AN ATTRACTIVE ϕ^4 THEORY IN LIGHT-FRONT COORDINATES

by

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To my family

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ABSTRACT

AN ATTRACTIVE ϕ^4 THEORY IN LIGHT-FRONT COORDINATES

We study an attractive ϕ^4 interaction using Tamm-Dancoff truncation with light-front coordinates in $3+1$ dimensions. The truncated theory requires a coupling constant renormalization, we compute its β function non-perturbatively, show that the model is asymptotically free, and find the corresponding Callan-Symanzik equations. The model supports bound states, we find the wave function of the ground state of the two-particle sector. We also give a bound for the N-particle ground state energy within a mean field approximation, including the corresponding result for the case of $2+1$ dimensions where the model does not require renormalization.

ÖZET

IŞIK CEPHESİ KOORDİNATLARINDA ÇEKİCİ BİR ϕ^4 KURAMI

Çekici bir ϕ^4 etkileşimi Tamm-Dancoff kesimiyle ışık cephesi koordinatlarında $3+1$ boyutta çalışılmıştır. Kesilmiş kuramın etkileşim katsayısı renormalizasyonu gerektirdiği görülmüş, β fonksiyonu pertürbasyonsuz hesaplanmış, modelin sonuşmazca serbest olduğu gösterilmiş ve buna karşılık gelen Callan-Symanzik denklemleri hesaplanmıştır. Model bağlı haller içermektedir, iki parçacık sektörünün temel durumunun dalga denklemi bulunmuştur. Ayrıca N parçacık temel durumunun enerjisi için ortalama alan yaklaşıklığında bir sınır hesaplanmış ve bu sonucun renormalizasyon gerektirmeyen $2+1$ boyuttaki karşılığı da verilmiştir.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF SYMBOLS

$a(\cdot)$	Particle annihilation operator
E_{gr}	Ground state energy of the N-particle sector
$\mathcal{F}_{\mathcal{B}}$	Fock space of bosons
g	Metric tensor
H	Hamiltonian
H_I	Interaction part of the Hamiltonian
H_R	Renormalized Hamiltonian
H_1	Truncated interaction part of the Hamiltonian
$\mathcal{J}$	Conserved current density
$J_0(\cdot)$	Bessel function of the first kind of order zero
$K_0(\cdot)$	Modified Bessel function of the second kind of order zero
$\mathcal{L}$	Lagrangian density
L^2	Square integrable functions
m	Mass of particles
N	Number of particles
p	Four-momentum of a particle
$\mathbf{p}$	Three-momentum of a particle
$\mathbf{p}^{\perp}$	Transverse components of the momentum of a particle
P	Total four-momentum of particles
$[\mathrm{d}p]$	Volume element of the momentum space
$R(\cdot)$	Resolvent
$R_R(\cdot)$	Renormalized resolvent
$\mathcal{S}$	Action
$w(\cdot)$	Eigenvalues of the renormalized principal operator
x	Four-position of a particle
$\mathbf{x}$	Three-position of a particle
$\mathbf{x}^{\perp}$	Transverse components of the position of a particle

$\beta(\cdot)$	Beta function
$\delta(\cdot)$	Dirac delta function
θ	Rapidity of boosts
$\theta(\cdot)$	Step function
λ	Coupling constant
μ	Ground state energy of the the two-particle sector
Π_0	Projector onto the no-orthofermion subspace
Π_1	Projector onto the one-orthofermion subspace
φ	Angle of rotations
$\Phi(\cdot)$	Principal operator
$\Phi_R(\cdot)$	Renormalized principal operator
$\chi(\cdot)$	Orthofermion annihilation operator
$\Psi(\cdot,\cdot)$	Ground state wave function of the two-particle sector
$\vert\Omega_0\rangle$	Variational ansatz for the principal operator for many particles

LIST OF ACRONYMS/ABBREVIATIONS

H. C.	Hermitian conjugate

1. INTRODUCTION

Dirac, in his paper [1], was the first to consider studying systems of particles by specifying their dynamical variables in surfaces of spacetime other than the instant

$$x^0 = 0\,, \tag{1.1}$$

where x^0 is the time coordinate. He called the dynamics associated with this usual choice the *instant form.* Motivated by Einstein's special principle of relativity [2] together with the need to express the equations of motion in the Hamiltonian form for quantum mechanics, he sought other forms of relativistic dynamics, thinking that other Hamiltonians may express atomic phenomena better.

Besides the condition that the world line of particles intersect this surface once and only once, choices for such surfaces are otherwise unrestricted. However, in order to keep the expressions for the dynamical variables corresponding to the elements of the Poincaré group as simple as possible, Dirac only considered surfaces that are left invariant by some subgroups of the Poincaré group, which are called the *stability groups* of the surface [3]. He then found two more forms: The point form

$$x \cdot x = a^2,\; x^0 > 0\,, \tag{1.2}$$

and the front form

$$x^0 - x^3 = 0\,, \tag{1.3}$$

where a is a constant and x is the four-position. For completeness, we note the two other forms found after a detailed study of the subgroups of the Poincaré group [4],

$$\begin{aligned} &x \cdot x + (x^3)^2 = a^2,\ x^0 > 0\,, \\ &x \cdot x + \mathbf{x}^{\perp} \boldsymbol{\cdot} \mathbf{x}^{\perp} = a^2,\ x^0 > 0\,. \end{aligned} \tag{1.4}$$

Subject to this work is the front form, which in its conventional form has the particularly simple dispersion relation

$$p_+ = \frac{m^2 + p_\perp^2}{4p_-}\,, \tag{1.5}$$

where p_+ and p_- are the momentum coordinates corresponding to the $x^+ := x^0 + x^3$ and $x^- := x^0 - x^3$ position coordinates respectively, $p_\perp$ is the norm of the momentum in the remaining directions and m is the mass of particles. In this work, we deviate from this convention by taking only x^- and keeping x^3, which results in the dispersion relation

$$p_- = \frac{m^2 + p_\perp^2 + p_3^2}{2p_3}\,, \tag{1.6}$$

which is similar to the conventional one in the lack of square roots, but different from it in that the term in the denominator, p_3, also appears in the numerator. This results in a difference in the location of the minima of the two dispersion relations, where, setting $p_\perp = 0$, ours has the minimum at the finite momentum of $p_3 = m$, which also gives the energy $p_- = m$, whereas the conventional form has the minimum at $p_- \to \infty$. This fact about our version adds to the advantages of the conventional one, making it another considerable tool for studying bound states and using variational arguments. We note that the longitudinal momentum in both versions of the front form is strictly non-negative, which results in the *triviality* of the front form vacuum. This is detailed in the next chapter. The main results of this work stand in part on these properties and those of another tool to be introduced.

In this setting of the front form, we study a problem of relativistic, attractive contact interactions for spinless, massive particles, with the effective Hamiltonian

$$H = \int [\mathrm{d}p] \frac{m^2 + p_\perp^2 + p_3^2}{2p_3} a^\dagger a(\mathbf{p}) - \lambda \int (\prod_{i=1}^{4} \frac{[\mathrm{d}p_i]}{\sqrt{2p_{i3}}}) a_1^\dagger a_2^\dagger a_3 a_4 \delta(\mathbf{p_1} + \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4}) , \quad (1.7)$$

where $\lambda > 0$ is the coupling constant of the interaction and $a^\dagger$ and a are the creation and annihilation operators. We are led to this effective Hamiltonian from a truncation of another interaction,

$$-\lambda \frac{(2\pi)^3}{6} \int \mathrm{d}x^3 \, \mathrm{d}^2 x^\perp \, \phi^4(x) \, . \quad (1.8)$$

Here, $\phi(x)$ is the field constructed out of the creation and annihilation operators in the standard, Lorentz invariant manner, and it is causal in our coordinates as well.

The said truncation is done with particle number conservation in mind, by dropping all terms in the normal ordering of ϕ^4 with unequal number of creation and annihilation operators. This is in line with the Tamm-Dancoff truncation [5,6], which Weinberg [7] notes to be more meaningful when done in the *infinite momentum frame*, where the observer is taken to the limit of infinite momentum in some direction. The correspondence of this frame to the front form is detailed in [8]. We are therefore motivated to study sectors of fixed particle numbers and bound states therein in this setting. We note that the non-relativistic case of the same problem was studied in [9–11].

The analysis of the resulting Hamiltonian proceeds after the construction of another tool that exchanges the search of the poles of the resolvent of the Hamiltonian with the search of the zeros of what is called the *principal operator*, $\Phi(E)$, introduced by Rajeev [9]. It works by splitting into two multiplicative terms the interaction

$$-\lambda \int (\prod_{i=1}^{4} \frac{[\mathrm{d}p_i]}{\sqrt{2p_{i3}}}) a_1^\dagger a_2^\dagger a_3 a_4 \delta(\mathbf{p_1} + \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4}) \, , \quad (1.9)$$

which is symmetric in its integrand in the indices up to a Hermitian conjugate. This splitting is made possible by the introduction of the *orthofermion* χ, which reproduces the Dirac delta in the interaction with its property

$$\chi(\mathbf{p})\chi^\dagger(\mathbf{q}) = \delta(\mathbf{p}-\mathbf{q})\Pi_0\,, \tag{1.10}$$

where again the symmetry in the indices up to a Hermitian conjugate makes the the said splitting possible. This procedure does more than change where to look for poles or zeros, it also splits the coupling constant λ to an isolated, additive term. This allows for the possibility to renormalize for the divergences in the theory by an *exact* redefinition of the coupling constant. Together with the flow property of the eigenvalues of the principal operator, which guarantees locating its zeros by adjusting E and the properties of our version of the front form, these are what this work depends on critically.

It is outside the scope of this work to give a full treatment of the various *forms of relativistic dynamics*. We point out for the interested reader the review by Heinzl [3] that treats the problem of choosing a surface in spacetime to parametrize the world lines of relativistic particles as a problem of gauge fixing for the reparametrization invariance of the action. The subsequent choice of a surface is then called a *foliation* of spacetime into space and time. In what follows, we drop the use of the word “form” and use the more commonly known “Minkowski coordinates” in the place of the instant form, and “light-front” in the place of the front form.

In this work, the light-front and the quantization therein is studied in the continuum. Thanks to the specific form of the truncated interaction, we are able to give non-perturbative results for the two-particle sector and variational estimates for the case of many particles. However, there is also a discretized formulation of the light-front theory that opens the way for numerical studies, which makes the problem of bound state analysis a problem of diagonalizing the Hamiltonian, in this discrete case a finite, but large matrix, and possibly much more complicated in form than the one in this work. We point to [12] and once again [3] for reviews on this subject. This method has seen success especially in $1+1$ dimensional problems [13–15].

The rest of this thesis is organized as follows. In Section 2.1, we review the Minkowski coordinates, to which we compare the advantages of both the conventional light-front coordinates, reviewed in Section 2.2, and our version, constructed in Section 2.3. Afterwards, in our version of light-front coordinates, we give the free field Lagrangian density in Section 3.1 and prescribe the quantization of the field in Section 3.2. We then study the question of whether our version of coordinates has a proper realization of the Poincaré group in terms of single particle operators in Section 3.3. We specify the interaction in Section 3.4. We review the *principal operator* and the *orthofermion* that are central to this work in Section 4.1, and construct the principal operator specific to our problem in Section 4.2. We find that this operator diverges and renormalize it in Section 4.3. We prove the flow of eigenvalues of the principal operator, which is critical for the bound state analysis and variational calculations in this work, in Section 4.4. We demonstrate certain scaling properties of this operator in Section 4.5. We find the wave function of the ground state of the two-particle sector in Chapter 5. For the case of a large number of particles we provide a lower bound for the ground state energy in Chapter 6 and include the case for $2 + 1$ dimensions in Section 6.1. We then conclude in Chapter 7.

This thesis is based on a joint work of O. T. Turgut and the author, under the same title, yet unpublished at the time of the completion of this thesis.

2. COORDINATES

We start with a review of the Minkowski coordinates. We refer to the results contained in this review during the construction of the conventional light-front coordinates and the version that we use. Note that throughout this work we use the notation

$$a \cdot b := a_\mu b^\mu \tag{2.1}$$

to denote the contraction of any two four-vectors a and b. For momentum, when multiple particles are involved, we use capital letters, such as P, to refer to their total momentum.

2.1. Minkowski coordinates

We give the components of the four-position with an upper index,

$$x^\mu = (x^0, x^1, x^2, x^3)\,, \tag{2.2}$$

where x^0 is time. We refer to the spatial components with

$$\mathbf{x} := (x^1, x^2, x^3)\,, \tag{2.3}$$

and call $\mathbf{x}$ the three-position. The four-position is equipped with the Lorentz invariant norm

$$\begin{aligned} x \cdot x &= g_{\mu\nu} x^\mu x^\nu \\ &= (x^0)^2 - \mathbf{x} \cdot \mathbf{x}\,. \end{aligned} \tag{2.4}$$

Here, g is the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.5}$$

Dual to the four-position is the four-momentum. We give its components with a lower index,

$$p_\mu = (p_0, p_1, p_2, p_3) . \tag{2.6}$$

We refer to the spatial components with

$$\mathbf{p} \coloneqq (p_1, p_2, p_3) , \tag{2.7}$$

and call $\mathbf{p}$ the three-momentum. As the Minkowski metric, Equation (2.5), is its own inverse, the norm of the four-momentum,

$$\begin{aligned} p \cdot p &= g^{\mu\nu} p_\mu p_\nu \\ &= (p_0)^2 - \mathbf{p} \cdot \mathbf{p} , \end{aligned} \tag{2.8}$$

is not different in form from that of the four-position, Equation (2.2). This norm is constrained for a particle with mass m that is *on-shell* with

$$m^2 = p \cdot p . \tag{2.9}$$

Alternatively, such a particle has its energy fixed by m and $\mathbf{p}$ with

$$p_0 = \sqrt{m^2 + \mathbf{p} \cdot \mathbf{p}} . \tag{2.10}$$

Note that we took the square root with the positive sign. This sign is preserved under *proper* Lorentz transformations.

We note the Lorentz transformations for completeness. We define the rapidity parameter θ,

$$v =: \tanh\theta \tag{2.11}$$

and for simplicity, assume a boost in the x^3 direction with speed v, which gives the transformation of the four-position as

$$\begin{aligned} x^0 &\mapsto \cosh\theta x^0 + \sinh\theta x^3\,, \\ x^3 &\mapsto \sinh\theta x^0 + \cosh\theta x^3\,, \\ x^1 &\mapsto x^1\,, \\ x^2 &\mapsto x^2\,. \end{aligned} \tag{2.12}$$

The transformations of the four-momentum follow from these by raising and lowering indices with the metric, Equation (2.5). Boosts in coordinate directions other than x^3 follow simply by relabelling these. Again, as the Minkowski metric is its own inverse, the four-momentum with lower indices transform the same as the four-position with upper indices.

The construction of fields from creation and annihilation operators with momentum labels requires a Lorentz invariant measure over the momentum space. Such a measure is

$$\mathrm{d}^4p\,\delta(p\cdot p - m^2)\theta(p_0)\,. \tag{2.13}$$

The overall Lorentz invariance of this follows from the invariance of d^4p and $p\cdot p$ together with fixing the sign of p_0 with Equation (2.10). The p_0 part of this measure can be integrated out by noting that

$$\delta(p \cdot p - m^2)\theta(p_0) = \frac{\delta(p_0 - \sqrt{m^2 + \mathbf{p} \cdot \mathbf{p}})}{2\sqrt{m^2 + \mathbf{p} \cdot \mathbf{p}}} . \tag{2.14}$$

Note that $\theta(p_0)$ gets rid of the solution with the negative square root. This gives a Lorentz invariant measure in the three-momentum space as

$$\frac{[\mathrm{d}p]}{2\sqrt{m^2 + \mathbf{p} \cdot \mathbf{p}}} , \tag{2.15}$$

where $[\mathrm{d}p]$ is a shorthand for $\mathrm{d}^3 p$, the volume element over the three-momentum.

2.2. Conventional light-front coordinates

The conventional light-front coordinates, as in the reviews of Harindranath [16] and Heinzl [3], result from the change of coordinates

$$(x^0, x^1, x^2, x^3) \mapsto (x^+, x^1, x^2, x^-) , \tag{2.16}$$

where the new coordinates x^+ and x^- are defined with

$$x^+ := x^0 + x^3 , \tag{2.17}$$

$$x^- := x^0 - x^3 . \tag{2.18}$$

Preserving the Lorentz invariant norm $x \cdot x$ requires that

$$x \cdot x = x^+ x^- - \mathbf{x}^\perp \cdot \mathbf{x}^\perp , \tag{2.19}$$

where we refer to the transverse parts of the three-position with

$$\mathbf{x}^\perp := (x^1, x^2) . \tag{2.20}$$

This inner product is accomplished with the metric

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} , \tag{2.21}$$

which has the inverse

$$g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} . \tag{2.22}$$

Therefore, on-shell four-momenta satisfy

$$p_+ = \frac{m^2 + p_\perp^2}{4p_-} , \tag{2.23}$$

where we refer to the transverse parts of the three-momentum with

$$\mathbf{p}_\perp \coloneqq (p_1, p_2) , \tag{2.24}$$

and its magnitude with

$$p_\perp^2 \coloneqq \mathbf{p}_\perp \cdot \mathbf{p}_\perp . \tag{2.25}$$

A Lorentz invariant measure in this case is

$$\frac{[\mathrm{d}p]}{4p_-} . \tag{2.26}$$

An immediate observation for the dispersion relation, Equation (2.23), is that it does not contain square roots. When studying many particles, this allows to change to the total and relative momenta coordinates, and do so without any Feynman parametrizations, which makes analysis of the sort in this work possible. The selection of the positive energy region is also coupled with choosing a kinematic region for the longitudinal momentum. That is, for negative p_-, the light-front energy, Equation (2.23) is negative and for positive p_- it is positive. This is in contrast with the Minkowski coordinates where the sign of energy, Equation (2.10), is set independent of momentum. Once energy is fixed to be positive for the light-front, it leads to the important consequence that the light-from vacuum is the unique state with zero longitudinal momentum,

$$P_- \left|0\right\rangle = 0\,, \tag{2.27}$$

as the addition of states with nonzero p_- while keeping the total longitudinal momentum P_- zero is not possible. For this reason, the light-front vacuum is called *trivial* [3]. However, in the case of the Minkowski coordinates, when interactions are present, states such as

$$a^\dagger(\mathbf{p})a^\dagger(-\mathbf{p}) \left|0\right\rangle \tag{2.28}$$

are possible, and they also have zero total momentum. Such states are said to *mix* with the free vacuum [12]. The corresponding vacuum is called *non-trivial*.

These facts make the light-front a considerable tool for the study of bound state problems. There is, however, the so called *zero modes* problem with the light-front, associated to the point $p_- = 0$. States with $P_- = 0$ can mix with the light-front vacuum, if such states can indeed be realized. This is considered to be one of the unsolved problems of the light-front [3]. Observing that the light-front energy, Equation (2.23), goes to infinity at $p_- = 0$, Burkardt [12] suggests the possibility that these "high energy" modes can be treated systematically to produce an effective light-front field theory and presents some examples. This issue is outside the scope of this work, here we consider

p_- to be a singularity to be avoided, and show that doing so is possible in a Lorentz invariant manner.

In order to avoid the said singularity, it is necessary to choose a region for p_- that does not contain this point. By convention we choose

$$p_- > 0\,. \tag{2.29}$$

This also fixes the sign of the energy-like coordinate p_+ in a Lorentz invariant manner, resulting in

$$p_+ > 0\,. \tag{2.30}$$

That these conditions are preserved under Lorentz transformations requires proof, we now show how they transform.

We study two cases, one where the boost is in the x^3 direction, and one where the boost is in a transverse direction. In all cases, the transformations are derived from Equation (2.12) with relabellings and raising and lowering of the indices with the metric, Equations (2.21) and (2.22).

In the case where the boost is in the x^3 direction, we get

$$p_- \mapsto e^{\theta} p_-\,, \tag{2.31}$$

$$p_+ \mapsto e^{-\theta} p_+\,. \tag{2.32}$$

Both p_- and p_+ preserve their signs. Furthermore, they do not mix, but only get scaled and do so with reciprocal scales.

The case where the boost is in a transverse direction is more complicated. Calling the magnitude of momentum in that direction $p_\perp$, we get

$$p_- \mapsto \sinh^2(\frac{\theta}{2})p_+ + \cosh^2(\frac{\theta}{2})p_- - \sinh\left(\frac{\theta}{2}\right)\cosh\left(\frac{\theta}{2}\right)p_\perp\,, \tag{2.33}$$

$$p_+ \mapsto \sinh^2(\frac{\theta}{2})p_- + \cosh^2(\frac{\theta}{2})p_+ - \sinh\left(\frac{\theta}{2}\right)\cosh\left(\frac{\theta}{2}\right)p_\perp\,. \tag{2.34}$$

This time p_- and p_+ mix, and their transformations contain terms with indefinite signs. However, what matters is the transformation of on-shell momenta, therefore we substitute p_+ with Equation (2.23), which leads to

$$p_- \mapsto \frac{1}{4p_-}\left(\left[\cosh\left(\frac{\theta}{2}\right)2p_- - \sinh\left(\frac{\theta}{2}\right)p_\perp\right]^2 + m^2\sinh^2(\frac{\theta}{2})\right)\,, \tag{2.35}$$

$$p_+ \mapsto \frac{1}{4p_-}\left(\left[\cosh\left(\frac{\theta}{2}\right)p_\perp - \sinh\left(\frac{\theta}{2}\right)2p_-\right]^2 + m^2\cosh^2(\frac{\theta}{2})\right)\,. \tag{2.36}$$

These are manifestly positive, therefore p_- and p_+ of on-shell momenta indeed preserve their sign.

2.3. An alternative light-front coordinate system

In the rest of this work, we use a different coordinate system where in the place of x^0 we use x^-, Equation (2.18), but keep x^3. That is,

$$(x^0, x^1, x^2, x^3) \mapsto (x^-, x^1, x^2, x^3)\,. \tag{2.37}$$

Preserving $x \cdot x$ requires that

$$x \cdot x = (x^-)^2 + 2x^-x^3 - \mathbf{x}^\perp \cdot \mathbf{x}^\perp\,. \tag{2.38}$$

This is accomplished with the metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.39}$$

which has the inverse

$$g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{2.40}$$

Therefore, on-shell four-momenta satisfy

$$p_- = \frac{m^2 + p_\perp^2 + p_3^2}{2p_3}. \tag{2.41}$$

A Lorentz invariant measure in this case is

$$\frac{[\mathrm{d}p]}{2p_3}. \tag{2.42}$$

The dispersion relation of our version of light-front coordinates, Equation (2.41), and the associated vacuum have the same advantages as the conventional light-front coordinates over the Minkowski case, listed in the corresponding parts in Section 2.2. Our version, while being a non-orthogonal coordinate system, has more advantages resulting from the appearance of p_3 in both the numerator and denominator of the dispersion relation.

We compare the two versions, Equations (2.23) and (2.41), at zero transverse momentum, $p_\perp = 0$. The conventional light-front energy, Equation (2.23), has its minimum at $p_- \to \infty$. The minimum of energy is not at a finite momentum in this case. Whereas in the version that we use, Equation (2.41), the minimum is at $p_3 = m$, with energy also taking the value $p_- = m$ there. In this case, the minimum of energy is at a finite momentum, and its value coincides with mass. This is an advantage of our version of light-front coordinates, it allows to find the ground the state wave function of the two-particle sector, and facilitates the variational calculations for the case of many particles.

In order to avoid the singularity of Equation (2.41) at $p_3 = 0$, we choose the region

$$p_3 > 0\,, \tag{2.43}$$

which also fixes the sign of the energy-like coordinate p_-,

$$p_- > 0\,. \tag{2.44}$$

Again, in order to prove that these conditions continue to hold under Lorentz transformations, we show how they transform.

In the case where the boost is in the x^3 direction, we get

$$p_3 \mapsto e^{-\theta} p_3\,, \tag{2.45}$$

$$p_- \mapsto e^{\theta} p_- - \sinh\theta p_3\,. \tag{2.46}$$

Here p_3 only gets scaled, keeping its sign, however p_- mixes with p_3 and its transformation contains a term with an indefinite sign. Again, we look at how it transforms on-shell by substituting p_-, Equation (2.41), which gives

$$p_- \mapsto e^{\theta}\left(\frac{m^2 + p_\perp^2}{2p_3}\right) + e^{-\theta}\frac{p_3}{2}\,. \tag{2.47}$$

This shows that for on-shell momenta, p_- does preserve its sign.

In the case where the boost is in a transverse direction, we get

$$p_3 \mapsto p_3 + (\cosh\theta - 1)p_- - \sinh\theta p_\perp\,, \tag{2.48}$$

$$p_- \mapsto \cosh\theta p_- - \sinh\theta p_\perp\,. \tag{2.49}$$

This time p_3 and p_- mix, and their transformations contain terms with indefinite signs. Again, looking at how they transform on-shell, we get

$$p_3 \mapsto \frac{1}{p_3}\left(\left[\cosh\left(\frac{\theta}{2}\right)p_3 - \sinh\left(\frac{\theta}{2}\right)p_\perp\right]^2 + m^2\sinh^2(\frac{\theta}{2})\right)\,, \tag{2.50}$$

$$p_- \mapsto \frac{1}{2p_3}\left(\frac{e^\theta}{2}(p_3 - p_\perp)^2 + \frac{e^{-\theta}}{2}(p_3 + p_\perp)^2 + m^2\cosh\theta\right)\,. \tag{2.51}$$

These are manifestly positive, therefore p_- and p_3 of on-shell momenta indeed preserve their sign.

3. FORMULATION OF THE PROBLEM

We now formulate the interaction in the coordinate system we derived. To do so, we present the Lagrangian formulation of the free field, quantize it, and add the interaction in the Hamiltonian formalism. We then address the generators of the Poincaré group corresponding to our choice of coordinates.

3.1. Lagrangian formulation of the free field

The scalar, spinless, massive free field $\phi(x)$ has the coordinate-free Lagrangian density

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \,. \tag{3.1}$$

In our coordinates, using the metric, Equation (2.40), it has the explicit form

$$\mathcal{L} = \partial_3 \phi \partial_- \phi - \sum_{i=1}^{3} \frac{1}{2} \partial_i \phi \partial_i \phi - \frac{1}{2} m^2 \phi^2 \,. \tag{3.2}$$

Under the action $\mathcal{S}$, by assuming boundary conditions on the field ϕ such that the difference of $\phi \partial_\mu \phi$ on the coordinate boundaries vanishes, this can be brought into the form

$$\begin{aligned} \mathcal{S} &= \int \mathrm{d}^4 x \, \mathcal{L} \\ &= \int \mathrm{d}^4 x \left(\phi(-\partial_3) \partial_- \phi - \frac{1}{2} \phi \left(-\partial_1^2 - \partial_2^2 - \partial_3^2 + m^2 \right) \phi \right) . \end{aligned} \tag{3.3}$$

The conserved four-current density $\mathcal{J}$ of this Lagrangian in coordinate-free form is

$$\mathcal{J}_\mu = g_{\mu\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \delta\phi \,, \tag{3.4}$$

where $\delta\phi$ is a variation of the field. In our coordinates, the corresponding conserved charge density is

$$J_- = \partial_-\phi\delta\phi\,. \tag{3.5}$$

3.2. Quantization of the field

We now construct the field $\phi(x)$ from creation and annihilation operators $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$ with

$$\phi^{(+)}(x) = \int \frac{[\mathrm{d}p]a(\mathbf{p})e^{-ipx}}{(2\pi)^{3/2}\sqrt{2p_3}}\,, \tag{3.6}$$

$$\phi^{(-)} = \left(\phi^{(+)}\right)^\dagger\,, \tag{3.7}$$

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)\,, \tag{3.8}$$

where

$$\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] = \delta(\mathbf{p}-\mathbf{q})\,. \tag{3.9}$$

We check the causal structure of this construction. Note the following commutator:

$$\left[\phi^{(+)}(x), \phi^{(-)}(y)\right] = \frac{1}{(2\pi)^3}\int \frac{[\mathrm{d}p]}{2p_3}e^{-ip(x-y)} =: \Delta(x-y)\,. \tag{3.10}$$

This is Lorentz invariant. We evaluate it at equal light-front time, $x^- = y^-$. With $b \coloneqq x-y$ we have $(b)^2 = -(\mathbf{b}^\perp)^2$. This gives

$$\begin{aligned}\Delta(b^- = 0, \mathbf{b}) &= \frac{1}{(2\pi)^3}\int_0^\infty \frac{\mathrm{d}p_3}{2p_3}e^{-ip_3b^3}\int \mathrm{d}^2p_\perp\, e^{-i\mathbf{p}_\perp\cdot\mathbf{b}^\perp} \\ &= \delta(\mathbf{b}^\perp)\frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}p_3}{2p_3}e^{-ip_3b^3}\,.\end{aligned} \tag{3.11}$$

The field commutator is

$$[\phi(x), \phi(y)] = \left[\phi^{(+)}(x), \phi^{(-)}(y)\right] - \left[\phi^{(+)}(y), \phi^{(-)}(x)\right] . \tag{3.12}$$

We get

$$\begin{aligned} \left[\phi(x^-, \mathbf{x}), \phi(x^-, \mathbf{y})\right] &= \delta(\mathbf{x}^\perp - \mathbf{y}^\perp)\frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}p_3}{2p_3}(-2i)\sin[p_3(x^3 - y^3)] \\ &= -\frac{i}{4}\,\mathrm{sign}(x^3 - y^3)\delta(\mathbf{x}^\perp - \mathbf{y}^\perp) . \end{aligned} \tag{3.13}$$

This vanishes for $\mathbf{x}^\perp \neq \mathbf{y}^\perp$. Therefore the field commutes with itself at space-like intervals, it is causal.

3.3. Fock space realizations of the Poincaré generators

We identify the Fock space realizations of the generators of the Poincaré group as the conserved charges, Equation (3.5), that result from the variations of the field $\phi(\mathbf{x})$ due to the variations of $\mathbf{x}$ by small Poincaré group actions. In what follows, we use the notation

$$\begin{aligned} \delta x^\mu[G] &\coloneqq (Gx)^\mu - x^\mu , \\ \delta\phi[G] &\coloneqq \phi(Gx) - \phi(x) , \end{aligned} \tag{3.14}$$

to refer to the variations of the four-position and the field respectively by an element G of the Poincaré group.

The actions of translations P are trivial. They result in the variations of the field as

$$\begin{aligned}
\delta\phi[P_-] &= \partial_-\phi\,,\\
\delta\phi[P_1] &= \partial_1\phi\,,\\
\delta\phi[P_2] &= \partial_2\phi\,,\\
\delta\phi[P_3] &= \partial_3\phi\,.
\end{aligned} \tag{3.15}$$

Rotations J with the small angle φ vary x in our coordinate system as

$$\delta x^\mu[J_3] = \begin{pmatrix} 0 \\ \varphi x^2 \\ -\varphi x^1 \\ 0 \end{pmatrix}, \quad \delta x^\mu[J_1] = \begin{pmatrix} \varphi x^2 \\ 0 \\ \varphi x^3 \\ -\varphi x^2 \end{pmatrix}, \quad \delta x^\mu[J_2] = \begin{pmatrix} -\varphi x^1 \\ -\varphi x^3 \\ 0 \\ \varphi x^1 \end{pmatrix}. \tag{3.16}$$

These result in the variations

$$\begin{aligned}
\delta\phi[J_3] &= (x^2\partial_1 - x^1\partial_2)\phi\,,\\
\delta\phi[J_1] &= (x^2\partial_- + x^3\partial_2 - x^2\partial_3)\phi\,,\\
\delta\phi[J_2] &= (-x^1\partial_- - x^3\partial_1 + x^1\partial_3)\phi\,.
\end{aligned} \tag{3.17}$$

Boosts K with the small rapidity θ vary x as

$$\delta x^\mu[K_3] = \begin{pmatrix} -\theta x^- \\ 0 \\ 0 \\ \theta(x^-{+}x^3) \end{pmatrix}, \quad \delta x^\mu[K_1] = \begin{pmatrix} \theta x^1 \\ \theta(x^-{+}x^3) \\ 0 \\ 0 \end{pmatrix}, \quad \delta x^\mu[K_2] = \begin{pmatrix} \theta x^2 \\ 0 \\ \theta(x^-{+}x^3) \\ 0 \end{pmatrix}. \tag{3.18}$$

These result in the variations

$$\begin{aligned}
\delta\phi[K_3] &= (-x^-\partial_- + (x^- + x^3)\partial_3)\phi\,,\\
\delta\phi[K_1] &= (x^1\partial_- + (x^- + x^3)\partial_1)\phi\,,\\
\delta\phi[K_2] &= (x^2\partial_- + (x^- + x^3)\partial_2)\phi\,.
\end{aligned} \tag{3.19}$$

Using these variations of the field in the conserved charge density, Equation (3.5), and integrating them over the three-position gives the Fock space realizations of the generators. The derivations are done by replacing position variables and derivatives with respect to position acting on the field, which are under position and momentum integrals, using

$$\begin{aligned} x^\mu &\mapsto \frac{\partial}{\partial p_\mu}\,, \\ \partial_\mu &\mapsto p_\mu\,, \end{aligned} \tag{3.20}$$

and normal ordering the results. We drop divergent integrals that appear after normal ordering, of type

$$\int [\mathrm{d}p] f(p)\delta(p)\delta(p)\,, \tag{3.21}$$

where f is an arbitrary function of the three-momentum. We now list the generators at light-front time $x^- = 0$.

The Hamiltonian is

$$H = \int [\mathrm{d}p] \frac{m^2 + p_\perp^2 + p_3^2}{2p_3} a^\dagger a\,. \tag{3.22}$$

The momenta are

$$\begin{aligned} P_3 &= \int [\mathrm{d}p] p_3 a^\dagger a\,, \\ P_1 &= \int [\mathrm{d}p] p_1 a^\dagger a\,, \\ P_2 &= \int [\mathrm{d}p] p_2 a^\dagger a\,. \end{aligned} \tag{3.23}$$

The angular momenta are

$$
\begin{aligned}
J_3 &= i\int[\mathrm{d}p]\left((p_1\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial p_1})a^\dagger\right)a\,,\\
J_1 &= i\int[\mathrm{d}p]\left((\frac{m^2+p_\perp^2+p_3^2}{2p_3}\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial p_3}-p_3\frac{\partial}{\partial p_2})a^\dagger\right)a\,,\\
J_2 &= i\int[\mathrm{d}p]\left((-\frac{m^2+p_\perp^2+p_3^2}{2p_3}\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial p_3}+p_3\frac{\partial}{\partial p_1})a^\dagger\right)a\,.
\end{aligned}
\tag{3.24}
$$

The boosts are

$$
\begin{aligned}
K_3 &= i\int[\mathrm{d}p]\left(p_3(\frac{\partial}{\partial p_3}a^\dagger)\right)a\,,\\
K_1 &= i\int[\mathrm{d}p]\left((\frac{m^2+p_\perp^2+p_3^2}{2p_3}\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial p_3})a^\dagger\right)a\,,\\
K_2 &= i\int[\mathrm{d}p]\left((\frac{m^2+p_\perp^2+p_3^2}{2p_3}\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial p_3})a^\dagger\right)a\,.
\end{aligned}
\tag{3.25}
$$

Following Harindranath [16], we define linear combinations that have closed subalgebras:

$$
\begin{aligned}
F_1 &= -K_1+J_2 = -2i\int[\mathrm{d}p]\left((\frac{m^2+p_\perp^2+p_3^2}{2p_3}\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial p_3}-\frac{1}{2}p_3\frac{\partial}{\partial p_1})a^\dagger\right)a\,,\\
F_2 &= -K_2-J_1 = -2i\int[\mathrm{d}p]\left((\frac{m^2+p_\perp^2+p_3^2}{2p_3}\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial p_3}-\frac{1}{2}p_3\frac{\partial}{\partial p_2})a^\dagger\right)a\,,\\
E_1 &= -K_1-J_2 = -i\int[\mathrm{d}p]\left((p_3\frac{\partial}{\partial p_1})a^\dagger\right)a\,,\\
E_2 &= -K_2+J_1 = -i\int[\mathrm{d}p]\left((p_3\frac{\partial}{\partial p_2})a^\dagger\right)a\,.
\end{aligned}
\tag{3.26}
$$

We list the commutators of these generators at Table 3.1. Those that are trivially zero are not noted there for the ease of reading. The calculations of these commutators are done using the identities

$$
\int[\mathrm{d}p]\left[a^\dagger a(\mathbf{p}),\frac{\partial a^\dagger}{\partial q_i}a(\mathbf{q})\right]g(\mathbf{p}) = a^\dagger a(\mathbf{q})\frac{\partial g(\mathbf{q})}{\partial q_i}\,,
\tag{3.27}
$$

$$
\int[\mathrm{d}p][\mathrm{d}q]\left[\frac{\partial a^\dagger}{\partial p_i}a(\mathbf{p}),\frac{\partial a^\dagger}{\partial q_j}a(\mathbf{q})\right]g(\mathbf{p})h(\mathbf{q}) = \int[\mathrm{d}k]\left(\frac{\partial a^\dagger}{\partial k_i}\frac{\partial g}{\partial k_j}h-\frac{\partial a^\dagger}{\partial k_j}\frac{\partial h}{\partial k_i}g\right)a\,,
\tag{3.28}
$$

where h and g are arbitrary functions of the three-momentum.

Table 3.1. Commutators of the Poincaré generators.

	P_3	P_1	P_2	K_3	E_1	E_2	J_3	F_1	F_2	H
P_3				iP_3	0	0	0	$-2iP_1$	$-2iP_2$	
P_1				0	$-iP_3$	0	$-iP_2$	$-2iH+iP_3$	0	
P_2				0	0	$-iP_3$	iP_1	0	$-2iH+iP_3$	
K_3	$-iP_3$	0	0		$-iE_1$	$-iE_2$	0	iF_1	iF_2	$-iP_3+iH$
E_1	0	iP_3	0	iE_1		0	$-iE_2$	$-2iK_3$	$-2iJ_3$	iP_1
E_2	0	0	iP_3	iE_2	0		iE_1	$2iJ_3$	$-2iK_3$	iP_2
J_3	0	iP_2	$-iP_1$	0	iE_2	$-iE_1$		iF_2	$-iF_1$	0
F_1	$2iP_1$	$2iH-iP_3$	0	$-iF_1$	$2iK_3$	$-2iJ_3$	$-iF_2$		0	iP_1
F_2	$2iP_2$	0	$2iH-iP_3$	$-iF_2$	$2iJ_3$	$2iK_3$	iF_1	0		iP_2
H				iP_3-iH	$-iP_1$	$-iP_2$	0	$-iP_1$	$-iP_2$	

It is important to note that the algebra is closed, no central terms appear. That is, commutators of the generators are expressed in terms of linear combinations of the generators themselves.

3.4. Interaction and its truncation

We have the free Hamiltonian

$$H_0 = \int [\mathrm{d}p]\omega(\mathbf{p})a^\dagger a(\mathbf{p})\,. \tag{3.29}$$

To this we add an *attractive* ϕ^4 interaction,

$$H_I \coloneqq -\lambda\frac{(2\pi)^3}{6}\int \mathrm{d}x^3\,\mathrm{d}^2x^\perp\,\phi^4(x)\,. \tag{3.30}$$

As proposed in [17], the *Tamm-Dancoff* approximation can be more reliable in the light-front formalism to calculate bound state energies and wave functions. We follow that idea in this work. To that end we normal order and then truncate this interaction, keeping only the $\phi^-\phi^-\phi^+\phi^+$ terms as an extreme limit of truncation. These are the only terms in the interaction that conserve particle number. We note the normal ordering:

$$\phi^4 = 6{\phi^-}^2{\phi^+}^2 + 12\Delta(0)\phi^-\phi^+ + 3\Delta^2(0) + ({\phi^-}^4 + 6\Delta(0){\phi^-}^2 + 4{\phi^-}^3\phi^+ + \mathrm{H.\,C.})\,. \tag{3.31}$$

Note that the terms ${\phi^-}^4$, ${\phi^-}^2$ and their Hermitian conjugates have no contributions to vacuum expectation values due to the definitiness of the sign of p_3. This is also the case with the conventional light-front coordinates.

We absorb the mass term, $\phi^-\phi^+$, into the definition of the mass. The resulting interaction Hamiltonian is

$$H_1(x^-) \coloneqq -\lambda(2\pi)^3\int \mathrm{d}x^3\,\mathrm{d}^2x^\perp\,\phi^{(-)}\phi^{(-)}\phi^{(+)}\phi^{(+)}(x)\,. \tag{3.32}$$

At zero light-front time, $x^- = 0$, this is

$$H_1 = -\lambda \int (\prod_{i=1}^{4} \frac{[\mathrm{d}p_i]}{\sqrt{2p_{i3}}}) a_1^\dagger a_2^\dagger a_3 a_4 \delta(\mathbf{p_1} + \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4})\,. \tag{3.33}$$

Here and for what follows a_1 is a shorthand for $a(\mathbf{p_1})$, ω_1 for $\omega(\mathbf{p_1})$ etc. We take as the approximate quantum Hamiltonian $H = H_0 + H_1$.

4. ANALYSIS OF THE HAMILTONIAN

We study the interaction Hamiltonian, Equation (3.33), using a method by Rajeev [9, 18], utilizing the orthofermion χ and connected to it the principal operator Φ. The method seperates the coupling constant to an isolated, additive term, which then facilitates the identification of both any divergences that can be renormalized and the connection of the coupling constant to physical quantitites like binding energies. The essential idea is to extend the bosonic Fock space into $\mathcal{F}_{\mathcal{B}} \oplus [\mathcal{F}_{\mathcal{B}} \otimes \mathrm{L}^2(\mathbf{R}^3)]$. Here $\mathrm{L}^2(\mathbf{R}^3)$ refers to the Hilbert space of the orthofermion.

4.1. Orthofermion and the principal operator

Following [9, 18] we introduce the orthofermion χ and its algebra. It obeys the extreme number statistics

$$\chi(\mathbf{p})\chi^\dagger(\mathbf{q}) = \delta(\mathbf{p}-\mathbf{q})\Pi_0\,, \quad \chi(\mathbf{p})\chi(\mathbf{q}) = 0\,. \tag{4.1}$$

The projections onto the no-orthofermion and one-orthofermion subspaces are

$$\Pi_0 = \int [\mathrm{d}p]\chi(\mathbf{p})\chi^\dagger(\mathbf{p})\,, \quad \Pi_1 = \int [\mathrm{d}p]\chi^\dagger(\mathbf{p})\chi(\mathbf{p})\,. \tag{4.2}$$

A Hermitian operator O that acts on this space, which is assumed to be an extension from the Fock space of bosons $\mathcal{F}_{\mathcal{B}}$ to the space contanining no orthofermion or one, $\mathcal{F}_{\mathcal{B}} \oplus [\mathcal{F}_{\mathcal{B}} \otimes \mathrm{L}^2(\mathbf{R}^3)]$, can be decomposed with respect to the orthofermion number

$$O =: \begin{pmatrix} a & b^\dagger \\ b & d \end{pmatrix}\,, \tag{4.3}$$

where

$$
\begin{aligned}
&a : \mathcal{F}_{\mathcal{B}} \mapsto \mathcal{F}_{\mathcal{B}} \ , \\
&b : \mathcal{F}_{\mathcal{B}} \mapsto \mathcal{F}_{\mathcal{B}} \otimes \mathrm{L}^2(\mathbf{R}^3) \ , \\
&d : \mathcal{F}_{\mathcal{B}} \otimes \mathrm{L}^2(\mathbf{R}^3) \mapsto \mathcal{F}_{\mathcal{B}} \otimes \mathrm{L}^2(\mathbf{R}^3) \ .
\end{aligned}
\tag{4.4}
$$

We have the inverse

$$
O^{-1} =: \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix} \ , \tag{4.5}
$$

where

$$
\begin{aligned}
\alpha &= (a - b^\dagger d^{-1} b)^{-1} \\
&= a^{-1} + a^{-1} b^\dagger (d - b a^{-1} b^\dagger)^{-1} b a^{-1} \ .
\end{aligned}
\tag{4.6}
$$

We define the principal operator Φ as

$$
\Phi := d - b a^{-1} b^\dagger \ . \tag{4.7}
$$

4.2. Principal operator of the problem

In order to separate the coupling constant from its multiplicative form in the interaction and remove the divergence additively, we define the operator

$$
\tilde{H} := H_0 \Pi_0 + \left(\int \frac{[\mathrm{d}p_1 \, \mathrm{d}p_2]}{\sqrt{4 p_{13} p_{23}}} a_1^\dagger a_2^\dagger \chi(\mathbf{p_1} + \mathbf{p_2}) + \mathrm{H.\,C.} \right) + \frac{\Pi_1}{\lambda} \ . \tag{4.8}
$$

The reason behind defining $\tilde{H}$ this way becomes clear in later steps. We decompose it with respect to its action on the orthofermion subspaces:

$$\begin{aligned}\tilde{H} - E\Pi_0 &= \begin{pmatrix} (H_0 - E)\Pi_0 & \int \frac{[\mathrm{d}p_1 \mathrm{d}p_2]}{\sqrt{4p_{13}p_{23}}} a_1^\dagger a_2^\dagger \chi(\mathbf{p_1} + \mathbf{p_2}) \\ \int \frac{[\mathrm{d}p_3 \mathrm{d}p_4]}{\sqrt{4p_{33}p_{43}}} a_3 a_4 \chi^\dagger(\mathbf{p_3} + \mathbf{p_4}) & \frac{\Pi_1}{\lambda} \end{pmatrix} \\ &=: \begin{pmatrix} a & b^\dagger \\ b & d \end{pmatrix} .\end{aligned} \tag{4.9}$$

This has the inverse

$$(\tilde{H} - E\Pi_0)^{-1} =: \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix} . \tag{4.10}$$

Using the first expression of Equation (4.6) we find

$$\begin{aligned}\alpha &= \left(H_0 - E - \lambda \int \frac{[\mathrm{d}p_1\, \mathrm{d}p_2]}{\sqrt{4p_{13}p_{23}}} a_1^\dagger a_2^\dagger \chi(\mathbf{p_1} + \mathbf{p_2}) \int \frac{[\mathrm{d}p_3\, \mathrm{d}p_4]}{\sqrt{4p_{33}p_{43}}} a_3 a_4 \chi^\dagger(\mathbf{p_3} + \mathbf{p_4}) \right)^{-1} \Pi_0 \\ &= (H_0 - E + H_1)^{-1}\Pi_0 \\ &= (H - E)^{-1}\Pi_0 \\ &=: R(E)\Pi_0 .\end{aligned} \tag{4.11}$$

For the second line Equation (4.1) is used. This convenient recombination of individual terms to reproduce the original Hamiltonian is the reason behind the particular form of $\tilde{H}$. Here $R(E)$ is the formal resolvent of the Hamiltonian H. Again using Equation (4.6) we have an alternate form of the same expression,

$$\alpha = \left(\frac{1}{H_0 - E} + \frac{1}{H_0 - E} b^\dagger \Phi(E)^{-1} b \frac{1}{H_0 - E} \right) \Pi_0 . \tag{4.12}$$

Here $\Phi(E)$, as defined in Equation (4.7), is the principal operator

$$\Phi(E) = \frac{\Pi_1}{\lambda} - \int (\prod_{i=1}^{4} \frac{[\mathrm{d}p_i]}{\sqrt{2p_{i3}}}) \chi^\dagger(\mathbf{p_3} + \mathbf{p_4}) a_3 a_4 (H_0 - E)^{-1} a_1^\dagger a_2^\dagger \chi(\mathbf{p_1} + \mathbf{p_2}) . \tag{4.13}$$

The possible zeros of this operator, as long as they are below the N-particle threshold of the free part, correspond to bound states, as all the other terms in the resolvent, Equation (4.12), are regular at E. For this reason, the eigenvalues $w(E)$ of the principal operator,

$$\Phi(E)\,|w(E)\rangle = w(E)\,|w(E)\rangle\ , \tag{4.14}$$

carry information about the bound states. In Section 4.4, these eigenvalues are shown to "flow" with E, that it is possible to adjust E, to find an E^* such that $w(E^*) = 0$. Also note that when $R(E)$ acts on a N-particle state, $\Phi(E)$ sees a $N-2$-particle state with one orthofermion, since b annihilates two particles and creates an orthofermion.

We realize that Equation (4.13) is not normal ordered. Normal ordering it gives

$$\Phi(E) = \frac{\Pi_1}{\lambda} - \left(2K(E) + 4U_1(E) + U_2(E)\right), \tag{4.15}$$

where

$$K(E) := \int \frac{[\mathrm{d}p_1\,\mathrm{d}p_2]}{4p_{13}p_{23}}\chi^\dagger(\mathbf{p_1}+\mathbf{p_2})\left(H_0 - E + \sum_{i=1}^{2}\omega_i\right)^{-1}\chi(\mathbf{p_1}+\mathbf{p_2})\,, \tag{4.16}$$

$$U_1(E) := \int \frac{[\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3]}{2p_{13}\sqrt{4p_{23}p_{33}}}\chi^\dagger(\mathbf{p_1}+\mathbf{p_3})a_2^\dagger\left(H_0 - E + \sum_{i=1}^{3}\omega_i\right)^{-1}a_3\chi(\mathbf{p_1}+\mathbf{p_2})\,, \tag{4.17}$$

$$U_2(E) := \int (\prod_{i=1}^{4}\frac{[\mathrm{d}p_i]}{\sqrt{2p_{i3}}})\chi^\dagger(\mathbf{p_3}+\mathbf{p_4})a_1^\dagger a_2^\dagger\left(H_0 - E + \sum_{i=1}^{4}\omega_i\right)^{-1}a_3a_4\chi(\mathbf{p_1}+\mathbf{p_2})\,. \tag{4.18}$$

Note that for $R(E)$ acting on a two-particle state, that is $\Phi(E)$ acting on the vacuum of the bosonic system and one orthofermion state, the terms U_1 and U_2 are irrelevant. They contain particle annihilation terms so they evaluate to zero acting on such a state. Therefore for a description of the two-particle sector $K(E)$ alone is in effect.

4.3. Renormalization

We note that there is a divergence in $K(E)$, therefore the operator $\Phi(E)$ is not well-defined as it is. We now demonstrate this divergence and then renormalize it by introducing a similar divergence to the reciprocal of the coupling constant. A remarkable aspect of this extended Fock space construction is the possibility to renormalize the truncated Hamiltonian non-perturbatively.

We make the following coordinate changes with unit Jacobian for the integral in $K(E)$:

$$\begin{aligned} P &\coloneqq p_{13} + p_{23}\,, \qquad Q \coloneqq \frac{p_{13} - p_{23}}{2}\,, \\ \boldsymbol{\xi} &\coloneqq \mathbf{p_1}_\perp + \mathbf{p_2}_\perp\,, \quad \boldsymbol{\eta} \coloneqq \frac{p_{23}\mathbf{p_1}_\perp - p_{13}\mathbf{p_2}_\perp}{p_{13} + p_{23}}\,. \end{aligned} \tag{4.19}$$

This gives

$$\begin{aligned} K(E) = \frac{1}{2}\int \mathrm{d}P\,\mathrm{d}Q\,\mathrm{d}^2\xi\,\mathrm{d}^2\eta\,\chi^\dagger(P,\boldsymbol{\xi}) \\ \left([2(H_0 - E) + P + \frac{\xi^2}{P}](\frac{P^2}{4} - Q^2) + m^2 P + \eta^2 P\right)^{-1} \chi(P,\boldsymbol{\xi})\,. \end{aligned} \tag{4.20}$$

We focus on the following part of this integral:

$$\begin{aligned} &\int \mathrm{d}Q\,\mathrm{d}^2\eta \left([2(H_0 - E) + P + \frac{\xi^2}{P}](\frac{P^2}{4} - Q^2) + m^2 P + \eta^2 P\right)^{-1} \\ &= \frac{1}{2}\int_{-1}^{1}\mathrm{d}\tau \int \mathrm{d}^2\eta \left([2(H_0 - E)P + P^2 + \xi^2]\frac{1-\tau^2}{4} + m^2 + \eta^2\right)^{-1}\,. \end{aligned} \tag{4.21}$$

Here we used the coordinate transformation $Q = \frac{P\tau}{2}$. We note that the η integral diverges logarithmically. It is possible to renormalize this divergence with a coupling constant redefinition. As expected the truncated model does not require a mass renormalization [19].

A possible choice for λ is

$$\frac{1}{\lambda} := \frac{1}{\lambda_R(M)} + \int \mathrm{d}^2\eta \left(M^2 + \eta^2\right)^{-1} . \tag{4.22}$$

This gives a finite combination

$$\frac{\Pi_1}{\lambda} - 2K(E) = \frac{\Pi_1}{\lambda_R(M)} - 2K_R(E;M)\,, \tag{4.23}$$

where

$$\begin{aligned} K_R(E;M) := -\frac{\pi}{4} \int & \mathrm{d}P\, \mathrm{d}^2\xi\, \chi^\dagger(P,\boldsymbol{\xi}) \\ & \int_{-1}^{1} \mathrm{d}\tau \ln\left(\frac{[2(H_0-E)P + P^2 + \xi^2](1-\tau^2) + 4m^2}{4M^2}\right) \chi(P,\boldsymbol{\xi})\,. \end{aligned} \tag{4.24}$$

Here M is an arbitrary scale for the system, and the renormalized coupling constant $\lambda_R(M)$ runs with it. It has the β function

$$\beta(\lambda_R(M)) = -2\pi\lambda_R^2\,, \tag{4.25}$$

where the β function is defined as

$$\beta(\lambda(M)) := \frac{\partial\lambda(M)}{\partial \ln M}\,. \tag{4.26}$$

This β function is absolutely negative, therefore the model is asymptotically free.

We remark that this is not the only possible renormalization scheme, we can also use a physical parameter like the binding energy in place of the arbitrary scale M. This is utilized for the bound state wave function calculation in Chapter 5.

The renormalized principal operator becomes

$$\Phi_R(E) = \frac{\Pi_1}{\lambda_R(M)} - \left(2K_R(E;M) + 4U_1(E) + U_2(E)\right)\,. \tag{4.27}$$

We assert that the full expression involving $\Phi_R(E)$ now defines a resolvent through Equation (4.12), to be called $R_R(E)$. However the corresponding renormalized Hamiltonian H_R cannot be written down explicitly.

As it stands we have a finite resolvent, however this doesn't guarantee that the theory is finite. To secure this, we need to show that the ground state energies of all particle sectors are finite. We give a bound for the ground state energy of the N-particle sector within a mean field theory approximation in Chapter 6. Our calculations depend on the flow of eigenvalues of the principal operator, we now concentrate on this property.

4.4. Eigenvalue flow

The resolvent, Equation (4.12), shows that the bound states below the spectrum of H_0 can only come from the zeros of $\Phi(E)$. In order to show that we can indeed find these zeros and locate the ground state, we prove that

$$\frac{\partial w(E)}{\partial E} = \left\langle \frac{\partial \Phi_R(E)}{\partial E} \right\rangle < 0 \,, \tag{4.28}$$

where $w(E)$ is an eigenvalue of $\Phi_R(E)$. That is, we prove that a given eigenvalue of $\Phi_R(E)$ decreases with increasing E. Recalling that the zero eigenvalues of $\Phi_R(E)$ correspond to possible bound states, this means that if we find any eigenvalue of $\Phi_R(E)$ below zero, we can increase E to make it zero and find the corresponding state. We assume that the flow of eigenvalues are differentiable and that no crossings occur. This also implies that the ground state energy of the system corresponds to the zero of the minimum eigenvalue of $\Phi_R(E)$, as it reaches zero with the smallest E. This observation is essential to obtain a mean field estimate of the ground state energy for large number of particles.

We remind that P_3 is always positive, therefore the minimum of the spectrum of the Hamiltonian is always the invariant mass for the corresponding state, higher values within the same composition give the translational energy increase of the system as a whole, as we see below explicitly for the two-particle sector.

To proceed with the proof we first show that the derivative of $\Phi(E)$ is negative definite. Then we show that the derivatives of $\Phi(E)$ and $\Phi_R(E)$ are exactly the same. Note that this only requires the derivatives of $K(E)$ and $K_R(E)$ to be equal.

We rewrite $\Phi(E)$ in terms of

$$I(E;s) := \int (\prod_{i=1}^{2} \frac{[\mathrm{d}p_i]}{\sqrt{2p_{i3}}}) e^{-\frac{s}{2}(H_0-E)} a_1^\dagger a_2^\dagger \chi(\mathbf{p_1}+\mathbf{p_2})\,. \tag{4.29}$$

Using this in Equation (4.13) we get

$$\begin{aligned}\Phi(E) &= \frac{\Pi_1}{\lambda} - \int (\prod_{i=1}^{4} \frac{[\mathrm{d}p_i]}{\sqrt{2p_{i3}}}) \chi^\dagger(\mathbf{p_3}+\mathbf{p_4}) a_3 a_4 \left(\int_0^\infty \mathrm{d}s\, e^{-s(H_0-E)}\right) a_1^\dagger a_2^\dagger \chi(\mathbf{p_1}+\mathbf{p_2}) \\ &= \frac{\Pi_1}{\lambda} - \int_0^\infty \mathrm{d}s\, I^\dagger I(E;s)\,.\end{aligned} \tag{4.30}$$

If we take the derivative formally, we get

$$\frac{\partial \Phi(E)}{\partial E} = -\int_0^\infty \mathrm{d}s\, s I^\dagger I(E;s)\,. \tag{4.31}$$

This operator is negative definite. We now compare the derivatives of $K(E)$ and $K_R(E;M)$.

The derivate of $K_R(E;M)$, using Equation (4.24), is

$$\begin{aligned}&-\frac{\pi}{4}\int \mathrm{d}P\,\mathrm{d}^2\xi\,\chi^\dagger(P,\boldsymbol{\xi})\int_{-1}^{1}\mathrm{d}\tau \\ &\qquad (-2P)(1-\tau^2)\left([2(H_0-E)P+P^2+\xi^2](1-\tau^2)+4m^2\right)^{-1}\chi(P,\boldsymbol{\xi})\,,\end{aligned} \tag{4.32}$$

and the derivative of $K(E)$, using Equation (4.21), is

$$\begin{aligned}&\frac{1}{4}\int \mathrm{d}P\,\mathrm{d}^2\xi\,\chi^\dagger(P,\boldsymbol{\xi})\int_{-1}^{1}\mathrm{d}\tau \\ &\quad \int \mathrm{d}^2\eta\,(2P)\frac{1-\tau^2}{4}\left([2(H_0-E)P+P^2+\xi^2]\frac{1-\tau^2}{4}+m^2+\eta^2\right)^{-2}\chi(P,\boldsymbol{\xi})\,,\end{aligned} \tag{4.33}$$

which is equal to Equation (4.32) after taking the η integral. Therefore the derivatives of K indeed agree. The other derivatives, those of U_1 and U_2, agree trivially. Therefore our claim on the eigenvalue flow of $\Phi_R(E)$ holds.

4.5. Scaling properties

We now demonstrate the scaling properties of the principal operator. Given a positive real number γ, one can construct a unitary operator acting on the Fock space. Here $U(\gamma)$ is a unitary operator that scales the momentum of the particle and orthofermion operators:

$$U(\gamma)a(\mathbf{p})U^\dagger(\gamma) = \gamma^{\frac{3}{2}}a(\gamma\mathbf{p})\,, \quad U(\gamma)\chi(\mathbf{p})U^\dagger(\gamma) = \gamma^{\frac{3}{2}}\chi(\gamma\mathbf{p})\,. \tag{4.34}$$

This in turn can be used to establish the following scaling properties of the principal operator $\Phi_R(E)$:

$$U^\dagger(\gamma)\Phi_R(\gamma E; M, \lambda_R(M), m)U(\gamma) = \Phi_R(E; \gamma^{-1}M, \lambda_R(M), \gamma^{-1}m) \tag{4.35}$$

$$= \Phi_R(E; M, \lambda_R(\gamma M), \gamma^{-1}m)\,. \tag{4.36}$$

These follow from such scalings of Equations (3.9) and (4.1). We derive Equation (4.35) now, Equation (4.36) follows afterwards. To get Equation (4.35), one can scale E and all momenta by γ in $\Phi_R(E)$ and insert $U(\gamma)U^\dagger(\gamma)$ as appropriate. We demonstrate this for $K_R(E; M)$, it can be shown for $U_1(E)$ and $U_2(E)$ precisely in the same manner:

$$\begin{aligned}
&U^\dagger(\gamma)K_R(\gamma E; M, m)U(\gamma) \\
&= -\frac{\pi}{4}U^\dagger(\gamma)\int \gamma\,\mathrm{d}P\,\gamma^2\,\mathrm{d}^2\xi\,\chi^\dagger(\gamma P, \gamma\boldsymbol{\xi}) \\
&\quad \int_{-1}^{1}\mathrm{d}\tau \ln\left(\frac{[2(\gamma\gamma^{-1}H_0 - \gamma E)\gamma P + \gamma^2P^2 + \gamma^2\xi^2](1-\tau^2) + \gamma^2 4(\gamma^{-1}m)^2}{4\gamma^2(\gamma^{-1}M)^2}\right) \\
&\quad \chi(\gamma P, \gamma\boldsymbol{\xi})U(\gamma)\,.
\end{aligned} \tag{4.37}$$

Rescaling the variables, this becomes

$$-\frac{\pi}{4}\int \mathrm{d}P\,\mathrm{d}^2\xi\,\chi^\dagger(P,\boldsymbol{\xi})$$
$$\int_{-1}^{1}\mathrm{d}\tau\ln\left(\frac{[2(H_0-E)P+P^2+\xi^2](1-\tau^2)+4(\gamma^{-1}m)^2}{4(\gamma^{-1}M)^2}\right)\chi(P,\boldsymbol{\xi})\,, \quad (4.38)$$

which is equal to

$$K_R(E;\gamma^{-1}M,\gamma^{-1}m)\,. \quad (4.39)$$

Note that here we used

$$U^\dagger(\gamma)H_0U(\gamma)=\gamma H_0\,. \quad (4.40)$$

Below, we obtain Equation (4.36) in a more conventional way by means of the renormalization group equation.

4.5.1. Callan-Symanzik equation

It is instructive to look at the *exact* scaling properties of the principal operator $\Phi_R(E)$ from a more conventional perspective. A well-known approach to scaling in field theories, which is particularly suitable for renormalized correlation functions, is given by the Callan-Symanzik equation [19, 20]. We obtain an analogous expression in our case directly for the principal operator.

Observe that the operator

$$\gamma\frac{\partial}{\partial\gamma}+M\frac{\partial}{\partial M}+m\frac{\partial}{\partial m}\,, \quad (4.41)$$

which can be thought of as a scale-invariant derivative, leads to

$$\left(\gamma\frac{\partial}{\partial\gamma} + M\frac{\partial}{\partial M} + m\frac{\partial}{\partial m}\right)\Phi_R(E;\gamma^{-1}M,\lambda_R(M),\gamma^{-1}m) = 0\,. \tag{4.42}$$

Therefore we get

$$\left(\gamma\frac{\partial}{\partial\gamma} - \beta\frac{\partial}{\partial\lambda_R} + m\frac{\partial}{\partial m}\right)\Phi_R(E;\gamma^{-1}M,\lambda_R(M),\gamma^{-1}m) = 0\,, \tag{4.43}$$

with β as found before, using the definition of the β function. If we consider this as an equation to be obeyed by the principal operator, we can look for a solution.

As a simple ansatz, we propose that

$$\Phi_R(E;\gamma^{-1}M,\lambda_R(M),\gamma^{-1}m) = f(\gamma)\Phi_R(E;M,\lambda_R(\gamma M),\gamma^{-1}m)\,. \tag{4.44}$$

Acting on this with the operator just defined we get

$$\left(\gamma\frac{\partial}{\partial\gamma} - \beta\frac{\partial}{\partial\lambda_R} + m\frac{\partial}{\partial m}\right)f(\gamma)\Phi_R(E;M,\lambda_R(\gamma M),\gamma^{-1}m) = 0\,. \tag{4.45}$$

This gives the condition

$$\frac{\partial f(\gamma)}{\partial\gamma} = 0\,, \tag{4.46}$$

which has the solution $f(\gamma) = 1$ with the initial condition $f(1) = 1$. We note that these results agree perfectly with the non-relativistic version of this theory [10, 21, 22]. This renormalization group point of view works along similar lines on a manifold as well, as shown in [23].

4.5.2. Exact scaling

The former derivation required an ansatz for the solution. However we can verify the same result directly as well since we renormalize the principal operator and solve the β function, Equation (4.25), non-perturbatively. It has the solution

$$\lambda_R(\gamma M) = \frac{\lambda_R(M)}{1 + 2\pi \ln \gamma \lambda_R(M)} \,. \tag{4.47}$$

We use this to verify Equation (4.36) directly. We need to show that

$$\frac{\Pi_1}{\lambda_R(M)} - 2K_R(E;\gamma^{-1}M) = \frac{\Pi_1}{\lambda_R(\gamma M)} - 2K_R(E;M)\,. \tag{4.48}$$

The remaining parts, $U_1(E)$ and $U_2(E)$ match trivially as they do not run with M. We get

$$\begin{aligned} &\frac{\Pi_1}{\lambda_R(\gamma M)} - 2K_R(E;M) \\ &= \frac{\Pi_1}{\lambda_R(M)} - 2K_R(E;\gamma^{-1}M) + 2\pi \ln \gamma \Pi_1 - 2K_R(E;M) + 2K_R(E;\gamma^{-1}M)\,, \end{aligned} \tag{4.49}$$

where we add and subtract the same term. With

$$\begin{aligned} -2K_R(E;M) + 2K_R(E;\gamma^{-1}M) =& -\frac{\pi}{2}\int \mathrm{d}P\,\mathrm{d}^2\xi\,\chi^\dagger(P,\boldsymbol{\xi}) \\ &\int_{-1}^{1}\mathrm{d}\tau \ln\left(\frac{4M^2}{4(\gamma^{-1}M)^2}\right)\chi(P,\boldsymbol{\xi}) \\ =& -2\pi \ln\gamma\Pi_1\,, \end{aligned} \tag{4.50}$$

we get the claimed equality.

5. WAVE FUNCTION OF THE GROUND STATE OF THE TWO-PARTICLE SECTOR

In order to find the wave function of the ground state of the two-particle sector we study the resolvent. For this purpose we follow an alternative but equivalent renormalization scheme. Looking at Equation (4.21), we see that choosing λ in terms of the binding energy of two-particles μ using

$$\frac{1}{\lambda} \coloneqq \frac{1}{2}\int_{-1}^{1} \mathrm{d}\tau \int \mathrm{d}^2\eta \left(-\mu^2 \frac{1-\tau^2}{4} + m^2 + \eta^2\right)^{-1} \tag{5.1}$$

leads to the following finite combination in the resolvent:

$$\begin{aligned}\frac{\Pi_1}{\lambda} - 2K(E) = \frac{\pi}{2} \int \mathrm{d}P\, \mathrm{d}^2\xi\, \chi^\dagger(P, \boldsymbol{\xi}) \\ \int_{-1}^{1} \mathrm{d}\tau \ln\left(\frac{[2(H_0-E)P + P^2 + \xi^2](1-\tau^2) + 4m^2}{-\mu^2(1-\tau^2) + 4m^2}\right) \chi(P, \boldsymbol{\xi})\,.\end{aligned} \tag{5.2}$$

Using this, we find the wave function of the two-particle bound state from the discontinuity of the resolvent above and below its continuum of eigenvalues. It obeys the following for small ϵ near an eigenvalue E:

$$\begin{aligned} R_R(E - i\epsilon) - R_R(E + i\epsilon) &= \frac{1}{H_R - (E - i\epsilon)} - \frac{1}{H_R - (E + i\epsilon)} \\ &= 2\pi i \delta(H_R - E) \\ &= 2\pi i \left|\psi(E)\middle\rangle\middle\langle\psi(E)\right| \,. \end{aligned} \tag{5.3}$$

Since the spectrum of H_0 begins at $2m$ for two-particle states and we seek $0 < E < 2m$, we can replace $\frac{1}{H_0 - E \pm i\epsilon}$ with $\frac{1}{H_0 - E}$ for the following, we are never near an eigenvalue of

H_0. We have, with Equation (4.12),

$$\begin{aligned} R_R(E-i\epsilon)-R_R(E+i\epsilon) &= \frac{1}{H_R-(E-i\epsilon)}-\frac{1}{H_R-(E+i\epsilon)} \\ &= \frac{1}{H_0-E}b^\dagger\left(\frac{1}{\Phi_R(E+i\epsilon)}-\frac{1}{\Phi_R(E-i\epsilon)}\right)b\frac{1}{H_0-E}\,. \end{aligned} \tag{5.4}$$

We expand $\frac{1}{\Phi_R}$ in terms of the eigenspace of Φ_R:

$$\frac{1}{\Phi_R(E-i\epsilon)}-\frac{1}{\Phi_R(E+i\epsilon)} = \int \mathrm{d}w\,\rho(w)\,|w\rangle\langle w|\left(\frac{1}{w(E-i\epsilon)}-\frac{1}{w(E+i\epsilon)}\right)\,. \tag{5.5}$$

Here $\rho(w)$ is the density of states for the eigenspace of Φ_R. We can expand a given eigenvalue w near its zero $E^\star(w)$, $w(E^\star)=0$:

$$w(E+i\epsilon)=\left.\frac{\partial w}{\partial E}\right|_{E^\star}(E+i\epsilon-E^\star)\,,\quad w(E-i\epsilon)=\left.\frac{\partial w}{\partial E}\right|_{E^\star}(E-i\epsilon-E^\star)\,. \tag{5.6}$$

This gives

$$\frac{1}{w(E-i\epsilon)}-\frac{1}{w(E+i\epsilon)}=\frac{2\pi i\delta(E-E^\star)}{-\left.\frac{\partial w}{\partial E}\right|_{E^\star}}=2\pi i\delta(w(E))\,. \tag{5.7}$$

The last equality follows from expanding a Dirac delta function with the zeros of its argument, and for a given w there is only one such zero. Therefore

$$\begin{aligned} \frac{1}{\Phi_R(E-i\epsilon)}-\frac{1}{\Phi_R(E+i\epsilon)} &= \int \mathrm{d}w\,\rho(w)\,|w\rangle\langle w|\,2\pi i\delta(w(E)) \\ &= 2\pi i\rho(w(E))\,|w(E)\rangle\langle w(E)|\,. \end{aligned} \tag{5.8}$$

Using Equation (4.9) for b and $b^\dagger$ and moving the $\frac{1}{H_0-E}$ terms inside, we have for a two-particle state

$$\begin{aligned} R_R(E-i\epsilon)-R_R(E+i\epsilon) = \int & \frac{[\mathrm{d}p_1\,\mathrm{d}p_2]}{\sqrt{4p_{1_3}p_{2_3}}} a_1^\dagger a_2^\dagger \frac{\chi(\mathbf{p_1}+\mathbf{p_2})}{\omega_1+\omega_2-E} \\ & \left(\frac{1}{\Phi_R(E-i\epsilon)}-\frac{1}{\Phi_R(E+i\epsilon)}\right) \\ & \int \frac{[\mathrm{d}p_3\,\mathrm{d}p_4]}{\sqrt{4p_{3_3}p_{4_3}}} a_3 a_4 \frac{\chi^\dagger(\mathbf{p_3}+\mathbf{p_4})}{\omega_3+\omega_4-E}\,. \end{aligned} \tag{5.9}$$

With Equation (5.8) this becomes

$$\int \frac{[\mathrm{d}p_1\,\mathrm{d}p_2]}{\sqrt{4p_{1_3}p_{2_3}}} a_1^\dagger a_2^\dagger \frac{\chi(\mathbf{p_1}+\mathbf{p_2})}{\omega_1+\omega_2-E} 2\pi i\rho(w(E))\,|w(E)\rangle\langle w(E)| \int \frac{[\mathrm{d}p_3\,\mathrm{d}p_4]}{\sqrt{4p_{3_3}p_{4_3}}} a_3 a_4 \frac{\chi^\dagger(\mathbf{p_3}+\mathbf{p_4})}{\omega_3+\omega_4-E}\,, \tag{5.10}$$

which can be compactly written as

$$2\pi i\left(\int \frac{[\mathrm{d}p_1\,\mathrm{d}p_2]}{\sqrt{4p_{1_3}p_{2_3}}} a_1^\dagger a_2^\dagger \frac{\chi(\mathbf{p_1}+\mathbf{p_2})}{\omega_1+\omega_2-E}\sqrt{\rho(w(E))}\,|w(E)\rangle\right)(\mathrm{H.\,C.})\,. \tag{5.11}$$

Using this with Equation (5.3) we get the wave function

$$|\psi(E)\rangle = \int \frac{[\mathrm{d}p_1\,\mathrm{d}p_2]}{\sqrt{4p_{1_3}p_{2_3}}} a_1^\dagger a_2^\dagger \frac{\chi(\mathbf{p_1}+\mathbf{p_2})}{\omega_1+\omega_2-E}\sqrt{\rho(w(E))}\,|w(E)\rangle\ . \tag{5.12}$$

For the ground state $E=\mu$ we have

$$\begin{aligned} |w(\mu)\rangle &= \int \mathrm{d}P\,\mathrm{d}^2\xi\,\chi^\dagger(P,\boldsymbol{\xi})\delta(P-\mu)\delta(\boldsymbol{\xi})\,|0\rangle \\ &= \chi^\dagger(\mu,0)\,|0\rangle\ . \end{aligned} \tag{5.13}$$

This is an eigenstate of $\Phi_R(E)$. This can be verified by acting on it with Equation (4.27). Note that due to translational invariance we expect a continuum of states. Using this we have

$$|\psi(\mu)\rangle = \int \frac{[\mathrm{d}p_1\,\mathrm{d}p_2]}{\sqrt{4p_{1_3}p_{2_3}}} a_1^\dagger a_2^\dagger \frac{\delta(p_{1_3}+p_{2_3}-\mu)\delta(\mathbf{p_1}_\perp+\mathbf{p_2}_\perp)}{\omega_1+\omega_2-\mu}\sqrt{\rho(w(\mu))}\,|0\rangle\ . \tag{5.14}$$

Therefore the momentum-space wave function is

$$\Psi(\mathbf{p_1},\mathbf{p_2}) = \sqrt{2\rho(w(\mu))}\frac{\delta(p_{1_3}+p_{2_3}-\mu)\delta(\mathbf{p_1}_\perp+\mathbf{p_2}_\perp)}{\omega_1+\omega_2-\mu}\,. \tag{5.15}$$

We take its Fourier transform to find the position-space wave function

$$\frac{\Psi(\mathbf{x_1},\mathbf{x_2})}{\frac{\sqrt{2\rho(w(\mu))}}{(2\pi)^3}} = \int [\mathrm{d}p_1\,\mathrm{d}p_2]\;\exp\Big(-i\big(p_{1_3}x_1^3+p_{2_3}x_2^3+\mathbf{p_1}_\perp\cdot\mathbf{x_1^\perp}+\mathbf{p_2}_\perp\cdot\mathbf{x_2^\perp}\big)\Big)$$
$$\frac{\delta(p_{1_3}+p_{2_3}-\mu)\delta(\mathbf{p_1}_\perp+\mathbf{p_2}_\perp)}{\omega_1+\omega_2-\mu}\,. \tag{5.16}$$

Applying the coordinate transformations of before, Equation (4.19), we get

$$\frac{\Psi(\mathbf{x_1},\mathbf{x_2})}{\frac{\sqrt{2\rho(w(\mu))}}{(2\pi)^3}} = \int \mathrm{d}P\,\mathrm{d}Q\,\mathrm{d}^2\xi\,\mathrm{d}^2\eta\;\exp\Big(-i\big(P\frac{x_1^3+x_2^3}{2}+Q(x_1^3-x_2^3)+\boldsymbol{\xi}\cdot\frac{\mathbf{x_1^\perp}+\mathbf{x_2^\perp}}{2}$$
$$+(\boldsymbol{\eta}+\frac{Q}{P}\boldsymbol{\xi})\cdot(\mathbf{x_1^\perp}-\mathbf{x_2^\perp})\big)\Big)\frac{\delta(P-\mu)\delta(\xi)}{\omega_1+\omega_2-\mu}\,. \tag{5.17}$$

Taking the integrals over P and ξ, and letting $Q=\frac{P\tau}{2}$, this becomes

$$\int_{-1}^{1}\mu\frac{\mathrm{d}\tau}{2}\int_0^\infty \eta\,\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\phi\exp\Big(-i\big(\mu\frac{x_1^3+x_2^3}{2}+\frac{\mu\tau}{2}(x_1^3-x_2^3)$$
$$+\eta\big|\mathbf{x_1^\perp}-\mathbf{x_2^\perp}\big|\cos\phi\big)\Big)\Big(\frac{m^2\mu+\eta^2\mu+\frac{\mu^3}{4}(1-\tau^2)}{2\frac{\mu^2}{4}(1-\tau^2)}-\mu\Big)^{-1}\,. \tag{5.18}$$

Taking the angular integral produces a Bessel function of the first kind,

$$\int_{-1}^{1}\mathrm{d}\tau\int_0^\infty \eta\,\mathrm{d}\eta\;\exp\Big(-i\big(\mu\frac{x_1^3+x_2^3}{2}+\frac{\mu\tau}{2}(x_1^3-x_2^3)\big)\Big)$$
$$\Big(\frac{m^2+\eta^2+\frac{\mu^2}{4}(1-\tau^2)}{\frac{\mu^2}{4}(1-\tau^2)}-2\Big)^{-1}2\pi J_0(\eta\big|\mathbf{x_1^\perp}-\mathbf{x_2^\perp}\big|)\,, \tag{5.19}$$

and taking the η integral we end up with a modified Bessel function of the second kind [24],

$$\frac{\pi}{2}\mu^2 \int_{-1}^{1} \mathrm{d}\tau \, \exp\Big(- i(\mu\frac{x_1^3 + x_2^3}{2} + \frac{\mu\tau}{2}(x_1^3 - x_2^3))\Big)$$
$$(1-\tau^2)K_0\left(\frac{1}{2}|\mathbf{x_1}^{\perp} - \mathbf{x_2}^{\perp}|\sqrt{4m^2 - (1-\tau^2)\mu^2}\right) . \quad (5.20)$$

A two-particle system is separable in the center-of-mass and relative coordinates. Therefore we change to the coordinates

$$\mathbf{x}_{\mathrm{CM}} = \frac{\mathbf{x_1} + \mathbf{x_2}}{2} , \quad \mathbf{x_r} = \mathbf{x_1} - \mathbf{x_2} , \quad (5.21)$$

and finally we get

$$\Psi(\mathbf{x}_{\mathrm{CM}}, x_r) = \mathcal{N} e^{-i\mu x^3_{\mathrm{CM}}} \int_{-1}^{1} \mathrm{d}\tau \, e^{-i\mu\tau x_r^3}(1-\tau^2)K_0\left(\frac{1}{2}\mathbf{x_r}^{\perp}\sqrt{4m^2 - (1-\tau^2)\mu^2}\right) . \quad (5.22)$$

Here $\mathcal{N}$ is a normalization constant. We recall that the bound state wave function of two non-relativistic particles interacting via a delta function potential is exactly $K_0(\sqrt{2m|E_b|}|\mathbf{x_1} - \mathbf{x_2}|)$, up to a normalization constant [9–11]. Here E_b refers to the binding energy. Note that for a relativistic system the absolute value of the binding energy is $\sqrt{4m^2 - \mu^2}$, in our case we have a convolution over all such possible differences. The convolution takes x_r^3 into account in a subtle way, in the transverse direction the system behaves very much like a two dimensional delta potential, whereas in the light-front direction we have an oscillatory superposition of this two dimensional wave function with a weighted energy difference.

5.1. Normalizability of the wave function

We integrate the square of the wave function in the relative coordinates and show that it is finite and positive. For what follows we drop the center of mass part of the wave function since that is what leads to a continuous spectrum and it is not expected to be normalizable:

$$\begin{aligned}
\int \mathrm{d}x_r^3\, \mathrm{d}^2x_r^\perp\, |\Psi(\mathbf{x_r})|^2 &\propto \int \mathrm{d}x_r^3 \int \mathrm{d}^2x_r^\perp \int_{-1}^{1} \mathrm{d}\tau \int_{-1}^{1} \mathrm{d}\tau'\, e^{-i2ma(\tau-\tau')x_r^3}(1-\tau^2)(1-\tau'^2) \\
&\quad \left\{ K_0(x_r^\perp m\sqrt{1-(1-\tau^2)a^2})K_0(x_r^\perp m\sqrt{1-(1-\tau'^2)a^2}) \right\} \\
&\propto \int_{-1}^{1} \mathrm{d}\tau \int_{-1}^{1} \mathrm{d}\tau'\, \delta(\tau-\tau')(1-\tau^2)(1-\tau'^2) \\
&\quad \left\{ \int_0^\infty \eta\, \mathrm{d}\eta\, K_0(\eta\sqrt{1-(1-\tau^2)a^2})K_0(\eta\sqrt{1-(1-\tau'^2)a^2}) \right\} \\
&\propto \int_{-1}^{1} \mathrm{d}\tau \int_{-1}^{1} \mathrm{d}\tau'\, \delta(\tau-\tau')\frac{(1-\tau^2)(1-\tau'^2)}{\tau^2-\tau'^2} \ln\left(\frac{1-(1-\tau^2)a^2}{1-(1-\tau'^2)a^2}\right) \\
&\propto \int_{-1}^{1} \mathrm{d}t\, \frac{(1-t^2)^2}{1-(1-t^2)a^2}\,,
\end{aligned} \tag{5.23}$$

where we made use of the dimensionless fraction $a := \frac{\mu}{2m}$. This result is clearly a positive, finite quantity, therefore the wave function is normalizable.

6. ESTIMATES FOR LARGE NUMBER OF PARTICLES

In order to get an idea on bound states for large number of particles, we propose that the mean field theory is a good approximation. We search for the smallest eigenvalue of $\Phi_R(E)$ using the following variational ansatz:

$$|\Omega_0\rangle := \int \frac{[\mathrm{d}q]}{\sqrt{2q_3}\sqrt{(N-2)!}} \left(\prod_{i=1}^{N-2} \frac{[\mathrm{d}p_i]}{\sqrt{2p_{i3}}} u(\mathbf{p_i}) a^\dagger(\mathbf{p_i}) \right) \psi(\mathbf{q}) \chi^\dagger(\mathbf{q}) \, |0\rangle \ . \tag{6.1}$$

Here we have the unknown wave functions u for the particles and ψ for the orthofermion. This ansatz is for the ground state wave function, with this wave function we can take the expectation value of the principal operator $\Phi_R(E)$,

$$\langle \Omega_0 | \Phi_R(E) | \Omega_0 \rangle =: \Omega(E) \,. \tag{6.2}$$

We can then minimize this eigenvalue by choosing u and ψ to find the smallest eigenvalue $\Omega_*(E)$ of $\Phi_R(E)$ and solve $\Omega_*(E) = 0$ for E to get a variational estimate on the bound state energy. This approach works thanks to the flow of eigenvalues of $\Phi_R(E)$ that we discuss in Section 4.4.

In principle, working out the variations with respect to the unknown functions u and ψ leads to variational equations to be solved. At present, we use a simpler approach and proceed by choosing $u(\mathbf{p})$, leaving $\psi(\mathbf{q})$ unknown, with both normalized:

$$u(\mathbf{p}) := \frac{4\alpha^2}{m^2\sqrt{\pi}} p_3 e^{-\alpha^2 p_3^2/m^2} e^{-\alpha^2 p_\perp^2/m^2} \,, \tag{6.3}$$

$$\int \frac{[\mathrm{d}p]}{2p_3} |u(\mathbf{p})|^2 = \int \frac{[\mathrm{d}q]}{2q_3} |\psi(\mathbf{q})|^2 = 1 \,. \tag{6.4}$$

Using Equations (4.15) to (4.18) and (5.2), we take the expectation value of the principal operator under the given variational ansatz $|\Omega_0\rangle$, Equation (6.1). Looking at

Equation (4.18), we note that the operator U_2 contains two pairs of ladder operators, which under this expectation value would come with N^2, whereas U_1, Equation (4.17), contains just one such pair, which would come with N. Therefore, since we consider the large-N limit, we drop terms coming from U_1. We also replace expectation values of functions of H_0 as follows: Given a function $f(H_0)$, we use the approximation

$$\langle f(H_0)\rangle \approx f(N\langle h_0\rangle)\,, \tag{6.5}$$

where

$$\langle h_0\rangle := \int [\mathrm{d}p]\omega(\mathbf{p})\frac{|u(\mathbf{p})|^2}{2p_3}\,. \tag{6.6}$$

We note that this can be improved for example with a cumulant expansion, since both functions have integral representations using the exponential of H_0.

Using these we get in the large-N limit

$$\begin{aligned}\Omega(E) \approx& \frac{\pi}{2}\int \mathrm{d}P\,\mathrm{d}^2\xi\,\frac{|\psi(P,\boldsymbol{\xi})|^2}{2P}\\ &\int_{-1}^{1}\mathrm{d}\tau\ln\left(\frac{2(N\langle h_0\rangle - E)P(1-\tau^2)+[(P^2+\xi^2)(1-\tau^2)+4m^2]}{-\mu^2(1-\tau^2)+4m^2}\right)\\ &-\frac{N^2}{16}\left(\frac{4\alpha^2}{m^2\sqrt{\pi}}\right)^4\int\frac{(\prod_{i=1}^4[\mathrm{d}p_i]e^{-\alpha^2(p_{i_3}^2+p_{i_\perp}^2)/m^2})}{N\langle h_0\rangle - E+[\sum_{i=1}^4\omega_i]}\frac{\psi^*(\mathbf{p_3}+\mathbf{p_4})\psi(\mathbf{p_1}+\mathbf{p_2})}{\sqrt{2(p_{3_3}+p_{4_3})}\sqrt{2(p_{1_3}+p_{2_3})}}\,.\end{aligned} \tag{6.7}$$

We drop the terms in square brackets, which are all positive, leading to

$$\begin{aligned}\Omega(E) >& \frac{\pi}{2}\int \mathrm{d}P\,\mathrm{d}^2\xi\,\frac{|\psi(P,\boldsymbol{\xi})|^2}{2P}\int_{-1}^{1}\mathrm{d}\tau\ln\left(\frac{2(N\langle h_0\rangle - E)P(1-\tau^2)}{-\mu^2(1-\tau^2)+4m^2}\right)\\ &-\frac{N^2\left(\frac{4\alpha^2}{m^2\sqrt{\pi}}\right)^4}{16(N\langle h_0\rangle - E)}\int(\prod_{i=1}^{4}[\mathrm{d}p_i]e^{-\alpha^2(p_{i_3}^2+p_{i_\perp}^2)/m^2})\frac{\psi^*(\mathbf{p_3}+\mathbf{p_4})\psi(\mathbf{p_1}+\mathbf{p_2})}{\sqrt{2(p_{3_3}+p_{4_3})}\sqrt{2(p_{1_3}+p_{2_3})}}\,.\end{aligned} \tag{6.8}$$

For the evaluation of the integral in the second term above, the following coordinate transformations of unit Jacobian are used:

$$\begin{aligned} \mathbf{P} &\coloneqq \mathbf{p_1} + \mathbf{p_2}\,, \\ \mathbf{Q} &\coloneqq \frac{1}{2}(\mathbf{p_1} - \mathbf{p_2})\,. \end{aligned} \tag{6.9}$$

We define

$$I \coloneqq \int \mathrm{d}^3P\,\mathrm{d}^3Q\,\frac{\psi(\mathbf{P})}{\sqrt{2P_3}} e^{-\alpha^2(P^2/2+2Q^2)/m^2}\,, \tag{6.10}$$

where $Q_3 \in (-\frac{1}{2}P_3, \frac{1}{2}P_3)$ is to be taken. With this, we have for the above integral

$$\int (\prod_{i=1}^{4}[\mathrm{d}p_i]e^{-\alpha^2(p_{i_3}^2+p_{i_\perp}^2)/m^2}) \frac{\psi^*(\mathbf{p_3}+\mathbf{p_4})\psi(\mathbf{p_1}+\mathbf{p_2})}{\sqrt{2(p_{3_3}+p_{4_3})}\sqrt{2(p_{1_3}+p_{2_3})}} = I^* I\,. \tag{6.11}$$

In order to bound this integral, observe that

$$\begin{aligned} |I| &\le \int \mathrm{d}^3P\,\mathrm{d}^3Q \left|\frac{\psi(\mathbf{P})}{\sqrt{2P_3}}\right| \left|e^{-\alpha^2(P^2/2+2Q^2)/m^2}\right| \\ &\le \left(\int \mathrm{d}^3P \left|\frac{\psi(\mathbf{P})}{\sqrt{2P_3}}\right|^2\right)^{\frac{1}{2}} \left(\int \mathrm{d}^3P\, e^{-\alpha^2P^2/m^2} \left(\int \mathrm{d}^3Q\, e^{-\alpha^2 2Q^2/m^2}\right)^2\right)^{\frac{1}{2}}\,. \end{aligned} \tag{6.12}$$

Therefore [25],

$$\begin{aligned} I^* I &\le m^9(2\pi)^3 \frac{\sqrt{2}\pi}{64\alpha^9} \int_0^\infty \mathrm{d}x\,\mathrm{erf}^2(x)e^{-2x^2} \\ &< \frac{m^9\pi^{\frac{9}{2}}}{72\alpha^9}\,. \end{aligned} \tag{6.13}$$

Here $\mathrm{erf}(x)$ is the error function,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t\, e^{-t^2}\,. \tag{6.14}$$

Now we work on the first term of Equation (6.8). Observe that

$$\pi \ln\left(\frac{N\langle h_0\rangle - E}{\sqrt{4m^2-\mu^2}}\right) + \frac{\pi}{2}\int \mathrm{d}P\,\mathrm{d}^2\xi\,\frac{|\psi(P,\boldsymbol{\xi})|^2}{2P}\int_{-1}^{1}\mathrm{d}\tau\ln\left(\frac{2P(1-\tau^2)\sqrt{4m^2-\mu^2}}{-\mu^2(1-\tau^2)+4m^2}\right)$$
$$= \frac{\pi}{2}\int \mathrm{d}P\,\mathrm{d}^2\xi\,\frac{|\psi(P,\boldsymbol{\xi})|^2}{2P}\int_{-1}^{1}\mathrm{d}\tau\ln\left(\frac{2(N\langle h_0\rangle - E)P(1-\tau^2)}{-\mu^2(1-\tau^2)+4m^2}\right). \quad (6.15)$$

Here the second term of the left hand side is positive and finite, we drop it in what follows. Combining these results, we get a lower bound for the expectation value of the principal operator,

$$\Omega(E) > \pi\ln\left(\frac{N\langle h_0\rangle - E}{\sqrt{4m^2-\mu^2}}\right) - \frac{2m\pi^{\frac{5}{2}}}{9\alpha}\frac{N^2}{N\langle h_0\rangle - E}. \quad (6.16)$$

Note that the expectation value $\langle h_0\rangle$ for the given trial function $u(\mathbf{p})$ can be calculated:

$$\langle h_0\rangle = m\frac{\sqrt{2\pi}}{8}(\frac{3}{\alpha} + 4\alpha). \quad (6.17)$$

We can minimize the lower bound for Ω by properly adjusting the parameter α. That leads to a complicated equation. We instead consider a simpler lower bound and estimate from below. Assuming $E < 0$ for simplicity, we get

$$\Omega(E) > \pi\ln\left(\frac{Nm\frac{\sqrt{6\pi}}{2} + |E|}{\sqrt{4m^2-\mu^2}}\right) - \frac{2m\pi^{\frac{5}{2}}}{9}\frac{N^2}{Nm\frac{3\sqrt{2\pi}}{8}}. \quad (6.18)$$

We now solve for the zero of this lower bound. The true ground state is bounded from below by this value, thanks to the flow of eigenvalues that we showed in Section 4.4. As a result we find

$$E_{\mathrm{gr}} > -\sqrt{4m^2-\mu^2}\,e^{\frac{8\sqrt{2\pi}}{27}N}. \quad (6.19)$$

6.1. Remarks on the $2+1$ dimensional model

We can follow the same steps of analysis for $2+1$ dimensions. Here, the kinetic term that diverges logarithmically in $3+1$ dimensions, Equation (4.20), is finite. We have instead

$$K(E) = \frac{1}{2}\int \mathrm{d}P\,\mathrm{d}Q\,\mathrm{d}\xi\,\mathrm{d}\eta\,\chi^\dagger(P,\boldsymbol{\xi}) \\ \left([2(H_0-E)+P+\frac{\xi^2}{P}](\frac{P^2}{4}-Q^2)+m^2P+\eta^2P\right)^{-1}\chi(P,\boldsymbol{\xi})\,. \quad (6.20)$$

As before, we focus on the following part:

$$\begin{aligned}&\int \mathrm{d}Q\,\mathrm{d}\eta\left([2(H_0-E)+P+\frac{\xi^2}{P}](\frac{P^2}{4}-Q^2)+m^2P+\eta^2P\right)^{-1}\\ &=\frac{1}{2}\int_{-1}^{1}\mathrm{d}\tau\int\mathrm{d}\eta\left([2(H_0-E)P+P^2+\xi^2]\frac{1-\tau^2}{4}+m^2+\eta^2\right)^{-1},\end{aligned} \quad (6.21)$$

Again we used the coordinate transformation $Q=\frac{P\tau}{2}$. Unlike Equation (4.21), the η integral here is finite.

We can find everything for the principal operator as before now without the need for renormalization. Using the same eigenstate, Equation (5.13), for the principal operator, we can search for the bound state energy within the two-particle sector. This gives the eigenvalue equation

$$\Phi(E)\chi^\dagger(\mu,0)\left|0\right\rangle = (\frac{\Pi_1}{\lambda}-2K(E))\chi^\dagger(\mu,0)\left|0\right\rangle\,, \quad (6.22)$$

where we then set $E=\mu$ to read the eigenvalue of Φ corresponding to the ground state,

$$\Phi(\mu)\chi^\dagger(\mu,0)\left|0\right\rangle = \left(\frac{1}{\lambda}-\frac{1}{2}\int_{-1}^{1}\mathrm{d}\tau\int\mathrm{d}\eta\left(-\mu^2\frac{1-\tau^2}{4}+m^2+\eta^2\right)^{-1}\right)\chi^\dagger(\mu,0)\left|0\right\rangle\,. \quad (6.23)$$

Setting this eigenvalue of Φ to zero then gives a relationship between the coupling constant λ and the binding energy μ. As a result, we find the equation

$$\lambda = \frac{1}{2\pi}\frac{\mu}{\sinh^{-1}\left(\frac{\mu}{\sqrt{4m^2-\mu^2}}\right)}\,. \tag{6.24}$$

In principle, we can replace the coupling constant with this expression, by specifying the two-particle binding energy.

We proceed with the variational calculations. The function $u(\mathbf{p})$ of before, Equation (6.3), normalized for this case is

$$u(\mathbf{p}) := \frac{2\sqrt{2}\alpha^{3/2}}{m^{3/2}(\frac{\pi}{2})^{1/4}}p_3 e^{-\alpha^2 p_3^2/m^2}e^{-\alpha^2 p_\perp^2/m^2}\,. \tag{6.25}$$

Note that in the large-N limit, in addition to the U_1 term, Equation (4.17) that we ignored, we can now ignore the K term, Equation (4.16), as well. This term comes with $\frac{1}{\sqrt{N}}$ and can be ignored compared to the U_2 term, Equation (4.18), which comes with N^2. However, unlike in the case of $3+1$ dimensions there is a separate positive term, $\frac{1}{\lambda}$, that we should keep. This gives the bound for the expectation value, complementary to the $3+1$ case of Equation (6.8), as

$$\Omega(E) > \frac{1}{\lambda} - \frac{N^2\left(\frac{2\sqrt{2}\alpha^{3/2}}{m^{3/2}(\frac{\pi}{2})^{1/4}}\right)^4}{16(N\langle h_0\rangle - E)}\int(\prod_{i=1}^{4}[\mathrm{d}p_i]e^{-\alpha^2(p_{i_3}^2+p_{i_\perp}^2)/m^2})\frac{\psi^*(\mathbf{p_3}+\mathbf{p_4})\psi(\mathbf{p_1}+\mathbf{p_2})}{\sqrt{2(p_{3_2}+p_{4_2})}\sqrt{2(p_{1_2}+p_{2_2})}}\,. \tag{6.26}$$

The integral here is bounded exactly in the same manner as Equation (6.11). Here, the corresponding bound is

$$I^*I < \frac{m^7\pi^{\frac{7}{2}}}{72\alpha^7}\,. \tag{6.27}$$

The expectation value of the free Hamiltonian, complementary to Equation (6.17), is

$$\langle h_0\rangle = m\frac{\sqrt{2\pi}}{8}(\frac{2}{\alpha} + 4\alpha)\,. \tag{6.28}$$

These lead to a lower bound for the lowest eigenvalue,

$$\Omega(E) > \frac{1}{\lambda} - \frac{2\pi^2}{9} \frac{N^2}{Nm\sqrt{\pi} + |E|} \,. \tag{6.29}$$

Again we solve for the zero of this lower bound of the eigenvalue to get an estimate for the ground state energy, as a result we find

$$E_{\mathrm{gr}} > -\frac{2\pi^2}{9} \lambda N^2 \,. \tag{6.30}$$

Due to our limited form of the trial wave function, this cannot be considered a definitive proof, but it gives us some indication that the $2+1$ dimensional theory behaves much better than its higher dimensional counterpart. A more careful analysis should give a better estimate of this lower bound to the ground state energy. We remark that in the exact result for the $1+1$ dimensional *non-relativistic* version [26], the ground state energy goes with

$$-\frac{\lambda^2}{48} N^3 \tag{6.31}$$

to leading order, so the present theory seems to behave even better in $2+1$ dimensions.

7. CONCLUSION

Equation (6.19) gives us some indication that for large number of particles, the ground state energy of our truncated theory may be bounded from below, which makes the theory well-defined as it stands. This result is in accordance with the previously found bound in the $2+1$ dimensional *non-relativistic* system [9].

Nevertheless, the bound that we present is far off from what we would like to find. A true relativistic theory with pair creation processes should not have bound state energies well below zero. That would mean that one can create more particles and by binding them reduce the total energy of the system, and the vacuum would then become unstable. Our crude estimates cannot answer this problem at the moment. The $2+1$ dimensional version of our truncation behaves much better, it may be possible to think of it as a consistent theory by itself describing some intermediate energy phenomena of light bosons. Alternatively, there is a possibility that the terms that we drop in the truncation can be added to this version as perturbations to gain a better approximation of the full version.

In any case, the present truncation can give us some insight about the spectrum of few-body systems with sufficiently weak attractive interactions. We established a resolvent free of divergences after an exact coupling constant renormalization. Thanks to this exact renormalization we showed that the truncated theory is asymptotically free and that the associated principal operator satisfies an operator analog of the Callan-Symanzik equations. We found the ground state wave function of the two-particle sector and provided estimates for the ground state energy of the many particle sectors. Whether any of these can be retained for the full non-truncated theory remains a future challenge.

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