

$$\eta(\gamma \otimes_T n) : p \mapsto \gamma(p) \otimes_T n$$

*where  $\gamma \in \text{Hom}_S(P, U)$ ,  $n$  in  $N$  and  $p$  in  $P$ . If  $P$  is finitely generated and projective, then  $\eta$  is an isomorphism.*

*Proof.* See [1, 20.10]. □

**Proposition 2.10.** *A finitely generated left module over a Noetherian ring is Noetherian.*

*Proof.* See [5, 3.3]. □

**Proposition 2.11.** *Let  $R$  be a semiperfect ring and consider only finitely generated  $R$ -modules. Let  $N = \text{Jac}(R)$ . Let  $f : P \rightarrow X$  be a surjection with  $P$  projective. Then  $f$  gives a projective cover if and only if  $\ker f \subseteq NP$ .*

*Proof.* See [5, 6.25(i)]. □

**Lemma 2.12.** *(Nakayama's Lemma) Let  $R$  be a commutative ring. Let  $I$  be an ideal of  $R$  which is contained in every maximal ideal of  $R$ . If  $M$  is a finitely generated  $R$ -module and  $MI = M$ , then  $M = (0)$ .*

*Proof.* See [10, X.4.1]. □

### 3. THE QUOTIENTS OF HOM-FUNCTORS

#### 3.1. The Ideal $J_\beta$

Let  $R$  be a commutative Noetherian ring and  $T$  be a semiperfect  $R$ -algebra which is finitely generated as an  $R$ -module. Assume that both  $T$  and  $R$  have multiplicative identities, and that  $T$  is unital as  $R$ -module. Let  $M$  be a finitely generated left  $T$ -module. Since  $T$  is semiperfect, there exists a projective presentation  $(\beta, P)$  of  $M$ . In this work, all modules are finitely generated unless stated otherwise. The set of  $R$ -module endomorphisms of  $M$  is denoted by  $\text{End}_R(M)$ .

**Notation.**  $(\text{End}_T(P))_\beta = \{\phi \in \text{End}_T(P) \mid \phi(\ker\beta) \subseteq \ker\beta\}$

$$J_\beta = \{\psi \in \text{End}_T(P) \mid \text{im}\psi \leq \ker\beta\}$$

In [6], it was stated that  $(\text{End}_T(P))_\beta/J_\beta$  and  $\text{End}_T(M)$  are isomorphic as  $R$ -algebras. Now, we prove this statement.

**Proposition 3.1.**  $J_\beta$  is an ideal of  $(\text{End}_T(P))_\beta$  and  $(\text{End}_T(P))_\beta/J_\beta \cong \text{End}_T(M)$  as  $R$ -algebra canonically .

*Proof.* Clearly, the set  $J_\beta$  is a subset of  $(\text{End}_T(P))_\beta$ . Also  $J_\beta$  is nonempty since 0 is an element of  $J_\beta$ . Let  $\psi_1$  and  $\psi_2$  be in  $J_\beta$ . For any  $p$  in  $P$ , we have

$$(\psi_1 - \psi_2)(p) = \psi_1(p) - \psi_2(p) \in \ker\beta$$

since  $\text{im}\psi_1$  and  $\text{im}\psi_2$  are submodules of  $\ker\beta$ . So, the set  $\text{im}(\psi_1 - \psi_2)$  is also a submodule of  $\ker\beta$ . Hence the element  $\psi_1 - \psi_2$  is in  $J_\beta$ . Let  $\psi$  be in  $J_\beta$  and  $\phi$  be in  $(\text{End}_T(P))_\beta$ . For  $p$  in  $P$ , we have  $\phi(\psi(p))$  is in  $\ker\beta$  since  $\text{im}\psi$  is a submodule of  $\ker\beta$  and  $\phi(\ker\beta)$  is a subset of  $\ker\beta$ . Hence  $J_\beta$  is an ideal of  $(\text{End}_T(P))_\beta$ .

Now, define  $\tilde{\beta} : (\text{End}_T(P))_\beta \rightarrow \text{End}_T(M)$  as

$$\tilde{\beta}(\phi)(m) := \beta(\phi(p))$$

for  $\phi$  in  $(\text{End}_T(P))_\beta$ , the element  $m$  in  $M$  and  $p$  in  $P$  such that  $\beta(p) = m$ . Such a  $p$  always exists since  $\beta$  is surjective.

For each  $\phi$  in  $(\text{End}_T(P))_\beta$ , the map  $\tilde{\beta}(\phi)$  is well-defined since for  $p_1$  and  $p_2$  in  $P$  such that  $p_1 \neq p_2$ , if  $\beta(p_1) = \beta(p_2)$  then  $p_1 - p_2$  is in  $\ker\beta$ . This implies  $\phi(p_1 - p_2)$  is in  $\ker\beta$  since  $\phi$  is in  $(\text{End}_T(P))_\beta$ . That means  $\beta(\phi(p_1 - p_2)) = 0$ . Then, we have  $\beta(\phi(p_1)) = \beta(\phi(p_2))$ , that is  $\tilde{\beta}(\phi)(\beta(p_1)) = \tilde{\beta}(\phi)(\beta(p_2))$ . Also  $\tilde{\beta}$  is well-defined as  $\beta$  is well-defined.

Now, we are to show that  $\tilde{\beta}$  is an  $R$ -algebra homomorphism. Let  $\phi, \phi_1$  and  $\phi_2$  be in  $(\text{End}_T(P))_\beta$ , the element  $r$  be in  $R$ , the element  $m$  be in  $M$  and  $p$  be in  $P$  such that  $\beta(p) = m$ . Then

$$\begin{aligned} \tilde{\beta}(\phi_1 + \phi_2)(m) &= \beta((\phi_1 + \phi_2)(p)) = \beta(\phi_1(p) + \phi_2(p)) = \beta(\phi_1(p)) + \beta(\phi_2(p)) \\ &= \tilde{\beta}(\phi_1)(m) + \tilde{\beta}(\phi_2)(m) \end{aligned}$$

and

$$\begin{aligned} \tilde{\beta}(\phi_1\phi_2)(m) &= \beta((\phi_1\phi_2)(p)) = \beta(\phi_1(\phi_2(p))) = \tilde{\beta}(\phi_1)(\beta(\phi_2(p))) \\ &= \tilde{\beta}(\phi_1)(\tilde{\beta}(\phi_2)(m)) = \tilde{\beta}(\phi_1)\tilde{\beta}(\phi_2)(m) \end{aligned}$$

and

$$\tilde{\beta}(r\phi)(m) = \beta(r\phi(p)) = r\beta(\phi(p)) = r\tilde{\beta}(\phi)(m)$$

since  $\beta$  is an  $R$ -module homomorphism. That proves  $\tilde{\beta}$  is an  $R$ -algebra homomorphism.

Now, we are to prove that  $\tilde{\beta}$  is surjective. To this end, let  $\psi$  be in  $\text{End}_T(M)$ . Since  $\psi\beta$  is an  $R$ -module homomorphism, the map  $\beta$  is surjective and  $P$  is projective, there exists a  $\phi$  in  $\text{End}_T(P)$  such that the diagram

$$\begin{array}{ccc}
 & P & \\
 & \searrow \phi & \downarrow \beta \\
 & & M \\
 P & \xrightarrow{\beta} & \downarrow \psi \\
 & & M
 \end{array}$$

commutes. That is, we have  $\beta\phi = \psi\beta$ . Then,

$$\beta(\phi(\ker\beta)) = \psi(\beta(\ker\beta)) = \psi(0) = 0.$$

So, the set  $\phi(\ker\beta)$  is a subset of  $\ker\beta$ . Hence, the map  $\phi$  is in  $(\text{End}_T(P))_\beta$ , and it is mapped to  $\psi$  under  $\tilde{\beta}$  since for  $m$  in  $M$  and  $p$  in  $P$  such that  $\beta(p) = m$ , we have

$$\tilde{\beta}(\phi)(m) = \beta(\phi(p)) = \psi(\beta(p)) = \psi(m).$$

Therefore, the map  $\tilde{\beta}$  is surjective.

Finally, we are to show that  $J_\beta = \ker\tilde{\beta}$ . Let  $\phi$  be in  $(\text{End}_T(P))_\beta$ . By definition,  $\phi$  is in  $J_\beta$  means  $\text{im}\phi$  is a submodule of  $\ker\beta$ , and that means  $\beta(\phi(p)) = 0$  for all  $p$  in  $P$ , and so  $\tilde{\beta}(\phi)(m) = 0$  for all  $m$  in  $M$ , or equivalently, the map  $\phi$  is an element of  $\ker\tilde{\beta}$ .

Therefore, using the First Isomorphism Theorem, we conclude that

$$\text{End}_T(P)/J_\beta \cong \text{End}_T(M).$$

That proves the proposition. □

### 3.2. Definition of the Functor $H$

Using a projective presentation  $(\beta, P)$  of  $M$ , a functor  $H$  from  $\text{mod}_T$  to  $\text{mod}_{\text{End}_T(M)}$  was defined in [6]. We state this definition and prove that  $H$  is a covariant functor.

**Proposition 3.2.** *Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . The mapping*

$$H := H^\beta := H_M^\beta : \text{mod}_T \rightarrow \text{mod}_{\text{End}_T(M)}$$

*defined for  $V \in \text{mod}_T$  by*

$$H(V) = \text{Hom}_T(P, V) / \text{Hom}_T(P, V)J_\beta$$

*is a covariant functor.*

*Proof.*  $\text{Hom}_T(P, V)$  is an  $\text{End}_T(P)$ -module via the action  $\delta\theta(p) = \delta(\theta(p))$  for  $\theta$  in  $\text{End}_T(P)$  and  $\delta$  in  $\text{Hom}_T(P, V)$  and  $p$  in  $P$ . Thus, there is an induced  $\text{End}_T(P)$ -action on  $H(V) = \text{Hom}_T(P, V) / \text{Hom}_T(P, V)J_\beta$ . Also, we have  $J_\beta(H(V)) = 0$  since for  $\delta + \text{Hom}_T(P, V)J_\beta$  in  $H(V) = \text{Hom}_T(P, V) / \text{Hom}_T(P, V)J_\beta$  and  $\theta$  in  $J_\beta$ , we have

$$(\delta + \text{Hom}_T(P, V)J_\beta)\theta = \delta\theta + \text{Hom}_T(P, V)J_\beta = \text{Hom}_T(P, V)J_\beta.$$

Then  $\text{End}_T(P)/J_\beta$  acts on  $H(V)$  via the action  $(\theta + J_\beta)\delta = \theta\delta$  for  $\theta$  in  $\text{End}_T(P)$  and  $\delta$  in  $\text{Hom}_T(P, V)$ . Hence  $H(V)$  is an  $\text{End}_T(P)/J_\beta$ -module. By Proposition 3.1, we have  $\text{End}_T(P)/J_\beta \cong \text{End}_T(M)$ , then  $H(V)$  is also an  $\text{End}_T(M)$ -module.

Let  $V$  and  $V'$  be in  $\text{mod}_T$ , and  $f : V \rightarrow V'$  be a  $T$ -module homomorphism. Then

$$f_* = \text{Hom}_T(P, f) : \text{Hom}_T(P, V) \rightarrow \text{Hom}_T(P, V'), \phi \mapsto f\phi$$

is an  $\text{End}_T(P)$ -homomorphism, so an  $(\text{End}_T(P))_\beta$ -homomorphism.

Also, the set  $f_*(\text{Hom}_T(P, V)J_\beta)$  is a subset of  $\text{Hom}_T(P, V')J_\beta$  since if  $\phi$  is an element

of  $\text{Hom}_T(P, V)J_\beta$ , then  $\phi = \alpha\gamma$  for some  $\alpha$  in  $\text{Hom}_T(P, V)$ , some  $\gamma$  in  $J_\beta$ , and so

$$f_*(\phi) = f\phi = f(\alpha\gamma) = (f\alpha)\gamma \in \text{Hom}_T(P, V')J_\beta.$$

Then  $f_*$  induces an  $\text{End}_T(M)$ -homomorphism  $H(f) : H(V) \rightarrow H(V')$  defined for  $\psi + \text{Hom}_T(P, V)J_\beta$  in  $H(V)$  as

$$H(f)(\psi + \text{Hom}_T(P, V)J_\beta) = f_*(\psi) + \text{Hom}_T(P, V')J_\beta.$$

Also, we have  $H(1_V) = 1_{H(V)}$  since for  $\psi + \text{Hom}_T(P, V)J_\beta$  in  $H(V)$  we have

$$\begin{aligned} H(1_V)((\psi + \text{Hom}_T(P, V)J_\beta) \in H(V)) &= 1_V\psi + \text{Hom}_T(P, V)J_\beta \\ &= 1_{H(V)}(\psi + \text{Hom}_T(P, V)J_\beta) \end{aligned}$$

and for elements  $V, V'$  and  $V''$  in  $\text{mod}_T$ , and morphisms  $f : V \rightarrow V'$  and  $g : V' \rightarrow V''$ , we have  $H(gf) = H(g)H(f)$  since for  $\psi + \text{Hom}_T(P, V)J_\beta$  in  $H(V)$  we have

$$\begin{aligned} H(gf)(\psi + \text{Hom}_T(P, V)J_\beta) &= (gf)\psi + \text{Hom}_T(P, V'')J_\beta \\ &= g(f\psi) + \text{Hom}_T(P, V'')J_\beta \\ &= H(g)(f\psi + \text{Hom}_T(P, V')J_\beta) \\ &= H(g)(H(f)(\psi + \text{Hom}_T(P, V)J_\beta)) \\ &= H(g)H(f)(\psi + \text{Hom}_T(P, V)J_\beta) \end{aligned}$$

Therefore  $H$  is a covariant functor from  $\text{mod}_T$  to  $\text{mod}_{\text{End}_T(M)}$ . □

### 3.3. The Functors for Different Projective Presentations

The functor  $H$  depends on the projective presentation  $(\beta, P)$  we choose. We are to investigate what happens if we change  $(\beta, P)$  with the minimal projective cover  $(\beta_1, P_1)$  of  $M$ . In [7], a necessary and sufficient condition for the equivalence of  $H^\beta$  and  $H^{\beta_1}$  was stated and a sketch of a proof for that statement was given. Here, we give a detailed

proof following the sketch in [7].

**Definiton 3.3.** *Let  $S$  be a ring. For  $S$ -modules  $V_1$  and  $V_2$  and for a submodule  $U$  of  $V_1$ , the  $S$ -module  $\text{Hom}_S(V_1, V_2)U$  defined as the submodule of  $V_2$  spanned by all images  $\text{im}\phi$  for restrictions  $\phi$  of  $U$  of homomorphisms from  $V_1$  to  $V_2$ . In the case  $U$  is equal to  $V_1$ , the  $T$ -module  $\text{Hom}_S(V_1, V_2)V_1$  is called trace of  $V_1$  in  $V_2$  and denoted as  $\text{tr}_{V_1}(V_2)$ .*

Since  $P$  and  $P_1$  are both projective, we have  $P = P_1 \oplus P_2$  where  $P_2 = \ker\beta/\ker\beta_1$  and  $\beta_2 : P_2 \rightarrow M$  is the zero map. Then  $\ker\beta = \ker\beta_1 \oplus P_2$  and we may express  $\beta$  as

$$\beta = \beta_1 \oplus 0.$$

Hence, we have the following short exact sequence:

$$0 \rightarrow \ker\beta_1 \oplus P_2 \rightarrow P_1 \oplus P_2 \xrightarrow{\beta_1 \oplus 0} M \rightarrow 0 \quad (3.1)$$

**Proposition 3.4.** *Let  $\beta = (\beta_1, 0) : P_1 \oplus P_2 \rightarrow M$  be given as in the exact sequence in (3.1). Then  $(\text{End}_T(P))_\beta = \text{End}_T(P)$  if and only if  $\text{tr}_{P_2}(P_1)$  is a submodule of  $\ker\beta_1$ . Moreover, for a  $T$ -module  $V$  we have*

$$H^\beta(V) \cong H^{\beta_1}(V)/\text{Hom}_T(P_2, V)\text{Hom}(P_1, P_2).$$

*Thus  $H^\beta = H^{\beta_1}$  if and only if every homomorphism from  $P_1$  to  $P_2$  factors through a linear combination of endomorphisms of  $P_1$  whose image is contained in  $\ker\beta_1$ , that is,  $H^{\beta_1}(P_2) = (0)$ .*

*If this condition does not hold, then  $H^\beta$  is a proper quotient of  $H^{\beta_1}$ .*

*Proof.* Since  $P = P_1 \oplus P_2$ , we can write the elements of  $P$  as column vectors with two components, the first one from  $P_1$  and the second one from  $P_2$ . Consequently, we can represent the endomorphisms of  $P$  as  $2 \times 2$  matrices with entries in the appropriate

Hom-spaces, hence we have

$$\text{End}_T(P) = \begin{pmatrix} \text{Hom}_T(P_1, P_1) & \text{Hom}_T(P_2, P_1) \\ \text{Hom}_T(P_1, P_2) & \text{Hom}_T(P_2, P_2) \end{pmatrix}.$$

Then, since  $\text{End}_T(P_1) = (\text{End}_T(P_1))_{\beta_1}$  and

$$\begin{aligned} (\text{End}_T(P_2))_{\beta_2} &= \{\phi \in \text{End}_T(P_2) \mid \phi(\ker \beta_2) \subseteq \ker \beta_2\} \\ &= \{\phi \in (\text{End}_T(P_2))_{\beta_2} \mid \phi(P_2) \subseteq P_2\} \\ &= \text{End}_T(P_2) \end{aligned}$$

we have

$$\text{End}_T(P) = \begin{pmatrix} (\text{End}_T(P_1))_{\beta_1} & \text{Hom}_T(P_2, P_1) \\ \text{Hom}_T(P_1, P_2) & (\text{End}_T(P_2))_{\beta_2} \end{pmatrix}.$$

Then we have

$$\begin{aligned} \text{End}_T(P)\ker\beta &= \begin{pmatrix} (\text{End}_T(P_1))_{\beta_1} & \text{Hom}_T(P_2, P_1) \\ \text{Hom}_T(P_1, P_2) & (\text{End}_T(P_2))_{\beta_2} \end{pmatrix} \begin{pmatrix} \ker\beta_1 \\ P_2 \end{pmatrix} \\ &= \begin{pmatrix} (\text{End}_T(P_1))_{\beta_1}\ker\beta_1 + \text{Hom}_T(P_2, P_1)P_2 \\ \text{Hom}_T(P_1, P_2)\ker\beta_1 + (\text{End}_T(P_2))_{\beta_2}P_2 \end{pmatrix} \\ &\subseteq \begin{pmatrix} \ker\beta_1 + \text{tr}_{P_2}(P_1) \\ P_2 \end{pmatrix} \end{aligned} \tag{3.2}$$

Now, we are to prove that  $(\text{End}_T(P))_{\beta} = \text{End}_T(P)$  if and only if  $\text{tr}_{P_2}(P_1)$  is a submodule of  $\ker\beta_1$ . If  $\text{tr}_{P_2}(P_1)$  is a submodule of  $\ker\beta_1$ , then  $\ker\beta_1 + \text{tr}_{P_2}(P_1) = \ker\beta_1$ , thus, by the inclusion in (3.2), we have  $(\text{End}_T(P))_{\beta} = \text{End}_T(P)$ . Conversely, assume that  $(\text{End}_T(P))_{\beta} = \text{End}_T(P)$ . Let  $\psi$  be in  $\text{Hom}_T(P_2, P_1)$ . Define

$$\phi : P_1 \oplus P_2 \rightarrow P_1 \oplus P_2, (\alpha_1, \alpha_2) \mapsto (\psi(\alpha_2), 0).$$

Clearly,  $\phi$  is well-defined since  $\psi$  is. Since  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ , the set  $\phi(\ker\beta)$  is a submodule of  $\ker\beta$ . Then  $\phi(\ker\beta_1 \oplus P_2)$  is a submodule of  $\ker\beta_1 \oplus P_2$ . That means for any  $\alpha_1$  in  $\ker\beta_1$  and  $\alpha_2$  in  $P_2$  we have  $\phi(\alpha_1, \alpha_2)$  in  $\ker\beta_1 \oplus P_2$ , that is,  $(\psi(\alpha_2), 0)$  in  $\ker\beta_1 \oplus P_2$ . Then we have  $\psi(\alpha_2)$  in  $\ker\beta_1$ . Since  $\alpha_2$  is arbitrary, we conclude that  $\text{im}\psi$  is a submodule of  $\ker\beta_1$ , and consequently  $\text{tr}_{P_2}(P_1)$  is a submodule of  $\ker\beta_1$ . Therefore  $(\text{End}_T(P))_\beta = \text{End}_T(P)$  if and only if  $\text{tr}_{P_2}(P_1)$  is a submodule of  $\ker\beta_1$ , or equivalently,  $(\text{End}_T(P))_\beta = \text{End}_T(P)$  if and only if  $\text{Hom}_T(P_2, P_1)$  is a subset of  $\text{Hom}_T(P_2, \ker\beta_1)$ . The first part of the proposition is proved.

Now, we are to prove the second part. Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Let  $\theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}$  be in  $J_\beta$ . Then for any  $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$  in  $\ker\beta$  we have

$$\begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \subseteq \ker\beta,$$

that is,

$$\begin{pmatrix} \theta_1(\gamma_1) + \theta_2(\gamma_2) \\ \theta_3(\gamma_1) + \theta_4(\gamma_2) \end{pmatrix} \subseteq \ker\beta = \ker\beta_1 \oplus P_2.$$

Then  $\theta_1(\gamma_1) + \theta_2(\gamma_2)$  is in  $\ker\beta_1$ . Since  $\theta_2$  is in  $\text{Hom}_T(P_2, \ker\beta_1)$ , we have  $\theta_2(\gamma_2)$  in  $\ker\beta_1$ . So  $\theta_1(\gamma_1)$  is in  $\ker\beta_1$ . As  $\gamma_1$  is arbitrary, we have  $\text{im}\theta_1$  a submodule of  $\ker\beta_1$ . Hence we obtain

$$J_\beta = \begin{pmatrix} J_{\beta_1} & \text{Hom}_T(P_2, \ker\beta_1) \\ \text{Hom}_T(P_1, P_2) & (\text{End}_T(P_2))_{\beta_2} \end{pmatrix}.$$

We represent homomorphisms from  $P$  into a  $T$ -module  $V$  as row vectors  $\delta = (\delta_1, \delta_2)$ , where  $\delta_i$  is in  $\text{Hom}_T(P_i, V)$  for  $i = 1, 2$ . Then, we have

$$\begin{aligned} \text{Hom}_T(P, V)J_\beta &= \left( \text{Hom}_T(P_1, V), \text{Hom}_T(P_2, V) \right) \begin{pmatrix} J_{\beta_1} & \text{Hom}_T(P_2, P_1) \\ \text{Hom}_T(P_1, P_2) & (\text{End}_T(P_2))_{\beta_2} \end{pmatrix} \\ &= \left( \text{Hom}_T(P_1, V)J_{\beta_1} + \text{Hom}_T(P_1, V)\text{Hom}_T(P_1, P_2), \text{Hom}_T(P_1, V)\text{Hom}_T(P_2, P_1) + \text{Hom}_T(P_2, V)(\text{End}_T(P_2))_{\beta_2} \right) \end{aligned}$$

Clearly, the  $T$ -module  $\text{Hom}_T(P_1, V)\text{Hom}_T(P_2, P_1) + \text{Hom}_T(P_2, V)(\text{End}_T(P_2))_{\beta_2}$  is a submodule of  $\text{Hom}_T(P_2, V)$ . Also, any  $\xi$  in  $\text{Hom}_T(P_2, V)$  can be written as  $\xi = \xi \circ \text{id}_{(\text{End}_T(P_2))_{\beta_2}}$ , hence it is an element of  $\text{Hom}_T(P_2, V)(\text{End}_T(P_2))_{\beta_2}$ . Thus, the set  $\text{Hom}_T(P_2, V)$  is a submodule of  $\text{Hom}_T(P_2, V)(\text{End}_T(P_2))_{\beta_2}$ . Then we have

$$\text{Hom}_T(P, V)J_\beta = \left( \text{Hom}_T(P_1, V)J_{\beta_1} + \text{Hom}_T(P_1, V)\text{Hom}_T(P_1, P_2), \text{Hom}_T(P_2, V) \right).$$

Now, we can write  $H^\beta(V)$  as

$$\begin{aligned} H^\beta(V) &= \frac{\text{Hom}_T(P, V)}{\text{Hom}_T(P, V)J_\beta} \\ &= \frac{(\text{Hom}_T(P_1, V), \text{Hom}_T(P_2, V))}{(\text{Hom}_T(P_1, V)J_{\beta_1} + \text{Hom}_T(P_2, V)\text{Hom}_T(P_1, P_2), \text{Hom}_T(P_2, V))} \\ &\cong \frac{\text{Hom}_T(P_1, V)}{(\text{Hom}_T(P_1, V)J_{\beta_1} + \text{Hom}_T(P_2, V)\text{Hom}_T(P_1, P_2))} \\ &\cong \frac{\text{Hom}_T(P_1, V)/\text{Hom}_T(P_1, V)J_{\beta_1}}{\text{Hom}_T(P_2, V)\text{Hom}_T(P_1, P_2)} \\ &\cong \frac{H^{\beta_1}(V)}{\text{Hom}_T(P_2, V)\text{Hom}_T(P_1, P_2)} \end{aligned} \quad (3.3)$$

Finally, we are to show that  $H^\beta = H^{\beta_1}$  if and only if  $H^{\beta_1}(P_2) = (0)$ . First, assume  $H^{\beta_1}(P_2) = (0)$ . Then  $\text{Hom}_T(P_1, P_2) = \text{Hom}_T(P_1, P_2)J_{\beta_1}$ . Hence, we have

$$\begin{aligned} \text{Hom}_T(P_2, V)\text{Hom}_T(P_1, P_2) &= \text{Hom}_T(P_2, V)\text{Hom}_T(P_1, P_2)J_{\beta_1} \\ &\subseteq \text{Hom}_T(P_1, V)J_{\beta_1} \end{aligned}$$

Then, by Equation 3.3, we have

$$H^\beta(V) \cong \text{Hom}_T(P_1, V)/\text{Hom}_T(P_1, V)J_{\beta_1} = H_1^\beta(V).$$

Conversely, assume  $H^\beta = H^{\beta_1}$ . Then, we have

$$H^{\beta_1}(V) = \frac{H^{\beta_1}(V)}{\text{Hom}_T(P_2, V)\text{Hom}_T(P_1, P_2)}$$

for any  $T$ -module  $V$ . Then, we obtain  $\text{Hom}_T(P_2, V)\text{Hom}_T(P_1, P_2) = (0)$ . For  $V = P_2$ , we have  $\text{Hom}_T(P_2, V) = \text{Hom}_T(P_2, P_2) \neq (0)$ . Hence  $\text{Hom}_T(P_1, P_2) = (0)$ . Therefore, we have  $H^{\beta_1}(P_2) = (0)$ .  $\square$

Now, we prove a relevant lemma. First we need a definition:

**Definiton 3.5.** *Let  $P$  and  $V$  be in  $\text{mod}_T$  and assume that  $P$  is projective.*

- (i) *The  $P$ -torsion submodule  $\text{tor}_P(V)$  is the sum of all submodules  $X$  of  $V$  with respect to the property  $\text{Hom}_T(P, X) = (0)$ . If  $\text{tor}_P(V) = (0)$ , then  $V$  is called  $P$ -torsionless.*
- (ii) *The kernel  $\ker_P$  is the full subcategory of  $\text{mod}_T$  whose objects are the  $T$ -modules  $V$  with  $\text{Hom}_T(P, V) = (0)$ . Therefore, the  $T$ -module  $V$  is in  $\ker_P$  if and only if  $\text{tor}_P(V) = (0)$ .*

**Lemma 3.6.** *Let  $\beta = (\beta_1, 0)$  be as in Lemma 3.4. Then  $(\text{End}_T(P))_\beta = \text{End}_T(P)$  if and only if  $\text{Hom}_T(P_2, M) = (0)$ . In this case,  $M$  is  $P$ -torsionless if and only if it is  $P_1$ -torsionless.*

*Proof.* Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Then by Lemma 3.4, we have  $\text{tr}_{P_2}(P_1)$  a submodule of  $\ker\beta_1$ . Let  $\phi$  be in  $\text{Hom}_T(P_2, M)$ . Since  $P_2$  is projective and  $\beta$  is surjective, there exists a homomorphism  $\psi$  in  $\text{Hom}_T(P_2, P_1)$  such that  $\phi = \beta\psi$ . Then we have  $\phi = 0$  as

$$\text{im}\psi \subseteq \text{tr}_{P_2}(P_1) \subseteq \ker\beta_1.$$

Conversely, assume that  $\text{End}_T(P) \neq (\text{End}_T(P))_\beta$ . Then  $\text{tr}_{P_2}(P_1)$  is not a submodule of  $\ker\beta_1$ . That means there exists a homomorphism  $\theta$  in  $\text{Hom}_T(P_2, P_1)$  whose image is not contained in  $\ker\beta_1$ . Then, the map  $\beta_1\theta$  in  $\text{Hom}_T(P_2, M)$  is nonzero. Hence  $\text{Hom}_T(P_2, M) \neq (0)$ .

Now, assume that  $\text{Hom}_T(P_2, M) = (0)$ . Let  $X$  be a submodule of  $M$ . Then  $\text{Hom}_T(P_2, X) = (0)$ . Hence, we have

$$\text{Hom}_T(P, X) = \text{Hom}_T(P_1 \oplus P_2, X) = \text{Hom}_T(P_1, X) \oplus \text{Hom}_T(P_2, X) = \text{Hom}_T(P_1, X).$$

Therefore, we conclude that  $M$  is  $P$ -torsionless if and only if it is  $P_1$ -torsionless, as claimed.  $\square$

**Corollary 3.7.** *Suppose  $P_2$  is a projective  $T$ -module such that  $\text{Hom}_T(P_1, P_2) = (0)$  and that  $\text{tr}_{P_2}(P_1)$  is a submodule of  $\ker \beta_1$ . Then, for*

$$\beta = \beta_1 \oplus 0 : P_1 \oplus P_2 \rightarrow M$$

*we have  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ , and  $H^\beta = H^{\beta_1}$ .*

*Proof.* If  $\text{Hom}_T(P_1, P_2) = (0)$  then  $H^{\beta_1}(P_2) = (0)$ , hence the result follows by Lemma 3.4.  $\square$

### 3.4. Right Inverse of $H$

The functor  $H$  has a right inverse. Before proving this statement, we need some definitions and lemma which were stated and proved in [6].

**Lemma 3.8.** *Let  $V$  and  $V'$  be in  $\text{mod}_T$  and let  $P$  be projective  $T$ -module. Then  $\text{tor}_P(V)$  is the unique maximal submodule  $X$  of  $V$  such that  $\text{Hom}_T(P, X) = (0)$ . Moreover,  $\text{tor}_P(V/\text{tor}_P(V)) = (0)$  and for a  $T$ -module homomorphism  $f : V \rightarrow V'$  we have  $f(\text{tor}_P(V))$  is a subset of  $\text{tor}_P(V')$ .*

*Proof.* First, we are to prove that  $\text{tor}_P(V)$  is the unique maximal  $P$ -torsion submodule of  $V$  with respect to the property  $\text{Hom}_T(P, X) = (0)$ . We need only to show the uniqueness part. Assume there exists another maximal submodule  $N$  of  $V$  satisfying the condi-

tion  $\text{Hom}_T(P, N) = (0)$ . Then, we have  $\text{Hom}_T(P, N/(\text{tor}_P(V) \cap N)) = (0)$  since, if there would exist a nonzero morphism in  $\text{Hom}_T(P, N/(\text{tor}_P(V) \cap N))$ , then, by projectivity of  $P$  and surjectivity of the natural projection from  $N$  onto  $N/(\text{tor}_P(V) \cap N)$ , there would exist a nonzero morphism in  $\text{Hom}_T(P, N)$ , which is not the case. Hence, by the Second Isomorphism Theorem, we conclude that  $\text{Hom}_T(P, (\text{tor}_P(V) + N)/\text{tor}_P(V)) = (0)$ . Then, for any homomorphism  $\phi$  in  $\text{Hom}_T(P, \text{tor}_P(V) + N)$ , we have the  $T$ -module  $\text{im}\phi$  is a submodule of  $\text{tor}_P(V)$ . Then, since  $\text{Hom}_T(P, \text{tor}_P(V)) = (0)$ , the map  $\phi$  must be the zero map. Hence, we obtain  $\text{Hom}_T(P, \text{tor}_P(V) + N) = (0)$ . However, this result is contradicting the maximality of  $\text{tor}_P(V)$  since  $\text{tor}_P(V)$  is a submodule of  $(\text{tor}_P(V) + N)$ . Therefore, we must have  $\text{tor}_P(V)$  as the unique maximal  $P$ -torsion submodule of  $V$ .

Next, we are to show that  $\text{tor}_P(V/\text{tor}_P(V)) = (0)$ . To this end, we have to prove that for any submodule  $W/\text{tor}_P(V)$  of  $V/\text{tor}_P(V)$ , we have  $\text{Hom}_T(P, W/\text{tor}_P(V)) \neq (0)$ . We prove by contradiction; assume that there exists a submodule  $W_0/\text{tor}_P(V)$  of  $V/\text{tor}_P(V)$  such that  $\text{Hom}_T(P, W_0/\text{tor}_P(V)) = (0)$ . Then we have  $\text{Hom}_T(P, W_0) = (0)$  since  $\text{Hom}_T(P, \text{tor}_P(V)) = (0)$ . But, that contradicts maximality of  $\text{tor}_P(V)$  since  $\text{tor}_P(V)$  is a subset of  $W_0$ . Hence, we obtain  $\text{tor}_P(V/\text{tor}_P(V)) = (0)$ .

Finally, we are to establish the last statement; for any  $T$ -module homomorphism  $f : V \rightarrow V'$ , we have  $f(\text{tor}_P(V))$  as a submodule of  $\text{tor}_P(V')$ . Our goal is to prove the equality  $\text{Hom}_T(P, f(\text{tor}_P(V))) = (0)$ , then, since  $\text{tor}_P(V')$  is maximal, the result follows. If there would be a nonzero homomorphism in  $\text{Hom}_T(P, f(\text{tor}_P(V)))$ , then by projectivity of  $P$  and surjectivity of the map  $f|_{\text{tor}_P(V)} : \text{tor}_P(V) \rightarrow f(\text{tor}_P(V))$  which is obtained by restricting  $f$  to  $\text{tor}_P(V)$ , there would exist a nonzero homomorphism in  $\text{Hom}_T(P, \text{tor}_P(V))$ , which is not the case. Therefore  $\text{Hom}_T(P, f(\text{tor}_P(V))) = (0)$ .  $\square$

Using the above result  $f(\text{tor}_P(V)) \subseteq \text{tor}_P(V')$ , we can conclude that for  $V$  and  $V'$  in  $\text{mod}_T$ , any  $T$ -module homomorphism  $f : V \rightarrow V'$  induces a  $T$ -module homomorphism from  $V/\text{tor}_P(V)$  to  $V'/\text{tor}_P(V')$ . Now, we define a functor  $A_P$  which has an intermediate role in the definition of right inverse of  $H$ :

**Definiton 3.9.** Define the functor

$$A_P : \text{mod}_T \rightarrow \text{mod}_T, V \mapsto V/\text{tor}_P(V)$$

for  $V$  in  $\text{mod}_T$  and define  $A_P(f)$  as the induced morphism from  $V/\text{tor}_P(V)$  to  $V'/\text{tor}_P(V')$  for any  $T$ -module homomorphism  $f : V \rightarrow V'$ .

Now, we are ready to define inverses of  $H$ :

**Definiton 3.10.** Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . We define four functors from  $\text{mod}_{\text{End}_T(M)}$  to  $\text{mod}_T$  as

$$\begin{aligned} F_M &= - \otimes_{\text{End}_T(M)} M \\ \tilde{F}_M &= A_P \circ (- \otimes_{\text{End}_T(M)} M) \\ G_M &= - \otimes_{\text{End}_T(P)} P \\ \tilde{G}_M &= A_P \circ (- \otimes_{\text{End}_T(P)} P) \end{aligned}$$

Before stating the proposition on inverses of  $H$ , we state a lemma which shall be used in the proof of that proposition. The sketch of the proof was given in [6]. Here, we give a detailed proof.

**Lemma 3.11.** Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Then  $\text{End}_T(M) \cong \text{Hom}_T(P, M)$  as  $\text{End}_T(M) - \text{End}_T(M)$  bimodules.

*Proof.* Firstly, we observe that, given a morphism  $\alpha$  in  $\text{Hom}_T(P, M)$ , by projectivity of  $P$  and surjectivity of  $\beta$ , we have a morphism  $\phi$  in  $\text{End}_T(P)$  such that  $\alpha = \beta\phi$ . Then, since we assume that  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ , by Proposition 3.1, there exists a morphism  $\psi$  in  $\text{End}_T(M)$  such that for  $m$  in  $M$   $\psi(m) = \beta(\phi(p))$  where  $\beta(p) = m$ . Combining these two results, we obtain for any  $\alpha$  in  $\text{Hom}_T(P, M)$ , a  $\psi$  in  $\text{End}_T(M)$  given by  $\psi(m) = \alpha(p)$  for  $m$  in  $M$  where  $p$  is in  $P$  such that  $\beta(p) = m$ .

Now, we define

$$\Phi : \text{Hom}_T(P, M) \rightarrow \text{End}_T(M)$$

mapping  $\alpha$  in  $\text{Hom}_T(P, M)$  to  $\psi$  in  $\text{End}_T(M)$  where  $\psi$  is defined as above. Well-definedness of  $\Phi$  is clear since even if different morphisms  $\phi_1$  and  $\phi_2$  satisfy the property  $\alpha = \beta\phi$ , the resulting morphisms  $\psi_1$  and  $\psi_2$  are the same, as we have

$$\psi_1(m) = \beta(\phi_1(p)) = \alpha(p) = \beta(\phi_2(p)) = \psi_2(m)$$

for any  $m$  in  $M$  and  $p$  in  $P$  such that  $\beta(p) = m$ . The map  $\Phi$  is an  $\text{End}_T(M)$ -module homomorphism since, for  $\psi \in \text{End}_T(M)$  and  $\alpha \in \text{Hom}_T(P, M)$ ,

$$\Phi(\psi\alpha)(m) = \psi\alpha(p) = \psi(\alpha(p)) = \psi\Phi(\alpha)(m).$$

Also  $\Phi$  is surjective since, by Proposition 3.1, for  $\psi$  in  $\text{End}_T(M)$ , there exists a  $\phi$  in  $\text{End}_T(P)$  such that  $\tilde{\beta}(\phi) = \psi$  and we have  $\beta\phi$  in  $\text{Hom}_T(P, M)$ , and for all  $m$  in  $M$  and  $p$  in  $P$  such that  $\beta(p) = m$  we have  $\Phi(\beta\phi)(m) = \beta\phi(p) = \psi(m)$ . Finally,  $\Phi$  is injective since if  $\alpha$  is in  $\ker\Phi$ , then  $\Phi(\alpha) = 0$ , and as  $\tilde{\beta}$  in the proof of Proposition 3.1 is an isomorphism, the corresponding  $\phi$  in  $(\text{End}_T(P))_\beta$  is in  $J_\beta$ , that is  $\text{im}\phi$  is a subset of  $\ker\beta$ , hence  $\alpha = \beta\phi = 0$ . Therefore  $\Phi$  is an isomorphism and the lemma follows.  $\square$

The following proposition gives right inverses for  $H$ . The proof is taken from [6].

**Proposition 3.12.** *Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Let  $\hat{H}$  be one of the four functors defined in Definition 3.10. Then  $\hat{H}$  is a right inverse of the functor  $H$ .*

*Proof.* Let  $X$  be an  $\text{End}_T(M)$ -module. Firstly, we observe that, by Proposition 3.1,  $\text{End}_T(M)$ -module  $M$  is also an  $\text{End}_T(P)/J_\beta$ -module. Besides, the ideal  $J_\beta$  acts on  $M$  trivially, that is  $J_\beta \cdot M = (0)$  since the action of any element of  $\text{End}_T(P)/J_\beta$  on  $M$  is defined as the action of the corresponding element in  $\text{End}_T(M)$  and  $J_\beta$  is mapped to

the zero element of  $\text{End}_T(M)$ , therefore

$$J_\beta \cdot M = 0 \cdot M = 0.$$

Hence, we have

$$X \otimes_{\text{End}_T(P)} M \cong X \otimes_{\text{End}_T(M)} M.$$

Also, by Proposition 2.9, we know that

$$\text{Hom}_T(P, X \otimes_{\text{End}_T(P)} M) \cong X \otimes_{\text{End}_T(P)} \text{Hom}_T(P, M).$$

Thus, using Lemma 3.11, we obtain

$$\begin{aligned} \text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M) &\cong \text{Hom}_T(P, X \otimes_{\text{End}_T(P)} M) \\ &\cong X \otimes_{\text{End}_T(P)} \text{Hom}_T(P, M) \\ &\cong X \otimes_{\text{End}_T(P)} \text{End}_T(M) \\ &\cong X \otimes_{\text{End}_T(M)} \text{End}_T(M) \\ &\cong X \end{aligned}$$

Besides, by Proposition 2.9, any morphism  $\phi$  in  $\text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M)$  can be written of the form  $\phi_{x,\beta}$  for some  $x$  in  $X$  such that  $\phi_{x,\beta}(p) = x \otimes \beta(p)$  for  $p$  in  $P$ . Thus we have

$$\text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M) J_\beta = 0$$

since for  $\psi$  in  $J_\beta$  and  $\phi_{x,\beta}$  in  $\text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M)$  and  $p$  in  $P$ ,

$$\phi_{x,\beta}\psi(p) = x \otimes \beta(\psi(p)) = x \otimes 0$$

as  $\text{im}\phi$  is a submodule of  $\ker\beta$ .

Therefore we have

$$\begin{aligned} H_M(F_M(X)) &= \text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M) / \text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M) J_\beta \\ &\cong \text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M) \\ &\cong X \end{aligned}$$

and hence  $F_M$  is a right inverse of  $H$ .

Obviously, for  $V$  in  $\text{mod}_T$ , we have  $\text{Hom}_T(P, V) \cong \text{Hom}_T(P, A_P(V))$ . Thus  $H(V) = H(A_P(V))$ . Then, using the above result we get

$$H(\tilde{F}_M(X)) = H(A_P(F_M(X))) = H(F_M(X)) = X.$$

Therefore  $\tilde{F}_M$  is also a right inverse of  $H$ . The statement can be proved similarly also for  $G_M$  and  $\tilde{G}_M$ .  $\square$

### 3.5. Correspondence between $(\text{Irr}T)_H$ and $\text{IrrEnd}_T(M)$

Our next aim is to maintain a correspondence between certain irreducible  $T$ -modules and non-isomorphic irreducible  $\text{End}_T(M)$ -modules. First, we need some lemma. Proofs of those lemma are taken from [6].

**Lemma 3.13.** *Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Let  $V$  be an irreducible  $T$ -module. Then  $H(V) = (0)$  or  $\text{Hom}_T(P, V) J_\beta = (0)$ , and  $H(V) = \text{Hom}_T(P, V) \neq (0)$  is an irreducible  $\text{End}_T(M)$ -module.*

*Proof.* First note that if  $\text{Hom}_T(P, V) \neq (0)$  then  $\text{Hom}_T(P, V)$  is an irreducible  $\text{End}_T(P)$ -module. For proof, see [2, 6.3].

Assume  $H(V) \neq 0$ . Then  $\text{Hom}_T(P, V) \neq 0$  and  $\text{Hom}_T(P, V) J_\beta \neq \text{Hom}_T(P, V)$ . Then

$\text{Hom}_T(P, V)J_\beta$  is a proper submodule of  $\text{Hom}_T(P, V)$ . But, since  $\text{Hom}_T(P, V)$  is an irreducible  $\text{End}_T(P)$ -module, we have  $\text{Hom}_T(P, V)J_\beta = (0)$ . So  $H(V) = \text{Hom}_T(P, V)$  is an irreducible  $\text{End}_T(P)$ -, hence  $\text{End}_T(M)$ -module.  $\square$

**Lemma 3.14.** *Let  $X$  be an  $\text{End}_T(M)$ -module. Then  $\text{tr}_P(X \otimes_{\text{End}_T(M)} M) = X \otimes_{\text{End}_T(M)} M$  and  $\text{tr}_P(X \otimes_{\text{End}_T(P)} P) = X \otimes_{\text{End}_T(P)} P$ .*

*Proof.* We know by Lemma 3.11 and Proposition 2.9 that  $X \cong \text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M)$  via the map  $x \mapsto \phi_{x,\beta}$  where  $x$  is in  $X$  and  $\phi_{x,\beta}$  is in  $\text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M)$  defined as  $\phi_{x,\beta}(p) = x \otimes \beta(p)$  for  $p$  in  $P$ . Let  $x \otimes m$  be an arbitrary generator of  $X \otimes_{\text{End}_T(M)} M$ . As  $\beta$  is surjective, there exists a  $p$  in  $P$  such that  $\beta(p) = m$ , hence  $\phi_{x,\beta}(p) = x \otimes m$ . Thus  $x \otimes m$  is in  $\text{tr}_P(X \otimes_{\text{End}_T(M)} M)$ . Then, since  $x \otimes m$  is arbitrary, we obtain  $X \otimes_{\text{End}_T(M)} M \subseteq \text{tr}_P(X \otimes_{\text{End}_T(M)} M)$ . Also we have, by definition,  $\text{tr}_P(X \otimes_{\text{End}_T(M)} M) \subseteq X \otimes_{\text{End}_T(M)} M$ . Therefore we proved

$$\text{tr}_P(X \otimes_{\text{End}_T(M)} M) = X \otimes_{\text{End}_T(M)} M.$$

Since  $X$  is also an  $\text{End}_T(P)$ -module, the same proof provided  $M$  replaced by  $P$  gives us the second statement,  $\text{tr}_P(X \otimes_{\text{End}_T(P)} P) = X \otimes_{\text{End}_T(P)} P$ .  $\square$

**Lemma 3.15.** *Let  $X$  be an irreducible  $\text{End}_T(M)$ -module. Then  $\tilde{F}_M(X) \neq (0)$ , and  $\tilde{F}_M(X)$  is an irreducible  $T$ -module, and we have  $\tilde{F}_M(X) = \tilde{G}_M(X)$ .*

*Proof.* By Proposition 3.12 we know that  $H_M(\tilde{F}_M(X)) \cong X \neq (0)$ . Hence, we have  $\tilde{F}_M(X) \neq (0)$ .

Now, we are to show that  $\tilde{F}_M(X)$  is irreducible. Let  $U$  be a submodule of  $F_M(X)$  which is equal to  $X \otimes_{\text{End}_T(M)} M$ . Then, the  $\text{End}_T(P)$ -module  $\text{Hom}_T(P, U)$  is canonically embedded into  $\text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M) \cong X$ . Since  $X$  is an irreducible  $\text{End}_T(M)$ -module, hence irreducible  $\text{End}_T(P)$ -module, we have either  $\text{Hom}_T(P, U) = (0)$  or  $\text{Hom}_T(P, U) = X \cong \text{Hom}_T(P, X \otimes_{\text{End}_T(M)} M)$ . If the latter holds, then the image of every homomorphism from  $P$  to  $X \otimes_{\text{End}_T(M)} M$  is contained in  $U$ . That means

$\text{tr}_P(X \otimes_{\text{End}_T(M)} M)$  is a submodule of  $U$ . Then, by Lemma 3.14  $X \otimes_{\text{End}_T(M)} M$  is a submodule of  $U$ . Hence  $U = X \otimes_{\text{End}_T(M)} M$ . This shows that if  $U$  is a proper submodule of  $X \otimes_{\text{End}_T(M)} M$ , then  $\text{Hom}_T(P, U) = (0)$ . Then, by definition of  $\text{tor}_P(X \otimes_{\text{End}_T(M)} M)$ , for all proper submodules  $U$  of  $X \otimes_{\text{End}_T(M)} M$ ,  $U$  is a submodule of  $\text{tor}_P(X \otimes_{\text{End}_T(M)} M)$ . So  $\text{tor}_P(X \otimes_{\text{End}_T(M)} M)$  is the unique maximal submodule of  $X \otimes_{\text{End}_T(M)} M$ , hence, we obtain that  $T$ -module  $\tilde{F}_M(X) = X \otimes_{\text{End}_T(M)} M / \text{tor}_P(X \otimes_{\text{End}_T(M)} M)$  is irreducible.

Since  $X$  is also irreducible as  $\text{End}_T(P)$ -module, substituting  $M$  with  $P$  in the above argument gives that  $\tilde{G}_M(X) = X \otimes_{\text{End}_T(P)} P / \text{tr}_P(X \otimes_{\text{End}_T(P)} P)$  is also irreducible.

Finally we are to show that  $\tilde{F}_M(X) = \tilde{G}_M(X)$ . Since the functor  $X \otimes_{\text{End}_T(P)} -$  is right exact, the morphism

$$1 \otimes \beta : X \otimes_{\text{End}_T(P)} P \rightarrow X \otimes_{\text{End}_T(P)} M \cong X \otimes_{\text{End}_T(M)} M$$

induced by  $\beta$  is an epimorphism. So the induced mapping

$$A_P(1 \otimes \beta) : \tilde{G}_M(X) \rightarrow \tilde{F}_M(X)$$

is also an epimorphism. As  $A_P(1 \otimes \beta)$  is nonzero and  $\ker A_P(1 \otimes \beta)$  is a submodule of the irreducible module  $\tilde{G}_M(X)$ , we have  $\ker A_P(1 \otimes \beta) = (0)$ . Therefore, by the First Isomorphism Theorem,  $\tilde{F}_M(X) \cong \tilde{G}_M(X)$ .

□

Now we state the correspondence theorem mentioned before. The proof is partly taken from [6].

**Notation.** Let  $R$  be a ring. The complete set of non-isomorphic irreducible  $R$ -modules is denoted by  $\text{Irr}R$ .

**Theorem 3.16.** *Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$  Define the set*

$$(\text{Irr}T)_H = \{V \in \text{Irr}T \mid H_M(V) \neq (0)\}.$$

*Then  $H_M$  induces a bijective correspondence*

$$H_M : (\text{Irr}T)_H \rightarrow \text{Irr}(\text{End}_T(M))$$

*and the inverse of  $H_M$  is*

$$\tilde{F}_M : \text{Irr}(\text{End}_T(M)) \rightarrow (\text{Irr}T)_H.$$

*On  $\text{Irr}(\text{End}_T(M))$ , the functors  $\tilde{F}_M$  and  $\tilde{G}_M$  coincide.*

*Proof.* We proved before in Proposition 3.12 that  $\tilde{F}_M$  is a right inverse for  $H_M$ . So if we show that  $\tilde{F}_M$  is also a left inverse for  $H_M$  we obtain the required correspondence. To this end, let  $V$  be in  $(\text{Irr}T)_H$ . Then  $X := H_M(V) \neq (0)$  and so, by Lemma 3.13, we have  $H_M(V) = \text{Hom}_T(P, V)$  and  $H_M(V)$  is irreducible. Now we define the map  $\phi : \text{Hom}_T(P, V) \otimes_{\text{End}_T(P)} P \rightarrow V$ , for  $f$  in  $\text{Hom}_T(P, V)$  and  $p$  in  $P$ , as  $\phi(f \otimes p) = f(p)$ . Clearly,  $\phi$  is well-defined. Now, we are to show that  $\phi$  is surjective. First, observe that any nonzero  $f$  in  $\text{Hom}_T(P, V)$  is surjective since otherwise  $\text{im} f$  is a proper submodule of  $V$  and that contradicts the irreducibility of  $V$ . Now let  $v$  be in  $V$ . As  $\text{Hom}_T(P, V) \neq (0)$  there exists a nonzero  $f$  in  $\text{Hom}_T(P, V)$  and since  $f$  is surjective there exists  $p$  in  $P$  such that  $f(p) = v$ . Then  $\phi(f \otimes p) = f(p) = v$ . Since  $v$  is arbitrary, we have  $\phi$  surjective.

Observe that, by Lemma 3.13, we have  $\text{tor}_P(V) = (0)$  as  $\text{Hom}_T(P, V) \neq (0)$  and  $V$  is irreducible. Also, by Lemma 3.8,  $\phi(\text{tor}_P(X \otimes_{\text{End}_T(P)} P)) \subseteq \text{tor}_P(V) = (0)$ . Then  $\text{tor}_P(X \otimes_{\text{End}_T(P)} P) = (0)$ . Then, since  $\tilde{G}_M(X)$  is irreducible,  $\phi$  induces an isomorphism, and by Lemma 3.15, we have

$$\tilde{F}_M(X) \cong \tilde{G}_M(X) = X \otimes_{\text{End}_T(P)} P / \text{tor}_P(X \otimes_{\text{End}_T(P)} P) \cong V / \text{tor}_P(V) \cong V. \quad \square$$

**Definiton 3.17.** Let  $Y \subseteq \text{End}_T(M)$ . Define  $YM$  as the ideal of  $M$  generated by the images of homomorphisms in  $Y$ , that is,  $YM = \langle \text{im}\phi : \phi \in Y \rangle$ . The below theorem was stated and partly proved in [6]. We give a full proof here.

**Theorem 3.18.** Let  $Y$  be a right ideal of  $\text{End}_T(M)$ . Then  $\tilde{F}_M(Y) = A_P(YM)$  and  $H_M(YM) = Y$ . In particular, if  $M$  is  $P$ -torsionless, then  $\tilde{F}_M(Y) = YM$ .

*Proof.* First, observe that, by general theory and Lemma 3.11, we have the isomorphisms

$$Y \cong Y \otimes_{\text{End}_T(M)} \text{End}_T(M) \cong Y \otimes_{\text{End}_T(M)} \text{Hom}_T(P, M)$$

via maps  $y \mapsto y \otimes 1 \mapsto y \otimes \beta$ , and by Proposition 2.9 the isomorphism

$$Y \otimes_{\text{End}_T(M)} \text{Hom}_T(P, M) \cong \text{Hom}_T(P, Y \otimes_{\text{End}_T(M)} M)$$

via the map  $y \otimes \beta \mapsto \phi_{y,\beta}$  where  $\phi_{y,\beta}(p) = y \otimes \beta(p)$  for  $p$  in  $P$ . Hence the elements of  $\text{Hom}_T(P, Y \otimes_{\text{End}_T(M)} M)$  are of the form  $\phi_{y,\beta}$ . By the definition of  $YM$  the map  $\gamma : Y \otimes_{\text{End}_T(M)} M \rightarrow YM$  defined by  $\gamma(y \otimes m) = y(m)$  is surjective.

So the induced map

$$\gamma_* : \text{Hom}_T(P, Y \otimes_{\text{End}_T(M)} M) \rightarrow \text{Hom}_T(P, YM)$$

is also surjective. Now we are to prove the injectivity of  $\gamma_*$ : For  $\phi$  in  $\ker\gamma_*$  we have  $\gamma_*(\phi) = 0$ , that is  $\gamma\phi(p) = 0$  for all  $p$  in  $P$ . Then, since any  $\phi$  in  $\text{Hom}_T(P, Y \otimes_{\text{End}_T(M)} M)$  can be written as  $y \otimes \beta$  for some  $y$  in  $Y$ , we have  $\gamma(y \otimes \beta(p)) = 0$  for some  $y$  in  $Y$ , for all  $p$  in  $P$ . Then, by surjectivity  $\beta$ , we have  $\gamma(y \otimes m) = 0$  for all  $m$  in  $M$ . By definition of  $\gamma$ , that means  $y(m) = 0$  for all  $m$  in  $M$ . Then  $y = 0$ , hence  $\phi = y \otimes \beta = 0$ . Therefore  $\ker\gamma_* = (0)$ . So  $\gamma_*$  is an isomorphism and we obtain  $Y \cong \text{Hom}_T(P, YM)$ .

In particular,  $\text{Hom}_T(P, YM)J_\beta = (0)$  since for  $\phi$  in  $J_\beta$ ,  $\psi$  in  $\text{Hom}_T(P, YM)$ ,  $p$  in  $P$ , using the isomorphism  $\gamma_*$ ,  $\psi$  can be written as  $y \otimes \beta$  for some  $y$  in  $Y$  and hence

$$\psi\phi(p) = \psi(\phi(p)) = y \otimes \beta(\phi(p)) = y \otimes 0 = 0.$$

Thus,

$$H_M(YM) = \text{Hom}_T(P, YM) / \text{Hom}_T(P, YM)J_\beta = \text{Hom}_T(P, YM) \cong Y.$$

Second, we are to prove  $\tilde{F}_M(Y) = A_P(YM)$ . Since  $\gamma : Y \otimes_{\text{End}_T(M)} M \rightarrow YM$  is surjective, it is enough to show that  $\ker\gamma$  is a submodule of  $\text{tor}_P(Y \otimes_{\text{End}_T(M)} M)$  and  $\gamma(\text{tor}_P(Y \otimes_{\text{End}_T(M)} M)) = \text{tor}_P(YM)$ . Then, by general theory, we can conclude that

$$\tilde{F}_M(Y) = Y \otimes_{\text{End}_T(M)} M / \text{tor}_P(Y \otimes_{\text{End}_T(M)} M) \cong YM / \text{tor}_P(YM) = A_P(YM).$$

Now, applying the functor  $\text{Hom}_T(P, -)$  to the exact sequence

$$0 \longrightarrow \ker\gamma \xrightarrow{\iota} Y \otimes_{\text{End}_T(M)} M \xrightarrow{\gamma} YM \longrightarrow 0$$

we obtain the exact sequence

$$0 \longrightarrow \text{Hom}_T(P, \ker\gamma) \xrightarrow{\iota_*} \text{Hom}_T(P, Y \otimes_{\text{End}_T(M)} M) \xrightarrow{\gamma_*} \text{Hom}_T(P, YM) \longrightarrow 0$$

Then, since  $\gamma_*$  is an isomorphism,  $\iota_*$  is the zero map. So  $\text{Hom}_T(P, \ker\gamma) = (0)$ . Hence  $\ker\gamma \subseteq \text{tor}_P(Y \otimes_{\text{End}_T(M)} M)$ .

Since the restriction  $\gamma|_{\text{tor}_P(Y \otimes_{\text{End}_T(M)} M)}$  of  $\gamma$  to  $\text{tor}_P(Y \otimes_{\text{End}_T(M)} M)$  is surjective and  $P$  is projective, the induced morphism

$$\begin{aligned} (\gamma|_{\text{tor}_P(Y \otimes_{\text{End}_T(M)} M)})_* : \text{Hom}_T(P, \text{tor}_P(Y \otimes_{\text{End}_T(M)} M)) \\ \rightarrow \text{Hom}_T(P, \gamma(\text{tor}_P(Y \otimes_{\text{End}_T(M)} M))) \end{aligned}$$

is also surjective. Then, we have  $\text{Hom}_T(P, \gamma(\text{tor}_P(Y \otimes_{\text{End}_T(M)} M))) = (0)$ . Now, assume there is a submodule  $W$  of  $YM$  such that  $\gamma(\text{tor}_P(Y \otimes_{\text{End}_T(M)} M))$  is a submodule of  $W$

and  $\text{Hom}_T(P, W) = (0)$ . Then there exists a submodule  $Z$  of  $Y \otimes_{\text{End}_T(M)} M$  such that  $\text{tor}_P(Y \otimes_{\text{End}_T(M)} M)$  is a submodule of  $Z$  and  $\gamma(Z) = W$  since  $\gamma$  is surjective. For any  $\eta$  in  $\text{Hom}_T(P, Z)$ , we have  $\eta$  in  $\text{Hom}_T(P, \ker \gamma)$  since  $\gamma\eta$  is in  $\text{Hom}_T(P, \text{tor}_P(Y \otimes_{\text{End}_T(M)} M))$  and  $\text{Hom}_T(P, \text{tor}_P(Y \otimes_{\text{End}_T(M)} M)) = (0)$ . However, we know that  $\text{Hom}_T(P, \ker \gamma) = (0)$ , hence we obtain  $\text{Hom}_T(P, Z) = (0)$ , but this contradicts the maximality of the  $P$ -torsion submodule  $\text{tor}_P(Y \otimes_{\text{End}_T(M)} M)$ . Hence we conclude that

$$\gamma(\text{tor}_P(Y \otimes_{\text{End}_T(M)} M)) = \text{tor}_P(YM).$$

Therefore we have  $\tilde{F}_M(Y) \cong A_P(YM)$ .

Finally, if  $M$  is  $P$ -torsionless, so is  $YM$  since otherwise, if there would exist a submodule  $W$  of  $YM$  such that  $\text{Hom}_T(P, W) = (0)$ , then as  $YM$  is a submodule of  $M$ ,  $W$  is also a submodule of  $M$  and that would contradict the assumption that  $M$  is  $P$ -torsionless. Thus, we have

$$\tilde{F}_M(Y) \cong A_P(YM) = YM/\text{tor}_P(YM) = YM.$$

□

The proof of the following theorem was given in [6].

**Theorem 3.19.** *Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Let  $X$  and  $Y$  be right ideals of  $\text{End}_T(M)$ . Suppose  $M$  is  $P$ -torsionless. Then  $H$  induces an isomorphism, also denoted by  $H$ , from  $\text{Hom}_T(XM, YM)$  onto  $\text{Hom}_{\text{End}_T(M)}(X, Y)$ .*

*Proof.* Using functoriality of  $H_M$  we define

$$H : \text{Hom}_T(XM, YM) \rightarrow \text{Hom}_{\text{End}_T(M)}(X, Y), \quad \phi \mapsto H(\phi)$$

where  $H(\phi) : H(XM) \rightarrow H(YM)$ . Since  $M$  is  $P$ -torsionless, we have  $H(XM) = X$  and  $H(YM) = Y$  by Theorem 3.18. Then  $H(\phi)$  is a map from  $X$  to  $Y$ . Since  $H$  is a

functor, for  $\phi$  and  $\psi$  in  $\text{Hom}_T(XM, YM)$ , we have  $H(\phi\psi) = H(\phi)H(\psi)$ . Hence  $H$  is a homomorphism. Clearly  $H$  is  $R$ -linear.

Now, we are to show that  $H$  is surjective. Let  $\alpha$  be an element of  $\text{Hom}_{\text{End}_T(M)}(X, Y)$ . Consider the map  $\alpha \otimes 1_M : X \otimes_{\text{End}_T(M)} M \rightarrow Y \otimes_{\text{End}_T(M)} M$ . This map induces a  $T$ -linear map

$$\gamma : \frac{X \otimes_{\text{End}_T(M)} M}{\text{tor}_P(X \otimes_{\text{End}_T(M)} M)} \rightarrow \frac{Y \otimes_{\text{End}_T(M)} M}{\text{tor}_P(Y \otimes_{\text{End}_T(M)} M)}$$

By Theorem 3.18, we have  $A_P(X \otimes_{\text{End}_T(M)} M) = XM$  and  $A_P(Y \otimes_{\text{End}_T(M)} M) = YM$ . Hence  $\gamma$  is a  $T$ -linear map from  $XM$  to  $YM$ . Also, we have

$$H(\gamma) = H(A_P(\alpha \otimes 1_M)) = H(A_P(F_M(\alpha))) = H(\hat{H}(\alpha)) = \alpha.$$

Therefore  $H$  is surjective.

$H$  is also injective: Let  $f : XM \rightarrow YM$  be a nonzero  $T$ -linear map. We are to show that  $H(f)$  is also nonzero. Set  $U := \text{im} f$ .  $U$  is a submodule of  $M$ . Then since  $M$  is  $P$ -torsionless, so is  $U$ . Then  $\text{Hom}_T(P, U) \neq (0)$ . By Lemma 3.11 we have  $\text{Hom}_T(P, M) \cong \text{End}_T(M)$ . Then, by projectivity of  $P$  and Proposition 3.1, we conclude that  $\text{Hom}_T(P, M)J_\beta = (0)$ . So, since  $\text{Hom}_T(P, U)$  is a submodule of  $\text{Hom}_T(P, M)$ , we have  $\text{Hom}_T(P, U)J_\beta = (0)$  as well. Hence  $H(U) = \text{Hom}_T(P, U)$ . Since  $P$  is projective and  $f : XM \rightarrow U$  is surjective, the map  $f_* : \text{Hom}_T(P, XM) \rightarrow \text{Hom}_T(P, U)$  is also surjective. Then, for each nonzero element  $\rho$  of  $\text{Hom}_T(P, U)$ , there exists an element  $\tau$  of  $\text{Hom}_T(P, XM) = H(XM)$  such that  $\rho = f\tau$ . Then, we have

$$H(f)(\tau) = \text{Hom}_T(P, f)(\tau) = f\tau = \rho \neq 0.$$

So  $H(f) : H(XM) \rightarrow H(YM)$  is not the zero map. Therefore  $H$  is injective.

Thus, we have shown that  $H$  is bijective, hence an isomorphism.  $\square$

### 3.6. Correspondence between $(\text{Irr}T)_H$ and Constituents of $hd(M)$

Now we are to give the exposition of the proof for one of the main theorems of this thesis. Firstly, we need a lemma which were proved in [6]:

**Lemma 3.20.** *Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Let  $X$  be an  $\text{End}_T(M)$ -module and  $Y$  be a maximal submodule of  $X$ . Let  $i : Y \rightarrow X$  be the canonical embedding. Let  $V$  denote  $F_M(X) = X \otimes_{\text{End}_T(M)} M$ , and  $U$  be the image of the  $T$ -linear map*

$$i \otimes 1_M : Y \otimes_{\text{End}_T(M)} M \rightarrow X \otimes_{\text{End}_T(M)} M.$$

*Then  $\text{tor}_P(V/U)$  is the unique maximal submodule of  $V/U$  and the factor module  $A_P(V/U)$  is canonically isomorphic to the irreducible  $T$ -module  $\tilde{F}_M(X/Y)$ .*

*Proof.* Applying the right exact functor  $- \otimes_{\text{End}_T(M)} M$  to the exact sequence

$$0 \rightarrow Y \xrightarrow{i} X \rightarrow X/Y \rightarrow 0$$

we obtain the exact sequence

$$Y \otimes_{\text{End}_T(M)} M \xrightarrow{i \otimes 1_M} X \otimes_{\text{End}_T(M)} M \rightarrow X/Y \otimes_{\text{End}_T(M)} M \rightarrow 0,$$

hence the exact sequence

$$0 \rightarrow U \rightarrow V \rightarrow X/Y \otimes_{\text{End}_T(M)} M \rightarrow 0.$$

Then we have  $V/U \cong X/Y \otimes_{\text{End}_T(M)} M$  and therefore

$$(V/U)/\text{tor}_P(V/U) \cong X/Y \otimes_{\text{End}_T(M)} M/\text{tor}_P((X/Y) \otimes_{\text{End}_T(M)} M).$$

By definition, the left hand side is equal to  $A_P(V/U)$  and the right hand side is equal to  $\tilde{F}_M(X/Y)$ . Hence, we obtain  $A_P(V/U) = \tilde{F}_M(X/Y)$ . Since  $Y$  is maximal, the quotient

module  $X/Y$  is irreducible. Then, by Lemma 3.15, we have  $\tilde{F}_M(X/Y)$  irreducible and hence  $\text{tor}_P(V/U)$  is maximal.  $\square$

We are to state a corollary of Lemma 3.20. Let  $X$  be an  $\text{End}_T(M)$ -module and  $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$  be a filtration of  $X$  such that  $Y_i := X_{i-1}/X_i$  is an irreducible  $\text{End}_T(M)$ -module for  $i \geq 1$ . Let  $V := F_M(X) = X \otimes_{\text{End}_T(M)} M$ , and let  $V_i$  be the canonical image of  $X_i \otimes_{\text{End}_T(M)} M$  in  $V$  for  $i \geq 0$ .

**Corollary 3.21.** *Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$  and let  $X_i$  be an  $\text{End}_T(M)$ -module for  $i \geq 0$  and  $X = X_0$ . For the induced filtration*

$$V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \dots$$

where  $V_i = F_M(X_i)$ , let  $U_i$  be the factor module  $V_{i-1}/V_i$ . Then  $\text{tor}_P(U_i)$  is the unique maximal submodule of  $U_i$  and the irreducible  $T$ -module  $A_P(U_i) = U_i/\text{tor}_P(U_i)$  is canonically isomorphic to  $\tilde{F}_M(Y_i)$  for all  $i \geq 0$ .

The next theorem gives us a correspondence between  $(\text{Irr}T)_H$  and constituents of  $\text{hd}(M)$ . In the proof, we follow the sketch given in [6].

**Theorem 3.22.** *Let  $R$  be a field. Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Then,  $(\text{Irr}T)_H$  is a complete set of non-isomorphic irreducible constituents of head of  $M$   $\text{hd}(M)$ . Every indecomposable direct summand of  $M$  has a simple head and factoring out the Jacobson radical induces a bijection between the isomorphism classes of indecomposable direct summands of  $M$  and the elements of  $(\text{Irr}T)_H$ .*

*Proof.* Let  $\{e_1, \dots, e_k\}$  be a complete set of non-conjugate, primitive idempotents of  $\text{End}_T(M)$ , that is  $e_i \text{End}_T(M) \neq e_j \text{End}_T(M)$  for  $i \neq j$ . Then, by Lemma 2.3, the set  $\{e_i M \mid 1 \leq i \leq k\}$  is a complete set of non-isomorphic, indecomposable direct summands of  $M$ . Hence, there is a bijective correspondence between the indecomposable direct summands of  $M$  and projective indecomposable  $\text{End}_T(M)$ -modules.

We may consider  $\beta : P \rightarrow M$  as the projective cover of  $M$ . Then  $\ker\beta$  is superfluous in  $P$ . Clearly, any submodule of  $\ker\beta$  is also superfluous. Then, since the module  $\text{im}\phi$  is a submodule of  $\ker\beta$  for any  $\phi$  in  $J_\beta$ , by Proposition 2.4(i), we have  $J_\beta$  is an ideal of  $\text{Jac}(\text{End}_T(P))$ . As  $\text{End}_T(P)$  is finitely generated, it is right Artinian. Then, by Proposition 2.4(ii), the Jacobson radical  $\text{Jac}(\text{End}_T(P))$  is nilpotent. Then,  $J_\beta$  is also nilpotent as it is an ideal of  $\text{Jac}(\text{End}_T(P))$ .

As  $\text{End}_T(P)/J_\beta \cong \text{End}_T(M)$  by Proposition 3.1, and  $J_\beta$  is nilpotent, the non-conjugate primitive idempotents of  $\text{End}_T(P)$  are in one-to-one correspondence with the non-conjugate primitive idempotents of  $\text{End}_T(M)$ . Hence, we can lift idempotents from  $\text{End}_T(M)$  to  $\text{End}_T(P)$ . Therefore, we have a one-to-one correspondence between the indecomposable direct summands of  $P$  and those of  $M$ , given by restricting  $\beta$  to indecomposable direct summands of  $P$ . In particular, the projective cover  $P_N$  of any indecomposable direct summand  $N$  of  $M$  is an indecomposable projective  $T$ -module. Then, by Proposition 2.5,  $P_N$  has a unique maximal submodule, namely  $\text{Jac}(P_N)$ .

As indecomposable direct summands of  $P$  are in one-to-one correspondence with those of  $M$ , hence with projective indecomposable  $\text{End}_T(M)$ -modules, the  $\text{End}_T(M)$ -modules

$$hd(e_i \text{End}_T(M)) = e_i \text{End}_T(M) / \text{Jac}(e_i \text{End}_T(M))$$

are simple for each  $i \in 1, \dots, k$ . Then the set

$$\{hd(e_i \text{End}_T(M)) \mid 1 \leq i \leq k\}$$

is a complete set of non-isomorphic irreducible  $\text{End}_T(M)$ -modules. Since this set is equal to  $\text{Irr}(\text{End}_T(M))$ , by Theorem 3.16, the set  $(\text{Irr}T)_H$  can be written as

$$(\text{Irr}T)_H = \{\tilde{F}_M(e_i \text{End}_T(M) / \text{Jac}(e_i \text{End}_T(M))) \mid 1 \leq i \leq k\}.$$

By the assumption in the theorem,  $R$  is a field,  $T$  a finite dimensional algebra, so, all

$T$ -modules and  $\text{End}_T(M)$ -modules being finitely generated, have composition series, and the multiplicities of  $Y \in \text{IrrEnd}_T(M)$  as a composition factor of  $e_i \text{End}_T(M)$  equals to the multiplicity of  $\tilde{F}_M(Y)$  as a composition factor of  $\tilde{F}_M(X)$ . Applying the functor  $F_M$  to the filtration

$$e_i \text{End}_T(M) \supseteq \text{Jac}(e_i \text{End}_T(M)) \supseteq \dots$$

we obtain

$$e_i M \supseteq \text{Jac}(e_i M) \supseteq \dots$$

Then, by Corollary 3.21, the irreducible  $T$ -module  $e_i M / \text{Jac}(e_i M)$  is canonically isomorphic to  $\tilde{F}_M(e_i \text{End}_T(M) / \text{Jac}(e_i \text{End}_T(M)))$ . Therefore, the set  $(\text{Irr}T)_H$  consists of precisely the direct summands of the head  $M / \text{Jac}(M)$  of  $M$ .  $\square$

Now, we state a more specific version of the previous theorem. In the proof, we use the sketch given in [7].

**Theorem 3.23.** *Let  $R$  be a field. Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ , the  $T$ -module  $M$  is  $P$ -torsionless, and  $\text{End}_T(M)$  is self-injective. Then*

- (i) *Every element of  $\text{Irr}(\text{End}_T(M))$  is isomorphic to  $XM$  for some minimal ideal  $X$  of  $\text{End}_T(M)$ .*
- (ii) *The set  $(\text{Irr}T)_H$  is up to isomorphism a complete set of irreducible constituents of  $\text{soc}(M)$  as well as  $\text{hd}(M)$ .*
- (iii) *Every indecomposable direct summand of  $M$  has a simple socle and a simple head, and taking socles, respectively heads, induce bijections between the isomorphism classes of indecomposable direct summands of  $M$  and the elements of  $(\text{Irr}T)_H$ .*
- (iv) *Socle and heads of the indecomposable direct summands of  $M$  are isomorphic if in addition  $\text{End}_T(M)$  is a symmetric algebra.*

*Proof.* As  $\text{End}_T(M)$  is self-injective, by Proposition 2.7, minimal ideals of  $\text{End}_T(M)$

correspond to principal indecomposable direct summands of the module  $\text{End}_T(M)$ . Thus, part (i) of the theorem follows from Theorem 3.18.

The parts (ii) and (iii) are proved for  $hd(M)$  in Theorem 3.22, so, it is enough to prove the statements for  $\text{soc}(M)$ . Consider the projective cover  $(\beta_1, P_1)$  of  $M$ . By definition of the projective cover,  $\ker\beta_1$  is superfluous. Then, by Proposition 2.8(i),  $\ker\beta_1$  is a submodule of  $Jac(P_1)$ . Then, since  $\beta_1$  is an epimorphism, by Proposition 2.8(ii), we have

$$\beta_1(Jac(P_1)) = Jac(M).$$

Then, we obtain

$$P_1/Jac(P_1) \cong M/Jac(M).$$

Thus, using Theorem 3.22, we conclude that the set  $(\text{Irr}T)_H$  is, up to isomorphism, precisely the set of the irreducible constituents of  $hd(P_1)$ .

Let  $S$  be a simple submodule of  $M$ . Then there exists a surjective map  $\phi$  from  $P$  onto  $S$ . Since  $\phi$  is surjective,  $\ker\phi$  is maximal. Then  $Jac(P)$  is a submodule of  $\ker\phi$ . Thus, we have

$$S \cong \frac{P}{\ker\phi} \subset \frac{P}{Jac(P)} = hd(P).$$

Therefore, every simple submodule of  $M$ , or equivalently, every constituent of  $\text{soc}(M)$  is isomorphic to an irreducible constituent of  $hd(P)$ .

We use the notation of (3.1). By assumption,  $M$  is  $P$ -torsionless. Then, by Lemma 3.6, it is  $P_1$ -torsionless, that is, we have  $\text{Hom}_T(P_2, M) = (0)$ . Since  $R$  is a field,  $T$  is a finite dimensional algebra. So, every  $T$ -module has a composition series. Since  $\text{Hom}_T(P_2, M) = (0)$ , no composition factor of  $hd(P_2)$  occurs as a composition factor of  $M$ . In particular, no simple submodule of  $hd(P_2)$  occurs as a simple submodule of  $M$ .

Then, we conclude that, every constituent of  $\text{soc}(M)$  is isomorphic to an irreducible constituent of  $hd(P_1)$ , hence to an element of  $(\text{Irr}T)_H$ .

Now, let  $S$  be an element of  $(\text{Irr}T)_H$ . Then, by Lemma 3.15 and Theorem 3.16 we have  $S = \hat{H}(X)$  for some irreducible  $\text{End}_T(M)$ -module  $X$ . By the part (i), we may assume  $X$  to be a minimal right ideal of  $\text{End}_T(M)$ .  $M$  is  $P$ -torsionless, hence by Theorem 3.18, we have  $S = XM$ . Then, the  $T$ -module  $S$  is contained in the socle of  $M$ . Therefore, the set  $(\text{Irr}T)_H$  is, up to isomorphism, a complete set of irreducible constituents of  $\text{soc}(M)$ .

Finally, for part (iv), we refer to [9, I.8.6] which enables us to obtain a correspondence between heads and socles of  $\text{End}_T(M)$ -modules in the case that  $\text{End}_T(M)$  is a symmetric algebra.  $\square$

Recall that the functor  $H$  depends on  $P$ . The functor  $\tilde{F}_M$  also depends on the choice of  $P$  since it is the composition of the functor  $-\otimes_{\text{End}_T(M)} M$  and the functor  $A_P$  that is determined by  $P$ . As in the case of  $H$  we are to compare functors  $\tilde{F}_M^\beta$  and  $\tilde{F}_M^{\beta_1}$  where  $\beta_1 : P_1 \rightarrow M$  is the minimal projective cover of  $M$ . The following lemma was stated in [7]. Here, we give a proof.

**Lemma 3.24.** *Let  $X$  be an  $\text{End}_T(M)$ -module. Then there is a natural epimorphism from  $\tilde{F}_M^{\beta_1}(X)$  onto  $\tilde{F}_M^\beta(X)$ .*

*Proof.* Since  $\tilde{F}_M^\beta(X) = A_P(X \otimes_{\text{End}_T(M)} M)$ , it is enough to show that for any  $T$ -module  $V$ ,  $A_{P_1}(V)$  is an epimorphic image of  $A_P(V)$ .

Firstly observe that if  $\text{Hom}_T(P, \text{tor}_P(V)) = (0)$ , then  $\text{Hom}_T(P_1, \text{tor}_P(V)) = (0)$ , hence  $\text{tor}_P(V) \leq \text{tor}_{P_1}(V)$ . Now, since  $1_V : V \rightarrow V$  is surjective and

$$1_V(\text{tor}_P(V)) = \text{tor}_P(V) \leq \text{tor}_{P_1}(V),$$

$1_V$  induces an epimorphism

$$\bar{1}_V : A_P(V) \rightarrow A_{P_1}(V) \quad v + \text{tor}_P(V) \mapsto v + \text{tor}_{P_1}(V)$$

where  $v$  is in  $V$ . Hence  $A_{P_1}(V)$  is an epimorphic image of  $A_P(V)$  and lemma follows.  $\square$

For later use, we are to show that the ideal  $J_{\beta_1}$  of the endomorphism ring  $(\text{End}_T(P_1))_{\beta_1}$  is contained in the Jacobson radical  $Jac((\text{End}_T(P_1))_{\beta_1})$  of  $(\text{End}_T(P_1))_{\beta_1}$ , under the assumption that  $T$  is Noetherian. The proof is taken from [7].

**Lemma 3.25.** *Suppose that  $T$  is Noetherian. Then  $J_{\beta_1}$  is a subset of  $Jac((\text{End}_T(P_1))_{\beta_1})$ .*

*Proof.* Since  $P_1$  is a finitely generated module over the Noetherian ring  $T$ , it is Noetherian. Then every surjective endomorphism of  $P_1$  is actually an isomorphism. For details see [5, 3.3] and [5, 5.8]. Using this fact, we observe that for a maximal submodule  $V$  of  $P_1$ , the set

$$\{\phi \in (\text{End}_T(P_1))_{\beta_1} \mid \text{im}\phi \subseteq V\}$$

is a maximal right ideal of  $(\text{End}_T(P_1))_{\beta_1}$ , and every maximal right ideal of  $V$  is obtained in this way.

Since  $\beta_1 : P \rightarrow M$  is a minimal projective cover of  $M$  and  $T$  is semiperfect, by [5, 6.25(i)], we conclude that  $\ker\beta_1$  is a submodule of  $Jac(P_1)$ . Then, we have

$$\begin{aligned} J_{\beta_1} &= \{\phi \in (\text{End}_T(P_1))_{\beta_1} \mid \text{im}\phi \leq \ker\beta_1\} \\ &\subseteq \{\phi \in (\text{End}_T(P_1))_{\beta_1} \mid \text{im}\phi \leq Jac(P_1)\} \\ &= Jac((\text{End}_T(P_1))_{\beta_1}) \end{aligned}$$

$\square$

Since  $\hat{H}$  has a left inverse, it is injective on objects. Moreover, decomposable  $\text{End}_T(M)$ -modules are taken to decomposable  $T$ -modules by  $\hat{H}$ . However, it does not preserve indecomposability in general. Next lemma concerns with these facts. In the proof, we use the sketch given in [7].

**Lemma 3.26.** *Let  $X$  be an indecomposable  $\text{End}_T(M)$ -module, and let  $\hat{H}(X) = V$ . Let  $V = V_1 \oplus V_2 \oplus \dots \oplus V_k$  be a decomposition of  $V$  into a direct sum of indecomposable  $T$ -modules. Then there is an index  $i$  in  $\{1, \dots, k\}$  such that the following holds (viewing  $X$  if needed as an  $\text{End}_T(P)$ -module via the epimorphism  $\tilde{\beta} : \text{End}_T(P) \rightarrow \text{End}_T(M)$  in the proof of Proposition 3.1):*

- (i)  $H(V_i) = X$  and  $H(V_j) = (0)$  for  $j \neq i$  in  $\{1, \dots, k\}$
- (ii)  $V_i \not\cong V_j$  for  $i \neq j$  in  $\{1, \dots, k\}$
- (iii)  $\text{Hom}_T(P_1, V_i) \neq (0)$  and  $\text{Hom}_T(P_1, V_j) = (0)$  for  $i \neq j$  in  $\{1, \dots, k\}$ .

*Proof.* Since  $H$  is a left inverse for  $\hat{H}(X) = V$ , we have  $H(V) = H(\hat{H}(X)) = X$ . Then, since  $X$  is indecomposable by assumption, so is  $H(V)$ . Observe that  $H$  preserves direct sums since the functor  $\text{Hom}_T(P, -)$  does. Then

$$X = H(V) = H(V_1 \oplus V_2 \oplus \dots \oplus V_k) = H(V_1) \oplus H(V_2) \oplus \dots \oplus H(V_k)$$

and since  $X$  is indecomposable, we have  $H(V_i) = X$  for some  $i$  and  $H(V_j) = (0)$  for all  $j \neq i$ . That proves part (i).

Part (ii) follows from part (i) as  $H$  is well-defined, so if it would be the case  $V_i \cong V_j$  for some  $j$  in  $\{1, \dots, k\}$  and  $j \neq i$ , then we must have had  $H(V_i) = H(V_j)$  which is not the case.

Part (iii) is proved first for the two choices of  $V$  obtained using the functors  $F_M$  and  $G_M$  stated in Definition 3.10, since they are mapped to  $X$  under the functor  $H^{\beta_1}$ . We have  $H^\beta(V) = H^{\beta_1}(V)$ . Then, by part (i), we have  $H^{\beta_1}(V_i) \neq (0)$ , and hence  $\text{Hom}_T(P_1, V_i) \neq (0)$ . Also, we obtain by part (i) that  $H^{\beta_1}(V_j) = (0)$  for all  $j \neq i$ . That

means

$$\mathrm{Hom}_T(P_1, V_j) / \mathrm{Hom}_T(P_1, V_j) J_{\beta_1} = (0).$$

Then by Lemma 3.25 and Lemma 2.12, we have  $\mathrm{Hom}_T(P_1, V_j) = (0)$ . Hence, we have  $\mathrm{Hom}_T(P_1, V_j) = (0)$  for all  $j \neq i$ .

Before proving part (iii) for the remaining two choices of  $V$  which obtained using the functors  $\tilde{F}_M$  and  $\tilde{G}_M$ , we need to show that the functor  $A_P$  preserves direct sums. Now, let  $W_1$  and  $W_2$  be  $T$ -modules. Assume that  $\mathrm{Hom}_T(P, W_1 \oplus W_2) \neq (0)$ . Since  $\mathrm{Hom}_T(P, \mathrm{tor}_P(W_1)) = (0)$  and  $\mathrm{Hom}_T(P, \mathrm{tor}_P(W_2)) = (0)$ , we have

$$\mathrm{Hom}_T(P, \mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2)) = \mathrm{Hom}_T(P, \mathrm{tor}_P(W_1)) \oplus \mathrm{Hom}_T(P, \mathrm{tor}_P(W_2)) = (0).$$

Then, we have  $\mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2)$  is a submodule of  $\mathrm{tor}_P(W_1 \oplus W_2)$ . Since we assume that  $\mathrm{Hom}_T(P, W_1 \oplus W_2) \neq (0)$ , we have one of the statements  $\mathrm{Hom}_T(P, W_1) \neq (0)$  and  $\mathrm{Hom}_T(P, W_2) \neq (0)$  true. Then, we have either  $\mathrm{Hom}_T(P, W_1 / \mathrm{tor}_P(W_1)) \neq (0)$  or  $\mathrm{Hom}_T(P, W_2 / \mathrm{tor}_P(W_2)) \neq (0)$ . Then, we have

$$\begin{aligned} (0) \neq \mathrm{Hom}_T \left( P, \frac{W_1}{\mathrm{tor}_P(W_1)} \oplus \frac{W_2}{\mathrm{tor}_P(W_2)} \right) \\ = \mathrm{Hom}_T \left( P, \frac{W_1 \oplus W_2}{\mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2)} \right) \end{aligned}$$

Hence, we conclude that  $\mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2) = \mathrm{tor}_P(W_1 \oplus W_2)$ . Then, we obtain

$$\begin{aligned} A_P(W_1 \oplus W_2) &= (W_1 \oplus W_2) / \mathrm{tor}_P(W_1 \oplus W_2) \\ &= (W_1 \oplus W_2) / \mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2) \\ &= W_1 / \mathrm{tor}_P(W_1) \oplus W_2 / \mathrm{tor}_P(W_2) = A_P(W_1) \oplus A_P(W_2). \end{aligned}$$

For  $V = \tilde{F}_M(X)$ , we have  $V = A_P(F_M(X)) = A_P(V_1) \oplus A_P(V_2) \oplus \dots \oplus A_P(V_k)$  for some decomposition of  $F_M(X) = V_1 \oplus V_2 \oplus \dots \oplus V_k$  of  $F_M(X)$  into a direct sum of indecomposable  $T$ -modules. Set  $V'_l = A_P(V_l)$  for all  $l \in \{1, \dots, k\}$ . Since there does not

exist any  $T$ -module homomorphism from  $P$  to  $\text{tor}_P(V_s)$  for all  $s \in \{1, \dots, k\}$ , we have

$$\text{Hom}_T(P, V_s) = \text{Hom}_T(P, V_s/\text{tor}_P(V_s)) = \text{Hom}_T(P, V'_s).$$

In particular, we have  $H(V'_s) = H(V_s)$ .

Since part (iii) is proved for the functor  $F_M$ , we have  $\text{Hom}_T(P_1, V_i) \neq (0)$  and  $\text{Hom}_T(P_1, V_j) = (0)$  for  $i \neq j$  in  $\{1, \dots, k\}$ . Then we obtain  $\text{Hom}_T(P_1, V'_j) = (0)$  for  $j \neq i$  in  $\{1, \dots, k\}$ . Even if  $V'_j$  for  $j \neq i$  is decomposable, for each indecomposable constituent  $W'$  of  $V'_j$ , we have  $\text{Hom}_T(P_1, W') = (0)$ .

Since  $\text{Hom}_T(P_1, V_i) \neq (0)$  for some  $i \in \{1, \dots, k\}$  and there does not exist any  $T$ -homomorphism from  $P_1$  to  $\text{tor}_P(V_i)$ , we have

$$\text{Hom}_T(P_1, V'_i) = \text{Hom}_T(P_1, A_P(V_i)) = \text{Hom}_T(P_1, V_i/\text{tor}_P(V_i)) \neq (0).$$

Now, assume  $V'_i$  is decomposable, say  $V'_i = W'_1 \oplus W'_2$ . Since  $H$  is a left inverse for  $\tilde{F}_M$ , and  $H(V'_j) = (0)$  for all  $j \neq i$ , we have

$$\begin{aligned} X &= H(\tilde{F}_M(X)) = H(V'_1 \oplus V'_2 \oplus \dots \oplus V'_k) = H(V'_1) \oplus H(V'_2) \oplus \dots \oplus H(V'_k) \\ &= H(V'_i) = H(W'_1) \oplus H(W'_2). \end{aligned}$$

Since  $X$  is indecomposable, we have either  $H(W'_1) = (0)$  or  $H(W'_2) = (0)$ . Assume  $H(W'_2) = (0)$ . Then we have

$$\frac{\text{Hom}_T(P, W'_2)}{\text{Hom}_T(P, W'_2)J_\beta} = (0).$$

We have shown in the proof of Theorem 3.22 that  $J_\beta$  is contained in  $\text{Jac}(\text{End}_T(P))$ . Then, using Lemma 2.12 we conclude that  $\text{Hom}_T(P, W'_2) = (0)$ . Hence we have  $\text{Hom}_T(P_1, W'_2) = (0)$ .

Therefore, for a decomposition of  $\tilde{F}_M(X) = V'_1 \oplus V'_2 \dots \oplus V'_{k'}$  into indecomposable  $T$ -modules,  $\text{Hom}_T(P, V'_i) \neq (0)$  for some  $i \in \{1, \dots, k'\}$  and  $\text{Hom}_T(P, V'_j) = (0)$  for all  $j \neq i$ . The proof for  $V = \tilde{G}_M(X)$  is exactly the same as the proof for  $\tilde{F}_M(X)$ , therefore is omitted.  $\square$

The previous lemma has a corollary stated in [7]. Here, we give a proof.

**Corollary 3.27.** *Under the assumption of Lemma 3.26 suppose that  $H^\beta = H^{\beta_1}$ , where  $T$  is a finite-dimensional algebra over some field. Then no composition factor of the head of  $P_1$ , hence of  $M$ , occurs as a composition factor of  $V_j$  for  $i \neq j \in \{1, \dots, k\}$ .*

*Proof.* We have shown in the proof of Theorem 3.23 that  $P_1$  and  $M$  have the same head. By Theorem 3.22, we know that the set of constituents of  $hd(M)$  isomorphic to the set  $\text{Irr}T_H$ . However, for any  $i \neq j \in \{1, \dots, k\}$   $H^{\beta_1}(V) = (0)$ . Hence,  $V_j$  is not an element of  $\text{Irr}T_H$ . The result follows.  $\square$

Now we state an application of Theorem 3.19. The proof is taken from [7].

**Corollary 3.28.** *Assume that  $M$  is  $P$ -torsionless. Let  $X$  be an indecomposable right ideal of  $\text{End}_T(M)$ . Then  $\hat{H}(X) = XM$  is indecomposable.*

*Proof.* Since  $M$  is  $P$ -torsionless, by Theorem 3.18, we have  $\hat{H}(X) = XM$ . Also, by Theorem 3.19, there exists an isomorphism between the endomorphism rings of  $X$  and  $XM$ . For a Noetherian  $R$ -module  $M$ , the endomorphism ring of  $M$  is local if and only if  $M$  is indecomposable, for details see [3, VII.1.27]. Then, since  $X$  is indecomposable, its endomorphism ring is local, hence the endomorphism ring of  $XM$  is local as well. Then, we conclude that  $XM$  is indecomposable.  $\square$

## 4. APPLICATION OF $H$ TO HARISH-CHANDRA THEORY

### 4.1. Harish-Chandra Induction and Truncation

Let  $G$  be a finite group and  $F$  be a field. For subgroups  $P$  and  $U$  of  $G$  with  $U$  normal in  $P$ , we define *Harish-Chandra induction* from  $F[P/U]$ -modules to  $FG$ -modules, denoted by  $R_{P/U}^G$ , as the functor that lifts an  $F[P/U]$ -module to an  $FP$ -module by letting  $U$  act trivially and then inducing it from  $P$  to  $G$ . The right adjoint functor of  $R_{P/U}^G$ , *Harish-Chandra truncation*, denoted by  $T_{P/U}^G$ , is defined as the functor that restricts an  $FG$ -module to  $FP$ -module and takes  $U$ -fixed points to yield an  $F[P/U]$ -module.

**Theorem 4.1.** (*Mackey Decomposition Theorem*) *Let  $P, Q, U, V$  be subgroups of  $G$  with  $U$  normal in  $P$  and  $V$  normal in  $Q$ . Suppose that the orders of  $U$  and  $V$  are invertible in  $F$ . Let  $M$  be an  $F[P/U]$ -module. Then*

$$T_{Q/V}^G \circ R_{P/U}^G(M) \cong \bigoplus_{x \in P \backslash G / Q} R_{(P^x \cap Q)/V}^{Q/V} C_{\substack{(Q \cap P^x)U^x \\ (V \cap P^x)U^x, (U^x \cap Q)V}}^\phi T_{(Q \cap P^x)U^x / (V \cap P^x)U^x}^{P^x/U^x}(M^x)$$

where

$$C_{\substack{(Q \cap P^x)U^x \\ (V \cap P^x)U^x, (U^x \cap Q)V}}^\phi : \frac{(Q \cap P^x)U^x}{(V \cap P^x)U^x} \rightarrow \frac{(P^x \cap Q)V}{(U^x \cap Q)V}$$

is an isomorphism, and  $M^x$  denotes the conjugate module for the conjugate factor group  $x(P/U)x^{-1}$ , and  $P \backslash G / Q$  is a set of  $P - Q$ -double coset of representatives in  $G$ .

We are to prove Theorem 4.1 using biset functors. At this section, we are to introduce the notion of bisets and prove some facts about bisets.

## 4.2. Biset Functors

**Definiton 4.2.** *Let  $H$  and  $K$  be groups.*

(i) *An  $(H, K)$ -biset  $X$  is both a left  $H$ -set and a right  $K$ -set such that the  $H$ -action and the  $K$ -action commute, that is, for any  $x \in X$ , for all  $h \in H$  and  $k \in K$ , we have*

$$(h \cdot x) \cdot k = h \cdot (x \cdot k).$$

(ii) *An  $(H, K)$ -biset  $X$  is called transitive if for any elements  $x, y$  in  $X$  there exists  $(h, k)$  in  $H \times K$  such that*

$$h \cdot x \cdot k = y.$$

(iii) *The stabilizer  $(H, K)_x$  of  $x$  in  $(H \times K)$  is the subgroup of  $H \times K$  defined by*

$$(H, K)_x = \{(h, k) \in H \times K \mid h \cdot x = x \cdot k\}.$$

**Lemma 4.3.** *Let  $H$  and  $K$  be groups, and  $X$  be an  $(H, K)$  biset. Choose a set  $H \backslash X / K$  of representatives of  $(H, K)$ -orbits of  $X$ . Then there is an isomorphism of  $(H, K)$ -bisets*

$$X \cong \bigsqcup_{x \in H \backslash X / K} \frac{(H \times K)}{(H, K)_x}.$$

*In particular, any transitive  $(H, K)$ -biset is isomorphic to  $(H \times K)/L$ , for some subgroup  $L$  of  $H \times K$ .*

*Proof.* See [4, 2.3.4].

□

Composition of bisets is defined as follows:

**Definiton 4.4.** Let  $G$ ,  $H$  and  $K$  be groups. If  $U$  is an  $(H, G)$ -biset, and  $V$  is a  $(K, H)$ -biset, the composition of  $V$  and  $U$  is the set of  $H$ -orbits on the cartesian product  $V \times U$ , where the right action of  $H$  is defined by

$$(v, u) \cdot h = (v \cdot h, h^{-1} \cdot u)$$

for all  $(v, u)$  in  $V \times U$ . It is denoted by  $V \times_H U$ .

Now, we are to state a lemma which provides us a useful formula for the composition of bisets:

**Lemma 4.5.** (*Mackey Formula for Bisets*) Let  $G$ ,  $H$  and  $K$  be groups. If  $L$  is a subgroup of  $H \times G$ , and if  $M$  is a subgroup of  $K \times H$ , then there is an isomorphism of  $(K, G)$ -bisets

$$\frac{K \times H}{M} \times_H \frac{H \times G}{L} \cong \bigsqcup_{x \in p_2(M) \backslash H / p_1(L)} \frac{K \times G}{M *^{(x,1)} L}$$

where  $p_2(M) \backslash H / p_1(L)$  is a set of representatives of double cosets and

$$M *^{(x,1)} L = \{(k, g) \in K \times G \mid (k, h) \in M \text{ and } (h, g) \in {}^{(x,1)}L \text{ for some } h \in H\}.$$

*Proof.* See [4, 2.3.24]. □

Let  $X$  be a  $G$ -set. We define the permutation  $FG$ -module with permutation basis  $X$  as  $FX$ . That is,

$$FX = \bigoplus_{x \in X} F \cdot x.$$

Then,  $G$  acts on  $FX$ , for  $g \in G$ ,  $x \in X$ , and  $\lambda_x \in F$ , as

$$g \left( \sum_{x \in X} \lambda_x x \right) = \sum_{x \in X} \lambda_x gx.$$

Similarly, for a  $(G, H)$ -biset  $X$ ,  $FX$  is an  $FG - FH$ -bimodule.

**Lemma 4.6.** *For a  $(G, H)$ -biset  $X$  and a  $(H, K)$ -biset  $Y$*

$$F(X \times_H Y) = FX \otimes_{FH} FY.$$

*Proof.* Firstly, we observe that,

$$\begin{aligned} FX \otimes_{FH} FY &= \left( \bigoplus_{x \in X} Fx \right) \otimes_{FH} \left( \bigoplus_{y \in Y} Fy \right) \\ &= \bigoplus_{(x,y) \in X \times Y} Fx \otimes_{FH} Fy \\ &= \bigoplus_{\substack{(x,y) \in X \times Y \\ h \in H}} \frac{Fx \times Fy}{\sim} \end{aligned}$$

where  $\sim$  is an equivalence relation on  $X \times Y$  relating the elements  $(xh, y)$  and  $(x, hy)$  for every  $h$  in  $H$ . Also, we have  $(xh, y) \sim (x, hy)$  if and only if  $(x, y) \sim (xh, yh^{-1})$ . Then we obtain

$$FX \otimes_{FH} FY = \bigoplus_{(x,y) \in X \times_H Y} F(x, y) = F(X \times_H Y).$$

□

**Definiton 4.7.** *Let  $G$  and  $K$  be groups. Let  $H$  be a subgroup of  $G$  and  $N$  be a normal subgroup of  $G$ .*

- (i) *The set  $G$  is a  $(H, G)$ -biset for the actions given by left and right multiplications in  $G$ . It is denoted by  $\text{res}_H^G$ .*
- (ii) *The set  $G$  is a  $(G, H)$ -biset for the actions given by left and right multiplications in  $G$ . It is denoted by  $\text{ind}_H^G$ .*
- (iii) *The set  $G/N$  is a  $(G, G/N)$ -biset for the left action of  $G$  by projection to  $G/N$ , and then left multiplication in  $G/N$ , and the right action of  $G/N$  by multiplication. It is denoted by  $\text{inf}_{G/N}^G$ .*

- (iv) The set  $G/N$  is a  $(G/N, G)$ -biset for the left action of  $G/N$  by multiplication, and the right action of  $G$  by projection to  $G/N$ , and then right multiplication in  $G/N$ . It is denoted by  $\text{def}_{G/N}^G$ .
- (v) If  $f : G \rightarrow K$  is a group isomorphism, then the set  $K$  is an  $(K, G)$ -biset for the left action of  $K$  by multiplication, and the right action of  $G$  given by taking image by  $f$ , and then multiplying on the right in  $K$ . It is denoted by  $c_{K,G}^f$ .

These five bisets defined in Definition 4.7 are transitive, therefore their orbit sets have cardinality 1. Then, using Lemma 4.3 we can rewrite those elementary bisets as follows:

$$\begin{aligned} \text{res}_H^G &= (H \times G)/R \text{ where } R = \{(h, h) \mid h \in H\} \\ \text{ind}_H^G &= (G \times H)/T \text{ where } T = \{(h, h) \mid h \in H\} \\ \text{inf}_{G/N}^G &= (G \times G/N)/I \text{ where } I = \{(g, gN) \mid g \in G\} \\ \text{def}_{G/N}^G &= (G/N \times G)/D \text{ where } D = \{(gN, g) \mid g \in G\} \\ c_{K,G}^f &= (K \times G)/C^f \text{ where } C^f = \{(f(g), g) \mid g \in G\} \end{aligned}$$

Now, using Lemma 4.6, we define five elementary biset functors:

**Definiton 4.8.** *Let  $G$  and  $K$  be groups. Let  $H$  be a subgroup of  $G$  and  $N$  be a normal subgroup of  $G$ .*

- (i) *For an  $FG$ -module  $V$ , the restriction functor is defined as*

$$\text{Res}_H^G V := F(\text{res}_H^G \times_G V) =_{FH} FG_{FG} \otimes_{FG} V.$$

- (ii) *For an  $FH$ -module  $V$ , the induction functor is defined as*

$$\text{Ind}_H^G(V) := F(\text{ind}_H^G \times_H V) =_{FG} FG_{FH} \otimes_{FH} V.$$

(iii) For an  $F[G/N]$ -module  $V$ , the inflation functor is defined as

$$\text{Inf}_{G/N}^G(V) := F(\text{inf}_{G/N}^G \times_{G/N} V) =_{FG} FG_{F[G/N]} \otimes_{F[G/N]} V.$$

(iv) For an  $FG$ -module  $V$ , the deflation functor is defined as

$$\text{Def}_{G/N}^G(V) := F(\text{def}_{G/N}^G \times_G V) =_{F[G/N]} F[G/N]_{FG} \otimes_{FG} V.$$

(v) For an  $FG$ -module  $V$  and an isomorphism  $f : G \rightarrow K$ , the transport of structure functor is defined as

$$\text{C}_{K,G}^f(V) := F(\text{c}_{K,G}^f \times_G V) =_{FK} FG_{FG} \otimes_{FG} V.$$

### 4.3. Mackey Decomposition Theorem

At this section, first, we are to prove Mackey Decomposition Theorem using the results of the previous section. Then, we are to show the adjointness of  $T_{P/U}^G$  and  $R_{P/U}^G$  on both sides.

*Proof of Theorem 4.1.* The equality in the statement of Theorem 4.1 can be rewritten as

$$\begin{aligned} & \text{Def}_{Q/V}^Q \text{Res}_Q^G \text{Ind}_P^G \text{Inf}_{P/U}^P(M) \\ &= \bigsqcup_{x \in P \backslash G/Q} \text{Ind}_{(P^x \cap Q)V}^{Q/V} \text{Inf}_{\frac{(P^x \cap Q)V}{(U^x \cap Q)V}}^{(P^x \cap Q)V} \text{C}_{\frac{(P^x \cap Q)V}{(U^x \cap Q)V}, \frac{(Q \cap P^x)U^x}{(V \cap P^x)U^x}}^\phi \text{Def}_{\frac{(Q \cap P^x)U^x}{(V \cap P^x)U^x}}^{(Q \cap P^x)U^x} \text{Res}_{(Q \cap P^x)U^x}^{P^x/U^x}(M^x) \end{aligned}$$

We can write

$$\text{Def}_{Q/V}^Q \text{Res}_Q^G \text{Ind}_P^G \text{Inf}_{P/U}^P = \frac{Q/V \times Q}{D} \times_Q \frac{Q \times G}{R} \times_G \frac{G \times P}{T} \times_P \frac{P \times P/U}{I}$$

where

$$D = \{(qV, q) \mid q \in Q\}, R = \{(q, q) \mid q \in Q\}, T = \{(p, p) \mid p \in P\}, I = \{(p, pU) \mid p \in P\}.$$

By Lemma 4.5, we know

$$\frac{Q/V \times Q}{D} \times_Q \frac{Q \times G}{R} = \bigsqcup_{q \in p_2(D) \setminus Q/p_1(R)} \frac{Q/V \times G}{D * {}^{(q,1)}R}$$

where

$$p_1(R) = \{q \in Q \mid (q, g) \in R \text{ for some } g \in G\},$$

$$p_2(D) = \{q \in Q \mid (q'V, q) \in D \text{ for some } q'V \in Q/V\},$$

$$D * {}^{(q,1)}R = \{(qV, g) \in Q/V \times G \mid (qV, q') \in D \text{ and } (q', g) \in {}^{(q,1)}R \text{ for some } q' \in Q\}.$$

For any  $q$  in  $Q$ , the element  $(q, q)$  is in  $R$ . Thus, we have  $p_1(R) = Q$ . Also, for any  $q$  in  $Q$ , the element  $(q, qV)$  is in  $D$ . So, we have  $p_2(D) = Q$ . Then, the set  $p_2(D) \setminus Q/p_1(R)$  contains only one coset and the union consists of only one biset. We can take the identity element 1 of  $Q$  as the coset representative  $q$ . Then we have

$$\frac{Q/V \times Q}{D} \times_Q \frac{Q \times G}{R} = \frac{Q/V \times G}{D * {}^{(1,1)}R}.$$

Clearly the set  ${}^{(1,1)}R$  is equal to  $R$ . Therefore

$$\begin{aligned} D * {}^{(q,1)}R &= D * R \\ &= \{(qV, g) \in Q/V \times G \mid (qV, q') \in D \text{ and } (q', g) \in R \text{ for some } q' \in Q\} \\ &= \{(qV, q) \in Q/V \times Q\} \\ &= \{(qV, q) \mid q \in Q\} \\ &= D \end{aligned}$$

Hence, we obtain

$$\text{Def}_{Q/V}^Q \text{Res}_Q^G = \frac{Q/V \times Q}{D} \times_Q \frac{Q \times G}{R} = \frac{Q/V \times G}{D}.$$

Similarly, we have

$$\frac{G \times P}{T} \times_P \frac{P \times P/U}{I} = \bigsqcup_{p \in p_2(T) \setminus P/p_1(I)} \frac{G \times P/U}{T * {}^{(p,1)}I}$$

where

$$p_1(I) = \{p \in P \mid (p, p'U) \in I \text{ for some } p'U \in P/U\},$$

$$p_2(T) = \{p \in P \mid (p, g) \in T \text{ for some } g \in G\},$$

$$T * {}^{(p,1)}R = \{(g, pU) \in G \times P/U \mid (g, p') \in T \text{ and } (p', pU) \in {}^{(p,1)}I \text{ for some } p' \in P\}.$$

For any  $p$  in  $P$ , the element  $(p, pU)$  is in  $I$ . Thus, we have  $p_1(I) = P$ . Also, for any  $p$  in  $P$ , the element  $(p, p)$  is in  $T$ . So, we have  $p_2(T) = P$ . Then, the set  $p_2(T) \setminus P/p_1(I)$  contains only one coset and the union consists of only one biset. We can take the identity element 1 of  $P$  as the coset representative  $p$ . Then we have

$$\frac{G \times P}{T} \times_P \frac{P \times P/U}{I} = \frac{G \times P/U}{T * {}^{(1,1)}I}.$$

Clearly the set  ${}^{(1,1)}I$  is equal to  $I$ . Therefore

$$\begin{aligned} T * {}^{(p,1)}I &= T * I \\ &= \{(g, pU) \in G \times P/U \mid (g, p') \in T \text{ and } (p', pU) \in I \text{ for some } p' \in P\} \\ &= \{(p, pU) \in P \times P/U\} \\ &= \{(p, pU) \mid p \in P\} \\ &= I \end{aligned}$$

Hence, we obtain

$$\text{Ind}_P^G \text{Inf}_P^P / U = \frac{G \times P}{T} \times_P \frac{P \times P/U}{I} = \frac{G \times P/U}{I}.$$

Now, by Lemma 4.5, we obtain

$$\frac{Q/V \times G}{D} \times_G \frac{G \times P/U}{I} = \bigsqcup_{g \in [p_2(D) \backslash G/p_1(I)]} \frac{Q/V \times P/U}{D * {}^{(g,1)}I}$$

where

$$\begin{aligned} p_1(I) &= \{p \in P \mid (p, p'U) \in I \text{ for some } p'U \in P/U\} \\ p_2(D) &= \{q \in Q \mid (q'V, q) \in D \text{ for some } q'V \in Q/V\} \end{aligned}$$

For any  $p$  in  $P$ , we have  $(p, pU)$  in  $I$ , and hence  $p$  in  $p_1(I)$ . Thus, we have  $p_1(I) = P$ . Also, for any  $q$  in  $Q$ , we have  $(qV, q)$  in  $D$ , and hence  $q$  in  $p_2(D)$ . Thus, we have  $p_2(D) = Q$ . Then, we obtain

$$p_2(D) \backslash G/p_1(I) = Q \backslash G/P.$$

Also, for any  $g$  in  $p_2(D) \backslash G/p_1(I)$ , we have

$$\begin{aligned} {}^{(g,1)}I &= (g, 1)I(g^{-1}, 1) = \{(g, 1)(p, pU)(g^{-1}, 1) \mid p \in P\} \\ &= \{(gpg^{-1}, pU) \mid p \in P\} \end{aligned}$$

and

$$\begin{aligned} L := D * {}^{(g,1)}I &= \{(qV, pU) \in Q/V \times P/U \mid (qV, g) \in D \text{ } (g, pU) \in {}^{(g,1)}I \text{ for some } g \in G\} \\ &= \{(qV, pU) \in Q/V \times P/U \mid q = gpg^{-1}\} \end{aligned}$$

Now, by [4, 2.3.25] and [4, 2.3.26] we have

$$\frac{Q/V \times P/U}{L} \cong \text{Ind}_{p_1(L)}^{Q/V} \text{Inf}_{p_1(L)/k_1(L)}^{p_1(L)} \text{C}_{p_1(L)/k_1(L); p_2(L)/k_2(L)}^f \text{Def}_{p_2(L)/k_2(L)}^{p_2(L)} \text{Res}_{p_2(L)}^{P/U}$$

where

$$p_1(L) = \{qV \in Q/V \mid (qV, pU) \in L \text{ for some } pU \in P/U\}$$

$$= \{qV \in Q/V \mid q = gpg^{-1} \text{ for some } p \in P\}$$

$$k_1(L) = \{qV \in Q/V \mid (qV, U) \in L\} = \{qV \in Q/V \mid q = gug^{-1} \text{ for some } u \in U\}$$

$$p_2(L) = \{pU \in P/U \mid (qV, pU) \in L \text{ for some } qV \in Q/V\}$$

$$= \{pU \in P/U \mid q = gpg^{-1} \text{ for some } q \in Q\}$$

$$k_2(L) = \{pU \in P/U \mid (V, pU) \in L\} = \{pU \in P/U \mid v = gpg^{-1} \text{ for some } v \in V\}$$

and

$$f : p_2(L)/k_2(L) \rightarrow p_1(L)/k_1(L), (pU)k_2(L) \mapsto (gpg^{-1}V)k_1(L)$$

On the other hand,

$$(P^g \cap Q)V = \{qV \in Q/V \mid q = g^{-1}pg \text{ for some } p \in P\} = p_1(L)$$

$$(U^g \cap Q)V = \{qV \in Q/V \mid q = g^{-1}ug \text{ for some } u \in U\} = k_1(L)$$

$$(Q \cap P^g)U^g = \{(pU)^g \mid g^{-1}pg = q \text{ for some } q \in Q\} = (p_2(L))^g$$

$$(V \cap P^g)U^g = \{(pU)^g \mid g^{-1}pg = v \text{ for some } v \in V\} = (k_2(L))^g$$

and, by Butterfly Lemma [10, 3.3], there is an isomorphism

$$\phi : (Q \cap P^g)U^g / (V \cap P^g)U^g \rightarrow (P^g \cap Q)V / (U^g \cap Q)V$$

Therefore, we obtain

$$\begin{aligned}
T_{Q/V}^G R_{P/U}^G(M) &= \text{Def}_{Q/V}^G \text{Res}_Q^G \text{Ind}_P^G \text{Inf}_{P/U}^P(M) \\
&= \bigsqcup_{g \in p_2(D) \backslash G/p_1(I)} \text{Ind}_{p_1(L)}^{Q/V} \text{Inf}_{\frac{p_1(L)}{k_1(L)}}^{p_1(L)} C_{\frac{p_2(L)}{k_2(L)}, \frac{p_1(L)}{k_1(L)}}^f \text{Def}_{\frac{p_2(L)}{k_2(L)}}^{p_2(L)} \text{Res}_{p_2(L)}^{P/U}(M) \\
&= \bigsqcup_{g \in p_2(D) \backslash G/p_1(I)} \text{Ind}_{p_1(L)}^{Q/V} \text{Inf}_{\frac{p_1(L)}{k_1(L)}}^{p_1(L)} C_{\left(\frac{p_2(L)}{k_2(L)}\right)^g, \frac{p_1(L)}{k_1(L)}}^f \text{Def}_{\left(\frac{p_2(L)}{k_2(L)}\right)^g}^{(p_2(L))^g} \text{Res}_{(p_2(L))^g}^{(P/U)^g}(M^g) \\
&= \bigsqcup_{g \in P \backslash G/Q} \text{Ind}_{(P^g \cap Q)V}^{Q/V} \text{Inf}_{\frac{(P^g \cap Q)V}{(U^g \cap Q)V}}^{(P^g \cap Q)V} C_{\frac{(Q \cap P^g)U^g}{(V \cap P^g)U^g}, \frac{(P^g \cap Q)V}{(U^g \cap Q)V}}^\phi \text{Def}_{\frac{(Q \cap P^g)U^g}{(V \cap P^g)U^g}}^{(p_2(L))^g} \text{Res}_{(V \cap P^g)U^g}^{(P/U)^g}(M^g) \\
&= \bigoplus_{g \in P \backslash G/Q} R_{\frac{(P^g \cap Q)V}{(U^g \cap Q)V}}^{Q/V} T_{\frac{(Q \cap P^g)U^g}{(V \cap P^g)U^g}}^{(P/U)^g}
\end{aligned}$$

□

Let  $P$  be a subgroup of  $G$  and  $U$  be a normal subgroup of  $P$ . The quotient  $P/U$  is called a *subquotient* of  $G$ .

**Lemma 4.9.** *For a subquotient  $P/U$  of  $G$ , if the order of  $U$  is invertible in  $F$ , the functor  $T_{P/U}^G$  is adjoint on both sides of  $R_{P/U}^G$ .*

*Proof.* By Definition 4.8, for an  $FG$ -module  $A$ , we have

$$T_{P/U}^G(A) = \text{Def}_{P/U}^P \text{Res}_P^G(A)$$

and, for an  $F[P/U]$ -module  $B$ , we have

$$R_{P/U}^G(B) = \text{Ind}_P^G \text{Inf}_{P/U}^P(B).$$

Clearly,  $\text{Ind}_P^G$  is adjoint on both sides of  $\text{Res}_P^G$ . So, to prove the statement, it is enough to examine left and right adjoints of  $\text{Inf}_{P/U}^P$ .

The left and right adjoints of  $\text{Inf}_{P/U}^P$  is not necessarily equal. However, in our

case, we are to prove that they are isomorphic. Firstly, we are to show that the functor

$$\text{Def}_{P/U}^P : \text{mod}_{FP} \rightarrow \text{mod}_{F[P/U]}, M \mapsto M^U$$

where  $M^U = \{m \in M \mid Um = m\}$ , is the left adjoint of  $\text{Inf}_{P/U}^P$ .

Let  $M$  be an  $FP$ -module and  $N$  be an  $F[P/U]$ -module. We define a map

$$\Phi : \text{Hom}_{FP}(M, \text{Inf}_{P/U}^P(N)) \rightarrow \text{Hom}_{F[P/U]}(\text{Def}_{P/U}^P(M), N), \phi \mapsto \tilde{\phi}$$

where  $\tilde{\phi}$  is defined as

$$\tilde{\phi} : \text{Def}_{P/U}^P(M) \rightarrow N, m \mapsto \phi(m).$$

$\tilde{\phi}$  is an  $F[P/U]$ -module homomorphism since

$$\tilde{\phi}(pUm) = \tilde{\phi}(pm) = \phi(pm) = p\phi(m) = pU\phi(m) = pU\tilde{\phi}(m)$$

for  $p$  in  $P$  and  $m$  in  $M^U$ . Now, we define a second map

$$\Psi : \text{Hom}_{F[P/U]}(\text{Def}_{P/U}^P(M), N) \rightarrow \text{Hom}_{FP}(M, \text{Inf}_{P/U}^P(N)), \psi \mapsto \hat{\psi}$$

where  $\hat{\psi}$  is defined as

$$\hat{\psi} : M \rightarrow \text{Inf}_{P/U}^P(N), m \mapsto \psi \left( \frac{1}{|U|} \sum_{u \in U} um \right).$$

The map  $\hat{\psi}$  is an  $FP$ -module homomorphism since, for  $p$  in  $P$  and  $m$  in  $M$ ,

$$\begin{aligned}\hat{\psi}(pm) &= \psi \left( \frac{1}{|U|} \sum_{u \in U} upm \right) = \psi \left( \frac{1}{|U|} \sum_{u \in U} pum \right) = \psi \left( p \frac{1}{|U|} \sum_{u \in U} um \right) \\ &= p\psi \left( \frac{1}{|U|} \sum_{u \in U} um \right) = p\hat{\psi}(m).\end{aligned}$$

Now, we are to show that  $\Psi$  is the inverse of  $\Phi$ . For  $\psi$  in  $\text{Hom}_{F[P/U]}(\text{Def}_{P/U}^P(M), N)$  and  $m$  in  $M^U$ , we have

$$\Phi\Psi(\psi)(m) = \Phi(\hat{\psi}(m)) = \tilde{\psi}(m) = \hat{\psi}(m) = \psi \left( \frac{1}{|U|} \sum_{u \in U} um \right) = \psi \left( \frac{1}{|U|} \sum_{u \in U} m \right) = \psi(m)$$

Also, for  $\phi$  in  $\text{Hom}_{FP}(M, \text{Inf}_{P/U}^P(N))$  and  $m$  in  $M$ , we have

$$\begin{aligned}\Psi\Phi(\phi)(m) &= \Psi(\tilde{\phi}(m)) = \hat{\phi}(m) = \tilde{\phi} \left( \frac{1}{|U|} \sum_{u \in U} um \right) = \phi \left( \frac{1}{|U|} \sum_{u \in U} um \right) \\ &= \frac{1}{|U|} \sum_{u \in U} u\phi(m) = \phi(m)\end{aligned}$$

Therefore, we obtain

$$\text{Hom}_{FP}(M, \text{Inf}_{P/U}^P(N)) \cong \text{Hom}_{F[P/U]}(\text{Def}_{P/U}^P(M), N),$$

that is, the functor  $\text{Def}_{P/U}^P$  is the left adjoint of  $\text{Inf}_{P/U}^P$ .

Secondly, we are to show that the functor

$$\text{Codef}_{P/U}^P : \text{mod}_{FP} \rightarrow \text{mod}_{F[P/U]}, M \mapsto M_U$$

where  $M_U = \{\sum_{u \in U} um \mid m \in M\}$ , is the right adjoint of  $\text{Inf}_{P/U}^P$ .

We define a map

$$\Theta : \text{Hom}_{FP}(\text{Inf}_{P/U}^P(N), M) \rightarrow \text{Hom}_{F[P/U]}(N, \text{Codef}_{P/U}^P(M)), \theta \mapsto \tilde{\theta}$$

where  $\tilde{\theta}$  is defined as

$$\tilde{\theta} : N \rightarrow \text{Codef}_{P/U}^P(M), n \mapsto \sum_{u \in U} u\theta(n).$$

The map  $\tilde{\theta}$  is an  $F[P/U]$ -module homomorphism since

$$\tilde{\theta}(pUn) = \sum_{u \in U} u\theta(pn) = \sum_{u \in U} pu\theta(n) = p \sum_{u \in U} u\theta(n) = p\tilde{\theta}(n) = pU\tilde{\theta}(n)$$

for  $p$  in  $P$  and  $n$  in  $N$ . Now, we define a second map

$$\Gamma : \text{Hom}_{F[P/U]}(N, \text{Codef}_{P/U}^P(M)) \rightarrow \text{Hom}_{FP}(\text{Inf}_{P/U}^P(N), M), \gamma \mapsto \hat{\gamma}$$

where  $\hat{\gamma}$  is defined as

$$\hat{\gamma} : \text{Inf}_{P/U}^P N \rightarrow M, n \mapsto \frac{1}{|U|} \gamma(n).$$

The map  $\hat{\gamma}$  is an  $F[P/U]$ -module homomorphism since

$$\hat{\gamma}(pn) = \frac{1}{|U|} \gamma(pn) = p \frac{1}{|U|} \gamma(n) = p\hat{\gamma}(n)$$

for  $p$  in  $P$  and  $n$  in  $N$ .

Now, we prove that  $\Gamma$  is the inverse of  $\Theta$ . For  $\gamma$  in  $\text{Hom}_{F[P/U]}(N, \text{Codef}_{P/U}^P(M))$

and  $n$  in  $N$ , we have

$$\begin{aligned}\Theta\Gamma(\gamma)(n) &= \Theta(\hat{\gamma}(n)) = \tilde{\hat{\gamma}}(n) = \sum_{u \in U} u\hat{\gamma}(n) = \sum_{u \in U} u \frac{1}{|U|} \gamma(n) \\ &= \frac{1}{|U|} \sum_{u \in U} u\gamma(n) = \frac{1}{|U|} \sum_{u \in U} \gamma(n) \\ &= \gamma(n).\end{aligned}$$

Also, for  $\theta$  in  $\text{Hom}_{FP}(\text{Inf}_{P/U}^P(N), M)$  and  $n$  in  $\text{Inf}_{P/U}^P(N)$ , we have

$$\begin{aligned}\Gamma\Theta(\theta)(n) &= \Gamma(\tilde{\theta}(n)) = \hat{\tilde{\theta}}(n) = \frac{1}{|U|} \tilde{\theta}(n) = \frac{1}{|U|} \sum_{u \in U} u\theta(n) \\ &= \frac{1}{|U|} \sum_{u \in U} \theta(un) = \frac{1}{|U|} \sum_{u \in U} \theta(n) = \theta(n)\end{aligned}$$

Therefore we obtain

$$\text{Hom}_{FP}(\text{Inf}_{P/U}^P(N), M) \cong \text{Hom}_{F[P/U]}(N, \text{Codef}_{P/U}^P(M)),$$

that is,  $\text{Codef}_{P/U}^P$  is the right adjoint of  $\text{Inf}_{P/U}^P$ .

Now, we are to show that, for an  $F[P/U]$ -module  $M$ , the  $FP$ -modules  $\text{Def}_{P/U}^P(M)$  and  $\text{Codef}_{P/U}^P(M)$  are isomorphic. To this end, we define two maps

$$\zeta : \text{Def}_{P/U}^P(M) \rightarrow \text{Codef}_{P/U}^P(M), \quad m \mapsto \sum_{u \in U} um$$

and

$$\xi : \text{Codef}_{P/U}^P(M) \rightarrow \text{Def}_{P/U}^P(M), \quad \sum_{u \in U} um \mapsto \frac{1}{|U|} \sum_{u \in U} um.$$

$\zeta$  and  $\xi$  are  $F[P/U]$ -module homomorphisms since for  $m$  in  $M^U$  and  $\sum_{u \in U} un$  in  $M_U$ ,

we have

$$\zeta(pUm) = \zeta(pm) = \sum_{u \in U} upm = p \sum_{u \in U} um = pU \sum_{u \in U} um = pU \zeta(m)$$

and

$$\begin{aligned} \xi(pU \sum_{u \in U} un) &= \xi(p \sum_{u \in U} un) = \xi(\sum_{u \in U} upn) = \frac{1}{|U|} \sum_{u \in U} u(pn) \\ &= p \left( \frac{1}{|U|} \sum_{u \in U} un \right) = pU \left( \frac{1}{|U|} \sum_{u \in U} un \right) = pU \xi(\sum_{u \in U} un) \end{aligned}$$

Also, we have

$$\begin{aligned} \zeta \xi(\sum_{u \in U} um) &= \zeta \left( \frac{1}{|U|} \sum_{u \in U} um \right) = \frac{1}{|U|} \sum_{u \in U} u \xi(m) = \frac{1}{|U|} \sum_{u \in U} u \sum_{u \in U} um \\ &= \frac{1}{|U|} \sum_{u \in U} \sum_{u \in U} um = \sum_{u \in U} um \end{aligned}$$

for  $m + UM$  in  $M_U$ , and

$$\xi \zeta(n) = \xi(n + UM) = \frac{1}{|U|} \sum_{u \in U} un = n$$

for  $n$  in  $M^U$ . Therefore  $\text{Def}_{P/U}^P(M)$  and  $\text{Codef}_{P/U}^P(M)$  are isomorphic.

Now, for an  $FG$ -module  $N$  and  $FP$ -module  $M$ , we have

$$\begin{aligned} \text{Hom}_{FP}(T_{P/U}^G(N), M) &= \text{Hom}_{FP}(\text{Def}_{P/U}^P \text{Res}_P^G(N), M) \\ &\cong \text{Hom}_{FP}(\text{Res}_P^G(N), \text{Inf}_{P/U}^P(M)) \\ &\cong \text{Hom}_{FP}(N, \text{Ind}_P^G \text{Inf}_{P/U}^P(M)) \\ &= \text{Hom}_{FP}(N, R_{P/U}^G(M)) \end{aligned}$$

and

$$\begin{aligned}
\mathrm{Hom}_{FG}(N, T_{P/U}^G(M)) &= \mathrm{Hom}_{FG}(N, \mathrm{Def}_{P/U}^P \mathrm{Res}_P^G(M)) \\
&\cong \mathrm{Hom}_{FG}(N, \mathrm{Codef}_{P/U}^P \mathrm{Res}_P^G(M)) \\
&\cong \mathrm{Hom}_{FG}(\mathrm{Inf}_{P/U}^P(N), \mathrm{Res}_P^G(M)) \\
&\cong \mathrm{Hom}_{FG}(\mathrm{Ind}_P^G \mathrm{Inf}_{P/U}^P(N), M) \\
&= \mathrm{Hom}_{FG}(R_{P/U}^G(N), M)
\end{aligned}$$

Thus,  $T_{P/U}^G$  is adjoint on both sides of  $R_{P/U}^G$ .  $\square$

#### 4.4. Mackey System

Recall that, for a subgroup  $P$  of  $G$  and a normal subgroup  $U$  of  $P$ , a subquotient of  $G$  is the quotient  $P/U$ . A system  $\mathbf{M}$  of subquotients of  $G$  is called a *Mackey system*, if it contains  $G$ , is closed under conjugation and the operation

$$P/U \sqcap Q/V = (P \cap Q)U / (P \cap V)U$$

for  $P/U$  and  $Q/V$  in  $\mathbf{M}$ .

For a prime  $p$ , the system  $\mathbf{M}$  is called *p-modular*, if for all  $P/U$  in  $\mathbf{M}$ ,  $U$  is  $p$ -regular, that is, the order of  $U$  is not divisible by  $p$ . Thus, if  $F$  has characteristic  $p$  and  $\mathbf{M}$  is  $p$ -modular, we may apply Theorem 4.1 to the elements of  $\mathbf{M}$ .

For  $P/U$  in  $\mathbf{M}$ , the set

$$\mathbf{M}_{P/U} = \{P/U \sqcap Q/V \mid Q/V \in \mathbf{M}\}$$

defines a Mackey system in  $P/U$ . We give a proof of this fact. Let  $P/U \sqcap Q/V$  and  $P/U \sqcap R/Y$  be two subquotients in  $\mathbf{M}_{P/U}$ . Then, since  $U \leq P$  and  $P \cap VU \leq P \cap QU$ ,

we have

$$\begin{aligned}
(P/U \sqcap Q/V) \sqcap (P/U \sqcap R/Y) &= \frac{(P \cap Q)U}{(P \cap V)U} \sqcap \frac{(P \cap R)U}{(P \cap Y)U} \\
&= \frac{[(P \cap Q)U \cap (P \cap R)U](P \cap V)U}{[(P \cap Q)U \cap (P \cap Y)U](P \cap V)U} \\
&= \frac{(P \cap QU \cap P \cap RU)(P \cap VU)}{(P \cap QU \cap P \cap YU)(P \cap VU)} \\
&= \frac{(P \cap QU \cap RU)(P \cap VU)}{(P \cap QU \cap YU)(P \cap VU)} \\
&= \frac{P \cap RU(P \cap VU) \cap QU}{P \cap YU(P \cap VU) \cap QU} \\
&= \frac{[P \cap RU(P \cap VU) \cap Q]U}{[P \cap YU(P \cap VU) \cap Q]U} \\
&= \frac{P}{U} \sqcap \frac{RU(P \cap VU) \cap Q}{YU(P \cap VU) \cap Q}
\end{aligned}$$

Hence,  $\mathbf{M}_{P/U}$  is closed under the operation  $\sqcap$ . Clearly, it is closed under conjugation.

Also, we have

$$P/U = PU/U = \frac{(P \cap P)U}{(P \cap U)U} = P/U \sqcap P/U.$$

So,  $P/U$  is an element of  $\mathbf{M}_{P/U}$ .

If  $\mathbf{M}$  is  $p$ -modular, so is  $\mathbf{M}_{P/U}$ . To show this statement, assume  $\mathbf{M}$  is  $p$ -modular. Let  $Q/V \sqcap P/U = (Q \cap P)V/(Q \cap U)V$  be an element of  $\mathbf{M}_{P/U}$ . If  $\mathbf{M}$  is  $p$ -modular, then  $U$  and  $V$  are  $p$ -regular. Then, the order of  $Q \cap U$  is not divisible by  $p$  since  $Q \cap U$  is a submodule of  $U$  and the order of  $U$  is not divisible by  $p$ . Then  $p$  does not divide the order of the module  $(Q \cap U)V$ . So, the Mackey system  $\mathbf{M}_{P/U}$  is also  $p$ -modular.

Now, assume  $\mathbf{M}$  is  $p$ -modular, where  $p$  is the characteristic of  $F$ . An  $FG$ -module  $M$  is called *cuspidal* with respect to  $\mathbf{M}$ , if  $T_{P/U}^G(M) = (0)$  for all subquotients  $P/U$  of  $G$  different from  $G$ . For a subquotient  $P/U$  in  $\mathbf{M}$ , an  $F[P/U]$ -module  $N$  is called *cuspidal* with respect to  $\mathbf{M}$  if it is cuspidal with respect to  $\mathbf{M}_{P/U}$ . If  $\mathbf{M}$  contains a proper subgroup  $P/1$ , then  $FG$  does not have any cuspidal modules since for any

nonzero  $FG$ -module  $M$ , we have

$$T_{P/1}^G(M) = \text{Res}_P^G(M) \neq (0).$$

Even if  $\mathbf{M}$  contains a proper subgroup and hence  $FG$  does not have any cuspidal modules with respect to  $\mathbf{M}$ , the same might not be true for  $\mathbf{M}_{P/U}$ , so  $F[P/U]$  might have cuspidal modules with respect to  $\mathbf{M}_{P/U}$ .

The following theorem establishes a relation between Harish-Chandra theory and the results of the first chapter. In [7], a sketch for the proof was given. Here, we give a full proof using this sketch.

**Theorem 4.10.** *Let  $F$  be of characteristic  $p$  where  $p > 0$ . Let  $\mathbf{M}$  be a  $p$ -modular Mackey system for  $G$ . For  $P/U$  in  $\mathbf{M}$ , let  $M$  be an irreducible cuspidal  $F[P/U]$ -module, and  $\beta : X \rightarrow M$  be a minimal projective cover of  $M$ . Then we have*

$$\text{End}_{FG}(R_{P/U}^G(X)) = (\text{End}_{FG}(R_{P/U}^G(X)))_{R_{P/U}^G(\beta)}$$

where  $R_{P/U}^G(\beta)$  is the map

$$R_{P/U}^G(\beta) : R_{P/U}^G(X) \rightarrow R_{P/U}^G(M)$$

induced from the map  $\beta : X \rightarrow M$ .

*Proof.* We apply the functor  $\text{Hom}_{FG}(R_{P/U}^G(X), -)$  to the map

$$R_{P/U}^G(\beta) : R_{P/U}^G(X) \rightarrow R_{P/U}^G(M)$$

to obtain the map

$$(R_{P/U}^G(\beta))_* : \text{End}_{FG}(R_{P/U}^G(X)) \rightarrow \text{Hom}_{FG}(R_{P/U}^G(X), R_{P/U}^G(M)), \phi \mapsto \beta\phi$$

Since  $X$  is projective it can be written as  $X = \bigoplus_{n \in \mathbb{N}} F[P/U]$ . Then we have

$$\begin{aligned}
R_{P/U}^G(X) &= \text{Ind}_P^G \text{Inf}_{P/U}^P F[P/U](X) = FG \otimes_P (FP \otimes_{F[P/U]} X) \\
&= FG \otimes_P (FP \otimes_{F[P/U]} \bigoplus_{n \in \mathbb{N}} F[P/U]) \\
&= FG \otimes_P \bigoplus_{n \in \mathbb{N}} (FP \otimes_{F[P/U]} F[P/U]) \\
&= \bigoplus_{n \in \mathbb{N}} (FG \otimes_P (FP \otimes_{F[P/U]} F[P/U])) \\
&= \bigoplus_{n \in \mathbb{N}} (\text{Ind}_P^G \text{Inf}_{P/U}^P F[P/U](F[P/U])) \\
&= \bigoplus_{n \in \mathbb{N}} R_{P/U}^G(F[P/U]).
\end{aligned}$$

Also we have

$$\text{Inf}_{P/U}^P F[P/U] = \text{Inf}_{P/U}^P \text{Ind}_{U/U}^{P/U} F = \text{Ind}_U^P \text{Inf}_{U/U}^U F = \text{Ind}_U^P F.$$

Therefore we obtain

$$R_{P/U}^G(F[P/U]) = \text{Ind}_P^G \text{Inf}_{P/U}^P F[P/U] = \text{Ind}_P^G \text{Ind}_U^P F.$$

Since  $|U|$  is invertible in  $F$ , the field  $F$  is a projective  $FU$ -module. Also, since induction preserves projectivity, we have  $R_{P/U}^G(F[P/U])$  projective, and hence, being the direct sum of projective modules,  $R_{P/U}^G(X)$  is projective.

Then, using projectivity of  $R_{P/U}^G(X)$  and surjectivity of  $\beta$ , we conclude that, the map  $(R_{P/U}^G(\beta))_*$  is surjective. Also  $\ker(R_{P/U}^G(\beta))_* = J_{R_{P/U}^G(\beta)}$  where

$$J_{R_{P/U}^G(\beta)} = \{\psi \in \text{End}_{FG}(R_{P/U}^G(X)) \mid \text{im} \psi \text{ is a submodule of } \ker R_{P/U}^G(\beta)\}.$$

Then, we have

$$\dim_F \text{Hom}_{FG}(R_{P/U}^G(X), R_{P/U}^G(M)) = \dim_F \text{End}_{FG}(R_{P/U}^G(X)) - \dim_F J_{R_{P/U}^G(\beta)}.$$

By Proposition 3.1, we have the isomorphism

$$\mathrm{End}_{FG}(R_{P/U}^G(M)) \cong (\mathrm{End}_{FG}(R_{P/U}^G(X)))_{R_{P/U}^G(\beta)} / J_{R_{P/U}^G(\beta)}.$$

Therefore, we obtain

$$\dim_F \mathrm{End}_{FG}(R_{P/U}^G(M)) = \dim_F (\mathrm{End}_{FG}(R_{P/U}^G(X)))_{R_{P/U}^G(\beta)} - \dim_F J_{R_{P/U}^G(\beta)}.$$

Then, these two equations imply that

$$\mathrm{End}_{FG}(R_{P/U}^G(X)) = \mathrm{End}_{FG}(R_{P/U}^G(X))_{R_{P/U}^G(\beta)}$$

if and only if

$$\dim_F \mathrm{Hom}_{FG}(R_{P/U}^G(X), R_{P/U}^G(M)) = \dim_F \mathrm{End}_{FG}(R_{P/U}^G(M)).$$

By Theorem 4.1, we have

$$T_{P/U}^G \circ R_{P/U}^G(M) = \bigoplus_{x \in P \backslash G/P} R_{\substack{(P^x \cap P)U \\ (U^x \cap P)U}}^{P/U} \circ T_{\substack{(P \cap P^x)U^x \\ (U \cap P^x)U^x}}^{P^x/U^x}(M^x).$$

Applying the functor  $\mathrm{Hom}_{FG}(M, -)$  to this equation, we obtain

$$\mathrm{Hom}_{FG}(M, T_{P/U}^G \circ R_{P/U}^G(M)) = \bigoplus_{x \in P \backslash G/P} \mathrm{Hom}_{F[P/U]}(M, R_{\substack{(P^x \cap P)U \\ (U^x \cap P)U}}^{P/U} \circ T_{\substack{(P \cap P^x)U^x \\ (U \cap P^x)U^x}}^{P^x/U^x}(M^x)).$$

Now, using adjointness of the functors  $R_{P/U}^G$  and  $T_{P/U}^G$  we get

$$\mathrm{Hom}_{FG}(R_{P/U}^G(M), R_{P/U}^G(M)) = \bigoplus_{x \in P \backslash G/P} \mathrm{Hom}_{F[P/U]}(T_{\substack{(P^x \cap P)U \\ (U^x \cap P)U}}^{P/U}(M), T_{\substack{(P \cap P^x)U^x \\ (U \cap P^x)U^x}}^{P^x/U^x}(M^x)).$$

Since  $M$  is cuspidal,  $T_{P/U}^G(M) = (0)$  for any proper subquotient of  $P/U$  in  $\mathbf{M}_{P/U}$ .

Then, we have

$$T_{\frac{(P^x \cap P)U}{(U^x \cap P)U}}^{P/U}(M) \neq (0)$$

if and only if  $\frac{(P^x \cap P)U}{(U^x \cap P)U}$  is equal to  $P/U$ , and

$$T_{\frac{(P \cap P^x)U^x}{(U \cap P^x)U^x}}^{P^x/U^x}(M^x) \neq (0)$$

if and only if  $\frac{(P \cap P^x)U^x}{(U \cap P^x)U^x}$  is equal to  $P^x/U^x$ . Therefore we have

$$\mathrm{Hom}_{FG}(R_{P/U}^G(M), R_{P/U}^G(M)) = \bigoplus_{x \in N_G(P,U) \cap (P \setminus G/P)} \mathrm{Hom}_{F[P/U]}(M, M^x)$$

where  $N_G(P,U) := \{x \in G \mid (P^x \cap P)U / (U^x \cap P)U = P/U\}$ .

Similarly, we have

$$\mathrm{Hom}_{FG}(R_{P/U}^G(X), R_{P/U}^G(M)) = \bigoplus_{x \in N_G(P,U) \cap (P \setminus G/P)} \mathrm{Hom}_{F[P/U]}(X, M^x)$$

Since  $M$  is irreducible,  $\dim_F \mathrm{Hom}_{FG}(M, M^x) = (0)$  unless  $M \cong M^x$  in which case that dimension equals to 1. Also since  $X$  is the minimal projective cover of  $M$ , similarly we have  $\dim_F \mathrm{Hom}_{FG}(X, M^x) = (0)$ , unless  $M \cong M^x$ , and it equals to 1 in that case. Therefore, we have

$$\dim_F \mathrm{Hom}_{FG}(M, M^x) = \dim_F \mathrm{Hom}_{FG}(X, M^x)$$

and hence

$$\mathrm{End}_{FG}(R_{P/U}^G(X)) = \mathrm{End}_{FG}(R_{P/U}^G(X))_{R_{P/U}^G(\beta)}$$

□

**Corollary 4.11.** *Let  $F$  be of characteristic  $p$  where  $p > 0$ . Let  $\mathbf{M}$  be a  $p$ -modular Mackey system for  $G$ . For  $P/U$  in  $\mathbf{M}$ , let  $M$  be a cuspidal  $F[P/U]$ -module, and  $\beta : X \rightarrow M$  be a minimal projective cover of  $M$ . The functors  $H^\beta$  and  $\hat{H}^\beta$  provide a bijection between the isomorphism classes of the irreducible  $FG$ -modules occurring in the head of  $R_{P/U}^G(M)$  and a set of representatives of the isomorphism classes of irreducible  $\text{End}_{FG}(R_{P/U}^G(M))$ -modules.*

## APPENDIX A: SUMMARY OF RESULTS

In this appendix, we restate some definitions and main theorems of the text to help the reader to understand the notation and terminology easily.

- $(\text{End}_T(P))_\beta = \{\phi \in \text{End}_T(P) \mid \phi(\ker\beta) \subseteq \ker\beta\}$   
 $J_\beta = \{\psi \in \text{End}_T(P) \mid \text{im}\psi \leq \ker\beta\}$
- $J_\beta$  is an ideal of  $(\text{End}_T(P))_\beta$  and  $(\text{End}_T(P))_\beta/J_\beta \cong \text{End}_T(M)$  as  $R$ -algebra canonically, (Proposition 3.1).
- Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . The mapping

$$H := H^\beta := H_M^\beta : \text{mod}_T \rightarrow \text{mod}_{\text{End}_T(M)}$$

defined for  $V \in \text{mod}_T$  by

$$H(V) = \text{Hom}_T(P, V)/\text{Hom}_T(P, V)J_\beta$$

is a covariant functor, (Proposition 3.2).

- Let  $S$  be a ring. For  $S$ -modules  $V_1$  and  $V_2$ , *trace of  $V_1$  in  $V_2$* ,  $\text{tr}_{V_1}(V_2)$ , is defined as the submodule of  $V_2$  spanned by images of all homomorphisms from  $V_1$  to  $V_2$ , (Definition 3.3).
- Let  $P$  and  $V$  be in  $\text{mod}_T$  and assume that  $P$  is projective. The  *$P$ -torsion submodule*  $\text{tor}_P(V)$  is the sum of all submodules  $X$  of  $V$  with respect to the property  $\text{Hom}_T(P, X) = (0)$ . The *kernel*  $\ker_P$  is the full subcategory of  $\text{mod}_T$  whose objects are the  $T$ -modules  $V$  with  $\text{Hom}_T(P, V) = (0)$ . Therefore, the  $T$ -module  $V$  is in  $\ker_P$  if and only if  $\text{tor}_P(V) = (0)$ , (Proposition 3.5).
- Define the functor

$$A_P : \text{mod}_T \rightarrow \text{mod}_T, V \mapsto V/\text{tor}_P(V)$$

for  $V$  in  $\text{mod}_T$  and define  $A_P(f)$  as the induced morphism from  $V/\text{tor}_P(V)$  to

$V'/\text{tor}_P(V')$  for any  $T$ -module homomorphism  $f : V \rightarrow V'$ , (Definition 3.9).

- Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . We define four functors from  $\text{mod}_{\text{End}_T(M)}$  to  $\text{mod}_T$  as

$$\begin{aligned} F_M &= - \otimes_{\text{End}_T(M)} M \\ \tilde{F}_M &= A_P \circ (- \otimes_{\text{End}_T(M)} M) \\ G_M &= - \otimes_{\text{End}_T(P)} P \\ \tilde{G}_M &= A_P \circ (- \otimes_{\text{End}_T(P)} P) \end{aligned}$$

Let  $\hat{H}$  be one of the four functors defined above. Then  $\hat{H}$  is a right inverse of the functor  $H$ , (Definition 3.10 and Proposition 3.12).

- Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Define the set

$$(\text{Irr}T)_H = \{V \in \text{Irr}T \mid H_M(V) \neq (0)\}.$$

Then  $H_M$  induces a bijective correspondence

$$H_M : (\text{Irr}T)_H \rightarrow \text{Irr}(\text{End}_T(M))$$

and the inverse of  $H_M$  is

$$\tilde{F}_M : \text{Irr}(\text{End}_T(M)) \rightarrow (\text{Irr}T)_H.$$

On  $\text{Irr}(\text{End}_T(M))$ , the functors  $\tilde{F}_M$  and  $\tilde{G}_M$  coincide, (Theorem 3.16).

- Let  $R$  be a field. Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ . Then,  $(\text{Irr}T)_H$  is a complete set of non-isomorphic irreducible constituents of head of  $M$   $hd(M)$ . Every indecomposable direct summand of  $M$  has a simple head and factoring out the Jacobson radical induces a bijection between the isomorphism classes of indecomposable direct summands of  $M$  and the elements of  $(\text{Irr}T)_H$ , (Theorem 3.22).
- Let  $R$  be a field. Assume  $(\text{End}_T(P))_\beta = \text{End}_T(P)$ , the  $T$ -module  $M$  is  $P$ -

torsionless, and  $\text{End}_T(M)$  is self-injective. Then

- (i) Every element of  $\text{Irr}(\text{End}_T(M))$  is isomorphic to  $XM$  for some minimal ideal  $X$  of  $\text{End}_T(M)$ .
- (ii) The set  $(\text{Irr}T)_H$  is up to isomorphism a complete set of irreducible constituents of  $\text{soc}(M)$  as well as  $hd(M)$ .
- (iii) Every indecomposable direct summand of  $M$  has a simple socle and a simple head, and taking socles, respectively heads, induce bijections between the isomorphism classes of indecomposable direct summands of  $M$  and the elements of  $(\text{Irr}T)_H$ .
- (iv) Socle and heads of the indecomposable direct summands of  $M$  are isomorphic if in addition  $\text{End}_T(M)$  is a symmetric algebra.

(Theorem 3.23).

- Let  $P, Q, U, V$  be subgroups of  $G$  with  $U$  normal in  $P$  and  $V$  normal in  $Q$ . Suppose that the orders of  $U$  and  $V$  are invertible in  $F$ . Let  $M$  be an  $F(P/U)$ -module. Then

$$T_{Q/V}^G \circ R_{P/U}^G(M) \cong \bigoplus_{x \in P \backslash G / Q} R_{\frac{(P^x \cap Q)V}{(U^x \cap Q)V}}^{Q/V} C_{\frac{(Q \cap P^x)U^x}{(V \cap P^x)U^x}, \frac{(P^x \cap Q)V}{(U^x \cap Q)V}}^\phi T_{\frac{(Q \cap P^x)U^x}{(V \cap P^x)U^x}}^{P^x/U^x}(M^x)$$

where

$$C_{\frac{(Q \cap P^x)U^x}{(V \cap P^x)U^x}, \frac{(P^x \cap Q)V}{(U^x \cap Q)V}}^\phi : \frac{(Q \cap P^x)U^x}{(V \cap P^x)U^x} \rightarrow \frac{(P^x \cap Q)V}{(U^x \cap Q)V}$$

is an isomorphism, and  $M^x$  denotes the conjugate module for the conjugate factor group  $x(P/U)x^{-1}$ , and  $P \backslash G / Q$  is a set of  $P - Q$ -double coset of representatives in  $G$ , (Theorem 4.1).

- A system  $\mathbf{M}$  of subquotients of  $G$  is called a *Mackey system*, if it contains  $G$ , is closed under conjugation and the operation

$$P/U \sqcap Q/V = (P \cap Q)U / (P \cap V)U, \text{ for } P/U \text{ and } Q/V \text{ in } \mathbf{M}.$$

- For  $P/U$  in  $\mathbf{M}$ , the set

$$\mathbf{M}_{P/U} = \{P/U \sqcap Q/V \mid Q/V \in \mathbf{M}\}$$

defines a Mackey system in  $P/U$ .

- Let  $F$  be of characteristic  $p$  where  $p > 0$ . Let  $\mathbf{M}$  be a  $p$ -modular Mackey system for  $G$ . For  $P/U$  in  $\mathbf{M}$ , let  $M$  be a cuspidal  $F[P/U]$ -module, and  $\beta : X \rightarrow M$  be a minimal projective cover of  $M$ . Then we have

$$\text{End}_{FG}(R_{P/U}^G(X)) = (\text{End}_{FG}(R_{P/U}^G(X)))_{R_{P/U}^G(\beta)}$$

where  $R_{P/U}^G(\beta)$  is the map

$$R_{P/U}^G(\beta) : R_{P/U}^G(X) \rightarrow R_{P/U}^G(M)$$

induced from the map  $\beta : X \rightarrow M$ , (Theorem 4.10).

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